Håkon Ruud

Topological Quantum Computation

Bachelor's project in Mathematics
Supervisor: Markus Szymik
May 2020

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NTNU
Norwegian University of Science and Technology
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Introduction

Quantum computation is a proposed model of computation that applies quantum mechanics to perform information processing and store information in quantum states. Quantum mechanics applies for many different phenomena, with many possible systems in which it is possible to model and manipulate the fundamental quantum information bit - the qubit - and thus there are hypothetically many ways to construct a quantum computer. One proposed way of quantum computation is to use non-abelian anyons to model qubits. These are exotic quasi-particles whose wave functions evolve non-trivially when permuting their positions. This allows for computation with qubits simply by permuting anyons, a process called braiding since their trajectories in spacetime resemble braids. The quantum states associated with the anyons evolve only when the positions of anyons are permuted and do not depend on the paths the anyons take. For this reason this model of quantum computation is called topological quantum computation (TQC). One of the main advantages of TQC is that computations are inherently fault tolerant: there is no noise due to anyons taking strange paths since the quantum evolution is path independent. The goal of this text is to investigate the mathematical framework for this proposed model of quantum computation. The main results are the possible gates that can be applied to two one-qubit topological computers. Given the key properties of the anyons used, Theorem 5.2 states the possible one-qubit gates in an *Ising* computer, and Theorem 7.1 states the possible one-qubit gates in a *Fibonacci* computer.

The text is structured in the following way. Section 1 is a brief introduction to quantum computing in general and introduces qubits and the operations that act on them known as quantum gates. Section 2 introduces anyons and discusses the properties of anyons that make them promising for quantum computation. Here we will also briefly give an overview of all the key points in TQC and illustrate how a one-qubit operation might manifest itself in spacetime (Figure 2.2). In Section 3 we go through Temperley-Lieb-Jones theory. Eventually we will arrive at the Jones category which models TQC. The objects of the category represent anyons and the morphisms represent the physical events that can take place. There are two important physical events that can take place: fusion and splitting, meaning that two anyons can fuse to one anyon and that one anyon can split to two anyons. Fusion and splitting are the central processes that are needed to initialize and measure qubits. Representation theory suffices to model these processes, but the reason to model anyon processes in categorial language is that we also want to braid anyon trajectories, a process that naturally has an interpretation in the Jones category. This allows for finding unitary representations of the braid group, which in fact will be the quantum gates of TQC. Section 4 describes how to construct a qubit and how braiding evolves the state of the qubit. In Section 5 we give an example of a specific topological one-qubit computer using Ising anyons. The section shows how to construct a qubit using these anyons and applies the theory developed in the previous sections to find all possible operations that can be applied to this qubit by braiding alone. In Section 6 we introduce the F-matrix and the R-symbols, which simplifies calculations when one tries to find the braid group representations on a set of anyons. Finally in Section 7 we present a model of TQC using Fibonacci anyons:

a theoretically proposed anyon species for which there is yet no experimental evidence. The Fibonacci anyons are superior to Ising anyons for TQC, since they allow for all possible quantum gates to be implemented.

1 Quantum Computing in General

In the following we go on to explain the mathematics of quantum computing in general. A detailed introduction to this topic can be found in the book by Nielsen and Chuang [1].

1.1 Qubits

Abstractly, a classical computer consists of bit strings and operations that manipulate such strings. A bit takes either the value 0 or 1 and lives in $\{0,1\}$, and we will use the convention that $\mathbb{Z}_2$ refers to this set. A bit string is a sequence of bits, for instance 100101 or 10. The operations on bit strings are then functions on the form

$$f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n. \tag{1}$$

In a quantum computer, the fundamental information quantity is not a bit, but a qubit. Just like a bit, a qubit also has a state. Two possible states are $|0\rangle$ and $|1\rangle$ which correspond to the classical bit states of 0 and 1. However, a qubit may also be in a state that is a linear combination of $|0\rangle$ and $|1\rangle$. Any state

$$|\psi\rangle \in \{\alpha|0\rangle + \beta|1\rangle \ : \ |\alpha|^2 \ + |\beta|^2 = 1\} \tag{2}$$

is also an allowed state of a qubit. The numbers α and β are complex numbers and the state of a qubit is a vector living in $\mathbb{C}^2$. The two states $|0\rangle$ and $|1\rangle$ are called a computational basis for the space $\mathbb{C}^2$, and they are orthogonal and have norm 1. Recall that the norms of quantum states are induced by the inner product associated with the space, since quantum states live in a Hilbert space. Since the allowed states have unit length, computations in quantum computers are unitary transformations

$$U : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}. \tag{3}$$

Although there are infinitely many states of a qubit, there are only two possible states that can be measured. If one were to examine a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, one loses information. Quantum mechanics tells us that during measurement the state collapses, and that the result is probabilistic. The probability amplitudes are the projections of $|\psi\rangle$ onto some basis. Therefore the act of examining a qubit is called a *projective measurement*. For instance, if one wants to examine $|\psi\rangle$, one can choose to project it to the basis $\{|0\rangle, |1\rangle\}$ and one either obtains $|0\rangle$ with probability $|\alpha|^2$ or $|1\rangle$ with probability $|\beta|^2$. One could also choose another basis to project the qubit onto, for instance $\left\{|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$, and one would obtain either $|+\rangle$ with probability $\frac{|\alpha+\beta|^2}{2}$ or $|-\rangle$ with probability $\frac{|\alpha-\beta|^2}{2}$.

1.2 Quantum Gates

A unitary transformation as in (3) is called a *quantum gate*, the analog of classical logic gates. Just as in a classical computer, all operations on a set of qubits can be performed by manipulating them repeatedly with a finite set of gates. For instance, the following are typical one-qubit gates with respect to the computational basis $\{|0\rangle, |1\rangle\}$:

$$\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \sigma_z^{\frac{1}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix} \tag{4}$$

where H is called the *Hadamard gate*. Let $|mk\rangle = |m\rangle \otimes |k\rangle$, then a two qubit gate with respect to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is for instance

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

and is called the *controlled* NOT gate since the state of the first qubit determines whether the NOT gate should act on the second qubit. All gates swapping two states is also called a CNOT, meaning that we also call the following gates CNOTs:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{6}$$

1.3 Universal Gate Sets

We now define the language to describe the computational power of a quantum computer:

Definition 1.1. 1. A **gate set** G is a set of quantum gates acting on a finite number of qubits. The gates $g \in G$ need not to act on the same number of qubits.

2. Suppose there is a system with n qubits and with a gate set G. Having n-qubits, there are 2^n basis states. Moreover, let $\mathbf{1}_k$ be the $k \times k$ identity matrix. An **n-qubit quantum circuit** is a composition of matrices of the form $\mathbf{1}_{2p} \oplus g \oplus \mathbf{1}_{2q}$, where $g \in G$ and p, q are natural numbers.

Example. For instance, a 3-qubit quantum circuit over $\{H, \text{CNOT}\}$ with respect to the basis $\{|abc\rangle\}_{abc\in\{0,1\}}$ is

$$(\text{CNOT} \oplus \mathbf{1}_4)(H \oplus \mathbf{1}_6)(\text{CNOT} \oplus \mathbf{1}_4)(\mathbf{1}_4 \oplus \text{CNOT}) \tag{7}$$

Quantum mechanical operations are in general unitary, but we may restrict ourselves to special unitary matrices. The reasons for this is that quantum states $|\phi\rangle$ have a U(1) part that is not measurable. All states can be written on the form

$$|\phi\rangle = \sum_{j=1} r_j e^{i\theta_j} |\phi_j\rangle = e^{i\theta_1} \left(r_1|\phi_1\rangle + \sum_{j=2} r_j e^{i(\theta_j - \theta_1)} |\phi_j\rangle \right) \tag{8}$$

and since it is only the square of the projection of $|\phi\rangle$ onto some basis state that is measurable, it is impossible to determine the global phase $e^{i\theta_1}$. We can therefor, choose the global phase as we desire. A quantum unitary gate can be written on the form $U = e^{i\theta}V$, and since

$$\det(U) = e^{i\theta}\det(V) = e^{i\theta}e^{i\phi_2} \tag{9}$$

we choose $e^{i\theta} = e^{-i\phi_2}$ such that $\det(U) = 1$. This means that we can restrict our discussion to the special unitary group.

Definition 1.2. A gate set G is said to be **universal** if for all integral n the set Q of all n-qubit quantum circuits is dense in $\mathrm{SU}(2^n)$ up to a global phase, meaning the quantum circuits in Q are allowed to carry an unimportant global phase $e^{i\theta}$.

Theorem 1.1. *The gate set* $\{H, \sigma_z^{\frac{1}{4}}, \mathrm{CNOT}\}$ *is universal.*

1.4 Proof of Theorem 1.1

The particular gate set in Theorem 1.1 is due to Boykin, Mor, Pulver, Roychowdhury and Vatan [2]. However, to prove the theorem one needs some intermediate results which are drawn from other sources as well.

Lemma 1.1. *The one qubit gates* H *and* $\sigma_z^{\frac{1}{4}}$ *generate a dense set in* $\mathrm{SU}(2)$ *up to a global phase.*

This statement appeared in [2] and the proof below is taken from that paper. In the following we will use the **Pauli** matrices that are extensively used in quantum mechanics:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

and define $\hat{n} \cdot \sigma \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, where $\hat{n}$ is a three dimensional vector. In general, an element $U \in \mathrm{SU}(2)$ can be written as

$$U = e^{i\phi_U \hat{n}_U \cdot \sigma}. \tag{11}$$

The key point is that one can think of this as a three dimensional rotation. In the rotation group SO(3), an element can be described by Euler angles as

$$R_{\hat{n}_U}(\theta) = R_z(\alpha)R_y(\beta)R_z(\gamma) \tag{12}$$

and one can rewrite (13) in the similar form

$$U = e^{i\phi_U \hat{n}_U \cdot \sigma} = e^{i\alpha\sigma_z}e^{i\beta\sigma_y}e^{i\gamma\sigma_z} \tag{13}$$

meaning that one can think of elements in SU(2) as two rotations around the z axis and one rotation around the y axis. These directions are arbitrary; one can choose any two directions that are not parallel. This is what we will use to prove that H and $\sigma_z^{1/4}$ form a dense set in SU(2) up to a global phase.

Proof. (Lemma 1.1)
We start by making the following definitions

$$\sigma_z^\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\alpha} \end{pmatrix}, \quad \sigma_x^\alpha = H\sigma_z^\alpha H, \quad \sigma_y^\alpha = \sigma_z^{\frac{1}{2}}\sigma_x^\alpha\sigma_z^{-\frac{1}{2}}, \quad H^\alpha = \sigma_y^{\frac{1}{4}}\sigma_z^\alpha\sigma_y^{-\frac{1}{4}}, \tag{14}$$

and using these we define

$$R_1 = e^{i\lambda\hat{m}\cdot\sigma} = \sigma_z^{-\frac{1}{4}}\sigma_x^{\frac{1}{4}}, \quad R_2 = e^{i\lambda\hat{n}\cdot\sigma} = H^{-\frac{1}{2}}R_1H^{\frac{1}{2}} \tag{15}$$

and using (15) one can calculate λ and $\hat{m}$ and $\hat{n}$. One can then show that λ is irrational [2], and that $\hat{m}$ and $\hat{n}$ are orthogonal. The irrationality of λ can be deduced by showing that $e^{2\pi i\lambda}$ is a root of the irreducible monic polynomial

$$x^4 + x^3 + \frac{1}{4}x^2 + x + 1. \tag{16}$$

This polynomial is not cyclotomic, and thus λ is irrational. Since λ is irrational, one can reach any number in $[0, 2\pi)$ by an integer multiple of λ modulo 2π. Hence, R_1 and R_2 reach all rotations around their respective axes, and every element in SU(2) can be approximated arbitrary close by

$$R_1^k R_2^l R_1^m \tag{17}$$

by integers $k, l, m \in \mathbb{N}$. □

Lemma 1.2. *Any unitary gates can be constructed by a combination of two level gates.*

Proof. The basic idea is that one takes the relevant unitary matrix U and multiplies it from the left with two level unitary gates U_i until the identity is obtained, that is, $U_k \dots U_2U_1U = I$. Since the inverse of unitary operations are given by their adjoints, the unitary gate U can be decomposed as $U = U_1^\dagger U_2^\dagger \dots U_k^\dagger$. We refer to [3] for a detailed proof. □

Lemma 1.3. *Up to a phase,* $\mathrm{SU}(2) \cup \{\mathrm{CNOT}\}$ *is universal.*

Proof. The following proof is taken from the book by Nielsen and Chuang[1, p. 191-193], and is due to [4]. Using Lemma 1.2, we only need to show that single qubit gates and CNOT suffice to construct any two level unitary gate. We do this by using *Gray codes.* Suppose we have a two level gate U acting non-trivially only on the space spanned by the computational basis states $|x\rangle$ and $|y\rangle$ where $x = x_1 \dots x_n$ and $y = y_1 \dots y_n$ are the binary expansions of x and y. Further, let $\tilde{U}$ by the unitary 2×2 submatrix of U acting on $|x\rangle$ and $|y\rangle$. A Gray code connecting x and y is a sequence g of n bit strings where each bit string g_i differs from the adjacent bit string by exactly one bit. This means that $x = g_1$ and $y = g_n$. For instance, a Gray code connecting 1001 and 1110 is

$$1001 \tag{18}$$
$$1000 \tag{19}$$
$$1010 \tag{20}$$
$$1110 \tag{21}$$

To idea is as follows: suppose g_1 and g_2 differ in the jth digit. We then swap the states $|g_1\rangle$ and $|g_2\rangle$ by performing a control bit flip on the jth digit. After this we swap $|g_2\rangle$ and $|g_3\rangle$, and continue to do this procedure until $|g_{n-2}\rangle$ and $|g_{n-1}\rangle$ are swapped. Now, suppose g_{n-2} and g_{n-1} differ in the kth bit. Now apply $\tilde{U}$ on the kth qubit, and undo the first step, that is: swap $|g_{n-1}\rangle$ and $|g_{n-2}\rangle$ and then $|g_{n-2}\rangle$ and $|g_{n-3}\rangle$ until $|g_2\rangle$ and $|g_1\rangle$ are swapped. □

Example. For clarity, we demonstrate the procedure described above on the two level unitary gate

$$U = \begin{pmatrix} a & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & d \end{pmatrix} \tag{22}$$

acting on the computationanl basis $\{|00\rangle\{|01\rangle\{|10\rangle\{|11\rangle\}$. We see that U acts non-trivially on $\{|00\rangle$ and $\{|11\rangle$ and where

$$\tilde{U} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{23}$$

is unitary. The relevant Gray code is

$$00 \quad 01 \quad 11 \tag{24}$$

Now, 00 and 01 differ in the last bit, so we swap the first and second qubit. This is done by applying a control-NOT gate on,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{25}$$

and then we do the second step and apply $\tilde{U}$ to the first qubit, and finally we swap the qubits again. This means that

$$\begin{pmatrix} a & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{26}$$

Proof. (Theorem 1.1) Combining Lemma 1.1 and Lemma 1.3 the result follows immediately. □

2 Anyons

The properties of anyons are the fundamental physical phenomena that enable TQC [5, 6, 7]. Anyons are exotic quasi-particles that differ from bosons and fermions in that they exhibit non-trivial exchange statistics. Recall that the statistics of a particle species is the relation that tells what happens when the positions of two particles are permuted. Let $\psi_B(r_a, r_b)$ be the wave function of two bosons a and b with positions r_a and r_b. Permuting the bosons makes no changes to the wave function, i.e.

$$\psi_B(r_a, r_b) = \psi_B(r_b, r_a). \tag{27}$$

However if $\psi_F(r_a, r_b)$ is the wave function of two fermions, then a permutation results in a sign difference

$$\psi_F(r_a, r_b) = -\psi_F(r_b, r_a). \tag{28}$$

Bosons and fermions are the particles that occurs in nature, and thus their statistics is what one observes in nature as well. However, there are other possible statistics. There is, *inter alia*, experimental evidence for particles that obey other statistics in the fractional quantum Hall effect. Such particles are called *anyons*. Let $\psi_A(r_a, r_b)$ be the wave function of two anyons, then permuting their positions results in

$$\psi_A(r_a, r_b) = e^{i\theta}\psi_A(r_b, r_a) \tag{29}$$

where $e^{i\theta}$ can be *any* phase, hence the name of the particles.
Some theoretical models also predict further possibilities of anyons. If the ground state is degenerate, that is if the state space for the lowest energy is spanned by two or more eigenvectors of the Hamiltonian, then the statistics is described by a matrix. Let $\{\psi_i(r_a, r_b)\}_{i=1}^n$ be a basis for the ground state manifold of two anyons. Also, let σ_{ab} be the operator that permutes the positions of anyon a and b. That is

$$\sigma_{ab}\psi_i(r_a, r_b) = \psi_i(r_b, r_a), \tag{30}$$

then the result of applying σ_{ab} to the wave function results in the statistics

$$\sigma_{ab}\psi_i(r_a, r_b) = \sum_j U_{ij}(\sigma_{ab})\psi_j(r_a, r_b) \tag{31}$$

where $U(\sigma_{ab})$ is allowed to be any unitary $n \times n$ matrix. For a system with three or more anyons these unitary matrices need not commute. That is, for a system with three anyons a, b and c the relation $U(\sigma_{ab})U(\sigma_{bc}) = U(\sigma_{bc})U(\sigma_{ab})$ does not hold in general. In that case the anyons are said to be non-abelian. It is the non-abelian anyons that have applications for quantum computing. For certain non-abelian anyons, the images of the different $U(\sigma)$ are dense in SU(2^n) and thus makes for the possibility of making a universal quantum computer.

2.1 Anyons from Topology

Anyons only live in two spatial dimensions, and the reason for this is that the class of trajectories in $2+1$-spacetime is topologically different from the class of trajectories in $3+1$ space time [5]. Consider Figure 2.1, where anyon a takes either path μ or λ. If there are three spatial dimensions, μ and λ are equivalent paths since each path can smoothly be contracted to a point at a. If there are only two spatial dimensions however, only λ can smoothly be contracted to a point at a since μ winds around b. This distinction is also manifested in that

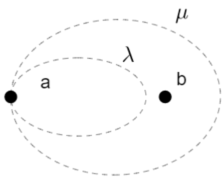


Figure 1: Two anyons a and b. μ and λ are two different possible paths a can take.

the fundamental group of the configuration space is not the same in dimension two and three. Denote the configurations of n particles in m dimensional space $\mathrm{C}_n(\mathbb{R}^m)$, then one can show that

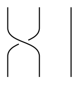
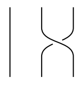
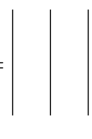
$$\pi_1(\mathrm{C}_n(\mathbb{R}^m)) \cong \begin{cases} 1, & m=1, \\ \mathcal{B}_n, & m=2 \\ \mathcal{S}_n, & m \geq 3. \end{cases} \tag{32}$$

where $\mathcal{S}_n$ is the symmetric group and $\mathcal{B}_n$ is the braid group generated by $\{\sigma_1, \sigma_2, ..., \sigma_{n-1}\}$ subject to the relations

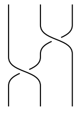
1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$ (far commutativity)
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, 2, ..., n-2$ (braid relation)

Remark. If one also requires that the generators be involutions, i.e. $\sigma_i = \sigma_i^{-1}$, then the resulting group is just the symmetric group.

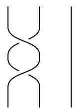
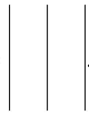
The braid group has a diagrammatic representation that look like braids. We now show this for $\mathcal{B}_3$, but it is easily generalized to any $\mathcal{B}_n$. Let

$\sigma_1 =$  $\sigma_2 =$  $\mathbf{1}_{\mathcal{B}_3} =$  (33)

where $\mathbf{1}_{\mathcal{B}_3}$ is the group unit. Then group multiplication is performed by stacking one diagram on top if the other, that is

$\sigma_2 \sigma_1 =$  (34)

and the inverse is given by undoing the braid by reflecting the braid along the horizontal axis:

$$\mathbf{1}_{\mathcal{B}_3} = \sigma_1\sigma_1^{-1} =$$  $$=$$  . (35)

This diagrammatic representation also captures what happens physically. The trajectories of anyons in $2+1$-spacetime form braids. If one permutes the position of two anyons, then their spacetime trajectories look exactly like the diagrammatic representation of $\sigma_1 \in \mathcal{B}_2$.

The key insight from our discussion is that the statistics of a particle species is a representation of the fundamental group of the configuration space. Thus, the statistics of anyons is a representation of the braid group. It is the fact that the braid group is infinite that allows for a representation of it to be dense in the special unitary matrices.

2.2 Quantum Computation using Anyons

A topological quantum computer is one way to realize a quantum computer. The idea is to construct qubits from anyons. Since braiding evolves the wave functions of the anyons, braiding corresponds to quantum gates. Recall that for non-abelian anyons, the ground state is degenerate and braiding allows for the wave function to evolve non trivially according to

$$\sigma_{ab}\psi_i(r_a, r_b) = \sum_j U_{ij}(\sigma_{ab})\psi_j(r_a, r_b). \tag{36}$$

Although the wave function changes during braiding, the energy associated to the wave function remains the same, and it is the energy that is measurable. This means that if one were to model a qubit by two anyons, one is not immediately able to measure what state the qubit is in. To experimentally distinguish the different states of anyons, one needs additional interactions. This is done by bringing the anyons close, a process called *fusion*. When bringing anyons close, they start acting like a single composite particle and the energy degeneracy lifts. The resulting composite anyon of two anyons that are fused is called the *fusion outcome*. There might be different fusion outcomes, and different fusion outcomes have different energies associated with it. The state of the wave function before fusion that describes the two anyons determines the fusion outcome. Similarly, one can start with a composite anyon and split it to new anyons. *Splitting* is in fact the time reversed process of fusion, and so analogous to fusion outcomes there are also different *splitting outcomes*. For instance, a composite anyon may split to two anyons a and b, or to two other anyons c and d.
There are three steps in TQC. **First**, one initialises the desired qubits. The template of each qubit is a composite anyon. Then one initializes each qubit by splitting them to more anyons. The state of a qubit is determined by a particular splitting outcome. **Second**, one applies gates to the qubits by braiding the anyons. This changes the wave functions of the system and alters the internal state of the anyons. **Third**, one measures the state of the qubits by fusing

the anyons. If the internal state is different after braiding, then the fusion outcome might be different from the composite anyons that one started with. As in the general case for measuring a quantum qubit, the fusion outcome is a projective measurement of the qubit. Since fusion lifts the degeneracy, different fusion outcomes give different energies. We illustrate a one-qubit computation in Figure 2.2.

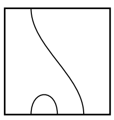
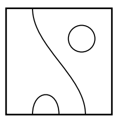
3 Temperley-Lieb-Jones Theories

This section introduces Temperley-Lieb categories, and the theory is covered in [8, 9]. The goal is to obtain a fusion category with objects modeling anyons and morphisms modeling the fusion and splitting processes of anyons. The motivation for constructing these categories is that they are examples of braided fusion categories, in which the morphisms also model braiding of anyons. When we have obtained a fusion category that models these processes, we can then go on to determine the quantum gates that act on the state space of anyons when braiding them.

Definition 3.1. Consider the rectangle $\mathbb{R}\times[0,1]$ with the sets $\{1,2,\dots m\}\times\{0\}$ with $\{1,2,\dots n\}\times\{1\}$ such that $m+n$ is even. A **Temperley-Lieb** diagram D is then a collection of $\frac{m+n}{2}$ arcs in the interior of the rectangle connecting $\{1,2,\dots m\}\times\{0\}$ and $\{1,2,\dots n\}\times\{1\}$ plus any number of closed loops inside the rectangle. All arcs and loops are disjoint and non intersecting. For brevity we call these diagrams TL diagrams.

Definition 3.2. Let $d\in\mathbb{C}$. If two TL diagrams D_1 and D_2 induce the same pairing the $|x|+|y|$ boundary points they are said to be ***d*-isotopic**. If there are n closed loops in a TL diagram, they may be factored out by d^n. I.e. if there are n more closed loops in D_2 then D_1 than $d^nD_1=D_2$.

Example. For instance, the following two diagrams D_1, D_2 are d-isotopic TL diagrams from 3 to 1 points.

$D_1 =$  $D_2 =$  (37)

Remark. There are finitely many TL-diagrams up to d-isotopy from n to m points, and the number of them is given by the Catalan number $c_k=\frac{1}{k+1}\binom{2k}{k}$ where $k=\frac{m+n}{2}$ [8, 10].

Definition 3.3. The **generic Temperley-Lieb category** TL has non-negative integers $n\in\mathbb{N}$ as objects. The morphisms in $\mathrm{Hom}(n,m)$ are given by the $\mathbb{C}$-linear span of all TL-diagrams from n to m points up to d-isotopy. Given two morphisms $f\in\mathrm{Hom}(n,m)$ and $g\in\mathrm{Hom}(m,k)$ the composition $f\circ g\in\mathrm{Hom}(m,k)$ is given by stacking the TL-diagram of f on top of the TL-diagram of g. The identity 1_n is the TL-diagram containing n vertical lines from n to n points.

Proposition 3.1. TL *is a $\mathbb{C}$-linear monoidal category. For two objects n, n' the tensor product is given by $n\otimes n' = n+n'$ and for morphisms $f: n\to m$ and $g: n'\to m'$ define $f\otimes g: n\otimes n'\to m\otimes m'$ by juxtaposition: place the* TL*-diagram of f to the left of the* TL*-diagram of g.*

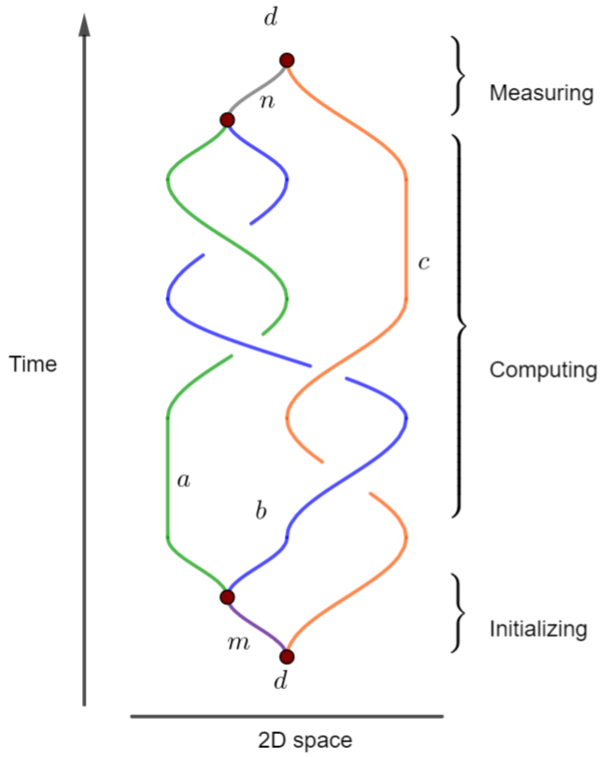
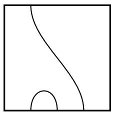
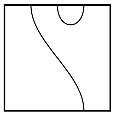
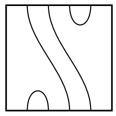


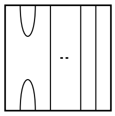
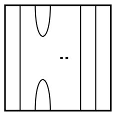
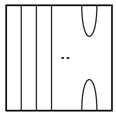
Figure 2: Visualization of computing with one qubit. A qubit is realised using three anyons. A composite anyons d splits to m and c, and m splits to a and b. The three anyons a, b and c are the qubit, and their internal state is defined to be so that if one were to fuse a and b one would obtain m with certainty. This internal state represents the qubit to be the state $|0\rangle$. If on the other hand a and b would fuse to n with certainty, the qubit would have been in the state $|1\rangle$. To perform a one-qubit operation, a, b and c are then braided. This changes the internal state of the anyons so that one is no longer guaranteed to obtain m from a and b. This means that the qubit is in a superposition of $|0\rangle$ and $|1\rangle$. In the last step, a projective measurement is performed onto the basis $\{|0\rangle, |1\rangle\}$, and one obtains n from the fusion of a and b instead of m. This means that when one examined the qubit one measured the qubit to be in the state $|1\rangle$.

Example.

 $\otimes$  $=$  (38)

Definition 3.4. Fix $A \in \mathbb{C}^{\times}$ and set $d = -A^2 - A^{-2}$. Then the **Tempelery Lieb algebra** $\mathrm{TL}_n(A)$ is the endomorphism space $\mathrm{Hom}(n, n) \in \mathrm{TL}$ consisting of the $\mathbb{C}$-linear span of TL-diagrams from n to n points.

The algebra $\mathrm{TL}_n(A)$ is generated by $n-1$ simple diagrams

$u_1 =$  $u_2 =$  $u_{n-1} =$  (39)

and the identity is the same as the identity in TL. One can check that the following relations hold:

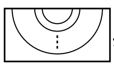
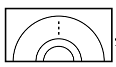
1. $u_i u_j = u_j u_i \quad |i - j| \geq 2$ (far commutativity)
2. $u_i u_{i\pm1} u_i = u_i$ (braid relation)
3. $u_i^2 = d u_i$ (Hecke relation)

which in fact defines the algebra if one does not provide the TL-diagrams [10, p. 12-13].

Remark. We call the second relation in the above definition a braid relation since setting $u_i = A\sigma_i - A^2$ recovers the braid relation in $\mathcal{B}_n$, except that it is set to zero, i.e. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} = 0$.

3.1 Markov Trace and Pairing

The Temperley-Lieb diagrams will eventually be used to model anyon trajectories, and we want to braid those trajectories and find a representation of the braid group. To do this, we will need an inner product, in particular the Markov pairing which is constructed from the Markov trace. In the following, let

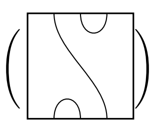
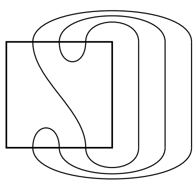
$\cup_n =$ , $\quad \cap_n =$ , (40)

where $\cup_n$ and $\cap_n$ both have n arcs.

Definition 3.5. The **Markov trace** $\mathrm{Tr} : \mathrm{TL}_n(A) \to \mathbb{C}$ is the linear form given by connecting the top and bottom n points in the diagram $D \in \mathrm{TL}_n(A)$ by n disjoint arcs and counting the number of loops which we denote as η. Then $\mathrm{Tr}(D) = d^{\eta}$. That is,

$$\mathrm{Tr}(D) = \cap_n (D \otimes \mathbf{1}_n) \cup_n . \tag{41}$$

Example.

Tr  $=$  $= d^1$ (42)

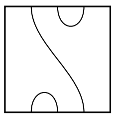
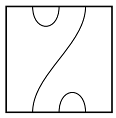
We denote the involution of D as $\bar{D}$ which is given by flipping the diagram through the middle horizontal line, as illustrated in (44). Then we define the following inner product:

Definition 3.6. The **Markov pairing** $\langle \cdot, \cdot \rangle : \mathrm{TL}_n(A) \times \mathrm{TL}_n(A) \to \mathbb{C}$ is given by

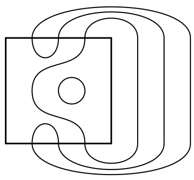
$$\langle D_1, D_2 \rangle = \mathrm{Tr}(\bar{D}_1 D_2) \tag{43}$$

where $\bar{D}_1$ is the involution of D_1.

Example.

$D =$  $\quad \bar{D} =$  (44)

then

$\langle D, D \rangle = \mathrm{Tr}(\bar{D}D) =$  $= d^3$ (45)

The Markov pairing is a sesquilinear inner product, meaning that for all $D_1, D_2, D_3 \in \mathrm{TL}_n(A)$ and for all $a, b \in \mathbb{C}$ the Markov pairing satisfies [10]

1. $\langle D_1 + D_2, D_3 \rangle = \langle D_1, D_3 \rangle + \langle D_2, D_3 \rangle$
2. $\langle aD_1, D_2 \rangle = a\langle D_1, D_2 \rangle$
3. $\langle D_1, D_2 \rangle = \overline{\langle D_2, D_1 \rangle}$

and is non-degenerate, meaning that if $\langle D_1, D_2 \rangle = 0$ for all D_2 then $D_1 = 0$.

3.2 Jones-Wenzl Projectors

We now introduce the Jones-Wenzl projectors that are special morphisms in the Temperley-Lieb categories that will be used to model anyons. A specific Jones-Wenzl projector corresponds to a particular anyon species.

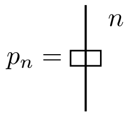
Proposition 3.2. *There exist a unique central idempotent $p_n \in \mathrm{TL}_n(A)$ characterized by*

1. $p_n \neq 0$,
2. $p_n^2 = p_n$,
3. $u_i p_n = p_n u_i = 0$ *for all* $1 \leq i \leq n-1$.

Proof. See [8]. □

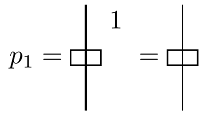
Definition 3.7. The idempotent p_n in Proposition 3.2 is called a **Jones-Wenzl projector**. By the third property we say that the idempotent kills any turnbacks.

Diagrammatically the Jones-Wenzl projectors are denoted

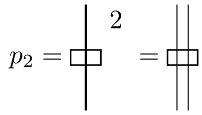


(46)

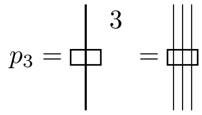
which means that the TL-diagram has n incoming and outgoing strands, meaning we can also write



(47)

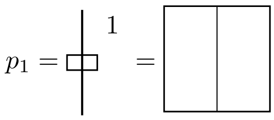


(48)

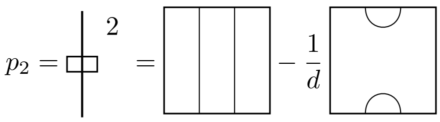


(49)

etc. The first two projectors are



(50)

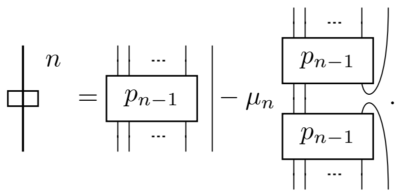


(51)

(52)

Higher order projectors are found by using the recurrence relation given below:

Proposition 3.3. *Let* $\mu_n = \frac{[n-1]_d}{[n]_d}$*, where* $[n]_d = \frac{d^n - d^{-n}}{d - d^{-1}}$*. The Jones-Wenzl projectors satisfy*



(53)

Proof. A proof can be found in [11] where the recurrence relation originally appeared. □

Remark. In general, d need not be invertible, and both $[n]_d$ and μ_n might not be defined if a division by zero occurs. However, we can avoid these issues by choosing A such that d^{-1}, $[n]_d$ and μ_n are defined, which we will do when using them later.

3.3 Temperley-Lieb-Jones Category

The Jones-Wenzl projectors allow us to construct trivalent vertices. The trivalent vertices will be used to model fusion and splitting of anyons, and the different anyons are modeled by Jones-Wenzl projectors: each anyon type corresponds to a particular projector [8, 10, 12]. We now go on to construct a new category from TL, called the *Temperley-Lieb-Jones* category TLJ.

Definition 3.8. The **Temperley-Lieb-Jones category** TLJ has as object sets of Jones-Wenzl projectors. Given $a, b \in \mathrm{TLJ}^0$ a morphism $f \in \mathrm{Hom}(a, b)$ is given by the $\mathbb{C}$-linear span of TL-diagrams connecting the projectors a and b. Composition of morphisms is the same as in TL. The identity morphism on an object $a \in \mathrm{TLJ}^0$ is the object itself since the objects themselves are morphisms. TLJ is also monoidal with a tensor product given by juxtaposition as in the case of TL.

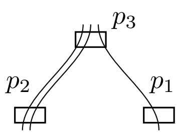
Remark. The generic TLJ is semi-simple since all objects can be written as a finite sum of Jones-Wenzl projectors, which are simple. However, this is not the case when A is a root of unity. For application to quantum computing, A will be a root of unity, and so we need to take a quotient of TLJ to make it semi-simple. Semi-simplicity is needed since we eventually want to construct a matrix algebra of the endomorphism spaces in TLJ.

Anyons can fuse and split, and Jones-Wenzl projectors will represent anyons. Thus, we need to be able to fuse and split the projectors if they are to represent anyons. There are special morphisms in TLJ that do this and they are called trivalent vertices. They model both fusion and splitting, by connecting three projectors. In the following we will call a triplet of natural numbers a, b, c admissible if

1. $a + b + c$ is even, and if
2. $a \leq b + c$, $b \leq a + c$ and $c \leq a + b$.

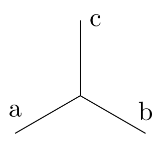
Definition 3.9. Given an admissible triplet a, b, c a **trivalent vertex** is a morphism in $\mathrm{Hom}(p_a \otimes p_b, p_c)$ or $\mathrm{Hom}(p_a, p_c \otimes p_b)$. For the triplet a, b, c, there are unique natural numbers k, l, m given by $a = k + l$, $b = k + m$ and $c = l + m$. Now, take k strands from p_a and attach them to k strands of b. Take the remaining l strands from p_a and the remaining m strands from p_b and attach them to the $l + m$ strands of p_c. We are now left with a morphism in either $\mathrm{Hom}(p_a \otimes p_b, p_c)$ or $\mathrm{Hom}(p_a, p_c \otimes p_b)$.

Example. For the triplet $1, 2, 3$ the trivalent vertex attaching p_1, p_2, p_3 is



. (54)

Although a trivalent looks like (54), we will for short write



. (55)

since the strands are uniquely determined for a given admissible triplet a, b, c.

Proposition 3.4. *The following holds for morphisms in* TLJ *[8]:*

1. $\mathrm{Hom}(p_a, p_b) \cong \begin{cases} \mathbb{C} & \textit{if } a{=}b \\ 0 & \textit{otherwise} \end{cases}$

2. $\mathrm{Hom}(p_a \otimes p_b, p_c) \cong \begin{cases} \mathbb{C} & \textit{if } a{,}b \textit{ and } c \textit{ are admissible} \\ 0 & \textit{otherwise} \end{cases}$

At certain roots of unity, some Jones-Wenzl projectors are no longer defined. This is seen in the recurrence relation for the projectors. If $[n]_d = 0$ then there is a division by zero and p_n is not defined. In that case TLJ is no longer semi-simple. To get around this we simply take a quotient of TLJ which is semi-simple. This is accomplished in the following way:
pick an integer $r \geq 3$ and choose $A \in \{\pm ie^{\pm 2\pi i/4r}\}$. Then the first projector that is not defined is p_r, and one can show that $p_{r-1} = 0$. Taking the quotient by p_{r-1} of TLJ yields the Jones category.

Definition 3.10. Pick an integer k and choose $A \in \{\pm ie^{\pm 2\pi i/4r}\}$ where $r = k+2$. This determines the **Jones category** $\mathrm{V_{A,k}}$. The objects of $\mathrm{V_{A,k}}$ are sets of Jones-Wenzl projectors labeled by a *label set* $L = \{0, 1, ..., k\}$. For two objects $a, b \in \mathrm{V_{A,k}}^0$, $\mathrm{Hom}(a, b)$ is the quotient space $\mathrm{Hom_{TLJ}}(a, b)/\mathrm{I}(a, b)$ where $\mathrm{I}(a, b)$ is the subspace of all homomorphisms in $\mathrm{Hom_{TLJ}}(a, b)$ on the form $g \circ p_{r-1} \circ h$ for $g, h \in \mathrm{Hom_{TLJ}}(a, b)$.

Remark. In the generic TLJ all Jones-Wenzl projectors are defined. Thus before picking any specific k, the projectors p_{r-1} is non-zero. We only choose our parameters after we have taken a quotient by p_{r-1}.

Remark. From the recurrence relation it follows that only the Jones-Wenzl projectors labeled by $L = \{0, 1, ..., k\}$ are present in $\mathrm{V_{A,k}}$.

The Jones category $\mathrm{V_{A,k}}$ is a fusion category and it can model fusion and splitting of anyons. However, there is no *a priori* notion of braiding in the category. It is braiding that evolves the internal quantum states of the anyons, and so to model TQC the morphisms in $\mathrm{V_{A,k}}$ should also describe braids. To make $\mathrm{V_{A,k}}$ a braided fusion category we introduce the Kauffman bracket that bridges the gap by providing a way to represent three dimensional braids as two dimensional diagrams.

3.4 Kauffman Bracket

Anyon trajectories are three dimensional braids since anyons live in 2 + 1-dimensional spacetime. We want to represent braids by TL-diagrams which are two dimensional. To resolve this, one can write three dimensional crossings ╳ as a linear combination of ⌣⌢ and)([13].

Theorem 3.5. *(Kauffman's theorem) There is a unique algebra morphism* $\langle \cdot \rangle : \mathbb{C}[\mathcal{B}_n] \to \mathrm{TL}_n(A)$ *given by the rule*

$$\langle \sigma_i \rangle = A\mathbf{1} + A^{-1} u_i$$

where σ_i *and* u_i *are the* i*th generators of* $\mathcal{B}_n$ *and* $\mathrm{TL}_n(A)$ *respectively and* $\mathbf{1}$ *is the identity in* $\mathrm{TL}_n(A)$*. Additionally,* $\langle \cdot \rangle$ *is surjective.*

Definition 3.11. The algebra morphism $\langle\cdot\rangle$ in Proposition 3.5 is called the **Kauffman bracket**.

Proof. Uniqueness: This is clear since the group algebra $\mathbb{C}[\mathcal{B}_n]$ is generated as an algebra by the generators σ_i of the braid group.
As for existence, we need to check that the images of $\langle\sigma_i\rangle$ of the σ_is are invertible and that they satisfy the braid relations.
Invertibility: Consider

$$(A\mathbf{1} + A^{-1}u_i)(A^{-1}\mathbf{1} + Au_i) = \mathbf{1} + A^{-2}u_i + A^2u_i + u_i^2$$

and insert the relation $u_i^2 = (-A^2 - A^{-2})u_i$. Then the above expression evaluates to $\mathbf{1}$. Hence $(A^{-1}\mathbf{1} + Au_i)$ is an inverse.
Far commutativity: We have

$$\begin{aligned}\langle\sigma_i\rangle\langle\sigma_j\rangle &= (A\mathbf{1} + A^{-1}u_i)(A\mathbf{1} + A^{-1}u_j)\\ &= A^2\mathbf{1} + u_i + u_j + A^{-2}u_iu_j\end{aligned}$$

but $A^{-2}u_iu_j = A^{-2}u_ju_i$ when $|i-j| \geq 2$, so the above evaluates to the same as

$$\langle\sigma_j\rangle\langle\sigma_i\rangle = A^2\mathbf{1} + u_i + u_j + A^{-2}u_ju_i.$$

Braid relation:

$$\begin{aligned}&\langle\sigma_i\rangle\langle\sigma_{i+1}\rangle\langle\sigma_i\rangle = (A\mathbf{1} + A^{-1}u_i)(A\mathbf{1} + A^{-1}u_{i+1})(A\mathbf{1} + A^{-1}u_i)\\ &= A^3\mathbf{1} + 2Au_i + Au_{i+1} + A^{-1}u_i^2 + A^{-1}(u_iu_{i+1} + u_{i+1}u_i) + A^{-3}u_iu_{i+1}u_i\end{aligned}$$

Using $u_i^2 = (-A^2 - A^{-2})u_i$ and $u_iu_{i+1}u_i = u_i$ one obtains

$$\langle\sigma_i\rangle\langle\sigma_{i+1}\rangle\langle\sigma_i\rangle = A^3\mathbf{1} + A(u_i + u_{i+1}) + A^{-1}(u_iu_{i+1} + u_{i+1}u_i)$$

which is invariant by change of indices $i \mapsto i+1$ and $i+1 \mapsto i$, hence the braid relation holds

$$\langle\sigma_i\rangle\langle\sigma_{i+1}\rangle\langle\sigma_i\rangle = \langle\sigma_{i+1}\rangle\langle\sigma_i\rangle\langle\sigma_{i+1}\rangle$$

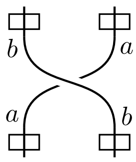
Surjectivity: Each u_i is an image of $\langle\cdot\rangle$ given by

$$\langle A\sigma_i - A^2\mathbf{1}_{B_n}\rangle = u_i$$

where $\mathbf{1}_{B_n}$ is the group unit of $\mathcal{B}_n$. Then surjectivity follows since $\{u_i\}$ generates $\mathrm{TL}_n(A)$. □

3.5 Braids in the Jones Category $\mathrm{V_{A,k}}$

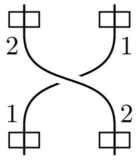
The morphisms in the Jones category $\mathrm{V_{A,k}}$ represent anyon trajectories, but there is yet no interpretation for trajectories that are braided. To braid anyons, we must braid Jones-Wenzl projectors, and we would like to have the diagram



(56)

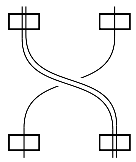
have an interpretation in the space $\mathrm{Hom}(p_a \otimes p_b, p_b \otimes p_a)$. The Kauffman bracket is the bridge that makes (56) an allowed morphism in $\mathrm{V_{A,k}}$. There are a strands that link the top and bottom p_a, and likewise b strands that link the top and bottom p_b, Each of the a strands from p_a cross over each of the b strands of p_b. Thus there are ab crossings in total. Now, resolve all ab crossings through the Kauffman bracket and one is left with a linear combination of 2^{ab} TL-diagrams from $p_a \otimes p_b$ to $p_b \otimes p_a$. We assign this result to be what we mean by (56).

Example. For clarity we show the case for the braid



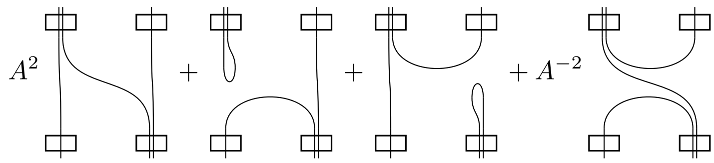
(57)

in $\mathrm{Hom}(p_1 \otimes p_2, p_2 \otimes p_1)$. First we draw all strands:



(58)

Resolving the 4 crossings then yields



(59)

4 Topological Qubits and Gates

We now have all we need to model qubits and quantum gates. The Jones category $\mathrm{V_{A,k}}$ models fusion and splitting of anyons, and the Kauffman bracket provides a way to model braiding of anyons by morphisms in $\mathrm{V_{A,k}}$. Fusion, splitting and braiding are all the events that happen in TQC. Anyons themselves are modeled as Jones-Wenzl projectors being simple objects in the semi-simple category $\mathrm{V_{A,k}}$. This means that objects in $\mathrm{V_{A,k}}$ can be written as a direct sum of Jones-Wenzl projectors. We will use this when we now go on to describe topological qubits. This section is based on [12, 14, 8, 10].

4.1 Fusion Rules

To fully describe an anyon model for quantum computing we need to know the possible fusion outcomes of two anyons that are fused, which is found by studying the underlying topological quantum field theory [6, 7]. The possible fusion outcomes are captured in a *fusion rule* for all anyon pairs. Given two anyons a and b, the result of fusing them is captured in

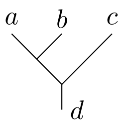
$$a \otimes b \cong \bigoplus_{c} N_{ab}^{c} c \tag{60}$$

where the sum is over all possible anyon types in the given anyon model. The fusion coefficients N_{ab}^{c} are integers telling the number of ways c can be made by fusing a and b. For abelian anyons, there will only be one non-zero fusion coefficient. For non-abelian anyons however, there will be multiple anyons c_i that a and b can fuse to. If this is the case then it is possible to define orthonormal energy eigenstates $|ab; c_i\rangle$ of the Hamiltonian such that if a and b are in the state $|ab; c_i\rangle$ one is guaranteed to obtain c_i when they are brought together. By exchanging the positions of a and b, the state evolves non-trivially to another state in the state space spanned by $|ab; c_i\rangle$. The idea of TQC is to identify this space with qubits, and that operations on qubits are quarried out by exchanging a and b. Since exchanges are braids in $2+1$-spacetime, the space in which anyons lives, quantum gates are thus manifested physically as braids.

4.2 Initializing and Measuring Qubits

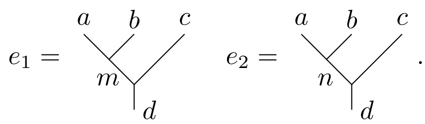
The states $|ab; c_i\rangle$ represent the internal configuration of a and b, such that if they were to be fused, one would obtain c_i. If there are two orthogonal states $|ab; c_1\rangle$ and $|ab; c_2\rangle$, then any normalized linear combination of the two would also be a configuration of a and b, and the weight of each $|ab; c_i\rangle$ would be the probability amplitude to obtain c_i after fusion. Although the two anyons can be in any such state, quantum mechanics tells us that one can only initialize a and b to either $|ab; c_1\rangle$ or $|ab; c_2\rangle$. Similarly, only those two states can be measured. To get a linear combination of these states, one must braid the anyons. We will come back to this in the next subsection.
We will use three anyons to represent a qubit. To initialize the qubit one starts with one composite anyon which is split to three. This is a process in $\mathrm{Hom}(d, a \otimes$

$b \otimes c$) in the Jones category $\mathrm{V}_{\mathrm{A,k}}$, and we represent it diagrammatically as



$$. \tag{61}$$

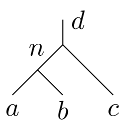
where we recall that the process goes from bottom to top. The space $\mathrm{Hom}(d, a \otimes b \otimes c)$ is isomorphic to $\mathbb{C}^2$ since there are only two fusion outcomes of a and b, that is c_1 and c_2. Thus this space allows for representing a qubit. If the anyon model has the fusion rules $a \otimes b = m \oplus n$, $m \otimes c = d$ and $n \otimes c = d$, then there are two allowed processes



$$. \tag{62}$$

which are orthogonal as vectors. Strictly speaking, any linear combination of them is also allowed, but it is only e_1 and e_2 one is able to measure by the postulates of quantum mechanics. Without loss of generality we may assume e_1 and e_2 to be normalized, since one can always multiply by some constant to make them have unit norm. If the composite anyon splits like in e_1, we identify that process as the initialization of the qubit in the state $|0\rangle$ and likewise e_2 as the initialization of the qubit in the state $|1\rangle$. For this reason we assign $|0\rangle = e_1$ and $|1\rangle = e_2$.

To measure the state of the qubit, a, b and c are brought together and fused. Fusion is the reversed process of splitting, hence diagrammatically one just flips the initialization diagrams around the horizontal line to get the diagram of measurement. For instance, measuring $|1\rangle$ means that the fusion process

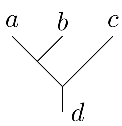


(63)

occurred when bringing the anyons together.

4.3 Quantum Gates and Braids

To obtain the possible gates that can be applied to topological qubits, one must find out the effect of braiding the anyons that constitute the qubit. From Section 3.4 we know that braiding the strands of a, b and c in



(64)

has an interpretation in $\mathrm{V}_{\mathrm{A,k}}$ as a linear combination of e_1 and e_2. Thus, we can define a representation of the braid group $\rho : \sigma \in \mathcal{B}_n \mapsto \rho(\sigma) \in \mathrm{SU}(2^n)$ for a system with n anyons. In the one-qubit case, e_1 and e_2 is a basis for the space. The amplitude of measuring e_k after braiding the i and $i+1$th strand of a qubit in the state e_j is then given by $\langle e_k, \sigma_i e_j \rangle$ where the innerproduct is the Markov pairing defined in Section 3. For this purpose we define ρ on a one-qubit computer by

$$\rho(\sigma_i) = \begin{pmatrix} \langle e_1, \sigma_i e_1 \rangle & \langle e_1, \sigma_i e_2 \rangle \\ \langle e_2, \sigma_i e_1 \rangle & \langle e_2, \sigma_i e_2 \rangle \end{pmatrix} \tag{65}$$

in the basis $\{e_1, e_2\}$. All one-qubit operations are then a finite combination of $\rho(\sigma_1)$ and $\rho(\sigma_2)$. We will refer to the braid group representation on the form in (65) as the **Jones representation** of the braid group.

Remark. For quantum computation to be useful one needs many qubits and two-qubit operations that can act on any pair of qubits. For an n-qubit system, align the qubits horizontally, and a two-qubit gate is given by braiding the strands from each of the qubits. For instance, if each qubit consists of three anyons, the two-qubit representation $\rho(\sigma_i)$ is a representation of $\mathcal{B}_6$.

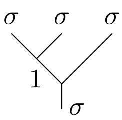
5 Ising Model

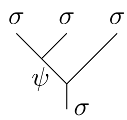
Let's set $k = 2$ in $\mathrm{V_{A,k}}$. Then $A = ie^{-2\pi i/16}, d = \sqrt{2}$ and the Jones-Wenzl projectors are $\{p_0, p_1, p_2\}$ in $\mathrm{V_{A,k}}$. This defines the *Ising model.* The Jones-Wenzl projectors correspond to the anyons in the set $\{1, \sigma, \psi\}$: 1 is the vacuum, σ is the Ising anyon and ψ is a Majorana fermion. We will use these labels interchangeably with the projectors. The fusion rules are the trivial rules and

$$\sigma \otimes \sigma = 1 \oplus \psi \quad \psi \otimes \sigma = \sigma, \tag{66}$$

where the non-abelian property is encoded in the first relation. The goal of this example is to describe a one-qubit system and the gates that can be applied to it by braiding. First we want to define the two states that will constitute a basis for the qubit. To get started, we must provide a computational basis for our qubit.

Proposition 5.1. *The states*

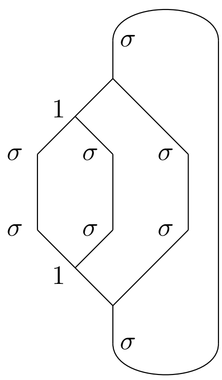
$e_1 = \frac{1}{\sqrt{2}}$ 

, $e_2 =$ 

(67)

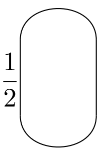
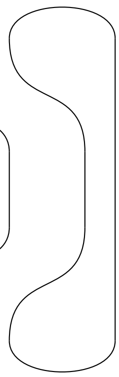
constitute an orthonormal basis for the one-qubit space $\mathbb{C}^2$.

Proof. We must check that $\langle e_i, e_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. First,

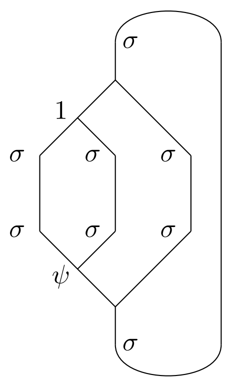
$\langle e_1, e_1 \rangle = \frac{1}{2}$ 

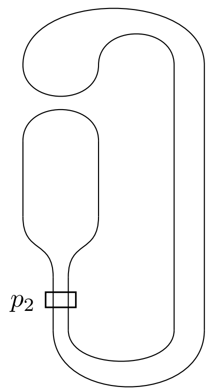
(68)

which reduces to

$\langle e_1, e_1 \rangle = \frac{1}{2}$   $= \frac{1}{2}d^2 = 1$ (69)

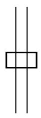
after inserting the respective Jones-Wenzl projectors. Similarly, we find

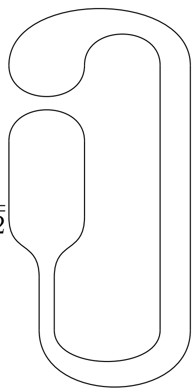
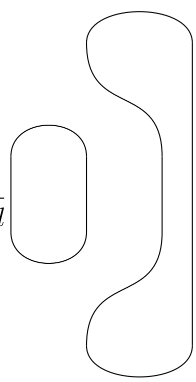
$$\langle e_1, e_2 \rangle = \frac{1}{\sqrt{2}}$$ 

$$= \frac{1}{\sqrt{2}}$$ 

(70)

where we have omitted drawing p_0 and p_1 since these are trivial. After expanding

$p_2 =$  in TL-diagrams we obtain

$$\langle e_1, e_2 \rangle = \frac{1}{\sqrt{2}}$$  $$- \frac{1}{\sqrt{2}d}$$  $$= \frac{d}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{d^2}{d} = 0. \quad (71)$$

By symmetry this also implies that $\langle e_1, e_2 \rangle = 0$. Finally, if one computes $\langle e_2, e_2 \rangle$ one finds that this is 1. $\square$

Having an orthonormal basis for $\mathbb{C}^2$ that can be identified with the basis states of a qubit, we are now interested in the quantum gates that can be applied to it.

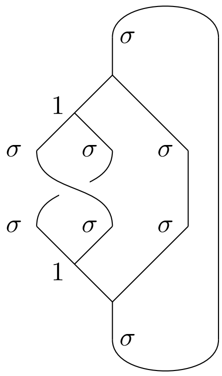
Theorem 5.2. *The one-qubit operations with respect to* $\{e_1, e_2\}$ *that can be applied by braiding the Ising anyons is given by the Jones representation* $\rho(\sigma_1), \rho(\sigma_2) \in \mathcal{B}_3$*. They are:*

$$\rho(\sigma_1) = e^{\frac{i\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}, \quad \rho(\sigma_2) = \frac{e^{\frac{i\pi}{8}}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{72}$$

The gate $\rho(\sigma_1)$ is the $\sigma_z^{\frac{1}{4}}$ gate up to an overall phase of $e^{\frac{i\pi}{8}}$ and the second gate implements the NOT-gate up to a phase by $\rho(\sigma_2)^2$.

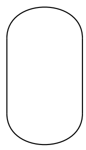
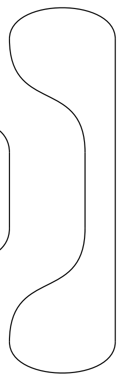
Proof. We show how to obtain the first matrix entry for $\rho(\sigma_1)$. The other entries and $\rho(\sigma_2)$ are found similarly. The amplitude of applying $\rho(\sigma_1)$ to e_1 and obtain e_1 is given by

$$\langle e_2, \sigma_1 e_1 \rangle = \frac{1}{2} \tag{73}$$

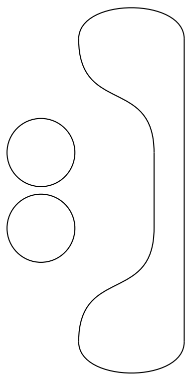


Resolving the braid through the Kauffman bracket yields

$$\langle e_2, \sigma_1 e_1 \rangle = \frac{A}{2} \tag{74}$$

$$+ \frac{A^{-1}}{2}$$



$$= \frac{Ad^2}{2} + \frac{A^{-1}d^3}{2} = e^{\frac{i\pi}{8}}$$

□

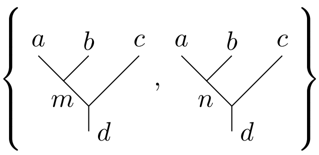
Although the Ising model implements the NOT-gate, it is not universal. One can show that the images of braiding in the Ising model is isomorphic to the Clifford group [15]. The simplest anyons that allows for universal quantum computations are the Fibonacci anyons presented in the last section.

6 Change of Basis Matrix and Braiding Eigenvalues

The theory developed up to now is sufficient for determining the gates that can be applied to a qubit built from anyons. However, there are two important concepts that simplify calculations. These are called the F-matrix and the R-symbols. For a more detailed treatment of the F-matrix and the R-symbols, see [8].

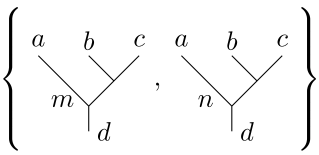
6.1 F-Matrix and Change of Basis

In the previous sections we used



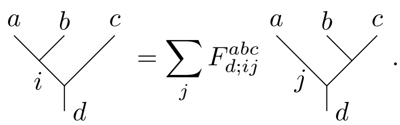
(75)

as a basis for $\mathrm{Hom}(a \otimes b \otimes c, d) \cong \mathbb{C}^2$, where a and b fuse first. But this choice is arbitrary. One could also choose to fuse b and c first, and instead use



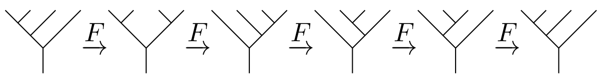
(76)

as a basis. Since the Jones category $\mathrm{V_{A,k}}$ is linear, they must be related by a linear transformation. Let F_d^{abc} be the change of basis matrix given by



(77)

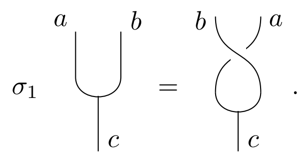
We refer to this matrix as the *F-matrix*. Since the F-matrix relates the fusion process of $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$ this matrix must be an associator of the Jones category which is monoidal, and hence, it must satisfy the pentagon axiom (125). Schematically, this means that the F-matrix satisfies the following cyclic relation:



(78)

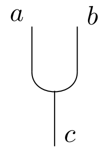
6.2 R-Symbols and Braiding Eigenvalues

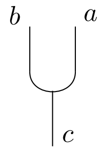
Consider $\mathrm{Hom}(a \otimes b, c)$ and let $\sigma_1 \in \mathcal{B}_2$ act on the trajectories of a and b by braiding them. That is,



(79)

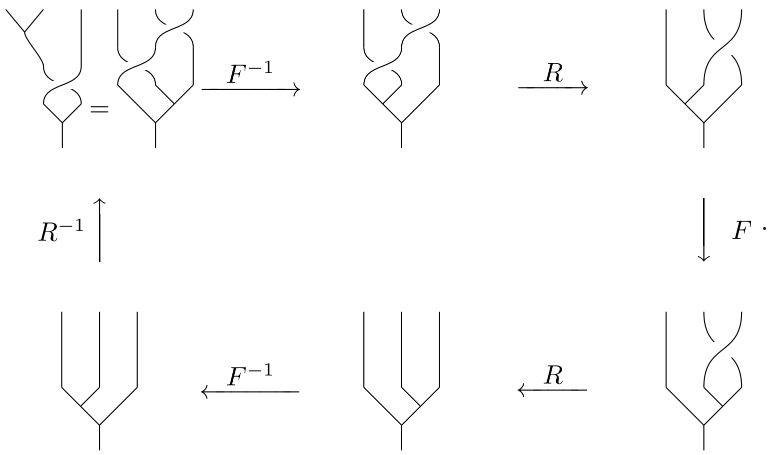
Since $\mathrm{Hom}(a \otimes b, c) \cong \mathbb{C}$ braiding can only be a change by a scalar, and so we write

σ_1 

$= R_c^{ab}$ 

. (80)

We refer to the braiding eigenvalue R_c^{ab} as an R-symbol. The scalar R_c^{ab} cannot be any scalar, it must by a phase $e^{i\theta_c^{ab}}$ since braiding is a unitary operation. Similarly to the F-matrix, the R-symbols follow a specific consistency rule, called the hexagon identity [8]. This requirement is rooted in that the Jones category is a braided category (consult the Appendix for a definition). Schematically, the requirement is that the R-symbols is consistent with the following:



(81)

There is a general formula for the R-symbols, given by

$$R_c^{ab} = (-1)^{\frac{a+b-c}{2}} A^{-\frac{c(c-2)-a(a-2)-b(b-2)}{2}}. \tag{82}$$

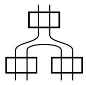
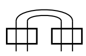
and we show this for the special cases R_0^{22} and R_2^{22} in the Fibonacci model in the next section.

7 Fibonacci Model

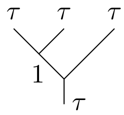
In the following we present the Fibonacci model. The goal is to build a one qubit system and find the gates that can be applied to it by braiding. The Fibonacci model has two anyons, the Fibonacci anyon τ and the vacuum 1. The Jones-Wenzl projectors that correspond to them are p_2 and p_0 respectively in the Jones category $\mathrm{V_{A,k}}$ where $A = ie^{\frac{i\pi}{5}}$ and $k = 2$. This sets $d = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The only non-trivial fusion rule is $\tau \otimes \tau = 1 \oplus \tau$.The naming comes from the amusing property that repeated fusion yields the Fibonacci sequence:

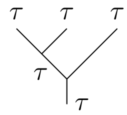
$$
\begin{aligned}
\tau \otimes \tau &= 1 \oplus \tau \\
\tau \otimes \tau \otimes \tau &= 1 \oplus 2\tau \\
\tau \otimes \tau \otimes \tau \otimes \tau &= 2 \oplus 3\tau \\
\tau \otimes \tau \otimes \tau \otimes \tau \otimes \tau &= 3 \oplus 5\tau \\
\tau \otimes \tau \otimes \tau \otimes \tau \otimes \tau \otimes \tau &= 5 \oplus 8\tau
\end{aligned} \tag{83}
$$

It is easily seen that two τ can fuse to both 1 and τ by seeing observing that $(2, 2, 0)$ and $(2, 2, 2)$ are admissible triplets for a trivalent vertex. To have a better feel for the calculations below, this is how it looks like when drawn:


$\in \mathrm{Hom}(\tau \otimes \tau, \tau)$,

$\in \mathrm{Hom}(\tau \otimes \tau, 1)$ (84)

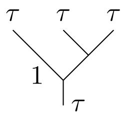
Our qubit is constructed from the two states

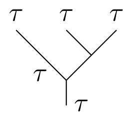
$e_1 = \phi^{-1}$


, $e_2 = \phi^{\frac{3}{2}}$


. (85)

One can check that $\langle e_i, e_j \rangle = \delta_{ij}$, where δ_{ij} is the usual Kronecker delta function. Let's also define another basis

$\tilde{e}_1 = \phi^{-1}$


, $\tilde{e}_2 = \phi^{\frac{3}{2}}$


(86)

We wish to find the Jones representation with respect to e_1 and e_2. We do this by acquiring the F-matrix and R-symbols.

7.1 F-Matrix

Lemma 7.1. *The F-matrix that relates $\{e_1, e_2\}$ with $\{\tilde{e}_1, \tilde{e}_2\}$ is*

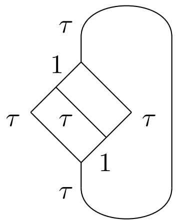
$$
F = \begin{pmatrix} \phi^{-1} & \phi^{-\frac{1}{2}} \\ \phi^{-\frac{1}{2}} & -\phi^{-1} \end{pmatrix}. \tag{87}
$$

Proof. The F-matrix is deduced from the property that

$$
e_1 = F_{11}\tilde{e_1} + F_{12}\tilde{e_2} \tag{88}
$$

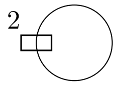
$$
e_2 = F_{21}\tilde{e_1} + F_{22}\tilde{e_2}. \tag{89}
$$

Using the orthogonality relation $\langle e_i, e_j \rangle = \delta_{ij}$ this means that $F_{ij} = \frac{1}{\langle e_i, \tilde{e_j} \rangle}$. We now calculate these matrix elements. First, let's calculate F_{11}.$\langle e_1, \tilde{e_1} \rangle^{-1}$ is ϕ^{-2} times



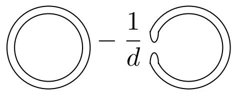
(90)

which after inserting the projectors reduces to



(91)

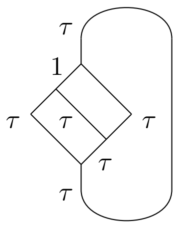
and expanding p_2 yields



(92)

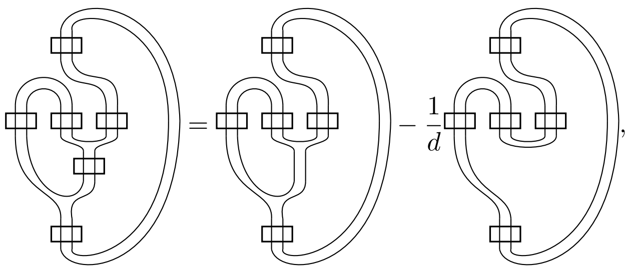
which is $d^2 - 1$. Since d is the golden ratio ϕ - and the golden ratio is defined to satisfy $\phi^2 - 1 = \phi$, this is also ϕ. Thus, $F_{11} = \phi^{-1}$.

Now, let's calculate F_{12} and F_{21} which must be equal by symmetry. The inner-product $\langle e_1, \tilde{e_2} \rangle$ is $\phi^{\frac{1}{2}}$ times



(93)

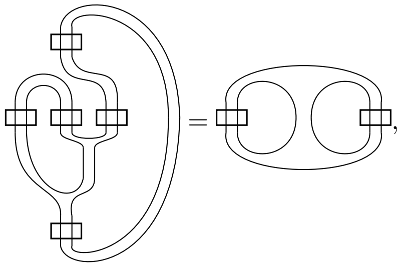
which can be written as



(94)

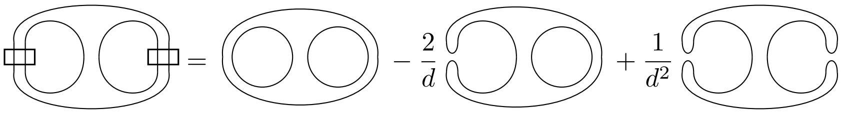
where the second term is the same as (92) since for projectors $p_n^k = p_n$ for all

$k \in \mathbb{N}$ holds. The first term can be reshaped to



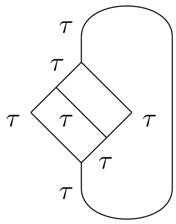
(95)

and expanding the projectors then yields



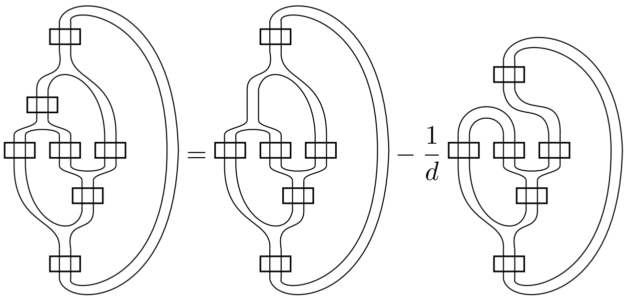
(96)

which evaluates to $d^3 - 2d + \frac{1}{d} = \phi$. Combining this with the above results one obtains $F_{12} = F_{21} = \phi^{-\frac{1}{2}}$.
Lastly, $\langle e_2, \tilde{e_2} \rangle$ is ϕ^3 times



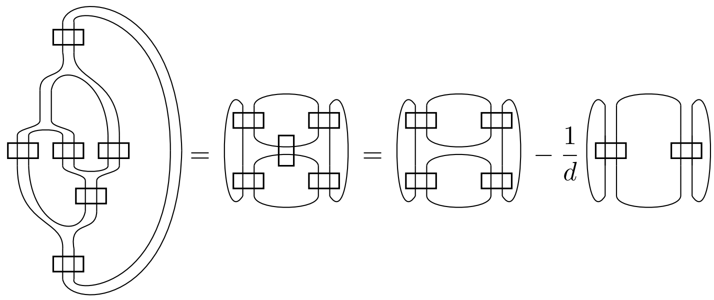
(97)

which is



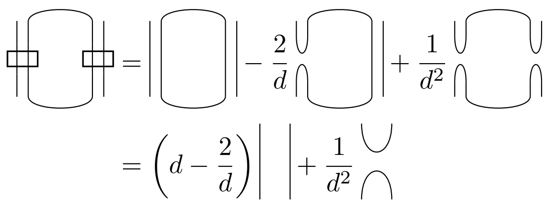
(98)

where the second term were found in the last steps to be 1. Further, we find by stretching that



(99)

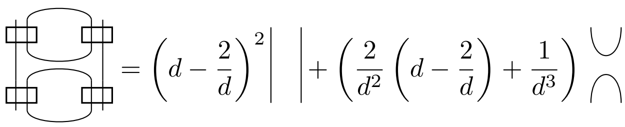
Since



(100)

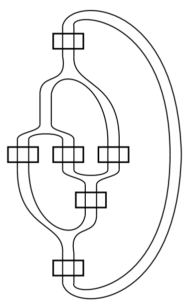
(101)

it follows that



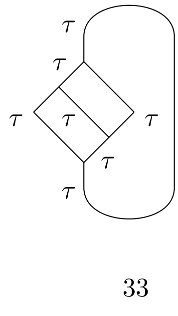
(102)

Inserting (100) and (102) into (99) then yields



$$= d^2\left(d-\frac{2}{d}\right)^2 + d\left(\frac{2}{d^2}\left(d-\frac{2}{d}\right)+\frac{1}{d^3}\right) - d\left(d-\frac{2}{d}\right) - \frac{1}{d^2} \tag{103}$$

which for $d = \phi$ evaluates to $\phi - 2 + \phi^{-1}$. Hence



$$= \phi - 2 \tag{104}$$

and

$$F_{22} = \frac{1}{\phi^3(\phi - 2)} = -\phi^{-1} \tag{105}$$

□

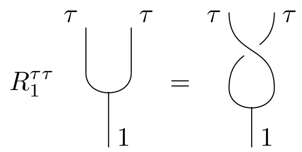
7.2 Braiding Eigenvalues

The two braiding eigenvalues we wish to find is $R_1^{\tau\tau}$ and $R_\tau^{\tau\tau}$.

Lemma 7.2. *The braiding eigenvalues of two Fibonacci anyons are*

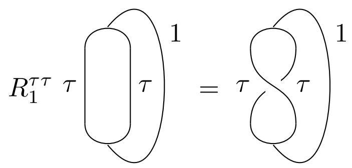
$$R_1^{\tau\tau} = A^{-8}, \quad R_\tau^{\tau\tau} = -A^{-4}. \tag{106}$$

Proof. Using the defining property that



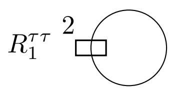
(107)

and taking the inner product with itself we find



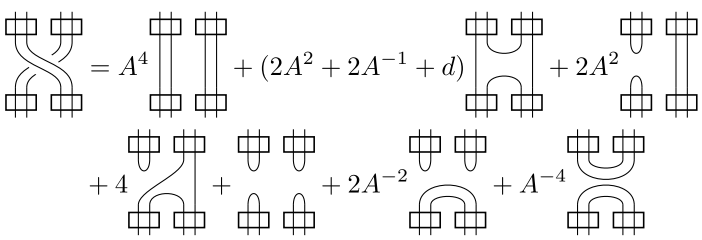
(108)

where the left hand side is is just



(109)

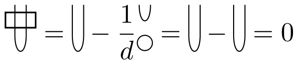
and a loop with a projector p_2 were previously found to be $d^2 - 1$, which is ϕ. The right hand side has a braid. Since each projector has two strands, there are four crossings to be resolved. Doing this one finds



(110)

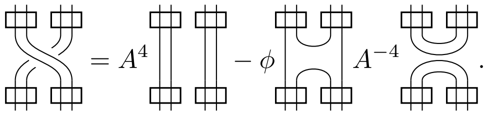
(111)

if one treats the mirror diagrams through the vertical and horizontal lines as the same diagram. We do this since mirror diagrams count the same in the following calculations. Using $d = \phi$ and



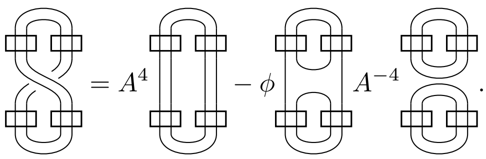
(112)

one is left with



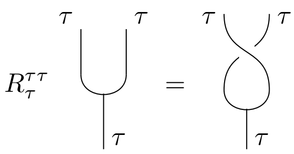
(113)

Hence the right hand side in (116) is



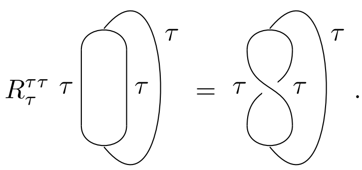
(114)

which is easily shown to be ϕA^{-8}. Hence $R_1^{\tau\tau} = A^{-8}$. The second braiding eigenvalue is found similarly by taking the inner product of the defining relation



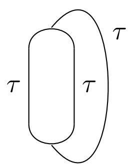
(115)

with itself, that is

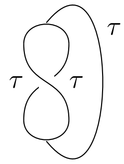


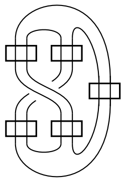
(116)

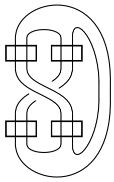
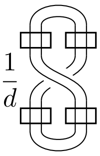
First,

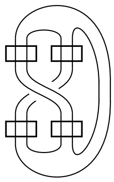


is actually identical to (93) which was found to be ϕ^{-1}. The right hand side is

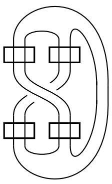
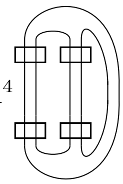
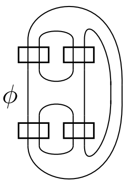
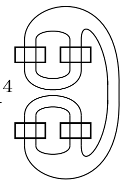


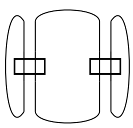
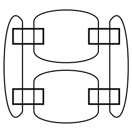
$=$

(117)

$=$

$- \frac{1}{d}$

(118)

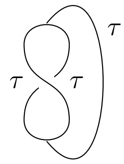
$=$

$- A^{-8}$ (119)

and using (114) we find


$= A^4$

$- \phi$

$+ A^4$

(120)

$= (A^4 + A^{-4})$

$- \phi$

(121)

which is easily shown to be -1. Hence



$$= -\frac{1}{d}A^{-4} \tag{122}$$

and $R_{\tau}^{\tau\tau} = -A^{-4}$. □

Having both the change of basis matrix and the braiding eigenvalues we can then easily calculate the Jones representation.

Theorem 7.1. *The Jones representation in the one-qubit Fibonacci model with respect to* $\{e_1, e_2\}$ *is*

$$\rho(\sigma_1) = \begin{pmatrix} e^{\frac{-4\pi i}{5}} & 0 \\ 0 & -e^{-\frac{\pi i}{5}} \end{pmatrix}, \quad \rho(\sigma_2) = \begin{pmatrix} \phi^{-1}e^{\frac{4\pi i}{5}} & -\phi^{-\frac{1}{2}}e^{\frac{2\pi i}{5}} \\ -\phi^{-\frac{1}{2}}e^{\frac{2\pi i}{5}} & -\phi^{-1} \end{pmatrix} \tag{123}$$

Proof. From the pentagon and hexagon identities the Jones representation must satisfy

$$\rho(\sigma_1) = \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix}, \quad \rho(\sigma_2) = F \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix} F^{-1}. \tag{124}$$

Inserting the F-matrix and R-symbols and substituting $A = ie^{\frac{i\pi}{5}}$ gives the result. □

It can be shown that the images of the braid group representations of the Fibonacci anyons are dense in $\mathrm{SU}(2^n)$ for n anyons, see [10] for details.

Appendix

In the following we define some notions in category theory that are used in this text. The definitions can be found in [16].

Definition 7.1. Let Λ^0 be the set of objects of a category Λ. The category Λ is said to be $\mathbb{F}$**-linear** if for any $x, y \in \Lambda^0$ the morphism set $\mathrm{Hom}(x, y)$ is a $\mathbb{F}$-vector space and if the composition of morphisms is bilinear.

Remark. Endomorphism spaces of $\mathbb{F}$-linear categories are $\mathbb{F}$-algebras with multiplication given by composition of morphisms.

Definition 7.2. A **monoidal** category Λ is equipped with the following structure:

1. a bifunctor $\otimes : \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{\Lambda}$ called the *tensor product.*
2. a object $\mathbf{1}$ called the *unit object*
3. a natural isomorphism α with components $\alpha_{xyz} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$ called the *associator.*
4. a natural isomorphism λ with components $\lambda_x : \mathbf{1} \otimes x \cong x$ called the *left unitor.*
5. a natural isomorphism ρ with components $\rho_x : x \otimes \mathbf{1} \cong x$ called the *right unitor.*

such that $\forall x, y, z, w \in \lambda$ the *pentagon identity*

$$\begin{array}{ccccc} ((w \otimes x) \otimes y) \otimes z & \xrightarrow{\alpha_{w,x,y} \otimes 1_z} & (w \otimes (x \otimes y)) \otimes z & \xrightarrow{\alpha_{w,x \otimes y,z}} & w \otimes ((x \otimes y) \otimes z) \\ \downarrow{\scriptstyle \alpha_{w \otimes x,y,z}} & & & & \downarrow{\scriptstyle 1_w \otimes \alpha_{x,y,z}} \\ (w \otimes x) \otimes (y \otimes z) & & \xrightarrow[\alpha_{w,x,y \otimes z}]{} & & w \otimes (x \otimes (y \otimes z)) \end{array} \tag{125}$$

and the *triangle identity*

$$\begin{array}{ccc} (x \otimes \mathbf{1}) \otimes y & \xrightarrow{\alpha_{x\mathbf{1}y}} & x \otimes (\mathbf{1} \otimes y) \\ & {\scriptstyle \rho_x \otimes \mathbf{1}_y} \searrow \quad \swarrow {\scriptstyle \mathbf{1}_x \otimes \lambda_y} & \\ & x \otimes y & \end{array}$$

both commute.

Definition 7.3. An object a in a category λ with a zero object is called **simple** if there are precisely two quotient objects of a: a and 0.

Definition 7.4. A monoidal linear category is called **semi-simple** if there is a collection of simple objects such that any object is a direct sum of finitely many simple objects.

Definition 7.5. A **braided monoidal** category is a monoidal category C equipped with a natrual isomorphism

$$B_{x,y} : x \otimes y \to y \otimes x$$

called the **braiding** such that the two **hegaxon** identities

$$\begin{array}{ccccc}
(x \otimes y) \otimes z \overset{a_{x,y,z}}{\to} & & x \otimes (y \otimes z) \overset{B_{x,y\otimes z}}{\to} & & (y \otimes z) \otimes x \\
\downarrow^{B_{x,y}\otimes Id} & & & & \downarrow^{a_{y,z,x}} \\
(y \otimes x) \otimes z \overset{a_{y,x,z}}{\to} & & y \otimes (x \otimes z) \overset{Id\otimes B_{x,z}}{\to} & & y \otimes (z \otimes x)
\end{array}$$

and

$$\begin{array}{ccccc}
x \otimes (y \otimes z) \overset{a^{-1}_{x,y,z}}{\to} & & (x \otimes y) \otimes z \overset{B_{x\otimes y,z}}{\to} & & z \otimes (x \otimes y) \\
\downarrow^{Id\otimes B_{y,z}} & & & & \downarrow^{a^{-1}_{z,x,y}} \\
x \otimes (z \otimes y) \overset{a^{-1}_{x,z,y}}{\to} & & (x \otimes z) \otimes y \overset{B_{x,z}\otimes Id}{\to} & & (z \otimes x) \otimes y
\end{array}$$

commute for all objects involved. Here $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the components of the associator of $C^{\otimes}$.

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