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The Riemann hypothesis, the Lindelöf hypothesis and the density hypothesis - consequences and relations

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Abstract

In this paper, we will discuss some important properties of the Riemann zeta function. We will discuss the famous Riemann hypothesis, as well as the Lindelöf hypothesis and the density hypothesis, and the connections between these. We begin by proving the prime number theorem and seeing how it is related to the zeta function, and then moving on to linking these hypotheses together.

1. Introduction

Our main object of study in this thesis will be the Riemann zeta function and the zeros of it. The Riemann Zeta-function is defined as

$$\zeta(s) = \zeta(\sigma + it) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $s \in \{s \in \mathbb{C} : \sigma = \Re(s) > 1\}$. Although named after Riemann, the zeta function was first studied by Leonard Euler, who established the product representation:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{2}$$

for $\sigma > 1$ and p runs over all the prime numbers. This product, since named the Euler product, proves to be important for a number of reasons. Firstly, it shows us a clear relationship between the primes and the zeta function, which we will spend most of our time discussing. Secondly, it follows immediately that $\zeta(s) \neq 0$ when $\sigma > 1$;

Consider $\zeta(s) \prod_p (1 - p^{-s}) = 1$. We know that $\sum p^{-s}$ converges absolutely, and hence $\prod_p (1 - p^{-s})$ converges absolutely, thus it is finite. Therefore, $\zeta(s)$ cannot be zero (as $0 \cdot c \neq 1$ for any finite number c).

Bernhard Riemann showed that this function has a meromorphic continuation to the complex plane with a simple pole at $s = 1$. He proved much more about the function, including the following functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s) \tag{3}$$

The gamma function has simple poles for non-negative integers, which cancel out the zeros of the sin function. Further, we know of the gamma function that it is never equal to zero, and therefore does not give rise to any other zeros of zeta. Altogether, the functional equation gives us $\zeta(s) = 0$ for -2, -4, -6,..., These points are called the *trivial* zeros of the zeta function. More importantly, it follows that all non-trivial zeros are symmetric about the *critical strip*, $\sigma = 1/2$. Since we know there are no zeros in the half plane $\sigma > 1$, this means there are no non-trivial zeros in the half plane $\sigma < 0$. We call the strip $0 \leq \sigma \leq 1$ the *critical strip*.

The *Riemann hypothesis* states that every non-trivial zero of the Riemann zeta function lies on the critical line. In spite of great efforts, no analytic proof of this claim exists. It has, however, been numerically confirmed that the first 100 billion zeros lie on the critical line. We will discuss some proven results related to the zeta function and some important consequences of the truth of the Riemann hypothesis.

2. The prime number theorem

Theorem 1 (The prime number theorem): Let $\pi(x)$ denote the number of primes smaller to or equal to x. Then:

$$\pi(x) \sim \frac{x}{\log x}. \tag{4}$$

To prove this, we will first need some machinery to put to use.

Theorem 2:[1] For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$. In other words, no zero of ζ has real part 1.

Proof: First we find the logarithmic derivative of ζ by means of the Euler product:

$$\begin{aligned}
\log(\zeta(s)) = -\sum_p 1 - p^{-s} \implies \frac{\zeta(s)}{\zeta'(s)} &= -\sum_p \frac{p^{-s}\log p}{1-p^{-s}} \\
&= -\sum_p \log p \left(\frac{1}{1-p^{-s}-1}\right) \\
&= -\sum_p \log p \sum_{n\geq 1}(p^{-ns}) \\
&= -\sum_{n\,\geq\,1} \Lambda(n) n^{-s}
\end{aligned}$$

where $\Lambda(n)$ is the Von Mangoldt function, defined by

$$\Lambda\left(n\right) = \begin{cases} \log\left(p\right) & : \exists m \in \mathbb{N},\, p\,\text{prime, such that}\, n = p^m \\ 0 & : \text{otherwise.} \end{cases}$$

We expand upon this further, to achieve

[1] [1] p. 28

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \sum_{n\geq 1} \Lambda(n) n^{-s} = \sum_{n\geq 1} \Lambda(n) n^{-\sigma} \exp(-it\log n) \\ &= \sum_{n\geq 1} \Lambda(n) n^{-\sigma} (\cos(t\log n) - i\sin(t\log n)). \end{aligned}$$

Therefore,

$$-\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n\geq 1} \Lambda(n) n^{-\sigma}\cos(t\log n) \tag{5}$$

Next, note that the inequality

$$3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \geq 0$$

holds for all real values of θ. For all $n \geq 1$, $\Lambda(n)n^{-\sigma} \geq 0$, and it follows by (5) that

$$\begin{aligned} 0 &\leq \sum_{n\geq 1} \Lambda(n) n^{-\sigma} \{3 + 4\cos(t\log n) + \cos(2t\log n)\} \\ &= -\Re\left(3\frac{\zeta'(\sigma)}{\zeta(\sigma)} + 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right). \end{aligned} \tag{6}$$

For convenience, define $\eta(s)$ by $\eta(s) = \zeta(s)^3 \cdot \zeta(s+it)^4 \cdot \zeta(s+2it)$. By the above computation, we know that the real part of $\frac{\eta'(s)}{\eta(s)}$ is always non-positive.

Now assume for the sake of a contradiction that ζ has a zero of order $d \geq 1$ at a point $1+it$. Then, η has a zero of order $k = 4d-3 \geq 0$. In other words, we have $\eta(s) \sim (s-1)^{4d-3}$ as $s \to 1^+$ along the real axis. Thus we have

$$\frac{\eta'(s)}{\eta(s)} \sim \frac{4d-3}{s-1}$$

as $s \to 1^+$. As $\Re(\frac{4d-3}{s-1}) \to +\infty$ as $s \to 1^+$, so does

$$\Re\left(\frac{\eta'(s)}{\eta(s)}\right) \to \infty.$$

But we already proved by (6) that $\Re\left(\frac{\eta'(s)}{\eta(s)}\right) \leq 0$. Thus we have a contradiction, and $1+it$ cannot be a zero of ζ. ■

Lemma 3: [2] We have the following bounds:

$$\begin{aligned} |\zeta(s)| &= O(\log t) \qquad (\sigma \geq 1, t \geq 2) && (7) \\ |\zeta'(s)| &= O(\log^2 t) \qquad (\sigma \geq 1, t \geq 2) && (8) \\ |\zeta(s)| &= O(t^{1-\delta}) \qquad (\sigma \geq \delta, t \geq 1) && (9) \end{aligned}$$

[2] [1] p. 27

Proof: Assume $t, X \geq 1$ and $\sigma \geq 0$. By applying partial summation to $\sum_{n\leq X}\frac{1}{n^s}$, we obtain (for $X \geq 1$)

$$\sum_{n\leq X}\frac{1}{n^s} = s\int_1^X \frac{[x]}{x^{s+1}}dx + \frac{[X]}{X^s}$$

Write $x = [x] + (x)$ to get

$$\sum_{n\leq X}\frac{1}{n^s} = \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int_1^X \frac{(x)}{x^{s+1}}ds + \frac{1}{X^{s-1}} - \frac{(X)}{X^s}. \tag{10}$$

By taking $X \to \infty$ on both sides, we get

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{(x)}{x^{s+1}}dx. \tag{11}$$

(This is an analytic continutation of ζ to the half plane $\sigma > 0$).

From here, we subtract our expressions for $\sum_{n\leq X}\frac{1}{n^s}$ from $\zeta(s)$ to get

$$\zeta(s) - \sum_{n\leq X}\frac{1}{n^s} = \frac{1}{(s-1)X^{s-1}} + \frac{(X)}{X^s} - s\int_X^\infty \frac{(x)}{x^{s+1}}dx.$$

Hence

$$\begin{aligned} |\zeta(s)| &< \sum_{n\leq X}\frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + |s|\int_X^\infty \frac{dx}{x^{\sigma+1}} \\ &< \sum_{n\leq X}\frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + (1+\frac{t}{\sigma})\frac{1}{X^\sigma} \end{aligned} \tag{12}$$

because $|s| < \sigma + t$. If $\sigma \geq 1$, it follows that

$$|\zeta(s)| < \sum_{n\leq X}\frac{1}{n} + \frac{1}{t} + \frac{1}{X} + \frac{1+t}{X} \leq (\log X + 1) + 3 + \frac{t}{X}.$$

Set $t = X$ and the first inequality (7) of the theorem is proved.

Now assume $\sigma \geq \delta$ where $0 < \delta < 1$. Then we have by (12)

$$\begin{aligned} |\zeta(s)| &< \sum_{n\leq X}\frac{1}{n^\delta} + \frac{1}{tX^{\delta-1}} + \left(2+\frac{t}{\delta}\right)\frac{1}{X^\delta} \\ &< \int_0^{[X]}\frac{dx}{x^\delta} + \frac{X^{1-\delta}}{t} + \frac{3t}{\delta X^\delta} \\ &\leq \frac{X^{1-\delta}}{1-\delta} + X^{1-\delta} + \frac{3t}{\delta X^\delta}. \end{aligned}$$

By again setting $X = t$, we deduce

$$|\zeta(s)| < t^{1-\delta}\left(\frac{1}{1-\delta}+1+\frac{3}{\delta}\right), \qquad (\sigma \geq \delta, t \geq 1) \tag{13}$$

which proves the third inequality (9) of the theorem.

To deduce the bound for $|\zeta'(s)|$, we consider any point $s_0 = \sigma_0 + t_0 i$ in the region $\sigma \geq 1, t \geq 2$ and denote by C the circle centered at s_0 with radious $\rho < \frac{1}{2}$. Then, by Cauchy's integral formula, we have

$$|\zeta'(s_0)| = \left|\frac{1}{2\pi i}\oint_C \frac{\zeta(s)ds}{(s-s_0)^2}\right| \leq \frac{1}{2\pi}\oint_C \frac{|\zeta(s)|}{|\rho^2|}ds \leq \frac{M}{\rho}$$

where M denotes, as usual, the maximum of $|\zeta(s)|$ on the circle. Now $\sigma \geq \sigma_0 - \rho \geq 1-\rho$ and $1 < t < 2t_0$ for all points on C. Therefore, by (13),

$$M < (2t_0)^\rho\left(\frac{1}{\rho}+1+\frac{3}{1-\rho}\right) < \frac{10t_0^\rho}{\rho}$$

since $1 < t < 2t_0$ and $\rho < 1-\rho$. Thus, so far, we have

$$|\zeta'(s)| < \frac{10t_0^\rho}{\rho^2} = 10e(\log t_0 + 2)^2 \tag{14}$$

by the substitution $\rho = (\log t_0 + 2)^{-1}$ and by $t_0^\rho = e^{\rho \log t_0} < e$.

Thus the lemma is concluded. ■

Next, we will define some important prime counting functions and discuss some properties of them.

Theorem 4:[3] Define Chebyshev's auxiliary functions $\vartheta(x)$ and $\psi(x)$ by the following:

$$\vartheta(x) = \sum_{p\leq x}\log(p), \qquad \psi(x) = \sum_{p^m\leq x}\log(p)$$

Then, if one of the three quotients

$$\frac{\vartheta(x)}{x}, \frac{\psi(x)}{x}, \frac{\pi(x)}{x/\log(x))}$$

converges to a limit as $x \to \infty$, then so do the others with the same limit.

Proof: First notice that by grouping together the terms of $\psi(x)$ by the values of m, we obtain

$$\psi(x) = \sum_{n=1}^{\infty}\vartheta(x^{1/n}). \tag{15}$$

For each x, this sum will only have a finite number of non-vanishing terms, as for $n > \log_2(x)$, we have $\vartheta(x^{1/n})) = 0$.

If we group together the terms of $\psi(x)$ by the values of p instead, we obtain

[3] [1] p. 13

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log(x)}{\log(p)}\right] \log(p) \tag{16}$$

where for any real $u, [u]$ denotes the largest integer not exceeding u. This expression comes from the fact that the values associated with a given p is equal to the number of positive integers m for which $m \log(p) \leq \log(x)$ holds, which is $[\frac{\log(x)}{\log(p)}]$.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ denote the upper limits and $\lambda_1, \lambda_2, \lambda_3$ denote the lower limits of the three quotients, respectively. By (15) and (16):

$$\vartheta(x) \geq \psi(x) \leq \sum_{p \leq x} \frac{\log(x)}{\log(p)} \log(p) \leq \pi(x) \log(x), \tag{17}$$

which implies $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3$.

Let $0 < \alpha < 1, x > 1$. Then

$$\vartheta(x) \leq \sum_{x^\alpha < p \leq x} \log(p) \geq [\pi(x) - \pi(x^\alpha)] \log(x^\alpha).$$

Thus, we get

$$\frac{\vartheta(x)}{x} \geq \alpha \left(\frac{\pi(x)\log(x)}{x} - \frac{\pi(x^\alpha)\log(x)}{x}\right) > \alpha \left(\frac{\pi(x)\log(x)}{x} - \frac{\log(x)}{x^{1-\alpha}}\right).$$

(since $\pi(x^\alpha) < x^\alpha$).

Fix α and let $x \to \infty$. $\log(x)/x^{1-\alpha} \to 0$, so we get that $\Lambda_1 \geq \alpha \Lambda_3$. Since we can choose α arbitrarily close to 1, we conclude that $\Lambda_1 = \Lambda_3$. Combined with the inequality $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3$, we conclude that the upper limits are equal. The same exact line of argument will work for the lower limits, and we conclude that the limits of the quotients, if they exist, are equal.■

We also notice, for future reference, that this theorem implies that the relations

$$\pi(x) \sim \frac{x}{\log(x)}, \quad \vartheta(x) \sim x, \quad \psi(x) \sim x \tag{18}$$

are equivalent. Thus, if we manage to prove either of the two latter, the prime number theorem will follow directly. We will work our way towards proving $\psi(x) \sim x$. Before we get there, we have to prove the following:

Lemma 5:[4] If k is a positive integer, $c > 0$ and $y > 0$, then

$$J = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{y^s ds}{s(s+1)...(s+k)} = \begin{cases} 0, & \text{if } y \leq 1 \\ \frac{1}{k!}(1-\frac{1}{y})^k, & \text{if} y \geq 1 \end{cases} \tag{19}$$

Proof: Denote by J_T the integral on the line segment from $c - Ti$ to $c + Ti$. Assume first $y \geq 1$. Denote by C the circle centered at 0 that passes trough the points $c \pm Ti$, and define its radius to be R, and make sure we pick a T large enough so that $R > 2k$.

[4] [1] p. 31

Finally, denote by C_1 the circle arc to the left of the line $\Re(s) = \sigma = c$. Cauchy's theorem of residues now gives us $J_T = S + J(C_1)$, where S denotes the sum of all residues of the integrand, and $J(C_1)$ denotes the integral along C_1. Note that the poles of the integrand are located at 0, -1, -2,...

On C_1, we have that $\sigma \leq c$ and thus $|y^s| = y^\sigma \leq y^c$, since $y \geq 1$. Also, for all $s \in C_1$, we have $|s - n| \geq R - k < R/2$ for $n = 0, 1, 2, ..., k$.

Thus, by the estimation lemma, we get

$$|J(C_1)| < \frac{1}{2\pi} \frac{y^c}{(R/2)^{k+1}} 2\pi R < \frac{2^{k+1} y^c}{R^k} < \frac{2^{k+1} y^c}{T^k} \longrightarrow 0 \tag{20}$$

as $T \to \infty$. Hence, by $J_T = S + J(C_1)$, we get $J_T \to S$ as T grows to infinity, i.e. $J = S$. A quick calculation yields

$$S = \sum_{n=0}^{k} \frac{y^{-n}}{(-1)^n n!(k-n)!} = \frac{1}{k!}(1 - y^{-1})^k \tag{21}$$

by the binomial theorem. The proof is analogous for $y \leq 1$, but the circle arc to the right of $\sigma = c$ is used instead of the C_1, so no poles are passed over and S vanishes. ■

Theorem 6 (fundamental formula): [5] Define the function $\psi_1(x)$ by the following:

$$\psi_1(x) = \int_0^x \psi(u)du = \int_1^x \psi(u)du = \sum_{n \leq x} (x - n)\Lambda(n) \tag{22}$$

(the equality in the last two expressions comes from partial summation, and $\Lambda(n)$ denotes as before the Von Mangoldt function)

Then:

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \qquad (x > 0, c > 1). \tag{23}$$

This is known as the *fundamental formula*.

Proof: For $x > 0$, we have, by lemma 5,

$$\left(1 - \frac{n}{x}\right) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{(x/n)^s}{(s+1)s} ds.$$

Thus we get

$$\frac{\psi_1(x)}{x} = \sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{(x/n)^s}{s(s+1)} ds. \tag{24}$$

Since $c > 1$, we get

$$\sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} \left| \frac{\Lambda(n)(x/n)^s}{s(s+1)} ds \right| < x^c \sum_{n=1}^{\infty} \frac{|\Lambda(n)|}{n^c} \int_{-\infty}^{\infty} \frac{dt}{c^2 + t^2}$$

[5] [1] p. 32

which is finite. Thus we can change the order of integration and summation in (24), and we achieve

$$\frac{\psi_1(x)}{x} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} ds = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds. \quad (25)$$

■

Theorem 7:[6]

$$\psi_1(x) \sim \frac{1}{2}x^2 \quad \text{as } x \to \infty$$

Proof Suppose without loss of generality (since we will eventually let x expand towards infinity) that $x > 1$. By the fundamental formula for $\psi_1(x)$, we have, with $c > 1$

$$\frac{\psi_1(x)}{x^2} = \int_{c-\infty i}^{c+\infty i} g(s)x^{s-1}ds$$

where g(s) is defined, for convenience, by

$$g(s) = \frac{1}{2\pi i}\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right).$$

Since we have proved $\zeta(1+it) \neq 0$ for all t, we know that $g(s)$ is holomorphic at all points where $\sigma \geq 1$ except at the point $s = 1$, at which ζ has a pole. Moreover, by lemma 3, we have the bound

$$|g(s)| < A_1 \cdot |t|^{-2} \cdot A_2 \log^{-2}|t| \cdot A_3 \log^{A_4}|t| < |t|^{-3/2} \quad (26)$$

when $\sigma \geq 1, |t| > t_0 \geq 2$. Take $\epsilon > 0$ and consider $L = L(\epsilon$ be the infinite broken line, consisting of the following segments:

$$\begin{aligned} L_1 &= (1-\infty i, 1-Ti) \\ L_2 &= (1-Ti, \alpha - Ti) \\ L_3 &= (\alpha - Ti, \alpha + Ti) \\ L_4 &= (\alpha + Ti, 1+Ti) \\ L_5 &= (1+Ti, 1+\infty i) \end{aligned}$$

where $T = T(\epsilon)$ satisfies $\int_T^\infty |g(1+ti)|dt < \epsilon$, and $0 < \alpha = \alpha(\epsilon) < 1$ such that the rectangle $\alpha \leq \sigma \leq 1, -T \leq t \leq T$ contains no zeros of $\zeta(s)$. This is possible, as we have shown ζ has no zeros with real part 1, and since it is meromorphic, it contains at most a finite number of zeros in the region $1/2 \leq \sigma \leq 1, -T \leq t \leq T$. Apply Cauchy's theorem of integrals to $\psi_1(x)/x^2$ to obtain

$$\frac{\psi_1(x)}{x^2} = c + \int_L g(s)x^{s-1}ds = 1/2 + J \quad (27)$$

[6] [1] p. 32

The constant 1/2 comes from the simple pole at $s = 1$ .By our construction of L, the integrand is holomorphic (except at $s = 1$) on and in-between the lines L and $\sigma = c$. By (26), we know J is absolutely convergent. Write $J = J_1 + ... + J_5$ where J_i denotes the integral along L_i. Because $\overline{g(s)x^{s-1}} = g(\overline{s})\overline{x}^{s-1}$, we have

$$|J_1| = |J_5| = \left| \int_T^\infty g(1+ti)x^t idt \right| \leq \int_T^\infty |g(1+ti)|dt < \epsilon$$

by our definition of T. Further, we have

$$|J_2| = |J_4| = \left| \int_\alpha^1 g(\sigma + Ti)x^{\sigma+Ti-1} d\sigma \right| \leq M \int_\alpha^1 x^{\sigma-1} d\sigma < \frac{M}{\log(x)},$$

$$|J_3| \leq 2Mx^{\alpha-1}T.$$

where $M = M(\epsilon)$ is the maximum of $|g(s)|$ on the line segments L_2, L_3 and L_4.

Adding together the line integrals, we get

$$\left| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \right| = |J| < 2\epsilon + \frac{2M}{\log(x)} + \frac{2MT}{x^{1-\alpha}} < 3\epsilon \tag{28}$$

for all $x > x_0 = x_0(\epsilon)$. Since ϵ is arbitarily small, this implies that $\frac{\psi_1(x)}{x^2} \to \frac{1}{2}$ as $x \to \infty$. ■

Lemma 8: [7] Let $c_1, c_2, ...$ be a given sequence of numbers and let

$$C(x) = \sum_{n \leq x} c_n, \quad C_1(x) = \int_0^x C(u)du = \sum_{n \leq x} (x-n)c_n \tag{29}$$

(where the last equality again comes from partial summation). If $c_n \geq 0$ and $C_1(x) \sim Cx^c$, where $C, c > 0$ are constants, then $C(x) \sim Ccx^{c-1}$.

Proof: Let $0 < \alpha < 1 < \beta$. Because $C(u)$ is a non-decreasing function, we have for $x > 0$,

$$C(x) \leq \frac{1}{\beta x - x} \int_x^{\beta x} C(u)du = \frac{C_1(\beta x) - C_1(x)}{\beta x - x}$$

$$\implies \frac{C(x)}{x^{c-1}} \leq \frac{1}{\beta - 1} (\frac{C_1(\beta x)}{(\beta x)^c} \beta^c - \frac{C_1(x)}{x^c}.$$

Keep β fixed and let $x \to \infty$. Since $\frac{C_1(x)}{x^c} \to C$ when $x \to \infty$ by assumption), we get:

$$\limsup_{x \to \infty} \frac{C(x)}{x^{c-1}} \leq C \frac{\beta^c - 1}{\beta - 1}. \tag{30}$$

Similarly, we consider the interval $(\alpha x, x)$:

$$C(x) \geq \frac{1}{x - \alpha x} \int_{\alpha x}^x C(u)du = \frac{C_1(x) - C_1(\alpha x)}{x - \alpha x}$$

[7] [1] p. 35

$$\implies \frac{C(x)}{x^{c-1}} \geq \frac{1}{1-\alpha}(\frac{C_1(x)}{(x)^c} - \frac{C_1(\alpha x)}{(\alpha x)^c}\alpha^c)$$

$$\implies \liminf_{x\to\infty} \frac{C(x)}{x^{c-1}} \geq C\frac{1-\alpha^c}{1-\alpha}. \tag{31}$$

By letting α and β approach 1, we get arbitrarily close to Cc (easily seen by using L'Hôpital's rule, for example) in both expressions. Thus, finally, we get

$$\limsup_{x\to\infty} \frac{C(x)}{x^{c-1}} = \liminf_{x\to\infty} \frac{C(x)}{x^{c-1}} = Cc. \tag{32}$$

which is equivalent to the theorem. ■

We are now finally in a position to prove the prime number theorem, using all our results so far.

Theorem 1 (The prime number theorem): Let $\pi(x)$ denote the number of primes smaller than or equal to x. Then:

$$\pi(x) \sim \frac{x}{\log(x)} \tag{33}$$

Proof: Since $\psi(x) = \sum_{n\leq x} \Lambda(n)$ and $\Lambda(n) \geq 0$, $\psi_1(x) \sim \frac{1}{2}x^2$ and theorem B (with $C = \frac{1}{2}, c = 2$) gives us $\psi(x) \sim x$ immediately. This, as we proved in lemma 8, is equivalent the the prime number theorem. ■

Notice that we used in our proof for theorem 7 that $\zeta(1+it) \neq 0$. This, as it turns out is an essential property needed for the prime number theorem to be true. This led many mathematicians to believe it to be impossible to prove the theorem without going through the theory of complex functions; however, in 1949, Selberg and Erdos together proved the theorem once and for all through elementary means.

Next, we will shift our gaze to the so-called Lindelöf hypothesis, and explain its relation to the Riemann hypothesis. In order to understand the exact connection, we need some more weapons in our arsenal.

3. Density of zeros

Before we go further, we need to know an important result about the density of zeros in the critical strip. Before we get there, we prove a useful lemma:

Lemma 9: [8] Suppose $f(z)$ is holomorphic in the cicle $|z - z_0| \leq R$, and has at least n zeros in $|z - z_0| \leq r < R$, and $f(z_0) \neq 0$. Then

$$\left(\frac{R}{r}\right)^n \leq \frac{M}{|f(z_0)|}$$

, where M is the maximum of $|f(z)|$ on the circumference of the larger circle.

Proof:

[8] [1] p. 49

Suppose without loss of generality that $z_0 = 0$ (the general case reduces to this case by the substition $z = z_0 + z'$). Denote the zeros of f in the smaller circle by $a_1, a_2, ..., a_n$. Note that by assumption, there can be more zeros than these, and any zero gets repeated according to their order of multiplicity. We can write f as

$$f(z) = \phi(z) \prod_{\nu=1}^{n} \frac{R(z - a_\nu)}{R^2 - \overline{a_\nu} z}$$

where $\phi(z)$ is regular in the larger circle. On the circumference of the larger circle, $|z| = R$, each factor in the product has modulus 1, so $|\phi(z)| = |f(z)| \leq M$ on the circle. The max modulus principle tells us that $|\phi|$ achieves its maximum value on the circle, and therefore $|\phi(0)| \leq M$. Hence:

$$|f(0)| = |\phi(0)| \prod_{\nu=1}^{n} \frac{a_\nu}{R} \leq M \left(\frac{r}{R}\right)^n .$$

Rearrange (and remember $f(0) \neq 0$, and the theorem is proven. ■

Next, we begin by defining a useful function, namely $\xi(s)$, by

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{34}$$

$\xi(s)$ is carefully designed to have the useful functional equation $\xi(s) = \xi(1-s)$, i.e. it is symmetric about the line $\sigma = \frac{1}{2}$. Moreover, we have $\xi(\overline{s}) = \overline{\xi(s)}$.

Recall Weierstrass' definition of the gamma function (here $\frac{s}{2}$ is used for obvious reasons),

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} = \frac{s}{2} e^{\gamma \frac{s}{2}} \prod_{n=1}^{\infty} \left(\left(1 + \frac{s}{2n}\right) e^{-\frac{s}{2n}} \right) . \tag{35}$$

Thus the factor $\Gamma(\frac{s}{2})$ in the definition of $\xi(s)$ cancels out all the trivial zeros $s = -2, -4, ...$. Because of this, the zeros of $\xi(s)$ are exactly the complex zeros of $\zeta(s)$! We shall see why this is useful to us in the following theorem.

Theorem 10: [9] Let $\rho = \beta + i\gamma$ denote the non-trivial zeros of $\zeta(s)$ (i.e. $\zeta(\rho) = 0$). Let $N(t)$ denote the total numbers of zeros in the critical strip with complex part $0 \leq \gamma \leq T$. Then, when $T \to \infty$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log(T)).$$

Proof: Suppose $T > 3$ and $T \neq \gamma$ for all ρ. Consider the rectangle C with vertices $2 \pm Ti, -1 \pm Ti$. Because ξ and ζ share the same complex zeros, and since $\xi(s)$ gives conjugate values for conjugate s, we know that $\xi(s)$ has exactly $N(T)$ zeros inside C and none on its boundary. By the double symmetry of $\xi(s)$ and by Cauchy's principle of argument, we get

$$4\pi i N(T) = [\arg \xi(s)]_C$$

[9] [1] p. 68

where $[\arg f(s)]_C$ denotes the increase of the argument of f along the rectangle C, going counter-clockwise.

We write this as

$$[\arg \xi(s)]_C = \left[\arg \frac{s(s+1)}{2}\right]_C + [\arg \phi(s)]_C$$

where $\phi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. The first term on the right is easily calculated to be 4π. Recall that $\xi(s) = \xi(1-s)$ and $\xi(\overline{s}) = \overline{\xi(s)}$. This implies that $[\arg \phi(s)]_C = 4[\arg \phi(s)]_L$, where L is the broken line $(2, 2+Ti) \cup (2+Ti, \frac{1}{2}+Ti)$.

Now, let's calculate $[\arg \phi(s)]_L = [\arg \pi^{-\frac{s}{2}}]_L + [\arg \Gamma(\frac{s}{2})]_L + [\arg \zeta(s)]_L$ termwise:

$$\left[\arg \pi^{-\frac{s}{2}}\right]_L = [\Im\left(\log \pi^{-\frac{s}{2}}\right)]_L = -\frac{T \log \pi}{2} \tag{36}$$

$$\left[\arg \Gamma\left(\frac{s}{2}\right)\right]_L = \left[\Im \log \Gamma\left(\frac{s}{2}\right)\right]_L = \Im \log \Gamma\left(\frac{1}{4} + \frac{1}{2}Ti\right)) - \Im \log \Gamma(1) \tag{37}$$

Stirling's formula can be fomulated as the following:

$$\log \Gamma(z+a) = \log z \cdot (z + a - \frac{1}{2}) - z + \frac{1}{2}\log 2\pi + O(|z|^{-1})$$

By setting $z = \frac{1}{2}Ti$ and $a = \frac{1}{4}$, we obtain from (37)

$$\begin{aligned}\left[\arg \Gamma\left(\frac{s}{2}\right)\right]_L &= \Im\left(-\frac{1}{4} + \frac{1}{2}Ti\right)\log\left(\frac{1}{2}Ti\right) - \frac{1}{2}Ti + \frac{1}{2}\log 2\pi + O(T^{-1}) \\ &= \frac{1}{2}T \log \frac{1}{2}T - \frac{1}{8}\pi - \frac{1}{2}T + O(T^{-1}).\end{aligned} \tag{38}$$

The most difficult step in this proof is calculating $S(T) = [\arg \zeta(s)]_L$.

We begin by letting m be the number of points $s' \in L$ such that $\Re\zeta(s') = 0$. We can bound $S(T)$ by $S(T) \leq (m+1)\pi$, since $\arg \zeta(s)$ cannot vary more than π between two adjacent such points (as $\Re\zeta(s)$ does not change sign on these segments).

No such points s' is located on $(2, 2+Ti)$, since

$$\Re\zeta(2+ti) \geq 1 - \sum_2^{\infty} \frac{1}{n^2} = 1 - (\frac{\pi^2}{6} - 1) = 0.35507...$$

Thus m describes the number of distinct points $\frac{1}{2} < \sigma < 2$ such that $\Re\zeta(\sigma + Ti) = 0$.

Define now the function $g(s)$ by $g(s) = \frac{1}{2}\zeta(s+Ti) + \frac{1}{2}\zeta(s-Ti)$ for $\frac{1}{2} < s < 2$. The number of points on this segment such that $g(s) = 0$ is now exactly equal to m. To see this, just notice that $g(\sigma) = \Re\zeta(\sigma + Ti)$ for real σ, because $\zeta(\overline{s}) = \overline{\zeta(s)}$.

Thus we have reduced the problem to counting the numbers of zeros m of an holomorphic (except at the points $s = 1 \pm Ti$ function. We bound m by applying theorem C to $g(s)$, using the circles $|s-2| \leq \frac{7}{4}$ and $|s-2| \leq \frac{3}{2}$. Since we assumed T to be larger than 3, the poles of $g(s)$ are outside of the circles. By theorem lemma 3 (iii), g also satisfies

$$|g(s)| < A_1 \frac{1}{2}(|t+T|^{\frac{3}{4}} + |t-T|^{\frac{3}{4}}) < A_1(T+2)^{\frac{3}{4}}$$

because $\sigma \geq \frac{1}{4}$ and $1 < |t \pm T| < 2 + T$ at all points in the larger circle. Lastly, we need that $g(2) = \Re\zeta(2 + Ti) \approx 0.36 > \frac{1}{4}$. Now we can apply theorem C:

$$\left(\frac{7}{6}\right)^m < A_1 \frac{(T+2)^{\frac{3}{4}}}{\frac{1}{4}} < T$$

for $T \geq T_0 > 3$, T_0 large enough. Thus $m < A_2 \log T$ for $T > T_0$. Since we have $S(T) \leq (m+1)\pi$, we can calculate

$$S(T) \leq (m+1)\pi = O(\log T) + \pi. \tag{39}$$

Adding together the estimates we've now found in (36), (37) and (39), we get

$$\begin{aligned}
[\arg \phi(s)]_L &= -\frac{1}{2} T \log \pi + \frac{1}{2} T \log \frac{1}{2} T - \frac{1}{8}\pi - \frac{1}{2} T + O(T^{-1}) + \pi + O(\log T) \\
&= \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} + O \log(T).
\end{aligned}$$

Since $4\pi N(T) = [\arg \xi(s)]_C = 4\pi + 4[\arg \phi(s)]_L$, the theorem is deduced. ■

Corollary 11: [10] Let h be any fixed positive number. Then, as $T \to \infty$,

$$N(T+h) - N(t) = O(\log T).$$

Proof: Write

$$P(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi}$$

Now write

$$P(T+h) - P(T) = hP'(T + \alpha h)$$

for some $0 < \alpha < 1$. By differentiating $P(t)$, we get

$$P'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} = O(\log t).$$

■

4. The Lindelöf hypothesis

Next we will discuss an important conjecture in number theory, namely the Lindelöf hypothesis. The Lindelöf hypothesis is the (unproven) claim that:

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\epsilon)$$

for all $\epsilon > 0$. A number of interesting consequences about the growth of the ζ-function along vertical lines within the critical strip can be made from this statement. For this, we define the Lindelöf $\mu(\sigma)$-function:

[10] [1] p. 70

$$\mu(\sigma) = \inf_{A \in} \{A | \zeta(\sigma + it) = \mathcal{O}(t^A).$$

The Lindelöf hypothesis is equivalent to the statement $\mu\left(\frac{1}{2}\right) = 0$. We see clearly from the convergence of the infinite sum $\sum_n^\infty \frac{1}{n^\sigma}$ that $\mu(\sigma) = 0$ for $\sigma \geq 1$. Next we will show how $\mu(\sigma)$ behaves on the negative real line.

Theorem 12:[11] We have, for all $\sigma \in \mathbb{R}$:

$$\mu(\sigma) = \mu(1-\sigma) - \sigma + \frac{1}{2}.$$

Also, for all $\sigma \leq 0$, we have:

$$\mu(\sigma) = \frac{1}{2} - \sigma.$$

This is known as the *convexity bound* of $\mu(\sigma)$, and is an improvement of (9) in lemma 3.

Proof:

Let us recall the $\xi(s)$-function we defined ealier:

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Using this in conjunction with Stirling's formula for $\Gamma(s)$, we get, as $t \to \infty$:

$$\begin{aligned}
\log|\xi(\sigma+it)| &= \Re \log \xi(\sigma+it) \\
&= \Re \log \Gamma\left(\frac{\sigma+it}{2}\right) - \frac{\sigma}{2}\log\pi + \frac{1}{2}\log|\sigma - 1 + it| \\
&\quad + \frac{1}{2}\log|\sigma+it| + \log|\zeta(\sigma+it)| \\
&\sim \frac{\sigma+it}{2}\log\left|\frac{s}{2}\right| - \frac{t}{2}\Im\log\frac{s}{2} - \frac{\sigma}{2} + \frac{1}{2}\log 2\pi - \frac{\sigma}{2}\log\pi \\
&\quad + \frac{1}{2}\log|t| + \frac{1}{2}\log|t| + \log|\zeta(\sigma+it)| \\
&\sim \frac{\sigma}{2}\left(\log\frac{t}{2} - 1 - \log\pi\right) - \frac{t}{2}\cdot\frac{\pi}{2} + \frac{3}{2}\log t - \frac{1}{2}\log 2 \\
&\quad + \frac{1}{2}\log 2\pi + \log|\zeta(\sigma+it)| \\
&\sim \frac{\sigma}{2}\log\frac{t}{2\pi e} - \frac{t\pi}{4} + \frac{3}{2}\log t + \frac{1}{2}\log\pi
\end{aligned}$$

Here we used that $|s| \to t$ and $\Im \log z = \arctan\frac{t}{\sigma} \to \frac{\pi}{2}$ as $t \to \infty$. From this, we get

$$\begin{aligned}
0 &= \log|\xi(\sigma+it)| - \log|\xi(1-\sigma+it)| \\
&\sim \frac{\sigma}{2}\log\frac{t}{2\pi e} - \frac{1-\sigma}{2}\log\frac{t}{2\pi e} + \log|\zeta(\sigma+it)| - \log|\zeta(|1-\sigma+it)|
\end{aligned}$$

[11] [2] p.185

whence

$$1 \sim \left(\frac{t}{2\pi e}\right)^{\sigma-\frac{1}{2}} \left|\frac{\zeta(\sigma+it)}{\zeta(1-\sigma+it)}\right|$$

From which the first part of the theorem follows. The second part follows directly from the first, since the term $\mu(1-\sigma)$ vanishes when $\sigma \leq 0$. ■

Theorem 13:[12] We have

$$\mu(\sigma) \leq \frac{1}{2}(1-\sigma)$$

for all $0 \leq \sigma \leq 1$. Moreover, if the Lindelöf hypothesis holds (i.e. $\mu(\frac{1}{2}) = 0$) then

$$\mu(\sigma) \begin{cases} 0, & \text{if } \sigma \geq \frac{1}{2} \\ \frac{1}{2} - \sigma, & \text{if } \sigma \leq \frac{1}{2} \end{cases}.$$

Proof:

Let $M(\sigma, T) = \max_{1 \leq t \leq T} |\zeta(\sigma + it)|$. Fix $0 \leq \sigma_1 < \sigma < \sigma_2 \leq 1$. Let C denote the rectangle with vertices

$$\sigma_2 - \sigma - \frac{iT}{2}, \sigma_2 - \sigma + \frac{iT}{2}, \sigma_1 - \sigma + \frac{iT}{2}, \sigma_1 - \sigma - \frac{iT}{2}.$$

Notice that both -1 and 1 lie outside of this contour.

Consider the contour integral:

$$\frac{1}{2\pi i} \oint_C \zeta(\sigma + it + w) \frac{x^w}{w(w+1)} dw = \zeta(\sigma + it) \tag{40}$$

The poles of the integrand are located at $-1, 0$, and 1. Only 0 lies inside the rectangle at the equality follows directly from Cauchy's theorem of residues.

Consider the contour integral as a sum of four line integrals. On the horizontal lines H_1, H_2 of the contour, we have by the estimation lemma

$$\left|\int_{H_i} \zeta(\sigma + it + w) \frac{x^w}{w(w+1)} dw\right| \leq \max_{\sigma_1 \leq \sigma \leq \sigma_2} \zeta\left(\sigma + \frac{iT}{2}\right) \frac{1}{\frac{T}{2}(\frac{T}{2}+1)} \to 0$$

as $T \to \infty$. On the two vertical lines V_1, V_2, we have, again by the estimation lemma,

$$\left|\int_{V_i} \zeta(\sigma + it + w) \frac{x^w}{w(w+1)} dw\right| \leq C_i M(\sigma, 2T) x^{\sigma_i - \sigma}$$

with some constants C_1, C_2.

Adding these line integrals together, we get

$$\sigma(\sigma + it) < C\left(M(\sigma_1, 2T)x^{\sigma_1 - \sigma} + M(\sigma_2, 2T)x^{\sigma_2 - \sigma}\right)$$

for some C. By setting $x = \left(\frac{M(\sigma_1, 2T)}{M(\sigma_2, 2T)}\right)^{\frac{1}{\sigma_2 - \sigma_1}}$, we get

[12] [3] p. 339

$$M(\sigma, T) = O\left(M(\sigma_1, 2T)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} M(\sigma_2, 2T)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}\right).$$

Letting $T \to \infty$ and taking logarithms, we get by the definition of the Lindelöf $\mu(\sigma)$-function

$$\mu(\sigma) \leq \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}\mu(\sigma_1) + \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}\mu(\sigma_2). \tag{41}$$

Since $\mu(0) = \frac{1}{2}$ and $\mu(1) = 0$ by theorem 12, the first part of the theorem is achieved by setting $\sigma_1 = 0$ and $\sigma_2 = 1$.

Now assume the Lindelöf hypothesis, and setting $\sigma_1 = \frac{1}{2}, \sigma_2 = 0$. Then $\mu(\sigma) = 0$ for $\frac{1}{2} \leq \sigma \leq 1$, and the statement follows from theorem 12. ■

Lemma 14: [13] If $f(s)$ is holomorphic, and satisfies

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \qquad (M > 1)$$

inside the circle centered at s_0 with radius r, then

$$\left|\frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s-\rho}\right| < \frac{AM}{r}, \qquad \left(|s - s_0| \leq \frac{1}{4}r\right) \tag{42}$$

where ρ denotes the zeros of $f(s)$ such that $|\rho - s_0| \leq \frac{1}{4}r$.

Proof:

Define the function $g(s) = f(s)\prod_\rho (s-\rho)^{-1}$. This function is holomorphic in the circle $|s - s_0| \leq r$, and has no zeros in the concentric circle with radius $\frac{1}{2}r$.

On the circumference of the circle, we have $|s - \rho| \geq \frac{1}{2}r \geq |s_0 - \rho|$, therefore we have

$$\left|\frac{g(s)}{g(s_0}\right| = \left|\frac{f(s)}{f(s_0}\prod_\rho\left(\frac{s_0-\rho}{s-\rho}\right)\right| \leq \left|\frac{f(s)}{f(s_0}\right| < e^M.$$

This inequality also holds on the inside of the circle, due to the maximum modulus principle. Hence, the function

$$h(s) = \log\frac{g(s)}{g(s_0)}$$

is regular for $|s - s_0| \leq \frac{1}{2}r$, $h(s_0) = 0$ and $\Re h(s) < M$.

Recall the Borel-Caratheodory theorem:

If $h(z)$ is holomorphic on a disk with radius R centered at z_0 and $0 < R' < R$, then:

$$\sup_{|z-z_0|\leq R'} h(z) < \frac{2R'}{R-R'}\sup_{|z-z_0|\leq R}\Re h(z) + \frac{R+R'}{R-R'}|h(z_0)|. \tag{43}$$

[13] [4] p. 56

Applying this to our $h(s)$-function, using $R = \frac{1}{2}r, R' = \frac{3}{8}r$ and $z_0 = s_0$, we find

$$|h(s)| < AM \qquad (|s - s_0| \leq \frac{3}{8}r).$$

By Cauchy's integral formula, we get

$$|h'(s)| = \left| \frac{1}{2\pi i} \oint_{|z-s|=\frac{1}{8}r} \frac{h(z)}{(z-s)^2} \right| < \frac{AM}{r}.$$

Since

$$h'(s) = \frac{g'(s)}{g(s)} = (\log g(s))' = \frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s-\rho}, \tag{44}$$

we get the desired result. ■

Theorem 15: [14] Let $\rho = \beta + i\gamma$ denote the non-trivial zeros of $\zeta(s)$. Then, for $-1 \leq \sigma \leq 2$, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log t).$$

Proof: Apply lemma 14 with $f(s) = \zeta(s), s_0 = 2 + iT, r = 12$. Since we have

$$\left| \frac{\zeta(s)}{\zeta(s_0)} \right| < T^A = e^{A \log T}$$

in the circle $|s - s_0| = 12$, we get $M = A \log T$. Thus, the lemma above gives us

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - s_0|\leq 6} \frac{1}{s-\rho} + O(\log T), \qquad (|s - s_0| \leq 3). \tag{45}$$

Since this holds in the whole circle $|s - s_0| \leq 3$, it naturally also holds on the line $-1 \leq \sigma \leq 2, t = T$. Next, we compare the two sums

$$\sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho}, \sum_{|\rho - s_0|\leq 6} \frac{1}{s-\rho}.$$

Any term that is included in the second sum and not in the first (denote these $h_j(s)$) will necessarily be bounded ($h_j(s) \leq C$). By corollary 11, we know that the amount of such terms cannot exceed

$$N(t+6) - N(t-6) = O(\log t).$$

Thus we get, from (45):

[14] [4] p. 217

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \sum_{|\rho - s_0|\le 6} \frac{1}{s-\rho} + O(\log t)\\ &= \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + \sum_j h_j(s) + O(\log t)\\ &\le \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + CO(\log T) + O(\log t)\\ &= \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + O(\log T).\end{aligned}$$

■

Theorem 16 (Backlund's reformulation of the Lindelöf hypothesis):
Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ such that $\sigma < \beta, 0 \le \gamma \le$. Then the Lindelöf hypothesis holds if and only if for every $\sigma \ge \frac{1}{2} + \delta$, we have

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T).$$

Proof:
We apply Jensen's formula to $\zeta(s)$ to the circle centered at $2 + it$ and radius $r = \frac{3}{2} - \frac{1}{4}\delta$:

$$\log \frac{r^n}{|a_1| \cdot \ldots \cdot |a_n|} = \frac{1}{2\pi} \int_0^{2\pi} \log |\zeta(re^{i\theta} + 2 + it)| d\theta - \log |\zeta(2 + ti)| \tag{46}$$

where a_i denotes the zeros of $\zeta(s + 2 + ti)$ in the circle $|s - 2 - ti| \le r$. On the Lindelöf hypothesis, the right hand side is clearly $o(\log t)$. Now, let N denote the number of zeros in the slightly smaller concentric circle with radius $r_0 = \frac{3}{2} - \frac{1}{2}\delta$. We also notice that

$$\log \frac{r^n}{|a_1| \cdot \ldots \cdot |a_n|} = \sum_{i=1}^{n} \log \frac{r}{|a_i|} > \sum_{n=1}^{N} \log \frac{r}{|a_i|} > N \log \frac{\frac{3}{2} - \frac{1}{4}\delta}{\frac{3}{2} - \frac{1}{2}\delta}.$$

Hence the number of zeros in the smaller circle is $o(\log t)$, since

$$N < \left(\log \frac{\frac{3}{2} - \frac{1}{4}\delta}{\frac{3}{2} - \frac{1}{2}\delta}\right)^{-1} \frac{1}{2\pi} \int_0^{2\pi} \log |\zeta(re^{i\theta} + 2 + it)| d\theta - \log |\zeta(2 + ti)| = o(\log T).$$

To get the statement, cover the rectangle in question with a sufficient amount of such circles (which depend only on δ).

Conversely, let us assume that $N(\sigma, T+1) - N(\sigma, T) = o(\log T)$. Let C_1 denote the circle with centre $2 + iT$ and radius $r_1 = \frac{3}{2} - \delta$, and let $\sum_1$ denote the summation over zeros ρ of $\zeta(s)$ in the circle C_1. Let C_2 be the slightly smaller concentric circle with radius $r_2 = \frac{3}{2} - 2\delta$, and C_3 the smaller again circle with radius $r_3 = \frac{3}{2} - 3\delta$. Notice that for each term which is in one of the sums

$$\sum_1 \frac{1}{s-\rho}, \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho}$$

but not in the other, we have $|s-\rho| \geq \delta$. From corollary 11, the number of such terms is $O(\log T)$. Therefore, from theorem 15, we have for $s \in C_2$:

$$g(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum_1 \frac{1}{s-\rho} = O\left(\frac{\log T}{\delta}\right). \tag{47}$$

Denote by C one last concentric circle with radius $\frac{1}{2}$. There are no zeros inside this circle, so clearly each term in the sum $\sum_1$ is bounded, and so is $\frac{\zeta'(s)}{\zeta(s)}$. By assumption, the number of terms in the sum is $o(\log T)$.

Recall Hadamard's three circle theorem:

Let $f(s)$ be holomorphic on the annulus $r_1 \leq |s| \leq r_3$. Let $M(r)$ be the maximum of $|f(s)|$ on the circle $|s| = r$. Then:

$$[M(r_2)]^{\log \frac{r_3}{r_1}} \leq [M(r_1)]^{\log \frac{r_3}{r_2}} [M(r_3)]^{\log \frac{r_2}{r_1}}$$

$$\implies [M(r_2)] \leq [M(r_1)]^{1-\lambda} [M(r_3)]^{\lambda}, \qquad (0 < \lambda < 1)$$

for any three concentric circles with radii $r_1 < r_2 < r_3$.

Apply this theorem, setting $f(s) = g(s)$, to the circles $C \subset C_3 \subset C_2$. We get, for s in C_3:

$$|g(s)| < [o(\log T)]^{1-\lambda} \left[O\left(\frac{\log T}{\delta}\right)\right]^{\lambda}$$

where λ depends only on δ. Thus, for all s in C_3 and for any given δ, we have

$$g(s) = o(\log T).$$

Now we can compute

$$\begin{aligned}\int_{\frac{1}{2}+3\delta}^{2} g(s) d\sigma &= \log \zeta(2+it) - \log \zeta(\frac{1}{2}+3\delta+it) - \sum_1 \log(2+it-\rho) + \sum_1 \log(\frac{1}{2}+3\delta+it-\rho) \\ &= O(1) - \log \zeta(\frac{1}{2}+3\delta+it) + o(\log T) + \sum_1 \log(\frac{1}{2}+3\delta+it-\rho)\end{aligned}$$

since there are $o(\log T)$ terms in $\sum_1$ and each term is bounded. Also, by setting $t = T$, the integral on the left hand side is $o(\log T)$. Taking real parts, we get

$$\log \left|\zeta\left(\frac{1}{2}+3\delta+iT\right)\right| = o(\log T) + \sum_1 \log \left|\frac{1}{2}+3\delta+iT-\rho\right|.$$

Again by using the fact that $\sum_1$ has $o(\log T)$ terms and each term is bounded, it follows that

$$\log \left|\zeta\left(\frac{1}{2}+3\delta+iT\right)\right| < o(\log T)$$

$$\implies \zeta\left(\frac{1}{2}+iT\right) = O(T^{\epsilon}). \tag{48}$$

■

5. The Density hypothesis

The density hypothesis is the unproven claim that the following bound holds

$$N(\sigma, T) = O(T^{2(1-\sigma)} \log^B T$$

for some B. We will see how this ties in with our previous results shortly.

Theorem 17: Let f be holomorphic in the vertical strip $\sigma \in [\sigma_1, \sigma_2]$. Let

$$J_\sigma = \int_{-\infty}^{\infty} |f(\sigma+it)|^2 \, dt$$

and assume J_σ is convergent for all σ in the strip. Assume also $lim_{t\to\infty} f(\sigma+it) = 0$. Then $\log J_\sigma$ is a convex function, i.e.

$$J_\sigma \leq J_{\sigma_1}^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \cdot J_{\sigma_2}^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}.$$

Proof:

It is enough to show that

$$J_{\frac{\sigma_1+\sigma_2}{2}} \leq J_{\sigma_1}^{\frac{1}{2}} \cdot J_{\sigma_2}^{\frac{1}{2}} \tag{49}$$

because a continuous function ϕ on an interval $[x, y]$ satisfying

$$\phi\left(\frac{x_0+y_0}{2}\right) \leq \frac{\phi(x_0)+\phi(y_0)}{2}$$

for all $x \leq x_0, y_0 \leq y$ is convex.

To prove (49), we define the midpoint $\sigma_0 = \frac{\sigma_1+\sigma_2}{2}$ and the function $f^*(s) = \overline{f(\sigma_0 - \bar{s})}$. Notice that $f^*(s) = \overline{f(s)}$ on the line $\sigma = \sigma_1$, and $f^*(s)$ is analytic in the same strip as f.

Let L_1 denote the broken line segment with vertices $\sigma_0 - iT, \sigma_1 - iT, \sigma_1 + iT, \sigma_0 + iT$ and similarly let L_2 denote the broken line segment with vertices $\sigma_0 - iT, \sigma_2 - iT, \sigma_2 + iT, \sigma_0 + iT$. Cauchy's integral theorem gives us

$$\int_{\sigma_0-iT}^{\sigma_0+iT} |f(s)|^2 ds = \int_{\sigma_0-iT}^{\sigma_0+iT} f(s)f^*(s)ds = \int_{L_2} f(s)f^*(s)ds.$$

The Cauchy-Schwarz inequality gives us

$$\begin{aligned}\left|\int_{\sigma_0-iT}^{\sigma_0+iT} |f(s)|^2 ds\right| &\leq \left(\int_{L_2} |f(s)|^2 |ds|\right)^{\frac{1}{2}} \left(\int_{L_2} |f^*(s)|^2 |ds|\right)^{\frac{1}{2}} \\ &= \left(\int_{L_2} |f(s)|^2 |ds|\right)^{\frac{1}{2}} \left(\int_{L_1} |f(s)|^2 |ds|\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{50}$$

By assumption, $\lim_{t\to\infty} f(\sigma+it)=0$. Hence

$$\int_{\sigma_0+iT}^{\sigma_2+iT} |f(s)|^2 |ds| \xrightarrow{T\to\infty} 0, .$$

Thus the two integrals in (50) converge to J_{σ_1} and J_{σ_2} respectively. Thus midpoint convexity is proven, and the theorem follows. ■

(Note that in the following theorem, both the Lindelöf $\mu(\sigma)$ function and the unrelated Möbius function $\mu(n)$ are used. The Möbius function is defined by $\mu(1)=1$, $\mu(n)=0$ if n divides the square of a prime number, and $\mu(n)=(-1)^k$ if $n=p_1p_2\cdot\dots\cdot p_k$ where p_i are distinct primes. The Möbius function is only defined for positive integer values, and will always be denoted accordingly.)

Lemma 18: [15] Let

$$f_X(s)=\zeta(s)\sum_{n<X}\frac{\mu(n)}{n^s}-1=\zeta(s)M_X(s)-1 \tag{51}$$

where $\mu(n)$ is the Möbius function. Assume the Lindelöf $\mu\left(\frac{1}{2}\right)=c$. Then:

$$\int_1^T |f_X(\sigma+ti)|^2 dt < C\frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}}(T+X)\log^4(T+X)$$

for $\frac{1}{2}\le\sigma\le 1$, $T,X>1$, $C>0$.

Proof:

Asumme without loss of generality that $X\ge 2$, because $f_X(s)=f_2(s)$ for $1<X<2$. We know from theorems 12 and 13 that $c\le\frac{1}{2}$.

For $\sigma>1$, we have

$$f_X(s)=\sum_{n\ge X}\frac{a_X(n)}{n^s}$$

where

$$a_X(n)=\sum_{d|n,d<X}\mu(d).$$

Notice that $a_X(1)=1$, $a_X(n)=0$ for $1<n<X$ and $|a_X(n)|\le\tau(n)$ for all n, where $\tau(n)$ is the number of divisors of n. Therefore, for $0<\delta<1$, we have

$$\begin{aligned}\int_0^T |f_X(1+\delta+it)|^2 dt &= \sum_{m,n\ge X}\frac{a_X(m)a_X(n)}{(mn)^{1+\delta}}\int_0^T\left(\frac{m}{n}\right)^{it}dt\\ =\sum_{m=n\ge X} +2\Re\sum_{n>m\ge X} &\le T\sum_{n\ge X}\frac{\tau^2(n)}{n^{2+2\delta}}+4\sum_{n>m\ge X}\frac{\tau(m)\tau(n)}{(mn)^{1+\delta}\log\left(\frac{n}{m}\right)}.\end{aligned}$$

[15] [5] Theorem 2

We can bound these sums using the known inequalities

$$\sum_{n\leq x} \tau(n)^2 \ll x \log^3 x \tag{52}$$

and

$$\sum_{m<n\leq x} \frac{\tau(m)\tau(n)}{(mn)^{\frac{1}{2}} \log\left(\frac{n}{m}\right)} \ll x \log^4 x. \tag{53}$$

Using (52) we deduce that

$$\begin{aligned} \sum_{n\geq X} \frac{\tau^2(n)}{n^{2+2\delta}} &= \sum_{n\geq X} \tau^2(n) \int_n^\infty \frac{2+\delta}{x^{3+2\delta}} dx = \int_n^\infty \frac{2+\delta}{x^{3+2\delta}} \sum_{X\leq n\leq x} \tau^2(n) dx \\ &\ll \int_X^\infty \frac{(2+2\delta)\log^3 x}{x^{2+2\delta}} dx \ll \frac{1}{X^{1+2\delta}} \log^3 x \end{aligned} \tag{54}$$

by repeated integration by parts. Further, from (53) along with the inequality $1 < \log x + \frac{1}{x} < \log x + \frac{1}{\sqrt{x}}$, we deduce

$$\begin{aligned} \sum_{n>m\geq X} \frac{\tau(m)\tau(n)}{(mn)^{1+\delta} \log\left(\frac{n}{m}\right)} &\ll \sum_{n>m\geq X} \frac{\tau(m)\tau(n)}{(mn)^{1+\delta}} + \sum_{n>m\geq X} \frac{\tau(m)\tau(n) n^{-\frac{1}{2}} m^{\frac{1}{2}}}{(mn)^{1+\delta} \log\left(\frac{n}{m}\right)} \\ &\ll \left(\sum_{n=1}^{\infty} \frac{\tau(n)}{n^{1+\delta}}\right)^2 + \sum_{n>m\geq 1} \frac{\tau(m)\tau(n)}{(mn)^{\frac{1}{2}} \log\left(\frac{n}{m}\right)} \int_n^\infty \frac{1+\delta}{x^{2+\delta}} dx \\ &\ll \zeta^4(1+\delta) + \int_1^\infty \frac{1+\delta}{x^{2+\delta}} \sum_{n>m\geq 1} \frac{\tau(m)\tau(n)}{(mn)^{\frac{1}{2}} \log\left(\frac{n}{m}\right)} (x) dx \\ &\ll \delta^{-4} + \int_1^\infty \frac{1+\delta}{x^{1+\delta}} \log^4 x dx \ll \delta^{-4}. \end{aligned}$$

Note that we have $\frac{\log^3 X}{X^{2\delta}} < \delta^{-3}$. Hence, we get the bound

$$\begin{aligned} \int_0^T |f_X(1+\delta+it)|^2 dt &\leq T \sum_{n\geq X} \frac{\tau^2(n)}{n^{2+2\delta}} + 4 \sum_{n>m\geq X} \frac{\tau(n)\tau(m)}{(mn)^{1+\delta} \log\left(\frac{n}{m}\right)} \\ &\ll \left(\frac{T}{X} + 1\right) \delta^{-4}. \end{aligned} \tag{55}$$

On the line $\sigma = \frac{1}{2}$, we use the inequality $\frac{1}{\log x} < \frac{x}{x-1} < 1 + \frac{\sqrt{x}}{x-1}$ to estimate

$$\begin{aligned}
\int_0^T |f_X(\frac{1}{2}+it)|^2 dt &\ll \int_0^T |\zeta(\frac{1}{2}+it)|^2 |M_X(\frac{1}{2}+it)|^2 dt + T \\
&\ll T^{2c} \int_0^T |M_X(\frac{1}{2}+it)|^2 dt + T \\
&\ll T^{1+2c} \sum_{n<X} \frac{\mu^2(n)}{n} + 4T^{2c} \sum_{m<n<X} \frac{|\mu(m)\mu(n)|}{(mn)^{\frac{1}{2}} \log\left(\frac{n}{m}\right)} \\
&\ll T^{1+2c} \log X + 4T^{2c} \sum_{m<n<X} \left(\frac{1}{(mn)^{\frac{1}{2}}} + \frac{1}{n-m} \right) \\
&\ll T^{2c}(T+X)\log X. \qquad (56)
\end{aligned}$$

We now have bounds for the integral on the lines $\sigma = \frac{1}{2}, \sigma = 1+\delta$, and from these we will deduce an inequality for all $\frac{1}{2} \leq \sigma \leq 1+\delta$. Begin by defining

$$I_\sigma(T) = \int_0^T |f_X(\sigma+it)|^2 dt, J_\sigma = \int_{-\infty}^{\infty} |\phi(\sigma+it)|^2 dt$$

where $\phi(s) = \phi_{X,\tau}(s) = \frac{s-1}{s\cos\left(\frac{s}{2\tau}\right)} f_X(s), \quad (\tau > \frac{3}{\pi})$.

In the strip $\frac{1}{2} \leq \sigma \leq 1+\delta$, $\phi(s)$ is holomorphic and satisfies

$$\phi(s)|^2 \ll e^{-\frac{|t|}{\tau}} |f_X(s)|^2 \qquad (57)$$

and therefore is bounded for fixed X and τ. Further, we have for $\frac{1}{2} \leq \sigma \leq 1+\delta, \sigma \neq 1$,

$$J_\sigma \ll 2\int_0^\infty e^{-\frac{t}{\tau}} |f_X(\sigma+it)|^2 dt = 2\int_0^\infty e^{-u} I_\sigma(\tau u) du$$

by partial integration and the substitution $t = \tau u$. By estimaes (55) and (56), it follows that

$$J_{1+\delta} \ll \int_0^\infty e^{-u} \left(\tau u X + 1\right) \delta^{-4} du \ll \left(\tau u X + 1\right) \delta^{-4},$$

$$J_{\frac{1}{2}} \ll \int_0^\infty e^{-u} (\tau u)^{2c} (\tau u + X) \log X du \ll \tau^{2c}(\tau+X)\log X.$$

By theorem 17, it follows that for $\frac{1}{2} \leq \sigma \leq 1+\delta$

$$\begin{aligned}
J_\sigma &\ll \left(\left(\frac{\tau}{X} + 1 \right) \delta^{-4} \right)^{\frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta}} \left(\tau^{2c}(\tau+X)\log X \right)^{\frac{1+\delta-\sigma}{\frac{1}{2}+\delta}} \\
&= X^{\frac{1-2\sigma}{1+2\delta}} \tau^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} (\tau+X)(\delta^{-4}+\log X). \qquad (58)
\end{aligned}$$

Now $|\phi(s)|^2 \gg e^{-\frac{t}{\tau}} |f_X(s)|^2 \quad (\frac{1}{2} \leq \sigma \leq 1+\delta, t \geq 1)$ and (58) gives us

$$e^{-\frac{T}{\tau}}\int_1^T |f_X(\sigma+it)|^2 dt \ll X^{\frac{1-2\sigma}{1+2\delta}}\tau^{\frac{4c(1+\delta-\sigma)}{1+2\delta}}(\tau+X)(\delta^{-4}+\log X).$$

Setting $\tau=\lambda T, \delta=\frac{\lambda'}{\log(T+X)}$, we get

$$X^{\frac{1-2\sigma}{1+2\delta}} \le X^{-(1-2\delta)(2\sigma-1)} \le X^{-(2\sigma-1)+2\delta} \ll X^{-(2\sigma-1)}$$

and

$$T^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} \le T^{4c(1+\delta-\sigma)} \le T^{4c(1-\sigma)2\delta} \ll T^{4c(1-\sigma)}.$$

Plug these estimates into (59) and the theorem is deduced. ■

Theorem 19:[16] Let $\mu\left(\frac{1}{2}\right)=c$. Then:

$$N(\sigma,T)=O\left(T^{2(1+2c)(1-\sigma)}\log^5 T\right) \tag{59}$$

for $\frac{1}{2}\le\sigma\le 1$ as $T\to\infty$.

Proof:

Define f_X as in the previous lemma. Define functions $g(s)=g_X(s), h_X(s)$ by

$$1-f_X^2=\zeta M_X(2-\zeta M_X)=\zeta g=h$$

We note that g,h and holomorphic for all $s\neq 1$ and

$$\log|h|\le\log(1+|f_X|^2)\le|f_X|^2. \tag{60}$$

Moreover, for $\sigma\ge 2$,

$$|f_X|^2\le\left(\sum_{x\ge X}\frac{\tau(n)}{n^2}\right)^2<\frac{1}{2X}<\frac{1}{2}$$

for X large enough, whence

$$-\log|h|\le-\log(1-|f_X|^2)\le 2|f_X|^2<X^{-1} \qquad (\sigma\ge 2, X>1). \tag{61}$$

Now, set $T>4$ and choose T_1 and T_2 such that $3<T_1<4<T<T_2<T+1$ and such that $h(s)$ does not vanish when $t=T_1$ or $t=T_2$ when $\frac{1}{2}\le\sigma\le 2$. Let $N_h(\sigma;T_1,T_2)$ denote the number of zeros $\beta+i\gamma$ of $h(s)$ such that $T_1\le\gamma\le T_2$.

Using Fubini's theorem of integrals and Cauchy's principal of argument, we get the integral

$$\begin{aligned}\int_\sigma^2 N_h(\sigma;T_1,T_2)d\sigma = & \int_{T_1}^{T_2}(\log|h(\sigma_o+it)|-\log|h(2+ti)|)\,dt \\ & +\int_{\sigma_0}^2(\arg h(\sigma+T_2 i)-\arg h(\sigma+T_1 i))\,d\sigma\end{aligned}$$

[16] [1] Theorem 3

for $\frac{1}{2} \leq \sigma_0 \leq 1$.

The first integral is easily estimated by (60), (61) and lemma 18:

$$\int_{T_1}^{T_2} (\log|h(\sigma_o + it)| - \log|h(2+ti)|)\, dt < \int_{T_1}^{T_2} |f_X(\sigma_0 + it)|^2 + X^{-1} dt$$
$$\ll (T+1)^{4c(1-\sigma_0)} X^{1-2\sigma_0}(T+1+X)\log^4(T+1+X) + TX^{-1} \qquad (62)$$

The second integral can be estimated in a similar fashion to a technique used in the proof of theorem 10.

Define m_r to be the total amount of points on the segment $t = T_r, \sigma_0 < \sigma < 2$ such that the real part of $h(s)$ is zero. Then, by a familiar argument,

$$|\arg h(\sigma + T_r i)| \leq (m_r + 1)\pi \qquad (r = 1, 2). \qquad (63)$$

Note that m_r is the number of zeros of the function

$$H_r(s) = \frac{1}{2}\left(h(s + T_r i) + h(s - T_r i)\right)$$

on the segment $\sigma_0 < \sigma < 2$, and can therefore not exceed the number of zeros in the circle $|s-2| \leq \frac{3}{2}$. Since $H_r(s)$ is holomorphic in the concentric circle with radius $\frac{7}{4}$, we get by lemma 9:

$$\left(\frac{\frac{7}{4}}{\frac{3}{2}}\right)^{m_r} = \left(\frac{7}{6}\right)^{m_r} \leq \max_{|s-2| \leq \frac{7}{4}} \left|\frac{H_r(s)}{H_r(2)}\right| \leq \max_{\sigma \geq \frac{1}{4}, 1 \leq t \leq T+3} \frac{|h(s)|}{\Re h(2 + T_r i)} < (T+X)^C$$

for some constant C, by definition of $h(s)$ and $\Re h(s) > \frac{1}{2}(\sigma \geq \frac{1}{2})$. Thus $m_r \ll \log(T+X)$, and therefore

$$\int_{\sigma_0}^{2} (\arg h(\sigma + T_2 i) - \arg h(\sigma + T_1 i))\, d\sigma \ll \log(T+X). \qquad (64)$$

Using the inequalities $TX^{-1} \leq TX^{1-2\sigma_0}$ and $\log(T+X) \leq X^{2(1-\sigma_0)} \log(T+X)$, we get, by combining the estimates for the two integrals:

$$\int_{\sigma_0}^{2} N_h(\sigma; T_1, T_2) d\sigma \ll T^{4c(1-\sigma_0)}(TX^{1-2\sigma_0} + X^{2(1-\sigma_0)})\log^4(T+X). \qquad (65)$$

Since $N_h \geq N_\zeta$ by definition of $h(s)$, we get

$$\int_{\sigma_0}^{2} N_h(\sigma; T_1, T_2) d\sigma \geq \int_{\sigma_0}^{\sigma_0 + \delta} N_\zeta(\sigma; T_1, T_2) d\sigma \geq \delta N_\zeta(\sigma_0 + \delta; T_1, T_2) \qquad (66)$$

for $0 < \delta < 1$. Set $\sigma = \sigma_0 + \delta$. Since $N_\zeta(\sigma, T) \ll N_\zeta(\sigma; T_1, T_2)$, we deduce from ((65) and (66):

$$N_\zeta(\sigma, T) \ll \frac{1}{\delta} T^{4c(1-\sigma+\delta)}(TX^{1-2\sigma+2\delta} + X^{2(1-\sigma+\delta)}\log^4(T+X)$$

for $\frac{1}{2} + \delta \leq \sigma \leq 1$. For $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \delta$, we have by theorem 10:

$$N_\zeta(\sigma, T) \ll T \log T \leq T^{2(1-\sigma+\delta)} \log T.$$

The theorem thus holds for all $\frac{1}{2} \leq \sigma \leq 1$ by setting $X = T > 4$ and $\delta = \frac{1}{\log T}$. ■

According to the Lindelöf hypothesis, we have $\mu\left(\frac{1}{2}\right) = c = 0$, which gives

$$N(\sigma, T) = O(T^{2(1-\sigma)} \log^5 T). \tag{67}$$

In other words, the density hypothesis follows from the Lindelöf hypothesis, which again follows from the Riemann hypothesis.

The density hypothesis is a much weaker assumption than the Riemann hypothesis, yet under the assumption of the density hypothesis, we are able to bound gaps between consecutive primes almost as well as under the Riemann hypothesis.

Let $g_n = p_{n+1} - p_n$ where p_n denotes the nth prime. Cramér showed that on the Riemann hypothesis, we have the bound [17]

$$g_n = (\sqrt{p_n} \log p_n),$$

which is to say that every interval $[x, c\sqrt{x} \log x]$ for some c when x large enough. This is still a much wider estimate than the proposed *Cramér's conjecture* which states $g_n = O(\log^2(n)$, which remains to be proven, even under the assumption of the Riemann hypothesis. Under the density hypothesis, g_n can be shown to be $O(p_n^{\frac{1}{2}+\epsilon})$

Hoheisel showed the following theorem (which we will not prove):

Theorem 20 (Hoheisel's theorem):[18] Suppose (i) that $\zeta(s)$ has no zeros in the domain

$$\sigma > 1 - A\frac{\log \log t}{\log t} \qquad (A > 0, t > t_0 > 3),$$

and (ii) that

$$N(\sigma, T) = O(T^{b(1-\sigma)} \log^B T)$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ as $T \to \infty$, where $b > 0$ and $B \geq 0$. Then, for any α satisfying

$$1 - \frac{1}{b + \frac{B}{A}} < \alpha < 1,$$

we have

$$\pi(x + x^\alpha) - \pi(x) \sim \frac{x^\alpha}{\log x} \tag{68}$$

as $x \to \infty$, and therefore also

$$g_n = p_{n+1} - p_n = O(p_n^\alpha). \tag{69}$$

[17] [6]

[18] [5] Theorem 1

Condition (i) of the theorem is met, as proven by Littlewood. The truth of the density hypothesis clearly implies condition (ii), with $b = 2$. Hence, under the density hypothesis, Hoheisel's theorem tells us that (68) and (69) are valid for all α such that

$$\lim_{A\to\infty} 1 - \frac{1}{b + \frac{B}{A}} = 1 - \frac{1}{b} = \frac{1}{2} < \alpha < 1. \tag{70}$$

Hence $g_n = O(p_n^{\frac{1}{2}+\epsilon})$ under the density hypothesis.

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