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# Underactuated mechanical systems: Whether orbital stabilization is an adequate assignment for a controller design?

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**Abstract:** The paper contributes to developing algorithms for motion planning and motion control for mechanical systems with two and more passive degrees of freedom by exploring a challenging example in details. As shown, some of arguments of motion planning methods developed for systems of underactuation degree one can be generalized for novel demanding settings, while corresponding arguments and concepts for controller design should be substantially reconsidered and updated. Rigorous theoretical results are well supported by numerical studies.



*Keywords:* Underactuared systems, Zhukovsky stability, nested representation of a motion

## 1. INTRODUCTION

This note is aimed at emphasizing structural limitations and challenges present in developing motion planning and controller design architectures for stabilization of motions of underactuated mechanical systems with two (and more) passive degrees of freedom (DoF). As argued below, both assignments – being primary explored for the mechanical systems of underactuation degree one, – possess new features for systems of underactuation degree two (and more), which are absent for the class of systems having only one passive DoF.

One of such generic peculiarity of a mechanical system with several passive DoF comes from an observation that if a feedforward input is chosen as an operator of the system states, then *most of feasible forced behaviors of such system are likely to be non-periodic* even though they are enforced to be bounded. This feature is well-known for control-free conservative mechanical systems: their bounded solutions are generically non-periodic and the corresponding orbits are not closed. Two noticeable examples here, see Arnold (1989), are

- rotations of a rigid body having a fixed point moving in absence of the gravitational field (the Euler case);
- planar rotations of a point mass moving in a central field, which potential function is different from the quadratic or Kepler cases.

For both examples, almost any bounded solution can be visualized as a dense winding of the corresponding configuration variables on a 2-dimensional torus $\mathbb{T}$, making it ambiguous to use a concept of orbital (Poincare) stability for examining asymptotic properties of such solutions. By definition, the notion of orbital stability of a motion relies on computing a distance between a perturbed solution and a subset of the phase space of the system defined by the orbit of a nominal behavior. However, due to the dense winding of the nominal solution on $\mathbb{T}$ such distance will only depend on velocities of the perturbed behavior. The part of the distance function measuring the deviation of configuration variables of the perturbed solution *is always equal to zero*. Indeed, since the nominal behavior is represented by a dense winding on $\mathbb{T}$, then the distance from any point of $\mathbb{T}$ to its orbit is zero, and thus so for any perturbed solution at any time moment.

The paper contributes to the topic by a comprehensive discussion of the mentioned & related features of motion planning and controller design assignments by exploring a nontrivial example of controlled mechanical systems with two passive DoF: it consists of a passive spherical pendulum put on a puck, which is allowed sliding without friction on the horizontal plane and which position can be controlled by two independent external (control) forces acting along $x_1$-and $x_2$-axes, see Fig. 1. The first contribution of the paper provided for that example allows both extending and illustrating one of possible generalizations of the commonly used method of motion planning for systems with one passive DoF. That method – emerged two decades ago as a specific tool for simultaneous planning and orbital stabilization of periodic gaits of walking machines, see Grizzle et al. (2001); Aoustin and Formal'sky (2003); Chevallereau et al. (2003) and others, – was developed under the assumption that a controller is used to enforce a sufficient number of geometric relations [1] in-between coordinates written as a nested parametrization of a nominal behavior. Such format simply means that along the motion all degrees of freedom of the system can be written as

[1] often referred to as *virtual holonomic constraints*


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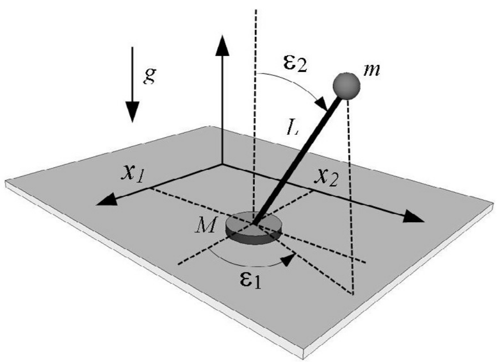


Fig. 1. A spherical pendulum on a puck. The coordinates $x_1$, $x_2$ represent the position of the puck on the horizontal plane; the angles $\varepsilon_1$ and $\varepsilon_2$ give the orientation of the pendulum with respect to the inertial frame.

smooth functions of one of coordinates. The representation of forced motions has been found instrumental for other classes of systems of underactuation degree one even though their descriptions have not necessarily included hybrid dynamics and controller design procedures have required new ideas, see e.g. Freidovich et al. (2009); Mettin et al. (2010); Shiriaev et al. (2010), and new computational tools, see Gusev et al. (2016) and others.

Despite the fact that the nested parametrization of feasible behaviors *cannot be literally used in planning motions of systems with two and more passive DoF*, the arguments elaborated for the example elucidate one of possible generalizations. In particular, a realizable part of such parametrization becomes interpreted as a feedback transform of the dynamics into an integrable control-free system. Consequently, solutions of that integrable control-free system are found and the nested parametrization – as it would be for the system with underactuation degree one – is re-established and successfully re-used. Without surprise and similarly to control-free mechanical systems with several DoF, most of derived in such a way feasible behaviors of the spherical pendulum on a puck with two passive DoF are found to be non-periodic.

The second contribution of the paper suggests novel tools for developing feedback controllers for the newly found non-periodic solutions of the underactuated mechanical system with two passive DoF. As commented above, the standard settings of *orbital stabilization as a concept* become deficient and *inadequate* for controlling the transverse dynamics in a vicinity of a non-periodic nominal solution. Instead, we suggest to invoke and explore another stability concept appropriate for the situation – the so-called Zhukovsky stability, see Leonov (2006); Shiryaev et al. (2019). The difference between two notions (Poincare vs. Zhukovsky stability) is in computing the distance from the perturbed behavior to the nominal one: for checking an orbital stability one has to compute a distance from a current state on the perturbed solution to the set defined as orbit of the nominal solution, while for Zhukovsky stability one measures such distance along a moving Poincare section defined as a smooth family of locally disjoint hypersurfaces, which are transverse to the nominal solution and parametrized by points on it.

In the paper, we suggest a constructive procedure for *defining a set of transverse coordinates* for the found non-periodic bounded solutions of the system *analytically!* This exceptional result for deriving explicitly *transverse coordinates* in a vicinity of a non-periodic trajectory *defined on an infinite interval of time at once*, allows realizing various numerically challenging steps including a computation of transverse linearizations of the system dynamics in symbolic form. Such auxiliary linear control systems are indispensable both for controller design and for analysis of a closed loop system by Lyapunov methods.

The paper is organized as follows. The settings and the problem formulation are given in Section 2. The main results illustrating steps in planning feasible behaviors of the spherical pendulum on a puck possessing two passive degrees of freedom are collected in Section 3. In turn, Section 4 provides the discussion of constructive procedures for deriving transverse coordinates and their use in analysis and feedback controller design for asserting Zhukovski stability of the nominal motion found for the case study. Sections 5 and 6 suggest some results of computer simulations and a collection of concluding remarks summarizing the contributions.

## 2. PROBLEM FORMULATION AND SYSTEM DYNAMICS

The main object of investigation of the note is an underactuated mechanical system consisting of a spherical pendulum affected by the gravity and put on a puck that can freely move on a horizontal table, see Fig. 1. It is used for examining the following tasks. Namely,

**Task 1:** What are feasible behaviors of the system when the puck is forced by a controller to follow a curve on a horizontal table? Do there exist behaviors such that the pendulum remains over the horizontal for all moments of time?

**Task 2:** How to introduce transverse coordinates for analysis of transverse dynamics and for design of a feedback controller that would provide local contraction or convergence of perturbed solutions to a nominal trajectory found in solving Task 1?

The status of the system under consideration shown on Fig. 1 is well defined by four generalized coordinates with $x = [x_1; x_2]$ being the position of the puck on the horizontal and $\epsilon = [\epsilon_1; \epsilon_2]$ being the precession and nutation angles of the pendulum with respect to the inertial frame. The dynamics of the system compactly written as the Euler-Lagrange equations have the form

$$\frac{d}{dt}\Big[\frac{\partial \mathcal{L}}{\partial \dot{\epsilon}}\Big] - \frac{\partial \mathcal{L}}{\partial \epsilon} = 0, \qquad \frac{d}{dt}\Big[\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big] = \tau, \tag{1}$$

where $\tau = [\tau_1; \tau_2]$ is the vector of external (control) forces acting just on a puck along $x_1$-and $x_2$-directions respectively. The lack of external torques in the first part of Eqn. (1) underlines the fact that the dynamics of the pendulum is passive, and, therefore, the pendulum's behavior is only affected by the gravity and the movement of the puck. With the parameters $M$ and $m$ being masses of the puck and the pendulum; $L$ being the distance to the center of mass of the pendulum from the suspension; $g$ being the acceleration due to gravity, the Lagrangian $\mathcal{L}(\cdot)$

in Eqn. (1) is defined as the difference between the kinetic and potential energies

$$\mathcal{L}(\cdot) = K_{puck}(\cdot) + K_{pend}(\cdot) - \Pi_{pend}(\cdot),$$

where $K_{puck} = \frac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2)$, $\Pi_{pend} = mgL\cos(\epsilon_2)$, and

$$K_{pend} = \frac{1}{2}m\left(\left[\frac{d}{dt}\{x_1 + L\cos(\epsilon_1)\sin(\epsilon_2)\}\right]^2 + \left[\frac{d}{dt}\{x_2 + L\sin(\epsilon_1)\sin(\epsilon_2)\}\right]^2 + \left[\frac{d}{dt}\{L\cos(\epsilon_2)\}\right]^2\right).$$

## 3. MAIN RESULT: STEPS IN MOTION PLANNING

In order to organize a search of feasible behaviors of the system with a limited travel of a puck on a table, it is convenient to analyze the dynamics of $x$-variables re-written in polar coordinates $(R, \psi)$ as

$$x_1 = R\cos(\psi), \qquad x_2 = R\sin(\psi).$$

New variables become suitable in characterizing yet unknown induced behavior by relating an arbitrary (unlimited) variation of a polar angle $\psi(\cdot)$ with the angles $\epsilon_1(\cdot)$ and $\epsilon_2(\cdot)$ of the pendulum along such a motion. Thus, limits imposed on a travel of the puck can be converted into constraints on the radius variable $R(\cdot)$.

By inducing relations between the angles, one implicitly calls upon the framework of a nested representation of a motion suggested in Grizzle et al. (2001); Aoustin and Formal'sky (2003); Chevallereau et al. (2003). However, that approach cannot be implemented here as it was originally described for systems of underactuation degree one. This is primarily due to the absence of a sufficient number of independent control inputs for the case study, which would be required for that settings. Indeed, the method assumes that the mechanical system of four degrees of freedom is equipped with at least three actuators, while the system under consideration has only two.

To overcome the limitation of the method and to enlighten one of its possible generalizations, let us start exploring the system dynamics provided that available control inputs $\tau(\cdot)$ ensure an invariance of only **one geometric relation** between precession and polar angles $\epsilon_1(\cdot)$ and $\psi(\cdot)$, which is for simplicity chosen as

$$\epsilon_1(t) \equiv \psi(t), \qquad \forall\, t. \tag{2}$$

The next three statements describe various properties of passive dynamics of the system (1) provided that the relation (2) is indeed kept invariant by feedback.

*Lemma 1.* If a forced solution

$$[\epsilon_1^*(t); \epsilon_2^*(t); R^*(t); \psi^*(t)],$$

of the system (1) defined for $t \in [0, T]$, satisfies the relation (2) for that interval of time, then, by necessity, these variables satisfy the following two differential equations

$$(R + L\sin(\epsilon_2))\ddot{\epsilon}_1 + 2\dot{R}\dot{\epsilon}_1 + 2L\cos(\epsilon_2)\dot{\epsilon}_1\dot{\epsilon}_2 = 0 \tag{3}$$

$$L\ddot{\epsilon}_2 - \cos(\epsilon_2)\big(R + L\sin(\epsilon_2)\big)\dot{\epsilon}_1^2 + \ddot{R}\cos(\epsilon_2) - g\sin(\epsilon_2) = 0 \tag{4}$$

defined w.r.t. three variables $\epsilon_1(\cdot)$, $\epsilon_2(\cdot)$ and $R(\cdot)$. ■

*To prove* Lemma 1, one needs to substitute the relations

$$x_1(t) = R(t)\cos\big(\epsilon_1(t)\big), \quad x_2(t) = R(t)\sin\big(\epsilon_1(t)\big) \tag{5}$$

and their first and second time derivatives into the passive dynamics of the system (1). Collecting the terms results in two equations (3)-(4). ■

*Lemma 2.* If three scalar $C^2$-smooth functions

$$\epsilon_1(t), \quad \epsilon_2(t), \quad R(t)$$

defined for $t \in [0, T]$, satisfy of the equation (3) for that interval of time, then the following function

$$I_1(t) := [L \cdot \sin(\epsilon_2(t)) + R(t)]^2 \cdot \dot{\epsilon}_1(t) \tag{6}$$

remains constant on the interval $[0, T]$, i.e. $I_1(t) \equiv I_1(0)$ $\forall\, t \in [0, T]$. Furthermore, in this case the time derivative of $\epsilon_1(\cdot)$ cannot change its sign on $[0, T]$ and the behavior of this variable is strictly monotonic. ■

*To prove* Lemma 2, one can observe that differentiating the function $I_1(\cdot)$ with respect to time, s/he obtains the expression

$$\frac{d}{dt}I_1 = [L\sin(\epsilon_2) + R] \times \big[\text{Left hand side of Eqn. (3)}\big].$$

It is equal zero due to the fact that the right hand side of Eqn. (3) for the chosen arguments is zero. Therefore, $I_1(\cdot)$ keeps its constant value for the time interval $[0, T]$, where the functions $\epsilon_1(\cdot)$, $\epsilon_2(\cdot)$, $R(\cdot)$ are defined and satisfy (3).

To prove the second assertion of Lemma, assume on contrary that $\dot{\epsilon}_1(t_k)$ approaches zero for some sequence of time moments $0 \le t_1 < \cdots < t_k < \cdots \le T$, i.e

$$\lim_{t_k \to t_*} \dot{\epsilon}_1(t_k) = 0.$$

Since $I_1(t) \equiv I_1(0)$ and $\text{sign}(I_1(t)) \equiv \text{sign}(\dot{\epsilon}_1(t))$ for any $t \in [0, T]$, then, by necessity, the factor $[L\sin(\epsilon_2(t)) + R(t)]^2$ of $\dot{\epsilon}_1(t)$ in the right-hand side of (6) should approach $+\infty$ on that sequence, i.e.

$$\lim_{t_k \to t_*} [L\sin(\epsilon_2(t_k)) + R(t_k)]^2 = +\infty.$$

However, the last property contradicts to the assumption that $R(\cdot)$ is a $C^2$-smooth function on $[0, T]$ and, therefore, bounded on $[0, T]$. Hence, there exists a constant $\delta > 0$ such that $\dot{\epsilon}_1^2(t) \ge \delta > 0$ for all $t \in [0, T]$ and the behavior of the variable $\epsilon_1(\cdot)$ is strictly monotonic on $[0, T]$. ■

To proceed further with the generic trajectory planning arguments devised originally for systems of underactuation degree one, one can search for **another geometric relation** between degrees of freedom of the system, which can be enforced by a controller on a motion, and which allow integrating the passive dynamics (3)-(4). The next statement describes one of such choices.

*Lemma 3.* Suppose the conditions of Lemma 1 are valid and, in addition, the control variable $\tau(\cdot)$ ensures that along a forced solution

$$q^*(t) = [\epsilon_1^*(t); \epsilon_2^*(t); R^*(t); \psi^*(t)], \quad t \in [0, T]$$

of the system, the behavior of the coordinate $R(\cdot)$ can be written as a $C^2$-smooth function of the coordinate $\epsilon_2(\cdot)$, i.e.

$$R^*(t) = r\left(\epsilon_2^*(t)\right), \qquad \forall \quad t \in [0, T] \tag{7}$$

If the function $r(\cdot)$ in Eqn. (7) is known, then the behavior $q^*(t)$ can be found by integrating the passive dynamics (3)-(4) analytically. ■

*To prove* Lemma 3 one can observe that the relations (2) and (7) allow computing time behaviors of coordinates $\psi^*(\cdot)$ and $R^*(\cdot)$ provided that time evolutions of $\epsilon_1^*(\cdot)$ and $\epsilon_2^*(\cdot)$ are given. In turn, $\epsilon_1^*(\cdot)$ can be computed from the invariant (6) provided that the function $\epsilon_2^*(\cdot)$ is available.

To this end, let us investigate the possibility to transform the equation (4) into the second order differential equation with respect to only one unknown variable $\epsilon_2(\cdot)$. For that purpose, observe that the relation (7) valid on the solution implies the corresponding identities for the first and second time derivatives of $R(\cdot)$ along the behavior

$$\dot{R}^*(t) = r'\left(\epsilon_2^*(t)\right)\dot{\epsilon}_2^*(t),$$
$$\ddot{R}^*(t) = r''\left(\epsilon_2^*(t)\right)\left[\dot{\epsilon}_2^*(t)\right]^2 + r'\left(\epsilon_2^*(t)\right)\ddot{\epsilon}_2^*(t).$$

Hence, the equation (4), which defines the behavior of unknown variable $\epsilon_2(\cdot)$, takes new equivalent form

$$\begin{aligned}\big(L + r'(\epsilon_2)\cos(\epsilon_2)\big)\ddot{\epsilon}_2 - \cos(\epsilon_2)\big(r(\epsilon_2) + L\sin(\epsilon_2)\big)\dot{\epsilon}_1^2 \quad (8)\\ + r''(\epsilon_2)\cos(\epsilon_2)\dot{\epsilon}_2^2 - g\sin(\epsilon_2) = 0.\end{aligned}$$

The derived system depends on $\dot{\epsilon}_1$, and to remove such dependence, one can take advantage of the invariant (6), which is available for that motion due to Lemma 2 and which allows rewriting Eqn. (8) further in the form

$$\begin{aligned}\big(L + r'(\epsilon_2)\cos(\epsilon_2)\big)\ddot{\epsilon}_2 + r''(\epsilon_2)\cos(\epsilon_2)\dot{\epsilon}_2^2 \quad (9)\\ - \cos(\epsilon_2)\frac{I_1(0)^2}{\big(r(\epsilon_2) + L\sin(\epsilon_2)\big)^3} - g\sin(\epsilon_2) = 0.\end{aligned}$$

The equation (9) becomes an ODE

$$\alpha(\epsilon_2)\ddot{\epsilon}_2 + \beta(\epsilon_2)\dot{\epsilon}_2^2 + \gamma(\epsilon_2) = 0, \quad (10)$$

with the coefficients given by

$$\alpha(\epsilon_2) = L + r'(\epsilon_2)\cos(\epsilon_2), \quad \beta(\epsilon_2) = r''(\epsilon_2)\cos(\epsilon_2),$$
$$\gamma(\epsilon_2) = -\cos(\epsilon_2)\frac{I_1(0)^2}{\big(r(\epsilon_2) + L\sin(\epsilon_2)\big)^3} - g\sin(\epsilon_2).$$

As known, the dynamics of (10) can be equivalently re-written as a mechanical system with one passive degree of freedom for any choice of the functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, i.e.

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}_\alpha}{\partial\dot{\epsilon}_2}\right] - \frac{\partial\mathcal{L}_\alpha}{\partial\epsilon_2} = 0 \text{ with } \mathcal{L}_\alpha = \frac{\mu(\epsilon_2)}{2}\dot{\epsilon}_2^2 - \Pi(\epsilon_2). \quad (11)$$

Therefore, the system (9) can be integrated. The explicit form of its first integral is given in Shiriaev et al. (2005).■

The conclusions derived in Lemmas 1-3 allow performing constructive steps in representing a set of feasible forced behaviors of a spherical pendulum of a puck (1), which are summarized in the next

*Theorem 1.* Consider a forced behavior

$$q^*(t) = [\epsilon_1^*(t); \epsilon_2^*(t); x_1^*(t); x_2^*(t)]$$

of the system (1), which is well defined for $t \in [0, T]$ and consistent with two kinematic relations

$$x_1^*(t) = r(\epsilon_2^*(t))\cos\epsilon_1^*(t),\ x_2^*(t) = r(\epsilon_2^*(t))\sin\epsilon_1^*(t) \quad (12)$$

ensured by a feedback controller $\tau$ for the same time interval, then $q^*(t)$ admits the following representations:

(1) The function $q^*(t)$ can be computed as a solution of the Euler-Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}_{aug}(q,\dot{q})}{\partial\dot{q}}\right] - \frac{\partial\mathcal{L}_{aug}(q,\dot{q})}{\partial q} = 0$$

with new Lagrangian $\mathcal{L}_{aug}(\cdot)$ defined by

$$\begin{aligned}\mathcal{L}_{aug} = \mathcal{L}_{red} \quad (13)\\ +\tfrac{1}{2}[\dot{x}_1 - r'(\epsilon_2)\cos(\epsilon_1)\dot{\epsilon}_2 + r(\epsilon_2)\sin(\epsilon_1)\dot{\epsilon}_1]^2\\ +\tfrac{1}{2}[\dot{x_2} - r'(\epsilon_2)\sin(\epsilon_1)\dot{\epsilon}_2 - r(\epsilon_2)\cos(\epsilon_1)\dot{\epsilon}_1]^2\end{aligned}$$

where the function $\mathcal{L}_{red}(\cdot)$ is

$$\mathcal{L}_{red} = \tfrac{1}{2}\Big(\big[L\sin(\epsilon_2) + r(\epsilon_2)\big]^2\dot{\epsilon}_1^2 + \mu(\epsilon_2)\dot{\epsilon}_2^2\Big) - \Pi(\epsilon_2)$$

(2) The function $q^*(t)$ can be computed based on the kinematic relations (12), where $\epsilon_2^*(t)$ is determined as a corresponding solution of the integrable system (11) and $\epsilon_1^*(t)$ is determined from the invariance of the function (6) combined with Eqn. (7). ■

*Proof* of the 1st part of Theorem 1 follows from the direct computations similar to those done in Shiriaev et al. (2014), while the second part of Theorem 1 constitutes the summary of calculations derived in Lemmas 1-3. ■

An integrability of the passive dynamics found for the the case study and reported in Theorem 1 has a number of consequences and implications, where the immediate one is the following: most of found forced behaviors $q_\star(t) = [x_\star(t); \epsilon_\star(t)]$ of the spherical pendulum on the puck are non-periodic even though they are bounded.

*Theorem 2.* Consider a forced bounded behavior

$$q^*(t) = [\epsilon_1^*(t); \epsilon_2^*(t); x_1^*(t); x_2^*(t)]$$

of the system (1) described in Theorem 1. If the solution is periodic, then the initial conditions for the spherical pendulum

$$[\epsilon_1^*(0); \epsilon_2^*(0); \dot{\epsilon}_1^*(0); \dot{\epsilon}_2^*(0)]$$

satisfy the following equation for some $k,\ l \in \mathbb{N}$

$$2\pi\frac{k}{l} = 2\int_{\epsilon_2^{min}}^{\epsilon_2^{max}} \frac{I_1(0)}{(r(\epsilon_2) + Lsin(\epsilon_2))^2\sqrt{\Phi}}d\epsilon_2 \quad (14)$$

where

$$\Phi(\epsilon_2, \epsilon_2^*(0), \dot{\epsilon}_2^*(0)) = \psi(\epsilon_2^*(0), \epsilon_2)(\dot{\epsilon}_2^*(0))^2 - \int_{\epsilon_2^*(0)}^{\epsilon_2}\psi(s, \epsilon_2)\frac{2\gamma(s)}{\alpha(s)}ds$$

Here the scalar function $\psi(\cdot,\cdot)$ is defined as

$$\psi(a, b) = \exp\left\{-2\int_a^b \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}$$

and $I_1(0)$ is calculated according to (6) at initial time.

In opposite, if a vector of initial conditions of a solution $q^*(\cdot)$ described in Theorem 1 meets the equation (14) for some $k,\ l \in \mathbb{N}$, then $q^*(\cdot)$ is periodic. ■

*Proof* of Theorem 2 is omitted due to the space limit and can be obtained by request. The main conclusion from Theorem 2 is that a set of initial conditions of periodic solutions listed for the case study in Theorem 1 is contained in a countable union of subsets of co-dimension one of the 4-dimensional state space of variables $[\epsilon_1; \epsilon_2; \dot{\epsilon}_1; \dot{\epsilon}_2]$. Therefore, for most of initial conditions the corresponding forced solutions of the system (1) listed in Theorem 1 are not periodic.

Meanwhile, it is relatively easy to single out behaviors from those given in Theorem 1, which are bounded. Indeed, choose any constant $I_1(0) \neq 0$ and a smooth scalar function $r(\cdot)$ such that solution of the system (10) is periodic. The corresponding counterpart of the pendulum configuration $\epsilon_1(\cdot)$ partly defined by the invariant value $I_1(0)$ will be bounded for any choice of $\epsilon_1(0)$ even when the equation (14) is faulty for any $k,\, l \in \mathbb{N}$! Therefore, obtained in such a way precession and nutation angles $\epsilon_1(\cdot)$ and $\epsilon_2(\cdot)$ representing a status of passive dynamics and considered separately on the motion from the rest of coordinates, will define a dense winding on $\mathbb{T}$ as it would be, for instance, in analysis of two mechanical control-free systems mentioned in Introduction.

## 4. MAIN RESULT: TRANSVERSE COORDINATES IN A VICINITY OF A NON-PERIODIC SOLUTION

The observation brought in the end of the last Section makes deficient some of classical settings used for stabilization and for analysis of stability if applied for non-periodic motions of underactuated mechanical systems. Indeed, restricting the attention to a behavior of the angles along any perturbed motion of the system, one can readily recognize that a dense winding on a torus of the nominal motion implies that the distance from any perturbation to the orbit of the nominal behavior is zero. This fact makes *any* feedback controller successful provided that it is requested to stabilize orbitally that part of the system variables for any of found non-periodic motions. The modified settings for controller design can be grounded on another concept for characterizing the stability of motion introduced N.E. Zhukovsky in 1882, which is relevant for the case study, see Leonov (2006).

*Definition 1.* Given a motion $x = x(t, x_0) \in \mathbb{R}^{2n}$ of a dynamic system $\dot{x} = f(x)$ well defined for all $t \geq 0$ and a set of homeomorphisms

$$Hom = \{\tau(\cdot) \mid \tau : [0, +\infty) \to [0, +\infty),\ \tau(0) = 0\},$$

the motion is said to be Zhukovsky stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any vector $y_0$, satisfying the inequality $|x_0 - y_0| \leq \delta(\varepsilon)$, there exists a function $\tau(\cdot) \in Hom$ such that the following inequality holds

$$|x(t, x_0) - x(\tau(t), y_0)| \leq \varepsilon, \quad \forall t \geq 0. \tag{15}$$

If, in addition, for some $\delta_0 > 0$ and any $y_0$ from the ball $\{y \mid |x_0 - y| \leq \delta_0\}$ there is $\tau(\cdot) \in Hom$ such that

$$\lim_{x \to \infty} |x(t, x_0) - x(\tau(t), y_0)| = 0, \tag{16}$$

then the motion $x(t, x_0)$ is said to be asymptotically stable in the sense of Zhukovsky. ∎

The next result (Leonov, 2006, Proposition 3, p. 285) becomes instrumental in synthesis of a feedback controller for achieving asymptotic Zhukovski stability of a nominal motion by taking advantage of linearization of the dynamics of the so-called transverse coordinates.

*Theorem 3.* Given a $C^2$-smooth nonlinear dynamic system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n} \tag{17}$$

and its motion $x = x(t, x_0)$ well defined for all $t \geq 0$, which belongs to a compact subset $\Omega \subset \mathbb{R}^n$, where the system (17) has no equilibrium, introduce for the motion a family of moving Poincare sections

$$S\left(x(t, x_0), \varepsilon\right) = \left\{y : (y - x(t, x_0))^T f(x(t, x_0)) = 0\right\}$$

and new smooth and mutually orthogonal coordinates $v_1$, $v_2$, $\ldots$, $v_{2n}$ in a vicinity of the motion such that the first component is aligned with $f(x(t, x_0))$ and others are orthogonal to $f(x(t, x_0))$. In this way, the $(2n-1)$ variables $v_2$, $\ldots$, $v_{2n}$ will be coordinates on the moving Poincare sections $S(\cdot)$ defined for the motion and are referred to as transverse. The linearization of the dynamics of transverse coordinates along the motion $x = x(t, x_0)$ is a linear time-varying system called a transverse linearization. Then asymptotic Zhukovsky stability of the motion $x = x(t, x_0)$ for the nonlinear system (17) follows from exponential stability of the origin of the transverse linearization computed for $x = x(t, x_0)$ . ∎

The statement suggests one of possible controller designs for achieving an asymptotic Zhukovsky stability of a motion for a nonlinear controlled system.

*Theorem 4.* Given a $C^2$-smooth nonlinear dynamic system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m \tag{18}$$

and its motion $x = x(t, x_0)$ obtained in response of a trivial input $u = 0$. Suppose the motion $x = x(t, x_0)$ is well defined for all $t \geq 0$ and belongs to a compact subset $\Omega \subset \mathbb{R}^n$, where the system (17) has no equilibrium. Introduce a family of moving Poincare sections, associated transverse coordinates as done in Theorem 3 and consider a smooth feedback controller

$$u = u(x) \quad \text{with} \quad u(x)|_{x = x(t, x_0)} \equiv 0, \tag{19}$$

which renders the origin of the transverse linearization computed for the closed loop system dynamics

$$\dot{x} = f(x) + g(x)u(x) = \tilde{f}(x) \tag{20}$$

exponentially stable. Then such feedback controller ensures asymptotic Zhukovsky stability of $x = x(t, x_0)$. ∎

Specific formats of moving Poincare sections and transverse coordinates used in Theorems 3 and 4 are convenient for establishing the link between properties of a nonlinear system and a linearization of its dynamics transverse to a given motion. However, the concepts of transverse linearization based analysis and feedback control can be equally used for other choices of moving Poincare sections and associated transverse coordinates provided that the linearization of dynamics of new quantities results in the linear systems equivalent to those that have been derived in Theorems 3 and 4.

To this end, it is worth mentioning the following candidates to serve as an alternative set of transverse coordinates $x_\perp(\cdot)$ associated with any of forced motion $q_\star(t) = [x_\star(t); \epsilon_\star(t)]$ of the system (1) found in the previous Section

$$\begin{aligned}
x_{1\perp} &:= \dot{\epsilon}_1^2 - \frac{\left[L \sin(\epsilon_{2\star}(0)) + r(\epsilon_{2\star}(0))\right]^4 \dot{\epsilon}_{1\star}^2(0)}{\left[L \sin(\phi(\epsilon_1)) + r(\phi(\epsilon_1))\right]^4} \\
x_{2\perp} &:= x_1 - r(\phi(\epsilon_1)) \cos \epsilon_1 \\
x_{3\perp} &:= x_2 - r(\phi(\epsilon_1)) \sin \epsilon_1 \\
x_{4\perp} &:= \epsilon_2 - \phi(\epsilon_1) \\
x_{5\perp} &:= \dot{x}_1 - r'(\phi(\epsilon_1))\phi'(\epsilon_1) \cos \epsilon_1 \dot{\epsilon_1} + r \sin \epsilon_1 \dot{\epsilon_1} \\
x_{6\perp} &:= \dot{x_2} - r'(\phi(\epsilon_1))\phi'(\epsilon_1) \sin \epsilon_1 \dot{\epsilon_1} - r \cos \epsilon_1 \dot{\epsilon_1} \\
x_{7\perp} &:= \dot{\epsilon}_2 - \phi'(\epsilon_1)\dot{\epsilon}_1
\end{aligned} \tag{21}$$

Here $\phi(\cdot)$ is the function that can be used for recomputing $\epsilon_2(\cdot)$ from $\epsilon_1(\cdot)$ for the given perturbed motion as

$$\epsilon_2(t) = \phi\left(\epsilon_1(t)\right), \quad \forall t.$$

The function $\phi(\cdot)$ is available for any of the found motions, since, as stated in Lemma 2, $\epsilon_1(\cdot)$ is always monotonic.

*Lemma 4.* The functions (21) constitute the full set of motion-dependent transverse coordinates for any of the found in Theorem 1 forced behaviors of the system (1). For a given behavior, they are parameterized by the initial conditions the motion and a scalar function $\phi(\cdot)$ relating behaviors of two angles of the pendulum. ∎

*To prove* Lemma 4, one can observe that all the seven functions defined by Eqns. (21) are equal to zero on the nominal motion. To verify that they are independent, we suggest to compute the Jacobian

$$\delta x_\perp = J(\cdot)\,[\,\delta q;\ \delta \dot{q}\,] \tag{22}$$

of the component-wise transformation of the system state defined by Eqn. (21). The direct calculations show that the following columns $J_i(\cdot)$ of the Jacobian are constant

$$[J_1\ J_2\ J_4\ J_5\ J_6\ J_8] = \begin{bmatrix} 0_{1\times 6} \\ I_{6\times 6} \end{bmatrix}$$

While the columns $J_7(\cdot)$ and $J_3(\cdot)$ will be equal to

$$J_7 = \begin{bmatrix} 2\dot{\epsilon}_1 \\ 0_{3\times 1} \\ r\sin(\epsilon_1) - r'\phi'\cos(\epsilon_1) \\ r\cos(\epsilon_1) - r'\phi'\sin(\epsilon_1) \\ -\phi' \end{bmatrix},\ J_3 = \frac{\partial x_\perp}{\partial \epsilon_1}.$$

Hence, to verify that the rank of $J(\cdot)$ is equal to 7, one can consider $7\times 7$ matrix function shaped by all the columns of $J(\cdot)$ except the third one. The determinant of such sub-matrix is equal to $-2\dot{\epsilon}_1$, and, since, by Lemma 2, the velocity of $\epsilon_1(\cdot)$ is separated from zero, then, the determinant is strictly separated from zero. ∎

## 5. EXAMPLE

To illustrate the contributions, let us search for those behaviors of the system for which the pendulum remains above the horizontal and the relation (7) is linear, i.e.

$$r(\epsilon_2) = R_0 + c\cdot\epsilon_2 \tag{23}$$

With such choice the passive dynamics (3)-(4) of the system take the form

$$\begin{aligned} \tfrac{d}{dt} I_1(t) = \tfrac{d}{dt}\left[\big(R_0 + c\cdot\cos(\epsilon_2) + L\sin(\epsilon_2)\big)^2\dot{\epsilon}_1\right] = 0 \\ (L + c\cdot\cos(\epsilon_2))\ddot{\epsilon}_2 - \frac{\cos(\epsilon_2)\cdot I_1(0)^2}{\big(R_0 + c\epsilon_2 + L\sin(\epsilon_2)\big)^3} \\ -g\sin(\epsilon_2) = 0 \end{aligned} \tag{24}$$

Exploring properties of (24), one can observe that for any $\dot{\epsilon}_1(0) \neq 0$ equation (24) has only one equilibrium ($\epsilon_2 = \epsilon_{20}$, $\dot{\epsilon}_2 = 0$). Indeed, to verify that one substitutes the relations $\epsilon_2 = \epsilon_{20}$, $\ddot{\epsilon}_2 = 0$ into (24), which after collecting the terms results in the equation for stationary points

$$\big(R_0 + c\cos(\epsilon_{20}) + L\sin(\epsilon_{20})\dot{\epsilon}_1(0) = -g\tan(\epsilon_{20}). \tag{25}$$

If $\epsilon_{20} \in [-\pi/2;\ \pi/2]$, then the function on the right hand side of Eqn. (25) is always decreasing and crossing the zero level when its argument is zero. At the same time, while the left hand side of Eqn. (25) is always increasing and is positive at 0. Consequently, the solution of Eqn. (25) and, therefore, the equilibrium $\epsilon_{20}$ is negative, $\epsilon_{20} < 0$.

To detect periodic solutions for (24), one can introduce family of auxiliary systems (26) with the property that some its solutions coincide with the solutions of (24)

$$\begin{aligned} (L + c\cos(e))\ddot{e} - \frac{I_1(0)^2\cos(e)}{\big(R_0 + ce + L\sin(e)\big)^3} \\ -g\sin(e) = 0. \end{aligned} \tag{26}$$

If new auxiliary ODE is initialized at $[e(0) = \epsilon_2(0), \dot{e} = \dot{\epsilon}_2(0)]^T$, then solutions of (24) with initial conditions $[\epsilon_2(0), \dot{\epsilon}_2(0)]^T$ and (26) are literally the same. In searching behaviors of ODE (26), one can start with identifying its equilibria. Here, it is worth noticing that depending on values of constant parameters $R_0$ and $c$, the system can have an asymptote, which presence results in discontinuity of the phase portrait at $e = e_{cr}$, where the function $(R_0 + ce + L\sin(e))$ is zero. Then by linearizing the dynamics at these stationary points, one can figure out the type of each equilibrium point, which can be either a saddle or a center.[2] Finally, one can analyze boundaries of regions of the phase portrait containing periodic solutions i.e. homoclinic and heteroclinic curves. Alternatively (and for the same purpose) one can analyze the properties of the energy of the auxiliary system

$$E_{aux}(e, \dot{e}) = \frac{1}{2}\dot{e}^2 + \int \Pi'_{aux}(e)de + \Pi_{aux,0}, \tag{27}$$

$$\Pi'_{aux}(e) = -\frac{I_1(0)^2\cos(e)}{(R_0 + c\ e + L\sin(e))^3} - g\sin(e). \tag{28}$$

Indeed, the auxiliary system has equilibriums at points when $\Pi'_{aux}(e)$ becomes zero, i.e. $\Pi'_{aux}(e_{eql}) = 0$. To draw conclusions about the type of equilibrium, sign of $\Pi''_{aux}(e)$ at these points should be evaluated, so that if $\Pi''_{aux}(e_{eql}) < 0$ then it is a saddle point and if $\Pi''_{aux}(e_{eql}) > 0$, then it is a center. Eqns. (27) and (28) can be also used to find the set of initial conditions such that (26) has periodic solution. For that the following inequality should hold

$$E_{aux}(e(0), \dot{e}(0)) < min\ E(e_{sdl,i}, 0), \tag{29}$$

where $e_{sdl,i}$ are saddle equilibriums of the system (26).

Coming to validation of theoretical contributions, we have perform the analysis when the parameters of the spherical pendulum on the puck and the kinematic relation (23) are equal to listed in Table 1. First, we have computed all equilibria of the auxiliary system (26), i.e. points at which $\Pi'_{aux}(e_{eql,i}) = 0$, that become equal to

$$e_{eql,1} \approx -0.0158,\ e_{eql,2} \approx -0.3463.$$

Actually chosen set of parameters correspond to the case when $\Pi'_{aux}(e)$ has discontinuity. The linearization-based analysis shows that $e_{eql,1}$ is the saddle point (since $\Pi''_{aux}(e_{eql,1}) < 0$), while $e_{eql,2}$ is the center (since $\Pi''_{aux}(e_{eql,2}) > 0$). The point $e = -0.2$ belongs to a region in-between two equilibria. Depending on the value of $\dot{e}(0)$ the system soltions will either get attracted to a periodic solution or become unstable. Interestingly, regardless of initial condition on velocity, the pendulum will never fall inwards, i.e. $\epsilon_2 = -\pi/2$. The reason for that is in existence of the invariant (6), that should be constant along the solution of (1). To find initial values of $\dot{e}(0)$ for a cycle, one can use the inequality (29):

[2] If the eigenvalues of the linearized system are real and of opposite signs then it is a saddle point, while if eigenvalues are strictly imaginary then it is a center Strogatz (2001).

Table 1. Parameters of the system

| $\dot\epsilon_1(0)$ | $\epsilon_2(0)$ | $L$ | $R_0$ | $c$ | $M$ | $m$ |
|---|---|---|---|---|---|---|
| 0.9 | $-0.2$ | 1 | 1.2 | 1.3 | 2 | 0.5 |

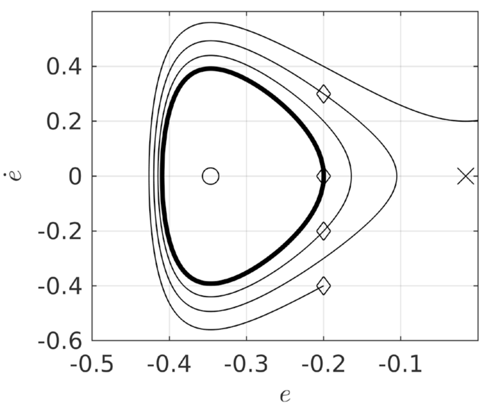


Fig. 2. Phase portrait of the auxiliary system

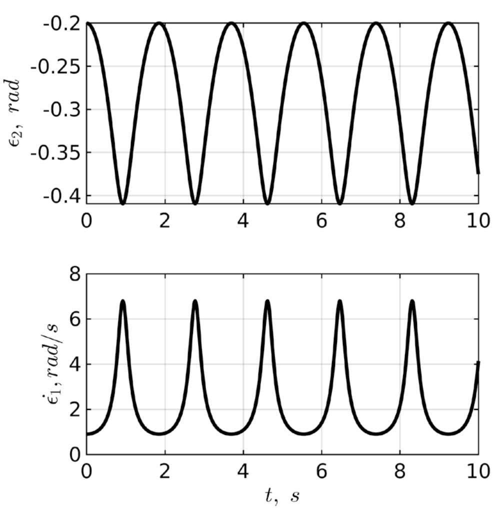


Fig. 3. Variables $\epsilon_2$ and $\dot\epsilon_1$ along the nominal trajectory

$$\tfrac{1}{2}\dot e^2(0) - 0.0595 < 0.0005 \Rightarrow \dot e(0) \in (-0.3464, 0.3464).$$

The phase portrait of the auxiliary system with behaviors of the system for different initial conditions on $\dot\epsilon_2(0)$ is shown on, Fig. 2. To visualize some of derived behaviors, the reduced dynamics have been simulated with the following initial conditions

$$\epsilon_1(0) = 0,\ \dot\epsilon_1(0) = 0.9,\ \epsilon_2(0) = -0.2,\ \dot\epsilon_2(0) = 0.$$

The behavior corresponds to a cycle depicted in bold on Fig. 2. The corresponding evolutions of $\epsilon_2(t)$ and $\dot\epsilon_1(t)$ are shown on Fig. 3. As seen, $\epsilon_2(t)$ oscillates within $[-0.41, -0.2]$ and is of period $T_{\epsilon_2} \approx 1.84$ $[sec]$. $\dot\epsilon_1(t)$ is of the same period, see (6), while the period of $\epsilon_1(t)$ is different $T_{\epsilon_1} \approx 2.76$ $[sec]$. The corresponding behaviors of $x_1$ and $x_2$ are shown on Fig. (5). And the corresponding feedforward control inputs $\tau_1$, $\tau_2$ are depicted on Fig. (6).

## 6. DISCUSSION AND CONCLUDING REMARKS

The following comments and remarks are in order

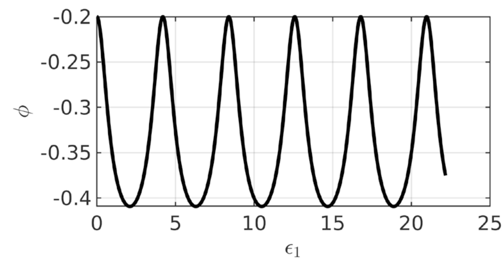


Fig. 4. The function $\phi(\cdot)$ that defines the synchronization $\epsilon_2 = \phi(\epsilon_1)$ on the nominal behavior

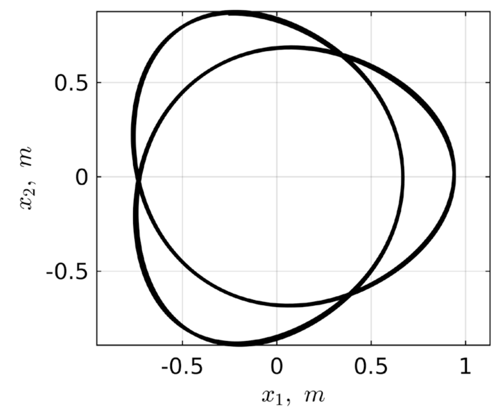


Fig. 5. The nominal trajectory of the puck coordinates

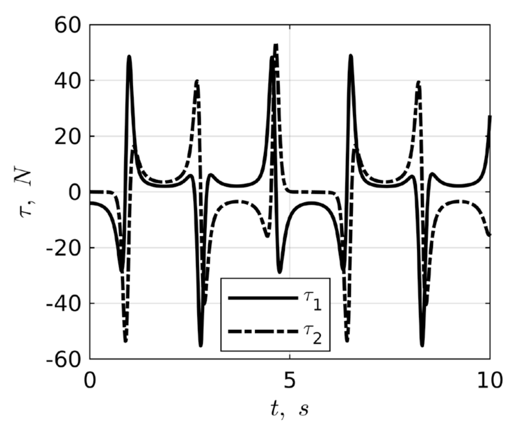


Fig. 6. Nominal forces required to move the puck along the nominal trajectory

- Developing scalable methods and constructive model-based procedures for searching feasible motions of underactuated system is needed for planning agile behaviors of robots and mechanisms with limited control authority. The paper suggests the detailed discussion of a controlled mechanical system with two passive degrees of freedom and emphasizes important steps in revealing sets of motion that can be generated by external inputs;
- The method exemplified in the paper is based on an idea of inducing by feedback those motion invariants that can lead to (partial) integration of the system dynamics. If successful, the scheme results in the compact representation of such behaviors and sets of

transverse coordinates that can be used for feedback controller design and the closed loop system analysis;

- The trivial relation (2) induced by feedback can be changed to more general one $\psi = \psi(\epsilon_1)$ and the most of points of the method will come through unless certain matching conditions for integrability of the first equation of the passive dynamics of the system (1) are in place. An interesting interpretation of that result is linked to a non-standard use of Noether theorem. Indeed, if one can find a new Euler-Lagrange system of equations, which has no external forces and one of which solutions coincides with a given forced behavior $q^*(\cdot)$ of the system (1), then the invariance of the dynamics with respect to the angle $\epsilon_1$ will lead to discovery of the corresponding conservation law for $q^*(\cdot)$. So that the result of Lemma 2 is the consequence of Theorem 1 and can be generalized for other cases;
- The main and distinctive contribution of the paper is in illustration of limits of the classical framework of orbital stability and stabilization settings for controlling motions of underactuated mechanical systems with two and more passive degrees of freedom. As shown by the example, a generic class of feasible bounded motions for the system (1) with two passive degrees of freedom reminds the description of bounded motions of a passive mechanical system with two degrees of freedom, which typically represent different windings on a torus. Since the most of such behaviors will not be periodic, then the concept of Poincare stability and stabilization for controlling such behaviors should be modified;
- The proposed alternative is based on a subtle notion of Zhukovsky stability, which has clear geometrical interpretation and relevance to the problem: if one searches for properties of a perturbed behavior negating the time, then the only way to compare it with the nominal motion is to use a moving Poncare sections and transverse coordinates to measure local deviation of one orbit from another. Surprisingly, such arguments re-used for analysis of motions of mechanical systems admit efficient analytic realization in computing motion-dependent transverse coordinates and in computing a linearization of their dynamics prior any feedback controller is defined;
- Clearly, the obtained set of transverse coordinates allows generating infinitely many equivalent representations of transverse dynamics by introducing smooth coordinate transformations in a vicinity of the nominal motion. The same comment is applied to the format of transverse linearization of the dynamics.

## ACKNOWLEDGEMENTS

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## REFERENCES

Aoustin, Y. and Formal'sky, A. (2003). Control design for a biped: Reference trajectory based on driven angles as functions of the undriven angle. *Journal of Computer and Systems Sciences International*, 42(4), 645–662.

Arnold, V. (1989). *Mathematical Methods of Classical Mechanics*, volume 60. Springer. doi:10.1115/1.1483350.

Chevallereau, C., Abba, G., Aoustin, Y., Plestan, F., Westervelt, E., Canudas De Wit, C., and J.W., G. (2003). Rabbit: A testbed for advanced control theory. *IEEE Control Systems Magazine*, 23(5), 57–79.

Freidovich, L., La Hera, P., Mettin, U., Robertsson, A., and Shiriaev, A. (2009). Shaping stable periodic motions of inertia wheel pendulum: theory and experiment. *Asian Journal of Control*, 11(5), 548–556.

Grizzle, J., Abba, G., and F, P. (2001). Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control*, 46(1), 51–64.

Gusev, S.V., Shiriaev, A.S., and Freidovich, L.B. (2016). SDP-based approximation of stabilising solutions for periodic matrix Riccati differential equations. *International Journal of Control*, 89(7), 1396–1405.

Leonov, G.A. (2006). Generalization of the Andronov-Vitt theorem. *Regular and chaotic dynamics*, 11(2), 281–289.

Mettin, U., La Hera, P., Freidovich, L., and Shiriaev, A. (2010). Parallel elastic actuators as a control tool for preplanned trajectories of underactuated mechanical systems. *International Journal of Robotics Research*, 29(9), 1186–1198.

Shiriaev, A.S., Freidovich, L.B., and Gusev, S.V. (2010). Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Transactions on Automatic Control*, 55(4), 893–906. doi: 10.1109/TAC.2010.2042000.

Shiriaev, A.S., Freidovich, L.B., and Spong, M.W. (2014). Controlled invariants and trajectory planning for underactuated mechanical systems. *IEEE Transactions on Automatic Control*, 59(9), 2555–2561. doi: 10.1109/TAC.2014.2308641.

Shiriaev, A., Perram, J.W., and Canudas-de Wit, C. (2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on automatic control*, 50(8), 1164–1176.

Shiryaev, A.S., Khusainov, R.R., Mamedov, S.N., Gusev, S.V., and Kuznetsov, N.V. (2019). On Leonov method for computing the linearization of transverse dynamics and analyzing Zhukovsky stability. *Vestnik St. Petersburg University, Mathematics*, 52(4), 334–341.

Strogatz, S. (2001). *Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering (Studies in Nonlinearity).* Westview Press.