Source Seeking With a Variable Leader Multi-Agent Fixed Topology Network

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***Abstract*— In this paper a source seeking method for multi-agent systems organized with the Leader-Follower scheme is studied. Our objective is to develop a controller for the headings of the agents. The group of agents is characterized by an initial leader, which steers all the agents towards an initial given heading. Another agent, an *active* follower, can take on the role as leader if it gets satisfactory measures from the environment. If the active follower becomes leader, it steers the whole group towards the source present in the field. We focus on a 2D case, and simulation results are reported to illustrate and validate theoretical results.**

I. Introduction

Artificial multi-agent systems have been widely studied during the few last years. They represent an important resource for the exploration of unstructured environments and spatially distributed tasks [7]–[9], [13], [16], [17], [23], [24]. Various applications are source seeking [1], [7]–[9], [13], target tracking [17], ice inspection [25].

When we deal with a multi-agent network there is a distinction between networks with or without a hierarchy among the agents. We may have a network with [1], [16], [17], [22] and without leaders [8], [9], [13], [24]. The Leader-Followers scheme has been investigated in depth in the last years. It is characterized by an agent, the *leader*, that is able to influence the states of the others, the *followers*.

One of the first studies about this scheme is given in [2]. In this work the author deals with a group of agents in which the *leader* is able to control the states of the other elements. The communication among the agents is characterized by a fixed topology graph. Necessary conditions on the structure of the communication graph are given under which the states of the followers are controllable by the leader. Further investigation about the connection between controllability of the Leader-Followers networks and the topology of the communication scheme is given in [11], [15], [18].

Other authors focused on the controllability of the Leader-Followers scheme considering a switching topology graph for communication [6], [14]. In particular [6] gives a thorough analysis of the stability of a network of agents with and without a leader. It is assumed that over a certain interval of time the neighbors of the $i-th$ agent change. Necessary conditions on the connectivity of the agents' graph over the time are given to having an agreement among the agents. In the case of the Leader-Followers scheme the followers agree to follow the leader's heading.

The common feature in the above articles is that fixed leaders are considered. The Leader-Followers scheme is typical for birds and fishes during migration. A motivation for the work in this paper is the observation that the animal which takes the role as a leader of the group changes with time. We believe that this feature may make multi-agent networks more efficient in search and survey operations.

The work [19] gives a model for the migration of the birds that is used in [3], [4] to study the evolution of leadership among the animals. According to this model, during the migration each animal spends energy in getting information from the environment and from its neighbors. This behavior is expressed by the investment parameter. If a bird spends a lot of energy in sensing the environment it is characterized by a strong investment parameter. On the other hand, if it gets information only from the neighbors, it is characterized by a low investment parameter. The leaders have a high investment parameter and influence the migration of the followers. The investment parameter changes over the evolution of the animals according to their fitness, so this change occurs over a long period of time. In [3], [4] a thorough bifurcation analysis gives the conditions under which stable migration is guaranteed.

For the particular task of source seeking, the multi-agent network has been investigated in depth in the last few years [1], [8], [9], [13], [16], [17], [23], [24]. In fact, it is important that generally the agents can only get scalar measures of the vector field, so distributed measures to compute the approximated gradient are generally needed. Source seeking tasks can be performed with the Leader-Followers scheme [1], [7], [16], [23] and with the leaderless scheme [8], [9], [24].

The work [1] deals with a source seeking mission performed using a multi-agent system in which only the leader gets scalar measures from the field. In order to get distributed measures, it moves due to a dither motion to compute the approximated gradient. The followers keep an assigned formation around the leader. Their relative position is controlled by a passivity based control. This controller guarantees that the followers move behind the leader filtering its dither motion. The asymptotic stability of the motion of the formation towards the source is proved.

In [7], [23], a group of full-actuated agents are used to perform source seeking using the scalar measure from each agent. In particular, some virtual leaders are used to achieve a prescribed formation using artificial potentials. Each real

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agent gets scalar measures that are used to compute the gradient of the environment. A Kalman filter is used to improve the quality of the gradient estimate. Asymptotic convergence to a region that contains local minima is proved.

In [16], a source seeking guidance law for UAVs agents based on a Leader-Followers scheme is studied. The followers constantly move on a circular trajectory around the leader. Only the followers are able to get scalar measures from the field. The leader receives the measures from the followers and uses them to compute the approximated gradient in order to move towards the source. It is necessary that each follower communicates with the leader during the motion. Asymptotic convergence towards the source is proved.

In this paper we consider the source seeking problem using under-actuated vehicles organized in a Leader-Followers scheme. Our model is inspired by the migration model in [4]. However in our work we consider the investment parameter as a function of the current scalar measures from the field. We want the followers to move along the trajectory imposed by the leader. But if a certain follower detects a strong scalar measure, it takes on the role as leader and imposes a new direction for the group, steering the whole group towards the source. The new direction is computed by only this agent. In order to achieve measurements of the gradient using only one agent, we assume that it is equipped with three sensors. In this way it can compute the gradient, and circular or dither motion is not needed. The gradient is computed using the method in [16], [17].

The main difference in this work compared to previous works is that in our model the leadership is exchanged according to the environment conditions, i.e. the proximity to the source. In particular, the agents move along a definite path assigned to the initial leader. But if during the motion one of the agents detects a source they deviate from the initial path. The sensing agent would take on the role as leader and would compute a new path that points towards the source. Being the new leader, it imposes the new trajectory to the rest of the agents. Which is the current leader depends on the current value of the investment parameter. This feature makes sure that the group follows the agent that has the best information available about the source they are seeking, and is expected to provide more efficient source seeking operations.

Another important feature of our approach is that it is not necessary that each follower communicates with the leader. It is only necessary that the graph for the communication scheme among the agents is connected.

The controlled agreement protocol from [11] for the Leader-Follower scheme is revisited with the addition of a variable leader in order to shift the agreement. That is, we have two leaders which can influence the other agents in order to achieve an agreement on a given trajectory. Switching between the leaders we switch the agreement among the followers and therefore the direction which the whole group follows.

The paper is organized as follows. Section II provides a brief description of the desired behavior of the system.

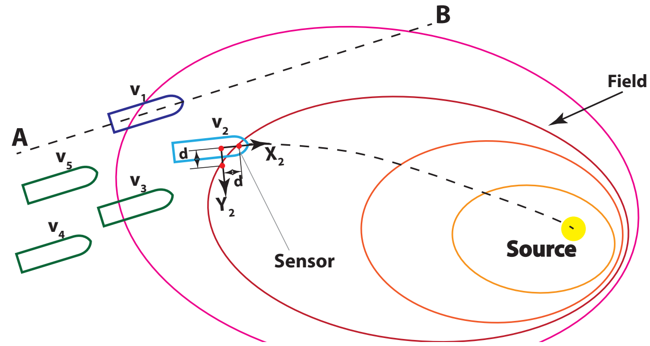

Fig. 1. Exchange of the leadership, the agent v_1 is the initial leader, the agent v_2 is the the active follower, the red dots are the sensors

Section III gives a short description of the graph theory. In Section IV the model is described. Section V presents the main results. In Section VI some simulations are presented. Section VII gives the conclusions.

II. Desired Behavior

The exchange of the leadership among the agents is conceived for exploration and survey operations. At the beginning of the mission, we want one agent to steer all the others on a fixed initial path. This is typically the path that the operators, based on their current information, expect that the source will lie along. We call this agent the *initial* leader. In the rest of the group there is at least one *active* follower that is able to sense the environment. When it detects a strong enough signal it takes on the role as leader. In other words, while the active follower is following the leader, it investigates the field. If it gets a relevant signal it will compute the direction that points towards the source of the signal and it will steer all the group in this new direction. A simple illustration is given in Figure 1.

III. Graph Theory

This section provides a brief description of the Graph Theory, further information is given in [11].

An undirected graph $\mathcal{G}$ consists of a set of n vertices $\mathcal{V}$ and an edge set $\mathcal{E}$ such that $\mathcal{E} = \{v_i, v_j\}$ with $i \neq j$, $i, j = 1, ..., n$. A graph $\mathcal{G}$ is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. If $x, y \in \mathcal{V}$ and $(x, y) \in \mathcal{E}$ then x, y are said to be *adjacent*. A graph is called *complete* if any two vertices are adjacent. A path of length r is a sequence of $r + 1$ distinct and adjacent vertices. If there is a path between any pair of vertices then the graph is said to be *connected*. The *adjacency* matrix of a graph $\mathcal{G}$ is the symmetric $[n \times n]$ matrix with 0-1 elements, such that $\mathcal{A}(i, j) = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $\mathcal{A}(i, j) = 0$ if $(v_i, v_j) \notin \mathcal{E}$. The *degree* matrix is the diagonal $[n \times n]$ matrix such that $\mathcal{D}(i, i) = d(v_i)$, where $d(v_i)$ gives the number of vertices adjacent to v_i. The *Laplacian* matrix is the $[n \times n]$ matrix defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. The matrix L is symmetric and *positive semi-definite*. Moreover its smallest eigenvalue $\lambda_1 = 0$ and if its second smallest eigenvalue $\lambda_2 > 0$ then the graph is connected.

Remark 1: In the following our considerations will be focused on *undirected graphs*. This ensures to have symmetric Laplacian matrix with real eigenvalues. We will not consider the more generic case of *directed graphs*, in which a direction is associated to each edge of $\mathcal{E}$. In the latter case, the Laplacian may not be symmetric and the eigenvalues may be complex.

IV. Model Description

In this section a model is proposed that describes the behavior of a fixed topology network of agents moving in a source field, given in Section II. The network follows a variable leader approach. That is, a certain agent takes on the role of a leader or a follower depending on its sensing of the strength of the field. In particular, the closer to the field maximum/minimum the agent is, the more the agent takes on the leader role. In order to make the group move into the area of interest, a fixed initial trajectory is given. The group should follow this path until any agent senses relevant information about the source field.

In the following we consider reaching the maximum of a given field $F(x, y)$. But analogous considerations hold to reach a minimum.

Each agent is characterized by a local coordinate system $\{b\}$. The velocity in the local frame is u_i along x_b, $\dot{\theta}$ around z and w_i along y_b. The velocity in the global frame $\{g\}$ is given by $V = [\dot{x}_i, \dot{y}_i, \dot{\theta}_i]^T$. An illustration is given in Figure 2 (By convention for marine vessels the z-axis points downwards into the sea). We assume that each agent is actuated only with a linear velocity u_i along its x_b axis, and with a heading velocity $\dot{\theta}_i$ around its z_b axis. The considered kinematic model is:

$$\begin{cases} \dot{x}_i = u_i \cos(\theta_i) \\ \dot{y}_i = u_i \sin(\theta_i) \\ \dot{\theta}_i = \omega_i \end{cases} \tag{1}$$

The works [16], [17], [24] use the same kinematic model. They control u_i and $\dot{\theta}_i$ to make the agents move in formation. However, in our work we decide not to focus on the formation of the group. Our main attention is given to the angular velocities $\vec{\dot{\theta}} = [\dot{\theta}_1, \dot{\theta}_2, \dots \dot{\theta}_n]^T$ in order to drive the agents towards a source only if a source is present in the area the agents are moving through. We decide to control the linear velocities $\vec{u} = [u_1, u_2, \dots, u_n]^T$ independently by $\vec{\theta}$, i.e. our method does not ensure that the agents reach the source keeping an assigned formation. Anyway our model require that the communication scheme among the agents has to be only connected, so even if they spread out over a larger area it is not necessary that the two most distant agents communicate directly.

In the following we consider a set of n agents defined by $\mathcal{V}$, that are indexed as v_i. We model them with a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Without loss of generality we index the agents such that the agent v_1 is the initial leader of the group. The initial leader's role is to make the group follow the fixed initial trajectory. This makes the agents move into the area of interest. As a first step, we consider the case where only one agent is able to sense the environment. This agent is denoted by v_2, and in the following is called the *active* follower. Finally the agents from v_3 to v_n are the *passive* followers. These cannot sense the environment and may represent agents with different sensors, for instance cameras. Moreover they might be communication nodes to allow communication among not directly connected agents. This is particularly useful in our case because as we have mentioned above, the model does not guarantee the agents keep a formation during the motion. But our model requires only a connected communication scheme among the agents, therefore more agents allow to have communication over larger areas. Because they cannot sense the environment, the concept of investment parameter k does not apply to them. In other words it is assumed that the investment parameter of the $n-2$ passive followers is always 0.

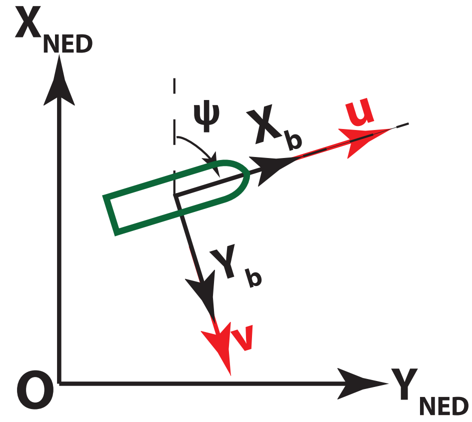

Fig. 2. Global and body frames

The following assumption is valid on $\mathcal{G}$:

Assumption 1: The graph $\mathcal{G}$, which characterizes the agents, is *connected* and has a controllable structure with respect to the nodes v_1, v_2.

Remark 2: Works [11], [18] demonstrate that connectivity is a necessary condition for a network based on leaders-followers scheme to be controllable. Furthermore, this kind of graph scheme does not require an all-to-all communication, which is an advantage for practical implementation. Conditions on the graph structure to be controllable are given in [2], [11], [18]. For instance, a graph has not to be complete or symmetric from the input node in order to be controllable.

Remark 3: We do not explicitly consider the case of directed graphs, which can have a non-symmetric Laplacian and complex conjugate eigenvalues. However, if the controllability condition with respect to v_1, v_2 is verified also for directed graph, then the following considerations hold also for this case.

Inspired by [3], [4], [19], we propose the following model for the heading velocities $\dot{\theta}_i(t)$ of the initial leader v_1 and the active follower v_2 :

$$\dot{\theta}_1(t) = k_1(t)(\theta_{d1}(t) - \theta_1(t)) - (1 - k_1(t))\frac{1}{d(v_1)}\vec{L}_1\vec{\theta}$$
$$\dot{\theta}_2(t) = k_2(t)(\theta_{d2}(t) - \theta_2(t)) - (1 - k_2(t))\frac{1}{d(v_2)}\vec{L}_2\vec{\theta} \quad (2)$$

where $d(v_i)$ is the degree number of the $i-th$ agent, $\vec{L}_i$ is the $i-th$ row of the Laplacian matrix. The parameters $k_1(t), k_2(t) \in [0, 1]$ are continuous functions that define the investment of the initial leader v_1 and the active follower v_2, respectively. If $k_i(t) = 1$ the agent is a full leader, if $k_i(t) = 0$ it becomes a pure follower. Equation (2) is similar to the one in [3], [4], [19], where the leadership is associated with $k_i(t)$. Except that in this paper $k_i(t)$ varies according to the scalar signal measured from the source in the field. When the agent v_2 takes measures that overtake a fixed threshold, it becomes the new leader. Finally θ_{d1} is the initial heading assigned to v_1, which is the initial leader. Then θ_{d2} is the desired heading of the active follower, and it depends on current position of the agent v_2.

In particular, we want to compute θ_{d2} by the gradient of the field, in fact it points towards the source. It is assumed that the *active* follower is equipped with three sensors that can get scalar measures of the field. One of the sensors is located at the center of the body of the vehicle. The other two are located at a distance d from this one on the axes x and y, respectively (Figure 1). Notice that the assumption that only one agent is equipped with several sensor in order to get distributed measurements does not guarantees a good computation of the gradient during real time operations. As a first step, however, we consider an ideal situation in which this kind of sensor is reliable.

We assume that at each instant the scalar measures are available. We assume that the scalar measure is $\delta(x, y) = hF(x, y)$, where h is a constant parameter which scale the measurements from the field $F(x, y)$ to the sensor. According to [16] it is possible to express the scalar measure of the field $\delta(x_i, y_i)$ at the generic position (x_i, y_i) according to the following Taylor expansion:

$$\delta(x_i, y_i) \approx \delta(x, y) + \frac{\partial\delta(x, y)}{\partial x}(x_i - x) + \frac{\partial\delta(x, y)}{\partial y}(y_i - y) \quad (3)$$

Considering that the active follower is equipped with three sensors it is possible to write:

$$\begin{bmatrix} \delta_{22} - \delta_{21} \\ \delta_{23} - \delta_{21} \end{bmatrix} = \begin{bmatrix} x_2^g - x_1^g & y_2^g - y_1^g \\ x_3^g - x_1^g & y_3^g - y_1^g \end{bmatrix} \begin{bmatrix} \frac{\partial\delta_{21}}{\partial x_1} \\ \frac{\partial\delta_{21}}{\partial y_1} \end{bmatrix} \quad (4)$$

where (x_i^g, y_i^g) are the coordinates of the sensors on the agent v_2 expressed in the *global* frame. The index $i \in \{1, 2, 3\}$ refers to the $i-th$ sensor. In particular, $i = 1$ indexes the sensor in the center of the body of v_2, $i = 2$ and $i = 3$ refer to the sensors on the x and y local-frame axis of v_2, respectively. Finally δ_{2i} is the scalar measure that corresponds to the $i-th$ sensor. From (4) it is possible to write:

$$\vec{G} = \begin{bmatrix} G_x \\ G_y \end{bmatrix} = \left(P^T P\right)^{-1} P^T \begin{bmatrix} \delta_2 - \delta_1 \\ \delta_3 - \delta_1 \end{bmatrix} \quad (5)$$

where:

$$P = \begin{bmatrix} x_2^g - x_1^g & y_2^g - y_1^g \\ x_3^g - x_1^g & y_3^g - y_1^g \end{bmatrix} \quad (6)$$

The vector $\vec{G}$ is directed towards the source, so it is possible to compute θ_{d2} for the active follower according to:

$$\theta_{d2} = \arctan 2(G_y, G_x) = 2 \arctan\left(\frac{G_y}{\sqrt{G_x^2 + G_y^2} + G_x}\right) \quad (7)$$

In this paper we consider a time-invariant field $F(x, y) : \mathbb{R}^2 \to \mathbb{R}^+$ that satisfies the following assumption:

Assumption 2: The field $F(x, y)$ is continuous and convex.

Remark 4: This assumption ensures that there is only one global maximum on $F(x, y)$, and therefore only one source in the field. A field with several local maxima is equivalent to having several sources in the environment.

Notice that (7) is not defined for $(G_x, G_y) = (0, 0)$ and $0 \geq \theta_{d2} \geq 2\pi$. We design the linear velocity such that the agents stop before arriving to the maximum of the field, where $(G_x, G_y) = (0, 0)$.

The investment parameters $k_1(t)$ and $k_2(t)$ in Equation (2) are chosen as follows:

$$k_1(t) = k(t), \quad k_2(t) = 1 - k(t) \quad (8)$$

This property ensures that the initial leader and the active follower cannot have their investment parameter equal to 1 or to 0 at the same time.

If $k_1 = k_2 = 1$ then they would both be leader. In this case they would not influence each other. Each one would follow its desired direction without caring about the direction of the other one. If $k_1 = k_2 = 0$ they would both be followers. In this case there would not be a leader in the group and all the agents would get an agreement on a common direction that would be the average of their initial direction of motion [11].

We define $k(t)$ as follows:

$$k(t) = \frac{1}{2}\tanh(\Delta(t)) + \frac{1}{2} \quad (9)$$

$$\Delta(t) = \alpha\frac{\delta_{21}(t) - \delta_m}{\delta_M - \delta_m} + \beta \quad (10)$$

where $\delta_m > 0$ defines the minimum relevant level for the measures, $\delta_M > \delta_m$ is the maximum thresholds over which v_2 becomes a full leader, $\delta_{21}(t)$ is the strength of the measurement signal that the agent v_2 currently receives to its central sensor. Then $\alpha < 0$ and $\beta > 0$ are two parameters that define the function Δ. Notice that $k(t) \to 1$ for $\Delta(t) \to -\infty$ while $k(t) \to 0$ when $\Delta(t) \to +\infty$. So β is chosen such that $\delta_{21}(t) = \delta_m \Rightarrow k = \tanh(\beta) \simeq 1$. Then it is obvious that $\delta_2 < \delta_m \Rightarrow k \simeq 1$. With the same reasoning α is chosen such that $\delta_{21}(t) \geq \delta_M \Rightarrow k \simeq 0$. This choice for $k(t)$ and $\Delta(t)$ ensures that $k(t)$ is bounded and continuous. Notice that δ_m, δ_M are not the minimum and the maximum of $F(x, y)$, but only two tuning parameters.

The derivative of k is:

$$\dot{k} = \frac{\alpha\dot{\delta}_{21}}{2\cosh^2(\Delta)} \quad (11)$$

where $\dot{\delta}_2$ is the velocity with which the signal from the field to the agent v_2 varies. We consider that the following Assumption holds for our model:

Assumption 3: The velocity with which the signal from the field to the agent v_2 varies is bounded, i.e. $\|\dot{\delta}_{21}\| \leq c$, where $c > 0$

Remark 5: This Assumption ensures that the agents move with a finite velocity over the field, and that the field itself has not discontinuities.

Following Assumption 3 $\dot{k}$ is bounded because $\min\{\cosh^2(\Delta)\} = 1$ for $\Delta = 0$.

Recalling that for the other $n-2$ passive followers the concept of investment parameter does not apply, their dynamics is given by the classic *agreement protocol* [11], that is:

$$\dot{\theta}_i = -\frac{1}{d(v_i)}\vec{L}_i\vec{\theta} \quad (12)$$

If the graph is connected the agents will agree on a common direction to follow [11]. In our case this direction is suggested by the current leader.

Equations (2) and (12) can be rewritten in matrix form:

$$\dot{\vec{\theta}} = K(\vec{\theta}_d - \vec{\theta}) - D^{-1}(I-K)L\vec{\theta} \quad (13)$$

where:

$$K = \left[\begin{array}{cc|c} k(t) & 0 & \\ 0 & 1-k(t) & O_{2\times(n-2)} \\ \hline & O_{(n-2)\times 2} & O_{(n-2)\times(n-2)} \end{array}\right] \quad (14)$$

$$\vec{\theta}_d = \begin{bmatrix} \theta_{d1} \\ \theta_{d2} \\ \vec{0}_{(n-2)\times 2} \end{bmatrix} \quad (15)$$

where $O_{a\times b}$ is a matrix $[a \times b]$ with all entries equal to zero. The vector $\vec{\theta}_d$ is the vector of the desired headings. Finally notice that the last $n-2$ elements of the vector are conventionally assumed to be zeros. In fact the first term $K(t)(\vec{\theta}_d(t) - \vec{\theta}(t))$ couples only the dynamics of the agents that can change their investment parameter k_i. The desired headings of the $n-2$ passive followers can assume whichever value possible, but they are ignored because their k_i is equal to zero. The dynamics of the $n-2$ passive followers are coupled with the ones of the agents v_1 and v_2 via the term $D^{-1}(I-K)L\vec{\theta}$. Notice that D^{-1} is the inverse of the degree matrix. Finally I is the $[n\times n]$ identity matrix. The equation can be rearranged and rewritten as:

$$\dot{\vec{\theta}} = -M\vec{\theta} + K\vec{\theta}_d, \qquad M = [K + D^{-1}(I-K)L] \quad (16)$$

From the matrix K the necessity of our choice in (8) is more clear. If both k_1 and k_2 were zero at the same time, the model given by (13) would become the classical *agreement protocol* $\dot{\vec{\theta}} = L\vec{\theta}$ [11]. If both $k_1 = k_2 = 1$, then both v_1 ad v_2 would be leaders, in this case each one follows its desired direction ignoring the other one. The passive followers follow a direction that depends on the characteristics of the connection graph.

In our approach the control of the linear velocities $\vec{u} = [u_1 \ u_2 \ \dots u_n]^T$ is done independently from the angular velocities $\dot{\vec{\theta}}$.

As we have already mentioned a drawback of this approach is that we cannot ensure the keeping of a formation during the motion. We choose the velocity controllers such that agent v_2 controls the linear velocity of the other agents during the mission. In particular, this means that the active follower v_2 does not necessarily take on the role as leader of the group with respect to controlling the headings, which is the case if its sensor measurements never exceeds the lower threshold. But v_2 will always be the leader as regards the linear velocity. The motivation for the choice of v_2 to decide the linear velocity of the group is that it is the only agent which obtains measures from the field, and it is desirable to adjust the linear velocity with the intensity of the signal from the field. The velocity for v_2 is chosen as:

$$\bar{u}_2 = u_0 - (1-k_c)u_0 \quad (17)$$

where u_0 is the initial assigned velocity, and $k_c = \frac{1}{2}\tanh(\Delta_c) + \frac{1}{2}$ is a function similar to the one used for k which takes into account the signal strength, and where:

$$\Delta_c = \alpha\frac{\delta_{21} - \delta_{mu}}{\delta_{Mu} - \delta_{mu}} + \beta \quad (18)$$

where $\alpha < 0$ and $\beta > 0$ are chosen with the same reasoning used for (10). Then δ_{21} is the current measure that v_1 gets, δ_{mu} defines the threshold which v_2 starts to slow down, and δ_{Mu} is the level measure which v_2 stops. Notice that we want to choose $\delta_{Mu} < \max(F(x,y))$. In fact if the agent v_2 arrived on $\max(F(x,y))$, then Equation (7) would not be defined.

The linear velocity of all the other agents are controlled by v_2, and we decide to use a model inspired by [2]:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = -\begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ 0 & 0 & \dots & 0 \\ L_{31} & L_{32} & \dots & L_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{u}_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (19)$$

where L_{ij} is the element (i,j) of the Laplacian matrix L, l_i is the measure of the distance which each agents has traveled. Finally $\bar{u}_2$ is the control input for the linear velocity of v_2.

The dynamics of the linear velocities of the agents $v_1, v_3, \dots, v_n$ is:

$$\vec{u}^* = -\bar{L}\vec{l}^* - \vec{r}\bar{u}_2 \quad (20)$$

where $\bar{L}$ is a matrix $[(n-1)\times(n-1)]$ obtained by the Laplacian deleting the second column and the second row, i.e. the row and the column that corresponds to v_2. The vector $\vec{l}^*$ is $\vec{l}^* = [l_1, l_3, \dots, l_n]^T$. Then the vector $\vec{r}$ is the $[(n-1)\times 1]$ vector obtained by the second column of the Laplacian deleting the second element. With this dynamic we want the agents $v_1, v_3, \dots, v_n$ move as long as the agent v_2 moves.

V. Results

In this section the conditions to have a stable exchange of the leadership within the group are stated and demonstrated. In the following $O_{a\times b}$ is used to indicate an $[a \times b]$ matrix with all the elements equal to zero. The Investment Parameter $k(t)$ is simply rewritten as k without neglecting its time variant behavior.

The following Lemma is important for the proof of the main Theorem.

Lemma 1: For a connected undirected graph $\mathcal{G}$ the matrix $M = K + D^{-1}(I-K)L$ is positive definite if $K \neq O_{n\times n}$.

Proof:

We want to demonstrate that M is positive definite for each value of k. We observe that M is not symmetric, but according to [21] we know that the positive-definiteness is a property that holds also for non-symmetric matrix. Furthermore, even in the case a non-symmetric matrix is positive definite, this guarantee that its eigenvalues are all positive. So we focus on the following quadratic form:

$$\vec{x}^T(K+D^{-1}(I-K)L)\vec{x} = \vec{x}^T K\vec{x} + \vec{x}^T D^{-1}(I-K)L\vec{x} \geq 0 \tag{21}$$

We do not know what are the eigenvalues of $K+D^{-1}(I-K)L$, but if we can prove that there is not a vector different from zero that makes (21) zero, then $K + D^{-1}(I-K)L$ is strictly greater than 0, and so positive definite [21].

Let us consider first the case $0 < k < 1$. The matrix K is diagonal and with $n-2$ null elements on its diagonal, so it is positive semi-definite because k and $1-k$ are non negative. Its eigenvalues are k, $1-k$ and 0. The eigenvalue 0 has algebraic and geometric multiplicity $n-2$, so there are $n-2$ eigenvectors that correspond to the eigenvalue $\lambda_k = 0$. We call $\vec{e_1}$ and $\vec{e_2}$ the eigenvectors that correspond to the eigenvalues $1-k$ and k. Then we call e_i, with $i \in \{3, ..., n\}$, the $n-2$ eigenvectors that correspond to $\lambda = 0$. The generic vector e_i has $n-1$ null elements and the value 1 in the $i-th$ position:

$$\vec{e}_3 = \{0, 0, 1, 0, \dots, 0\}^T \tag{22}$$

$$\vec{e}_i = \{0, 0, \dots, 1, \dots, 0\}^T \tag{23}$$

$$\vec{e}_n = \{0, 0, 0, \dots, 1\}^T \tag{24}$$

Now it is sufficient to show that none of the eigenvectors $\vec{e}_i$ makes the quadratic form (21) zero. For this purpose, we consider (21) with $\vec{x} = \vec{e_i}$ where $i \in \{3, \dots, n\}$. We obtain:

$$\vec{e}_i^T D^{-1}(I-K)L\vec{e}_i = L_{ii} = d(v_i) > 0 \tag{25}$$

where L_{ii} is the $i-th$ element on the diagonal of L that is equal to $d(v_i)$, that is the number of neighbors of v_i. Since we have a connected graph, each agent has to have at least one neighbor, so $d(v_i) > 0$.

When $k = 0$ we have $e_1 = \{1, 0, \dots, 0\}^T$ and $\vec{e}_1^T M\vec{e}_1 = d(v_1) > 0$.

When $k = 1$ we have $e_2 = \{0, 1, 0, \dots, 0\}^T$ and $\vec{e}_2^T M\vec{e}_2 = d(v_2) > 0$.

So we can conclude that there is not a vector that is different by $\vec{0}$, which can make zero both the terms of (21) at the same time. Consequently we have proved that M is positive definite, and so its eigenvalues are all positive [21]. ■

Now it is possible state the main theorem:

Theorem 1: Consider a group of agents that move according to (16) in a time-invariant field F(x,y), and assume that Assumptions 1-3 are satisfied. Then (16) is Input to State Stable (ISS) with respect to the input $K\vec{\theta}_d$. And if the eigenvalues of $\bar{L}$ are all distinct and the corresponding eigenvectors are not orthogonal to $\vec{r}$, then (19) is controllable by u_2.

Proof: We want to demonstrate that (16) is Input to State Stable (ISS) with input $K\vec{\theta}_d$. To this purpose we know that M is positive definite according to Lemma 1 and we want to focus on the norms of M and $\dot{M}$. First we focus on M:

$$M = \begin{bmatrix} 1 & -\frac{L_{12}(k-1)}{d_1} & -\frac{L_{13}(k-1)}{d_1} & \dots & -\frac{L_{1n}(k-1)}{d_1} \\ \frac{L_{21}k}{d_2} & 1 & \frac{L_{23}k}{d_2} & \dots & -\frac{L_{2n}k}{d_2} \\ \frac{L_{31}}{d_3} & \frac{L_{32}}{d_3} & 1 & \dots & \frac{L_{3n}}{d_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{L_{n1}}{d_n} & \frac{L_{n2}}{d_n} & \frac{L_{n3}}{d_n} & \dots & 1 \end{bmatrix} \tag{26}$$

where L_{ij} are the $(i,j)-th$ elements of L. According to [11] the sum of the elements on the rows of L is zero. So using this property, the sum of the elements on the rows of M are:

$$r_1 = 1 + \sum_{i=2}^{n} -\frac{L_{1i}(k-1)}{d_1} = 2 - k \tag{27}$$

$$r_2 = \frac{L_{21}k}{d_2} + 1 + \sum_{i=3}^{n} \frac{L_{2i}}{d_2} = 1 - k \tag{28}$$

$$r_j = \sum_{i=1}^{j-1} \frac{L_{ji}}{d_j} + 1 + \sum_{i=j+1}^{n} \frac{L_{ji}}{d_j} = 0 \tag{29}$$

where r_i gives the sum of the elements of the $i-th$ row, and $j \in [3, 4, \dots, n]$. According to this we have:

$$\|M\|_\infty = \max\{2-k, 1-k, 0\} \leq 2 \tag{30}$$

because $0 \geq k \geq 1$.

Now we focus on $\dot{M}$:

$$\dot{M} = (I - D^{-1})\dot{K} = \begin{bmatrix} 0 & -\frac{\dot{k}L_{12}}{d_1} & 0 & \dots & 0 \\ \frac{\dot{k}L_{21}}{d_2} & 0 & 0 & \dots & 0 \\ \frac{\dot{k}L_{31}}{d_3} & -\frac{\dot{k}L_{32}}{d_3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\dot{k}L_{n1}}{d_n} & -\frac{\dot{k}L_{n2}}{d_n} & 0 & \dots & 0 \end{bmatrix} \tag{31}$$

According to the definition of the Laplacian we have $L_{ij} \in \{0, -1\}$ for $i \neq j$. So considering also Assumption 3 we have:

$$\|\dot{M}\|_\infty = \max\left\{-\frac{\dot{k}L_{12}}{d_1}, \frac{\dot{k}L_{21}}{d_2}, \frac{\dot{k}L_{31}-\dot{k}L_{32}}{d_3}, \dots \right. \tag{32}$$

$$\left. \dots, \frac{\dot{k}L_{n1}-\dot{k}L_{n2}}{d_n}\right\} \tag{33}$$

$$< \max\left\{\frac{\dot{k}}{d_1}, -\frac{\dot{k}}{d_2}, \pm\frac{\dot{k}}{d_3}, \dots, \pm\frac{\dot{k}}{d_n}, 0\right\} \tag{34}$$

$$\leq \frac{\dot{k}}{\min\{d_i\}} \quad \text{for} \quad i \in [1, 2, \dots, n] \tag{35}$$

$$\leq \frac{\alpha c}{2\cosh^2(\Delta)\min\{d_i\}} \quad \text{for} \quad i \in [1, 2, \dots, n] \tag{36}$$

where c is the bound on $\dot{\delta}_2$. According to the properties of M and $\dot{M}$ we can conclude that:

$$\dot{\vec{\theta}} = -M\vec{\theta} \tag{37}$$

is Globally Exponentially Stable (GES), according to Theorem [8.7] in [12]. Moreover $0 \geq \theta_{d2} \geq 2\pi$ because of its definition. And we have:

$$\|K\|_\infty = \max\{1-k, k, 0\} \leq 1 \quad \forall k \in [0, 1] \tag{38}$$

because of the definition of K. So (16) is ISS according to Lemma [4.6] in [10].

As regards the linear velocities, the system (19) is controllable by u_2 if Theorem [IV.1] in [2] applies. According to [2], [11] if Assumption 1 holds then the eigenvalues of $\bar{L}$ are all distinct and the corresponding eigenvectors are not orthogonal to $\vec{r}$. Therefore the conditions of Theorem [IV.1] in [2] are satisfied and v_2 can control the velocities of the other agents. ■

The solution of (16) is $\vec{\theta} = [K + D^{-1}(I-K)L]^{-1}K\vec{\theta}_d$. Considering the chosen function for k we know that it evolves over the field between the values 1 and 0. So when $k = 1$ and v_2 is the leader the matrix $[K + D^{-1}(I - K)L]^{-1}K$ gives $\vec{\theta} = [\theta_{d2}, \dots, \theta_{d2}]^T$. When we have $k = 0$ the agent v_1 is the leader and the solution is $\vec{\theta} = [\theta_{d1}, \dots, \theta_{d1}]^T$. The direction θ_{d2} points towards the source [16]. It is important notice that the convergence towards the source depends on the tuning parameter $\alpha, \beta, \delta_M, \delta_m$. It is important to notice that the convergence to the source strongly depends on the tuning parameters $\alpha, \beta, \delta_m, \delta_M$.

VI. Simulation

In this section simulation results are presented to illustrate the behavior of the presented model.

We consider a group of five agents described by a graph $(\mathcal{G}, \mathcal{V})$. The graph is connected but without a directed connection between v_1 and v_2. In other words we assume a graph topology in which v_1 and v_2 do not directly exchange the information about their relative position. The graph topology for the communication scheme of the relative position is illustrated in Figure 3. The following Laplacian matrix corresponds to the illustrated topology:

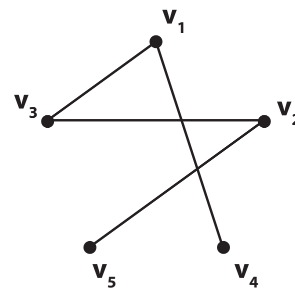

Fig. 3. Topology of the agents' graph

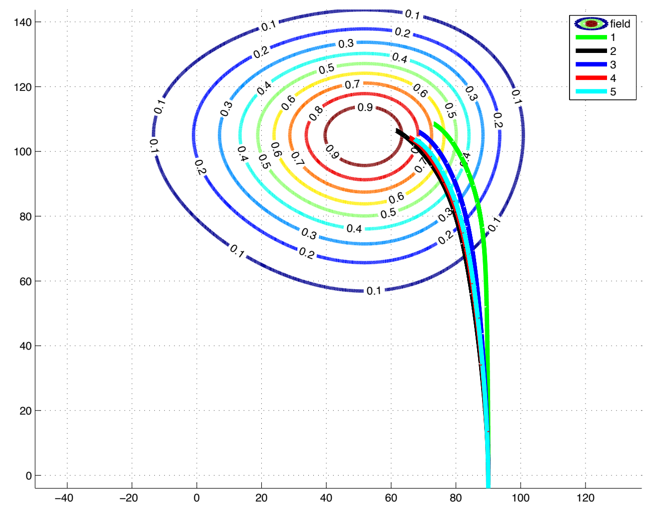

Fig. 4. Path of the agents

$$L = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

We assign $u_0 = 1\ m/s$. The considered field is:

$$F(x,y) = e^{(-0.1A(0.055(x+130)-10)^2 - 0.2B0.04(y+170)-11)^2} \tag{40}$$

z where $A = e^{0.005(x+130)}, B = e^{(0.005(y+170))}$. Notice that $F(x,y)$ respects Assumption 2. The thresholds for k are $\delta_m = 0.08$ and $\delta_M = 0.5$. The thresholds for the variation of the velocity u are $\delta_{mu} = 0.8$ and $\delta_{Mu} = 0.95$. The initial desired direction for the initial leader v_1 is θ_{d1} is 90°. This is the direction to which each agent agrees at the beginning. The initial desired direction is assigned to steer the agents towards an area that is supposed to be of interest. In our case an area is defined of interest if it contains the source of a vector field. Figure 4 shows the path that the agents follow.

VII. Conclusion

The objective of our work was to develop a kinematic model to improve the performance for source seeking operations using a multi-agent system. We have proposed a

Leader-Followers approach, where the leader changes according to the condition of the environment. The group is characterized by an initial leader and a follower that can vary its leadership condition. In particular, we have considered that one of the agents is able to sense the environment. When it gets satisfactory measures of the source it takes on the role as a leader and the initial leader becomes a simple follower.

Our result is based on a revision of the classic controlled agreement protocol with the addition of a variable leader in order to steer towards a source changing the direction of the agreement among the agents.

In our model we assume that one agent is equipped with three sensors and no specific formations are required. A future development for our work will be the definition of a proper linear velocities controller in order to have a coordinated motion of the agents to keep a formation during the motion. In this way it would be possible to equip more agents with one sensor in order to compute the gradient of the field by shared information among the agents. Furthermore, equipping each agent with a sensor will give more reliable measurements of the field $F(x, y)$.

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