Lecture notes on quantitative unique continuation for solutions of second order elliptic equations

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Abstract. In these lectures we present some useful techniques to study quantitative properties of solutions of elliptic PDEs. Our aim is to outline the proof of a recent result on propagation of smallness. The ideas are also useful in the study of the zero sets of eigenfunctions of the Laplace–Beltrami operator. Some basic facts about second order elliptic PDEs in divergent form are collected in the Appendix at the end of the notes.

Contents

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1. Eigenfunctions of Laplace–Beltrami operators

1.1. Definition Let M be an oriented Riemannian manifold with metric tensor $g = (g_{ij})$, let $|g|$ denote the absolute value of the determinant of the matrix (g_{ij}), and let $g^{-1} = (g^{ij})$ be the inverse tensor. The gradient of a C^1 function f on M is a vector field locally given by

$$\operatorname{grad}_M f = \sum_{i,j} (g^{ij} \partial_j f) \partial_i.$$

The Laplace–Beltrami operator on functions on M is defined as the divergence of the gradient. In local coordinates, it becomes

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \operatorname{div}(\sqrt{|g|} g^{-1} \nabla f),$$

where $\nabla f = (\partial_1 f, \dots, \partial_n f)$ in chosen coordinates.

The following Green formula holds for functions $f, h \in W_0^{1,2}(M)$

$$\int_M h \Delta_M f dV_M = - \int_M \langle \operatorname{grad}_M f, \operatorname{grad}_M h \rangle_g dV_M,$$

where the volume form dV_M is defined as $dV_M = \sqrt{|g|} dx_1 \wedge \dots \wedge dx_n$ in local coordinates .

Assume now that M is a compact manifold without boundary. We consider eigenfunctions ϕ_λ of the Laplace–Beltrami operator, such that

$$\Delta_M \phi_\lambda + \lambda \phi_\lambda = 0.$$

Then

$$\int_M |\operatorname{grad}_M \phi_\lambda|_g^2 dV_M = \lambda \int_M |\phi_\lambda|^2 dV_M.$$

All eigenvalues of $-\Delta_M$ are real and non-negative. Eigenfunctions corresponding to distinct eigenvalues are orthogonal since

$$\lambda \int_M \phi_\lambda \phi_\mu dV_M = -\int_M (\Delta_M \phi_\lambda)\phi_\mu dV_M = \mu \int_M \phi_\lambda \phi_\mu dV_M.$$

The eigenvalues form an increasing sequence that tends to infinity,

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots .$$

The first eigenfunction ϕ_0 is a constant. There is an orthonormal basis of eigenfunctions for $L^2(M)$. We refer the reader to [6, Chapter 1] for details.

Example 1.1.1. (Dirichlet Laplacian for a domain in $\mathbb{R}^d$) Instead of a compact manifold, we may also consider a bounded domain Ω in $\mathbb{R}^d$ and the Laplace operator with the Dirichlet boundary condition

$$\Delta\phi + \lambda\phi = 0, \quad \phi|_{\partial\Omega} = 0.$$

The first eigenvalue is given by the variational formula

$$\lambda_1(\Omega) = \min_\phi \int_\Omega |\nabla\phi|^2,$$

where the minimum is taken over all functions $\phi \in W_0^{1,2}(\Omega)$ such that $\int_\Omega |\phi|^2 = 1$. This formula implies that if $\Omega_1 \subset \Omega_2$ then

$$\lambda_1(\Omega_1) \geqslant \lambda_1(\Omega_2).$$

The first eigenfunction does not change sign and can be chosen positive in Ω, while all other eigenfunctions are orthogonal to the first one and therefore change sign in Ω. The eigenvalues can be determined by the min-max formula

$$\lambda_k(\Omega) = \min_{A_k} \max_{\phi \in A_k} \frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega |\phi|^2}, \tag{1.1.2}$$

where the minimum is taken over all k-dimensional subspaces of $W_0^{1,2}(\Omega)$. Alternatively, there is an inductive description of eigenvalues (and eigenfunctions),

$$\lambda_k(\Omega) = \min_\phi \frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega |\phi|^2}, \tag{1.1.3}$$

where the minimum is taken over all $\phi \in W_0^{1,2}(\Omega)$ which are orthogonal to the first $k-1$ eigenfunctions $\phi_{\lambda_1}, \dots, \phi_{\lambda_{k-1}}$.

The variational characterization of the eigenvalues, (1.1.2) and (1.1.3), also hold for the eigenvalues of the Laplace-Beltrami operator on compact manifolds.

1.2. Courant nodal domain theorem The zero set $Z(\phi)$ of a function ϕ is

$$Z(\phi) = \{x : \phi(x) = 0\},$$

and we also refer to it as the nodal set of ϕ. The connected components of $M \setminus Z(\phi)$ are called the nodal domains of the function ϕ.

The simplest example of a compact manifold is the unit circle $\mathbb{T} \simeq [0, 2\pi)$. Eigenfunctions of the Laplace operator are 2π-periodic solutions of the eigenvalue problem

$$\phi'' + \lambda\phi = 0.$$

This equation has a 2π periodic solution when $\lambda = n^2$ for some integer n. The first eigenfunction, corresponding to $n = 0$ is a constant. For $n > 0$ the eigenfunctions are linear combinations of $\phi_{n,1}(\theta) = \cos(n\theta)$ and $\phi_{n,2}(\theta) = \sin(n\theta)$. Each of them has $2n$ zeros on the circle.

The Courant nodal domain theorem gives an upper bound for the number of nodal domains of eigenfunctions on manifolds of arbitrary dimension. Let M be a compact manifold as above and ϕ_{λ_n} be an eigenfunction of the Laplace-Beltrami operator corresponding to the n-th smallest eigenvalue.

Theorem 1.2.1 (Courant)**.** *The number of connected components of* $M \setminus Z(\phi_{\lambda_n})$ *is at most* n.

For the proof we refer the reader to [8, Chapter 6] and [6]. The proof is beautiful and short except for one non-trivial result on weak unique continuation property of solutions of second order elliptic PDEs. The result says that a non-zero Laplace-Beltrami eigenfunction cannot vanish on an open subset of a manifold. The aim of these notes is to give a new quantitative sharpening of this uniqueness result.

1.3. More examples A first intuition on the geometry of zero sets of eigenfunctions comes from the pictures of nodal domains on the unit sphere and the standard torus, see [18].

Example 1.3.1. The eigenfunctions on the unit sphere S^d in $\mathbb{R}^{d+1}$ are restrictions of the homogeneous harmonic polynomials which are called spherical harmonics. If P is a polynomial of $d+1$ variables, $\Delta P = 0$ and $P(x) = |x|^n Y(x/|x|)$, where Y is a function on $S = S^d$, then

$$\Delta_S Y + n(n+d-1)Y = 0.$$

There is a basis of spherical harmonics for $L^2(S^d)$. Therefore there are no other eigenfunctions of the Laplace-Beltrami operator on the sphere, further details are given in Exercise 1.8.3.

Example 1.3.2. Another standard compact manifold, on which we can compute eigenfunctions explicitly, is the torus.

Let $\mathbb{T}^d$ be the d-dimensional torus which we will identify with the rectangle $\prod_{j=1}^d [-\pi, \pi]$ glued along each pair of opposite sides. Then we have a basis of eigenfunctions of the form

$$\phi(x) = \exp\left(i \sum_{j=1}^d n_j x_j\right), \quad \Delta_{\mathbb{T}^d} \phi + \sum_{j=1}^d n_j^2 \phi = 0,$$

where $n_j \in \mathbb{Z}$.

We notice that if dimension $d > 1$, there are eigenvalues for the Laplace–Beltrami operators on S^d and $\mathbb{T}^d$ with arbitrary large multiplicities. This is a source of interesting examples of eigenfunctions.

The zero sets of standard spherical harmonics and eigenfunctions on the torus are not hard to visualize, but the structure of the zero sets of linear combinations of these functions (corresponding to the same eigenvalue) may be complicated.

1.4. Bessel functions and Helmholtz equation Another classical example of eigenfunctions are bounded solutions of the Helmholtz equation in $\mathbb{R}^n$,

$$\Delta\phi + \lambda\phi = 0.$$

For $\lambda \leqslant 0$ the maximum principle holds and there are no non-trivial bounded solutions. Hence we are interested in the case $\lambda > 0$ and, rescaling the variable, we may assume that $\lambda = 1$.

The Laplace operator in polar coordinates can be written as

$$\Delta\phi = \partial_r^2\phi + \frac{d-1}{r}\partial_r\phi + \frac{1}{r^2}\Delta_S\phi.$$

We look for solutions of the equation $\Delta\phi + \phi = 0$ of the form $\phi(x) = f(|x|)Y(x/|x|)$. Separating the variables, one can check that Y is an eigenfunction of the Laplace–Beltrami operator on the unit sphere. The eigenvalues on the sphere are given in Example 1.3.1 (see also Exercise 1.8.3 below). Then we find a family of solutions of the Helmholtz equation of the form

$$\phi(x) = f_n(|x|)Y\left(\frac{x}{|x|}\right), \quad \Delta_S Y = -n(n+d-2)Y,$$

where $f_n(r)$ satisfies the following ordinary differential equation

$$r^2f'' + (d-1)rf' + (r^2 - n(n+d-2))f = 0.$$

Writing $f_n(r) = r^{1-d/2}g_n(r)$ we see that $g_n(r)$ satisfies the Bessel equation

$$r^2g'' + rg' + (r^2 - (n+d/2-1)^2)g = 0.$$

This is a second order ODE with analytic coefficients with a solution $J_{n+d/2-1}$ called the Bessel function (of the first kind) which is continuous at the origin. The solution is of the form

$$J_{n+d/2-1}(r) = r^{n+d/2-1}h_{n+d/2-1}(r)$$

where $h_{n+d/2-1}(r)$ is an analytic function of r and $h_{n+d/2-1}(0) \neq 0$ (see for example [35, Chapter 4.2]); the second solution has a singularity at $r = 0$. Thus we get

$$f_n(r) = r^{1-d/2}J_{n+d/2-1}(r) = r^n h_{n+d/2-1}(r).$$

We consider positive zeros of J_ν (they are simple, since J_ν is a non-zero solution do second order ODE) and enumerate them $0 < j_{\nu,1} < j_{\nu,2} < \cdots$.

Using the obtained description of the solutions of the Helmholtz equation, we can compute eigenfunctions and eigenvalues of the Dirichlet Laplace operator for the unit ball in $\mathbb{R}^d$, see Exercise 1.8.4 below.

1.5. Yau's conjecture Examples of eigenfunctions on the torus and sphere show that the number of nodal domains may vary, but is bounded from above as shown by Courant nodal domain theorem. At the same time, there exist eigenfunctions with large eigenvalues and just two nodal domains as was shown already in 1925 in the dissertation of Antonie Stern; see [3] for historical details and references.

On the other hand, these examples show that nodal lines become more complicated and dense as the eigenvalue grows. We give a proof of a well known result on the density of the zero sets of eigenfunctions in the next section. First we formulate a deep conjecture of Yau [37].

Conjecture (Yau)**.** *Let M be a smooth compact d-dimensional Riemannian manifold. There exist constants C_1 and C_2, which depend on M, such that*

$$C_1\sqrt{\lambda} \leqslant \mathcal{H}^{d-1}(Z(\phi_\lambda)) \leqslant C_2\sqrt{\lambda},$$

for any eigenfunction ϕ_λ satisfying $\Delta_M\phi_\lambda + \lambda\phi_\lambda = 0$.

The singular set of a function is the set where both the function and its gradient equal zero. The singular sets of an eigenfunction has Hausdorff dimensions $d-2$ and its nodal sets is the union of smooth hypersurfaces with finite $(d-1)$-dimensional Hausdorff measure and the singular set. The finiteness of the Hausdorff measure of the nodal set is a non-trivial fact; see [17] for details.

The Yau conjecture was proved for the case of real analytic metrics by Donnelly and Fefferman in 1988, [9]. We outline some of the ideas in Section 2.6.

1.6. Lift of eigenfunctions The following lifting trick is used intensively in the study of eigenfunctions. Let M be a d-dimensional manifold and ϕ_λ be an eigenfunction, $\Delta_M\phi_\lambda + \lambda\phi_\lambda = 0$, we define the function

$$h(x,t) = \phi_\lambda(x)e^{\sqrt{\lambda}t},$$

on the product manifold $M' = M \times \mathbb{R}$. Then $\Delta_{M'}h = 0$. Locally we view h as a solution of an elliptic equation in divergence form on a subdomain of $\mathbb{R}^{d+1}$.

The first application of the lifting trick is the proof of the result on the density of the zero sets of eigenfunctions.

Proposition 1.6.1. *Suppose that M is a compact Riemannian manifold. There exists $\rho = \rho(M)$ such that for any eigenfunction ϕ_λ with $\lambda > 0$ and any $x \in M$ the distance from x to the zero set $Z(\phi_\lambda)$ is less than $\rho\lambda^{-1/2}$.*

Proof. Suppose that ϕ_λ does not change sign in some ball $B_r \subset M$. We assume that r is small enough and consider a chart for M that contains B_r. Then the function $h(x,t) = \phi_\lambda(x)\exp(\sqrt{\lambda}t)$ is a solution of a second order elliptic equation in divergence form and h does not change sign in $B_r \times [-r, r]$. By the Harnack inequality, (see Theorem 5.1.6 below)

$$\sup_D |h| \leqslant C(M) \inf_D |h|,$$

where $D = B_{r/2} \times [-r/2, r/2]$. That $r < \rho\lambda^{-1/2}$ then follows from

$$\sup_D |h| = \sup_{B_{r/2}} |\phi_\lambda| \exp(r\sqrt{\lambda}/2) \geqslant \exp(r\sqrt{\lambda}) \inf_D |h|. \qquad \square$$

The zero set of $h = \phi_\lambda(x)\exp(\sqrt{\lambda}t)$ is the cylinder over $Z(\phi_\lambda)$, hence questions about $Z(\phi_\lambda)$ can be restated in terms of $Z(h)$. One advantage is that h is a solution of an elliptic second order PDE in divergence form with no lower order terms.

1.7. A question of Nadirashvili Suppose that h is a harmonic function in the unit disc $\mathbb{D} \subset \mathbb{R}^2$ such that $h(0) = 0$. The zero set of h is the union of analytic curves and by the maximum principle it has no loops. We assume that $h(0) = 0$ then an elementary geometric argument implies that

$$\mathcal{H}^1(Z(h) \cap \mathbb{D}) \geqslant 2.$$

Nadirashvili asked whether a higher dimensional version of this statement holds.

Conjecture (Nadirashvili)**.** *There is a constant* $c > 0$ *such that for any harmonic function* h *in the unit ball* B *of* $\mathbb{R}^3$ *such that* $h(0) = 0$*, the following inequality holds*

$$\mathcal{H}^2(Z(h) \cap B) \geqslant c.$$

The question was formulated for harmonic functions in $\mathbb{R}^n$ and remained open for many years. The proof given recently in [26] by the first author is complicated (and beyond the scope of these lectures), it gives the affirmative solution to the version of the Nadirashvili conjecture for solutions of second order elliptic equation in divergence form with smooth coefficients.

Theorem 1.7.1 ([26])**.** *Suppose that* $Lu = \operatorname{div}(A\nabla u)$ *is a uniformly elliptic operator in the unit ball* $B \subset \mathbb{R}^d$ *with smooth coefficients. There exists a constant* $c = c(A)$ *such that for any solution of* $Lu = 0$ *with* $u(0) = 0$ *satisfies*

$$\mathcal{H}^{d-1}(Z(u) \cap B) \geqslant c.$$

A corollary, also shown in [26], is the lower bound in Yau's conjecture on compact Riemannian manifolds with smooth metric. A polynomial upper bound

$$\mathcal{H}^{d-1}(Z(\phi_\lambda)) \leqslant C\lambda^{A_d},$$

where A_d depends only on the dimension of the manifold and C depends on the manifold and the metric was obtained in [25].

1.8. Exercises

Exercise 1.8.1 (Harnack inequality). Let $L = \operatorname{div}(A\nabla\cdot)$ be a uniformly elliptic operator with bounded coefficients. Use the Harnack inequality (Theorem 5.1.6) to prove the following statements.

(1) If u is a bounded solution of $Lu = 0$ in $\mathbb{R}^d$ then u is a constant.
(2) Let $\mathcal{C}_t$ denote the cylinder

$$\mathcal{C}_t = \{x = (x_1, \dots x_d) \in \mathbb{R}^d : x_1^2 + \dots + x_{d-1}^2 \leqslant t^2\}.$$

Suppose that $Lu + cu = 0$, $c \in \mathbb{R}$ and u is positive in the cylinder $\mathcal{C}_1$ and let $M(R) = \max\{u(x) : x \in \mathcal{C}_{1/2}, |x_d| \leqslant R\}$. Then there exists C such that $M(R) \leqslant u(0)e^{CR}$.

Exercise 1.8.2. Suppose that $\Delta_M u + \lambda u = 0$ and Ω is a connected component of $M \setminus Z(u)$. Assume that Ω is a domain with piece-wise smooth boundary and prove that the first Dirichlet Laplace eigenvalue of Ω is

$$\lambda_1(\Omega) = \lambda.$$

Remark: Careful details can be found in [6], see also [7].

Exercise 1.8.3 (Harmonic polynomials). The restrictions of homogeneous harmonic polynomials on the unit sphere $S \subset \mathbb{R}^{d+1}$, called spherical harmonics, are the eigenfunctions of the Laplace–Beltrami operator. We denote the eigenspace that corresponds to the eigenvalue $\lambda = n(n+d-1)$ by $E_{n,d}$. If $Y \in E_{n,d}$ then the function $P(x) = |x|^n Y(x/|x|)$ is harmonic.

(1) Apply Green's formula in $\mathbb{R}^d$ to show that if $Y_n \in E_{n,d}$ and $Y_m \in E_{m,d}$ with $n \neq m$ then
$$\int_S Y_n Y_m = 0.$$
(2) Consider the following inner product on the space $\mathcal{P}_{n,d}$ of homogeneous polynomials of degree n in d variables,
$$[P, Q] = P(D)(Q) = \sum_{|\alpha|=n} \alpha! P_\alpha Q_\alpha,$$
where $P(x) = \sum_{|\alpha|=n} P_\alpha x^\alpha$, $Q(x) = \sum_{|\alpha|=n} Q_\alpha x^\alpha$. Show that the space of harmonic polynomials $\mathcal{H}_{n,d} \subset \mathcal{P}_{n,d}$ is the orthogonal complement of
$$\mathcal{Q}_{n,d} = \{P \in \mathcal{P}_{n,d} : P(x) = |x|^2 P_1(x), P_1 \in \mathcal{P}_{n-2,d}\}$$
with respect to this inner product.
(3) Show that any homogeneous polynomial F of degree n in $\mathbb{R}^d$ can be written as
$$F(x) = H_n(x) + |x|^2 H_{n-2}(x) + \cdots |x|^{2k} H_{n-2k},$$
where $k = \lfloor n/2 \rfloor$ and H_j is a homogeneous harmonic polynomial of degree j. This implies that spherical harmonics form a basis for $L^2(S)$ and there no other eigenfunctions.
(4) Deduce that if $Y \in \mathcal{H}_{n,d}$ and F is a polynomial of degree less than n then $\int_S YF = 0$.
(5) Suppose that $P(x) \in \mathcal{H}_{n,d}$ and Q is a factor of P, $P = QF$ for some polynomial F. Show that Q changes sign in $\mathbb{R}^d$.

Exercise 1.8.4 (Dirichlet eigenfunctions for balls). Let J_n be the Bessel function such that

$$u(re^{i\theta}) = J_n(r)(a \cos n\theta + b \sin n\theta)$$

satisfies $\Delta u + u = 0$ in $\mathbb{R}^2$, i.e., J_n is a solution of the second order ODE

$$r^2 J'' + rJ + (r^2 - n^2)J = 0.$$

Furthermore, let $0 < j_{n,1} < j_{n,2} < \cdots$ be the positive zeros of J_n.

(1) Show that there is a constant c such that $n \leqslant j_{n,1} \leqslant cn$. (Hint: you may use the equation for the lower bound and the density of zero sets of eigenfunctions for the upper bound.)

(2) Show that the following functions

$$\phi_{n,k}(re^{i\theta}) = J_n(j_{n,k}r)(a\cos n\theta + b\sin n\theta)$$

are eigenfunctions of the Dirichlet Laplacian on the unit ball of $\mathbb{R}^2$, and that the smallest eigenvalue is $j_{0,1}^2$.

Remark 1: A classical and deep result of Siegel implies that two distinct Bessel functions J_n and J_m with integer n and m have no common zeros and thus all eigenvalues of a disk are simple.

Remark 2: Let $\lambda_{d,k}$ be the kth eigenvalue of the Dirichlet Laplace operator on the unit ball $B_0 \subset \mathbb{R}^d$. Suppose that M is a smooth d-dimensional Riemannian manifold, $x \in M$ and let $B = B(x, r)$ be the ball on M of radius r and center x. Let $\lambda_k(B)$ be the kth eigenvalue of the Dirichlet Laplace-Beltrami operator for B. Then one can show that (see [6])

$$\lambda_k(B) \sim r^{-2}\lambda_{d,k}, \quad r \to 0.$$

Exercise 1.8.5 (Yau's conjecture). Prove the lower bound $\mathcal{H}^{d-1}(Z(u)) \geqslant c\sqrt{\lambda}$ in the Yau conjecture in dimensions one and two. *Hint:* for the case $d = 2$, first use Exercise 1.8.2, then the inequality $\lambda_1(\Omega_1) \geqslant \lambda_1(\Omega_2)$ for $\Omega_1 \subset \Omega_2$, and finally Remark 2 above.

2. Doubling index and frequency function

An important tool to study nodal sets of eigenfunctions and growth properties of solutions of elliptic PDEs is the so-called frequency function. The idea goes back to the works of Almgren [2] and Agmon [1], where it was introduced for the case of harmonic functions in $\mathbb{R}^n$. It was generalized to solutions of second order elliptic equations by Garofalo and Lin [13], see also [20] and [31].

2.1. Frequency function Let $A(x)$ be a symmetric uniformly elliptic matrix with Lipschitz coefficients defined on some ball B_r centered at the origin and such that $A(0) = I$. Define the function μ by

$$\mu(x) = \frac{(A(x)x, x)}{|x|^2},$$

then $\mu(0) = 1, \quad \Lambda^{-1} \leqslant \mu(x) \leqslant \Lambda$. Moreover, since A has Lipschitz coefficients, we have

$$A(x) = I + O(|x|) \quad \text{and} \quad \mu(x) = 1 + O(|x|).$$

Let u be a solution to the equation $\operatorname{div}(A(x)\nabla u(x)) = 0$. We consider weighted averages of $|u|^2$ over spheres:

$$H(r) = r^{1-d}\int_{\partial B_r}\mu(x)|u(x)|^2 ds(x).$$

Denoting by $\nu = x/|x|$ the unit outer normal vector for the sphere and applying the divergence theorem, we obtain

$$H(r) = r^{-d}\int_{\partial B_r}(|u|^2 A(x)x,\nu)ds = r^{-d}\int_{B_r}\operatorname{div}(|u|^2 A(x)x).$$

In the case of the Laplace operator, $A = I$ and $\mu(x) = 1$, the function $t \mapsto H(e^t)$ is convex, i.e.,

$$H(r) \leqslant H(r_1)^{\alpha} H(r_2)^{1-\alpha}, \quad \text{when} \quad r = r_1^{\alpha} r_2^{1-\alpha},\ \alpha \in (0,1).$$

This can be proved either using the decomposition of harmonic functions into series of spherical harmonics, or by integration by parts as below, the computations are slightly simplified in this case, see [15].

A similar convexity property was discovered for solution of elliptic equations in [13], we provide a calculation that is a small variation of the one given in [20].

First we compute the derivative of H,

$$H'(r) = -dr^{-1}H(r) + r^{-d}\int_{\partial B_r}\operatorname{div}(|u|^2 A(x)x). \tag{2.1.1}$$

We rewrite the integral in the second term as

$$\int_{\partial B_r}\operatorname{div}(|u|^2 A(x)x) = \int_{\partial B_r} 2u(\nabla u, A(x)x) + \int_{\partial B_r}|u|^2 \operatorname{trace}(A(x)) + \int_{\partial B_r}|u|^2 A_D(x),$$

where $A_D(x) = \sum_{i,j}(\partial_i a_{ij})x_j$. We also note that

$$\mu(x) = 1 + O(|x|),\ \operatorname{trace}(A) = d + O(|x|),\ \text{and } A_D(x) = O(|x|).$$

This implies

$$\int_{\partial B_r}\operatorname{div}(|u|^2 A(x)x) = \int_{\partial B_r} 2u(\nabla u, A(x)x) + d\int_{\partial B_r}|u|^2\mu(x) + O(r^d H(r)). \tag{2.1.2}$$

We rewrite the first integral in the right-hand side of (2.1.2) using the symmetry of A and then apply the divergence theorem once again to obtain

$$\int_{\partial B_r} 2u(\nabla u, A(x)x) = \int_{\partial B_r} 2u(A(x)\nabla u, x) = 2r\int_{B_r}\operatorname{div}(uA(x)\nabla u).$$

Next, using the equation $\operatorname{div}(A\nabla u) = 0$, we have

$$\int_{\partial B_r} 2u(\nabla u, A(x)x) = 2r\int_{B_r}(A(x)\nabla u, \nabla u). \tag{2.1.3}$$

Finally, combining (2.1.1), (2.1.2), and (2.1.3), we get

$$H'(r) = 2r^{1-d}\int_{B_r}(A\nabla u, \nabla u) + O(H(r)).$$

Following [13] and [20], define

$$I(r) = r^{1-d}\int_{B_r}(A\nabla u, \nabla u) = r^{-d}\int_{\partial B_r}(uA\nabla u, x),$$

and the frequency function of u

$$N(r) = \frac{rI(r)}{H(r)}.$$

Then

$$H'(r) = 2I(r) + O(H(r)), \quad N(r) = \frac{rH'}{2H} + O(1). \tag{2.1.4}$$

Proposition 2.1.5. *There exists a constant* C *that depends only on the ellipticity and Lipschitz constants of matrix* $A(x)$ *such that for any solution* u *to* $\operatorname{div}(A\nabla u) = 0$ *in a ball* B_R *centered at the origin, the function* $F(r) = e^{Cr}N(r)$ *is increasing on* $(0, R)$.

Proof. We compute $N'(r)$, taking into account that the first derivatives of the coefficients of A are bounded. We already know that

$$H'(r) = 2I(r) + O(H(r)).$$

Next we estimate $(rI(r))'$. If w is a vector field in B_r with $(w, x) = r^2$ on ∂B_r, then

$$\begin{aligned}(rI(r))' &= (2-d)I(r) + r^{2-d}\int_{\partial B_r}(A\nabla u, \nabla u) \\ &= (2-d)I(r) + r^{1-d}\int_{B_r}\operatorname{div}(w(A\nabla u, \nabla u)) \\ &= (2-d)I(r) + r^{1-d}\int_{B_r}\operatorname{div}(w)(A\nabla u, \nabla u) + r^{1-d}\int_{B_r}(w, \nabla(A\nabla u, \nabla u)).\end{aligned} \tag{2.1.6}$$

We used the divergence theorem in the second equality above. To simplify the last term we note that

$$(w, \nabla(A\nabla u, \nabla u)) = 2(w, \mathrm{Hess}(u)(A\nabla u)) + (A_{D,w}\nabla u, \nabla u), \tag{2.1.7}$$

where $A_{D,w}(x) = \{\sum_k(\partial_k a_{ij})w_k\}_{i,j}$. Furthermore, the Hessian is a symmetric matrix and

$$\mathrm{Hess}(u)(w) = \nabla(\nabla u, w) - (Dw)\nabla u.$$

Thus, we obtain,

$$\begin{aligned}\int_{B_r}(\mathrm{Hess}(u)w, A\nabla u) &= \int_{B_r}(\nabla(\nabla u, w), A\nabla u) - \int_{B_r}((Dw)\nabla u, A\nabla u) \\ &= \int_{B_r}\operatorname{div}((\nabla u, w)A\nabla u) - \int_{B_r}((Dw)\nabla u, A\nabla u) \\ &= r^{-1}\int_{\partial B_r}(\nabla u, w)(A\nabla u, x) - \int_{B_r}((Dw)\nabla u, A\nabla u).\end{aligned} \tag{2.1.8}$$

We used the equation $\operatorname{div}(A\nabla u) = 0$ for the second identity and the divergence theorem for the third one.

Now we choose $w(x) = \mu(x)^{-1}A(x)x$. Then

$$(w(x), x) = |x|^2, \quad Dw = I + O(|x|), \quad \operatorname{div}(w) = d + O(|x|).$$

We proceed to work with (2.1.8) and rewrite the first integral as

$$\int_{\partial B_r} (\nabla u, w)(A\nabla u, x) = \int_{\partial B_r} \mu(x)^{-1}(A\nabla u, x)^2.$$

Now combine the second term in (2.1.6) and the second term in (2.1.8), taking into account the inequalities for Dw and $\operatorname{div}(w)$, we get

$$r^{1-d}\int_{B_r} \operatorname{div}(w)(A\nabla u, \nabla u) - 2r^{1-d}\int_{B_r}((Dw)\nabla u, A\nabla u) = (d-2)I(r) + O(rI(r)).$$

Moreover, we have

$$r^{1-d}\int_{B_r} |(A_{D,w}\nabla u, \nabla u)| \leqslant Cr^{1-d}\int_{B_r} r|\nabla u|^2 = O(rI(r)),$$

where C depends on the ellipticity and Lipschitz constants of A and on the dimension of the space. Now (2.1.6), (2.1.7), (2.1.8) and the last two inequalities imply

$$(rI(r))' = 2r^{-d}\int_{\partial B_r} \mu(x)^{-1}(A\nabla u, x)^2 + O(rI(r)).$$

Finally, the last inequality and (2.1.4) give

$$\begin{aligned} N'(r)(N(r))^{-1} &= (rI(r))'(rI(r))^{-1} - (H'(r))(H(r))^{-1} \\ &= \frac{2r^{-2d}}{I(r)H(r)}\left(\int_{\partial B_r} \frac{(A\nabla u, x)^2}{\mu(x)}\int_{\partial B_r} \mu(x)|u|^2 - \left(\int_{\partial B_r}(uA\nabla u, x)\right)^2\right) + O(1). \end{aligned}$$

The first term is positive by the Cauchy-Schwarz inequality. Therefore

$$N'(r) \geqslant -CN(r)$$

and the proposition follows. □

Corollary 2.1.9. *Suppose that* $\operatorname{div}(A(x)\nabla u(x)) = 0$ *in* B_{R_0}*, where* $A(x) = I + O(x)$ *as above. Let also* $N(r)$ *be the frequency of* u*. Then there exists* D_N *that depends on* R_0*,* $N(R_0/2)$*, the ellipticity and Lipschitz constants of the operator, and the dimension of the space, such that*

$$\int_{B_{2r}} |u|^2 \leqslant D_N \int_{B_r} |u|^2$$

for any $r \in (0, R_0/4)$*.*

Proof. For any $r < R_0/2$ we write (2.1.4) and apply the proposition

$$H'(r)H(r)^{-1} \leqslant 2I(r)H(r)^{-1} + c = 2r^{-1}N(r) + c \leqslant 2r^{-1}N(R_0/2)e^{C(R_0-2r)}.$$

Integrating $H'(r)/H(r)$ over an interval $[\rho, 2\rho]$ for $\rho < R_0/4$, we get

$$\int_{\partial B_{2\rho}} \mu(x)|u(x)|^2 ds(x) \leqslant C_N \int_{\partial B_\rho} \mu(x)|u(x)|^2 ds(x),$$

where $C_N = \exp(C_1 + C_2 N(R_0/2))$ with $C_2 = C_2(R_0)$. Finally, integrating the inequality with respect to ρ from 0 to r, and using that $\Lambda^{-1} \leqslant \mu \leqslant \Lambda$ we obtain the required estimate. $\square$

2.2. Three spheres theorem for elliptic PDEs Another consequence of the monotonicity of the frequency function is the three sphere theorem. Its simplest version is the classical Hadamard three circle theorem for analytic functions. It states that if f is an analytic function on the unit ball in $\mathbb{C}$ and

$$M(r) = \max\{|f(z)| : |z| = r\},$$

then the following inequality holds

$$M(r) \leqslant M(r_1)^\alpha M(r_2)^{1-\alpha}, \quad \text{where} \quad r = r_1^\alpha r_2^{1-\alpha},\ r, r_1, r_2 < 1.$$

The classical proof is based on the fact that the logarithm of the modulus of an analytic function is subharmonic. It turns out that even without analyticity a version of the Hadamard inequality holds for harmonic functions and more generally for solutions to uniformly elliptic equations. One of the first general results is due to Landis [21].

We derive the three spheres theorem from the properties of the frequency function, following [13]. Proposition 2.1.5 implies the inequality $e^{Cr}N(2r) \geqslant N(r)$, which, combined with (2.1.4), gives

$$\frac{rH'(r)}{H(r)} \leqslant \left(c + \frac{2rH'(2r)}{H(2r)}\right) e^{Cr}.$$

Then integrating from r to $2r$ with respect to dr/r we obtain

$$\log H(2r) - \log H(r) \leqslant (c \log 2 + \log H(4r) - \log H(2r)) e^{2Cr}. \tag{2.2.1}$$

Proposition 2.2.2. *Assume that* $L = \operatorname{div}(A\nabla\cdot)$ *is a uniformly elliptic operator,* A *is symmetric and has Lipschitz entries in a domain* Ω. *Suppose also that* $A(0) = I$ *and* $B(0, 4r) \subset \Omega$. *There exist* $\alpha > 0$ *and* $C > 0$ *such that for any solution* u *of* $Lu = 0$ *the following inequality holds*

$$\int_{\partial B_{2r}} |u|^2 \leqslant C \left(\int_{\partial B_r} |u|^2\right)^\alpha \left(\int_{\partial B_{4r}} |u|^2\right)^{1-\alpha}.$$

Proof. We collect similar terms in (2.2.1) and take the exponent of both sides to obtain

$$\int_{\partial B_{2r}} \mu|u|^2 ds \leqslant C_1 \left(\int_{\partial B_r} \mu|u|^2 ds\right)^\alpha \left(\int_{\partial B_{4r}} \mu|u|^2 ds\right)^{1-\alpha}$$

with $\alpha = (1 + e^{4Cr})^{-1}$ so that α can be chosen close to $1/2$ as $r \to 0$. This inequality and bounds on μ imply the required estimate. $\square$

Assume that A is as above with $A(0) = I$. Proposition 2.2.2 and the equivalence of L^p-norms for solutions of elliptic equations (see Corollary 5.1.4 below) imply the following three ball inequality for supremum norms

$$\sup_{B_{2r}} |u| \leqslant C \left(\sup_{B_r} |u|\right)^{\alpha_1} \left(\sup_{B_{8r}} |u|\right)^{1-\alpha_1},$$

for some C and $\alpha_1 \in (0, 1)$ depending on A and r but not on u.

We can drop the assumption that $A(0) = I$ applying a local change of variables, balls are replaced by ellipses. Applying the inequality several times and inscribing ellipses in balls we obtain the following statement. (We omit some technical details required for an accurate argument.)

Corollary 2.2.3. *Let $L = \operatorname{div}(A\nabla\cdot)$ be a uniformly elliptic operator with Lipschitz coefficients in a domain Ω. There exist $r_0 > 0$, k large enough, C and $\beta \in (0, 1)$ such that if $B = B_r$ is a ball with $r < r_0$ and $B_{k^2 r} \subset \Omega$ then*

$$\sup_{B_{2r}} |u| \leqslant C \left(\sup_{B_r} |u|\right)^{\beta} \left(\sup_{B_{kr}} |u|\right)^{1-\beta},$$

for any u that solves the equation $Lu = 0$ in Ω.

The general version of this result can be obtain by the chain argument.

Corollary 2.2.4. *Let L be as above and $B \subset K \subset\subset \Omega$, where B is open and K is compact. There exist C and $\gamma \in (0, 1)$ that depend only on K, Ω, B and the ellipticity and Lipschitz constants of L such that for any solution u to $Lu = 0$ in Ω the following inequality holds*

$$\sup_{K} |u| \leqslant C \left(\sup_{B} |u|\right)^{\gamma} \left(\sup_{\Omega} |u|\right)^{1-\gamma}.$$

Proof: Chain argument. Assume that $\sup_\Omega |u| = 1$. For each point $x \in K$ there is a curve γ connecting x to some fixed point in B. We then can find a finite sequence of balls $\{B_j\}_{j=1}^J$ such that $r(B_j) < r_0$, $B_1 \subset B$, $B_{j+1} \subset 2B_j$, $k^2 B_j \subset \Omega$ and $x \in B_J = B(x)$. Applying the previous corollary we see that

$$\sup_{B_{j+1}} |u| \leqslant \sup_{2B_j} |u| \leqslant C(\sup_{B_j} |u|)^{\beta}.$$

Iterating this estimate we obtain

$$\sup_{B_J} |u| \leqslant C_1 (\sup_{B} |u|)^{\beta_1},$$

for some C_1 and β_1 that depend on C, β and the number J of the iteration steps. Finally, we take a finite cover of K by balls $B(x)$ and get the required estimate. □

2.3. Doubling index We prefer to replace the frequency function by a comparable but more intuitive quantity that we call the doubling index. Let $h \in C(\Omega)$, such that h does not vanish on any open subset of Ω. For any closed ball B such that $2B \subset \Omega$ we define

$$\mathcal{N}_h(B) = \log \frac{\max_{2B} |h|}{\max_B |h|}.$$

Note that if p is a homogeneous polynomial of degree n and a ball B is centered at the origin than $\mathcal{N}_p(B) = n \log 2$. At the same time if we compute the frequency function $N_p(r)$ of this polynomial (defined for the case of the Laplace operator, $A = I$), we get $N_p(r) = n$. In general, if h is a solution to $Lh = 0$ in the ball B_{R_0} then, using the equivalence of norms (Corollary 5.1.4) and the estimate in the proof of Corollary 2.1.9, we obtain that for $r < R_0/4$

$$C_1^{-1} N_h(r) - C_2 \leqslant \mathcal{N}_h(B_r) \leqslant C_1 N_h(4r) + C_2.$$

The inequality above and the almost monotonicity of the frequency implies the following almost monotonicity for the doubling index when $4r < R < R_0$,

$$\mathcal{N}_h(B_r) \leqslant C(\mathcal{N}_h(B_R) + 1)\,. \tag{2.3.1}$$

2.4. Doubling index for eigenfunctions The monotonicity of the doubling index and three sphere theorem hold for solutions of second order elliptic equations of the form $\operatorname{div}(A\nabla h) = 0$. For eigenfunctions $\phi_\lambda(x)$ on compact manifolds there is no monotonicity of the doubling index and the three sphere inequality gets a constant that depends on the eigenvalue. As above, we consider the lift $h(x,t) = e^{\sqrt{\lambda} t}\phi_\lambda(x)$ and then apply the results of the previous sections to h that solves an equation of the form $\operatorname{div}(A\nabla h) = 0$.

Donnelly and Fefferman used the doubling indices in their study of nodal sets of eigenfunctions on smooth manifolds. One of their celebrating results for general smooth compact Riemannian manifolds is the following.

Proposition 2.4.1. *Let M be a smooth compact Riemannian manifold. There exists r_0 and C depending on M such that for any eigenfunction $\phi = \phi_\lambda$,*

$$\Delta_M \phi_\lambda + \lambda \phi_\lambda = 0,$$

the doubling index $\mathcal{N}_\phi(B) \leqslant C\sqrt{\lambda}$ when B is a ball on M with radius $r \leqslant r_0$.

Proof. Let $B = B(x,r)$ be a ball on M. We consider the ball B' on $M \times [-R, R]$, $R > r$, such that the center of B' is $(x, 0)$ and the radius on B' is r. We say that B' is the lift of B. We note that $\mathcal{N}_\phi(B) \leqslant \mathcal{N}_h(B') + C\sqrt{\lambda}$. It is enough to prove the estimate for the doubling index of h on $M \times [-R, R]$. Assume that $\max_M |\phi| = |\phi(x_0)| = 1$ and fix r such that for each point $x \in M$ the geodesic ball $B_r(x)$ is contained in a chart.

Let B be any ball of radius $r/2k$ on M and B' be its lift in $M \times \mathbb{R}$. Consider a finite chain $\{B_j\}_{j=1}^J$ of geodesic balls in $M \times [-r, r]$ with centers on $M \times 0$ and equal radii $r/2k$ chosen so that $B_1 = B'$, $B_{j+1} \subset 2B_j$ and $(x_0, 0) \in B_J$. Then, since $\sup_{kB_j} |h| \leqslant e^r$, Corollary 2.2.3 implies

$$\sup_{B_j} |h| \geqslant c(\sup_{2B_j} |h|)^{1/\beta} e^{-Cr} \geqslant c(\sup_{B_{j+1}} |h|)^{1/\beta} e^{-Cr}.$$

It implies that $\sup_{B'} |h| \geqslant c_1$, where where c_1 depends on r and M (which also determine the number of balls in a chain). Then,

$$e^{r\sqrt{\lambda}} \sup_B |\phi| \geqslant c_1,$$

Thus for any ball B of radius at least r and for the corresponding lifted ball B' we obtain $\mathcal{N}_\phi(B) \leqslant C(\sqrt{\lambda}+1)$ and $\mathcal{N}_h(B') \leqslant C(\sqrt{\lambda}+1)$. Finally, the almost monotonicity of the doubling index for h implies similar estimate for balls of radius less than r. $\square$

2.5. Cubes A version of the (maximal) doubling index for cubes is used in the next sections. For a given cube $Q \subset \mathbb{R}^d$ we denote its side length by $s(Q)$. Then the volume of the cube is $|Q| = (s(Q))^d$.

Assume that u is a solution to the equation $Lu = 0$ in a domain $\Omega \subset \mathbb{R}^d$ and for each cube Q with $2Q \subset \Omega$ define

$$\mathcal{N}^*_u(Q) = \sup_{q \subset Q} \log \frac{\max_{2q} |u|}{\max_q |u|}. \tag{2.5.1}$$

We claim that the almost monotonicity of the usual doubling index implies that the supremum above is finite. By the definition, we have now that if $q \subset Q$ then $\mathcal{N}^*_u(q) \leqslant \mathcal{N}^*_u(Q)$.

We want to compare the maximal doubling index $\mathcal{N}^*_u(Q)$ defined above to the doubling index $\log \max_{2Q} |u| - \log \max_Q |u|$. Take a cube $q \subset Q$. If q is small, $s(q) < c_d s(Q)$, we first apply almost monotonicity inequality for the doubling index (2.3.1). Let b be the largest ball inscribed in q then $2q \subset k_d b$, where $k_d = 2\sqrt{d}$ and we have

$$\log \frac{\max_{2q} |u|}{\max_q |u|} \leqslant \log \frac{\max_{k_d b} |u|}{\max_b |u|} \leqslant C_1 \log \frac{\max_{k_d B} |u|}{\max_B |u|} + C_2,$$

where B is a ball concentric with b such that $k_d B \subset Q$, $R = R(B) \sim s(Q)$. This implies

$$\frac{\max_{2q} |u|}{\max_q |u|} \leqslant C_3 \left(\frac{\max_{k_d B} |u|}{\max_B |u|} \right)^{C_1}.$$

Now, using that $R(B)$ is comparable to $s(Q)$, we repeat the chain argument from the proof of Corollary 2.2.4 to obtain the inequality

$$\max_Q |u| \leqslant C \left(\max_B |u| \right)^\gamma \left(\max_{2Q} |u| \right)^{1-\gamma}.$$

with C and $\alpha \in (0,1)$ which does not depend on B (for B with $R(B) \sim s(Q)$ the number of balls in the chain is uniformly bounded). Finally,

$$\frac{\max_Q |u|}{\max_{2Q} |u|} \leqslant C \left(\frac{\max_B |u|}{\max_{k_d B} |u|} \right)^\gamma \leqslant C \left(\frac{\max_q |u|}{\max_{2q} |u|} \right)^{\gamma/C_1}.$$

For large cubes q with $s(q) \geqslant c_d s(Q)$ the last inequality follows directly from the three balls inequality and the chain argument. Thus we obtain

$$\log \frac{\max_{2Q} |u|}{\max_Q |u|} \geqslant a_1 \mathcal{N}^*_u(Q) - a_2, \tag{2.5.2}$$

where a_1 and a_2 depend on the ellipticity and Lipschitz constants of the operator only when we assume that $s(Q) \leqslant 1$.

We also consider eigenfunctions on manifolds and define the doubling index for eigenfunctions over cubes in a similar way, to prove that the supremum is finite for this case we can use the monotonicity for the lifted function.

2.6. Remarks on the size of the zero sets of eigenfunctions and the doubling index In this section we first formulate some results that were proved by Donnelly and Fefferman [9]. We assume that M is a real-analytic Riemannian manifold (or that coefficients of the corresponding elliptic operator are real-analytic, see also [23].)

Lemma 2.6.1. *Let* $L = \operatorname{div}(A\nabla\cdot)$ *be a uniformly elliptic operator with real analytic coefficients defined in the unit cube* $Q_0 \subset \mathbb{R}^{d+1}$. *There is constant* $C = C(L)$ *such that if* $Lh = 0$ *in* Q_0 *then for any* Q_1 *such that* $4Q_1 \subset Q_0$ *the size of the zero set of* h *in* Q_1 *admits the following estimate*

$$\mathcal{H}^d(Z(h) \cap Q_1) \leqslant CN\, s(Q_1)^d,$$

where $N = \max\{1, \mathcal{N}_h^*(2Q_1)\}$.

We don't know if this lemma remains true for non-analytic case.

Suppose that ϕ_λ is an eigenfunction on a compact manifold M with real-analytic metric. Applying Lemma 2.6.1 to the function $h(x,t) = \phi_\lambda(x)\exp(\sqrt{\lambda}t)$ on charts and having in mind the bound for the doubling index of h, one obtains the upper bound for $\mathcal{H}^d(Z(h) \cap M \times [-1,1])$. Moreover, since $Z(h)$ is the cylinder over $Z(\phi)$, the upper bound in Yau's conjecture follows

$$\mathcal{H}^{d-1}(Z(\phi_\lambda)) \leqslant C\sqrt{\lambda}.$$

This part of the conjecture is open for non-analytic manifolds. The best known result, see [25], is based on a non-analytic version of the lemma above, the estimate is

$$\mathcal{H}^d(Z(h) \cap Q_1) \leqslant CN^A s(Q_1)^d$$

for some $A = A(d)$. It implies a polynomial bound in Yau's conjecture.

To obtain the lower bound in Yau's conjecture on manifolds with real analytic metric, Donnelly and Fefferman proved the following statement.

Lemma 2.6.2. *Suppose that* M *is a real-analytic manifold. There exists* N_0 *such that the following is true. If* $\phi = \phi_\lambda$ *is an eigenfunction on* M *and* M *is partitioned into cubes with side length* $\approx \sqrt{\lambda}^{-1}$, $M = \cup q$, *then for at least half of these cubes* q *the doubling index of* ϕ *in* q *is bounded by* N_0, $\mathcal{N}_\phi^*(q) \leqslant N_0$.

This lemma can be combined with the next one (applied for the lifted function) to give the conjectured lower bound for the size of the zero set of eigenfunctions on real-analytic manifolds.

Lemma 2.6.3. *Let* $L = \operatorname{div}(A\nabla\cdot)$ *be a uniformly elliptic operator with smooth coefficients in the unit cube* $Q_0 \subset \mathbb{R}^{d+1}$. *There exists a function* $f(N)$ *that depends only on* L *such* *that if* $Lh = 0$ *in* Q_0, $h(0) = 0$ *and* $\mathcal{N}^*_h(Q_1) \leqslant N$, *where* $Q_1 = 1/4Q_0$, *then*

$$\mathcal{H}^d(Z(h) \cap Q_1) \geqslant f(N)s(Q_1)^d.$$

The last lemma does not require analyticity of the coefficients. A simple quantification of this estimate is known (see remarks in [27]); the statement of 2.6.3 is weaker than Theorem 1.7.1. Detailed discussion of the current state of Yau's conjecture and related open problems can be found in [29].

We conclude this lecture by formulating an estimate for the size of the zero set from above which is not as precise as the polynomial bound in [25]. It follows from earlier results of Hardt and Simon [17].

Lemma 2.6.4. *Let* $L = \mathrm{div}(A\nabla\cdot)$ *be a uniformly elliptic operator with smooth coefficients in the unit cube* $Q_0 \subset \mathbb{R}^{d+1}$. *There exists a function* $F(N)$ *that depends only on* L *such that if* $Lh = 0$ *in* Q_0, *and* $\mathcal{N}^*_h(Q_1) \leqslant N$, *where* $Q_1 = 1/4Q_0$, *then*

$$\mathcal{H}^d(Z(h) \cap Q_1) \leqslant F(N)s(Q_1)^d.$$

2.7. Exercises

Exercise 2.7.1. For h harmonic on $\mathbb{R}^d$, define the frequency function of h by

$$N(r) = \frac{rH'(r)}{2H(r)},$$

where $H(r) = r^{1-d}\int_{|x|=r} |h(x)|^2 ds(x)$.

1. Show that if h is a homogeneous polynomial of degree n then $N(r) = n$.
2. Let $h = \sum_{k=l}^{L} p_k$, where p_k is a homogeneous harmonic polynomial of degree k and $p_l, p_L \neq 0$. Show that
$$\lim_{r\to 0} N(r) = l \quad \text{and} \quad \lim_{r\to\infty} N(r) = L.$$
Remark: l is called the vanishing order of h at the origin.
3. Use the fact that $N(r)$ is a non-decreasing function to prove that
$$\left(\frac{R}{r}\right)^{2N(r)} \leqslant \frac{H(R)}{H(r)} \leqslant \left(\frac{R}{r}\right)^{2N(R)}.$$

Exercise 2.7.2 (Applications of the three ball inequality).
Suppose that h is a non-constant harmonic function in $\mathbb{R}^d$ such that $|h| \leqslant 1$ on a half-space $\{x = (x_1, x_d, x_d > 0\}$. Let $m(R) = \max_{|x|<R} |h|$.

1. Show that there exist $c > 0$ and $\alpha \in (0,1)$ such that for any $R > 0$
$$m(R) \leqslant Cm(5R)^\alpha.$$
2. Show that $m(R) \geqslant c\exp(R^\beta)$ for some $\beta > 0$.

Exercise 2.7.3 (Log-convex functions).
Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. We say that m is log-convex if $f(t) = \ln(m(\exp(t)))$ is a convex function. (For example if $m(x) = x^a, a > 0$ then $f(t) = at$ and m is log-convex.) *Warning:* usually a positive function g is called logarithmically convex if $\log(g)$ is a convex function.

(1) Show that if a_k are non-negative numbers then
$$m(x) = \sum_{k=1}^{n} a_k x^k$$
is log-convex. Hint: The sum of two log-convex functions is log-convex.

(2) Let u be a harmonic function in the unit ball of $\mathbb{R}^d$, we know that
$$u(x) = \sum_{k=0}^{\infty} |x|^k Y_k(x/|x|),$$
where Y_k is an eigenfunction of the Laplace-Beltrami operator on the unit sphere $S \subset \mathbb{R}^d$. Show that
$$m(r) = \int_S |u(ry)|^2 ds(y)$$
is log-convex.

(3) Let $K(x,t)$ be the heat kernel in $\mathbb{R}^d$,
$$K(x,t) = (4\pi t)^{-d/2} \exp(-|x|^2/(4t)),$$
and it satisfies the equation $\Delta K(x,t) = \partial_t K(x,t)$. Suppose that u is a harmonic function in $\mathbb{R}^d$ such that $u(x)\exp(-c|x|^2) \in L^2(\mathbb{R}^d)$ for any $c > 0$. Define
$$M(t) = \int_{\mathbb{R}^d} |u(x)|^2 K(x,t) dt.$$
Compute $M'(t)$ and show that $M^{(m)}(t) \geqslant 0$ for any m.

Remark: The positivity of all derivatives implies that $M(t)$ is a log-convex function. This convexity was studied by Lippner and Mangoubi [24] for the case of discrete harmonic functions.

Exercise 2.7.4 (Reverse Hölder inequality for solutions of elliptic equations). Show that if u is a solution of a uniformly elliptic equation with Lipschitz coefficients, $\operatorname{div}(A\nabla u) = 0$ in a ball B_0 then for some (any) $q > 1$ there exists $C_q(u)$ such that for any ball $B \subset 1/2B_0$
$$\left(|B|^{-1} \int_B |u|^{2q}\right)^{1/q} \leqslant C_q(u)|B|^{-1} \int_B |u|^2.$$
Remark: It implies that $|u|^2$ is a Muckenhoupt weight and therefore the zero set has zero Lebesgue measure, $|Z(u)| = 0$. A similar inequality holds for the function $u - |B|^{-1}\int_B u$ and together with the Caccioppoli inequality it implies that $|\nabla u|^2$ is also a Muckenhoupt weight (see [13] for details).

3. Small values of polynomials and solutions of elliptic PDEs

Let P be a non-constant polynomial of one complex variable with complex coefficients, $P \in \mathbb{C}[z]$,
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_j \in \mathbb{C}$ and $a_n \neq 0$. As $|z|$ goes to infinity the behavior of $P(z)$ resembles that of the highest degree term $a_n z^n$. As we know $P(z)$ has n zeros counting multiplicities and the set $\{z : |P(z)| < C\}$ is bounded and contains the zeros. We use the following notation

$$E_a(P) = \{z : |P(z)| < e^{-a}\}.$$

3.1. Classical results of Cartan and Polya A classical result on the size of the set where a polynomial takes small values is due to H. Cartan. Let $\mathcal{P}_n$ denote the set of all polynomials of degree n with leading coefficient 1,

$$\mathcal{P}_n = \{p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]\}.$$

Lemma 3.1.1 (Cartan, 1928)**.** *Let $p \in \mathcal{P}_n$ then for any $a, \alpha > 0$ there exist a finite collection of balls $\{B_j\}$ such that $E_{na}(p) \subset \cup_j B_j$ and $\sum_j r_j^\alpha \leqslant e(2e^{-a})^\alpha$, where r_j is the radius of B_j.*

In particular, taking $\alpha = 2$, one obtains that $|E_{na}(p)| \leqslant 4\pi e^{1-2a}$. This estimate is not sharp as the next result shows.

Lemma 3.1.2 (Polya, 1928)**.** *Let $p \in \mathcal{P}_n$ then $|E_{na}(p)| \leqslant \pi e^{-2a}$ for any $a > 0$.*

The last inequality is sharp, the equality is obtained when $p(z) = z^n$.

Lemmas of Cartan and Polya deal with polynomials for which the leading coefficient is equal to one and provide estimates of the set of all points of the complex plane where the polynomial is small, the proofs of both lemmas and related results can be found in [30]. We are interested in a local version of such estimates.

3.2. Remez inequality for polynomials Now we consider polynomials with real coefficients on the real line and we do not normalize the leading coefficient.

Lemma 3.2.1 (Remez, 1936)**.** *Let E be a measurable subset of an interval I of positive measure, $|E| > 0$. Then for any polynomial $P_n \in \mathbb{R}[x]$ of degree n*

$$\max_{x\in I} |P_n(x)| \leqslant \left(\frac{4|I|}{|E|}\right)^n \max_{x \in E} |P_n(x)|$$

More precise inequality and its proof is outlined in the exercises below, see Exercise 3.5.3. The original reference in [33], a proof is also given in a more accessible paper [4].

We reformulate the inequality in the following way

$$|E| \leqslant 4|I| \left(\frac{\max_{x\in E} |P_n(x)|}{\max_{x\in I} |P_n(x)|}\right)^{1/n},$$

for any measurable subset $E \subset I$. We normalize P_n such that $\max_I |P_n| = 1$ and use the notation

$$E_{an}(P_n) = \{x \in \mathbb{R} : |P_n(x)| < e^{-an}\}.$$

Then the Remez inequality can be written as

$$|E_{an}(P_n) \cap I| \leqslant 4|I|e^{-a}.$$

There are interesting generalizations of the Remez inequality, in particular the measure of the set can be replaced by another geometric characteristic; higher dimensional version are also known, we refer the reader to [5,12].

3.3. Propagation of smallness result The main result we prove in these lectures is the following quantitative propagation of smallness for solutions of elliptic equation in divergence form. As above we assume that $\mathrm{div}(A\nabla\cdot)$ is a uniformly elliptic operator, A is a symmetric matrix with Lipschitz coefficients on some domain in $\mathbb{R}^d$. We know that a solution to $\mathrm{div}(A\nabla h) = 0$ cannot vanish on a set of positive measure (see for example Remark after Exercise 2.7.4) and look for a quantitative version of this result.

Theorem 3.3.1 ([28]). *Let h be a solution of $\mathrm{div}(A\nabla h) = 0$ in Ω. Assume that*

$$|h| \leqslant \varepsilon \quad \text{on} \quad E \subset \Omega,$$

where $|E| > 0$. Let K be a compact subset of Ω then

$$\max_K |h| \leqslant C_0 \sup_\Omega |h|^{1-\alpha}\varepsilon^\alpha, \tag{3.3.2}$$

where $C_0 > 0$ and $\alpha \in (0,1)$ depend on $A, |E|, \mathrm{dist}(E, \partial\Omega)$, and K.

The inequality (3.3.2) can be considered as a version of the three balls theorem where the smallest ball is replaced by a measurable set. The constants in the inequality depend on the measure of the set and the distance from this set to the boundary of Ω but not on the set itself, which could be an arbitrarily wild measurable set. The question whether such inequality holds was asked by Landis, weaker quantitative estimates were obtained by Nadirashvili [32] and Vessella [36].

First, we formulate the following result (Remez inequality for solutions of elliptic PDE, [28]):

Claim: *Let Q_0 be the unit cube in $\mathbb{R}^d$. Assume h is a solution to the equation $\mathrm{div}(A\nabla h) = 0$ in $2Q$ $\dot{T}$hen for any cube $Q \subset Q_0$ and any measurable subset E of Q of positive Lebesgue measure, the following inequality holds*

$$\sup_Q |h| \leqslant C \sup_E |h| \left(C\frac{|Q|}{|E|} \right)^{CN}, \tag{3.3.3}$$

where C depends on A only, and $N = \mathcal{N}^*_h(Q)$ is the doubling index defined in (2.5.1).

This statement confirms that in some sense solutions of elliptic equations locally behave as polynomials with degree bounded by the multiple of the doubling index. In particular (the lift of) an eigenfunction corresponding to eigenvalue λ behaves as a polynomial of degree $C\sqrt{\lambda}$.

This phenomenon was pointed out in the works of Donnelly and Fefferman, see for example [11], where, among other results, an interesting Bernstein type inequality for eigenfunctions is obtained.

Let us show that (3.3.3) implies Theorem 3.3.1. First we remind that by (2.5.2)

$$\exp(a_1 N) \leqslant e^{a_2} \sup_{2Q} |h| (\sup_{Q} |h|)^{-1},$$

for some $a_1, a_2 > 0$. Suppose that (3.3.3) holds with some constant C, choose $C_1 = C_1(|E|)$ such that

$$\left(C \frac{|Q|}{|E|} \right)^{C} = e^{a_1 C_1}, \quad \text{i.e. } C_1 = C a_1^{-1} \log(C|Q||E|^{-1}).$$

Then

$$\sup_{Q} |h| \leqslant C \sup_{E} |h| \exp(a_1 C_1 N) \leqslant C_2 \sup_{E} |h| \left(\sup_{2Q} |h| \right)^{C_1} \left(\sup_{Q} |h| \right)^{-C_1}.$$

This implies the inequality in the theorem for the case $\Omega = 2Q$ and $K = Q$ with $\alpha = (C_1+1)^{-1}$ and C_0 that depends on $|E|$ and on A but not on h. To obtain the statement of the theorem we use the standard chain argument as in the proof of Corollary 2.2.4.

In its turn, the inequality (3.3.3) is equivalent to the following local estimate of the volume of sub-level sets.

Lemma 3.3.4. *Suppose that* $\operatorname{div}(A\nabla h) = 0$ *in* $2Q$ *and that* $\sup_Q |h| = 1$. *Write* $N = \mathcal{N}_h^*(Q) \geqslant 1$ *and*

$$E_a(h) = \{x \in Q : |h(x)| < e^{a}\}.$$

Then

$$|E_a(h)| \leqslant C e^{-\beta a/N} |Q|, \tag{3.3.5}$$

for some positive C *and* β *that depend on* A *only.*

3.4. Base of induction We prove Lemma 3.3.4 in the next section using double induction on a and N. Now we check the base of the induction, considering two cases $a \leqslant c_0 N$ and $N \leqslant N_0$.

Our aim is to prove the inequality (3.3.5). First we note that for $a/N < c_0$ the inequality holds trivially. Indeed if we choose the constant $C = C(\beta)$ large enough, we get

$$C e^{-\beta a/N} \geqslant C e^{-\beta c_0} \geqslant 1.$$

Now we want to show that (3.3.5) holds for some β and C if we assume that N is small enough. The lemma below is the base of our induction on N.

Lemma 3.4.1. *Assume that* h *satisfies* $\operatorname{div}(A\nabla h) = 0$ *in* $k_d Q$, $\sup_Q |h| = 1$ *and* $\mathcal{N}_h^*(Q) \leqslant N_0$. *Let* $E_a = \{x \in Q : |h(x)| < e^{-a}\}$. *Then*

$$|E_a| \leqslant C e^{-\gamma a} |Q|, \tag{3.4.2}$$

for some $\gamma = \gamma(N_0, A)$ *and* $C = C(N_0, A)$.

The estimate on the doubling index implies that $\sup_{1/2Q} |h| \geqslant C(N_0)$. We combine this inequality with the oscillation theorem (see Theorem 5.1.5 in Appendix). Recall that $\operatorname{osc}_Q h = \sup_Q h - \inf_Q h$.

Theorem 3.4.3. *Let* $L = \operatorname{div}(A\nabla\cdot)$ *be a uniformly elliptic operator in* Ω *and* $Lh = 0$. *There exists* $\tau = \tau(s) < 1$, *depending on* s *and on the ellipticity constant, such that for any cube* $Q \subset \Omega$

$$\operatorname{osc}_{sQ} h < \tau(s) \operatorname{osc}_Q h \quad \textit{and } \tau(s) \to 0 \textit{ as } s \to 0.$$

Corollary 3.4.4. *Assume that* h *satisfies* $\operatorname{div}(A\nabla h) = 0$ *in* $2Q$, $\sup_Q |h| = 1$ *and* $\mathcal{N}^*_h(Q) \leqslant N_0$. *There exist an integer* K, *and positive* b *and* m *that depend on* N_0, *on the ellipticity constants of* A *and on the dimension* d *such that if* Q *is partitioned into* K^d *smaller equal cubes,* $Q = \cup q$, *then*

$$\sup_q |h| \geqslant b \quad \textit{for any } q$$

and there exists one cube q_0 *in the partition such that* $\inf_{q_0} |h| \geqslant m$.

Proof. Since $\mathcal{N}^*_h(Q) \leqslant N_0$, we get a lower bound on the supremum of $|h|$ on each small cube q. Furthermore, assume that $h(x_0) = \max_{Q/2} |h| \geqslant c(N_0)$ (we replace h by $-h$ if necessary) and that K is chosen large enough. We take q_0 such that $x_0 \in q_0$. Clearly $\operatorname{osc}_Q h \leqslant 2$ and since $K/2q_0 \subset Q$ by the oscillation theorem we have $\operatorname{osc}_{q_0} h \leqslant 2\tau(2/K)$. Then we conclude

$$\inf_{q_0} h = \sup_{q_0} h - \operatorname{osc}_{q_0} h \geqslant c(N_0) - 2\tau(2K^{-1}) \geqslant m,$$

when $m < c(N_0)/2$ and K is large enough. □

In particular, the corollary implies that $|\{x \in Q : |h| < m\}| \leqslant (1 - K^{-d})|Q|$. Dividing each q once again into smaller cubes, we get on each new cube the supremum of $|h|$ is at least b^2 and

$$|\{x \in Q : |h| < mb\}| \leqslant (1 - K^{-d})^2 |Q|.$$

Iterating the corollary we see that

$$|\{x \in Q : |h| < mb^l\}| \leqslant (1 - K^{-d})^{l+1} |Q|,$$

when $\sup_Q |h| = 1$. Thus the estimate (3.4.2) holds for $e^{-a} = b^l m$ and γ such that $b^\gamma = 1 - K^{-d}$, it completes the proof of the Lemma 3.4.1.

3.5. Exercises

Exercise 3.5.1. Let $f \in L^2(\mathbb{T}^2)$, $\|f\|_{L^2} = 1$. We define the L^2-doubling index of f on a square q by

$$n(f, q) = \log \frac{\int_{2q} |f|^2}{\int_q |f|^2}.$$

Assume that $\mathbb{T}^2$ is partitioned into K^2 equal squares we say that a square is good if $n(f, q) < 100$. Show that

$$\sum_{q \text{ good}} \int_q |f|^2 \geqslant 1/2.$$

Remark: Here the 1/2 is a very rough estimate. Can you can find a better one?

Exercise 3.5.2 (Discrete version of Remez inequality). Use the Remez inequality to show that if P is a polynomial of degree n and $S \subset I \cap \mathbb{Z}$ contains $n + m$ points then

$$\max_I |P| \leqslant \left(\frac{4|I|}{m}\right)^n \max_S |P|.$$

Exercise 3.5.3 (Remez inequality for polynomials). Let $T_n(x)$ be the Chebyshev polynomial or degree n, such that $T_n(\cos\theta) = \cos(n\theta)$. This sequence can be defined by

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Clearly for each n there is a sequence $-1 = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = 1$ such that $T_n(x_{n,k}) = (-1)^{n-k}$.

Suppose that $c > 0$ and $E \subset I = [-1, 1+c]$ is a measurable set with $|E| = 2$. In this exercise we prove that for any polynomial P of degree n

$$\max_I |P| \leqslant T_n(1+c) \max_E |P|.$$

The equality is obtained for example when $E = [-1, 1]$ and $P = T_n$. To prove the inequality it is enough to assume that E is open and show that

$$P(1+c) \leqslant T_n(1+c) \max_E |P|.$$

(1) Show that there are points y_k in E such that $|x_{n,k} - x_{n,j}| \leqslant |y_k - y_j|$ and $1 + c - x_{n,k} \geqslant 1 + c - y_k$ for $k = 0, \dots, n$.
(2) Use the Lagrange interpolation formula and the properties of the Chebyshev polynomials to show that $P(1+c) \leqslant T_n(1+c) \max_E |P|$.
(3) Let $x > 1$, show that $T_n(2x-1) \leqslant (4x)^n$.

Remark: This gives a proof of the Remez inequality formulated in the lecture notes.

Exercise 3.5.4 (Quantitative unique continuation for harmonic functions). We will use Remez inequality to show the quantitative unique continuation form sets of positive measure for harmonic functions.

(1) Suppose that h is a bounded harmonic function in the unit ball B_0. Let $r < r_0(d)$ be small enough. Show that there exists $q(r) < 1$ and C such that for any integer n there is a polynomial p_n having degree at most n such that

$$\max_{|x| \leqslant r} |h(x) - p_n(x)| \leqslant Cq(r)^n \max_{|x| \leqslant 1} |h(x)|.$$

Moreover $q(r) \to 0$ as $r \to 0$.
(2) Prove that there is $r_1 = r_1(d)$ such that if E is a measurable subset of $r_1 B_0$ of positive measure, $m = |E|$, and h is a harmonic function in B_0 then

$$\max_{r_1 B_0} |h| \leqslant C(\max_E |h|)^\alpha (\max_{B_0} |h|)^{1-\alpha},$$

where α depends on m and r_1.

Exercise 3.5.5 (logarithmic capacity). The logarithmic capacity of a compact subset of the complex plane is defined by

$$\operatorname{cap}(K) = \lim_{n\to\infty} \left(\min_{p\in\mathcal{P}_n} \max_K |p(x)| \right)^{1/n}.$$

(1) Show that the limit exists.
(2) Prove that $\operatorname{cap}(E_{na}(p)) = e^{-a}$ for any $p \in \mathcal{P}_n$.
(3) Use Polya's lemma to show that $|K| \leqslant \pi \operatorname{cap}(K)^2$ for any compact set $K \subset \mathbb{C}$.

4. Proof of propagation of smallness result

We now prove Lemma 3.3.4 using double induction on a and N and some iterative argument. We start with some preliminary result on the distribution of the doubling index that will help us to carry on the induction step.

4.1. On distribution of the doubling indices The results on the doubling index that we formulate below are crucial for the proof. Let Q_0 be the unit cube in $\mathbb{R}^d$.

Assume that $f \in C(Q_0)$ and for any cube q such that $2q \subset Q_0$ define

$$N_f(q) = \log \frac{\max_{2q}|f|}{\max_q |f|}.$$

Warning: We have used the notation $N_h(r)$ for the frequency of h in the ball $B(0,r)$ in Section 2. But for the rest of the notes we do not refer to the frequency function and use $N_f(q)$ for the doubling constant of f in a cube q as defined above.

Lemma 4.1.1. *Let a cube* $Q \subset Q_0$ *be partitioned into* K^d *equal cubes* q_i, $K \geqslant 8$. *Let* $N_{\min} = \min_i N_f(q_i)$ *and assume that* $N_{\min}$ *is large enough,* $N_{min} \geqslant N_0(d)$. *Then*

$$N_f(Q/2) \geqslant \frac{K}{8} N_{\min}.$$

Proof. Let $\max_{Q/2}|f| = |f(x_0)|$, $x_0 \in q_i$ for some i. Then, since $N_f(q_i) \geqslant N_{min}$, there exists $x_1 \in 2q_i$ such that $|f(x_1)| \geqslant e^{N_{min}}|f(x_0)|$. At this point, we have $x_1 \in 2q_i \subset (1/2 + 2/K)Q$. We find one of the cubes in the partition for which $x_1 \in q$ and repeat the step. We end up with a sequence $\{x_j\}_j$ such that

$$|f(x_j)| \geqslant e^{jN_{min}}|f(x_0)|$$

and $x_j \in (1/2 + 2j/K)Q$. We repeat this $J = \lfloor K/4 \rfloor$ times, such that the last x_J is still in Q. Then

$$\max_Q |f| \geqslant e^{KN_{min}/8} \max_{Q/2} |f|.$$

which implies the required estimate. □

For solutions of elliptic equations we can formulate the above result using the monotonicity of the doubling index and the maximal doubling index $\mathcal{N}_h^*(q)$ defined by (2.5.1).

Corollary 4.1.2. *Let* $L = \operatorname{div}(A\nabla\cdot)$ *be a uniformly elliptic operator in* $2Q_0$*. There exist constants* N_0 *and* J_0 *such that if* $Lh = 0$ *in* $2Q_0$*,* $Q \subset Q_0$*,* Q *is partitioned into* J^d *equal cubes* q_i *and* $J \geqslant J_0$ *and* $\mathcal{N}^*(Q) \geqslant N_0$*, then for at least one cube* q *in the partition*

$$\mathcal{N}^*_h(q) \leqslant \mathcal{N}^*_h(Q)/2.$$

We rewrite the inequality (2.5.2) in the following form

$$N_h(cq) \leqslant \mathcal{N}^*_h(q) \leqslant A_1 N_h(q) + A_2.$$

Then Corollary 4.1.2 follows immediately from Lemma 4.1.1.

Our aim in induction argument is to divide the cube into small cubes and find a sub-cube with small doubling index.

4.2. Choosing the right notation We fix the ellipticity constant $\Lambda > 1$ and the Lipschitz constant C and consider second order elliptic operator $L = \operatorname{div}(A\nabla\cdot)$ in the cube $2Q_0$, where Q_0 is the unit cube in $\mathbb{R}^d$. We vary the parameters $N > 1$ and $a > 0$ and aim at proving the estimate (3.3.5).

Let

$$m(u, a) = |\{x \in Q_0 : |u(x)| < e^{-a} \sup_{Q_0} |u|\}|$$

and

$$M(N, a) = \sup_{*} m(u, a),$$

where the supremum is taken over all elliptic operators $\operatorname{div}(A\nabla\cdot)$ and functions u satisfying the following conditions in $2Q_0$:

(i) $A(x) = [a_{ij}(x)]_{1\leqslant i,j\leqslant d}$ is a symmetric uniformly elliptic matrix with Lipschitz entries and ellipticity and Lipschitz constants bounded by Λ and C respectively,
(ii) u is a solution to $\operatorname{div}(A\nabla u) = 0$ in $2Q_0$,
(iii) $\mathcal{N}^*_u(Q_0) \leqslant N$.

Our aim is to show that

$$M(N, a) \leqslant Ce^{-\beta a/N}. \tag{4.2.1}$$

The constant $\beta > 0$ will be chosen later and will not depend on N.

As we remarked in Section 3.4 we can assume that $a/N > c_0$. By Lemma 3.4.1 we can also assume that N is sufficiently large. The proof now contains two main steps. First, with the help of Corollary 4.1.2 we prove a recursive inequality for $M(N, a)$. Then we show that the recursive inequality implies the exponential bound (3.3.5) by a double induction argument on a and N.

4.2.1. Recursive inequality. We show that for some $a_0 > 0$ and $s < 1$

$$M(N, a) \leqslant (1 - s)M(N/2, a - Na_0) + sM(N, a - Na_0). \tag{4.2.2}$$

Fix a solution u to the elliptic equation $\operatorname{div}(A\nabla u) = 0$ with $\mathcal{N}^*_u(Q_0) \leqslant N$. Divide Q_0 into J^d equal subcubes q. Then by Corollary 4.1.2 at least one cube q_0

satisfies $\mathcal{N}^*_u(q_0) \leqslant N/2$. We have

$$m(u,a) = \sum_q |\{x \in q : |u(x)| < e^{-a} \sup_{Q_0} |u|\}|.$$

By the definition of the doubling constant we see that

$$\sup_q |u| \geqslant c_1 J^{-C_1 N} \sup_{Q_0} |u|.$$

Since N is sufficiently large, we can forget about c_1 above by increasing C_1 and we have

$$\sup_q |u| \geqslant e^{-a_0 N} \sup_{Q_0} |u|.$$

Define $\mathrm{size}(q) := |\{x \in q : |u(x)| < e^{-a+a_0 N} \sup_q |u|\}|$. We continue to estimate $m(u,a)$ in terms of these sizes

$$m(u,a) \leqslant \sum_q \mathrm{size}(q) = \mathrm{size}(q_0) + \sum_{q \neq q_0} \mathrm{size}(q).$$

We can estimate the first term by $\mathrm{size}(q_0) \leqslant J^{-d} M(N/2, a - a_0 N)$ using the fact that the restriction of u to the cube $2q$ corresponds to a solution of another elliptic PDE which can also be written in divergence form with some coefficient matrix which has the same bounds for ellipticity and Lipschitz constants.

For the second term, we have

$$\sum_{q \neq q_0} \mathrm{size}(q) \leqslant (J^d - 1) J^{-d} M(N, a - a_0 N) = s M(N, a - a_0 N),$$

where $s = (J^d - 1) J^{-d} < 1$. Adding the inequalities for the first and second terms and taking the supremum over u, we obtain the recursive inequality (4.2.2) for $M(N,a)$.

4.3. Recursive inequality implies exponential bound We will now prove that

$$M(N,a) \leqslant C e^{-\beta a/N} \tag{4.3.1}$$

for some C large enough and $\beta > 0$ small enough by a double induction on N and a. Without loss of generality we may assume $N = 2^l$, where l is an integer number. Suppose that we know (4.3.1) for $N = 2^{l-1}$ and all $a > 0$ and now we wish to establish it for $N = 2^l$. By Lemma 3.4.1 we may assume l is sufficiently large. For a fixed l we argue by induction on a with step $a_0 2^l$. We may assume that $a/N > k_0 a_0$, where $k_0 > 0$ will be chosen later. For $a \leqslant k_0 a_0 N$ the inequality is true if we choose the constant C large enough. The induction base implies the inequality for $k = k_0$. We describe the step of the induction from $a = (k-1) a_0 2^l$ to $a = k a_0 2^l$.

By the induction assumption we have

$$M(2^l, (k-1) a_0 2^l) \leqslant C e^{-\beta (k-1) a_0}$$

and

$$M(2^{l-1}, (k-1) a_0 2^l) \leqslant C e^{-2\beta (k-1) a_0}.$$

We apply the recursive inequality (4.2.2)

$$M(2^l, ka_0 2^l) \leqslant C(1-s)e^{-2\beta(k-1)a_0} + Cse^{-\beta(k-1)a_0}.$$

Our goal is to obtain the following inequality

$$(1-s)e^{-2\beta(k-1)a_0} + se^{-\beta(k-1)a_0} \leqslant e^{-\beta k a_0}$$

for $k > k_0$ and some $\beta > 0$. Dividing by $e^{-ka_0\beta}$ we reduce it to

$$(1-s)e^{-\beta a_0(k-2)} + se^{\beta a_0} \leqslant 1.$$

The last inequality holds with the proper choice of the parameters: once $s < 1$ and a_0 are fixed, we choose β to be small enough so that the second term is less than $(1+s)/2$ and then choose k_0 sufficiently large that the first term is smaller than $(1-s)/2$ when $k \geqslant k_0$. This concludes the induction step and the proof of our main result.

More delicate propagation of smallness from sets of codimension smaller then one is discussed in [28].

4.4. Exercises

Exercise 4.4.1. Suppose that $Lu = 0$ in the unit cube Q_0.

(1) Use the oscillation theorem to show that there exists a constant K which depends on the Lipschitz and ellipticity constants for L such that if q is a small cube with $Kq \subset Q$ and $Z(u) \cap q \neq \emptyset$ then

$$\log \frac{\max_{Kq} |u|}{\max_q |u|} \geqslant 2.$$

(2) Show that there exists c and B_0 such that if Q_0 is partitioned into B^d cubes q, $B > B_0$ and $Z(u) \cap q \neq \emptyset$ for each q then

$$N_u(Q/2) = \log \frac{\max_Q |u|}{\max_{Q/2} |u|} \geqslant cB,$$

where c depends on K from (1).

Exercise 4.4.2. Assume that $m : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ satisfies

$$m(k,j) \leqslant C \text{ for } j < 4, \quad m(1,j) \leqslant e^{-j},$$

and

$$m(k,j) \leqslant m(k-1, 2(j-1)) + \frac{1}{4} m(k, j-1).$$

Prove that $m(k,j) \leqslant Ce^{-j}$.

Remark: A similar argument is used to derive the estimate in the lecture notes from the iterative inequality.

Exercise 4.4.3 (Remez inequality for eigenfunctions). (1) Let M be a compact manifold. Use the lift and the Remez inequality for solutions of elliptic equations to show that there exists a constant $C = C(M)$ such that for any eigenfunction ϕ_λ and any compact set $E \subset M$, we have the following inequality.

$$\max_E |\phi_\lambda| \geqslant C^{-1} \max_M |\phi_\lambda| \left(\frac{|E|}{C|M|}\right)^{C\sqrt{\lambda}}.$$

(2) Let $M = S^2$ and B be a small ball on S^2, construct a sequence of eigenfunctions ϕ_λ on the sphere with $\lambda \to \infty$ such that $\sup_B |\phi_\lambda| / \sup_M |\phi_\lambda|$ decays as $e^{-c\sqrt{\lambda}}$.

Exercise 4.4.4. Apply the Remez inequality for solutions of elliptic equations to show that if h is a solution of $Lh = 0$ in kQ_0 then $g = \log|h|$ is in BMO and $\|g\|_{BMO(Q_0)} \leqslant C_L \mathcal{N}_h^*(Q_0)$. *Reminder*: A function g is said to have bounded mean oscillation if there exists a constant C such that

$$\frac{1}{|Q|}\int_Q |g - c_Q| \leqslant C$$

for any cube Q and some constants c_Q. The smallest C for which the inequality holds is called the BMO-norm of g.
In particular if a function g satisfies

$$|\{x \in Q : |g(x) - c_Q| > \gamma\}| \leqslant C \exp(-A\gamma)|Q|,$$

for some c_Q then $g \in BMO$ and $\|g\|_{BMO} \leqslant c/A$.

5. Appendix: Second order elliptic equations in divergence form

5.1. Elliptic operator in divergence form: regularity We study solutions of second order elliptic equations in divergence form

$$Lu := \operatorname{div}(A\nabla u) + cu = 0,$$

where $u \in W^{1,2}(\Omega)$, i.e., $|\nabla u| \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$. The matrix $A = A(x)$ is symmetric and uniformly elliptic, i.e.,

$$\Lambda^{-1}|v|^2 \leqslant (A(x)v, v) \leqslant \Lambda|v|^2$$

for any $x \in \Omega$ and any $v \in \mathbb{R}^d$.

First we assume that the elements of $A(x)$ are measurable bounded functions (the boundedness follows from the uniform ellipticity condition). We will assume that c is measurable and bounded, weaker integrability assumptions on c are sufficient for some of the results below. The equation $Lu = 0$ is understood in the integral sense, similarly, we consider the inequalities $Lu \geqslant 0$ and $Lu \leqslant 0$. The first classical result is the maximal principle, see for example [14, Theorem 8.1]. We use here the standard notation, $u^+ = \max(u, 0)$.

Theorem 5.1.1 (Maximal principle). *Suppose that* $c \leqslant 0$ *and* $u \in W^{1,2}(\Omega)$ *satisfies* $Lu \geqslant 0$. *Then*

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+.$$

We also use the following classical inequality for gradients of solutions of general elliptic PDEs in divergence form.

Theorem 5.1.2 (Caccioppoli inequality)**.** *Suppose that* $Lu = 0$ *in* Ω*,* $B_R \subset \Omega, r < R$*. Then*

$$\int_{B_r} |\nabla u|^2 \leqslant C\left(\frac{1}{(R-r)^2} + \|c\|_{L^\infty}\right)\int_{B_R} |u|^2,$$

where $C = C(d, \Lambda)$*.*

Classical iteration methods of De Giorgi and Moser imply the following estimates, see [16, Chapter 4]

Theorem 5.1.3 (Local boundedness)**.** *Suppose that* $Lu \geqslant 0$ *in* Ω*,* $2B \subset \Omega$*, then* $u^+ \in L^\infty_{loc}(\Omega)$ *and*

$$\sup_B u^+ \leqslant C\left(|2B|^{-1}\int_{2B} |u^+|^2\right)^{1/2},$$

where C *depends on* d, Λ *and* $\|c\|_\infty$*.*

This gives immediately the equivalence of norms

Corollary 5.1.4. *Suppose that* $Lu = 0$ *in* $2B_0$*, where* B_0 *is the unit ball of* $\mathbb{R}^d$*, then*

$$C_1\|u\|_{L^2(B)} \leqslant \|u\|_{L^\infty(B)} \leqslant C_2\|u\|_{L^2(2B)},$$

where C *depends on* d, Λ *and* $\|c\|_\infty$*.*

Another part of the regularity theory that goes back to De Giorgi and Moser is the following oscillation theorem (seee [16, Chapter 4]).

Theorem 5.1.5 (Oscillation inequality)**.** *Let* $L = \operatorname{div}(A\nabla\cdot)$ *be a uniformly elliptic operator in* Ω*. There exists* $q = q(\Lambda) < 1$ *such that for any ball* B *such that* $2B \subset \Omega$

$$\sup_B u - \inf_B u < q(\sup_{2B} u - \inf_{2B} u).$$

The difference $\sup_B u - \inf_B u$ is called the oscillation of the function u in B and denoted by $\operatorname{osc}_B u$.

A different way to obtain regularity was discovered by Landis (see [21] for details) and developed to elliptic equations is non-divergence form with bounded coefficients by Krylov and Safonov, see [19,21,22,34]. This approach also leads to the oscillation inequality.

Finally, we formulate the Harnack inequality of Moser for solutions of elliptic equations in divergence form, see for example [16, Chapter 4].

Theorem 5.1.6 (Harnack inequality)**.** *Let* u *be a non-negative solution to elliptic equation* $\operatorname{div}(A\nabla u) = 0$ *in* Ω*,* $2B \subset \Omega$*. Then*

$$\sup_B u \leqslant C \inf_B u, \quad C = C(d, \Lambda).$$

There is a nice proof of the Harnack inequality for solutions of elliptic equations in divergence form that bypasses the classical iteration methods can be found in [34]. Note that in all of the results in this section the constants depend on the ellipticity constant only, thus we may apply the inequalities on small or big scales.

5.2. Comparison to harmonic functions We turn now to elliptic PDEs in divergence form with Lipschitz coefficients. This smoothness assumption allows us to freeze the coefficients and consider the equation as a perturbation of the equation with constant coefficients. Changing coordinates, we can think about constant coefficient elliptic operator as a simple transformation of the usual Laplace operator. More precisely, let u be a solution to

$$\operatorname{div}(A\nabla u) = 0,$$

where $A = \{a_{ij}(x)\}$, $x \in \Omega$ and

$$|a_{ij}(x) - a_{ij}(y)| \leqslant C|x - y|.$$

Then for any $x_0 \in \Omega$, we may choose first a ball $B_r(x_0)$ and then a linear transformation $S : B_\rho(0) \to B_r(x_0)$ such that $f = u \circ S$ is a solution of elliptic equation $\operatorname{div}(\tilde{A}\nabla f) = 0$ with

$$\tilde{A}(0) = I, \quad |\tilde{a}_{ij}(y) - \delta_{ij}| \leqslant C|y|.$$

Moreover r/ρ is bounded, the bound depends on the ellipticity and Lipschitz constants for A.

We mostly study local properties of solutions and then reduce the problem to equation of this specific form. Note that when we apply this idea we get inequalities that hold on small scales, the constants depend on the Lipschitz constants of the coefficients and may grow when we consider large balls.

A classical regularity result implies that if $u \in W^{1,2}(\Omega)$ is a weak solution of the divergence form elliptic equation as above (with Lipschitz coefficients) and $\Omega' \subset\subset \Omega$ then $u \in W^{2,2}(\Omega')$ and then if $\partial\Omega'$ is smooth then by the trace property $u, |\nabla u| \in L^2(\partial\Omega')$.

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