Instrumentation Schemes for Solving Systems of Linear Equations with Dynamic Geometry Software

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In this paper we focus on the link between the use of dynamic geometry software and student understanding for the solution of systems of linear equations from an instrumental genesis perspective. Three task-based interviews were conducted with an undergraduate linear algebra student majoring in mathematics education and proficient in using dynamic geometry software. Data included video recordings, student written work and screen recordings. The data analysis was guided by the theoretical lens of instrumental genesis, to elaborate on how student thinking was shaped by the use of the digital artefacts. The digital technology supported the participant's solution steps and over time the student developed four interrelated instrumentation schemes, which are parameter scheme, combined algebra and geometry scheme, the intersection of figures scheme and echelon form scheme.

1. INTRODUCTION

The topic of solving systems of linear equations appears in various levels of education, mainly from secondary school to university mathematics. This is because, in addition to its central role in the applications of different disciplines (Lay, 2006), from a didactical point of view, studying systems of linear equations provides a context for reasoning on variables and interpretation of the solution. In undergraduate-level linear algebra, the systems of linear equations go beyond and have a unique role in conceptualising linear systems. This could include $A\mathbf{x}=\mathbf{b}$ notation, echelon form of a matrix and associated geometric interpretation (Larson & Zandieh, 2013). However, connecting different representations and generalising to linear systems are not easy for students. Therefore, the teaching and learning of this phenomenon has received particular attention from researchers (Andrews-Larson, 2015; Sierpinska, 2000; Trigueros, Oktaç, & Manzanero, 2007; Trigueros & Possani, 2013).

In the related literature, several epistemological issues for learning on the systems of linear equations have appeared. As Trigueros et al. (2007) address, the lack of synergy among unknown, functional relationship and understanding the solution may form an epistemological barrier for making generalisations on linear systems. Similarly, students are not good at moving between different contexts; for example, moving from an algebraic context to a geometric one. They overgeneralise algebra and geometry of $\mathbb{R}^2$ to $\mathbb{R}^n$ (Oktaç, 2018), where they tend to think practically rather than theoretically (Sierpinska, 2000). Consequently, as a general problem, such issues can be also viewed as a barrier for moving from contexts of $\mathbb{R}^2$ and $\mathbb{R}^3$ to abstract (i.e., non-geometric) vector spaces. However, hypothetically, the use of parameters in the system of linear equations could be a heuristic tool for establishing synergy among different contexts. For instance, if a system of linear equations includes a parameter (as the coefficient of unknowns), the value of this specific variable characterises the solution: no solution, single solution or infinitely many solutions. Moreover, such solution characteristics correspond to several geometric interpretations in $\mathbb{R}^3$ (as well as in $\mathbb{R}^n$). In order to establish such synergy, we refer to dynamic geometry software (DGS), as will be explained later. Because DGS does not only have synchronic (parallel) windows showing algebra and geometry (of $\mathbb{R}^2$ and $\mathbb{R}^3$) synchronously, but also tools and functions that enable students to manipulate objects, explore different cases, establish conjectures and validate them mathematically (Gol Tabaghi, 2014; Gol Tabaghi & Sinclair, 2013).

In the present paper, we focus on a specific DGS, in particular, the role of functions and tools of GeoGebra in student thinking on the system of linear equations including specific parameters. We consider a compelling case including one proficient GeoGebra user's work for three proposed tasks and elaborate the function of digital artefacts in mathematical thinking within an instrumental genesis perspective. Since we exploit a geometry context to understand student thinking in depth, we first briefly focus on the role of geometry in teaching and learning linear algebra in the theoretical framework section. This is followed by a short mathematical and conceptual description of the system of linear equations and the role of parameters with DGS. This section also refers to the theory of instrumental genesis (Artigue, 2002; Vérillon & Rabardel, 1995). The third section provides details regarding the followed methods, while the fourth section gives our findings. The paper ends with conclusions and a discussion section in which we address some conjectures and limitations.

2. THEORETICAL FRAMEWORK

This section consists of three subsections. First, the use of geometry in linear algebra is briefly presented. Second, the mathematical concepts at stake are elaborated; a brief description of the solution of systems of linear equations and the notion of the parameter with DGS availability is provided. Third, the theory of instrumental genesis is expressed before the research question is formulated.

2.1 The Role of Geometry in Teaching Linear Algebra

Linear algebra is applied in various disciplines other than mathematics, such as statistics, engineering and computer science. As a consequence, in 1990, sixteen mathematicians from different departments of universities in US met to discuss two major points: student understanding and learning challenges in linear algebra, and how to improve the existing linear algebra curricula. The Linear Algebra Curriculum Study Group (LACSG) recommended that the first course on linear algebra should be matrix-oriented:

> … A matrix-oriented linear algebra course should proceed from concrete, and in many cases practical, examples to the development of general concepts, principles and the concomitant theory simplifies and clarifies and makes linear algebra so powerful and useful. (Carlson, Johnson, Lay & Porter, 1996, p. 42)

Along this direction, LACSG suggested to start with matrix algebra, systems of linear equations, determinants and other properties (linear combinations, bases, subspaces etc.) of $\mathbb{R}^n$. From an epistemological point of view, these topics connect to geometry and invite the use of concrete tools and students' experience with 2D and 3D geometry.

The use of geometry in linear algebra is well-explained from the perspective of mathematicians by Gueudet-Chartier (2004), and teaching principles and student learning are explained by Harel (2000, 2019). The common view is that geometry could be a pedagogical context to introduce linear algebra and to prepare the ground for abstract concepts. Still, we should avoid students making an overgeneralization of linear algebra being entirely geometry (Harel, 2000). Moreover, LACSG and Harel (2019) highlight the use of digital technologies in teaching. In incorporating geometry, many researchers refer to digital technologies, to design effective teaching environments and to discuss how digital tools shape the learner's mathematical thinking (Caglayan, 2019; Gol Tabaghi, 2014; Gol Tabaghi & Sinclair, 2013, Turgut, 2019). For example, Caglayan (2019) shows how digital tools mediate the coordination between algebraic and geometric representations.

To connect algebraic and geometric representations as suggested by researchers (Trigueros et al., 2007; Oktaç, 2018), we use DGS in the present paper. We focus on a student's use of DGS over time, and on how this shapes her way of thinking while solving system of linear equations, in which the interplay between algebraic and geometric representations is key.

2.2 Approaching System of Linear Equations with DGS

Interpretation of the reduced row echelon form of the augmented matrix could yield the type solution of systems of linear equations and gives clues regarding associated geometric meanings such as the position of (hyper)planes. However, we will focus on the case of 3×4 augmented matrix as B and the positions of three planes in $\mathbb{R}^3$ that are more *experientially real* (Gravemeijer, 2004) for undergraduate linear algebra students. Table 1 summarises exemplary cases regarding reduced row echelon form of B matrix, associated solution types and geometric interpretations.

Reduced Row Echelon Form	*Solution Type*	*Associated Geometric Interpretation*
$B \sim B' = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{pmatrix}$	Exact (single) solution. The system is consistent.	The planes represented by linear equations intersect along with single point.
$B \sim B' = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Infinitely many solutions; the need for a free variable. The system is consistent.	The planes intersect along a line. Values of free variable form the intersection line.
$B \sim B' = \begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Infinitely many solutions; the need for two free variables. The system is consistent.	The planes intersect along a plane. In fact, the given planes are coincident.
$B \sim B' = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & u \end{pmatrix}$ for $u \neq 0$.	No solution. The system is inconsistent.	The planes never intersect.
$B \sim B' = \begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & u \end{pmatrix}$ for $u \neq 0$, and $v \neq 0$.	No solution. The system is inconsistent.	The planes never intersect. All of them are parallel.

Table 1. Exemplary cases of the reduced row-echelon form in the context of the paper (inspired by Lay, 2006)

In linear algebra textbooks, the left-hand side column of Table 1 is at the fore and a number of cases are commonly presented in a static way. However, DGS has tools and functions that could provide a dynamic context bridging the columns of Table 1. For instance, at first, GeoGebra (6th version, as well as the previous versions) has parallel windows showing algebraic and (2D, 3D or together) geometric representations. In order to explore the solution of a system of linear equations, one can enter equations one-by-one and then obtain corresponding planes. Secondly, through the spreadsheet function, the user can obtain an augmented matrix) in terms of components of linear equations. Further, GeoGebra has a specific command to compute echelon form that can be activated by entering it into the Input line. Therefore, the user can obtain a reduced row echelon form of an augmented matrix, and the user can also explore intersections of the planes by using the 'Intersect' tool, which can be activated through the Input line. This could be a useful tool for establishing the epistemological link between algebraic and geometric views of the solution of systems of linear equations, and this is *why* we consider DGS context for developing student understanding regarding the solution of systems of linear equations with associated geometry.

A parameter can be conceived as an implicit and special variable in an equation, but it is different from ordinary variables, because a parameter could have different meanings. In other words, as Drijvers (2003) addresses, a parameter could have roles for 'reification of expressions' and 'a means for generalisation' (p. 59). Assigning parameters to coefficients of a system of linear equations could be a key point for linear algebra students to interpret the solution of a system of linear equations. I.e., this way approaching the solution of a system could open a door to having different algebraic and geometric cases when the assigned parameter(s) change(s).

As well as articulating algebraic and geometric meanings, GeoGebra has a specific tool, *slider*, which can be defined as a parameter, for instance in an equation. With this tool, the user can attach slider(s) to equations and explore the effects of parameters by changing the slider's values. All tools and functions could evoke an understanding of the epistemological link between algebraic and geometric views of the solution of systems of linear equations. Of course, it should be noted here, we only introduce associated functions and tools of a DGS the user might refer to, which can be considered as 'hypothetical schemes' (Drijvers et al., 2010, p. 113) regarding DGS use. However, the use of such artefacts is in relation to the user's *utilisation schemes*, as it is mainly described in the theory of instrumental genesis as follows.

2.3 Theory of Instrumental Genesis

The theory of Instrumental Genesis (TIG) is based on the distinction between artefacts and instruments (Artigue, 2002; Rabardel, 2002; Vérillon & Rabardel, 1995). The artefact is the "thing" that users – students in an educational setting – use. In our case, this is GeoGebra, its 3D Graphics and Algebra windows in particular. We speak of an instrument if the user developed one or more schemes while using the artefact for a specific class of instrumented activity situations, in our case solving systems of linear equations. Such a scheme is considered a more or less stable way to deal with specific situations or tasks (Piaget, 1985). Vergnaud (1987) speaks of 'a functional and organized sequence of rule-governed actions, a dynamic totality whose efficiency requires both sensorimotor skills and cognitive competencies' (p. 47). Here, *techniques* (in problem solving activity) have a central role in scheme development. Techniques are "manners" of solving a task (Artigue, 2002) and are defined as 'the observable part of the students' work on solving a given type of tasks' (Drijvers, Godino, Font, & Trouche, 2013, p. 27). We note that schemes are implicit and invisible entities, while techniques are observable instances (Jupri, Drijvers, & van den Heuvel-Panhuizen, 2016).

In our case, schemes involve different steps in the process of solving systems of linear equations. Such a scheme integrates both technical and conceptual elements. The technical elements are the more or less stable sequences of technical interactions between the user and the artefact; the conceptual elements concern the students' reasoning that on the one hand guide the technical interactions, but may also be shaped by the artefact's opportunities and constraints (Drijvers, 2019; Drijvers et al., 2013). In this paper, since the participant in this study is a proficient GeoGebra user, we focus on instrumentation schemes, and neglect the instrumentalization aspect.

TIG is a powerful framework to study the intertwinement of students' thinking and mathematical reasoning on the one hand, and students' techniques for using the artefact on the other. However, to date instrumentation schemes regarding the solution of systems of linear equations and associated geometry in $\mathbb{R}^3$ with DGS have not yet been elaborated.

The above leads to the following research question addressed in this paper: *Which instrumental schemes are developed while solving systems of linear equations with parameters using dynamic geometry software?*

3. CASE STUDY SET-UP

The participant, Ela (pseudonym) was a twenty-year-old female undergraduate linear algebra student enrolled in a mathematics teacher education program of a state university located in central Turkey. In a four-year mathematics teacher education program, the students follow pedagogical courses (i.e., the psychology of learning, teaching methods of mathematics, etc.), in addition to general mathematical courses such as calculus, abstract and discrete mathematics, linear algebra and differential equations. Linear algebra is taught over two semesters and covers the topics of linear systems, matrix algebra, determinants, vector spaces, Eigenvalues and Eigenvectors, inner product spaces, and diagonalization. The first author of this paper was the lecturer of the course, and the main textbook was *Elementary Linear Algebra* (Anton & Rorres, 2014).

Ela was a sophomore-level student, i.e., she already took a number of mathematical courses; calculus, abstract mathematics and geometry. Therefore, Ela knew 2D and 3D geometric elements in addition to fundamental topics such as set theory, relations, functions and the notion of limit. She had experience in proving techniques from abstract mathematics courses, in which she performed moderately. Ela also took an elective course where she learned how to use DGS, specifically GeoGebra. The elective course's name is Computer Assisted Geometry Instruction. The aim of this course is to introduce students to different DGS to provide an environment for them to explore and discuss several geometric problems. However, the main aim is to prepare students for a didactics of geometry course at lower secondary level, where the students design their own courses with digital and concrete tools/materials.

Ela volunteered to participate in the interviews after linear algebra course ended. Therefore, regarding the context of this paper, she learned how to solve linear equations, augmented matrices, Gauss elimination method, and geometric interpretations (position of lines in $\mathbb{R}^2$, planes in $\mathbb{R}^3$ and hyper-planes in $\mathbb{R}^n$, for $n \geq 4$) of the solution of linear systems. Ela also learned matrix operations, elementary and invertible matrices. But the whole class never used any digital resource for solving problems in the topics above.

In regular class lectures, Ela always asked critical questions when she did not understand a particular point, and also tried to establish conjectures to generalise the proposed content. Ela frequently discussed the solutions of classroom problems with digital tools, particularly referring to functions and tools of GeoGebra. Therefore, Ela was selected as a participant for our study, since she had extensive knowledge on the use of GeoGebra, including how to use it in calculus and linear algebra problems. She also knew how to solve linear systems and had good communication and self-expression skills.

3.1 Task Sequence

A task sequence with three tasks was prepared to explore Ela's instrumentation schemes regarding the solution of linear systems within GeoGebra. The aim of the first task was to elaborate on Ela's schemes regarding the use of a single parameter in a system of linear equations. It was proposed (Anton & Rorres, 2014, p. 102): *For which value(s) of a does the following system has zero solutions? One solution? Infinitely many solutions? Find the solution set if it is possible*.

$$\begin{cases} x+y+z=4 \\ z=2 \\ (a^2-4)\cdot z = a-2 \end{cases}$$

In the task presented above, the participant is expected to construct a slider as *a*, and considering in the third equation, and possibly to explore (synchronic) algebraic and geometric variations on the screen. Moreover, it is expected that the participant would understand the role of the parameter that only affects the third equation and position of the associated plane through her use of tools and functions of DGS. Here (as well as in Task 2) we note that $Ax+By+Cz=D$ is not always an equation of a plane. If $(A,B,C)=(0,0,0)$, then it may be the space (for $D=0$) or the empty set (otherwise).

The aim of the second task is to explore student thinking in the inclusion of two specific parameters to a system of linear equations. Similar to the first task, the second task is borrowed from Anton and Rorres (2014, p. 102), and formulated: *Let the following augmented matrix of a system of linear equations be given*

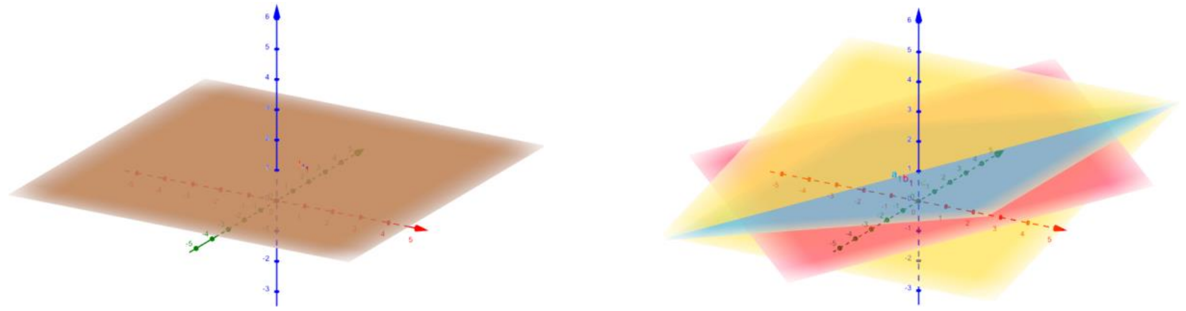
$$\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix}.$$

For which values of a and b, does the system have

(i) a unique solution,
(ii) a one-parameter solution,
(iii) a two-parameter solution,
(iv) no solution?

Regarding the second task, it is expected that the participant explores dynamic variations of *a* and *b*, which affect the type of solution as well as positions of the associated planes defined by $a\cdot x+b\cdot z=2$, $a\cdot x+a\cdot y+4\cdot z=4$, and $a\cdot y+2\cdot z=b$, since there are a number of specific cases. For instance, if the user selects $a=0$ with $b=2$, then all planes coincide (see Figure 1a), which algebraically means a two-parameter solution (e.g., the case in the third row of Table 1).

However, if it is selected as $a=0$ and $b\neq 2$, then all the planes are parallel, which implies no solution. Moreover, if $a\neq 0$ and $b=2$, then planes intersect along a line, i.e., the results are a one-parameter (e.g., the second row of Table 1) solution (see Figure 1b). Therefore, it is expected that at first, the user would define two parameters *a* and *b*, and thereafter would enter (plane) equations through the GeoGebra input line. By dragging sliders, the user could observe the changes in the positions of the planes and interpret the types of solution possible by using the tools and functions of DGS.



(a) (b)

Figure 1. (a) The case of $a = 0$ and $b = 2$ and (b) the case of $a \neq 0$ and $b = 2$ in the augmented matrix with coloured planes

Because the first and second task include specific parameters as coefficients of equations, we hypothetically thought that the user could make an *overgeneralisation*: inclusion and specific values of parameters in a system of linear equations always change the type of solution. To protect the user from such misconceptions, the third task was prepared and formulated as follows: *For the real numbers of a, b and c, explore the solution of the following system*

$$\begin{cases} x+y+z=a \\ 2x+2z=b \\ 3y+3z=c \end{cases}.$$

In the third task, the parameters do not change the position of the planes since they are not defined as coefficients of the system given above. After exploring the third task, it is expected that the user would arrive at a generalisation regarding the role of parameters in a system of linear equations*: parameters, as coefficients* or *not*.

3.2 Data Collection Process and Analysis

To elaborate student thinking together with techniques for using DGS, the data was collected through a series of task-based interviews with Ela. Each task was implemented in a different week and each lasted around 75 min. The data was triangulated with video-camera recordings, student production and screen recorder software. The first author of the paper acted as the interviewer (by asking a number of questions in relation to the student's employed steps) to understand the participant's techniques and justifications in-depth. The data is collected through task-based interviews where a laptop facing the student and screen recorder software were synchronously used. All data from interviews underwent an analysis to elaborate conceptual and technical elements (Drijvers, 2019; Drijvers et al., 2013) during the student's work with DGS.

4. FINDINGS

Below, we subsequently present the student's work on each of the three tasks and identify the conceptual and technical elements in the scheme development.

4.1 Findings Regarding Task 1

After the aim of the research and anonymity of the participant were addressed once more, the interviewer introduced the first task. The interviewer asked Ela to think out loud when possible to do so. Ela read the task and opened GeoGebra immediately. She entered $x+y+z=4$ and $z=2$ into the interface, and the software assigned these as a: $x+y+z=4$ and $b: z=2$ (similar to the Algebra window in Figure 1). Next, she analysed the (3D) Graphics window and said "*ok these planes intersect along a line, but what about a* [meaning the parameter *a*] *here*? *Should I use a slider here? Let's try it…*". She clicked on the slider tool and this time the software assigned *c* as the slider. She stated, "*I have to change this to a*" and entered the third equation of Task 1 to the software. Finally, she obtained the following figure (Figure 2).

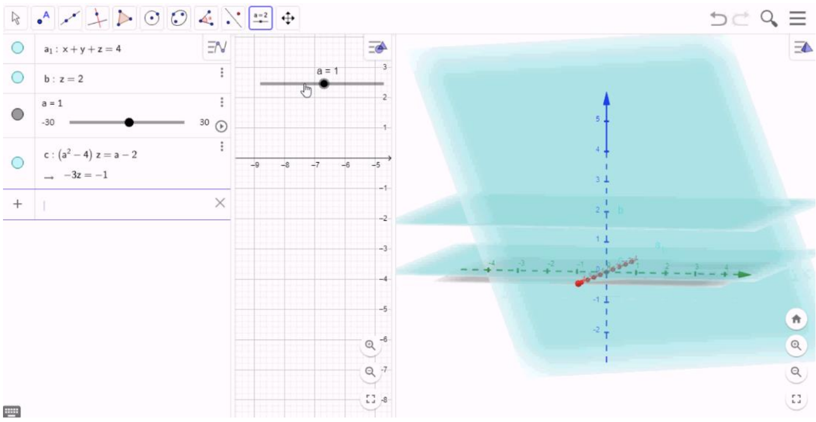


Figure 2. Ela's use of slider regarding the first task

Next, she dragged the slider from 1 to negative values and she expressed her initial observations, "*there are two parallel planes and the third plane intersecting both along a line*". In order to explore other cases, Ela activated the animation function of the slider. Because the slider moved fast, she adjusted the increment of the slider to 0.1. Then Ela began to drag the slider again and explored different cases.

While dragging slider *a*, Ela immediately realised that only the plane with the equation resulting from *c* moves, but the planes from a_1 and *b* never move. Then she said, "*I will try to intersect the planes given by b and c, and also I will try to coincide all planes*". She dragged the slider and also used the Rotate 3D Graphics View tool to change her viewpoint for a while. At the end she expressed, "*… only planes b and c and* a_1 *intersect, they never coincide and never intersect along a point, if we take z equals to two, then we obtain x plus y equals to two. By taking t as a parameter of y, we have x equals to two minus t, which means a line …*". Though she spoke about the intersections, she did not interpret any solution. Then Ela dragged the slider and tested her interpretation. After a while, the interviewer reminded her of the task. Because she observed lines (as the intersection of planes), she claimed that the system of given linear equations has *infinitely many solutions* and because the planes never intersected along a point. Ela also claimed that "*… about the first and second equation, I have to use parameter which means a line, but whatever I change a, I cannot intersect them along a point …*". She did not have a clear conclusion because she only focused on the movements (and positions) of the planes.

After some dragging practises and thinking about it, Ela was unable to decide, so could not proceed. Then she remembered classroom practice and thought out loud: "*What if I write the echelon form of this system? …*". As a next step, Ela tried to form a matrix through the Spreadsheet window by entering coefficients of the system (Figure 3).

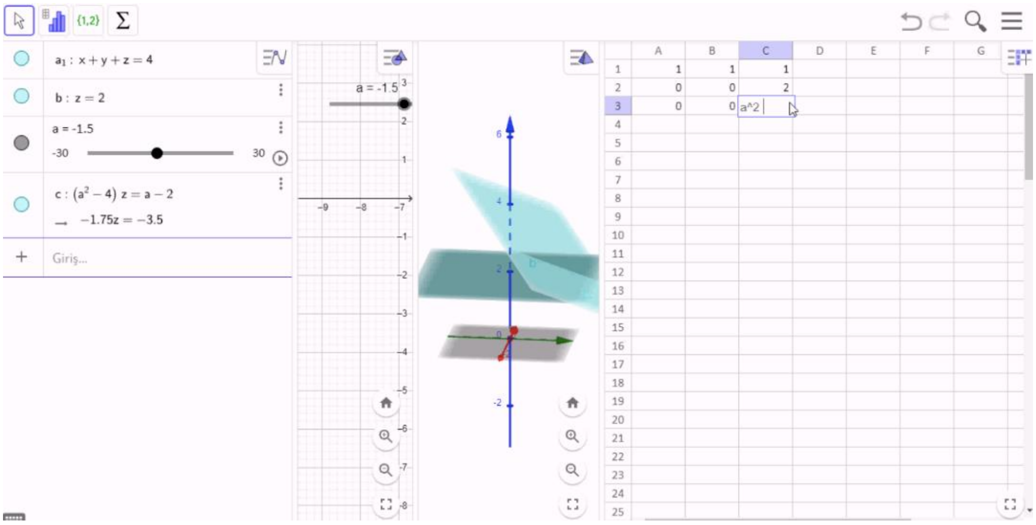


Figure 3. Ela's use of the Spreadsheet window to form the matrix

As seen from Figure 3, Ela only entered the coefficients of the equations and forgot the given values. She also made a mistake when she entered 2 as the coefficient of the linear equation $z = 2$. Next, she dragged the slider from $a = -1$ to $a = -3$, to check whether components of the matrix varied and went on using the matrix tool on Spreadsheet window, where she obtained a matrix

$$m_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{pmatrix}.$$

Ela dragged the slider and explored what was happening and carefully checked the rows of the m_1 and tried to remember the meaning of the case: "*Should I assign a parameter again? But how… I try to remember the meaning of the echelon form? Yeah… I have to transform this* [meaning m_1] *into a unit matrix …*". Next, she opened the tools and functions of GeoGebra and checked which command she would use, but after a while, she clicked on ReducedRowEchelonForm(<Matrix>) (in Turkish 'İndirgenmişSatırEşelonBiçimi(<Matris>)'), obtaining this row reduced echelon form of m_1

$$m_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ela dragged the slider, and this time saw there were two key values of the planes changing positions: $a = -1.5$ and $a = 2$. She skipped $a = 2$ since the software gave feedback that plane *c* (as well as the equation) would be undefined in that case. She dragged the slider again and realised that planes *b* and *c* overlapped for the value $a = -1.5$ but the echelon form did not change (Figure 4).

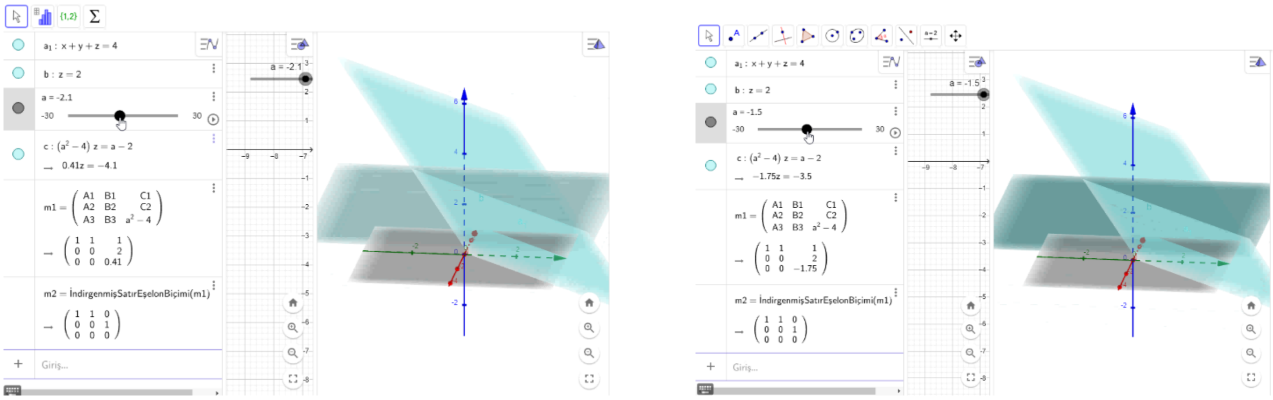


Figure 4. Different values of a where the echelon form does not change

Based on her knowledge from regular lectures, she realised that there was a problem in echelon form: "*… ok, there is a row with zeros, this means a parameter and a consistent system with infinitely many solutions. But I could not understand the solution type could change …*". Next, she thought for a while and went on dragging and realised the missing (given) values in the echelon form, then added one more column (i.e., the D column of the Spreadsheet window) to matrix m_1 and finally obtained a new matrix (as in Figure 5).

$$m_3 = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1.75 & -3.5 \end{pmatrix}.$$

But again, she did not realise that the second row was still wrong. As a next step, she computed the row reduced form of the matrix m_3 as

$$m_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Ela immediately dragged the slider to explore whether rows of the m_4 changed in the current case by taking the key value of $a = -1.5$. She was confused again, because, when she dragged the slider for a specific value, the planes b and c overlapped and plane a_1 intersected them. So, she thought that somehow the echelon form of the matrix should change (Figure 5), but it remained the same.

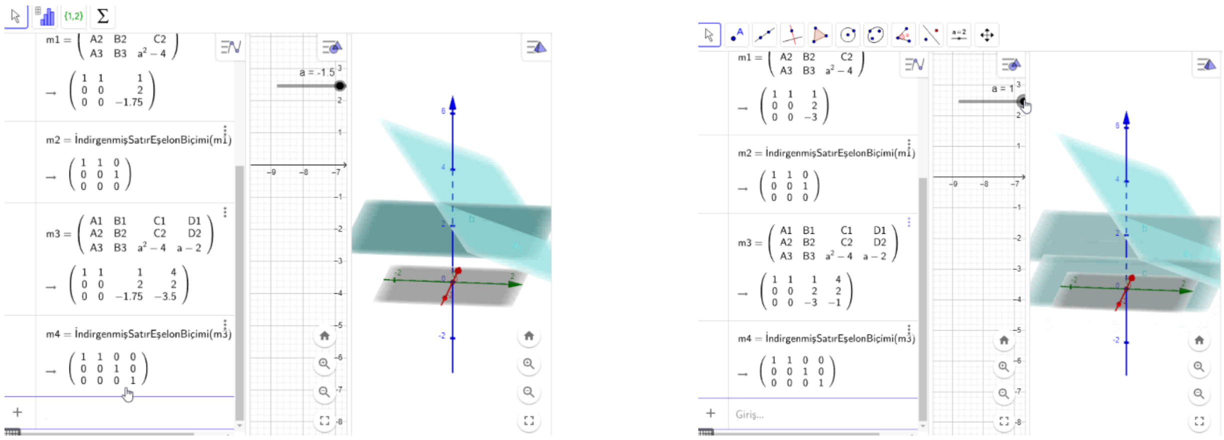


Figure 5. The cases of $a = -1.5$ (planes b and c overlap) and $a = 1$ (planes b and c are parallel)

Ela realises that there was an issue again and said "*how this is possible? The echelon form is always inconsistent, because of the second row, actually, it never changes, even planes b and c overlap … I must use a parameter in this case … but …*". She thought for a while and requested to save the GeoGebra file she had been working on and wanted to start again from the beginning.

Ela opened a new GeoGebra file, and, entering two equations, she sketched planes, and by using the slider she obtained the third equation and associated planes. Then she formed the augmented matrix (m_1) correctly and computed the row reduced form of m_1. She dragged the slider and obtained the following cases in Figure 6. Ela immediately took $a = -1.5$ and saw that the planes b and c overlapped. She was happy that this time the system was consistent for the value of $a = -1.5$, and the third row was completely zero: "*now … there are infinitely many solutions and I can assign a parameter, z equals to two and I have x plus y equals to two. If I denote y with t, I have an equation with that parameter … rather than* $a = -1.5$ *there is no solution because two planes are parallel then* [drags slider], *the system is inconsistent, and zero solution is impossible*".

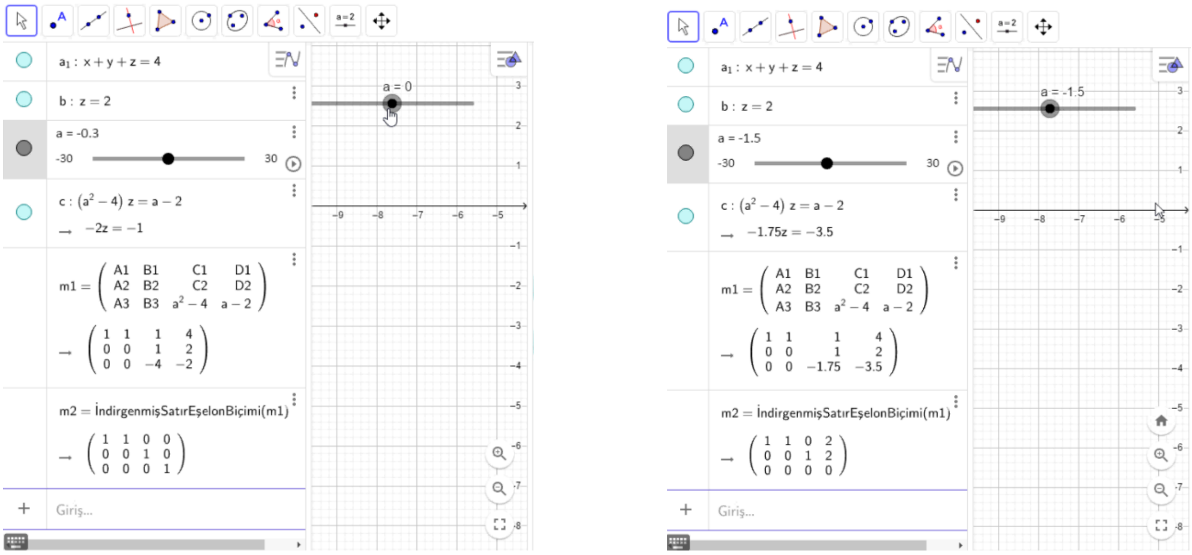


Figure 6. New computations regarding linear equations, augmented matrix and row reduced form

The interviewer asked why a zero solution was impossible. She opened the tools of the 3D Graphics section and thought for a while. She picked up the intersect tool and clicked on the planes, but she did not manage to obtain a line as an intersecting set of points. She thought out loud: "*Maybe the tool does not work because two planes overlap here*". Next, she tried to write its command to the Input line as Intersect(<Object>, <Object>) (in Turkish 'Kesiştir(<nesne>, (<nesne>)'). She entered *a*1 and *b* to the line, then the software gave an error for an undefined variable "a1". She understood that this was because of a_1 she wanted to change a_1 to *t*. Then, she obtained Figure 7.

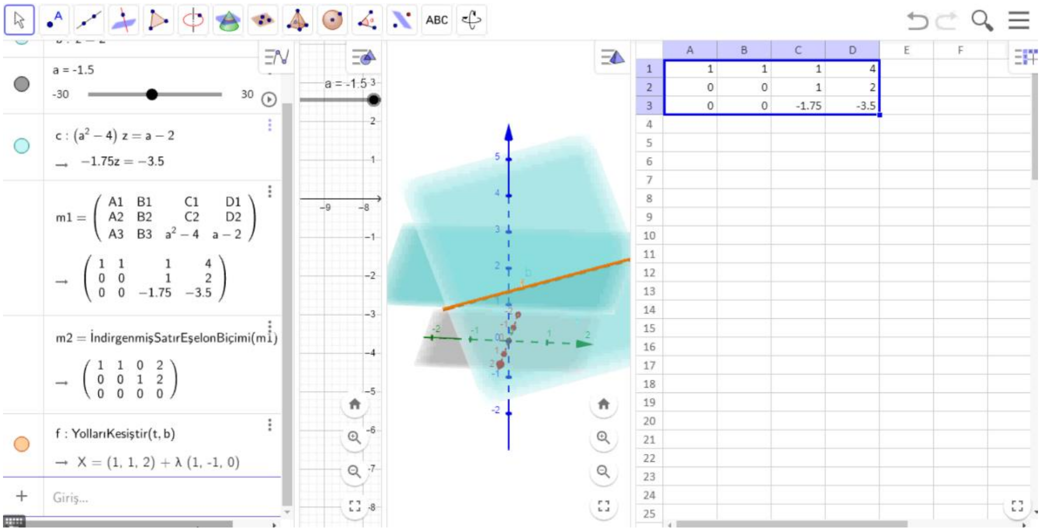


Figure 7. Ela's solution for the given system of linear equations for the case $a = -1.5$

Ela dragged the slider and saw the line did not move, and explained that she only intersected *t* with *b*. She repeated, regarding other values than $a = -1.5$ that the system had no solution because the three planes had no common point. She closed her statements by "… *here is the line equation with parameter lambda… zero solution cannot be here … because regarding the first equation* [meaning $x + y + z = 4$] *I cannot take zero for all variables …*". Then the session ended.

4.2 Findings Regarding Task 2

Ela read Task 2, and when opening GeoGebra, she immediately defined two sliders on the Graphics window. Next, she entered three given equations with their parameters, and she used the Rotate 3D Graphics view tool in the 3D Graphics window for a while, searching for intersections of the planes, specifically for a single intersection point.

Ela could not think to change the colours of the planes and stated that three planes seem to be intersected through single point: "… *I cannot see clearly in this window, but I can use the intersect tool, then I will be able to see* …". Ela clicked on the Intersect tool in the 3D Graphics window, and even this time, when she had no overlapping planes, she did not click on the planes. She used Input line and tried to write a command, but the Intersect tool worked only with two objects: Intersect(<Object>, <Object>). Ela had no idea how to proceed.

After a while, she thought that she could intersect them as pairs. Using the Input line, she first intersected planes *c* and *d*, and obtained a line defined as *f*. Next, Ela intersected *c* and *e* and obtained line *g*. Ela went on and this time intersected *f* and *g* and obtained a screen as in Figure 8.

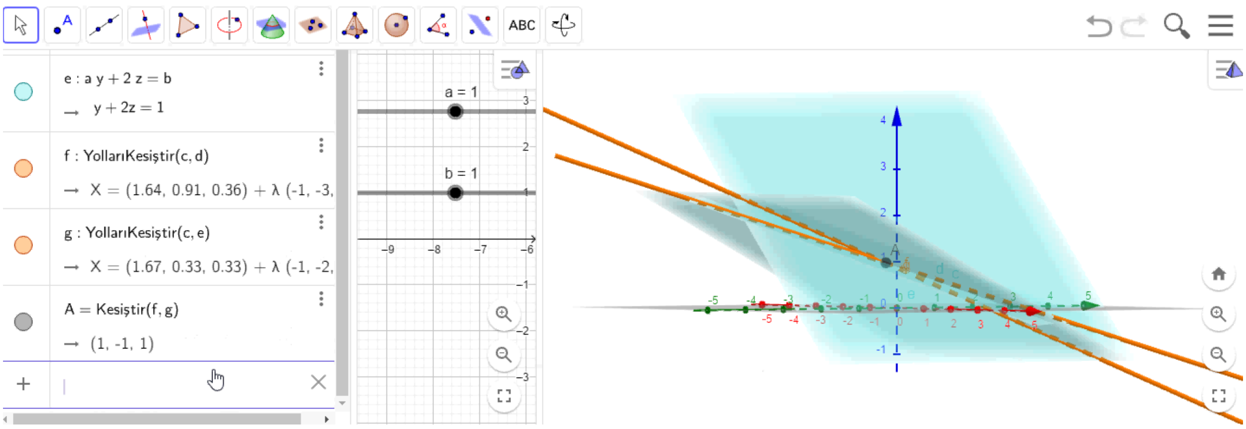
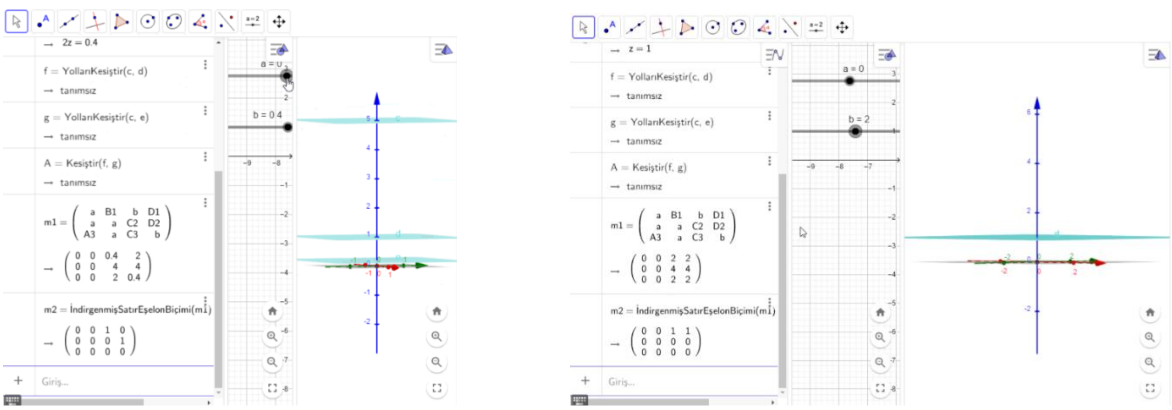


Figure 8. Screenshot for the intersection of lines *f* and *g*

Ela found a point $A=(1,-1,1)$ as the intersection of *f* and *g*, where she thought that she found the set of the solution. She was aware, though, that the situation was only valid for $a=1$ and $b=1$. Next, Ela decided to make a classification, and wrote down the case she obtained on paper. She dragged sliders and checked for any change. However, she overgeneralised immediately by stating that "… *whatever I change a and b, these planes never overlap* …". She dragged sliders for a while and also used the Rotate 3D Graphics view tool and realised something was happening if she changed both sliders. Because she could not manage to find any other related cases, after thinking for a while, she decided to compute the echelon form of the system.

Using the Spreadsheet window, she obtained an augmented matrix corresponding to the system of linear equations given in Task 2, and found its echelon form. As soon as she saw the echelon form: "… *it is over here* [pointing echelon form] *this part transformed into a unit matrix, this means we can find x, y and z separately*. *That is, it is obvious from here what the solution is* [pointing $x=4.25,\ y=-4.25,\ z=1$] …". Next, she dragged the sliders and explored the screen carefully, and got a result when $a=0$ and the intersection lines disappeared. She obtained parallel planes (see Figure 9a) and the software reacted with undefined (in Turkish 'tanımsız') for the intersection lines and for point *A*.



(a) (b)

Figure 9. Ela's computations regarding the case of (a) $a=0$ and $b=0.4$ and (b) $a=0$ and $b=2$

Ela went back to the equations and reanalysed the situation, and while exploring the 3D Graphics window she said that "*this is an inconsistent case … three equations have different z values and the second row is ending with 0 and 1 … means parallel … also here f, g and A are undefined, I am sure*…".

Next, when $a = 0$, Ela began to drag slider *b*. Then she saw that three planes overlapped when $b = 2$ (Figure 9b). She immediately arrived at a conclusion that the planes overlapped. She compared the 3D Graphics window with intersection lines and the echelon form of the matrix by stating "… *these overlap* [goes up to equations in Algebra window] *all equations equal to one and the last two rows of echelon are completely zero, I will use two parameters as free variables which means the solution set will be a plane*…". Next, she wrote down her findings regarding the last interpretation (Figure 10).

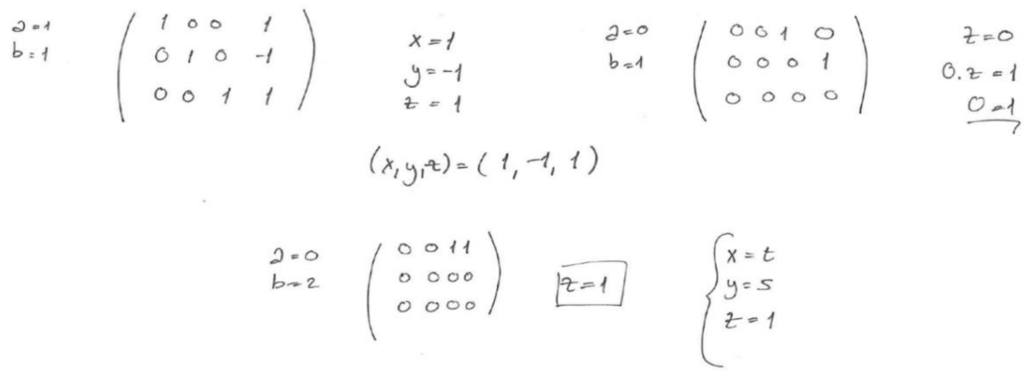


Figure 10. Ela's paper-and-pencil summary regarding key values of *a* and *b*

After her analysis for the case of $a = 0$ and $b = 2$, Ela thought that she completed the steps of the task. The interviewer requested her to go back to the task. She read one more and thought that one case was still missing in her solution. Therefore, Ela thought for a while and next says, "*Oh ok, I forgot to change a*". After Ela dragged slider *a* for a while, she observed that she had a new situation (Figure 11).

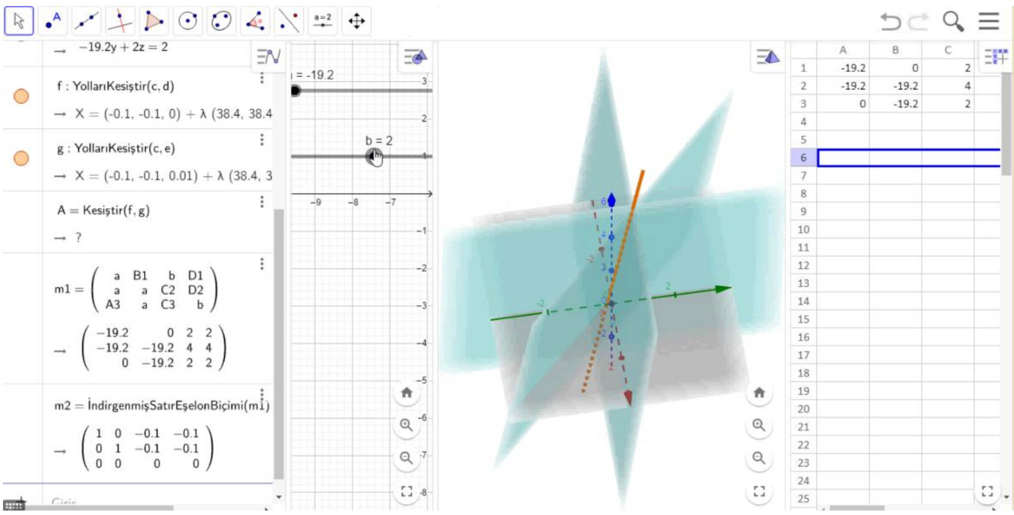


Figure 11. Screenshot for the case of $a = -19.2$ and $b = 2$

Ela looked at intersections defined as *f*, *g* and *A* while dragging *a* and *b*. She saw that if $b = 2$ and $a \neq 0$, three planes intersected along a line. But she was a bit confused because *f* and *g* seemed to be overlapped from the 3D Graphics window, but the software responded with "?" (see A in the algebra window in Figure 11) regarding the intersection of *f* and *g*. She says "… *ok here there are infinitely many solutions again, echelon form says that I will use one parameter, then I can obtain a line, ok but why a question mark there*?" She had no idea how to proceed, because she expected the lines were the same, so she thought she should have a line equation instead of "?"[1]. Next, she saw that the line equations also seemed to be different. After she wrote down her conclusions, she thought that GeoGebra did not manage to write intersection of lines as a set of points, but she never referred to the mathematics behind the lines, which is $\mathbf{u} = (38.4,\ 38.4,\ 368.64)$ as the direction vector and with two different points $K = (-0.1,\ -0.1,\ 0)$ and $L = (-0.1,\ -0.1,\ 0.01)$ (see *f* and *g* in the Algebra

[1] GeoGebra (of the 6th classic version) returns "?" if one tries to intersect two overlapping lines (either in 2D or 3D) through intersect command.

window in Figure 11). Because the direction vectors of the lines were the same, only the points that they went through were different. However, she only expressed what she saw.

After she completed the task, she expressed her feelings, "*… this problem was quite different to me. In the classroom, I always try to imagine when working with the echelon form and formulate the solution according to completely zero rows, but here I saw it is already dynamic and easy to articulate the different cases by sliders that I created ... there is no rule here, I have found all myself…*". The session ended afterwards.

4.3 Findings Regarding Task 3

The task-based interview of Task 3 started with Ela's initial expressions after she read the task. She stated "*… three linear equations with three parameters, the solution will be based on the values of parameters. I will sketch them and refer to echelon form to consolidate my interpretations …*". Next, Ela opened GeoGebra and constructed one slider as *a*, but this time she changed the interval of the slider sets from –50 to 50. Ela forgot to define other parameters *b* and *c*, and entered the second equation, but the software reacted with "undefined". Then Ela changed the plane's title to d_1 in order to define a new slider *b*.

Ela finally managed to sketch planes as linear equations with sliders of *a*, *b* and *c*. She explored the 3D Graphics window with the Rotate 3D View tool for a while and dragged three sliders spontaneously. She thought for a while and arrived at an immediate conclusion: "*… these planes are always intersecting along with a single point … they are not parallel or overlap …*". She thought that she missed something and decided to use the intersect tool to intersect planes two by two. Next, she repeated this step for obtained intersection lines and had the following screen (Figure 12).

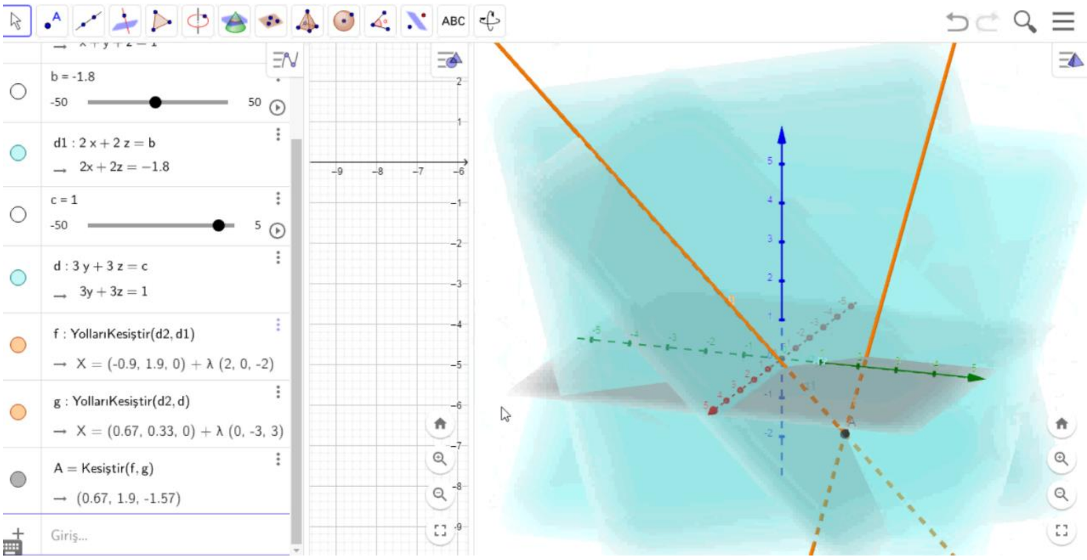


Figure 12. Screenshot for the case of the intersection of three planes along with single point *A*

Then she dragged three sliders again and explored the 3D graphics with (the given intersections of) the algebra window. She was a bit confused and stated, "*I am changing all sliders, but they always intersect along a point … I always have two lines as the intersection of planes so … a single point …*". She thought that was because she set slider intervals from –50 to 50, so then she set them to –10 and 10. Thereafter, she went on dragging three different sliders, for instance, she set $a = b = c = 0$ and saw that now point *A* was obtained with $A = (0,0,0)$, that is, three planes intersected along the origin. She thought for a while and remembered to go on with the echelon form of the augmented matrix associated with the given system of linear equations.

Ela opened the Spreadsheet window and formed the augmented matrix of the system and next she obtained its echelon form. She dragged sliders and compared the echelon form with other windows. She realised that a unit matrix was included in the echelon form and never changed. The screenshot she worked with is presented in Figure 13.

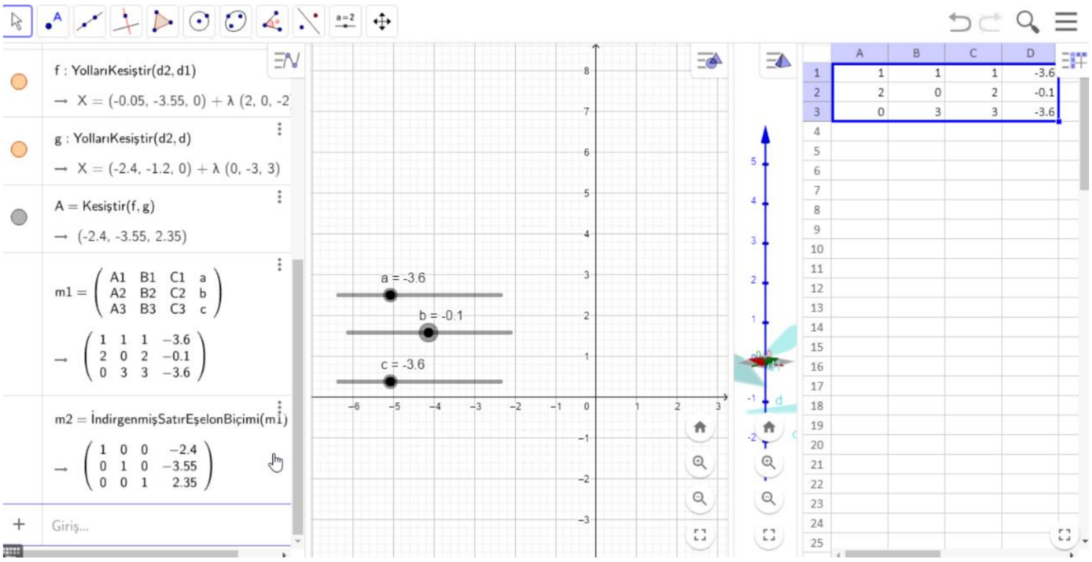


Figure 13. Ela's work on the echelon form and sliders

Then she stated, "… *I always obtain x, y and z as a single solution. There is no other possibility. This means three planes always intersect along a point … single solution and consistent … no need for a parameter … now I understand, here parameters a, b and c are not defined as coefficients of the linear system. This means the coefficient matrix is always the same and transforms to the unit matrix in echelon form … this is why my dragging sliders only changed the coordinates of the intersection point, not the positions of the plane …*". The session ended after her conclusions including the mathematical point of view of the task.

4.4 Overview of the Schemes, and their Conceptual and Technical Elements

Table 2 provides an overview of the identified instrumentation schemes, employed techniques, and conceptual and technical elements that the student referred to task sequence.

Instrumentation Scheme	*Conceptual Elements*	*Technique(s)*	*Technical Elements*
Parameter Scheme	– Slider tool can be conceived as an algebraic parameter. [1, 2, 3]* –A parameter provides dynamic variation in a linear equation as well as in associated geometry. [1, 2, 3] –Dissociation of parameters in a system: as coefficients of the linear system or *not*. [3]	–Drag slider(s). [1, 2, 3] –Attach slider as a parameter. [1, 2, 3]	–Use the animation function. [1] –Change the title. [1] –Change the increment. [1] –Click on slider tool (on Graphics window) and attach it to the equations. [1, 2, 3] –Use the parameter in Spreadsheet window. [1, 2, 3] –Change interval of sliders. [3]
Combined Algebra and Geometry Scheme	–The intersection of two planes yields a line. [1] –Intersection and other positions of the planes are associated with the solution of systems of linear equations. [1] –A linear equation corresponds to a plane in 3D Euclidean space. [1, 2, 3] –Articulate different cases. [2, 3]	–Enter mathematical language to plot geometry of given linear equations. [1, 2, 3] –Refer to commands. [1, 2, 3]	–Use of intersect tool. [1] –Write intersection command. [1] –Enter variables and numbers. [1, 2, 3] –Attach parameters to the equation. [1, 2, 3] –Drag slider and refer to the synergy of Algebra and 3D Graphics windows. [1, 2, 3] –Click on the figures in 3D Graphics window and scroll the mouse. [1, 2, 3]
The intersection of Figures Scheme	–The intersection of two planes yields a line but the intersection of three planes could be a line, a plane or single point. [2, 3] –Intersection and other positions of the planes are associated with the solution of	–Use the Intersect tool. [2, 3] –Write intersection command. [2, 3]	–Use the Input line, write command and activate it for pairs of planes and lines. [2, 3] –Scroll the 3D Graphics window. [2, 3]

	systems of linear equations. [2, 3]	–Explore Algebra window. [2, 3]	
Echelon Form Scheme	–The system is inconsistent if a row ends with 0 and 1. [1] –Planes overlap somewhere since the solution and echelon form must change. [1] –The system is inconsistent if a row ends with 0 and 1. [2] –A point, a line or a plane as a set of solutions. [2] – A linear system can be represented by a (i.e., an augmented) matrix. [1, 2, 3] –The solution of a system can be interpreted through the echelon form of the associated matrix. [1, 2, 3] –A completely-zero row means the use of the parameter. [1, 2, 3] –Two parameters as free variables. [2, 3] –Transformation into a unit matrix. [2, 3] –Parameters as coefficients in echelon form. [3]	–Use paper-and-pencil. [2] –Open Spreadsheet window. [1, 2, 3] –Use form matrix tool. [1, 2, 3] –Use reduced row echelon form command. [1, 2, 3]	–Add a new column to the matrix and mark the coefficients to create a new matrix. [1] –Summarise conclusions on paper. [2] –Enter coefficients into Spreadsheet window and mark them to create a matrix. [1, 2, 3] –Drag slider and refer to the synergy of Algebra and 3D Graphics windows. [1, 2, 3]

*Task numbers are given in brackets.

Table 2. Overview of Ela's instrumentation schemes, techniques, conceptual and technical elements

As shown in Table 2, regarding Task 1, we identify three interrelated schemes, called parameter scheme, combined algebra and geometry scheme, and echelon form scheme. At the beginning of Task 1, Ela immediately referred to her understanding of linear equations and associated geometry through functions and tools of the DGS. In other words, her algebra and geometry scheme appeared to guide her way of thinking at first, but the scheme became intertwined with the parameter scheme, in which the slider has a central role to understand the movements and categorisation of the figures. Next, the geometry was dominated by dragging the slider, while she did not clearly interpret the solution of systems or benefited from the echelon form of a matrix. Though she made a mistake while entering the coefficients, she was aware that the interpretation of the echelon form of the associated matrix would give more consistent results.

The echelon form scheme initially was algebra-oriented, since she used a number of expressions (i.e., 'completely-zero row', 'the use of parameter') that refer to algebra. Next, she tried to generalise the situation through the rows of the echelon form. However, later, she referred to the geometric view (i.e., 'planes overlap', 'echelon form must change') again through the functions and tools of the DGS. This progress can be considered as a trace of intertwinement of the three instrumentation schemes.

However, in the second task, we observed / identified four interrelated instrumentation schemes. Ela first referred to creating sliders, because she was more confident about parameters that can be represented by sliders. Her parameter scheme appeared to guide Ela to create an environment for the dynamic variation to observe the movement of geometric figures. However, such movement in the 3D Graphics window limited her thinking to the geometric view of the problem, rather than the algebraic view. Next, her intersection scheme appeared to be interlaced with the parameter and combined algebra and geometry schemes, but she overgeneralised the situation and could not come to a clear conclusion. This uncertainty evoked the emergence of the echelon scheme that appeared to provide her with a way to synthesize algebraic and geometric views in the context of generalisation. Even with the software's feedback of "?" regarding the intersection of two lines in $\mathbb{R}^3$, Ela was sure that an error occurred because other digital resources such as the echelon form on the Algebra window and the positions of the figures in the 3D Graphics window were consistent. This suggests a synergy between her developed instrumentation schemes and also evidences key progress in her reasoning.

Regarding Task 3, like in Task 2, we also define four interrelated instrumentation schemes. Initially, the parameter scheme came into play and Ela thought that there were three parameters *a*, *b* and *c*, and the solution of the system would be based on these three. Though she still had problems to assign sliders, her intersection of figures scheme guided Ela to move forward to explore and analyse different cases. Because the findings conflicted with her previous experiences from Task 1 and Task 2 – in which she changed parameters, but planes never overlapped, and were never parallel – she thought that she could not see the movements of planes very well. She changed the interval of the sliders, when the parameter scheme still guided her thinking. However, Ela believed that the echelon form would consolidate her interpretations, and, as shown in Table 2, all four schemes were intertwined when she referred to the change in the echelon form and the position of the planes. This opened a door to a mathematical conclusion regarding parameters in a system of linear equations.

4.5 Reflections on Schemes, and their Conceptual and Technical Elements

In the first task, there appeared three interrelated but stable instrumentation schemes; combined algebra and geometry scheme, parameter scheme and echelon form scheme, where Ela also referred to them in the subsequent tasks. Ela's preferred main scheme was observed as combined algebra and geometry scheme, since she interpreted and exploited the synergy between two contexts. However, initially, she mostly tended to think geometrically, and this guided her to develop a new scheme on the intersections of figures, as Ela thought that the intersection of figures could play a heuristic role to analyse the position of the planes.

At the end of the second task, a new conceptual aspect appeared in the combined algebra and geometry scheme; articulation of different cases. This came into play when Ela analysed the position of the planes, intersection results in the algebra window and the echelon form of the system through dragging sliders. This conceptual component and the employed techniques were stable in Task 3. And regarding the parameter scheme, a number of conceptual elements remained the same throughout Task 1 and 2. However, a new characterisation for the parameter used in a system of linear equations appeared at the end of Task 3.

Overall, the echelon form scheme was at the core of her findings; Ela solved three tasks thanks to the echelon form's synchronic function in the geometry context. At the beginning, Ela was looking for an indirect relationship between overlapping planes and the echelon form. However, in the end, she reinvented the epistemological link (i.e., the transformation into a unit matrix) between the two. This enabled her to add a new conceptual aspect to her repertoire.

5. CONCLUSIONS AND DISCUSSION

In this paper, we investigated the research question 'Which instrumental schemes are developed while solving systems of linear equations with parameters using dynamic geometry software?' A task sequence including three specific problems provided us with a general overview of the participant's instrumentation schemes. They are parameter scheme, combined algebra and geometry scheme, the intersection of figures scheme and the echelon form scheme. The development of these schemes showed how her thinking with digital artefacts evolved over time. Even though it was not linear, her way of thinking in solving systems of linear equations followed: enter algebraic equations to DGS, obtain geometry of equations, explore synchronic windows, intersect figures, refer to echelon form of the matrix, compare and contrast variations on different windows, articulate findings and make interpretations.

Of course, the study comes with limitations, and we now address four of them. The first is about the described case which was limited to a single student. However, we point out here the role of carefully designed tasks which have a core function in teaching-learning linear algebra of inviting students into meaningful mathematics (Andrews-Larson, Wawro, & Zandieh, 2017; Trigueros & Possani, 2013, Turgut, 2019). In our case, we underline that the key process was the coordination between algebraic and geometric representations in linear algebra (Caglayan, 2019). Our findings are in line with Zandieh and Andrews-Larson (2019), who report that students' symbolisation mainly shifted through variable renaming, variable creation, and reasoning on parameter.

The second limitation is about the participant in the study. Ela has used GeoGebra for a long time and knows its functions and tools in depth. In other words, she is a proficient GeoGebra user in two directions; she knows how to use tools to do mathematics, and she also knows how to integrate GeoGebra for teaching mathematics at the secondary school level. Another fact about the participant is that she knows how to solve a system of linear equations, geometry of solution types, and interpretations of the echelon system. These all together can be considered as the weakness of the study.

The third limitation is about tool use in learning mathematics. As Drijvers (2019) points out, digital tools come into play with their own affordances and constraints. In our case, affordances and constraints have somehow interacted. For instance, regarding affordances, Ela referred to different contexts that provided her progressive thinking and at the end, the echelon form of a linear system conveyed her to move between parameter use and geometry of solution. However, though she was faster at the beginning, she spent much time for technical issues. But one explicit constraint was the software's feedback regarding intersection of overlapped (3D) lines, which quite confused her. Thanks to the interrelation between algebra and geometry, she managed to arrive at a mathematical conclusion regarding the notion of parameters.

Before Ela's opted to use the echelon form, she mainly worked with the geometry of the mathematical situation through the dragging of sliders, and the context she worked was limited into $\mathbb{R}^3$. The use of geometry in teaching-learning linear algebra could be problematic, as addressed by researchers (Gueudet-Chartier, 2004; Harel, 2000, 2019). Our research context was based on geometry and this can be considered as a fourth limitation. However, such geometry-based problems could follow in further research elaboration of higher dimensional applications by the combination of Algebra and Spreadsheet windows.

Regarding the first limitation, it would be meaningful to discuss the same tasks within the classroom environment. Because Ela's instrumentation schemes, conceptual and technical elements provided clues for possible reasoning steps, a redesign of a task sequence preventing users from pitfalls could be elaborated in a classroom laboratory environment with moderate level GeoGebra users.

In the present paper, we provided a case regarding formulation of a DGS-based task sequence for developing student understanding on parameters in systems of linear equations. On the one hand, our conclusions underlined that it is possible to move between different lines and columns of Table 1 and how the tools and functions of GeoGebra are collection of practical artefacts to do so. Actually, the software acted as a heuristic tool for a combined view for algebra and geometry in solving linear equations. On the other hand, the TIG perspective provided a fruitful understanding on student thinking, particularly on how tools and functions of DGS shaped the user's knowledge over time.

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BIOGRAPHICAL NOTES

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