Reinventing Geometric Linear Transformations in a Dynamic Geometry Environment: Multimodal Analysis of Student Reasoning

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Abstract

This paper reports the multimodal resources that attached to students' reasoning in the reinvention of specific geometric linear transformations (like reflections according to the axes, projections onto axes and composition of reflections) in a dynamic geometry environment. Following the design heuristics of Realistic Mathematics Education, we design a task (in $\mathbb{R}^2$) referring to specific tools and functions of GeoGebra. Task-based interviews were conducted with a pair of linear algebra students, by way of a computer and a teacher. Data was collected using a video camera observing the students' working environment, screen recorder software, student production and field notes. The data was analysed according to the multimodal paradigm focusing on all semiotic resources, such as gestures and artefact use, in addition to written signs. According to the findings, the artefact use, verbal and written mathematical expressions all interlaced with the emergence of gestures. The students mostly gestured when they faced a new reflection situation and when describing associated geometric actions. Finally, a shared environment with action, production and communication conveyed student reasoning and they managed to reinvent a number of geometric linear transformations.

Keywords Learning linear algebra · Linear transformation · Dynamic geometry environment · Multimodal paradigm

Introduction

Linear algebra is an extensive catalogue, which includes different mathematical objects and representations, where it is not easy for students to build interconnections among them. For instance, the notions of function and linear transformation are strongly

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connected in this catalogue. However, as has been shown in a number of research papers (Bagley, Rasmussen, & Zandieh 2015; Zandieh, Ellis, & Rasmussen 2012, 2017), it is not an easy or trivial task for students to construct a mathematical link between a linear (matrix) transformation and a function, even if the students are aware of the classic Dirichlet-Bourbaki notion of function that is often formulated as $f: A \rightarrow B$ for two non-empty sets A and B.

In order to establish a link between two core notions, an emphasis on reflections according to the axes, projections onto axes and composition of the reflections as geometric linear transformations (GLT) in $\mathbb{R}^2$, could be a (twofold) heuristic tool for establishing the link between GLT and functions. First, students' (re)invention or (re)formulation of specific GLT as matrix transformations may consolidate the link between the matrix representations of associated functions. Second, a composition of reflections (as well as compositions of geometric transformations) on the Cartesian plane could enable students to construct a link between the composition of functions and the composition of matrix transformations, and could provide a better understanding of the link between a matrix transformation and a function.

In order to create an environment for the emergence of such a twofold view, we refer to the dynamic geometry environment (DGE) which provides users with a context for exploring, conjecturing, validating and creating linear algebraic notions (Gol Tabaghi, 2014; Gol Tabaghi & Sinclair, 2013). Along this direction, we adopt Realistic Mathematics Education (Freudenthal, 1973; Van den Heuvel-Panhuizen & Drijvers, 2014) as an instructional theory and Action, Production and Communication space (Arzarello, 2008) as a lens for analysing student thinking. Therefore, the next section starts with a brief analysis of the reflections according to the axes, projections onto axes and composition of the reflections (which we will shortly use GLT for all) in $\mathbb{R}^2$ and their DGE (particularly GeoGebra) availability and the theoretical framework of the paper. The third section provides the methods followed in the present paper, while the fourth section presents the findings. The final section provides conclusions and a discussion of the results with a number of recommendations.

Theoretical Framework

This section is organised under three subsections. First, we recall the mathematics of GLT to show DGE availability, followed by a brief description of Realistic Mathematics Education and Action, Production and Communication space with multimodality.

GLT as Matrix Transformations in $\mathbb{R}^2$

A geometric transformation can be expressed as a specific function $M: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with the idea of a geometric view (i.e. a point transforms to another) or with the idea of an algebraic view (i.e. an n-tuple goes or transforms to an m–tuple) (Lay, 2006). By taking $m = n = 2$ here, in a Cartesian plane context, several geometric applications associated with the matrix transformations can be defined (as GLT): specific reflections (according to the x-axis, y-axis, origin, and $y = x$ and $y = -x$ lines), and projections onto axes. A classification of these applications is summarised within Table 1.

Table 1 Classification of GLT in $\mathbb{R}^2$ (Lay, 2006, pp. 85–87)

Transformation (function)	Matrix transformation	Associated (geometric) interpretation
$M_1 : (x,y) \rightarrow (x,-y)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Reflection according to the x-axis
$M_2 : (x,y) \rightarrow (-x,y)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Reflection according to the y-axis
$M_3 : (x,y) \rightarrow (-x,-y)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Reflection according to the origin
$M_4 : (x,y) \rightarrow (y,x)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Reflection according to the $y=x$ line
$M_5 : (x,y) \rightarrow (-y,-x)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Reflection according to the $y=-x$ line
$M_6 : (x,y) \rightarrow (x,0)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Projection onto the x-axis
$M_7 : (x,y) \rightarrow (0,y)$	$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	Projection onto the y-axis

As linear algebra instructors, we generally introduce the notions in Table 1 with the applications of specific figures to create a means for associated matrix transformations. This way of approaching the topic is commonly static, i.e. not eligible for exploration and/or a combination of different situations, and it is dominantly based on the orientation of the teacher. For instance, as Molina and Oktaç (2007) examine, students tend to search for prototypical examples coming from their previous experience when they are provided with a figure and its transformation. This is in line with existing research results (Montiel, Wilhelmi, Vidakovic & Elstak, 2012; Sierpinska, 2000). This issue could be since the students are only working on static (paper-and-pencil) tasks. However, GLT and its compositions could be presented through a dynamic context, as will be explained later, and the notions in Table 1 have an extensive DGE availability to explore a number of cases together.

Recent studies have pointed out that the use of dynamic representations through GeoGebra applets provides students with a better understanding for conceptualising different representations of linear transformations (Oktaç, 2018; Romero Félix & Oktaç, 2015). Following this, we refer to (design heuristics of) Realistic Mathematics Education in order to create a meaningful context for students with digital tools for their exploration of GLT.

Realistic Mathematics Education

Realistic Mathematics Education (RME) is a domain-specific instructional theory rooted in Freudenthal's (1973) notion of 'mathematics as a human activity'. Here, the word realistic does not necessarily refer to real-world problems, but refers to contexts that students could experience and find meaningful. In other words, proposed contexts must be experientially real to the students (Gravemeijer, 1999). RME emphasises the importance of guiding students for their reinvention of mathematics, where the task design and the teacher's role are of crucial importance. Regarding these views, RME offers three (interrelated) design heuristics for developing meaningful teaching-learning environments (Gravemeijer, 1999; Van den Heuvel-Panhuizen & Drijvers,

2014): guided reinvention, didactic phenomenology and emergent modelling. The first point is that paradigmatic task situations must enable students to reinvent proposed mathematics under the teacher's guidance. To do so, the teacher must take into consideration the students' backgrounds, their needs and his/her own learning goals, and further elaborate a context (through a step-by-step plan) where the students create their own mathematics with his/her careful orientation, probably with provocative questions. At this point, in order to start designing a plan, the second heuristic (didactic phenomenology) comes into play. Didactic phenomenology means analysing and creating (and/or borrowing) experientially real phenomena that invite students to make their own mathematics. The task context must invite and enable students to explore, discuss and finally reinvent mathematics. The notion of emergent modelling is regarded as a synergy between informal and formal mathematical models, where the task context must include different situations to convey students from informal models to models that reflect the reinvention of mathematics (Doorman, Drijvers, Gravemeijer & Reed, 2012).

Recently, the RME perspective and the reinvention of mathematics view have received robust attention from linear algebra educators at university level (Andrews-Larson, Wawro & Zandieh, 2017; Wawro, Rasmussen, Zandieh & Andrews-Larson, 2013). Even though an RME perspective is not specific to the integration of digital technologies, it could help educational designers and researchers to design meaningful technology-rich environments (Drijvers, 2019). Bearing this in mind, in this paper, we adopt an RME perspective for guided reinvention of the link among geometric linear transformations, matrix representations and functions.

Action, Production and Communication Space and Multimodality

Learning mathematics is quite a complex process and occurs within reasoning on different mathematical concepts. As expressed in the previous section, mathematical conceptualisation is the production of experiencing with an experientially real phenomenon and, in this way, guided reinvention is possible. Furthermore, three design heuristics of RME together imply the creation of a rich set of situations for students to engage in mathematics, such as making graphs, using tools, exploring different cases, analysing, conjecturing and generalising. All of this engagement in mathematics yields different classroom resources that attach to students' reasoning and mathematical thinking.

Taking a semiotic perspective, Arzarello (2008) defines Action, Production and Communication (APC) space in order to frame all classroom resources while the teaching and learning of mathematics occur. According to an APC space view, learning mathematics is not independent of its teaching-learning environment; it is shaped, for instance, by the task situation, teacher guidance, communication with the teacher and peers, and artefact use. In other words, APC space points out that the learning of mathematics has different semiotic resources and it is a multimodal process. In fact, APC space not only follows an embodied cognition view, which claims 'that cognitive processes are rooted in interactions of the human body with the physical world' (Alibali, Boncoddo & Hostetter, 2014, p. 150), but it also embraces a Vygotskian view to learning mathematics (Maffia & Sabena, 2015). Therefore, APC space has three dimensions: the body, the physical world and the cultural environment (Arzarello,

2008). In summary, APC space and the multimodal paradigm here refer to a unitary system, as a lens, for classroom dynamics having the task context, artefacts/tools, communication taking place and the teacher's guiding (i.e. orchestrating and/or mediating) role for student learning.

In order to analyse semiotic resources under a multimodal paradigm, all signs (gestures, mimics, drawings, speech, interaction with digital tools and so on) produced by the sensory-motor functions of the classroom community come into play. Under APC space, Arzarello and colleagues (Arzarello, 2008; Arzarello, Paola, Robutti & Sabena, 2009; Arzarello & Robutti, 2008) introduce the notion of a semiotic bundle (grounding by Peircean semiotics) in order to frame all classroom production:

> … A semiotic bundle is a system of signs, with Peirce's comprehensive notion of sign, that is produced by one or more interacting subjects and that evolves over time. Typically, a semiotic bundle is made up of signs that are produced by a student or by a group of students while solving a problem and/or while discussing a mathematical question. The teacher may also participate in this production, and the semiotic bundle may, therefore, also include signs produced by teacher. (Arzarello et al., 2009, p. 100)

Along this perspective, the notion of semiotic bundle includes a two-dimensional analytical tool for making a fine-grained sign analysis of resources: a synchronic analysis which refers to relationships between different signs that are produced by the classroom community at a certain moment, and a diachronic analysis which refers to an evolution of signs for mathematical thinking, but in successive moments (Arzarello, 2008; Arzarello et al., 2009).

Recently, multimodal resources to understand student thinking for concepts related to function have received particular attention from researchers (Arzarello et al., 2009; Arzarello, Robutti & Thomas, 2015; Yoon, Thomas & Dreyfus, 2011). However, to date, no research has appeared in the available literature looking at multimodal resources for learning linear algebra with digital tools. Apart from consolidating the link between GLT and functions, this paper also aims to explore the usability of APC space in a learning linear algebra setting. To sum up, adopting both RME and APC space perspectives, in this paper, we investigate two interrelated research questions: (1) *Which multimodal resources emerge while students solve a GLT task in a DGE?* (2) *Which reasoning steps do students follow while reinventing GLT?*

Methods

This paper is part of an extensive design-based research project (Bakker & van Eerde, 2015) aimed at designing didactic cycles, piloting, refining and re-piloting them in order to ameliorate and elaborate a fine-tuned teaching-learning context with digital technologies. Therefore, a task sequence, including nine tasks for teaching-learning linear transformations in $\mathbb{R}^2$, was designed and a number of task-based interviews were implemented to pilot and ameliorate the tasks. The results presented here are the third-time pilot implementation of the second task, where we present the results of the first task by Turgut (2019) in the task sequence described above.

The participants, Hazel and Iris (pseudonyms), were two linear algebra students enrolled on a mathematics education program at a state university located in central Turkey. The participants were 20-year-old females at a sophomore level of the program, and they were selected from a group of volunteers. They were selected to work in pairs, Hazel having performed better compared to Iris in regular class paper-and-pencil assignments, together with classroom question-and-answer dialogues. However, Iris had good communicative skills, using mathematical language while describing a situation. Therefore, following APC space, they were paired to work together in front of a computer, installed with GeoGebra, after getting their consent to work as a pair. The students had taken several courses at the freshman level, such as Euclidean geometry, fundamental calculus, abstract and discrete mathematics, and a pilot study which took place after the students had learned linear systems and the Gaussian elimination method, vector spaces and subspaces, determinants and matrix transformations. These topics were taught by the author using a course book (Lay, 2006) with selected topics. Moreover, the students had experienced the matrix representation of linear transformations, and they also knew how to represent geometric linear transformations in $\mathbb{R}^2$ through matrices and the transition between function representation and matrix representation (Turgut, 2019):

$$M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ \ M(x,y) = (ax+by, cx+dy) = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In addition, the students had learned how to apply a (basic) matrix transformation to line segments, but not more. The students had not learned reflections according to the axes, projections onto axes and composition of specific reflections in $\mathbb{R}^2$, and their matrix representation before the pilot study was conducted. In other words, they had no experience with the tools or functions of GeoGebra regarding reflections according to the axes, projections onto axes and composition of specific reflections in the context of matrix transformations.

The Data and Analysis

The data was collected through a series of task-based interviews (approximately 3h with three break times) with the teacher in front of a laptop, installed with GeoGebra, facing the students. Following a multimodal paradigm, the data was triangulated and all of the data sources were considered, while the students' production was collected: field notes, a video camera overlooking the students' working environment recording the experiment to follow body signs connected to the students' meaning-making, and screen recorder software used to capture the students' techniques for accomplishing the task. The interviews were conducted in Turkish and translated into English by the author and checked by proof-readers. Following the notion of the semiotic bundle, all of the data sources (as semiotic representations) were analysed by synchronic and diachronic analysis techniques. Regarding synchronic analysis, we refer to a table inspired by Troup (2019), whose columns include verbal, gestural, artefact/tool use and written signs with specific screenshots. Regarding diachronic analysis, we refer to the evolution of mathematical thinking by depicting a

whole picture showing students' reasoning step-by-step. In other words, we consider diachronic analysis as a larger picture of (i.e. process of) students' reinvention of GLT regarding the second research question.

Design Heuristics of RME and Task Design

How to find a starting point for a context where the students involve themselves in mathematics and work on such a context consolidating the conceptualisation of geometric linear transformations as functions? Considering didactic phenomenology, we thought that the topics of reflections in a DGE would be meaningful for the students along two directions. The first was reflections, which was known by the students from primary school, thinking that the topic was experientially real for them. The second was the DGE advantage. Because the DGE context includes a set of situations where certain cases are reflections, at the same time, there is a composition of the reflections. Considering the emergent modelling heuristic, to create a rich context for students to explore and articulate, but in different situations, we refer to DGE's specific tools and functions as will be explained later. In this way, we believed that the students would explore and discuss interrelated situations, and that they could create their own models and arrive at mathematical conclusions. Considering the guided reinvention heuristic, the teacher analysed DGE for his aim and prepared a plan (Table 2), following his experience from previous rounds of the pilot study, the goal, the students' pre-knowledge and his possible provocative questions to orchestrate their learning.

The second line of Table 2 indicates specific tools and functions. The dragging tool is the main component of any DGE, enabling the user to move and manipulate objects on the screen. On opening the GeoGebra interface, a Cartesian plane appears with a grid function, with such a function establishing

Table 2 The teacher's plan to introduce GLT in DGE

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Students' pre-knowledge	Linear systems, Matrix representation of linear transformations, transition between function and matrix representation.
Which tools and functions and why	The dragging tool for exploration and possibly establishing conjectures and argumentation, the Slider tool for dynamic variation, the Grid function, the Checkbox tool for different situations and the ApplyMatrix construction tool to apply matrix transformation.
Exemplary task	Move sliders and explain what is happening on the windows synchronously. Use the paper-and-pencil form to express related mathematics behind the DGE.
Exemplary questions	What is the role of the sliders? How do you classify such reflections? How do you express your conclusion mathematically?
Expected steps in DGE	Dragging sliders, dragging objects, exploring different situations with the Checkbox tool, articulating paper-and-pencil with the DGE interface.
Goals	First, reinvention of specific reflections in $\mathbb{R}^2$ as matrix transformations; second, reinvention of the link between the composition of reflections and composite functions. As a result, consolidation for the link between linear transformations and functions in $\mathbb{R}^2$.
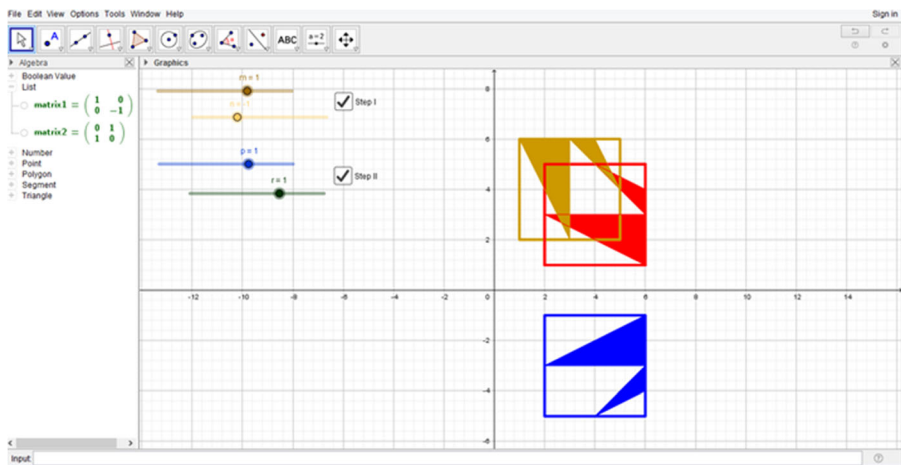
Fig. 1 Two exemplary cases ($m = 1$, $n = -1$ and $p = 1$, $r = 1$) of the task

(faded) parallel lines to the x- and y-axes. In the context of this paper, the grid function and axes create an environment where the user may explore GLT, and mathematical relationships between figures, comparing their lengths, areas and so on. The slider tool assigns parameters for an interval of real numbers, and such parameters can be attached to any function or tool of the software, thereby providing a dynamic variation. Furthermore, the ApplyMatrix construction tool makes it possible to apply a matrix transformation to any point, line and/or any figure. Considering the tools and functions above, first, we constructed four sliders that were each defined as the (different) coefficients of two matrices (see Fig. 1) as follows:

$$matrix1 = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}, matrix2 = \begin{pmatrix} 0 & p \\ r & 0 \end{pmatrix}.$$

Next, we drew the red figure[1] and, by using the ApplyMatrix construction tool, we applied two matrix transformations (matrix1 and matrix2 in the Algebra window of Fig. 1) to a red figure, a brown figure as matrix transformation under matrix1 and a blue figure as matrix transformation under matrix2. Two exemplary cases are provided in Fig. 1.

In Fig. 1, if the user clicks on step I, only sliders m and n and the associated matrix1 are visible. Similarly, if the user clicks on step II, all of the sliders are visible. In Fig. 1, the case of $m = 1$, $n = -1$ corresponds to a reflection according to the x-axis, while the case of $p = 1$, $r = 1$ corresponds to a reflection according to the $y = x$ line. Beyond reflections, if the user takes $m = 1$, $n = 0$, then the blue figure transforms into a line segment on the x-axis, which is a projection onto the x-axis (Fig. 2a). Similarly, if the

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[1] A red figure (a square including two rectangles) was intentionally designed to enable the user to compare/ explore the given figure, and the figures appear through matrix transformations. For example, in the case of reflection according to the $y = x$ line, we expected that the user would easily visualise/find to the reflection (see Fig. 1).
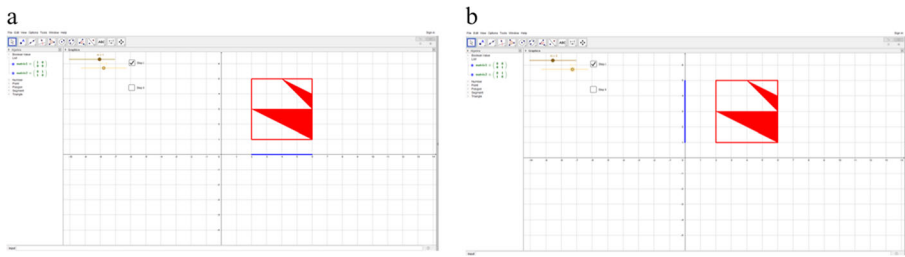
Fig. 2 (**a**) Projection onto the *x*-axis. (**b**) Projection onto the *y*-axis

user takes $m = 0$, $n = 1$, then the blue figure turns into a line segment on the *y*-axis, which is a projection onto the *y*-axis (Fig. 2b).

Moreover, if the user takes $p = -1$ and $r = 1$, then the brown figure in Fig. 3a is obtained. In fact, this is a specific case including composition of two reflections: at first, a reflection according to the *y*-axis, and then, a reflection of transformed figures according to $y = -x$ line (see Fig. 3b).

All such possible cases would contribute to students classifying GLT with associated matrices, which were providing a dynamic context and interrelated situations, as shown in Table 1. However, in order to shield the users from a cognitive load, we decided to divide the task into two steps using the Checkbox tool. Finally, the task was formulated as follows:

- Step 1. *Click on step I on the GeoGebra interface then drag the sliders and check a number of cases for different values. Explore and explain what happens on the Algebra and Graphics windows systematically.*
- Step 2. *Unclick step I and click on step II. Explore and explain what happens on the Algebra and Graphics windows systematically.*
- Step 3. *Explore and discuss for which values the constructed figures transform into different figures. Make a classification.*

Findings

Considering the notion of the semiotic bundle, we decided to present verbal, gestured and written signs with artefact/tool use and screenshots. In order to picture a step-by-step student reasoning, we present our findings under two interrelated subsections: (1)
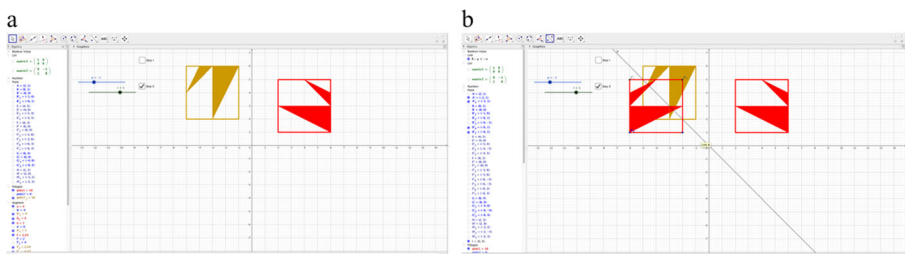
Fig. 3 (**a**) The case of $p = -1$ and $r = 1$. (**b**) A sketch of the composition of two reflections

reinvention of basic reflections and projections onto axes and (2) reinventions of compositions of reflections and reflections according to axes.

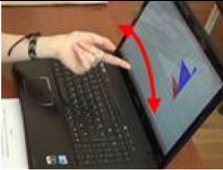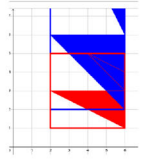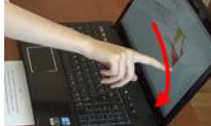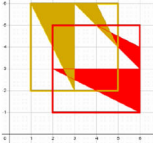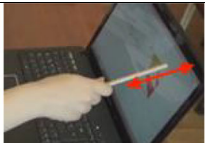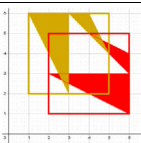### Synchronic Analysis of Reinventions of Basic Reflections and Projections onto Axes
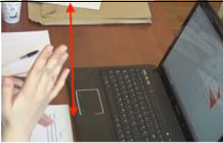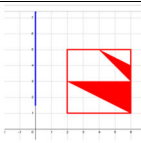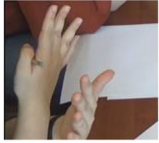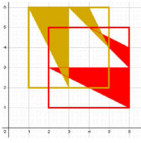
This subsection includes students' initial interaction with digital tools and their discussion on reflections according to the $y = x$ line, reflection according to the origin and their reasoning on projections onto axes.

At the beginning of the discussion, the teacher introduces the task to Hazel and Iris, providing a separate paper that includes the steps of the task. The students read the steps and Hazel starts by clicking on step I and begins to drag sliders m and n with Iris carefully looking at the screen. Together, they try a number of cases changing m and n. At that moment, different semiotic resources appear, and Table 3 summarises the synchronic analysis regarding the initial discussion and the students' reinvention of unit transformation and reflection according to the $y = x$ line.

At the beginning of the students' interaction with the DGE, the students express a number of specific verbal signs that attach to their observation and thinking (e.g. #13 and #16), 'moves along axis', and their gestures reflect their immediate analyses regarding the situation. In fact, the students first try to understand what the effects of the sliders on the movement and variations of the figures are. Finally, they manage to characterise the effects of the sliders with m and n 'expand' the red figure (#16). After this, Iris expresses 'the figures overlap' when $m = n = 1$ with a mathematical formula. The students make more dragging practice, and, after a while, in order to move them forward, the teacher asks them to analyse the Algebra window with the Graphics window as indicated in the task. After this, they easily relate the situation with matrices and matrix transformations (#24) by stating that m and n are the first and fourth components of matrix1 and express mathematical representations (as a transformation and as a function) regarding a unit transformation. They together discuss their findings. Here, the produced signs reflect that the students begin to reason the interpretative link among the components of the matrix, geometric transformations as arriving at a unit transformation and associated function. Because they start to blend their own knowledge as unit transformation (matrix transformations in general) with the situation, it results in the dragging of the sliders.

Next, the students start to discuss the second step of the task (#41). When clicking on step II, the software assigned $p = r = 1$, Hazel immediately thinks that the red and brown figures are symmetrical and that they seem to be similar figures, but never overlap. Hazel refers to the previous experience of writing unit transformation where she had written the mathematical representation of the case $p = r = 1$. After a number of dragging practices, the students gesture to describe reflection and also to express (#45, #46) mathematically ('x goes to y and y goes to x'). At this time, Hazel does not think that matrices and reflections are unrelated, but Iris's pencil gesture (#47) opens a door to finding reflection according to the $y = x$ line. They again take notes about the obtained case. It is obvious that dragging practices contribute to the students' reinvention of specific reflection and its mathematical representation. Produced signs show here that the students start to reinvent the relationships among the matrix entries, matrix transformation and reflection of the figures.
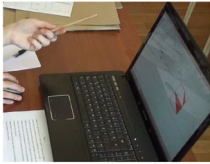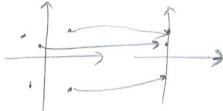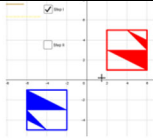
Table 3 Summary of synchronic semiotic resources regarding unit transformation and reflection according to the $y = x$ line

Verbal signs	Gestures	Artefact- Tool use	Written Signs or Screenshot
13 Iris: [while Hazel drags the slider n] *Now the line segment moves along the y axis… [gestures]*		-dragging	
16 Hazel: [dragging sliders] *A change of m expands the given figure along the x-axis while n does so along the y-axis… [gestures] In fact, the figure expands with a scale factor m and n, when they are greater than one…*		-dragging	
24 Iris: [the *case is m = n = 1*] *This is a matrix transformation and matrix1 is a unit matrix… Therefore, this is a unit transformation.* [writes]	No gesture	-dragging -paper and pencil	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $f: (x,y) \to (x,y)$
41 Hazel: [regarding the case $p = r = 1$, asks herself] *Is this a reflection? Let's write …* [writes matrix transformation]	No gesture	-dragging -paper and pencil	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ $f: (x,y) \to (y,x)$
45 Iris: [regarding the case $p = r = 1$] *… Now x goes to y and y goes to x… If I rotate 90 degrees anticlockwise and then reflect it … the brown figure… [gestures]*		-dragging -paper and pencil	
46 Hazel: *… to me, there is no relationship between matrices and reflections and rotations. Just a second… Is this a rotation?* [finger tracing] *or reflection according to origin?*		-dragging -paper and pencil	
47 Iris: [gestures with pencil] *Is this a reflection according to the y equals to x line?*		-dragging -paper and pencil	

As a next step, the students start by exploring the case $m = n = p = r = 0$. Thereafter, the students try to classify all reflections through the movements of the sliders. Table 4 summarises the semiotic resources emerging at that moment.

Iris explains this situation because all points would go to the origin, since there is no obtained figure (#70). She also points out that the brown figure disappears when all entries of the matrix are zero and she writes the mathematics of the case. Then, Hazel drags slider p and $r = 0$. Pointing to the screen, she explains mathematically why, in some cases, they get moving line segments on the screen (#79). She also formulates the associated matrix transformations with functions. Starting with Iris (#80), the students refer to two Cartesian systems and sketch a general view for the associated transformation that transforms all points on the plane onto the y-axis, where they reinvent the

Table 4 Summary of synchronic semiotic resources regarding exploration of reflections and projections onto axes

Verbal signs	Gestures	Artefact- Tool use	Written Signs or Screenshot
70 Iris: *Ha, if all sliders are zero, then we have here* [pointing the origin] *… its formula …*[writes] *whatever I have the outputs will be zero and zero.*		-dragging -paper and pencil	
79 Hazel: *…* [pointing to the screen] *when I take one slider as zero and move the other, the brown figure transforms to a line segment… If … the y value is always zero, and the only x varies. If p is zero, then x will be always zero. Therefore, the figure moves along the y-axis…* [writes]		-dragging -paper and pencil	
80 Iris: [draws two Cartesian systems] *… yes, movements depend on entries as p and r, and therefore it turns into a line segment on the y-axis…*	No gesture	-paper and pencil	
103 Hazel: [takes $m = n = -1$] *… I think this is a reflection according to the origin.*		-dragging -paper and pencil	

projections onto axes. The students speak for a while about their sketches and think that they have completed the task. Then, the teacher asks, 'What about the Algebra window and its relationship to the Graphics Window, and what about negative values for the sliders?' in order to move the discussion to negative-valued sliders and their variation. Then, the students realise that they had never analysed the negative values. Hazel goes back to step I and selects $m = n = -1$. She points to the screen with a pencil and immediately expresses that this is a reflection according to origin (#103). Then, Hazel and Iris work together because they had not referred to different negative values in step I and step II. They work for a while and make a list as shown in Fig. 4.

As can be seen from Fig. 4, the students do not consider negative values for p and r; rather, they only refer to $m = n = -1$. However, they sketch other situations with associated matrix transformations and functions, where synchronic signs imply that the students reinvent and start to make meaning-makings on specific geometric linear transformations.

Diachronic Analysis of Reinventions of Basic Reflections and Projections onto Axes

Regarding the evolution of the semiotic resources in the reinventions of basic reflections and projections onto axes within the perspective of student learning, even if it is not linear, we briefly sketch the students' (interrelated) reasoning steps. Figure 5 overviews our proposal.
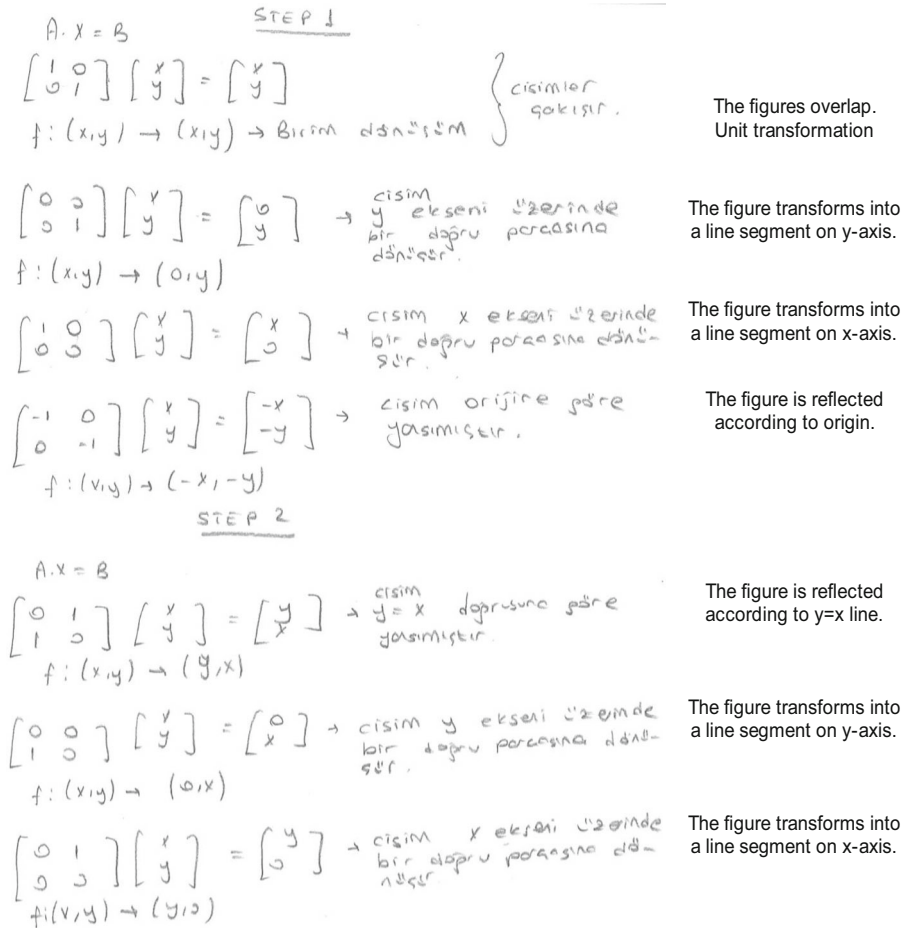
STEP 1

A. X = B

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$f: (x,y) \rightarrow (x,y) \rightarrow$ Birim dönüşüm

cisimler çakışır.

The figures overlap.
Unit transformation

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \rightarrow$$ cisim y ekseni üzerinde bir doğru parçasına dönüşür.

$f: (x,y) \rightarrow (0,y)$

The figure transforms into a line segment on y-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \rightarrow$$ cisim x ekseni üzerinde bir doğru parçasına dönüşür.

The figure transforms into a line segment on x-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \rightarrow$$ cisim orijine göre yansımıştır.

$f: (x,y) \rightarrow (-x,-y)$

The figure is reflected according to origin.

STEP 2

A. X = B

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \rightarrow$$ cisim y=x doğrusuna göre yansımıştır.

$f: (x,y) \rightarrow (y,x)$

The figure is reflected according to y=x line.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} \rightarrow$$ cisim y ekseni üzerinde bir doğru parçasına dönüşür.

$f: (x,y) \rightarrow (0,x)$

The figure transforms into a line segment on y-axis.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} \rightarrow$$ cisim x ekseni üzerinde bir doğru parçasına dönüşür.

$f: (x,y) \rightarrow (y,0)$

The figure transforms into a line segment on x-axis.

Fig. 4 The students' classification regarding different values of *m*, *n*, *p* and *r* (translated into English)

Figure 5 implies that the students' reasoning steps start with their observation and descriptions of what they see on the screen. First, the students focus on the link between tool use (i.e. dragging) and dynamic variation (i.e. similar figures/overlapping figures) on the screen. The teacher's guidance opens a door to an exploration of algebra of the situation in addition to a dynamic variation on the geometry window. Consequently, through a combined view, they reinvent the link between geometric variations and change of (positive) matrix entries and associated geometric linear transformations like unit and zero transformations, projection onto axes. Through one more teacher orientation, the students reinvent the mathematical descriptions of reflection, according to the $y=x$ line and reflection to the origin, but also the projections onto axes.

Synchronic Analysis of Reinventions of Compositions of Reflections and Reflections According to Axes

After the previous session, the teacher is aware of the need to orient students to explore other remaining cases, such as negative-valued sliders and their variations. Therefore,
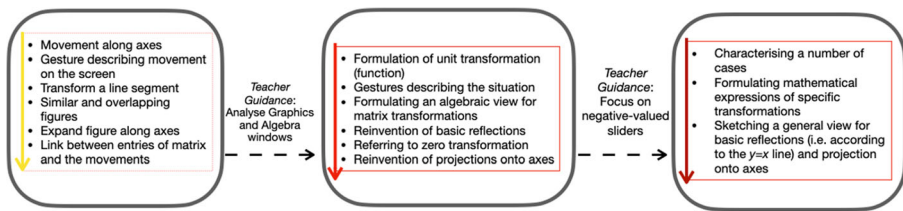
Fig. 5 The students' reasoning steps and associated signs with the teacher's guidance regarding basic reflections and projections onto axes
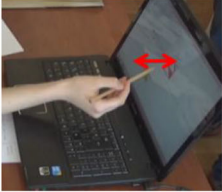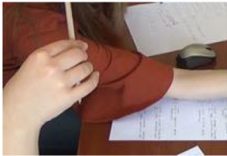
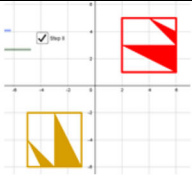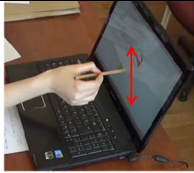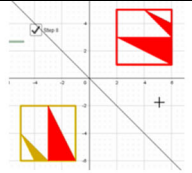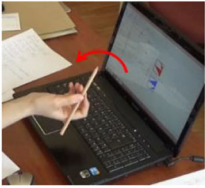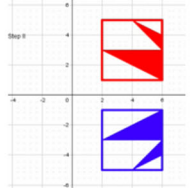again, the teacher, in particular, asks the students, 'What would happen with negative-valued p and r values'. Then, the students read the steps of the task again and Hazel starts to drag the sliders p and r. However, this time, the position of the blue figure seems different to them. They explore for a while, but decide to write the associated transformation comparing the Algebra and Graphics windows. They compare their findings with previous findings, but cannot find a clear way to proceed. Corresponding to student work in successive moments, Table 5 summarises the semiotic resources attached to their reasoning.

At the beginning of the discussion regarding reflection according to the $y = -x$ line, the students spend much time characterising. Through her spatial perception, even though Hazel first conjectures that there is a two-step reflection (#135), Iris helps her to express the right reflection line (#136). At that moment, Hazel's existing GeoGebra knowledge comes into play, and she sketches the $y = -x$ line and reflects a piece of the red figure (#137). Thereafter, the students check their conjecture and reinvent and reformulate the reflection according to the $y = -x$ line. As a next step, the teacher again is aware of the remaining values of the sliders, such as $m = 1$, $n = -1$ or $m = -1$, $n = 1$. The teacher asks, 'Do you believe that you have already explored all possible cases with four sliders?' The students think for a while and then focus on the cases of $m = 1$, $n = -1$ and $m = -1$, $n = 1$. First, Hazel drags the sliders and gestures that it is a reflection according to the x-axis (#159). Iris also expresses the mathematics of the reflection according to the y-axis (#160).

Thereafter, the students focus on step II and select different values of p and r. Hazel first selects $p = 1$ and $r = -1$. Referring spatial perception and mathematical formulas of associated transformation, the students analyse a number of cases. However, the cases $p = 1$ and $r = -1$ are quite different for them, and even though they easily find the algebraic formulation of the transformation, they do not manage to find the geometry of the reflection. Table 6 summarises the semiotic resources attached to this and the proceeding moments.

In this part of the discussion, the students begin to articulate meanings for the composition of reflections with matrix transformations. At first, in terms of dragging and the role of the sliders, they manage to formulate a matrix transformation, but not with a two-step view (#181). Iris reminds Hazel of the incorrect conjecture concerning a two-step reflection, and they then begin to discuss the possibility of this (#182). Iris asks Hazel to employ two reflections using the GeoGebra tools and functions (#183), and Hazel manages to obtain the overlapped figures (#184). After they discuss the issues for a while, by comparing this situation with their initial findings, Hazel formulates a two-step matrix transformation (#207). At that moment, the teacher asks

Table 5 Summary of synchronic semiotic resources regarding the exploration of reflections according to the axes and the $y = -x$ line
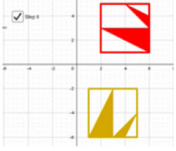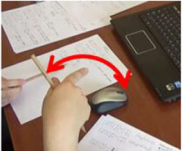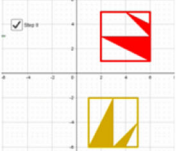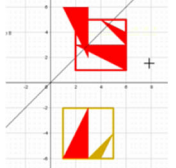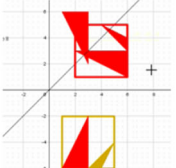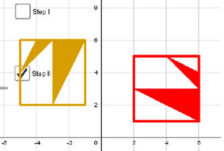
Verbal signs	Gestures	Artefact- Tool use	Written Signs or Screenshot
135 Hazel: … [pointing to the screen and comparing the points on the figure] … *it is first reflected according to the origin, and next it is reflected according to y equals to the x line?* …		-dragging -paper and pencil	
136 Iris: *Can we express two-step reflection here? But it seems to be a reflection according to y equals the minus x line* [gestures with pencil] …		-dragging -paper and pencil	
137 Hazel: … *May I draw y equals to minus x line here?'* [gestures with pencil] [she sketches the y=-x line and, using the reflect tool, she reflects a certain part of the red figure and obtains a new screen] *Yes overlapped*…		-dragging -paper and pencil -reflect tool	
159 Hazel: [selects *m*=1 and *n*= –1] … *very interesting now it is reflected according to the x axis* [gestures with pencil] …		-dragging -paper and pencil	
160 Iris: *Ha then, if m is minus one and n is one, we will have … [writes]…* No gesture		-paper and pencil	

how this two-step matrix transformation is related to their initial findings (i.e. the transformation in the first line of Table 6). The students explore the sliders and compare their findings for a while, and they summarise their findings regarding the situation, referring to the production of two matrices and the composition of functions (#215). Hazel also retests Iris's conjecture with the case of $p = -1$ and $r = 1$. Hazel explains her conclusion and conjectures verbally (#222) by generalising. The students reinvent the interrelation among matrices, the composition of reflections and functions.

Diachronic Analysis of Reinventions of Compositions of Reflections and Reflections According to Axes

Bearing in mind all of the semiotic resources, we articulate and overview the students' reasoning steps that intertwine with gestures, artefact use and verbal signs. In summary, by gesturing, they first describe the physical actions of the transformations. However,

Table 6 Summary of synchronic semiotic resources regarding the composition of reflections

Verbal signs	Gestures	Artefact- Tool use	Written Signs or Screenshot
181 Iris: [formulates] *x values go to y, but y values go to minus x... How can this be?... at the moment you mentioned a two-step reflection...*	No Gesture	-dragging -paper and pencil	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ $f: (x,y) \to (y,-x)$
182 Hazel: *Ha... This is reflected according to the origin at first* [gestures]*, then reflected again according to the y-axis? Right?*		-dragging -paper and pencil	
183 Iris: *Yes, there is a two-step reflection here, but not according to the origin. At first, could it reflect y equals the x line?* [gestures] *Could you try it?*		-paper and pencil	
184 Hazel: [draws the *y=x* line on the Graphics window, then reflects a certain part of the figure; at first according to the *y=x* line and then reflects again according to the *x*-axis] *Hah overlapped...*	No Gesture	-dragging -paper and pencil -reflect tool	
207 Hazel: *I understand, at first, it is a reflection according to the y=x line* [gestures with pencil] *matrix and... The second matrix is in the reflection according to the x-axis...*[writes]		-dragging -paper and pencil -reflect tool	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y \\ x \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$
215 Iris: *Ha... When I apply two or more reflections, in fact, two or more matrix transformations, I can formulate them as the production of matrices...* [showing] *it gives me our first formulation* [showing]... *ha, like composite functions... and also the order is important...*	No Gesture	-dragging -paper and pencil	
222 Hazel: *... this is like composite functions, I have tried and employed two-step reflections... In fact, if we have n-time matrix transformations, then it will be productions of the n matrices, but the order is important. This is quite interesting...*	No Gesture	-dragging -paper and pencil	

Hazel's actions in using tools and functions open a door to new phenomenology: conjecture and testing. When they are faced with new situations, they refer to gestures, and these gestures help them both to establish and test new conjectures. Figure 6 summarises how student thinking evolves over time.

Figure 6 shows that, regarding the composition of reflections, the students immediately transfer their initial findings to new cases with negative-valued sliders. They first focus on reflections according to the $y = -x$ line and reflections according to the axes

through the teacher's guiding role. When the students focus on the case of composition of the reflections, first, they manage to express the transformation algebraically, but they struggle to find associated geometry. Then, they refer to a two-step reflection as matrix transformations, which helps students to construct a mathematical link between composite functions and composition of the reflections. Their phenomenological experiences, under the teacher's guidance, turn to new mathematical situations, such as matrix transformations as the production of matrices, composition of reflections and conjecture with n-time matrix transformations.

Conclusions and Discussion

In this work, we focus on two interrelated research questions: Which multimodal resources emerge while students solve the GLT task in a DGE and which reasoning steps do students follow while reinventing GLT? Regarding the first question, we briefly address a number of interrelated and non-linear processes: artefact/tool use; a focus on the algebraic view of (matrix) transformation; gestures describing the geometric situation; establishing and testing conjectures with digital artefacts/tools; and reinventing the geometric and algebraic relationships between geometric linear transformations and functions. Regarding the second question, the tools and functions of DGE provide a context, where the students reason on the link between given sliders' values and matrix entries. The context also helps students arrive at the specific matrices of a number of GLT, like unit and zero transformations, projections onto axes and reflection according to origin and to the $y = x$ line, even if one student thinks that no link exists between reflections and matrices at the beginning of the discussion. Finally, the students extend their knowledge, not only by reinventing a number of key reflections and projections onto axes and their representation as matrix transformations, but also by relating the composition of functions with the composition of reflections and their matrix transformation as a multiplication of matrices. In other words, in the end, the students move from a DGE context to a combined view of the notions of geometric matrix transformation and function. This could be considered as a consolidation for the link between the notion of matrix (linear) transformation and the notion of function, where students have issues connecting them (Zandieh et al., 2017). In summary, the conclusions here address how a DGE context can be exploited to move between different lines and columns of Table 1, as earlier hypothesised.

The conclusions can be discussed regarding a number of main points; firstly, the setting, and secondly, the role of teacher. Regarding the setting, the students' starting point for moving to mathematical generalisations was largely based on their experience
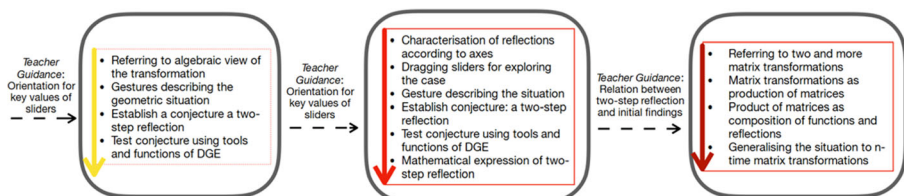

Fig. 6 The students' reasoning steps and associated signs with the teacher's guidance regarding the composition of reflections and projections onto axes

in the matrix representation of geometric transformations in $\mathbb{R}^2$, and also student knowledge in the use of the DGE (particularly experience in dragging, slider and reflect tools). Therefore, in order to consider the designed task in a classroom setting, a combination of such skills could be needed. However, as recommended by the Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay & Porter, 1993), utilisation computers may contribute to student learning in linear algebra and, therefore, a computer laboratory application for the designed task could be elaborated in further research studies to extend and discuss the results limited to a pair of students that cannot be generalised. The second main point, as a limitation, the role of the teacher, is crucial in the present paper. For this pair of students, it seems to have been easy, and the students' learning followed a number of clear steps. However, in a classroom setting, such a role of the teacher could be challenging and may need further strategies.

The Function of APC Space Paradigm

The processes above are possible through the tools and functions of GeoGebra and the teacher's guidance. Following an APC space paradigm, we focus on all of the semiotic resources from tool use, gestures and to verbal and mathematical expressions that are attached to student thinking. Such a way of approaching the data provides a detailed understanding and allows us to see how gestures convey to geometric thinking and how they emerge when an individual tries to describe his/her mental images (Alibali, 2005; Gol Tabaghi & Sinclair, 2013; Ng & Sinclair, 2013). In our study, the students even refer to the algebraic properties of the transformations at first, and they mostly gesture when they are faced with a new reflection situation and to describe associated geometric actions. A strong link exists between the gestures and the movements of the figures and capturing the reflection. Air gestures, along with pencil and tracing on the screen, build a shared environment for their descriptions associated transformation. In this way, in the APC space, the students exchange their ideas and they manage to arrive at mathematical conjectures. Thereafter, they mostly refer to verbal and written mathematical expressions.

Regarding the students' shared environment, one student always uses the computer, most likely due to her experience in GeoGebra. In the light of the produced gestures, the student uses the reflect tool of GeoGebra (even when not expected), and such tool use carries the students' discussion to new experiences, such as testing their conjectures and/or referring to mathematical formulas. We can see how tool use, in other words, how the instrumentation of the tools is intertwined with the emergence of embodied resources, as recently addressed by Drijvers (2019).

The Role of the Design Heuristics of RME

Through the design the heuristics of RME, the designed DGE creates where students can make their own mathematical explorations and conjecturing. The students move from a DGE context (i.e. model of) to their own models (i.e. model for) by generalising the link between matrix transformations and functions, which is possible under the teacher's guiding role. Another point is that the task steps are always with the students, but they always think that they have finished characterising all possible situations, where the teacher always needs to intervene to guide. The present work replicates how a teacher's

plan (like Table 2) and his/her careful orientations with provocative questions enable students' reinvention of key notions in linear algebra, as appears in the literature (Andrews-Larson et al., 2017). This is because the teacher is aware of the bridge between the potentiality of the tools and functions and the students' existing knowledge, orienting them to follow his didactical aim. In the present case, where the teacher even tries on several occasions to move the dialogue to different values of sliders, the students always focus on the values 0, 1 and −1. They never refer to, for instance, $m = 2$ and $n = -3$. This is an indicator to re-elaborate the task by adding extra steps.

Tool Use and the Synergy Between RME and APC Space

The coordination between an RME perspective and APC space provides us with dialectics between the task design and semiotic resources that are produced by digital tools. The use of a number of tools and functions of GeoGebra evokes the emergence of progressive mathematical thinking. On the one hand, this confirms the role and importance of dynamic representations in meaning-making for the geometric feature of linear algebra, which is in line with existing research results (Oktaç, 2018; Romero Félix & Oktaç, 2015; Turgut, 2018, 2019). On the other hand, the design heuristics of RME create a meaningful working environment. For instance, the slider tool is the most effective and critical tool in the present case, providing dynamic variation, and, in this way, the students explore different cases, establish several conjectures and validate them, which is in line with emergent modelling heuristic. It should be noted here that the ApplyMatrix construction tool works silently with the slider tool, because the four sliders are defined as coefficients of the associated matrix and all the matrix transformations are possible through the ApplyMatrix construction tool, which provides an extensive and dynamic environment for learning specific geometric linear transformations as appears in the literature (Turgut, 2018, 2019). However, the dragging, grid function and checkbox tool are referred to, creating a meaningful context for the point of departure: didactic phenomenology. Using a 'reflection' context works well and acts in producing all semiotic resources attached to student thinking.

Limitations and Further Research

There are a few points to be mentioned as contextual limitations. In the present work, the focus is on the geometric linear transformations from matrix transformation to reflections, and projections and composition of reflections in $\mathbb{R}^2$, which can be considered as a 'piece' of the notion of linear transformation. Students manage to reinvent certain key notions, but they do not discuss (and/or ask the teacher) the cases in $\mathbb{R}^n$, $n \geq 3$. Therefore, more elaboration could be included according to the audience level.

Regarding implementation of the proposed task, a contrary view could be included, that is, providing reflections, projections and composition of reflections without the Algebra window (without matrix transformation information), and asking to search for matrix transformations in a DGE context could be a subsequent research problem. Moreover, it is possible to define a matrix transformation in an $\mathbb{R}^3$ context (Turgut, 2018) that connects the Algebra and 3D Graphics windows, where such an extensive context could be a key point for designing tasks to reinvent reflections according to planes. Projections onto planes could also be explored in future research settings.

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