Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Natural Sciences
Department of Physics

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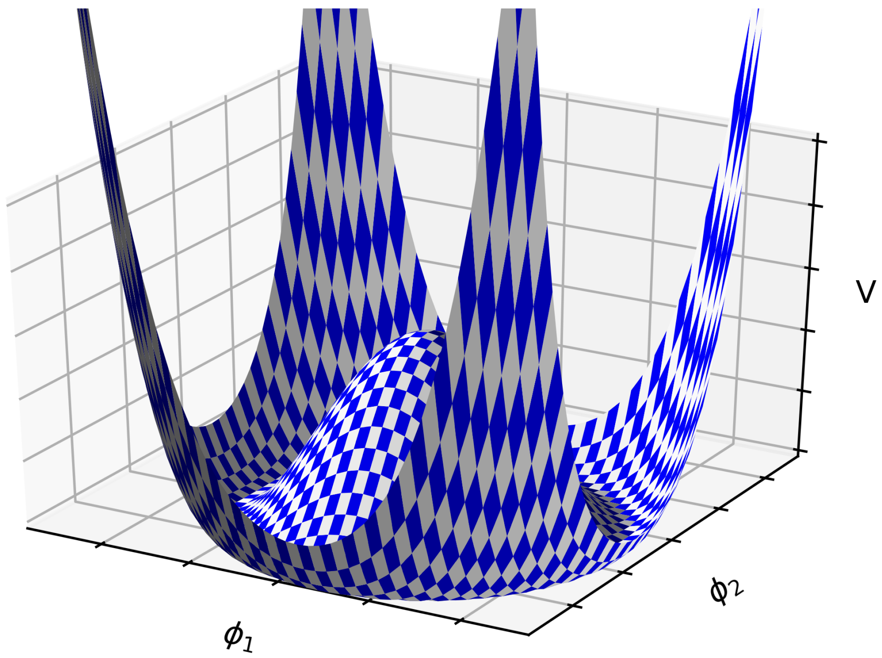
A field theoretical study of particles in an external magnetic field

Setting up the foundations to study QCD in a strong magnetic field

Master's thesis in Physics
Supervisor: Jens Oluf Andersen
May 2021



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Acknowledgments

First of all, I would like to thank my supervisor professor Jens Oluf Andersen for his excellent supervision throughout my two semesters of working on this thesis. I am grateful for his thorough feedback on my work and the enlightening discussions we shared.

I would also like to thank professor Michael Kachelriess for the valuable lessons he provided me in quantum field theory and particle physics. He provided me with some important foundations that transitioned to my thesis.

I thank Alexander Refset Krumsvik and Martin Spillum Grønli for important collaborations in both particle physics and quantum field theory which was a source of inspiration towards pursuing my degree in the field of astroparticle physics.

Finally, I would like to thank my family for providing me with their love and support throughout the entirety of my work. For that, I am forever grateful.

Abstract

In this thesis we study the interaction of different types of particles with a constant magnetic field. The purpose of this is to gather a basic understanding and develop fundamental tools in order to explore QCD in a strong magnetic field. We thus start with exploring free theories of both spin-0 and spin-1/2 particles. We use the imaginary time formalism to calculate the partition function of the respective theory which is expressed as a path integral. Using this expression for the partition function, we can calculate any thermodynamic quantity such as the energy density of the respective system. We see that the energy density has both a vacuum contribution and a finite-temperature contribution. In this thesis we only focus on calculating the vacuum contribution to the energy density, namely when temperature is zero.

Next, we investigate the interaction of a spin-0 particle and a spin-1/2 particle with a constant magnetic field. We can find the field equations of the respective theories and represent the wavefunctions as the Landau eigenfunctions of the corresponding Landau energy levels. Having found the wavefuntions, we can use them to calculate the propagator for the respective theory using the Schwinger proper time formalism. Finally, we use these propagators to calculate the vacuum energy density of the field theories. As we see, the vacuum energy density is divergent which we regularize using both dimensional regularization and a cutoff scheme. Finally, we renormalize our result to obtain a physically reasonable expression.

Having investigated free theories, we then proceed with interacting theories using the ϕ^4 model. We explore spontaneous symmetry breaking of systems with quartic interactions. We focus here only on scalar particles, find a generalization to N real scalars using the linear sigma model, and specialize to the case of two complex scalar fields, which is equivalent to a theory of four real scalars, namely $N = 4$. Proceeding, we calculate the effective potential of such systems which corresponds to the vacuum energy density of the system. Again, we obtain divergent results. We regularize the results using dimensional regularization and the $\overline{\text{MS}}$ renormalization scheme.

Contents

Conventions and notation

Units

We use natural units where $\hbar = c = k_B = 1$ unless stated otherwise in the text.

Metric

We have defined the Minkowski metric as

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{0.1}$$

We can then define two four-vectors in Minkowski space $a^\mu = (a^0, \boldsymbol{a})$ and $b^\mu = (b^0, \boldsymbol{b})$ where a^0 is the time component of the four-vector and $\boldsymbol{a}$ represents its three spatial components. The same applies for b^μ. Using the Minkowski metric, the scalar product of two four-vectors can be defined as

$$a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^i b^i. \tag{0.2}$$

Finally, we note that repeated Latin indices follows Einsteins summation convention and is simply summed over.

Gamma matrices and Clifford algebra

We have introduced the gamma matrices, γ^μ, in their Dirac representation, namely

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{0.3}$$

and

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{0.4}$$

where $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$ and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the respective Pauli-spin matrices. It is therefore important to note that each element in Eq. (0.3) and (0.4) represents a 2×2-matrix and γ^μ itself is a 4×4-matrix. The gamma matrices satisfy the Clifford algebra, namely

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{0.5}$$

1 Introduction

1.1 A brief description of QCD and the Standard Model of particle physics

Quantum chromodynamics (QCD) is the quantum field theory which describes the strong interactions between the family of elementary particles known as quarks [1]. This is one of three types of interaction which together form what is known as the Standard Model of particle physics. The other two types of interactions are electromagnetic interactions which are described by quantum electrodynamics (QED) and weak interactions. The weak interactions are described through the unified theory of electromagnetic interactions and weak interactions known as the Glashow-Weinberg-Salam (GWS) theory. In 1979 Glashow, Weinberg, and Salam received the Nobel prize for their contribution to create this theory [2]. The Standard Model thus describes the interaction between all the different types of elementary particles which have been discovered through experiments performed in particle accelerators. Of course there are limitations to the standard model as it for instance does not consider gravitational forces. However it is so far the best model that has been constructed as it has had large success in providing us with correct experimental predictions and is theoretically self-consistent [3]. It is convenient to group the elementary particles in the Standard Model according to the types of interactions which they partake in and organize them in separate generations according to the masses of the particles, Table. 1.

	Fermion generations			**Gauge bosons**	**Scalar bosons**
	I	**II**	**III**		
Quarks	u up	c charm	t top	g gluon	H Higgs
	d down	s strange	b bottom	γ photon	
Leptons	e electron	μ muon	τ tau	Z^0 Z boson	
	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	$W^\pm$ W boson	

Table 1: Elementary particles of the Standard Model

As we can see in Table. 1 the Standard Model contains three fundamentally different classes of particles, namely fermions, gauge bosons, and the Higgs boson. Fermions are spin-1/2 particles and the building blocks for all the current observable matter in the universe. As we move up the generations of fermions, the particle masses increase relative to the mass of the particle in the previous generation of the respective family. For instance, electrons which are first generation leptons have less mass than the muons which are second generation leptons. Furthermore, the six different particles in each family are distinguished by a property known as flavor. The gauge bosons are known as the force carriers of the Standard Model since they help mediate the different types of forces between the particles. Finally, the Higgs boson, or the Higgs field participates in a phenomenon known as the Higgs mechanism which allows for some fundamental particles, such as the fermions, to acquire mass. Not all the elementary particles can participate in all types of interactions. For instance, the neutrinos in the family of leptons can only interact through the weak forces. All the other leptons, namely the electron, muon, and tau can interact both through the weak and the electromagnetic forces. Finally, the family of fermions known as quarks are able to interact through all the different types of forces, in particular the strong force which the leptons are excluded from.

Despite being two completely different quantum field theories, the theory of QCD for strong interactions was made as an analogy to QED of electromagnetism. To see how this manifested itself, we must go back in time to the year 1911 when Ernest Rutherford [4] discovered the atomic nucleus which at the time he believed only consisted of protons. Of course, physicists at the time were familiar with the electromagnetic forces. Thus they suspected that there must exist another force that holds together the positively charged protons in the nucleus, which should have otherwise repelled one another due to them having the same charge. In 1932 James Chadwick [5] discovered that the nucleus also contains neutrons and shortly after, it was concluded in the physics community that the electromagnetic forces are not responsible for holding together the nucleus. Instead they theorized that there are two types of nuclear forces, which we now know as the strong and the weak force, which are responsible for holding the nucleus intact. However, no appropriate theory for such interactions was suggested until Yukawa's discovery of the meson in 1935 [6] which he believed was the virtual particle which transmitted the nuclear forces. As a result of this discovery, many new particles such as the pion was discovered. In 1964, Gell-Mann [7] and Zweig [8] made further improvements to Yukawas theory by suggesting that the mesons which Yukawa had previously discovered were made up of even smaller particles known as quarks. Following this discovery in the same year, it was shown by Greenberg [9] and Nambu [10][11] that quarks must be distinguished by something other than spin, mass and charge since there exists for instance a composite structure of quarks with the same flavor which would disobey the Pauli exclusion principle. This new property was called color of which there existed three states namely red, green, and blue with the respective anti-matter states namely anti-red, anti-green, and anti-blue. Quarks can only interact with one another in such a way that the final composite structure has zero net-color. Furthermore, the force mediator of the strong interactions was called the gluon. This was made analogous to the photon of QED. Whereas the photon mediates the interaction between particles which carry electric charge, the gluon mediates the interaction between particles which carry the charge of the strong interactions, namely the different color states. Consequently, only particles which holds this type of charge can participate in the strong interactions. Finally, in the early 1970's the quantum field theory of strong interactions, namely QCD, was deviced by Gell-Mann, Fritzsch, and Leutwyler [12]. This is the most accurate theory for strong interactions we have so far.

One of the main reasons why it took so long to discover quarks and come up with the theory of QCD was due to the fact that quarks cannot be observed freely in nature at the energies which we are observing them at. More correctly, only color singlet states can exist as free particles, which is to say that the net color charge of the particle must be zero. This means that all hadrons which have so far been observed must be color singlets or "colorless". This phenomenon is known as confinement. Another important property of QCD is asymptotic freedom. Asymptotic freedom is a property of certain gauge theories which results in the interactions between the particles to become weaker as the energy scale increases. Consequently, at low energies, the interactions are very strong. In 1973 Gross and Wilzcek [13] and in the same year Politzer [14] managed to show that QCD indeed possesses this property. Thus, at high energies resembling those of the early universe or some highly dense neutron stars, the confined structure of quarks known as hadrons undergo a phase transition and exist in a deconfined phase known as quark matter. In the case of the early universe, when both temperature and energy density was very high, the type of quark matter was quark-gluon plasma. On the other hand, in highly dense neutron stars where temperatures are not even close to that of the early universe (T much less than 10^{12}K), the quark matter is expected to be a color superconductor, which is a degenerate Fermi gas of quarks with a condensate of Cooper pairs near the Fermi surface that induces color Meissner effects [15].

The theory of confinement is however still incomplete since at the energy scales where confinement occurs, namely lower than approximately 1GeV, perturbation theory breaks down [16]. Of course, attempts have been made to study QCD at such energy scales using computational methods called lattice QCD, however there are limitations on the accuracy of such methods. In many of the systems where deconfinement occurs, and even at lower energy scales, there exists strong magnetic fields which the particles are affected by. Thus, it would be useful to understand QCD in a strong magnetic field and explore different models that considers such systems. We will look at some examples of such systems in the next section.

1.2 Applications of QCD in a strong magnetic field

In the field of high-energy physics, there are at least three areas where strong magnetic fields play an important role. This includes noncentral heavy-ion collisions, compact stars, and the early universe [17]. We shall look more closely at one of these examples, namely compact stars. There is a certain type of neutron star known as a magnetar which is different than a regular neutron star from the fact that it has a much larger magnetic field and a lower rotation frequency [18]. The magnetic field strength on the surface of such a star is believed to be around $10^{14} - 10^{15}$G while in the center of the star, it can go up to $10^{16} - 10^{19}$G. This is due to fact that the magnetic field strength is expected to increase as the density increases. Similar to other neutron stars, a magnetar is only 20 km in diameter and is more massive than the sun. To understand this mass-radius relation we must therefore have a complete understanding of the equation of state for systems with highly interacting particles in a magnetic field with field strength matching those of the magnetar. However, in order to find the different thermodynamic quantities such as the energy density of such a system, we must first have an appropriate model which describes it. There are many models that have been constructed so far to describe QCD in a strong magnetic field. These are known as low-energy effective models. To understand these models however, it is crucial to have a solid grasp of how to calculate the partition function of different systems consisting of different particles, how to introduce an external magnetic field to a system of such particles, and finally how the different particles interact in this magnetic field. More importantly, one must know how to calculate the energy density of such systems as it explains the physically observable features such as, for instance, the mass-radius relation of a neutron star.

1.3 Layout of thesis

In this work, we have developed the necessary tools in order to study the low-energy effective models of QCD in a strong magnetic field. In Section. 2 we have used the imaginary time formalism to first calculate the partition function in quantum mechanics. These calculations help us later on in this section when we calculate the partition function for both spin-0 and spin-1/2 particles. In the case of spin-1/2 particles, the fields are represented as Grassmann variables with their respective Grassman algebra. An explanation of such algebra is provided in Appendix. A. Using the partition function, we then calculate the energy density of the different systems. Furthermore, we show how to perform dimensional regularization on the vacuum energy density of a neutral scalar field in Appendix. E. In Section 3 we consider spin-0 and spin-1/2 particles in a constant magnetic field. We start by introducing a gauge field to the Lagrangian of a spin-0 and a spin-1/2 particle respectively by adding the Maxwell Lagrangian and requiring local gauge invariance. Next, we specialize to the case of a constant magnetic field and calculate the field equations which gives us the Klein-Gordon equation and the Dirac equation coupled to a constant magnetic field, for the spin-0 and spin-1/2 particles respectively. We choose the Landau gauge and find the wavefunctions which are the Landau eigenfunctions of the corresponding Landau energy levels. Furthermore, it proves useful to know the wavefunctions and energy levels of the harmonic oscillator which we calculate in Appendix. B. The propagator for a spin-0 particle in a constant magnetic field is calculated in Appendix. C using the respective wavefunctions. Using the propagator in the coincidence limit, we then calculate the vacuum energy density of such a system. The divergent results which we obtain are both regularized using dimensional regularization and momentum cutoff scheme and renormalized in Appendix. D. Finally, in Section. 4, we explore the theory of interacting scalar fields using the ϕ^4 model. We explore the linear sigma model which generalizes to interacting theories of N real scalars and find the effective potential which is equivalent to the vacuum energy density of the system. We have showed the derivation for the effective potential of a single scalar field in Appendix. F. This is easily generalized to N real scalars in Section. 4. Furthermore, we use the same methods of dimensional regularization as in Appendix. D to regularize our result and further use the $\overline{\text{MS}}$ renormalization scheme. Finally, we specialize the linear sigma model to $N = 4$ real scalars.

2 Quantum field theory at finite temperature

When it comes to quantizing a field theory, there are two ways to accomplish this. The first way is using canonical quantization, a method which strictly follows a quantum mechanical formalism. The second method is one which we will closely examine and was first introduced by Richard Feynman, namely the path-integral formalism. We start by first looking at the path-integral representation of the propagator in quantum mechanics, which is nothing more than that of a harmonic oscillator. Next, we use the imaginary-time formalism on the propagator to find an expression for the partition function, from which all thermodynamic quantities of a theory can be obtained. We then recycle some of these procedures towards getting a general path-integral representation of the propagator and thus the partition function in quantum field theory. Once we have accomplished this, we can specify the results to bosons and fermions. Due to the periodic operators of bosons and anti-periodic operators of spin-1/2 fermions, there will be some minor differences in our calculations. However, the overall approach is very similar as for the case of quantum mechanics.

2.1 Path-integral representation of the propagator in quantum mechanics

The work in this section is inspired from [19]. A quantum-mechanical system has properties which can be determined through its Hamiltonian, which in the case of a non-relativistic scalar particle (spin-0) in one dimension takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = \hat{T} + \hat{V}. \tag{2.1}$$

Here, m represents the mass of the particle and $\hat{p}$ is the momentum operator acting in one dimension. The Hamiltonian describes the dynamics of a system where the state $|\psi\rangle$ is governed by the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle. \tag{2.2}$$

The Schrödinger equation can be solved using the time-evolution operator $\hat{U}(t;t_0)$, which satisfies the equation

$$|\psi(t)\rangle = \hat{U}(t;t_0)|\psi(t_0)\rangle. \tag{2.3}$$

If we now act to the left on each side of (2.3) with a bra vector in position basis, we obtain

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|\hat{U}(t;t_0)|\psi(t_0)\rangle. \tag{2.4}$$

It is now useful to recall the completeness relation in position basis,

$$\int dx\,|x\rangle\langle x| = 1, \tag{2.5}$$

and also the completeness relation in momentum basis

$$\int \frac{dp}{B}|p\rangle\langle p| = 1. \tag{2.6}$$

where B is some constant. It will also be useful to recall some basic identities of quantum mechanics namely

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle = -i\hbar\partial_x\langle x|p\rangle. \tag{2.7}$$

This partial differential equation can be solved giving

$$\langle x|p\rangle = Ae^{\frac{ipx}{\hbar}}, \tag{2.8}$$

where A is another constant. We have included $\hbar$ in what follows for the sake of understanding its origin in the calculations. This will however be dropped in other sections as we implement natural units. We can now find A and B by noting that

$$
\begin{aligned}
1 &= \int dx \int \frac{dp}{B} \int \frac{dp'}{B} \left|p\right\rangle \left\langle p|x\right\rangle \left\langle x|p'\right\rangle \left\langle p'\right| \\
&= \int dx \int \frac{dp}{B} \int \frac{dp'}{B} \left|p\right\rangle |A|^2 e^{\frac{i(p'-p)x}{\hbar}} \left\langle p'\right| \\
&= \int \frac{dp}{B} \int \frac{dp'}{B} \left|p\right\rangle |A|^2 2\pi\hbar\delta(p'-p) \left\langle p'\right| \\
&= \frac{2\pi\hbar|A|^2}{B} \int \frac{dp}{B} \left|p\right\rangle \left\langle p\right| = \frac{2\pi\hbar|A|^2}{B}.
\end{aligned}
\tag{2.9}
$$

Thus, A and B are dependent on one another which means we can pick one of the constants arbitrarily and obtain the value for the other. If we for simplicity set $B = 1$, then we have that $A = \frac{1}{\sqrt{2\pi\hbar}}$. This will become handy in future calculations.

We are now ready to apply Eq. (2.5) to Eq. (2.4) to obtain

$$\psi(x,t) = \int dx_0 \left\langle x\right| \hat{U}(t;t_0) \left|x_0\right\rangle \left\langle x_0|\psi(t_0)\right\rangle. \tag{2.10}$$

We define the propagator

$$K(x,t;x_0,t_0) = \left\langle x\right| \hat{U}(t;t_0) \left|x_0\right\rangle, \tag{2.11}$$

which is simply the matrix elements of the time-evolution operator in position basis, also known as transition amplitudes. Substituting Eq. (2.11) into Eq. (2.10) we obtain the integral representation of $\psi(x,t)$ with respect to the propagator, namely

$$\psi(x,t) = \int dx_0 K(x,t;x_0,t_0)\psi(x_0,t_0). \tag{2.12}$$

Thus, we see that the knowledge of the propagator allows us to calculate the state of the system $\psi(x,t)$ given that the initial state of the system $\psi(x_0,t_0)$ is known.

We now turn our focus to the time-evolution operator which for a system with a time-independent Hamiltonian takes the form

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}(\hat{T}+\hat{V})t}. \tag{2.13}$$

It is important to note that since $\hat{T}$ and $\hat{V}$ do not commute in (2.13), we cannot simply express it as $e^{-\frac{i}{\hbar}\hat{T}t}e^{-\frac{i}{\hbar}\hat{V}t}$. That is unless we split the total time interval into small increments which yields the correct result to an approximation. Therefore, we take the total time interval $[0,t_f]$ and divide it into N smaller intervals of width $\epsilon = \frac{t_f}{N}$ where t_f is the final time after N small time intervals. We can then utilize the property of the time-evolution operator, namely

$$\hat{U}(t_f;0) = \hat{U}(t_f;t_f-\epsilon)\hat{U}(t_f-\epsilon;t_f-2\epsilon)...\hat{U}(t_f-(N-1)\epsilon;0). \tag{2.14}$$

The time-evolution operator from $t = 0$ to a later time $t = t_f$ can therefore be written as

$$\hat{U}(t_f;0) = [\hat{U}(\epsilon)]^N. \tag{2.15}$$

Next, we can Taylor expand $U(\epsilon)$ about $\epsilon = 0$ to first order in ϵ

$$\hat{U}(\epsilon) = 1 - \frac{i\epsilon}{\hbar}(\hat{T}+\hat{V}) + \mathcal{O}(\epsilon^2). \tag{2.16}$$

Furthermore, we can expand the individual exponential factors in $\hat{U}(t)$ to obtain

$$e^{-\frac{i}{\hbar}\hat{T}\epsilon}e^{-\frac{i}{\hbar}\hat{V}\epsilon} = [1 - \frac{i\epsilon}{\hbar}\hat{T} + \mathcal{O}(\epsilon^2)][1 - \frac{i\epsilon}{\hbar}\hat{V} + \mathcal{O}(\epsilon^2)] = 1 - \frac{i\epsilon}{\hbar}(\hat{T} + \hat{V}) + \mathcal{O}(\epsilon^2), \tag{2.17}$$

where we immediately see that

$$\hat{U}(\epsilon) = e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}} + \mathcal{O}(\epsilon^2). \tag{2.18}$$

Inserting this into Eq. (2.15) gives

$$\hat{U}(t_f;0) = \left[e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}} + \mathcal{O}(\epsilon^2)\right]^N = \left[e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}\right]^N + \mathcal{O}(\epsilon^2). \tag{2.19}$$

Since we are working in the limit where $\epsilon \to 0$, we also have that $N \to \infty$. Therefore, the transition amplitude can be written as

$$\langle x|\hat{U}(t_f;0)|x_0\rangle = \lim_{N\to\infty} \langle x|\left[e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}\right]^N|x_0\rangle. \tag{2.20}$$

We can now apply a trick where we insert the completeness relation previously defined in Eq. (2.5) between the N exponential factors in Eq. (2.20) giving us

$$\langle x|\hat{U}(t_f;0)|x_0\rangle = \lim_{N\to\infty}\int dx_1...dx_{N-1}\langle x_N|e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}|x_{N-1}\rangle\langle x_{N-1}|...|x_1\rangle\langle x_1|e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}|x_0\rangle. \tag{2.21}$$

We have renamed the variable x such that $x \equiv x_N$ for clarity. Next, we pick one of the transition amplitude factors and apply the completeness relation in momentum basis Eq. (2.6) between the exponential factors. This gives us

$$\langle x_{j+1}|e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}|x_j\rangle = \int dp\,\langle x_{j+1}|e^{-\frac{i\epsilon\hat{p}^2}{2m\hbar}}|p\rangle\langle p|e^{-\frac{i\epsilon V(\hat{x})}{\hbar}}|x_j\rangle = \int dp\,\langle x_{j+1}|p\rangle e^{-\frac{i\epsilon p^2}{2m\hbar}}\langle p|x_j\rangle e^{-\frac{i\epsilon V(x_j)}{\hbar}}. \tag{2.22}$$

Recycling our result from Eq. (2.8), we obtain

$$\langle x_{j+1}|e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}|x_j\rangle = \int\frac{dp}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\left[-\frac{\epsilon p^2}{2m} + p(x_{j+1} - x_j) - \epsilon V(x_j)\right]\right\}, \tag{2.23}$$

which is a Gaussian integral that can be solved using the formula

$$\int_{-\infty}^{\infty} dp e^{-(ap^2+2bp+c)} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2-ac}{a}}. \tag{2.24}$$

In our case, $a = \frac{i\epsilon}{2m\hbar}$ and $b = \frac{-i(x_{j+1}-x_j)}{2\hbar}$ which means Eq. (2.23) becomes

$$\langle x_{j+1}|e^{-\frac{i\epsilon}{\hbar}\hat{T}}e^{-\frac{i\epsilon}{\hbar}\hat{V}}|x_j\rangle = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\epsilon}(x_{j+1} - x_j)^2 - \epsilon V(x_j)\right]\right\}. \tag{2.25}$$

Having evaluated one of the matrix element factors, the rest is trivial, thus Eq. (2.21) becomes

$$\langle x|\hat{U}(t_f)|x_0\rangle = \lim_{N\to\infty}\left(\frac{m}{2\pi\hbar i\epsilon}\right)^{N/2}\int dx_1...dx_{N-1}\exp\left\{\frac{i\epsilon}{\hbar}\sum_{j=0}^{N-1}\left[\frac{1}{2}m\frac{(x_{j+1} - x_j)^2}{\epsilon^2} - V(x_j)\right]\right\}. \tag{2.26}$$

We have thus obtained a discretized representation for the path integral in non-relativistic quantum mechanics. Before proceeding, let us first understand the physical meaning behind the integral in Eq.

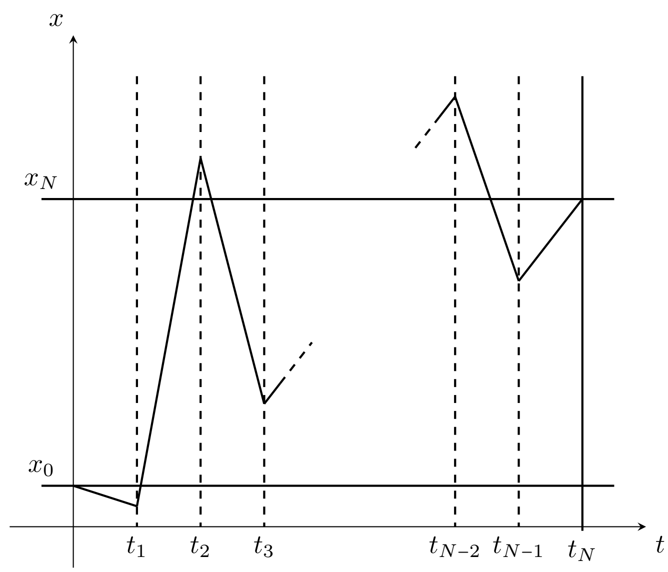

Figure 2.1: A spacetime diagram representing one of the infinitely many discretized paths that exist in the propagator in Eq. (2.26). The initial point (t_0, x_0) and final point (t_N, x_N) are always fixed while the intermediate points (aligned on the dotted lines) can take on any value from $-\infty$ to ∞.

(2.26) which is also presented pictorially in Figure. 2.1. We first turn our focus to the points $(x_0, ..., x_N)$. These points are considered relative to a path $x(t)$. They must cross the initial point (x_0) and final point (x_N) which are fixed for all the different paths. Furthermore, the path consists of straight lines between each point thus crossing each point x_j at time $t_j = j\epsilon$. Another way to view this is in reference with Eq. (2.21), where we see that the path integral is simply the integral over the product of transition amplitudes starting from point x_0 and finishing at x_N. Thus, it represents the transition from one point to the next with the square of each transition amplitude (e.g. for coordinate-point $j+1$) representing the probability that the system is at position x_{j+1} at time t_{j+1} given that the system was at position x_j at time t_j at the immediate step before. The product of these probabilities thus represents the probability that a particle ends up at position x_N at time t_f given that it starts at position x_0 at time t_0 and has passed through the points $(x_1, ..., x_{N-1})$ at corresponding times $(t_1, ..., t_{N-1})$. Finally, we note that as the number of points in configuration space approaches infinity, i.e $N \to \infty$, the path $x(t)$ will have crossed the points in our set $(x_0, ..., x_N)$ at all possible times between t_0 and t_f. In other words, we have integrated over all possible paths in configuration space connecting the starting point x_0 to end-point x_N. The path integral is therefore the sum over all possible paths connecting the points x_0 and x_N weighted by the transition amplitudes of the respective paths.

We will now proceed by simplifying the expression we have obtained for the propagator by first setting $\Delta t = \epsilon$ and $\Delta x_j = x_{j+1} - x_j$ thus transforming the exponent of Eq. (2.26) to

$$\exp\left\{\frac{i\epsilon}{\hbar}\sum_{j=0}^{N-1}\left[\frac{1}{2}m\frac{(x_{j+1}-x_j)^2}{\epsilon^2} - V(x_j)\right]\right\} \to \exp\left\{\frac{i\Delta t}{\hbar}\sum_{j=0}^{N-1}\left[\frac{1}{2}m\left(\frac{\Delta x_j}{\Delta t}\right)^2 - V(x_j)\right]\right\}. \tag{2.27}$$

We can see that in the limit where $N \to \infty$, the terms in the exponent of Eq. (2.27) resemble the classical Hamiltonian action, namely

$$\frac{i}{\hbar}S[x(t)] = \frac{i}{\hbar}\int_0^{t_f} dt\left[\frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2 - V(x(t))\right] = \frac{i}{\hbar}\int_0^{t_f} dt L(x(t), \dot{x}(t)), \tag{2.28}$$

where we have defined L as the Lagrangian in quantum mechanics given by

$$L(x(t), \dot{x}(t)) = \frac{1}{2}m\dot{x}^2(t) - V(x(t)). \tag{2.29}$$

We have therefore obtained the compact form for the propagator, namely

$$\langle x|\hat{U}(t)|x_0\rangle = C\int \mathcal{D}x(t)\exp\left\{\frac{i}{\hbar}\int_0^{t_f} dt L(x(t), \dot{x}(t))\right\}. \tag{2.30}$$

where we have defined C as the normalization constant which also appears in Eq. (2.26) and $\mathcal{D}x(t)$ is a simplified notation denoting the fact that we are integrating over all possible paths in configuration space.

2.2 Partition function in quantum mechanics

We will first find a path-integral representation for the partition function in quantum mechanics and then extend it also to scalar fields in quantum field theory. Some of the calculations in this section follows those of [19]. To start, we will assume that the particles in our system are of bosonic nature. This means that their respective operators obey the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \tag{2.31}$$

where a and $a^\dagger$ are annihilation and creation operators respectively. We now employ the grand canonical ensemble, where the partition function $\mathcal{Z}$ is a function of T. The partition function is given by

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta\hat{H}}] = \int dx\,\langle x|e^{-\beta\hat{H}}|x\rangle. \tag{2.32}$$

We can quickly see that the partition function is very similar to the propagator we defined in Eq. (2.11). In order to obtain the path-integral representation of the partition function in quantum mechanics, we take the expression we obtained for the path-integral representation of the propagator in quantum mechanics namely Eq. (2.30), and carry out the following steps:

i. Perform a Wick rotation, introducing $\tau \equiv it$. This is known as imaginary time and it means we are going from Minkowski to Euclidean space since we are transforming the Minkowski metric to a Euclidean one through a change in sign.

ii. Introduce the Euclidean Lagrangian

$$L_E \equiv -L(\tau = it) = \frac{1}{2}m\dot{x}^2(\tau) + V(x(\tau)), \tag{2.33}$$

which follows directly from step i. as we have gone from Minkowski to Euclidean space.

iii. Perform the τ integral over the interval $(0, \beta)$. This is equivalent to how we performed the integral over the interval $(0, t_f)$ when using the time-evolution operator.

iv. Require periodicity of $x(\tau)$, i.e. $x(\beta) = x(0)$. This is due to the cyclic nature of the trace operator introduced in the definition for the partition function Eq. (2.32) and the fact that we are considering a bosonic system.

Carrying out the above steps and noting that $idt = d\tau$ we have

$$\mathcal{Z} = C \int_{x(\beta)=x(0)} \mathcal{D}x \exp\left\{-\int_0^\beta d\tau L_E\right\}, \tag{2.34}$$

where C is the same as in Eq. (2.30). Because we used imaginary time in the first step, Eq. (2.34) is commonly known as the imaginary-time formalism of the partition function. It is important to note that this procedure also applies in quantum field theory as we shall see later on.

We will now proceed with evaluating the path integral for the partition function in quantum mechanics. The main purpose of this is to familiarize ourselves with the steps necessary for such calculations which can be later extended to quantum field theory. The calculations we will carry out will be in Fourier space with respect to the time coordinate τ. We start first by representing an arbitrary function $x(\tau)$, $0 < \tau < \beta$ with periodic boundary conditions $x(\beta^-) = x(0^+)$ as a Fourier sum

$$x(\tau) \equiv T \sum_{n=-\infty}^{\infty} x_n e^{i\omega_n \tau}, \tag{2.35}$$

where we have introduced the factor T due to convention. Since we have imposed periodicity, we have that $e^{i\omega_n \beta} = 1$ which means $\omega_n \beta = 2\pi n, n \in \mathbb{Z}$. The values $\omega_n = 2\pi n T$ are known as the Matsubara frequencies with the corresponding Matsubara modes, x_n. Next, we also require that $x(\tau)$ is real namely

$$x(\tau) \in \mathbb{R} \Rightarrow x^*(\tau) = x(\tau) \Rightarrow x_n^* = x_{-n}. \tag{2.36}$$

We can further write $x_n = a_n + ib_n$ thus obtaining the following relations

$$x_n^* = a_n - ib_n = x_{-n} = a_{-n} + ib_{-n} \Longrightarrow \begin{cases} a_n = a_{-n} \\ b_n = -b_{-n} \end{cases}, \tag{2.37}$$

where it will be also useful to note that $b_0 = 0$ and $x_{-n}x_n = a_n^2 + b_n^2$. Thus we have the expression

$$x(\tau) = T\left\{a_0 + \sum_{n=1}^{\infty}\left[(a_n + ib_n)e^{i\omega_n\tau} + (a_n - ib_n)e^{-i\omega_n\tau}\right]\right\}, \tag{2.38}$$

where a_0 is known as the zeroth Matsubara mode. Using Eq. (2.35), an integral quadratic in the paths can be written as

$$\begin{aligned} \int_0^\beta d\tau x(\tau)y(\tau) &= T^2 \sum_{m,n} x_n y_m \int_0^\beta d\tau e^{i\tau(\omega_n+\omega_m)} \\ &= T^2 \sum_{m,n} x_n y_m \frac{1}{T}\delta_{n,-m} = T\sum_n x_n y_{-n} \quad , \end{aligned} \tag{2.39}$$

where we in the second to last step used the integral representation of the Kronecker delta function namely,

$$\delta_{n,-m} = \frac{1}{\beta}\int_0^\beta d\tau e^{i\tau(\omega_n+\omega_m)}. \tag{2.40}$$

Furthermore, the potential we are working with is that of a harmonic oscillator, namely

$$V(x(\tau)) = \frac{1}{2}m\omega^2 x^2(\tau). \tag{2.41}$$

Thus, we are ready to evaluate the integral in the exponent of the partition function in Eq. (2.34) namely,

$$-\int_0^\beta d\tau L_E = -\int_0^\beta d\tau \frac{1}{2}m\left[\frac{dx(\tau)}{d\tau}\frac{dx(\tau)}{d\tau} + \omega^2 x(\tau)x(\tau)\right]. \tag{2.42}$$

Using Eq. (2.39) and the fact that

$$\frac{dx(\tau)}{d\tau} = \frac{d}{d\tau}(T\sum_{n=-\infty}^{\infty} x_n e^{i\omega_n\tau}) = Ti\sum_{-\infty}^{\infty}\omega_n x_n e^{i\omega_n\tau}, \tag{2.43}$$

we can write Eq. (2.42) as

$$\begin{aligned}
-\int_0^\beta d\tau L_E &= -\frac{1}{2}mT\sum_{n=-\infty}^{\infty} x_n[i\omega_n i\omega_{-n} + \omega^2]x_{-n} \\
&= -\frac{1}{2}mT\sum_{n=-\infty}^{\infty}(a_n^2 + b_n^2)(\omega_n^2 + \omega^2) \\
&= -\frac{1}{2}mTa_0^2\omega^2 - mT\sum_{n=1}^{\infty}(a_n^2 + b_n^2)(\omega_n^2 + \omega^2),
\end{aligned} \tag{2.44}$$

where in the second last step, we used that $\omega_n = -\omega_{-n}$ and in the last step, we split the sum.

Next, we must take into consideration the integration measure $\mathcal{D}(x(\tau))$. The first step will be to make a change of variables from $x(\tau)$, $x \in (0, \beta\hbar)$ to the Fourier components a_n and b_n. We have already seen from Eq. (2.38) that the independent variables are a_0 and $\{a_n, b_n\}$, $n \geq 1$. Thus, we can write the integration measure in terms of the Fourier components using the Jacobian of the transformation, namely

$$\mathcal{D}(x(\tau)) = \left|\det\left[\frac{\partial x(\tau)}{\partial x_n}\right]\right| da_0\left[\prod_{n\geq 1} da_n db_n\right]. \tag{2.45}$$

Next, we note that since the change of basis is independent of the potential $V(x)$, we are allowed to define for simplicity

$$C' \equiv C\left|\det\left[\frac{\partial x(\tau)}{\partial x_n}\right]\right|, \tag{2.46}$$

where C' is now our unknown coefficient which remains to be determined.

We are now in position to write out the partition function in terms of its Fourier components.

$$\begin{aligned}
\mathcal{Z} = C'\int_{-\infty}^{\infty} da_0 \int_{-\infty}^{\infty}\Big[\prod_{n\geq 1} da_n db_n\Big] \\
\exp\left\{-\frac{1}{2}mT\omega^2 a_0^2 - mT\sum_{n\geq 1}^{\infty}(\omega_n^2 + \omega^2)(a_n^2 + b_n^2)\right\}.
\end{aligned} \tag{2.47}$$

We next note that the integrals above are merely Gaussian integrals of the form

$$\int_{-\infty}^{\infty} dx e^{-cx^2} = \sqrt{\frac{\pi}{c}}, \tag{2.48}$$

which means Eq. (2.47) can be evaluated to be

$$\mathcal{Z} = C'\sqrt{\frac{2\pi}{mT\omega^2}}\prod_{n=1}^{\infty}\frac{\pi}{mT(\omega_n^2+\omega^2)}, \quad \omega_n = 2\pi Tn, \tag{2.49}$$

All that is left to do now is to determine the value of C'. In order to achieve this, we must first gather an understanding of the nature of C'. First of all, we must note that C' is independent of ω so we can set $\omega = 0$ in our calculations to obtain C'. Next, we should note that in the limit $\omega \to 0$, the factor $\sqrt{\frac{2\pi}{mT\omega^2}}$ is divergent which we can call for infrared divergence since the zeroth mode is the lowest-energy mode. This might seem problematic as we desire to take the limit $\omega \to 0$ but do not want to deal with a divergence. A solution to this problem is to momentarily regulate the integration over the zeroth mode. One way to do this is to note that

$$\frac{1}{\beta}\int_0^{\beta} d\tau x(\tau) = Ta_0, \tag{2.50}$$

which means Ta_0 is the average value of $x(\tau)$ over the τ-interval. This allows us to regulate the system by placing it in a "periodic box" or mathematically speaking, to restrict the average value of $x(\tau)$ to some interval (L) which is large but not infinite.

Having established these features in the limit where $\omega \to 0$ we can now set up two different computations and find C' by comparing the two. The first computation takes into account the regulator in the limit $\omega \to 0$ which we call the "Effective theory computation". Using this method, we obtain

$$\begin{aligned}\lim_{\omega\to 0}\mathcal{Z}_{Regulated} &= C'\int_{L/T} da_0 \int_{-\infty}^{\infty}\Big[\prod_{n\geq 1} da_n db_n\Big] \\ &\exp\Big\{-mT\sum_{n\geq 1}^{\infty}\omega_n^2(a_n^2+b_n^2)\Big\} \\ &= C'\frac{L}{T}\prod_{n=1}^{\infty}\frac{\pi}{mT\omega_n^2}, \quad \omega_n = 2\pi Tn.\end{aligned} \tag{2.51}$$

The second method of computation uses a regulator as well however treats the initial problem as if absent of $V(x)$ in the limit where $\omega \to 0$. We call this method the "Full theory computation" which gives us

$$\begin{aligned}\lim_{\omega\to 0}\mathcal{Z}_{Regulated} &= \int_L dx \,\langle x| e^{\frac{-\hat{p}^2}{2mT}} |x\rangle \\ &= \int_L dx \int_{-\infty}^{\infty}\frac{dp}{2\pi}\langle x| e^{\frac{-\hat{p}^2}{2mT}} |p\rangle \langle p|x\rangle \\ &= \int_L dx \int_{-\infty}^{\infty}\frac{dp}{2\pi} e^{\frac{-p^2}{2mT}} \langle x|p\rangle \langle p|x\rangle \\ &= \frac{L}{2\pi}\sqrt{2\pi mT},\end{aligned} \tag{2.52}$$

where in the second step, we inserted a completeness relation in momentum basis as defined in Eq. (2.6) and in the final step, we have used the definition of $\langle x|p\rangle$ as in Eq. (2.8) where we have set $A = 1$ and $B = 2\pi$. We can now compare the results from the two computations and isolate C' on one side to obtain

$$C' = \frac{T}{2\pi}\sqrt{2\pi mT}\prod_{n=1}^{\infty}\frac{mT\omega_n^2}{\pi}. \tag{2.53}$$

Since L cancels out in the calculations, C' can be regarded as an "ultraviolet" matching coefficient. We can now substitute the value of C' back into our expression for the partition function in Eq. (2.49) to obtain our final result, namely

$$\begin{aligned}\mathcal{Z} &= \frac{T}{\omega}\prod_{n=1}^{\infty}\frac{\omega_n^2}{(\omega_n^2+\omega^2)} \\ &= \frac{T}{\omega}\frac{1}{\prod_{n=1}^{\infty}\left[1+\frac{(\omega/2\pi T)^2}{n^2}\right]} \\ &= \frac{1}{2\sinh\left(\frac{\omega}{2T}\right)},\end{aligned} \tag{2.54}$$

where in the final step, we used the relation

$$\frac{\sinh \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right). \tag{2.55}$$

Next, we want to compare this result with that of a harmonic oscillator using its exact energy eigenvalues. A harmonic oscillator is known to describe a quantum mechanical system defined by the potential in Eq. (2.41) where the non-degenerate energy eigenstates $|n\rangle$ can be found explicitly and the corresponding energy eigenvalues are

$$E_n = \omega\left(n+\frac{1}{2}\right), \quad n = 0,1,2,... \tag{2.56}$$

Since we have the exact energy eigenvalues for the harmonic oscillator, we can quickly calculate the partition function for a harmonic oscillator in energy basis giving us

$$\mathcal{Z} = \sum_{n=0}^{\infty}\langle n|\, e^{-\beta\hat{H}}\,|n\rangle = \sum_{n=0}^{\infty} e^{-\beta\omega\left(n+\frac{1}{2}\right)} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} = \frac{1}{2\sinh\frac{\omega}{2T}}. \tag{2.57}$$

We thus see that the expression we found for the partition function in position basis is exactly the same as that for the harmonic oscillator proving that the path-integral representation of quantum mechanics indeed yields the correct result.

It is finally important to discuss some physical features of the partition function. In quantum mechanics, all observables including the partition function $\mathcal{Z}$ are finite functions of the parameters T, m, and ω [19, p. 8]. When transitioning to quantum field theory, we will encounter "Ultraviolet" (UV) divergences which are dealt with through renormalization. By solving the functional integral in quantum mechanics however, we can see that this divergence is independent of the "seemingly divergent" form of the matching coefficient C'. In fact, in many physically relevant observables the C' factor naturally drops out in the calculations.

2.3 Path-integral representation of the bosonic propagator in quantum field theory

In this section we will take the path-integral formalism obtained in quantum mechanics and generalize it to that of a quantum field theory. In order to do this, we first need to establish the key differences between quantum mechanics and quantum field theory. In quantum mechanics, one must use operators which acts on the Hilbert space of a system that has undergone quantization. The vectors which we usually deal with in such space is either in position or momentum basis. Once the operator acts on such a vector in Hilbert space, we obtain the value for the measurable quantity of interest.

Quantum field theory is very similar to quantum mechanics in the sense that there exists operators which act on a Hilbert space to provide a value for a measurable quantity. The difference between quantum field theory and quantum mechanics lies in the nature of the operator itself. As can be guessed from the name, quantum field theory uses field operators instead of regular operators which then act on eigenfunctions in field basis [20, p. 13]. For this reason, the path-integral formalism of quantum field theory follows closely the steps of quantum mechanics except we are now interested in finding the matrix elements of the time-evolution operator in field basis as opposed to position basis. Similar to quantum mechanics Eq. (2.3), the time-evolution operator for quantum field theory satisfies the relation

$$|\phi\rangle = \hat{U}(t_f;0)\,|\phi_0\rangle = e^{-i\hat{H}t_f}\,|\phi_0\rangle\,, \tag{2.58}$$

where $|\phi_0\rangle$ is the state of the system at time $t=0$ and $|\phi\rangle$ is the state of the system at a later time $t=t_f$. Similar to before, we act on the left of Eq. (2.58) with $\langle\phi|$ to obtain

$$\langle\phi|\,\hat{U}(t_f;0)\,|\phi_0\rangle = \langle\phi|\,e^{-i\hat{H}t_f}\,|\phi_0\rangle\,. \tag{2.59}$$

The matrix elements above represents therefore the transition amplitude from a state $|\phi_0\rangle$ at time $t=0$ to a state $|\phi\rangle$ at a later time $t=t_f$. In order to proceed any further with Eq. (2.59) we must first establish what the Hamiltonian of a field theory looks like. This can be accomplished by first noting that the Hamiltonian we will be interested in must be dependent on the fields and their derivatives as opposed to position and momentum. Secondly, we proceed similarly to quantum mechanics where we assume the form of the Hamiltonian in classical mechanics and proceed with quantizing the desired variables such as the canonical coordinates and the conjugate momenta to operators which satisfy their respective commutation relations.

In classical mechanics the Hamiltonian is defined by

$$H = p\dot{x} - L, \tag{2.60}$$

which is very similar to that in quantum field theory except the Hamiltonian (H) is now transformed to the Hamiltonian density ($\mathcal{H}$), the Lagrangian (L) becomes the Lagrangian density ($\mathcal{L}$), canonical coordinates (x) transform to fields (ϕ) and the conjugate momenta (p) becomes conjugate momenta density (π). We note that all of these transformations are due to the definition of a field which is the mapping from spacetime $\mathcal{M}$ with coordinates x, y, z and t to a field space $\mathcal{T}$, namely

$$\phi : \mathcal{M} \to \mathcal{T}. \tag{2.61}$$

The Hamiltonian density is then defined in analogy with the classical case, namely

$$\mathcal{H} = \pi_i\dot{\phi}_i - \mathcal{L}, \tag{2.62}$$

where

$$\pi_i = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}, \tag{2.63}$$

and the index i runs over the different fields under consideration. The field operators in Schrödinger-picture at time $t=0$ satisfy the eigenvalue problem

$$\hat{\phi}(0,\boldsymbol{x})\,|\phi\rangle = \phi(\boldsymbol{x})\,|\phi\rangle\,, \tag{2.64}$$

where $\phi(\boldsymbol{x})$ is the eigenvalue and $|\phi\rangle$ the corresponding eigenfunction. Next, we have the completeness and orthogonality relation in field basis

$$\mathbb{1} = \int \mathcal{D}\phi(\boldsymbol{x}) \, |\phi\rangle \langle\phi| \,, \tag{2.65}$$

$$\langle\phi_a|\phi_b\rangle = \prod_x \delta(\phi_a(\boldsymbol{x}) - \phi_b(\boldsymbol{x})). \tag{2.66}$$

where we have adopted a similar notation as in Eq. (2.30). $\mathcal{D}$ represents the field integration at discretized points in position space with the assumption that position space takes the structure of a square lattice [16, p. 285]. We then let the spacing between each point on said lattice go to zero thus giving us the representation

$$\mathcal{D}\phi(\boldsymbol{x}) = \prod_i d\phi(\boldsymbol{x}_i). \tag{2.67}$$

Similarly, the conjugate momentum density operator satisfies the eigenvalue problem

$$\hat{\pi}(0, \boldsymbol{x}) \, |\pi\rangle = \pi(\boldsymbol{x}) \, |\pi\rangle \,, \tag{2.68}$$

with the respective completeness and orthogonality relations

$$\mathbb{1} = \int \frac{\mathcal{D}\pi(\boldsymbol{x})}{B} \, |\pi\rangle \langle\pi| \,, \tag{2.69}$$

$$\langle\pi_a|\pi_b\rangle = \prod_x \delta(\pi_a(\boldsymbol{x}) - \pi_b(\boldsymbol{x})). \tag{2.70}$$

Following the same procedure as in Eq. (2.6)-Eq. (2.9), where we this time around set $A = 1$ for simplicity of calculations, meaning that $B = 2\pi$. It is important to note that this is allowed as one of the constants can be arbitrarily picked of which the second constant depends on. Furthermore, we want to replace $\langle x|p\rangle$ with the equivalent field theory version

$$\langle\phi|\pi\rangle = \exp\left\{ i \int d^3x \pi(\boldsymbol{x})\phi(\boldsymbol{x}) \right\}, \tag{2.71}$$

and define our new Hamiltonian with respect to the fields and conjugate momentum density, namely

$$H = \int d^3x \mathcal{H}(\hat{\phi}, \hat{\pi}). \tag{2.72}$$

We are now ready to continue from Eq. (2.59) where we again split the total time t_f into N small intervals of size $\epsilon = t_f/N$ and let the number of intervals approach infinity. We now proceed in a slightly different manner than in the case of quantum mechanics where we insert N completeness relation factors on each side of the exponential factors except now in both field and conjugate momentum density basis and only one completeness relation of differing basis to the left of the leftmost and to right of the rightmost exponential factor respectively. Finally, it is important to note that in field theory, the transition amplitudes of interest are those where the system begins and ends in the same state, thus we want our exponential factors (time-evolution operator) to be sandwiched between $\langle\phi_0|$ and $|\phi_0\rangle$. Thus, the transition amplitude can be written as

$$\begin{aligned}\langle\phi_0| e^{-i\hat{H}t_f} |\phi_0\rangle = \langle\phi_0| e^{-i\hat{H}\epsilon} ... e^{-i\hat{H}\epsilon} |\phi_0\rangle = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \mathcal{D}\phi_i \frac{\mathcal{D}\pi_i}{2\pi} \right) & \\ \times \langle\phi_0|\pi_N\rangle \langle\pi_N| e^{-i\hat{H}\epsilon} |\phi_N\rangle \langle\phi_N|\pi_{N-1}\rangle & \\ \times \langle\pi_{N-1}| e^{-i\hat{H}\epsilon} |\phi_{N-1}\rangle ... & \\ \times \langle\phi_2|\pi_1\rangle \langle\pi_1| e^{-i\hat{H}\epsilon} |\phi_1\rangle \langle\phi_1|\phi_0\rangle \,. &\end{aligned} \tag{2.73}$$

We can further simplify the above expression by utilizing Eq. (2.66) and Eq. (2.71) and making the Taylor expansion about $\epsilon = 0$ to first order in ϵ

$$\langle\phi_i| e^{-i\hat{H}_i\epsilon} |\pi_i\rangle \approx \langle\phi_i| (1 - i\hat{H}_i\epsilon) |\pi_i\rangle = \langle\phi_i|\pi_i\rangle (1 - i\hat{H}_i\epsilon) = (1 - i\hat{H}_i\epsilon) \exp\left\{-i \int d^3x \pi_i(\boldsymbol{x})\phi_i(\boldsymbol{x})\right\}, \tag{2.74}$$

where

$$H_i = \int d^3x \mathcal{H}(\hat{\phi}_i(\boldsymbol{x}), \hat{\pi}_i(\boldsymbol{x})). \tag{2.75}$$

Taking this into consideration the transition amplitude becomes

$$\langle\phi_0| e^{-i\hat{H}t_f} |\phi_0\rangle = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \mathcal{D}\phi_i \frac{\mathcal{D}\pi_i}{2\pi}\right) \delta(\phi_1 - \phi_0) \exp\left\{-i\epsilon \sum_{j=1}^{N} \int d^3x \left[\mathcal{H}(\pi_j, \phi_j) - \frac{\pi_j(\phi_{j+1} - \phi_j)}{\epsilon}\right]\right\}. \tag{2.76}$$

Finally, if we let $N \to \infty$, the sum over conjugate momentum density and field integration elements (similarly to the procedure in quantum mechanics) transforms to an integral where the limits for the field integration is simply from the field at the initial time $t = 0$ to the field representing the final state at a later time $t = t_f$. Similar to the case in quantum mechanics, once $N \to \infty$, the "points" $(\phi_0, \phi_1, ..., \phi_N)$ which are associated with the path $\phi(t, \boldsymbol{x})$ exist essentially at every possible time step $t_j = j\epsilon$ in our total time interval $[0, t_f]$. Again, we can say we are integrating over all possible field paths connecting the initial "point" $\phi_0(\boldsymbol{x})$ at time $t = 0$ and the final "point" $\phi_N(\boldsymbol{x})$ at a final time $t = t_f$. Finally, we note that since we are considering bosons, we have that the fields are periodic in the time variable, i.e. $\phi(0, \boldsymbol{x}) = \phi(t_f, \boldsymbol{x}) = \phi_0(\boldsymbol{x})$. Thus, we obtain the final expression for the transition amplitude of bosons in quantum field theory, namely

$$\begin{aligned}\langle\phi_0| e^{-i\hat{H}t_f} |\phi_0\rangle =& C \int [\mathcal{D}\pi(t, \boldsymbol{x})] \int_{\phi(0,\boldsymbol{x})=\phi_0(\boldsymbol{x})}^{\phi(t_f,\boldsymbol{x})=\phi_0(\boldsymbol{x})} [\mathcal{D}\phi(t, \boldsymbol{x})] \\ &\times \exp\left\{i \int_0^{t_f} dt \int d^3x \left[\pi(t, \boldsymbol{x}) \frac{\partial\phi(t, \boldsymbol{x})}{\partial t} - \mathcal{H}(\pi(t, \boldsymbol{x}), \phi(t, \boldsymbol{x}))\right]\right\},\end{aligned} \tag{2.77}$$

where $[\mathcal{D}\pi(t, \boldsymbol{x})]$ and $[\mathcal{D}\phi(t, \boldsymbol{x})]$ represents integration over all possible paths in the field and momentum density space of the system. It is also important to note that $C = \left(\frac{1}{2\pi}\right)^N$ causes divergence in the limit $N \to \infty$ but can be ignored as it must drop out of any physical quantities.

2.4 Partition function for bosons in quantum field theory

Our goal now is to use the result we obtained for the transition amplitude of quantum field theory in 3+1-dimensions to obtain an expression for the partition function. We first start by defining the partition function in quantum field theory, namely

$$\mathrm{Tr} e^{-\beta(\hat{H} - \mu_i \hat{N}_i)} = \sum_{\text{all states}} \int \mathcal{D}\phi_0 \langle\phi_0| e^{-\beta(\hat{H} - \mu_i \hat{N}_i)} |\phi_0\rangle, \tag{2.78}$$

where ϕ_0 describes our initial and final state. Furthermore, μ_i represents the chemical potential and N_i the number of fields of type i if the system were to interact with a particle reservoir. We can now implement the imaginary-time formalism similar to the case in quantum mechanics to obtain a path-integral representation for the partition function. Furthermore, if we were to assume a more general Hamiltonian, for instance one describing a system which admits a conserved charge, we would have to make the transformation $\mathcal{H}(\pi,\phi) \to \mathcal{H}(\pi,\phi) - j^0$ where j^0 is the conserved charge density from which the conserved charge can be defined, namely

$$Q = \int d^3x j^0. \tag{2.79}$$

We therefore have that

$$\begin{aligned}\langle\phi_0| e^{-iH(-i\tau)} |\phi_0\rangle =& C \int [\mathcal{D}\pi] \int_{\text{periodic}} [\mathcal{D}\phi] \\ &\times \exp\left\{\int_0^\beta d\tau \int d^3x \left[i\pi(\tau,\boldsymbol{x})\frac{\partial\phi(\tau,\boldsymbol{x})}{\partial\tau} - \mathcal{H}(\pi(\tau,\boldsymbol{x}),\phi(\tau,\boldsymbol{x})) + \mu j^0\right]\right\},\end{aligned} \tag{2.80}$$

where we note that we have removed the limits of integration on the integral over the fields and constrained the integration to be over fields which are periodic in the time variable. This comes directly from the cyclic nature of the trace operator used to define the partition function in Eq. (2.78). Due to this property, we can have our initial state and final state be described by any possible field in configuration space as long as the endpoints are equally separated as the original boundary condition, namely $\phi(0,\boldsymbol{x}) = \phi(\beta,\boldsymbol{x}) = \phi_0(\boldsymbol{x})$.

2.5 Partition function for a neutral scalar field

Having established a general path-integral representation for the partition function for bosons, we now want to extend this formulation to a neutral scalar field. The calculations in this section are inspired by [20]. We first start by writing out the Lagrangian density of a neutral scalar field namely,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi), \tag{2.81}$$

where $U(\phi)$ are the interaction terms and for our purpose goes up to fourth order in the fields, i.e.

$$U(\phi) = g\phi^3 + \lambda\phi^4. \tag{2.82}$$

Furthermore, we note that g and λ are coupling constants and $\lambda \geq 0$ in order for the potential to be bounded from below such that the system has a stable vacuum. We substitute this into Eq. (2.62). Furthermore, we note that the conjugate momentum density is simply the derivative of the Lagrangian density with respect to the time derivative of ϕ which in our case is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}. \tag{2.83}$$

Thus, the Hamiltonian density can be written as

$$\mathcal{H} = \pi^2 - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi). \tag{2.84}$$

We also note that since we are considering a neutral scalar field, the Hamiltonian density will not include any conserved charge terms since the Lagrangian does not remain invariant under a phase transformation in the fields, i.e. the system has no continuous symmetries.

In order to proceed with evaluating the path-integral representation for the partition function, we must return to the discretized version of Eq. (2.80). We then have

$$\begin{aligned}
\mathcal{Z} &= \lim_{N\to\infty}\left(\prod_{i=1}^{N}\frac{1}{2\pi}\int\mathcal{D}\pi_i\int_{\text{periodic}}\mathcal{D}\phi_i\right)\exp\left\{\epsilon_E\sum_{j=1}^{N}\int d^3x\left(i\pi_j\frac{\phi_{j+1}-\phi_j}{\epsilon_E}-\mathcal{H}(\pi_j,\phi_j)\right)\right\}\\
&= \lim_{N\to\infty}\left(\prod_{i=1}^{N}\frac{1}{2\pi}\int\mathcal{D}\pi_i\int_{\text{periodic}}\mathcal{D}\phi_i\right)\exp\left\{\sum_{j=1}^{N}\int d^3x\left[i\pi_j(\phi_{j+1}-\phi_j)-\epsilon_E\left(\frac{1}{2}\pi_j^2+\frac{1}{2}(\nabla\phi_j)^2+\frac{1}{2}m^2\phi_j^2+U(\phi_j)\right)\right]\right\},
\end{aligned} \tag{2.85}$$

where ϵ_E is equivalent to the small time-step (ϵ) defined in Eq. (2.73), except we are now working in Euclidean space. We first turn our attention to the momentum integrals which can be easily evaluated as they are of Gaussian nature. This can be clearly seen by rearranging the terms in Eq. (2.85) to obtain

$$\begin{aligned}
\mathcal{Z} = \lim_{N\to\infty}\left(\frac{1}{2\pi}\right)^N\int_{\text{periodic}}\left(\prod_{j=1}^{N}\mathcal{D}\phi_j\right)\exp\left\{\sum_{k=1}^{N}\int d^3x\left[-\epsilon_E\left(\frac{1}{2}(\nabla\phi_k)^2+\frac{1}{2}m^2\phi_k^2+U(\phi_k)\right)\right]\right\}\\
\times\left(\prod_{i=1}^{N}\int\mathcal{D}\pi_i\exp\left\{\int d^3x\left[i\pi_i(\phi_{i+1}-\phi_i)-\frac{1}{2}\epsilon_E\pi_i^2\right]\right\}\right).
\end{aligned} \tag{2.86}$$

We are therefore first looking to evaluate the integral,

$$\mathcal{I}_i = \int\mathcal{D}\pi_i\exp\left\{\int d^3x\left[i\pi_i(\phi_{i+1}-\phi_i)-\frac{1}{2}\epsilon_E\pi_i^2\right]\right\}. \tag{2.87}$$

For simplicity, we turn the three-dimensional spatial integral into a Riemann sum by turning space into an infinite volume cube with sides $L = aM$ and dividing this total volume into infinitely many smaller cubes each with sides of length a. Thus we have that as $a\to 0$, the total number of small cubes (M^3) goes to infinity. Thus an integral of the form in the exponent of Eq. (2.87) can be expressed as

$$\int d^3x f(\boldsymbol{x}) = \lim_{M\to\infty}a^3\sum_{n=1}^{M^3}f(\boldsymbol{x}_n), \tag{2.88}$$

where each n represents a different small cube. We can therefore write Eq. (2.87) as

$$\begin{aligned}
\mathcal{I}_i &= \lim_{M\to\infty}\int\mathcal{D}\pi_i\exp\left\{\frac{V}{M^3}\sum_{n=1}^{M^3}\left[i\pi_i(\boldsymbol{x}_n)(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))-\frac{1}{2}\epsilon_E\pi_i(\boldsymbol{x}_n)^2\right]\right\}\\
&= \lim_{M\to\infty}\prod_{n=1}^{M^3}\int_{-\infty}^{\infty}d\pi_i(\boldsymbol{x}_n)\exp\left\{\frac{V}{M^3}\left[i\pi_i(\boldsymbol{x}_n)(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))-\frac{1}{2}\epsilon_E\pi_i(\boldsymbol{x}_n)^2\right]\right\},
\end{aligned} \tag{2.89}$$

where we have used that $a^3 = V/M^3$ and V is the total volume representing all of space, namely $V = L^3 = a^3M^3$. We can now use the same formula for Gaussian integrals as we used previously in Eq. (2.24) where we now have that $a = \frac{1}{2}\frac{V}{M^3}$, $b = -\frac{i}{2}\frac{V}{M^3}(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))$, and $c = 0$. We thus have

$$\begin{aligned}
\mathcal{I}_i &= \lim_{M\to\infty}\prod_{n=1}^{M^3}\sqrt{\frac{2\pi M^3}{V\epsilon_E}}\exp\left\{-\frac{1}{2}\frac{V}{M^3}\frac{(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))^2}{\epsilon_E}\right\}\\
&= \lim_{M\to\infty}\left(\frac{2\pi M^3}{V\epsilon_E}\right)^{\frac{M^3}{2}}\exp\left\{-\frac{\epsilon_E V}{M^3}\sum_{n=1}^{M^3}\frac{1}{2}\left(\frac{(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))}{\epsilon_E}\right)^2\right\}.
\end{aligned} \tag{2.90}$$

Substituting this back into Eq. (2.86) we obtain

$$\begin{aligned}
\mathcal{Z} = &\lim_{M,N\to\infty}\left(\frac{1}{2\pi}\right)^N\left(\frac{2\pi M^3}{V\epsilon_E}\right)^{\frac{M^3N}{2}}\int_{\text{periodic}}\left(\prod_{j=1}^N\mathcal{D}\phi_j\right)\prod_{i=1}^N\exp\left\{-\frac{\epsilon_E V}{M^3}\sum_{n=1}^{M^3}\frac{1}{2}\left(\frac{(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))}{\epsilon_E}\right)^2\right\}\\
&\times\exp\left\{\sum_{k=1}^N\int d^3x\left[-\epsilon_E\left(\frac{1}{2}(\nabla\phi_k)^2+\frac{1}{2}m^2\phi_k^2+U(\phi_k)\right)\right]\right\}\\
&=\lim_{M,N\to\infty}\left(\frac{1}{2\pi}\right)^N\left(\frac{2\pi M^3}{V\epsilon_E}\right)^{\frac{M^3N}{2}}\int_{\text{periodic}}\left(\prod_{j=1}^N\mathcal{D}\phi_j\right)\\
&\times\exp\left\{-\epsilon_E\sum_{i=1}^N\left[\frac{V}{M^3}\sum_{n=1}^{M^3}\frac{1}{2}\left(\frac{(\phi_{i+1}(\boldsymbol{x}_n)-\phi_i(\boldsymbol{x}_n))}{\epsilon_E}\right)^2+\int d^3x\left(\frac{1}{2}(\nabla\phi_k)^2+\frac{1}{2}m^2\phi_k^2+U(\phi_k)\right)\right]\right\}\\
&=C'\int_{\text{periodic}}\mathcal{D}\phi\exp\left\{-\int_0^\beta d\tau\int d^3x\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2+U(\phi)\right]\right\},
\end{aligned}\tag{2.91}$$

where in the last step, we carried out the limits for N and M thus converting the summation over M to an integral over position space and the summation over N to a time integral. For simplicity, we now proceed with assuming that there are no interactions present between the fields, thus the interaction terms in the potential can be set to zero, i.e. $U(\phi) = 0$. Furthermore, it is important to remember that the normalization constant, regardless of its divergent nature, can be ignored since multiplying $\mathcal{Z}$ by any constant does not affect the thermodynamics of the system [20, p. 17]. We therefore have our simplified path-integral representation of the partition function, namely

$$\mathcal{Z} = C'\int_{\text{periodic}}\mathcal{D}\phi\exp\{-S_E\} = C'\int_{\text{periodic}}\mathcal{D}\phi\exp\left\{-\int_0^\beta d\tau\int d^3x\mathcal{L}_E\right\},\tag{2.92}$$

where the subscript E denotes that we are working with a Euclidean metric, i.e. $\mathcal{L}_E = -\mathcal{L}(\tau = it)$. Next, we want to evaluate the path-integral above by representing it in terms of its respective Fourier components. We start by representing the field $\phi(\tau,\boldsymbol{x})$ as a Fourier series, namely

$$\phi(\tau,\boldsymbol{x}) = \sqrt{\frac{T}{V}}\sum_{\omega_n}\sum_{\boldsymbol{k}}\tilde{\phi}(\omega_n,\boldsymbol{k})e^{i\omega_n\tau-i\boldsymbol{k}\cdot\boldsymbol{x}},\tag{2.93}$$

where $\omega_n = 2\pi Tn$. Furthermore, the factor $\sqrt{\frac{T}{V}}$ is a matter of normalization which ensures the Fourier coefficients are dimensionless. Next, we follow the same procedure as in Eq. (2.36) where we now require that $\phi(\tau,\boldsymbol{x})$ is real. Thus we have

$$[\tilde{\phi}(\omega_n,\boldsymbol{k})]^* = \tilde{\phi}(-\omega_n,-\boldsymbol{k}).\tag{2.94}$$

A constraint of the above nature and the symmetric nature of the sums in Eq. (2.93) leave us with only half of the total Fourier modes being independent. We continue in a similar manner as in Eq. (2.39) except now we want integrals with an integrand quadratic in the fields as opposed to position variables. Following the same procedure as before we have

$$\int_0^\beta d\tau\int d^3\boldsymbol{x}\phi_1(\tau,\boldsymbol{x})\phi_2(\tau,\boldsymbol{x}) = \beta^2\sum_{\omega_n}\sum_{\boldsymbol{k}}\tilde{\phi}_1(\omega_n,\boldsymbol{k})\tilde{\phi}_2(-\omega_n,-\boldsymbol{k}),\tag{2.95}$$

where we have used that

$$\frac{1}{\beta}\int_0^\beta d\tau e^{i\tau(\omega_n^{(1)}+\omega_n^{(2)})} = \delta_{\omega_n^{(1)},-\omega_n^{(2)}}, \tag{2.96}$$

and

$$\frac{1}{V}\int d^3\boldsymbol{x} e^{-i\boldsymbol{x}(\boldsymbol{k}_1+\boldsymbol{k}_2)} = \delta\left(\boldsymbol{k}_1+\boldsymbol{k}_2\right). \tag{2.97}$$

Thus, the exponent in Eq. (2.92) becomes

$$\begin{aligned}\exp\{-S_E\} &= \exp\left\{-\int_0^\beta d\tau\int d^3\boldsymbol{x}\mathcal{L}_E\right\}\\ &= \exp\left\{-\frac{\beta^2}{2}\sum_{\omega_n}\sum_{\boldsymbol{k}}\tilde{\phi}(\omega_n,\boldsymbol{k})\left(\omega_n^2+\boldsymbol{k}^2+m^2\right)\tilde{\phi}(-\omega_n,-\boldsymbol{k})\right\}\\ &= \exp\left\{-\frac{\beta^2}{2}\sum_{\omega_n}\sum_{\boldsymbol{k}}\left(\omega_n^2+\boldsymbol{k}^2+m^2\right)\left|\tilde{\phi}(\omega_n,\boldsymbol{k})\right|^2\right\},\end{aligned} \tag{2.98}$$

To evaluate this expression any further, we must split the Fourier modes $\tilde{\phi}(\omega_n,\boldsymbol{k})$ into a real and an imaginary part, namely

$$\tilde{\phi}(\omega_n,\boldsymbol{k}) = a(\omega_n,\boldsymbol{k})+ib(\omega_n,\boldsymbol{k}). \tag{2.99}$$

Furthermore, we also want to represent the integration element $\mathcal{D}\phi$ in terms of the Fourier components. This is done in a similar manner to Eq. (2.45). Specifically, we can write the continuous space-time variable of which ϕ is a function of as discretized points in the limit where the spacing, l, goes to zero. This allows us to perform a change of variables in the integration measure using the Jacobian of the transformation, namely

$$\mathcal{D}\phi = \lim_{l=0}\prod_i d\phi(x_i) = \left|\det\left[\frac{\partial\phi(x_i)}{\partial\tilde{\phi}(\omega_n,\boldsymbol{k})}\right]\right|\left[\prod_{n>0}\prod_{\boldsymbol{k}}da(\omega_n,\boldsymbol{k})db(\omega_n,\boldsymbol{k})\right] = \left|\det[J]\right|\left[\prod_{n>0}\prod_{\boldsymbol{k}}da(\omega_n,\boldsymbol{k})db(\omega_n,\boldsymbol{k})\right]. \tag{2.100}$$

where J is the Jacobian matrix. The Jacobian matrix for this transformation is orthogonal which means the determinant goes to unity. Thus, we have that

$$\mathcal{D}\phi(\tau,\boldsymbol{k}) = \prod_{n>0}\prod_{\boldsymbol{k}}da(\omega_n,\boldsymbol{k})db(\omega_n,\boldsymbol{k}). \tag{2.101}$$

Thus, Eq. (2.92) can be written as

$$\begin{aligned}\mathcal{Z} &= C'\int_{\text{periodic}}\left(\prod_{n>0}\prod_{\boldsymbol{k}}da(\omega_n,\boldsymbol{k})db(\omega_n,\boldsymbol{k})\right)\exp\left\{-\frac{\beta^2}{2}\sum_{\omega_n}\sum_{\boldsymbol{k}}\left(\omega_n^2+\boldsymbol{k}^2+m^2\right)\left|\tilde{\phi}(\omega_n,\boldsymbol{k})\right|^2\right\}\\ &= C'\prod_{n>0}\prod_{\boldsymbol{k}}\int_{\text{periodic}}da(\omega_n,\boldsymbol{k})db(\omega_n,\boldsymbol{k})\exp\left\{-\frac{\beta^2}{2}\left(\omega_n^2+\boldsymbol{k}^2+m^2\right)\left(a(\omega_n,\boldsymbol{k})^2+b(\omega_n,\boldsymbol{k})^2\right)\right\}\\ &= C'\prod_{n>0}\prod_{\boldsymbol{k}}\sqrt{\frac{2\pi}{\beta^2\left(\omega_n^2+\omega^2\right)}}\sqrt{\frac{2\pi}{\beta^2\left(\omega_n^2+\omega^2\right)}}\\ &= C'\prod_{\text{all n}}\prod_{\boldsymbol{k}}\sqrt{\frac{2\pi}{\beta^2\left(\omega_n^2+\omega^2\right)}},\end{aligned} \tag{2.102}$$

which is obtained in a similar manner in [16, p. 286]. It is important to note that all thermodynamic quantities emerge from the logarithm of the partition function. This means that any constant, independent of thermodynamic quantities, which is either multiplied by the partition function or equivalently added to the logarithm of the partition function can be dropped. The logarithm of the partition function therefore becomes

$$\ln(\mathcal{Z}) = \sum_n \sum_{\boldsymbol{k}} \ln\left(\frac{\beta^2(\omega_n^2+\omega^2)}{2\pi}\right)^{-1/2} = -\frac{1}{2}\sum_n \sum_{\boldsymbol{k}} \ln\left(\beta^2(\omega_n^2+\omega^2)\right), \tag{2.103}$$

where we dropped a constant term. To evaluate this expression further, we consider the integral

$$\int_1^{\beta^2\omega^2} \frac{d\theta^2}{\theta^2+(2\pi n)^2} = \ln[(2\pi n)^2+\beta^2\omega^2] - \ln[1+(2\pi n)^2]. \tag{2.104}$$

Dropping the constant term, we can write Eq. (2.103) as

$$\ln(\mathcal{Z}) = -\frac{1}{2}\sum_n \sum_{\boldsymbol{k}} \int_1^{\beta^2\omega^2} \frac{d\theta^2}{\theta^2+(2\pi n)^2}. \tag{2.105}$$

Next, we can perform the sum over n by noting that

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{\theta}{2\pi}\right)^2+n^2} = \frac{2\pi^2}{\theta}\coth\frac{\theta}{2} = \frac{2\pi^2}{\theta}\frac{e^\theta+1}{e^\theta-1} = \frac{2\pi^2}{\theta}\left(1+\frac{2}{e^\theta-1}\right). \tag{2.106}$$

We then have

$$\ln(\mathcal{Z}) = -\frac{1}{2}\sum_{\boldsymbol{k}} \int_1^{\beta^2\omega^2} \frac{d\theta^2}{(2\pi)^2}\sum_n \frac{1}{(\frac{\theta}{2\pi})^2+n^2} = -\sum_{\boldsymbol{k}} \int_1^{\beta\omega}\left(\frac{1}{2}+\frac{1}{e^\theta-1}\right)d\theta = \sum_{\boldsymbol{k}}\left(-\frac{1}{2}\beta\omega - \ln\left(1-e^{-\beta\omega}\right)\right), \tag{2.107}$$

where we again dropped terms which are constant and independent of β. Finally, we must evaluate the sum over $\boldsymbol{k}$. We can do this by approximating the sum as an integral. To do so we must first assume that the space we are working in has a large volume V. We then write the three momentum vector as

$$\boldsymbol{k} = \frac{2\pi\boldsymbol{n}}{L}, \quad \boldsymbol{n} = (n_1, n_2, n_3). \tag{2.108}$$

The spacing between neighboring values of $\boldsymbol{k}$ is

$$\Delta k_i = \frac{2\pi}{L}, \quad i = 1,2,3. \tag{2.109}$$

We can thus write Eq. (2.107) as

$$\ln(\mathcal{Z}) = \frac{\Delta k_1 \Delta k_2 \Delta k_3}{(2\pi/L)^3}\sum_{\boldsymbol{k}}\left(-\frac{1}{2}\beta\omega - \ln\left(1-e^{-\beta\omega}\right)\right). \tag{2.110}$$

In the limit $V \to \infty$ and $\Delta k_i \to 0$, we obtain

$$\ln(\mathcal{Z}) = \lim_{V\to\infty} \frac{\Delta k_1 \Delta k_2 \Delta k_3}{(2\pi/L)^3}\sum_{\boldsymbol{k}}\left(-\frac{1}{2}\beta\omega - \ln\left(1-e^{-\beta\omega}\right)\right) = \lim_{V\to\infty} V\int \frac{d^3k}{(2\pi)^3}\left(-\frac{1}{2}\beta\omega - \ln\left(1-e^{-\beta\omega}\right)\right). \tag{2.111}$$

This is the final expression for the logarithm of the partition function of a neutral scalar field. Using this, we can for instance find the free energy density of a neutral scalar field which is given by

$$\mathcal{F} = -\lim_{V\to\infty} \frac{T \ln \mathcal{Z}}{V}. \tag{2.112}$$

Thus, the free energy density of a neutral scalar field at finite temperature is

$$\mathcal{F} = \int \frac{d^3k}{(2\pi)^3} \left[\frac{\omega}{2} + T \ln\left(1 - e^{-\beta\omega}\right)\right]. \tag{2.113}$$

We close this section by turning our focus briefly to the final expression we obtained for the free-energy density of a neutral field, Eq. (2.113). The first term in this expression is known as the vacuum energy density which can be found by taking the vacuum expectation of the Hamilton density of the respective field theory [21, p. 41], namely

$$\rho = \langle 0| \mathcal{H} |0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2}\omega. \tag{2.114}$$

It is important to note that this integral is ultraviolet (UV) divergent and thus cannot be evaluated normally. In order to evaluate such an integral, one must therefore introduce a regularization scheme as can be seen in Appendix. E. The second term in Eq. (2.113), unlike the first term, is convergent and thus can be evaluated exactly. This term is known as the finite-temperature contribution to the free-energy density of a neutral scalar field.

2.6 Partition function for fermions

Having found a representation for the partition function of bosons, particularly that of a neutral scalar field with spin-0, we now want to express the partition function of spin-1/2 particles. The calculations in this section follows closely to those in [20].

We first start by writing the Lagrangian appropriate for this field theory, namely the Dirac Lagrangian density which takes the form

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{2.115}$$

where we have defined $\bar{\psi} \equiv \gamma^0 \psi^\dagger$. Next, we want to find the expression for the Hamiltonian density Eq. (2.62). The conjugate momenta density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i\psi^\dagger. \tag{2.116}$$

Substituting this into Eq. (2.62), the Hamiltonian density can be expressed as

$$\mathcal{H} = i\psi^\dagger \dot{\psi} - i\bar{\psi}\dot{\psi}\gamma^0 - i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + \bar{\psi}m\psi = \bar{\psi}(-i\boldsymbol{\gamma}\cdot\nabla + m)\psi. \tag{2.117}$$

In the case of a neutral scalar field, we were dealing with fields which were periodic in the time variable due to the commuting nature of the field operators. Now, we are dealing with spin-1/2 particles which obey the Pauli exclusion principle meaning that there can only be one particle with the specific quantum numbers in a respective quantum state. Thus, we want to decide the nature of our fields such that the system operates under the appropriate Fermi-Dirac statistics. It can be shown ([22, p. 285]) that this condition is met if the fields possess the following anti-commuting relations

$$
\begin{aligned}
\{\psi_\alpha(x), \psi_\beta(y)\} &= \left\{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\right\} = 0 \\
\left\{\psi_\alpha(x), \psi_\beta^\dagger(y)\right\} &= \delta_{\alpha\beta}\delta^3(x-y).
\end{aligned}
\tag{2.118}
$$

Therefore, in order to proceed with this theory we must take into consideration anti-commuting field variables which are also known as Grassmann variables. A description of Grassmann variables and their respective algebra can be found in Appendix A. The starting procedure for setting up an expression for the partition function of such fermions is very similar to that of bosons. We start by taking the trace of the time-evolution operator in imaginary-time. This corresponds to the integral over the transition amplitudes for the system to return to its initial state $|\psi(0,\boldsymbol{x})\rangle$ after a time t. The only difference now is the fact that we are dealing with anti-commuting fields and that the Hamiltonian takes on a slightly different form. We thus have

$$
\begin{aligned}
\mathcal{Z} &= \int_{\text{antiperiodic}} \mathcal{D}\pi(\tau,\boldsymbol{x})\mathcal{D}\psi(\tau,\boldsymbol{x}) \exp\{S[\pi,\psi]\} \\
&= \int_{\text{antiperiodic}} \mathcal{D}\pi(\tau,\boldsymbol{x})\mathcal{D}\psi(\tau,\boldsymbol{x}) \exp\left\{\int_0^\tau d\tau \int d^3x \left[i\pi\frac{\partial\psi}{\partial\tau} - \mathcal{H}(\pi,\psi) + \mu j^0\right]\right\},
\end{aligned}
\tag{2.119}
$$

where the "antiperiodic" subscript of the integral simply means that $\psi(0,\boldsymbol{x}) = -\psi(\beta,\boldsymbol{x})$ and $\pi(0,\boldsymbol{x}) = -\pi(\beta,\boldsymbol{x})$. Furthermore, j^0 is the conserved charge density which must now be considered as the system now has a continuous symmetry. The conserved current can be found using Noethers theorem and noting that the Lagrangian density remains invariant under a phase transformation of the form

$$
\psi \to \psi' = e^{-i\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}' = e^{i\alpha}\bar{\psi}.
\tag{2.120}
$$

We can therefore use the equation for the conserved current, namely

$$
j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\Delta\psi^\dagger,
\tag{2.121}
$$

where $\Delta\psi$ can be found by expanding Eq. (2.120) and also noting that

$$
\psi' = \psi + \alpha\Delta\psi, \quad \bar{\psi}' = \bar{\psi} + \alpha\Delta\bar{\psi},
\tag{2.122}
$$

where we assume that α is very small. We therefore have that

$$
j^\mu = \bar{\psi}\gamma^\alpha\psi = \psi^\dagger\gamma^0\gamma^\alpha\psi.
\tag{2.123}
$$

Thus, the conserved charge density can be expressed as

$$
j^0 = \psi^\dagger\psi.
\tag{2.124}
$$

We can therefore express the partition function as

$$
\mathcal{Z} = \int_{\text{antiperiodic}} \mathcal{D}\pi(\tau,\boldsymbol{x})\mathcal{D}\psi(\tau,\boldsymbol{x}) \exp\left\{\int_0^\tau d\tau \int d^3x\bar{\psi}\left[-\gamma^0\frac{\partial}{\partial\tau} + i\boldsymbol{\gamma}\cdot\nabla - m + \gamma^0\mu\right]\psi\right\}.
\tag{2.125}
$$

We can further evaluate the above integral by expressing the fields as a Fourier series, namely

$$\psi(\tau, \boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\omega_n} \sum_{\boldsymbol{k}} \tilde{\psi}(\omega_n, \boldsymbol{k}) e^{i\omega_n \tau + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.126}$$

where

$$\omega_n = \frac{(2n+1)\pi}{\beta}, \quad n \in \mathbb{Z}, \tag{2.127}$$

due to the anti-periodic nature of the fields in τ, i.e. $\psi(0, \boldsymbol{x}) = -\psi(\beta, \boldsymbol{x})$. The action in the exponent of Eq. (2.125) can then be expressed as

$$S = \frac{1}{V} \int_0^\beta d\tau \int d^3x \sum_{\omega_n, \omega_n'} \sum_{\boldsymbol{k}, \boldsymbol{k}'} \bar{\tilde{\psi}}(\omega_n, \boldsymbol{k})(-i\gamma^0 \omega_n' - \boldsymbol{\gamma}\cdot\boldsymbol{k}' - m + \gamma^0\mu)\tilde{\psi}(\omega_n', \boldsymbol{k}') e^{i(\omega_n' \tau + \boldsymbol{k}'\cdot\boldsymbol{x})} e^{-i(\omega_n \tau + \boldsymbol{k}\cdot\boldsymbol{x})}. \tag{2.128}$$

We now use a similar Fourier representation for the delta function as we used in Eq. (2.96) and Eq. (2.97) to obtain

$$\begin{aligned} S &= \beta \sum_{\omega_n} \sum_{\boldsymbol{k}} \bar{\tilde{\psi}}(\omega_n, \boldsymbol{k})(-i\gamma^0\omega_n - \boldsymbol{\gamma}\cdot\boldsymbol{k} - m + \gamma^0\mu)\tilde{\psi}(\omega_n, \boldsymbol{k}) \\ &= -\beta \sum_{\omega_n} \sum_{\boldsymbol{k}} i\tilde{\psi}^\dagger(\omega_n, \boldsymbol{k})(\omega_n - i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{k} - i\gamma^0 m + i\mu)\tilde{\psi}(\omega_n, \boldsymbol{k}) \\ &= -\sum_{\omega_n} \sum_{\boldsymbol{k}} \tilde{\pi}(\omega_n, \boldsymbol{k})(i\beta[(-i\omega_n + \mu) - \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{k} - \gamma^0 m])\tilde{\psi}(\omega_n, \boldsymbol{k}) \\ &= -\sum_{\omega_n} \sum_{\boldsymbol{k}} \tilde{\pi}(\omega_n, \boldsymbol{k}) D(\omega_n, \boldsymbol{k})\tilde{\psi}(\omega_n, \boldsymbol{k}), \end{aligned} \tag{2.129}$$

where we have defined

$$D(\omega_n, \boldsymbol{k}) = i\beta[(-i\omega_n + \mu) - \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{k} - \gamma^0 m] = i\beta \begin{pmatrix} -i\omega_n + \mu - m & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & -i\omega_n + \mu + m \end{pmatrix}. \tag{2.130}$$

The partition function then becomes,

$$\begin{aligned} \mathcal{Z} &= \int_{\text{antiperiodic}} \mathcal{D}\tilde{\pi}(\omega, \boldsymbol{k}) \mathcal{D}\tilde{\psi}(\omega_n, \boldsymbol{k}) \exp\left\{ -\sum_{\omega_n} \sum_{\boldsymbol{k}} \tilde{\pi}(\omega_n, \boldsymbol{k}) D(\omega_n, \boldsymbol{k}) \tilde{\psi}(\omega_n, \boldsymbol{k}) \right\} \\ &= \prod_{\omega_n} \prod_{\boldsymbol{k}} \left[\int_{\text{antiperiodic}} d\tilde{\pi}(\omega, \boldsymbol{k}) d\tilde{\psi}(\omega_n, \boldsymbol{k}) \exp\left\{ -\tilde{\pi}(\omega_n, \boldsymbol{k}) D(\omega_n, \boldsymbol{k}) \tilde{\psi}(\omega_n, \boldsymbol{k}) \right\} \right]. \end{aligned} \tag{2.131}$$

Referring to Appendix. A, we can use Eq. (A.23) and the fact that there is an implicit summation involving the components of the field/momentum vector (4-vector) and the D matrix (4×4 matrix). Thus, we have that

$$\mathcal{Z} = \prod_{\omega_n} \prod_{\boldsymbol{k}} \det D(\omega_n, \boldsymbol{k}). \tag{2.132}$$

In order to find the determinant of the D matrix, we separate the matrix into four block matrix components which allows us to use the formula [23, p. 134]:

$$\det D(\omega_n, \boldsymbol{k}) = \det \begin{pmatrix} A & B \\ C & E \end{pmatrix} = \det\left(AE - BE^{-1}CE \right), \tag{2.133}$$

where $A = i\beta(-i\omega_n + \mu - m)$, $B = C = i\beta(-\boldsymbol{\sigma}\cdot\boldsymbol{k})$, and $E = i\beta(-i\omega_n + \mu + m)$. We further note that $\boldsymbol{\sigma}\cdot\boldsymbol{k} = \boldsymbol{k}^2\boldsymbol{I}_2$ where $\boldsymbol{I}_2$ is the 2×2 identity matrix. As a result, we obtain

$$\det D(\omega_n, \boldsymbol{k}) = \beta^4[(-i\omega_n + \mu)^2 - \omega^2]^2. \tag{2.134}$$

Substituting this into Eq. (2.132) and taking the natural logarithm, we obtain

$$\begin{aligned}
\ln \mathcal{Z} = \ln\left(\prod_{\omega_n}\prod_{\boldsymbol{k}} \beta^4[(-i\omega_n+\mu)^2 - \omega^2]^2\right) &= \sum_{\omega_n}\sum_{\boldsymbol{k}} \ln\left(\beta^2[(\omega_n + i\mu)^2 + \omega^2]\right)^2 \\
&= \sum_{\omega_n}\sum_{\boldsymbol{k}} \ln\left(\beta^4[(\omega_n + i\mu)^2 + \omega^2][(\omega_n - i\mu)^2 + \omega^2]\right) \\
&= \sum_{\omega_n}\sum_{\boldsymbol{k}} \left(\ln\beta^2[\omega_n^2 + (\omega-\mu)^2] + \ln\beta^2[\omega_n^2 + (\omega+\mu)^2]\right).
\end{aligned} \tag{2.135}$$

We can evaluate the above expression further by considering the following integral,

$$\int_1^{\beta^2(\omega\pm\mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2\pi^2} = \ln[(2n+1)^2\pi^2 + \beta^2(\omega\pm\mu)^2] - \ln[1 + (2n+1)^2\pi^2]. \tag{2.136}$$

Using this and the expression for ω_n in Eq. (2.127), we get

$$\ln \mathcal{Z} = \sum_n\sum_{\boldsymbol{k}}\left\{\int_1^{\beta^2(\omega+\mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2\pi^2} + \int_1^{\beta^2(\omega-\mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2\pi^2} + 2\ln[1 + (2n+1)^2\pi^2]\right\}. \tag{2.137}$$

Similar to previous section, any constant independent of thermodynamic quantities which is either multiplied by the partition function or equivalently added to the logarithm of the partition function can be dropped. Therefore, we drop the $2\ln[1 + (2n+1)^2\pi^2]$ term from the above equation. Thus, we have

$$\ln \mathcal{Z} = \sum_n\sum_{\boldsymbol{k}}\left\{\int_1^{\beta^2(\omega+\mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2\pi^2} + \int_1^{\beta^2(\omega-\mu)^2} \frac{d\theta^2}{\theta^2 + (2n+1)^2\pi^2}\right\}. \tag{2.138}$$

Next, we use the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)(n-y)} = \frac{\pi[\cot(\pi x) - \cot(\pi y)]}{y - x}, \tag{2.139}$$

to evaluate the sum over the frequencies. In our case, we let $x = -\frac{1}{2} - i\frac{\theta}{2\pi}$ and $y = -\frac{1}{2} + i\frac{\theta}{2\pi}$ which gives us

$$\sum_{n=-\infty}^{\infty} \frac{1}{\theta^2 + (2n+1)^2\pi^2} = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)(n-y)} = \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{1+e^\theta}\right). \tag{2.140}$$

The logarithm of the partition function can therefore be expressed as

$$\ln \mathcal{Z} = \sum_{\boldsymbol{k}}\left\{\int_1^{\beta^2(\omega+\mu)^2} d\theta^2 \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{1+e^\theta}\right) + \int_1^{\beta^2(\omega-\mu)^2} d\theta^2 \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{1+e^\theta}\right)\right\}. \tag{2.141}$$

The integrals in the above expression can be easily evaluated to obtain

$$\ln \mathcal{Z} = 2\sum_{\boldsymbol{k}}\left[\beta\omega + \ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{2.142}$$

This discretized momentum sum can be made continuous by considering the infinite-volume limit, thus giving our final result

$$\lim_{V\to\infty} \ln \mathcal{Z} = \lim_{V\to\infty} 2V \int \frac{d^3k}{(2\pi)^3} \left[\beta\omega + \ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{2.143}$$

The corresponding free-energy density is then

$$\lim_{V\to\infty} \frac{\mathcal{F}}{V} = 2 \int \frac{d^3k}{(2\pi)^3} \left[-\omega - \frac{1}{\beta} \ln\left(1 + e^{-\beta(\omega-\mu)}\right) - \frac{1}{\beta} \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{2.144}$$

We conclude by first turning our attention to the factor 2 in front of the integral. This factor is directly due to the fact that we are considering a spin-1/2 particle which has two possible spin configurations, namely spin "up" or spin "down". This was not existent in the expression we found for the neutral scalar field as the particles only had one possible spin configuration, namely spin-0. Next, we look at the first term in the above expression. Similar to the case of a neutral scalar field, this term is the zero energy contribution to the free energy density and is highly divergent. Finally, the last two terms are the energy contributions from a particle and an antiparticle with chemical potential μ and $-\mu$ respectively.

3 Relativistic particles in a constant magnetic field

We investigate the interaction of both scalar and spin-1/2 particles with a constant magnetic field. We start with the free Lagrangian of our desired theory and then introduce coupling to an electromagnetic field by requiring local gauge invariance of our full Lagrangian. We accomplish this by replacing the partial derivatives with covariant ones and also including the Maxwell field Lagrangian to our total Lagrangian. This can then be extended to both spin-0 and spin-1/2 relativistic particles which are represented by the Klein-Gordon and Dirac equation respectively. Furthermore, we will simplify the problem to only include a constant magnetic field allowing us to find the energy spectrum associated with these interactions and the respective wavefunctions for the different systems. By choosing the Landau gauge, we shall see that the wavefunctions correspond to the Landau eigenfunctions of the respective Landau energy levels. Using the Landau eigenfunctions, we then calculate the propagators for the respective theories. Finally, using the propagators, we calculate the one-loop vacuum energy density of the system of interest.

3.1 Complex scalar field and coupling to electromagnetic field

To formulate a theory which describes the interaction between a scalar particle and an electromagnetic field, we first start by writing the Lagrangian density of the charged scalar field theory, i.e. the Lagrangian of a complex scalar field,

$$\mathcal{L} = \partial_\mu \Phi (\partial^\mu \Phi)^* - m^2 \Phi \Phi^*, \tag{3.1}$$

where Φ can be written in terms of two real scalar fields namely,

$$\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad \Phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}. \tag{3.2}$$

We note that Eq. (3.1) has a $U(1)$ global symmetry, i.e. it is invariant under the transformation,

$$\Phi \to \Phi' = \Phi e^{-iq\alpha}, \tag{3.3}$$

where α is a space-time independent phase parameter and q is the charge of the particle. We now want to couple a system described by the Lagrangian in Eq. (3.1) to an electromagnetic field. The Lagrangian for such an interaction is the sum of the complex scalar field Lagrangian, the Lagrangian of the Maxwell field, and an interaction Lagrangian describing the coupling between the scalar particle and the electromagnetic field. Thus we have,

$$\mathcal{L}_{\text{SEM}} = \mathcal{L}_{\text{S}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{I}}, \tag{3.4}$$

where $\mathcal{L}_{\text{S}}$ is the Lagrangian for the scalar field, $\mathcal{L}_{\text{EM}}$ is the Lagrangian of the Maxwell field, and $\mathcal{L}_{\text{I}}$ is the interaction Lagrangian. We can find $\mathcal{L}_{\text{EM}}$ by requiring a Lagrangian which is gauge invariant, i.e. invariant under the transformation

$$A^\mu \to A^\mu + \partial^\mu \Lambda(x), \tag{3.5}$$

and also Lorentz invariant. We therefore construct $\mathcal{L}_{\text{EM}}$ with the help of the anti-symmetric and gauge invariant field strength tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{3.6}$$

which we must contract with itself to ensure it is Lorentz invariant. Thus, we have

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{3.7}$$

where the sign and the $\frac{1}{4}$ in front ensures we get the correct Maxwell equations when solving the equations of motion for Eq. (3.7).

All that is left now is to find an expression for $\mathcal{L}_{\mathrm{I}}$. To do this, we require that the interaction term be such that when solving the field equations for the Maxwell field, that we obtain the correct Maxwell equations, namely

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{3.8}$$

where j^ν is the electric current which couples to the Maxwell field. Thus, we have

$$\mathcal{L}_{\mathrm{I}} = A_\nu j^\nu. \tag{3.9}$$

We next note that the current we want to use must be conserved which can be seen if we act on Eq. (3.8) with a differential operator

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0, \tag{3.10}$$

where in the last step, we used that

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\mu \partial_\nu F^{\nu\mu} = \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu} = 0. \tag{3.11}$$

The only conserved current we have encountered so far is that which arises from the global $U(1)$ symmetry of the complex scalar field Lagrangian namely,

$$j^\nu = \frac{\partial \mathcal{L}_S}{\partial \partial_\nu \Phi} \Delta\Phi + \frac{\partial \mathcal{L}_S}{\partial \partial_\nu \Phi^*} \Delta\Phi^* = iq(\Phi^* \partial^\nu \Phi - \Phi \partial^\nu \Phi^*). \tag{3.12}$$

Naively, one might attempt at using this current in Eq. (3.9). However, this will not be correct as this current is only conserved for $\mathcal{L}_{\mathrm{S}}$ and fails when considering the full Lagrangian $\mathcal{L}_{\mathrm{SEM}}$. We can see this since

$$\partial_\nu j^\nu = iq(\Phi^* \Box \Phi - \Phi \Box \Phi^*). \tag{3.13}$$

We can evaluate this further by using the equations of motion for both Φ and Φ^* which gives us

$$\partial_\mu \frac{\partial \mathcal{L}_{SEM}}{\partial \partial_\mu \Phi} - \frac{\partial \mathcal{L}_{SEM}}{\partial \Phi} = 0. \tag{3.14}$$

Thus, we have

$$\Box \Phi^* = -2iqA_\mu \partial^\mu \Phi^* - iq\Phi^* \partial^\mu A_\mu - m^2 \Phi^*. \tag{3.15}$$

Doing the same for Φ^* we obtain,

$$\Box \Phi = 2iqA_\mu \partial^\mu \Phi + iq\Phi \partial^\mu A_\mu - m^2 \Phi. \tag{3.16}$$

Multiplying Eq. (3.15) from the left by Φ and Eq. (3.16) from the left by Φ^*, we can write Eq. (3.13) as

$$\begin{aligned} \partial_\nu j^\nu &= iq(-m^2\Phi\Phi^* + 2iqA_\nu(\partial^\nu\Phi)\Phi^* + iq\Phi\Phi^*(\partial^\nu A_\nu) + m^2\Phi\Phi^* + 2iqA_\nu(\partial^\nu\Phi^*)\Phi + iq\Phi\Phi^*(\partial^\nu A_\nu)) \\ &= -2q^2\partial^\nu(A_\nu\Phi\Phi^*) \neq 0. \end{aligned} \tag{3.17}$$

Thus, we see that this current is indeed not conserved when considering the full Lagrangian of the system. To rectify this, we therefore want to impose another type of symmetry on the Lagrangian to obtain a true conserved current. Previously, our scalar Lagrangian had only a global $U(1)$ symmetry. We now want to impose a local $U(1)$ symmetry where the phase parameter is now dependent on space-time coordinates $\alpha \to \alpha(x)$. Furthermore, we also want our full Lagrangian to now be invariant under gauge transformations. Thus, we want a Lagrangian which is invariant under the transformations,

$$\Phi \to \Phi e^{-iq\alpha(x)}, \quad A^\mu \to A^\mu + \partial^\mu \Lambda(x). \tag{3.18}$$

We can immediately see that the Lagrangian in Eq. (3.1) does not have a local $U(1)$ symmetry due to the derivative terms. To fix this, we must therefore replace the derivative with a covariant derivative, namely

$$\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu. \tag{3.19}$$

Thus, the Lagrangian density of the system becomes

$$\mathcal{L}_{\text{SEM}} = D_\mu \Phi (D^\mu \Phi)^* - m^2 \Phi \Phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{3.20}$$

which is invariant under the transformations in Eq. (3.18). Furthermore, the current associated to such symmetry is

$$j^\nu = iq(\Phi^* D^\nu \Phi - \Phi D^\nu \Phi^*), \tag{3.21}$$

which, following the same steps as Eq. (3.13)-Eq. (3.17), can be shown to be conserved, namely

$$\partial_\nu j^\nu = 0. \tag{3.22}$$

We therefore obtained the complete Lagrangian density for a system of scalar particles coupled to an electromagnetic field by starting with the Lagrangian density of the scalar fields, adding the Lagrangian density of the Maxwell field, and transforming the derivatives in the scalar Lagrangian density to covariant ones. Finally, we note that this recipe can be extended to spin-1/2 particles also as we shall see later on.

3.2 Klein-Gordon equation in a constant magnetic field: Landau levels and wavefunctions

Having obtained the Lagrangian density for a system of scalar fields interacting with an electromagnetic field, we now want to simplify the problem further by considering a scalar particle in the presence of a uniform magnetic field. As a starting point we first solve the equation of motion using the Lagrangian we obtained in Eq. (3.20) with respect to Φ^* (or Φ). Thus from

$$\partial_\mu \frac{\partial \mathcal{L}_{\text{SEM}}}{\partial \partial_\mu \Phi^*} - \frac{\partial \mathcal{L}_{\text{SEM}}}{\partial \Phi^*} = 0, \tag{3.23}$$

we obtain

$$\Box \Phi + 2iqA_\mu \partial^\mu \Phi + iq\Phi \partial^\mu A_\mu - q^2 A_\mu A^\mu \Phi + m^2 \Phi = 0. \tag{3.24}$$

This can be written explicitly as

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + iq\frac{\partial A^0}{\partial t}\Phi + iq\nabla \cdot \boldsymbol{A}\Phi + 2iqA^0 \frac{\partial \Phi}{\partial t} + 2iq\boldsymbol{A} \cdot (\nabla \Phi) - q^2 (A^0)^2 \Phi + q^2 \boldsymbol{A}^2 \Phi + m^2 \Phi = 0, \tag{3.25}$$

where we used the explicit expression of the four derivative $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$ and the four vector potential $A^\mu = (A^0, \boldsymbol{A})$. Rearranging terms and writing it more compactly, Eq. (3.25) can be written as

$$\left(i\frac{\partial}{\partial t} - qA^0\right)^2 \Phi = \left[(i\nabla + q\boldsymbol{A})^2 + m^2\right] \Phi. \tag{3.26}$$

We can now narrow down our problem to a scalar particle interacting with a uniform magnetic field without the presence of an electric field $\boldsymbol{E}(x) = 0$ which means we can set the scalar potential to zero, $A^0 = 0$. Furthermore, we can consider the field Φ as a wavefunction which we split into spatial and time dependent part, namely

$$\Phi(\boldsymbol{x}, t) = \Phi(\boldsymbol{x})e^{-iEt}. \tag{3.27}$$

Upon doing this, we are therefore assuming that the Hamiltonian of our system is time-independent. Thus, $\Phi(\boldsymbol{x})$ can be considered as the eigenfunctions of the Hamiltonian operator satisfying the eigenvalue problem

$$\hat{H}\Phi(x) = E\Phi(x). \tag{3.28}$$

We can then write Eq. (3.26) as,

$$E^2\Phi = \left[\left(i\nabla + q\boldsymbol{A}\right)^2 + m^2\right]\Phi. \tag{3.29}$$

where we set $\Phi(\boldsymbol{x}) = \Phi$ for brevity and time-independence should be assumed.

To simplify the problem further, we now assume that the constant magnetic field is pointing in the z-direction such that $\boldsymbol{B} = B\boldsymbol{e}_z = (0, 0, B)$. We now have to pick a certain gauge which satisfies, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, to which we pick the Landau gauge, namely $\boldsymbol{A} = (0, Bx, 0)$. To proceed any further and be able to obtain the energy spectrum for this problem, we must pick a suitable ansatz as the solution to the wavefunction. To do this, we can imagine inserting $\boldsymbol{A}$ into the Lagrangian Eq. (3.20). Upon doing so we can immediately see that the y and z coordinates are cyclic due to their absence in the Hamiltonian which means their respective momenta p_y and p_z are conserved i.e. the Hamiltonian operator of our system commutes with the momentum operator in both y and z direction, $\hat{p}_y$ and $\hat{p}_z$ respectively. This of course means that the ansatz we pick must therefore allow for the eigenstates of the Hamiltonian operator $\hat{H}$ to also be eigenstates of $\hat{p}_y$ and $\hat{p}_z$. In other words, considering the operator formalism of the momentum operators $\hat{p}_y \to -i\frac{\partial}{\partial y}$, we must make an ansatz which satisfies

$$\hat{p}_y\Phi = p_y\Phi, \quad \hat{p}_z\Phi = p_z\Phi. \tag{3.30}$$

We therefore set our wavefunction to be

$$\Phi = \frac{1}{2\pi}\chi(x)e^{i(p_y y + p_z z)}, \tag{3.31}$$

to satisfy our requirements. The $\frac{1}{2\pi}$ factor is due the fact that we want our plane waves to be correctly normalized. Inserting our ansatz into Eq. (3.29), applying our Landau gauge, and specifying the charge of the particle $q = -e$ we obtain,

$$E^2\chi = -\chi'' + p_z^2\chi + p_y^2\chi + 2eBp_y x\chi + e^2B^2x^2\chi + m^2\chi = -\chi'' + \left[2m\frac{1}{2}m\omega^2(x - x_0)^2\right]\chi + m^2\chi + p_z^2\chi, \tag{3.32}$$

where we have defined the relativistic cyclotron frequency as $\omega = \frac{B|e|}{m}$ and $x_0 = -\frac{p_y}{Be}$. Rearranging the above equation such that all the constants are on one side we get,

$$\frac{(E^2 - m^2 - p_z^2)}{2m}\chi = E'\chi = -\frac{1}{2m}\chi'' + \frac{1}{2}m\omega^2(x - x_0)^2\chi, \tag{3.33}$$

where we have defined $E' = \frac{(E^2 - m^2 - p_z^2)}{2m}$. The above equation is nothing more than the Schrodinger equation for a harmonic oscillator oscillating about the point $x = x_0$, with frequency $\omega = \frac{B|e|}{m}$ and energy spectrum $E' = \omega(n + \frac{1}{2})$, Appendix. B. Substituting back the value for E' we obtain our final result for the energy spectrum of a relativistic scalar particle interacting with a constant magnetic field, namely

$$E = \pm\sqrt{2m\omega\left(n + \frac{1}{2}\right) + m^2 + p_z^2}. \tag{3.34}$$

We can see that both positive and negative energy values are accepted for this system. This is because the positive energy values are associated with a particle whereas the negative energy values are associated with an anti-particle. The eigenfunctions associated with the quantum number n for the particles can be found using Appendix. B and are

$$\Phi_n = \frac{1}{2\pi}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\lambda) e^{-\frac{\lambda^2}{2}} e^{i(p_y y + p_z z)}. \tag{3.35}$$

where

$$\lambda = \sqrt{m\omega}(x - x_0). \tag{3.36}$$

We can similarly find the eigenfunctions for the anti-particles by solving the equation of motion with respect to Φ in Eq. (3.23). Furthermore, we turn our attention towards the degeneracy in the energy levels. If we compare the expression for the energy spectrum Eq. (3.34) with the wavefunction Eq. (3.35), we see that the wavefunction is dependent on p_y where as the energy spectrum is not. This means the energy levels have a continuous degeneracy in p_y. We also note that since $x_0 = -\frac{p_y}{Be}$, the continuous degeneracy in p_y corresponds to a harmonic oscillator oscillating about all possible points x_0 as p_y takes on all possible values.

Finally, having found the eigenfunctions of the system, we can use these wavefunctions to find a Landau representation, using the proper-time formalism, of the scalar propagator in the presence of a constant magnetic field. The calculations have been done in Appendix. C and the result is

$$\tilde{D}(\omega, p_x, p_y, p_z) = \int_0^\infty ds \frac{e^{-is[-\omega^2 + p_z^2 + m^2 + \frac{p_\perp^2}{eBs}\tan(eBs)]}}{\cos(eBs)}. \tag{3.37}$$

3.3 Spin-1/2 particles and coupling to electromagnetic field

We now want to turn our attention to relativistic spin-1/2 particles and their interaction with an electromagnetic field. The equation which describes a relativistic spin-1/2 particle is the Dirac equation which, in contrast with the Klein-Gordon equation, also takes into consideration the intrinsic spin and the magnetic moments of the particles [1, p. 82]. We thus, similar to the scalar particle case, start with the Lagrangian density of our particle of interest, namely

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{3.38}$$

To obtain the full Lagrangian of the system of spin-1/2 particles interacting with an electromagnetic field, we must then add to Eq. (3.38) the Maxwell Lagrangian density, Eq. (3.7), and promote the derivatives to covariant ones Eq. (3.19). Thus, we have

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi, \tag{3.39}$$

which is also known as the Lagrangian density of quantum electrodynamics (QED) [24]. The conserved current can be found by solving the field equation for the Maxwell field,

$$\partial_\mu \frac{\partial\mathcal{L}_{\text{QED}}}{\partial\partial_\mu A^\nu} - \frac{\partial\mathcal{L}_{\text{QED}}}{\partial A^\nu} = 0, \tag{3.40}$$

which gives us,

$$\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi. \tag{3.41}$$

Thus, we have that the current is

$$j^\nu = q\bar{\psi}\gamma^\nu\psi, \tag{3.42}$$

which must be conserved, namely

$$\partial_\nu j^\nu = 0. \tag{3.43}$$

3.4 Dirac equation in a constant magnetic field: Landau levels and wavefunctions

We now proceed in a similar manner as for the scalar particle case, where we must first find the equation of motion with respect to $\bar{\psi}$ (or ψ) using the QED Lagrangian density. We thus have,

$$\partial_\mu \frac{\partial \mathcal{L}_{\text{QED}}}{\partial \partial_\mu \bar{\psi}} - \frac{\partial \mathcal{L}_{\text{QED}}}{\partial \bar{\psi}} = 0, \tag{3.44}$$

which gives

$$(i\not{D} - m)\psi = 0. \tag{3.45}$$

Eq. (3.45) can be written explicitly as

$$(i\not{D} - m)\psi = [i(\partial_\mu\gamma^\mu + iqA_\mu\gamma^\mu) - m]\psi = \left[\left(i\frac{\partial}{\partial t} - qA^0\right) - \gamma^0\boldsymbol{\gamma}\cdot(-i\nabla - q\boldsymbol{A}) - \gamma^0 m\right]\psi = 0, \tag{3.46}$$

where in the last step, we used that $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$ and $A^\mu = (A^0, \boldsymbol{A})$. We now simplify our problem further by considering a constant and uniform magnetic field where there is no electric field. This means we can set the scalar potential to zero, i.e. $A^0 = 0$. Again, we assume that the Hamiltonian of our system is time-independent which allows us to split the wavefunction up into a spatial dependent and time dependent part

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-iEt}, \tag{3.47}$$

where ϕ represents the first two components of the Dirac spinor and χ the last two. We will see that they are associated with the positive energy and negative energy solutions respectively. Substituting this expression for the wavefunction into Eq. (3.46) and using the matrix representation for the gamma matrices we get,

$$\begin{aligned} E\begin{pmatrix} \phi \\ \chi \end{pmatrix} &= \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}) \\ \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}) & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} + \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}) \\ \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}) & -m\mathbb{1} \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix}. \end{aligned} \tag{3.48}$$

We obtain two linear equations from the above matrix equation, namely

$$E\phi = m + \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A})\chi, \tag{3.49}$$

and

$$E\chi = \boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A})\phi - m\chi. \tag{3.50}$$

Rearranging these equations, we can write them as a function of χ and ϕ

$$\phi = \frac{\boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A})}{E - m}\chi, \tag{3.51}$$

and

$$\chi = \frac{\boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A})}{E + m}\phi. \tag{3.52}$$

Substituting Eq. (3.52) into Eq. (3.51).

$$\begin{aligned}(E^2 - m^2)\phi &= [\boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A})] [\boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A})] \phi \\ &= [(-i\nabla - q\boldsymbol{A}) \cdot (-i\nabla - q\boldsymbol{A})] \phi + i\boldsymbol{\sigma} \cdot [(-i\nabla - q\boldsymbol{A}) \times (-i\nabla - q\boldsymbol{A})] \phi,\end{aligned} \tag{3.53}$$

where in the last step, we used the well known identity relating the dot product and cross product of a vector with the Pauli-vector, i.e.

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b}) + (a \cdot b)\mathbb{1}. \tag{3.54}$$

We now, similarly to the case of a scalar particle, make the assumption that the constant magnetic field which we are considering is pointing in the z-direction, i.e. $\boldsymbol{B} = (0, 0, B)$. We thus must pick our gauge accordingly to satisfy $B = (\nabla \times \boldsymbol{A})_z = \partial_x A_y - \partial_y A_x$. Again, we pick our gauge to be the Landau gauge where $\boldsymbol{A} = (0, Bx, 0)$. With this choice of gauge, the first term in Eq. (3.53) becomes

$$[(-i\nabla - q\boldsymbol{A}) \cdot (-i\nabla - q\boldsymbol{A})] \phi = -\nabla^2\phi + qA^2\phi + iq(\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla)\phi = -\nabla^2\phi + qA^2\phi + 2iq(\boldsymbol{A} \cdot \nabla)\phi, \tag{3.55}$$

where in the last step we used that

$$(\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla)\phi = (\nabla \cdot \boldsymbol{A})\phi + \boldsymbol{A} \cdot \nabla\phi + \boldsymbol{A} \cdot \nabla\phi = 2\boldsymbol{A} \cdot \nabla\phi. \tag{3.56}$$

For the second term in Eq. (3.53), we have

$$\begin{aligned}i\boldsymbol{\sigma} \cdot [(-i\nabla - q\boldsymbol{A}) \times (-i\nabla - q\boldsymbol{A})] \phi &= i\boldsymbol{\sigma} \cdot [(-i\nabla) \times (-i\nabla) + (i\nabla) \times (q\boldsymbol{A}) + (q\boldsymbol{A}) \times (i\nabla) + (q\boldsymbol{A}) \times (q\boldsymbol{A})] \phi \\ &= -q\boldsymbol{\sigma} \cdot [(\nabla \times \boldsymbol{A} + \boldsymbol{A} \cdot \nabla)] = -q\boldsymbol{\sigma} \cdot (\nabla \times \boldsymbol{A})\phi,\end{aligned} \tag{3.57}$$

where in the last step, we used that

$$(\nabla \times \boldsymbol{A} + \boldsymbol{A} \times \nabla)\phi = (\nabla \times \boldsymbol{A})\phi - \boldsymbol{A} \times (\nabla\phi) + \boldsymbol{A} \times (\nabla\phi) = (\nabla \times \boldsymbol{A})\phi. \tag{3.58}$$

Combining these two terms Eq. (3.53) becomes

$$\begin{aligned}(E^2 - m^2)\phi &= -\nabla^2\phi + q^2\boldsymbol{A}^2\phi + 2iq(\boldsymbol{A} \cdot \nabla)\phi - q\boldsymbol{\sigma} \cdot (\nabla \times \boldsymbol{A})\phi \\ &= \left[-\nabla^2 + q^2\boldsymbol{A}^2 + 2iq(\boldsymbol{A} \cdot \nabla) - q\boldsymbol{\sigma} \cdot \boldsymbol{B}\right] \phi.\end{aligned} \tag{3.59}$$

Finally, setting the charge of our particle to $q = -e$, inserting the expression for $\boldsymbol{A}$, and dividing both sides of Eq. (3.59) by $2m$, we obtain

$$\begin{aligned}\frac{(E^2-m^2)}{2m}\phi &= \left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{B^2e^2x^2}{2m}-\frac{2eiBx}{2m}\frac{\partial}{\partial y}+\frac{eB}{2m}\sigma_z\right]\phi \\ &= \left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}m\omega^2x^2-i\omega x\frac{\partial}{\partial y}+\frac{1}{2}\omega\sigma_z\right]\phi \\ &= \left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}m\omega^2x^2-i\omega x\frac{\partial}{\partial y}+\omega\hat{S}_z\right]\phi,\end{aligned} \tag{3.60}$$

where in the second step we defined the cyclotron frequency as $\omega \equiv \frac{eB}{m}$ and in the last step, used the definition for the spin operator, namely $\hat{S}_z \equiv \frac{\sigma_z}{2}$.

Similar to the scalar particle case, we now pick an ansatz for the wavefunction ϕ which represents the spinor elements of the positive energy particles. The cyclic coordinates are again y and z which means p_y and p_z are both conserved quantities. Thus, we can write our ansatz as

$$\phi = \frac{1}{2\pi}f(x)e^{i(p_y y+p_z z)}\Gamma, \tag{3.61}$$

where Γ is the spinorial function which accounts for the spin interaction of the particle with the magnetic field which was not present in the scalar particle problem. The spinorial function is defined as

$$\Gamma = \begin{cases}\begin{pmatrix}1\\0\end{pmatrix}, & m_s = +\frac{1}{2}\\ \\ \begin{pmatrix}0\\1\end{pmatrix}, & m_s = -\frac{1}{2}\end{cases}, \tag{3.62}$$

where $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$ is the spin quantum number for the spin up and spin down particle respectively. The spinor elements for the positive energy particles can therefore be represented as the combination of $f(x)$ and Γ. We can define them as $U^{(+)}_{m_s}(x)$ where

$$U^{(+)}_{+\frac{1}{2}}(x) = \begin{pmatrix}f(x)\\0\end{pmatrix}, \quad U^{(+)}_{-\frac{1}{2}}(x) = \begin{pmatrix}0\\f(x)\end{pmatrix} \tag{3.63}$$

which satisfies the relation

$$\hat{S}_z U^{(+)}_{m_s}(x) = m_s U^{(+)}_{m_s}(x). \tag{3.64}$$

Using this relation and inserting our ansatz into Eq. (3.60), we obtain

$$\begin{aligned}\frac{(E^2-m^2)}{2m}f &= -\frac{1}{2m}f''+\frac{p_y^2}{2m}f+\frac{p_z^2}{2m}f+\frac{1}{2}m\omega^2x^2f+\omega p_y xf+\omega m_s f \\ &= -\frac{1}{2m}f''+\frac{1}{2}m\omega^2\left(x^2+\frac{2p_y}{m\omega}x+\frac{p_y^2}{m^2\omega^2}\right)f+\frac{p_z^2}{2m}f+\omega m_s f \\ &= -\frac{1}{2m}f''+\frac{1}{2}m\omega^2\left(x-x_0\right)^2f+\frac{p_z^2}{2m}f+\omega m_s f.\end{aligned} \tag{3.65}$$

where we have defined $x_0 = -\frac{p_y}{m\omega} = -\frac{p_y}{eB}$. Moving all the constants on one side, we get

$$\left(\frac{(E^2 - m^2)}{2m} - \omega m_s - \frac{p_z^2}{2m}\right) f = E' f = -\frac{1}{2m} f'' + \frac{1}{2} m\omega^2 (x - x_0)^2 f, \tag{3.66}$$

where

$$E' \equiv \frac{(E^2 - m^2)}{2m} - \omega m_s - \frac{p_z^2}{2m}. \tag{3.67}$$

Again, we obtain the Schrodinger equation for a harmonic oscillator oscillating with frequency $\omega = \frac{eB}{m}$ about a point $x_0 = -\frac{p_y}{eB}$ which has the energy, Appendix. B,

$$E' = \omega(n + \frac{1}{2}). \tag{3.68}$$

Finally, we substitute back in our expression for E' to obtain the energy spectrum for a relativistic spin-1/2 particle interacting with a constant magnetic field, namely

$$E = \pm\sqrt{2m\omega(n + \frac{1}{2} + m_s) + p_z^2 + m^2}. \tag{3.69}$$

Furthermore, the corresponding eigenfunctions can be found using Appendix. B. Using Eq. (B.30), the wavefunction for the particles can be expressed as

$$\phi_n = \frac{1}{2\pi}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\lambda) e^{-\frac{\lambda^2}{2}} e^{i(p_y y + p_z z)}\Gamma, \tag{3.70}$$

where

$$\lambda = \sqrt{m\omega}(x - x_0). \tag{3.71}$$

We can similarly find the wavefunction elements for the anti-particles χ. To conclude, we note that similar to the scalar particle case, we are again dealing with both positive and negative energies representing particle and anti-particle interactions with the magnetic field respectively. Furthermore, we can see that if we ignore the spin contribution, i.e. $m_s = 0$, we obtain the exact same energy spectrum as the case for a scalar particle Eq. (3.34). Next, we look at the degeneracy of the energy levels. Similar to the scalar particle case, the energy levels have a continuous degeneracy in p_y. However, there is now also a discrete degeneracy due to the spin quantum number. For example, we can have the same energy value for quantum number $n = 1$ and quantum number $n = 2$ if $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$ respectively. Similar to the case of scalar particles, the continuous degeneracy in p_y means we are dealing with a harmonic oscillator that oscillates about all possible points $x_0 = -\frac{p_y}{eB}$ based on the value of p_y. Finally, we turn our attention to the lowest Landau level, namely when $n = 0$ and $m_s = -\frac{1}{2}$. We can see that in this case, the energy becomes $E = \sqrt{p_z^2 + m^2}$ which is independent of the magnetic field B since $\omega = \frac{eB}{m}$. This is unique for the spin-1/2 particle interaction and arises due to energy being dependent on the spin quantum number. Looking back at the scalar particle energy spectrum Eq. (3.34), we see that the lowest landau level, i.e. $n = 0$, is $E = \sqrt{m\omega + p_z^2 + m^2}$ which is clearly dependent on the magnetic field B in contrast to the spin-1/2 particle case.

Finally, we can calculate the spin-1/2 propagator in a similar manner to that of a scalar particle, Appendix. C, with some minor differences (cf. [25]). We have borrowed the result in this paper, namely

$$\begin{aligned}\tilde{S}(\omega, p_x, p_y, p_z) = &\int_0^\infty ds e^{-is[-\omega^2 + p_z^2 + m^2 + \frac{\boldsymbol{p}_\perp^2}{eBs}\tan(eBs)]} \\ &\times \left[\gamma^0\omega - (\boldsymbol{\gamma}\cdot\boldsymbol{p} + m + (p_x\gamma^2 - p_y\gamma^1)\tan(eBs))\right][1 - \gamma^1\gamma^2\tan(eBs)].\end{aligned} \tag{3.72}$$

3.5 Vacuum energy density of a scalar particle in a constant magnetic field

We now want to find the one-loop vacuum energy density of a particle in a constant magnetic field. The vacuum energy density of such a system can be found using the respective propagator in the coincidence limit. For simplicity, we shall explore this relation only for bosons, namely spin-0 particles. The one-loop vacuum energy density of a free scalar field theory in Minkowski space is equivalent to the first quantum correction of the effective potential, whose form was calculated in Appendix. F for a single real scalar field. The difference now is that instead of one real scalar field, we have a complex scalar field, or equivalently two real scalars. This type of generalization is shown in more detail in Section. 4.4 where we generalize the effective potential to N real scalars. We are also now dealing with a non-interacting theory where our fields have a zero vacuum expectation value. Thus, we start by expressing the one-loop free energy density of a complex scalar field theory as

$$V = -\frac{i}{\Omega}\ln\det\left(\Box + m^2\right) = -\frac{i}{\Omega}\operatorname{tr}\ln\left(\Box + m^2\right) = -\frac{i}{\Omega}\int d^4x \int \frac{d^4k}{(2\pi)^4}\ln\left(-k^2 + m^2\right) = -i\int \frac{d^4k}{(2\pi)^4}\ln\left(-k^2+m^2\right), \tag{3.73}$$

where we in the last step wrote the space-time integral as the four-dimensional volume factor, Ω. Differentiating both sides with respect to m^2, we obtain

$$\frac{\partial V}{\partial m^2} = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} = D(0), \tag{3.74}$$

where

$$D(0) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2}, \tag{3.75}$$

is the Fourier transform of the scalar propagator in the coincidence limit. Eq. (3.73) can be extended to include coupling to a constant magnetic field by promoting the partial derivatives to covariant ones and choosing the Landau gauge similar to what we did in Appendix. C. Thus, we can use Eq. (3.74) except instead of the free scalar propagator, we use the free scalar propagator coupled to a constant magnetic field which we found in Appendix. C. We now recall that the propagator we obtained in Appendix. C was the product of a translationally invariant part, whose representation in momentum space was given in Eq. (C.22), and a translationally non-invariant Schwinger phase factor $e^{i\Phi}$ where Φ is the Schwinger phase given in Eq. (C.12). In the coincidence limit of the propagator, the Schwinger phase becomes zero which means the Schwinger phase factor goes to unity. Thus, we are left with only the translationally invariant part of the propagator. Eq. (3.74) then becomes

$$\frac{\partial V}{\partial m^2} = \int \frac{d^4p}{(2\pi)^4}\tilde{D}(\omega, p_x, p_y, p_z) = \int \frac{d^4p}{(2\pi)^4}\int_0^\infty ds \frac{e^{-is[-\omega^2 + p_z^2 + m^2 + \frac{\boldsymbol{p}_\perp^2}{eBs}\tan(eBs)]}}{\cos(eBs)}. \tag{3.76}$$

The current expression we have for the propagator is in Minkowski space. It is however simpler to solve the momentum integral on the right hand side of Eq. (3.76) if we work in Euclidean space. This can be done by performing a Wick rotation where we introduce the new temporal component of the momentum 4-vector $\omega_E = -i\omega$. Thus, starting from the representation of the propagator in terms of the Laguerre polynomials, Eq. (C.19), the momentum integral in Euclidean space becomes,

$$
\begin{aligned}
&\int \frac{d^4p_E}{(2\pi)^4} 2e^{-\frac{\boldsymbol{p}_\perp^2}{eB}} \sum_{n=0}^{\infty} \frac{(-1)^n L_n\left(2\frac{\boldsymbol{p}_\perp^2}{eB}\right)}{\omega_E^2 + p_z^2 + 2eB\left(n+\frac{1}{2}\right) + m^2} \\
&= \int \frac{d^4p_E}{(2\pi)^4} 2e^{-\frac{\boldsymbol{p}_\perp^2}{eB}} \int_0^\infty ds \sum_{n=0}^{\infty} e^{-s[\omega_E^2+p_z^2+2eB\left(n+\frac{1}{2}\right)+m^2]} L_n\left(2\frac{\boldsymbol{p}_\perp^2}{eB}\right) \\
&= \int \frac{d^4p_E}{(2\pi)^4} 2e^{-\frac{\boldsymbol{p}_\perp^2}{eB}} \int_0^\infty ds e^{-s[\omega_E^2+p_z^2+m^2]} \frac{e^{-seB}}{1+e^{-2seB}} \exp\left(\frac{2\frac{\boldsymbol{p}_\perp^2}{eB}e^{-2seB}}{1+e^{-2seB}}\right) \\
&= \int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds \frac{e^{-s[\omega^2+p_z^2+m^2+\frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]}}{\cosh(eBs)},
\end{aligned}
\tag{3.77}
$$

where in the penultimate step we used Eq. (C.21). We dropped the superscript in the last step as it can be assumed from the relative signs of the terms in the exponent that we are working with a Euclidean metric. Thus, Eq. (3.76) becomes

$$
\frac{\partial V}{\partial m^2} = \int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds \frac{e^{-s[\omega^2+p_z^2+m^2+\frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]}}{\cosh(eBs)}. \tag{3.78}
$$

It is worth noting that the calculations in Eq. (3.77) is equivalent to making the substitution $s = -is_E$ and $\omega = i\omega_E$ in Eq. (3.76) which allows us to go directly from Eq. (3.76) to Eq. (3.78).

We can now move the ∂m^2 to the right hand side and integrate both sides giving us

$$
\begin{aligned}
V &= -\int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds \frac{e^{-s[\omega^2+p_z^2+m^2+\frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]}}{s\cosh(eBs)} + C \\
&= -\int_0^\infty ds \frac{1}{s\cosh(eBs)} \int \frac{d^4p}{(2\pi)^4} e^{-s[\omega^2+p_z^2+m^2+\frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]} + C \\
&= -\int_0^\infty ds \frac{1}{s\cosh(eBs)} \mathcal{I} + C,
\end{aligned}
\tag{3.79}
$$

where we have defined

$$
\mathcal{I} = \int \frac{d^4p}{(2\pi)^4} e^{-s[\omega^2+p_z^2+m^2+\frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]}, \tag{3.80}
$$

and C is some constant of integration. The integration constant will however be dropped in the calculations that follow as we can always pick our reference energy such that it is discarded. We can now begin to evaluate $\mathcal{I}$. We start by noting that this integral has an ultraviolet divergence, which is to say it is divergent in the large energy limit. This can be seen more clearly if we evaluate the Gaussian momentum integrals in Eq. (3.80) giving us

$$
V = -\frac{eB}{(4\pi)^2} \int_0^\infty ds \frac{e^{-sm^2}}{s^2\sinh(eBs)}. \tag{3.81}
$$

This integral is clearly divergent for $s = 0$. Consequently, we know that we are dealing with an ultraviolet divergence since the dimension of s is inversely proportional to m^2 as seen from a dimensional analysis of the exponential term, e^{-sm^2}, in the integrand. Therefore, we must evaluate this integral using a regularization scheme. The regularization allows us to isolate the divergences in our result. Finally, we must renormalize this result in order to eliminate the divergences and obtain a finite expression for our vacuum energy density. The calculations and details of these procedures are given in Appendix. D. The final expression we obtain for the one-loop vacuum energy density of a scalar particle in a constant magnetic field is thus

$$\begin{aligned} V = \frac{1}{2}B^2\left[1 + \frac{e^2}{3(4\pi)^2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] - \frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2}\right] \\ + \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2} - \ln\left(\frac{m^2}{2eB}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[24\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m^2}{2eB}\right) + 1 + \ln\left(\frac{m^2}{2eB}\right)\right]. \end{aligned} \tag{3.82}$$

4 Spontaneous symmetry breaking and the ϕ^4 theory

So far, our focus has been on free theories in the presence of a constant magnetic field. However, in most systems of interest, the particles in the system also interact with each other. Therefore, we have to develop theories where the particle interactions are also accounted for. In what follows, we will be looking at quartic interactions which are introduced via a ϕ^4 term in the potential. Furthermore, we will be looking at an opposite signed mass term relative to the quartic interaction term. As we shall see, the potential has several minima which represent the ground state of the system. A particle represented by such a theory must thus have its ground state be at one of the solutions which minimizes the potential. By specifying the ground state to be at one of these solutions, we obtain a Lagrangian where the symmetry of the original Lagrangian is no longer apparent. We also see that the original symmetry of the Lagrangian is no longer a symmetry of the chosen ground state. The symmetry of our system is then said to be spontaneously broken. We shall explore this phenomena for a self-interacting real scalar field and then proceed to a complex scalar field theory consisting of two real scalar fields. Consequently, we will see the difference between breaking of a discrete symmetry versus a continuous symmetry. We will then generalize this for a theory with N real scalars and use this generalization for a theory with $N = 4$ real scalars. Finally, we compute the effective potential for the ϕ^4 theory of four real scalar fields.

4.1 Real scalar field

We will first look at the spontaneous symmetry breaking of a real scalar field in the ϕ^4 model. The Lagrangian of such a theory can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{4.1}$$

where λ is the coupling constant of the particle interactions. Next, we change the sign of the mass term by making the substitution $m^2 = -\mu^2$. Thus, we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{4.2}$$

where we have defined our potential as

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{4.3}$$

The form of the potential can be seen in Fig. 4.1. We note that the Lagrangian in Eq. (4.2) has a Z_2 symmetry which is to say it is invariant under the transformation $\phi \to -\phi$. If we now make the assumption that our fields are independent of space-time coordinates, the Lagrangian in Eq. (4.2) becomes

$$\mathcal{L} = -V(\phi). \tag{4.4}$$

Thus the equation of motion becomes

$$\frac{\partial V}{\partial \phi} = 0, \tag{4.5}$$

which gives the solutions that minimizes the potential V. The solutions are

$$\phi_0 = \pm\sqrt{\frac{6}{\lambda}}\mu, \tag{4.6}$$

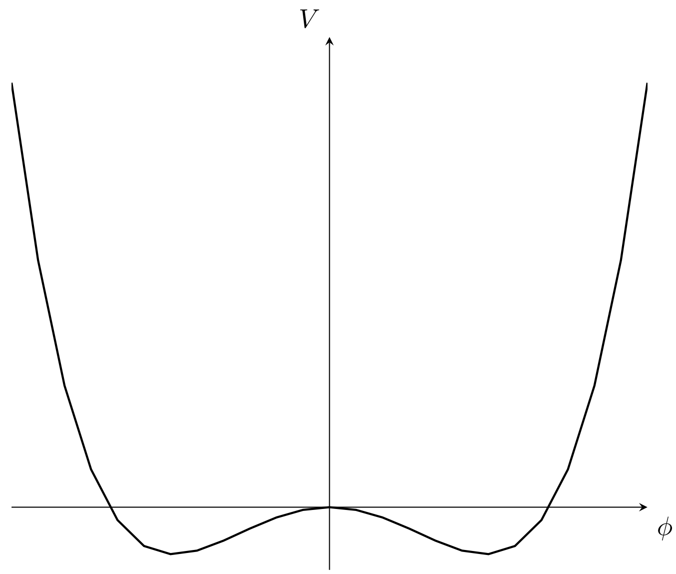

Figure 4.1: Potential of a self-interacting real scalar field in the ϕ^4 model and breaking of discrete symmetry

which are also known as the vacuum expectation values of ϕ. We should also note that the value $\phi_0 = 0$ also minimizes the potential. However, this does not represent a stable ground state of our system as the particle is able to move classically to a ground state with lower potential, namely Eq. (4.6). In the following, we will pick the positive vacuum expectation value to work with. We can now introduce a shifted field

$$\phi = \phi_0 + \eta(x) = \sqrt{\frac{6}{\lambda}}\mu + \eta(x). \tag{4.7}$$

Substituting this back into Eq. (4.2), we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{1}{2}(2\mu^2)\eta^2 - \sqrt{\frac{\lambda}{6}}\mu\eta^3 - \frac{\lambda}{4!}\eta^4, \tag{4.8}$$

where we dropped all constant terms. We have thus obtained a new Lagrangian in terms of the field $\eta(x)$ of mass $m_\eta = \sqrt{2}\mu$, with cubic and quartic interactions. We can see that the previous Z_2 symmetry is no longer apparent in this new Lagrangian. It is therefore said that the symmetry of the system has been spontaneously broken. Thus, by picking a specific solution which minimized the potential (we picked the positive solution) and introducing the shifted field Eq. (4.7) the original symmetry of our system was spontaneously broken. Let us close this section by looking at the second derivative of the potential with respect to the field at one of the minima. We shall stick to the positive solution, $\phi_0 = \sqrt{\frac{6}{\lambda}}\mu$, for consistency. Thus, we have

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\phi_0} = 2\mu^2. \tag{4.9}$$

We can see that the second derivative of the potential with respect to the field at a specific minimum yields the mass squared of the resulting massive field $\eta(x)$, namely $m_\eta^2 = 2\mu^2$. As we can see in Fig. 4.1, the second derivative of the potential is indeed positive at $\phi = \phi_0$ corresponding to a non-zero mass term. Thus, we can see that oscillations in ϕ at the point ϕ_0 corresponds to the massive $\eta(x)$ field. We shall next see what happens when we break a continuous symmetry of a system.

4.2 Complex scalar field

In the previous section, we looked at how the discrete symmetry of a self-interacting real scalar field theory was spontaneously broken. We shall now look at the interaction theory between two real scalar fields. The Lagrangian for this theory can be written as

$$\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^* + \mu^2 \Phi\Phi^* - \lambda(\Phi\Phi^*)^2 = \partial_\mu \Phi \partial^\mu \Phi^* - V(\Phi, \Phi^*), \tag{4.10}$$

where

$$V(\Phi, \Phi^*) = -\mu^2 \Phi\Phi^* + \lambda(\Phi\Phi^*)^2, \tag{4.11}$$

and

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{4.12}$$

Here, ϕ_1 and ϕ_2 are real scalar fields. This Lagrangian has a continuous global symmetry, known as a global $U(1)$ symmetry. This is to say that the Lagrangian is invariant under the transformation

$$\Phi \to \Phi e^{-i\alpha}. \tag{4.13}$$

where α is a constant independent of space-time coordinates as we previously encountered in Section. 3.1. Writing the Lagrangian out in terms of the real scalar fields, we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 + \mu^2 \phi_1^2) + \frac{1}{2}(\partial_\mu \phi_2 \partial^\mu \phi_2 + \mu^2 \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{4.14}$$

Following the same procedure as in previous section, we assume that the fields are independent of space-time coordinates, thus Eq. (4.14) becomes

$$\mathcal{L} = -V(\phi_1, \phi_2) = \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{4.15}$$

Again, the equation of motion gives the solutions which minimizes the potential. The equation of motion with respect to ϕ_1 (or ϕ_2) is

$$\frac{\partial V(\phi_1, \phi_2)}{\partial \phi_1} = \phi_1 \mu^2 - \lambda \phi_1^3 - \lambda \phi_1 \phi_2^2 = 0. \tag{4.16}$$

The solutions which minimizes the potential is therefore the set of points forming a circle in the ϕ_1-ϕ_2 plane satisfying the equation

$$\phi_1^2 + \phi_2^2 = \frac{\mu^2}{\lambda}. \tag{4.17}$$

We proceed by picking one specific solution out of this set, namely

$$\phi_1 = \frac{\mu}{\sqrt{\lambda}}, \quad \phi_2 = 0. \tag{4.18}$$

Continuing similarly to the previous section, we now introduce the shifted fields, namely

$$\phi_1 = \frac{\mu}{\sqrt{\lambda}} + \eta(x), \quad \phi_2 = \pi(x). \tag{4.19}$$

Substituting our shifted fields into Eq. (4.14) we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{1}{2}(2\mu^2)\eta^2 - \sqrt{\lambda}\mu\eta^3 - \sqrt{\lambda}\mu\pi^2\eta - \frac{\lambda}{2}\pi^2\eta^2 - \frac{\lambda}{4}\eta^4 - \frac{\lambda}{4}\pi^4, \tag{4.20}$$

where we again dropped the constant terms. We can see that the original $U(1)$ symmetry of the system is no longer apparent in this new Lagrangian. Thus, by picking a specific solution which minimizes the potential, the initial $U(1)$ symmetry of the system has been spontaneously broken. Furthermore, we note that in contrast to the self-interacting scalar field, we now not only have a massive field $\eta(x)$ but also a massless field $\pi(x)$. This is a direct consequence of breaking a continuous symmetry whereas in the case of the self-interacting scalar field, we were only breaking a discrete Z_2 symmetry. The resulting massless field is known as a Goldstone boson. The interacting theories for the ϕ^4 model can be generalized to N number of real scalar fields, known as the linear sigma model. In doing so, one can see that for every broken continuous symmetry of a system, there exists a respective Goldstone boson as we shall explore in the next chapter.

Finally, let us again look at the second derivative of the potential at a specific minimum of the potential. It will be easier if we write the fields in terms of polar coordinates, namely

$$\phi_1 = r\cos(\theta), \quad \phi_2 = r\sin(\theta), \tag{4.21}$$

where r is the radial coordinate in ϕ_1-ϕ_2 plane and θ is the angular coordinate. Our solution Eq. (4.18) in terms of the polar coordinates becomes

$$r = \frac{\mu}{\sqrt{\lambda}}, \quad \theta = 0. \tag{4.22}$$

Now we can find the second derivative of the potential with respect to both the radial and the angular coordinates at our desired minimum. With respect to the radial coordinates, we have

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\frac{\mu}{\sqrt{\lambda}},\theta=0} = 2\mu^2. \tag{4.23}$$

This is again the mass squared of the massive field $\eta(x)$. Thus, oscillations in the radial direction of the ϕ_1-ϕ_2 plane at the solution, Eq. (4.22), corresponds to a massive field. If we take the double derivative of the potential with respect to θ however, we see that

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{r=\frac{\mu}{\sqrt{\lambda}},\theta=0} = 0, \tag{4.24}$$

which means that $m_\theta^2 = 0$ for the corresponding field. Thus, we see that angular oscillations in the ϕ_1-ϕ_2 plane at the solution, Eq. (4.22), corresponds to the massless Goldstone boson.

4.3 Linear sigma model

We now want to generalize the ϕ^4 theory to N real scalar fields. We start by writing the appropriate Lagrangian for such a theory, namely

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2 = \frac{1}{2}(\partial_\mu \phi^i)^2 - V(\phi^i), \tag{4.25}$$

where $i = 1, ..., N$ and we have an implicit sum over the indices i in each factor of $(\phi^i)^2$. This Lagrangian has an $O(N)$ symmetry, i.e. it is invariant under the transformation $\phi^i \to T^{ij}\phi^j$ where T is an $N \times N$ orthogonal matrix. Similar to before, we can find the solutions which minimizes the potential by choosing fields which are independent of space-time coordinates and solving the field equations. Thus, we have that

$$\frac{\partial V}{\partial \phi^i} = -\mu^2\phi^i + \lambda(\phi_i)^3 = 0, \tag{4.26}$$

giving the solutions which correspond to the classical ground state of the system, namely

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}. \tag{4.27}$$

For simplicity we pick the solution

$$\phi_0^i = (0, 0, ..., \phi_0). \tag{4.28}$$

where $\phi_0 = \frac{\mu}{\sqrt{\lambda}}$. Again, we define the shifted fields as

$$\phi^i(x) = (\pi^k(x), \phi_0 + \eta(x)). \tag{4.29}$$

where $k = 1, ..., N-1$. Substituting the shifted fields back into the Lagrangian in Eq. (4.25) we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \eta)^2 - \frac{1}{2}(2\mu^2)\eta^2 - \mu\sqrt{\lambda}\eta(\pi^k)^2 - \mu\sqrt{\lambda}\eta^3 - \frac{\lambda}{2}(\pi^k)^2\eta^2 - \frac{\lambda}{4}[(\pi^k)^2]^2 - \frac{\lambda}{4}\eta^4, \tag{4.30}$$

where we again dropped all constant terms. Again, the original $O(N)$ symmetry of the system is no longer apparent. Furthermore, we have now one massive scalar field and $N-1$ massless Goldstone bosons. In the theory for a complex scalar field, we had one continuous symmetry which was broken corresponding to rotations in a single plane. For an $O(N)$ symmetric theory, we now have rotations in $N(N-1)/2$ planes. The resulting Lagrangian after the spontaneous symmetry breaking however only has an $O(N-1)$ symmetry, the symmetric rotations between the π fields, which corresponds to $(N-1)(N-2)/2$ planes of rotation. Taking the difference, we see that in breaking the $O(N)$ symmetry, we have $N-1$ broken continuous symmetries. Thus, we see that for every broken continuous symmetry, we have a Goldstone boson. This is known as the Goldstone theorem.

4.4 Effective potential of the linear sigma model

We have so far investigated the solutions which minimizes the classical potential of a field theory with quartic interactions. However, when we take into consideration quantum effects and look at higher order loop corrections to the potential, we see the possibility of deviations from the classical vacuum expectation value. To account for these shifts, we need a function which, once minimized, not only gives the correct vacuum expectation value at a classical level, but also accounts for quantum corrections at higher orders of perturbation theory. This function is known as the effective potential and is obtained in Appendix. F. We can thus start with Eq. (F.12), namely

$$V_{\text{eff}}(\phi_{\text{cl}}) = V(\phi_{\text{cl}}) - \frac{i}{2\Omega}\ln\det\left[-\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right]. \tag{4.31}$$

We note that this expression was originally found for the theory of a single real scalar field. To extend this for N real scalar fields, we note that we now have N product of Gaussian integrals which gives N products of functional determinants. Taking the natural logarithm of this product gives thus the sum of N logarithms of functional determinants. Furthermore, recall that all the linear terms vanish as we saw in Appendix. F. We can thus write the effective potential as

$$V_{\text{eff}}(\phi^i_{\text{cl}}) = V(\phi^i_{\text{cl}}) - \frac{i}{2\Omega}\ln\det\left[-\frac{\delta^2\mathcal{L}}{\delta\phi^1\delta\phi^1}\right] - \frac{i}{2\Omega}\ln\det\left[-\frac{\delta^2\mathcal{L}}{\delta\phi^2\delta\phi^2}\right] - . \quad . \quad . - \frac{i}{2\Omega}\ln\det\left[-\frac{\delta^2\mathcal{L}}{\delta\phi^N\delta\phi^N}\right]. \tag{4.32}$$

To evaluate this expression, we must thus start with the Lagrangian for the linear sigma model Eq. (4.25) and introduce the shifted fields $\phi^i = \phi^i_{\text{cl}} + \eta^i$ where we assume that the classical fields ϕ^i_{cl} are independent of space-time coordinates. Inserting these fields into the Lagrangian we obtain

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2}\mu^2(\phi^i_{\text{cl}})^2 - \frac{\lambda}{4}[(\phi^i_{\text{cl}})^2]^2 + [\mu^2 - \lambda(\phi^i_{\text{cl}})^2]\phi^i_{\text{cl}}\eta^i + \frac{1}{2}(\partial_\mu\eta^i)^2 + \frac{1}{2}\mu^2(\eta^i)^2 \\ & - \frac{\lambda}{2}[(\phi^i_{\text{cl}})^2(\eta^i)^2 + 2(\phi^i_{\text{cl}}\eta^i)^2] - \frac{\lambda}{4}[(\eta^i)^2]^2 - \lambda\phi^i_{\text{cl}}\eta^i(\eta^i)^2,\end{aligned} \tag{4.33}$$

where we dropped all constant terms. Next, we can perform an expansion in the Lagrangian about $\phi^i = \phi^i_{\text{cl}}$. Doing so, we can read off

$$\frac{\delta^2\mathcal{L}}{\delta\phi^i\delta\phi^j} = -\Box\,\delta_{ij} + \mu^2\delta_{ij} - \lambda[(\phi^k_{\text{cl}})^2\delta_{ij} + 2\phi^i_{\text{cl}}\phi^j_{\text{cl}}], \tag{4.34}$$

where we again have an implicit sum over the k index in the factor $(\phi^k_{\text{cl}})^2$. For simplicity, we now orient the classical fields to point in the Nth direction, namely

$$\phi^i_{\text{cl}} = (0, 0, ..., \phi_{\text{cl}}). \tag{4.35}$$

Thus, Eq. (4.34) becomes

$$\frac{\delta^2\mathcal{L}}{\delta\phi^m\delta\phi^n} = (-\Box + \mu^2 - \lambda\phi^2_{\text{cl}})\delta_{mn}, \qquad \frac{\delta^2\mathcal{L}}{\delta\phi^N\delta\phi^N} = -\Box + \mu^2 - 3\lambda\phi^2_{\text{cl}}. \tag{4.36}$$

where $m, n = 1, ..., N-1$. The effective potential in Eq. (4.32) then becomes

$$V_{\text{eff}}(\phi^i_{\text{cl}}) = V(\phi^i_{\text{cl}}) - \frac{i}{2\Omega}(N-1)\ln\det\left[\Box - \mu^2 + \lambda\phi^2_{\text{cl}}\right] - \frac{i}{2\Omega}\ln\det\left[\Box - \mu^2 + 3\lambda\phi^2_{\text{cl}}\right]. \tag{4.37}$$

We can now use the identity

$$\ln\det A = \text{Tr}\ln A = \int d^4x\,\langle x|\ln A\,|x\rangle\,. \tag{4.38}$$

to write

$$\begin{aligned}\ln\det\big(\Box+m^2\big) &= \int d^4x\,\langle x|\ln\big(\Box+m^2\big)|x\rangle = \int d^4x\int\frac{d^4k}{(2\pi)^4}\,\langle x|\ln\big(\Box+m^2\big)|k\rangle\,\langle k|x\rangle \\ &= \Omega\int\frac{d^4k}{(2\pi)^4}\ln\big(-k^2+m^2\big),\end{aligned} \tag{4.39}$$

where we have in the last step used the fact that the fields are independent of space-time coordinates to write the space-time integral as a four-volume factor Ω. It is easier to evaluate this integral in Euclidean space, thus we make the Wick rotation and define $k^0 = ik_E$. We then obtain

$$\ln\det\big(\Box+m^2\big) = i\Omega\int\frac{d^4k_E}{(2\pi)^4}\ln\big(k_E^2+m^2\big) = -i\Omega\,\frac{\partial}{\partial\alpha}\left(\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m^2)^\alpha}\right)\bigg|_{\alpha=0}, \tag{4.40}$$

We can evaluate the integral inside of the brackets by going to d dimensions and using Euler's Beta function. This is exactly the same process as the calculations in Appendix. E. Consequently, we have

$$\begin{aligned}-i\Omega\,\frac{\partial}{\partial\alpha}\left(\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m^2)^\alpha}\right)\bigg|_{\alpha=0} &= -i\Omega\,\frac{\partial}{\partial\alpha}\left(\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m^2)^\alpha}\right)\bigg|_{\alpha=0} \\ &= -i\Omega\,\frac{\partial}{\partial\alpha}\left(\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\frac{1}{(m^2)^{-d/2+\alpha}}\right)\bigg|_{\alpha=0} \\ &= \left(\frac{\Gamma'\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)^2}\Gamma\left(\alpha-\frac{d}{2}\right)\right)\frac{1}{(m^2)^{-d/2+\alpha}}\bigg|_{\alpha=0} \\ &\quad - e^{\frac{d}{2}\ln m^2}\ln m^2 e^{-\alpha\ln m^2}\frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma(\alpha)}\bigg|_{\alpha=0} \\ &= -i\Omega\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(-\frac{d}{2})}{(m^2)^{-d/2}},\end{aligned} \tag{4.41}$$

where we in the last step used that

$$\lim_{\alpha\to0}\frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma(\alpha)} = 0, \quad \lim_{\alpha\to0}\frac{\Gamma'\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)} = 0, \quad \lim_{\alpha\to0}\frac{\Gamma'(\alpha)}{\Gamma(\alpha)^2} = -1. \tag{4.42}$$

The effective potential then becomes,

$$V_{\rm eff}(\phi^i_{\rm cl}) = V(\phi^i_{\rm cl}) - \frac{N-1}{2(4\pi)^{d/2}}\frac{\Gamma(-\frac{d}{2})}{(-\mu^2+\lambda\phi^2_{\rm cl})^{-d/2}} - \frac{1}{2(4\pi)^{d/2}}\frac{\Gamma(-\frac{d}{2})}{(-\mu^2+3\lambda\phi^2_{\rm cl})^{-d/2}}. \tag{4.43}$$

At this point, we note that this expression is divergent and thus, we want to add counterterms which eliminate the divergences. On that note we have that

$$V_{\rm eff}(\phi^i_{\rm cl}) = V(\phi^i_{\rm cl}) - \frac{N-1}{2(4\pi)^{d/2}}\frac{\Gamma(-\frac{d}{2})}{(-\mu^2+\lambda\phi^2_{\rm cl})^{-d/2}} - \frac{1}{2(4\pi)^{d/2}}\frac{\Gamma(-\frac{d}{2})}{(-\mu^2+3\lambda\phi^2_{\rm cl})^{-d/2}} + A + \frac{1}{2}B\phi^2_{\rm cl} + \frac{1}{4}C\phi^4_{\rm cl}. \tag{4.44}$$

where A, B, and C are to be determined. We now isolate the divergences which occur in the second term of Eq. (4.43) by using dimensional regularization scheme with $d = 4-\epsilon$. Further details on such calculations can be seen in Appendix. D.1. The result is simply

$$V_{\text{eff}}(\phi^i_{\text{cl}}) = V(\phi^i_{\text{cl}}) - \frac{(-\mu^2+\lambda\phi^2_{\text{cl}})^2(N-1)}{4(4\pi)^2}\left[\frac{2}{\epsilon}+\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+\lambda\phi^2_{\text{cl}}}\right)\right] - \frac{(-\mu^2+3\lambda\phi^2_{\text{cl}})^2}{4(4\pi)^2}\left[\frac{2}{\epsilon}+\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+3\lambda\phi^2_{\text{cl}}}\right)\right] + A + \frac{1}{2}B\phi^2_{\text{cl}} + \frac{1}{4}C\phi^4_{\text{cl}}, \tag{4.45}$$

where Λ is a mass scale we introduced using the MS scheme and $\bar{\Lambda}^2 = \Lambda^2 4\pi e^{-\gamma}$ allowed us to further obtain the $\overline{\text{MS}}$ scheme. As we can see, the divergences exist in the form of the $\frac{2}{\epsilon}$ factors. We must therefore pick our counterterms such that these divergences are eliminated i.e. we want to pick A, B, and C such that

$$-\frac{(-\mu^2+\lambda\phi^2_{\text{cl}})^2(N-1)}{4(4\pi)^2}\frac{2}{\epsilon} - \frac{(-\mu^2+3\lambda\phi^2_{\text{cl}})^2}{4(4\pi)^2}\frac{2}{\epsilon} + A + \frac{1}{2}B\phi^2_{\text{cl}} + \frac{1}{4}C\phi^4_{\text{cl}} = \text{finite}. \tag{4.46}$$

With the use of algebra, it is then easy to show that

$$A = \frac{N\mu^4}{4(4\pi)^2}\frac{2}{\epsilon} + C_1, \quad B = -\frac{\lambda\mu^2(N+2)}{4(\pi)^2}\frac{2}{\epsilon} + C_2, \quad C = \frac{\lambda^2(N+8)}{4(\pi)^2}\frac{2}{\epsilon} + C_3 \tag{4.47}$$

where C_1, C_2, and C_3 are finite terms which depend on the nature of the renormalization conditions. Thus, we have

$$V_{\text{eff}}(\phi_{\text{cl}}) = -\frac{1}{2}\mu^2\phi^2_{\text{cl}} + \frac{\lambda}{4}\phi^4_{\text{cl}} - \frac{(-\mu^2+\lambda\phi^2_{\text{cl}})^2(N-1)}{4(4\pi)^2}\left[\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+\lambda\phi^2_{\text{cl}}}\right)\right] - \frac{(-\mu^2+3\lambda\phi^2_{\text{cl}})^2}{4(4\pi)^2}\left[\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+3\lambda\phi^2_{\text{cl}}}\right)\right] + C_1 + \frac{1}{2}C_2\phi^2_{\text{cl}} + \frac{1}{4}C_3\phi^4_{\text{cl}}. \tag{4.48}$$

In what follows, we shall use the $\overline{\text{MS}}$ renormalization scheme which allows us to set $C_1 = C_2 = C_3 = 0$. However, with such a choice of renormalization, we must pick our renormalization scale appropriately to satisfy the renormalization conditions which we are working with. One condition which we shall consider is

$$\left.\frac{\partial V_{\text{eff}}}{\partial\phi_{\text{cl}}}\right|_{\phi_{\text{cl}}=\frac{\mu}{\lambda}} = 0. \tag{4.49}$$

If we apply this condition to Eq. (4.48), after some simple algebra, we obtain

$$\frac{12\sqrt{\lambda}\mu}{4(4\pi)^2}(2\mu^2) + \frac{12\sqrt{\lambda}\mu}{4(4\pi)^2}(2\mu^2)\ln\frac{\bar{\Lambda}^2}{2\mu^2} = 0 \tag{4.50}$$

To satisfy this equation we can set the renormalization scale as

$$\bar{\Lambda}^2 = 2\mu^2\exp\left(-\frac{12\sqrt{\lambda}\mu}{4(4\pi)^2}(2\mu^2)\right) \tag{4.51}$$

With this choice of renormalization scale, we can now write the final expression for the effective potential of the linear sigma model, namely

$$V_{\text{eff}}(\phi_{\text{cl}}) = -\frac{1}{2}\mu^2\phi^2_{\text{cl}} + \frac{\lambda}{4}\phi^4_{\text{cl}} - \frac{(-\mu^2+\lambda\phi^2_{\text{cl}})^2(N-1)}{4(4\pi)^2}\left[\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+\lambda\phi^2_{\text{cl}}}\right)\right] - \frac{(-\mu^2+3\lambda\phi^2_{\text{cl}})^2}{4(4\pi)^2}\left[\frac{3}{2}+\ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+3\lambda\phi^2_{\text{cl}}}\right)\right]. \tag{4.52}$$

Finally, we can specify the linear sigma model to the case of two complex scalar fields, namely $N = 4$. Doing so, the effective potential for the theory of two complex scalar fields in the ϕ^4 model can be written as

$$\begin{aligned} V_{\text{eff}}(\phi_{\text{cl}}) = -\frac{1}{2}\mu^2\phi_{\text{cl}}^2 + \frac{\lambda}{4}\phi_{\text{cl}}^4 - \frac{3(-\mu^2+\lambda\phi_{\text{cl}}^2)^2}{4(4\pi)^2}\left[\frac{3}{2} + \ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+\lambda\phi_{\text{cl}}^2}\right)\right] \\ - \frac{(-\mu^2+3\lambda\phi_{\text{cl}}^2)^2}{4(4\pi)^2}\left[\frac{3}{2} + \ln\left(\frac{\bar{\Lambda}^2}{-\mu^2+3\lambda\phi_{\text{cl}}^2}\right)\right]. \end{aligned} \tag{4.53}$$

To close this section, we note that the spin-0 particles which we have considered in this section, and in particular in the linear sigma model with $N = 4$, can infact represent the two charged pions and the one netural pion of QCD with an additional field which we call the sigma field. Thus we can say that the linear sigma model with $N = 4$ scalar particles is an effective model of QCD. To understand this better, we note that an effective model is an approximation for an underlying physical theory such as the theory of QCD. Furthermore, an effective model must include the appropriate degrees of freedom in correspondence with the physical theory of interest up to some specified energy scale. The linear sigma model is therefore an effective model of QCD in the sense that it provides us with an appropriate Lagrangian, yet not the exact QCD Lagrangian, which has the relevant degrees of freedom for the pions and can thus give us an approximation to QCD at a given energy scale.

5 Conclusion and outlook

5.1 Summary and conclusion

In this work, we have studied the behavior of different particles in an external magnetic field. We started by finding the energy density of both a spin-0 and a spin-1/2 particle. This was done with the help of the partition function of the respective theory, which was calculated using the imaginary time formalism and expressed as a path integral. We found that for both cases, the energy density has both a vacuum contribution and a finite temperature contribution. We have focused in this work on the vacuum energy density and regularized our result using dimensional regularization.

Next, we have explored the interaction of both a spin-0 and a spin-1/2 particle with a constant magnetic field. We first introduced a gauge field to the Lagrangian for both a spin-0 particle and a spin-1/2 particle by adding the Maxwell Lagrangian and requiring local gauge invariance. Next, we specialized to the case of a constant magnetic field and found the field equations which was the Klein-Gordon equation and the Dirac equation coupled to a constant magnetic field, for the spin-0 and spin-1/2 particle respectively. We picked the Landau gauge and represented the wavefunctions as Landau eigenfunctions of the corresponding Landau energy levels. Using the wavefunctions, we calculated the propagator of a spin-0 particle in a constant magnetic field. Finally, we calculated the one-loop free vacuum energy density of the spin-0 particle in a constant magnetic field using the respective propagator in the coincidence limit. We regularized our result using both dimensional regularization and a momentum cutoff and renormalized to eliminate the divergences.

After having worked with free theories, we then explored interacting theories using the ϕ^4 model. We explored spontaneous symmetry breaking of both discrete and continuous symmetries for spin-0 fields. We then investigated spontaneous symmetry breaking for N real scalars using the linear sigma model. We calculated, to lowest order in quantum corrections, the effective potential of the linear sigma model and specialized to the case of $N = 4$ real scalars, or equivalently to two complex scalar fields. Finally, we regularized our result using dimensional regularization and renormalized using the $\overline{\text{MS}}$ renormalization scheme.

We have thus in this work developed the fundamental tools that are necessary to study QCD in a strong magnetic field. We calculated the free energy densities for free theories, free theories interacting with a constant magnetic field, and interacting theories with quartic interactions and spontaneous symmetry breaking. The spin-0 particles which we have considered in fact can represent the pions of QCD. Thus, as a next step, we could explore spontaneous symmetry breaking of the $N = 4$ linear sigma model in a constant magnetic field. We shall discuss this concept further in the outlook.

5.2 Outlook

We will now look at a few ways which we could continue this work.

Spontaneous symmetry breaking and the linear sigma model with $N = 4$ real scalars in a constant magnetic field

Having explored the spontaneous symmetry breaking of the linear sigma model specialized to $N = 4$ real scalars, which is equivalent to the theory of two complex scalar fields, the next logical step is to investigate the spontaneous symmetry breaking of such a system in a constant magnetic field. The outline for this procedure is as follows.

We would start with the Lagrangian for the two complex scalar fields in the ϕ^4 model. Next, similar to what we did for the case of a complex scalar field in a constant magnetic field, we would add the Maxwell

Lagrangian and impose local gauge invariance on the total Lagrangian. Next, we let the symmetry spontaneously break and as a result the three massless goldstone bosons which we found previously for the linear sigma model with $N = 4$ become the two charged pions ($\pi^{\pm}$) and the one neutral pion (π^0). The remaining scalar field, we call the sigma field σ. Thus, we are left with two neutral scalar fields and two charged scalar fields. The two charged scalars are complex scalar fields while the two neutral fields are individual real scalars. In order to find the effective potential of the system we see that for the two neutral scalar fields, namely the neutral pion and the sigma field, the magnetic field does not couple to the respective Lagrangian of the neutral particles by adding a Maxwell field Lagrangian and requiring local gauge invariance. This means the effective potential is simply the effective potential for two real scalar fields, without an external magnetic field, which we computed in Section. 2 of this thesis. For the complex scalar field, we can recycle the result we got for the vacuum energy density of a complex scalar field in a constant magnetic field Section. 3. It is important to note a minor change in detail however since the mass of the charged particles are now dependent on the magnetic field. This will mean that when we regularize the final result, the renormalization parameters which we pick will be different. Finally, by specifying the problem to low energies, we obtain a low-energy effective model for QCD in an external magnetic field specialized to the case of the two charged pions, one neutral pion, and one sigma field.

As a final note it is important to understand the meaning of an effective model since we have claimed that the linear sigma model is an effective model of QCD. An effective model is an approximation for an underlying physical theory such as the theory of QCD. Furthermore, an effective model must include the appropriate degrees of freedom in correspondence with the physical theory of interest up to some specified energy scale. The linear sigma model is therefore an effective model of QCD in the sense that it provides us with an appropriate Lagrangian, yet not the exact QCD Lagrangian, which has the relevant degrees of freedom for the pions and can thus give us an approximation to QCD at a given energy scale.

Exploring other low-energy effective models of QCD

Our calculations in this work was aimed towards one specific low-energy effective model, namely the linear sigma model specialized to $N = 4$ real scalars. However, there are numerous models which exist with each considering QCD in a strong magnetic field in their own way. More importantly these theories now consider the energy density of QCD in an external magnetic field as a funcion of temperature and magnetic field strength. This way, we would be able to explore some of the phenomena and behavior of QCD in an external magnetic field for varying temperatures and magnetic field strengths. Consequently, we would have a more appropriate representation for real physical systems where magnetic field and temperature is not a constant. These low-energy effective models of interest include the Massachusetts Institute of Technology (MIT) bag model [26], chiral perturbation theory (χPT) [27][28][29], the Nambu–Jona-Lasinio (NJL) model [30][31], the quark- meson (QM) model, and the hadron resonance gas model [32]. Furthermore, since we are now considering varying field strengths and temperatures, we could also explore the phase diagram of QCD. In particular we could look at the regime where deconfinement of hadrons occur. If we consider systems of zero chemical potential $\mu_f = 0$ we can compare the results from the effective models to that of exact numerical approach such as lattice QCD for temperatures where phase transition occurs. Furthermore, we could also explore the phase diagram of QCD for non-zero chemical potentials using the effective models. However, such calculations would be purely based on approximations as no exact approach is possible.

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A Grassmann variables and Grassmann algebra

A.1 Grassmann variables

The work in this appendix follows closely to that of [33]. We start by considering N Grassmann variables which are the generators of a Grassmann algebra. We define these variables as $x_i, i = 1, 2, ..., N$ which satisfy the anti-commutation relation

$$\{x_i, x_j\} = x_i^2 + x_j^2 = 0. \tag{A.1}$$

For the case where $i = j$, we have that $x_i^2 = 0$ which means that Grassmann variables are 2nd degree nilpotent. We next consider the dimension of such algebra as a vector space. We can do this by creating a linear combination of all possible monomials involving the product of the different Grassmann variables. We note that each monomial can have at most one unique Grassmann variable due to the anti-symmetry described in Eq. (A.1). We therefore have exactly 2^N possible monomials. Thus, the most general function of the generators can be described using a linear combination of these monomials, namely

$$f(x_1, x_2, ..., x_N) = f_0 + \sum_k f_1(k) x_k + \sum_{k_1, k_2} f_2(k_1, k_2) x_{k_1} x_{k_2} + ... + f_N(1, 2, ..., N) x_1 x_2 ... x_N. \tag{A.2}$$

The dimension of such an algebra with N generators is therefore the linear combination of all the monomials, namely 2^N.

A.2 Differentiation

Due to the anti-commuting nature of Grassmann variables it is important to define the direction of which the derivatives act in. We will use the convention of the differential operator acting on the right. This is important as differential operators with respect to Grassmann variables also anti-commute with Grassmann variables. If we have two Grassmann variables x and y, then a function of these variables can be represented according to Eq. (A.2), namely

$$f(x, y) = f_0 + f_x x + f_y y + f_{xy} xy = f_0 + f_x x + f_y y - f_{xy} yx. \tag{A.3}$$

Differentiating with respect to x first we have that

$$\frac{\partial f(x, y)}{\partial x} = f_x + f_{xy} y, \tag{A.4}$$

whereas differentiating with respect to y yields

$$\frac{\partial f(x, y)}{\partial y} = f_y - f_{xy} x. \tag{A.5}$$

A second order partial derivative can thus be written as

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = -\frac{\partial^2 f(x, y)}{\partial y \partial x} = -f_{xy}. \tag{A.6}$$

Finally, the chain rule of differentiation is the same for Grassmann variables as for normal calculus. However, this is not the case for the product rule due to the anti-symmetry in the fields. For two functions of Grassmann variables (f_1 and f_2) the equation

$$\frac{\partial (f_1 f_2)}{\partial x_a} = \frac{\partial (f_1)}{\partial x_a} f_2 + f_1 \frac{\partial (f_2)}{\partial x_a}, \tag{A.7}$$

only holds if f_1 is of even order in the Grassmann variables. Otherwise, we have to introduce a minus sign to the second term.

A.3 Berezin integration

Integration that involves Grassmann variables is known as Berezin integration. To no surprise, the anti-commutative nature of the Grassmann variables also extend to the integration elements where the variables themselves also commute with the integration measures, namely

$$\{x_i, dx_i\} = \{x_i, dx_j\} = \{dx_i, x_j\} = \{dx_i, dx_j\} = 0. \tag{A.8}$$

It turns out that the properties for definite Riemann integrals over normal variables holds for indefinite integrals over Grassmann variables. These properties include translation invariance,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x+y)dx, \tag{A.9}$$

and linearity,

$$\int_{-\infty}^{\infty} [a + bf(x)]dx = a\int_{-\infty}^{\infty} dx + b\int_{-\infty}^{\infty} f(x)dx. \tag{A.10}$$

We first turn our focus to Eq. (A.9) and set $f(x) = a + bx$ where a, b are complex variables and x, y are Grassmann variables. We then have

$$\int f(x+y)dx = \int a + b(x+y)dx = \int (a+bx)dx + by\int dx = \int f(x)dx + by\int dx, \tag{A.11}$$

which means

$$\int f(x+y)dx - \int f(x)dx = by\int dx = 0. \tag{A.12}$$

We therefore have that

$$\int dx = 0. \tag{A.13}$$

Conventionally, we also set

$$\int x dx = 1. \tag{A.14}$$

From Eq. (A.13) and Eq. (A.14), it becomes evident that Berezin integration is very similar to normal differentiation in the sense that the derivative of 1 with respect to x is zero and the derivative of x with respect to x is 1. This can be seen more clearly if we consider a change in variables when performing integration. In normal Riemann integrals, we would have

$$\int f(ax)dx = \frac{1}{a}\int f(x)dx. \tag{A.15}$$

In Berezin integration however, we have that

$$\int f(ax)dx = a\int f(x)dx, \tag{A.16}$$

which can be easily shown by setting $y = ax$, where x and y are both Grassmann variables, and remembering that

$$\int y dy = \int x dx = 1. \tag{A.17}$$

We next move on to Gaussian Berezin integrals. Before proceeding we note that multiple integrals are considered as iterated over in respective order, namely

$$\int x_j x_i dx_i dx_j = \int x_j \left(\int x_i dx_i \right) dx_j. \tag{A.18}$$

The trivial Gaussian Berezin integral is that of which there exists a product of two or more identical Grassman variables in the exponent. Using the nilpotent nature of the Grassmann variables, this yields

$$\int e^{x^2} dx = 0. \tag{A.19}$$

We thus look at terms in the exponential where no two term is alike. We first look at an exponential as a function of two Grassmann variables

$$\int e^{-axy} dx dy = \int (1 - axy) dx dy = a \int yx dx dy = a, \tag{A.20}$$

where we Taylor expanded the exponential factor in the second term. Finally, we consider the Gaussian Berezin integral where the exponential is a function of N Grassmann variables represented by x_i and N Grassmann variables represented by y_i where we also define

$$dx dy = dx_1 dy_1 dx_2 dy_2 ... dx_N dy_N. \tag{A.21}$$

The integral we are interested in thus takes the form,

$$\mathcal{I} = \int e^{-x_i A_{ij} y_j} dx dy. \tag{A.22}$$

We can now perform a change in basis in order to perform a unitary transformation on A_{ij} to diagonalize it. We call this diagonalized matrix A'. This transforms the exponential term in Eq. (A.22) to a product of exponential factors of the form $e^{-x'_i A'_{ii} y'_i}$. Therefore, we have that

$$\mathcal{I} = \int e^{-x_i A_{ij} y_j} dx dy = \prod_i \int e^{-x'_i A'_{ii} y'_i} dx'_i dy'_i = \prod_i A'_{ii} = \det(A') = \det(A), \tag{A.23}$$

where in the last step, we used that a unitary transformation to a matrix leaves the determinant of that matrix invariant.

B Harmonic oscillator: Energy eigenvalues and eigenfunctions

The following calculations follow closely [34] and [35]. The Schrodinger equation for the one-dimensional harmonic oscillator oscillating with frequency ω about a point x_0 is given by

$$E\psi = -\frac{1}{2m}\psi'' + \frac{1}{2}m\omega^2(x - x_0)^2\psi. \tag{B.1}$$

We can rearrange this to obtain

$$\psi'' + (2Em - m^2\omega^2(x - x_0)^2)\psi = 0. \tag{B.2}$$

This is a second order differential equation that can be solved using a power series. Introducing the dimensionless variable

$$\lambda = \sqrt{m\omega}(x - x_0), \tag{B.3}$$

Eq. (B.2) becomes

$$\frac{\mathrm{d}^2\psi(\lambda)}{\mathrm{d}\lambda^2} + \left(\frac{2E}{\omega} - \lambda^2\right)\psi = 0. \tag{B.4}$$

We thus want an ansatz for $\psi(\lambda)$ such that the double derivative term gives us λ^2 plus some constant. To that end, we pick

$$\psi(\lambda) = u(\lambda)e^{-\frac{1}{2}\lambda^2}, \tag{B.5}$$

which when inserted back into Eq. (B.4) gives us the following second order linear differential equation,

$$u'' - \lambda u' + \left(\frac{2E}{\omega} - 1\right)u = 0. \tag{B.6}$$

We now write the general solution of u as a power series in λ, namely

$$u(\lambda) = \sum_{j=0}^{\infty}\alpha_j\lambda^j. \tag{B.7}$$

The first derivative of this is

$$u'(\lambda) = \sum_{j=0}^{\infty}j\alpha_j\lambda^{j-1}, \tag{B.8}$$

and the second derivative is

$$u''(\lambda) = \sum_{j=0}^{\infty}j(j-1)\alpha_j\lambda^{j-2} = \sum_{j=0}^{\infty}(j+2)(j+1)\alpha_{j+2}\lambda^j, \tag{B.9}$$

where in the last step, we set $j = j + 2$ which is allowed since the first two terms ($j = 0$ and $j = 1$ term) in the series expansion of u'' vanishes. Inserting this into Eq. (B.6), we obtain

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty}(j+2)(j+1)\alpha_{j+2}\lambda^j - 2\lambda\sum_{j=0}^{\infty}j\alpha_j\lambda^{j-1} + \left(\frac{2E}{\omega} - 1\right)\sum_{j=0}^{\infty}\alpha_j\lambda^j \\ &= \sum_{j=0}^{\infty}\left[(j+2)(j+1)\alpha_{j+2} + \left(\frac{2E}{\omega} - 1 - 2j\right)\alpha_j\right]\lambda^j. \end{aligned} \tag{B.10}$$

Due to the uniqueness of power series expansions [34], we therefore have that the coefficient of each power of λ must vanish which means

$$(j+2)(j+1)\alpha_{j+2} + \left(\frac{2E}{\omega} - 1 - 2j\right)\alpha_j = 0. \tag{B.11}$$

We can rearrange this to obtain

$$\alpha_{j+2} = \frac{\left(\frac{2E}{\omega} - 1 - 2j\right)}{(j+2)(j+1)}\alpha_j. \tag{B.12}$$

From this recursion formula, we can generate all the even j and odd j coefficients starting from a_0 and a_1 respectively. Thus our complete solution for $u(\lambda)$ can be written in terms of the even and odd coefficients, namely

$$u(\lambda) = u_{\text{even}}(\lambda) + u_{\text{odd}}(\lambda), \tag{B.13}$$

where

$$u_{\text{even}} = a_0 + \lambda^2 a_2 + \lambda^4 a_4 + ... \tag{B.14}$$

and

$$u_{\text{odd}} = \lambda a_1 + \lambda^3 a_3 + \lambda^5 a_5 + ... \tag{B.15}$$

However, the recursion formula Eq. (B.12) in the limit $j \to \infty$ yields approximately

$$\frac{a_{n+2}}{a_n} \approx \frac{j}{2}, \tag{B.16}$$

which is the same ratio as that of the series for $\lambda^j e^{\lambda^2}$ [35]. Thus, in the $j \to \infty$ limit,

$$\psi(\lambda) \propto e^{\frac{\lambda^2}{2}}, \tag{B.17}$$

which clearly diverges in the case when $\lambda \to \infty$. This is of course not desirable as such a wavefunction would not have a physical meaning. In order to avoid this issue, we therefore want the recursion formula Eq. (B.12) to be truncated at some maximum value of value, call it n. Thus, we want $a_{n+2} = 0$ which we obtain if we set the numerator in Eq. (B.12) to zero. Furthermore, we must set $a_1 = 0$ for even values of n and $a_0 = 0$ for odd values to ensure we are only considering u_{even} or u_{odd} respectively. We then have

$$\frac{2E}{\omega} - 1 - 2n = 0, \tag{B.18}$$

which when rearranged gives us the energy spectrum for a harmonic oscillator, namely

$$E = \omega\left(n + \frac{1}{2}\right). \tag{B.19}$$

Next we want to find the eigenfunctions associated with each energy level. To do this, we consider first the expressions for $u(\lambda)$ when n is even, namely

$$\begin{aligned}
&u(\lambda) = a_0, \quad n = 0 \\
&u(\lambda) = a_0(1 - 2\lambda^2), \quad n = 2 \\
&u(\lambda) = a_0\left(1 - 4\lambda^2 + \frac{4}{3}\lambda^4\right), \quad n = 4 \\
&\quad ...
\end{aligned} \tag{B.20}$$

Similarly, for odd values of n, we obtain

$$
\begin{aligned}
&u(\lambda) = \lambda a_1, \quad n = 1 \\
&u(\lambda) = a_1\left(\lambda - \frac{2}{3}\lambda^3\right), \quad n = 3 \\
&u(\lambda) = a_1\left(\lambda - \frac{4}{3}\lambda^3 + \frac{4}{15}\lambda^5\right), \quad n = 5 \\
&\quad \ldots
\end{aligned}
\tag{B.21}
$$

These functions can be represented using Hermite polynomials, H_n, which are defined as

$$
H_n(\lambda) = (-1)^n e^{\lambda^2} \frac{\partial^n}{\partial \lambda^n} e^{-\lambda^2}. \tag{B.22}
$$

The first six Hermite polynomials are given below:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= 2\lambda \\
H_2 &= 4\lambda^2 - 2 \\
H_3 &= 8\lambda^3 - 12\lambda \\
H_4 &= 16\lambda^4 - 48\lambda^2 + 12 \\
H_5 &= 32\lambda^5 - 160\lambda^3 + 120\lambda.
\end{aligned}
\tag{B.23}
$$

where it is important to distinguish between the even and odd polynomials. Thus, the wavefunction of the harmonic oscillator can be written as

$$
\psi_n(\lambda) = N_n H_n(\lambda) e^{-\frac{\lambda^2}{2}}, \tag{B.24}
$$

where N_n is a normalization constant. We can find the normalization constant by expressing H_n in terms of the functional,

$$
S(\lambda, j) = e^{\lambda^2 - (j-\lambda)^2} = e^{-j^2 + 2j\lambda} = \sum_{n=0}^{\infty} \frac{H_n(\lambda)}{n!} j^n, \tag{B.25}
$$

and using the general property for a normalized wavefunction, namely

$$
\int_{-\infty}^{\infty} |\psi_n(x)|^2 \, dx = \frac{|N_n|^2}{\sqrt{m\omega}} \int_{-\infty}^{\infty} H_n^2(\lambda) e^{-\lambda^2} d\lambda = 1. \tag{B.26}
$$

Using Eq. (B.25), we can express $\int_{-\infty}^{\infty} H_n^2(\lambda) d\lambda$ by considering the integral

$$
\int_{-\infty}^{\infty} e^{-j^2+2j\lambda} e^{-k^2+2k\lambda} e^{-\lambda^2} d\lambda = \sum_{n=0}^{\infty} \frac{j^n k^n}{n! n!} \int_{-\infty}^{\infty} H_n^2(\lambda) e^{-\lambda^2} d\lambda = \sqrt{\pi} e^{2jk} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2jk)^n}{n!}, \tag{B.27}
$$

where in the penultimate step, we used the usual rules for Gaussian integration. Thus, we have that

$$
\int_{-\infty}^{\infty} H_n^2(\lambda) e^{-\lambda^2} d\lambda = \sqrt{\pi} 2^n n!, \tag{B.28}
$$

which we can insert back into Eq. (B.26) to obtain the correct expression for the normalization constant, namely

$$N_n = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}}. \tag{B.29}$$

Thus, our final expression for the properly normalized wavefunction of the one-dimensional harmonic oscillator is

$$\psi_n(\lambda) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\lambda) e^{-\frac{\lambda^2}{2}}. \tag{B.30}$$

C Scalar propagator in the presence of a constant magnetic field

We want to obtain a Landau representation of the propagator for a scalar particle in the presence of a constant magnetic field. Most of the calculations in this section follows closely those in [36],[25], and [37]. To begin, we first start with the formal definition for the scalar propagator in a constant magnetic field, namely

$$D(X,X') = i\left[(i\partial - qA)^2 - m^2\right]^{-1}\delta^4(X-X'), \tag{C.1}$$

where $X = (t,x,y,z) = (t,\boldsymbol{x})$ and the indices on the 4-vectors have been dropped for brevity. Furthermore, we note that we are now assuming the presence of a constant magnetic field in the z-direction with the choice of Landau gauge $\boldsymbol{A} = (0,Bx,0)$. We proceed by performing a Fourier transform of the propagator in the coordinates parallel to the direction of the magnetic field, namely t and z. Thus, we want to find $D(\omega,p_z,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp)$. We begin by using the fact that the propagator in Eq. (C.1) can be written as,

$$D(t-t',z-z',\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}e^{ip_z(z-z')}D(\omega,p_z,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp), \tag{C.2}$$

where $\boldsymbol{x}_\perp = (x,y)$ represents the spatial directions perpendicular to the magnetic field. Next, we can write the Fourier transform of the four-dimensional delta function as

$$\delta^4(X-X') = \delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}e^{ip_z(z-z')}. \tag{C.3}$$

Inserting Eq. (C.2) and Eq. (C.3) into Eq. (C.1), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}e^{ip_z(z-z')}D(\omega,p_z,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp) &= i\left[(i\partial - qA)^2 - m^2\right]^{-1}\delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)\\ &\times\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}e^{ip_z(z-z')}.\end{aligned} \tag{C.4}$$

Writing this expression out explicitly with our choice of gauge and using the fact that $\partial = (\frac{\partial}{\partial t},\boldsymbol{\nabla})$, we have that

$$D(\omega,p_z,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp) = i(\omega^2 - p_z^2 - \boldsymbol{\pi}_\perp^2 - m^2)^{-1}\delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp), \tag{C.5}$$

where we have defined $\boldsymbol{\pi}_\perp \equiv (-i\frac{\partial}{\partial x} - qA_x, -i\frac{\partial}{\partial y} - qA_y)$. In order to obtain a Landau representation for the scalar propagator Eq. (C.5) we must utilize the complete set of eigenstates of our system Eq. (3.35), or more specifically, of the $\boldsymbol{\pi}_\perp^2$ operator. Setting $q = -e$ with $e > 0$, we thus have

$$\psi_{n,p_y}(\boldsymbol{x}_\perp) = \frac{1}{\sqrt{2\pi}}\left(\frac{eB}{\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\lambda)e^{-\frac{\lambda^2}{2}}e^{ip_y y}, \tag{C.6}$$

where $\lambda = \sqrt{eB}(x + \frac{p_y}{eB})$. The $\boldsymbol{\pi}_\perp^2$ operator satisfies the eigenvalue equation

$$\boldsymbol{\pi}_\perp^2\psi_{n,p_y}(\boldsymbol{x}_\perp) = 2eB(n + \frac{1}{2}), \tag{C.7}$$

which can be seen by following the same steps as in Appendix. B. Furthermore, the wavefunctions satisfy the completeness relation,

$$\int_{-\infty}^{\infty}dp_y\sum_{n=0}^{\infty}\psi_{n,p_y}(\boldsymbol{x}_\perp)\psi^*_{n,p_y}(\boldsymbol{x}'_\perp) = \delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp). \tag{C.8}$$

Thus, we can write Eq. (C.5) as

$$
\begin{aligned}
D(\omega, p_z, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp) &= \int_{-\infty}^{\infty} dp_y \sum_{n=0}^{\infty} i\left(\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2\right)^{-1} \psi_{n,p_y}(\boldsymbol{x}_\perp)\psi^*_{n,p_y}(\boldsymbol{x}'_\perp) \\
&= \int_{-\infty}^{\infty} dp_y \sum_{n=0}^{\infty} i\left(\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2\right)^{-1} \frac{1}{2\pi}\left(\frac{eB}{\pi}\right)^{\frac{1}{2}} \frac{1}{2^n n!} \\
&\quad \times H_n\left(\sqrt{eB}\left(x + \frac{p_y}{eB}\right)\right) H_n\left(\sqrt{eB}\left(x' + \frac{p_y}{eB}\right)\right) e^{-\frac{eB}{2}\left(x+\frac{p_y}{eB}\right)^2} e^{-\frac{eB}{2}\left(x'+\frac{p_y}{eB}\right)^2} e^{ip_y(y-y')}.
\end{aligned}
\tag{C.9}
$$

We can solve the p_y integral by introducing the variables

$$
u = \frac{\sqrt{eB}}{2}\left[(x + x') - i(y - y') + \frac{p_y}{\sqrt{eB}}\right], \quad a = \frac{\sqrt{eB}}{2}\Big[(x - x') + i(y - y')\Big], \quad b = -\frac{\sqrt{eB}}{2}\Big[(x - x') - i(y - y')\Big].
\tag{C.10}
$$

Thus, we have that

$$
\begin{aligned}
D(\omega, p_z, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp) &= \int_{-\infty}^{\infty} dp_y \sum_{n=0}^{\infty} i\left(\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2\right)^{-1} \frac{1}{2\pi}\left(\frac{eB}{\pi}\right)^{\frac{1}{2}} \frac{1}{2^n n!} \\
&\quad \times H_n(u + a)H_n(u + b)e^{-\frac{1}{2}(u+a)^2} e^{-\frac{1}{2}(u+b)^2} e^{ip_y(y-y')} \\
&= \int_{-\infty}^{\infty} dp_y \sum_{n=0}^{\infty} i\left(\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2\right)^{-1} \frac{1}{2\pi}\left(\frac{eB}{\pi}\right)^{\frac{1}{2}} \frac{1}{2^n n!} \\
&\quad \times H_n(u + a)H_n(u + b)e^{-u^2} e^{-i\frac{eB}{2}(x+x')(y-y')} e^{-\frac{eB}{4}\left[(x-x')^2+(y-y')^2\right]} \\
&= i\frac{eB}{2\pi} e^{i\Phi} e^{-\frac{\eta}{2}} \sum_{n=0}^{\infty} \frac{L_n(\eta)}{\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2},
\end{aligned}
\tag{C.11}
$$

where we have defined the Schwinger phase as

$$
\Phi = -\frac{eB}{2}(x + x')(y - y'),
\tag{C.12}
$$

and

$$
\eta = \frac{eB}{2}\left[(x - x')^2 + (y - y')^2\right].
\tag{C.13}
$$

Furthermore, in the last step of Eq. (C.11), we used the table integral

$$
\int_{-\infty}^{\infty} du e^{-u^2} H_m(u + a)H_n(u + b) = 2^n\sqrt{\pi} m! b^{n-m} L_m^{n-m}(-2ab),
\tag{C.14}
$$

where L_α^β are the generalized Laguerre polynomials and we set $L_\alpha^0 = L_\alpha$ for brevity. The propagator in Eq. (C.11) can be considered as the product of a translationally non-invariant factor $e^{i\Phi}$ and a translationally invariant factor. Thus, the full propagator is not translationally invariant due to the Schwinger phase factor. Physically, this means that the momentum of the scalar particles in the x and y direction, i.e. directions perpendicular to the magnetic field, are not conserved quantities. We will proceed by considering only the translationally invariant part of the propagator, namely

$$
\tilde{D}(\omega, p_z, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp) = i\frac{eB}{2\pi} e^{-\frac{\eta}{2}} \sum_{n=0}^{\infty} \frac{L_n(\eta)}{\omega^2 - p_z^2 - 2eB\left(n + \frac{1}{2}\right) - m^2}.
\tag{C.15}
$$

We can now complete the Fourier transform of the translationally invariant part of the propagator in the x and y direction, namely

$$\begin{aligned}\tilde{D}(\omega, p_x, p_y, p_z) &= i\frac{eB}{2\pi}\int_{-\infty}^{\infty} dx e^{-ip_x(x-x')}\int_{-\infty}^{\infty} dy e^{-ip_y(y-y')}\tilde{D}(\omega, p_z, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp)\\ &= i\frac{eB}{2\pi}\sum_{n=0}^{\infty}\frac{1}{\omega^2 - p_z^2 - 2eB\left(n+\frac{1}{2}\right) - m^2}\int_0^\infty\int_0^{2\pi} dr d\theta r e^{-\frac{eB}{4}r^2} L_n\left(\frac{eB}{2}r^2\right) e^{-ip_x r\cos\theta - ip_y r\sin\theta},\end{aligned} \tag{C.16}$$

where in the last step, we have made the substitutions $u = x - x'$ and $v = y - y'$ and then switched to polar coordinates in the $u - v$ plane, namely $u = r\cos\theta$ and $v = r\sin\theta$. We can evaluate the θ integral using for instance a table integral or in our case, Mathematica giving us

$$\int_0^{2\pi} d\theta e^{-ip_x r\cos\theta - ip_y r\sin\theta} = 2\pi J_0\left(r\sqrt{p_x^2 + p_y^2}\right). \tag{C.17}$$

where J_0 is the zeroth order Bessel function. Using this expression together with the table integral [38]

$$\int_0^\infty d\xi \xi^{\frac{\lambda}{2}} e^{-p\xi} J_\lambda(b\sqrt{\xi}) L_n^\lambda(c\xi) = \left(\frac{b}{2}\right)^\lambda \frac{(p-c)^n}{p^{\lambda+n+1}} e^{-\frac{b^2}{4p}} L_n^\lambda\left(\frac{b^2 c}{4pc - 4p^2}\right), \tag{C.18}$$

we can evaluate the r integral as well. Thus, Eq. (C.16) becomes

$$\tilde{D}(\omega, p_x, p_y, p_z) = 2ie^{-\frac{\boldsymbol{p}_\perp^2}{eB}}\sum_{n=0}^{\infty}\frac{(-1)^n L_n\left(2\frac{\boldsymbol{p}_\perp^2}{eB}\right)}{\omega^2 - p_z^2 - 2eB\left(n+\frac{1}{2}\right) - m^2}, \tag{C.19}$$

where we have defined $\boldsymbol{p}_\perp \equiv (p_x, p_y)$. Finally, we want to represent the propagator in Eq. (C.19) in what is the known as the proper time formalism. We do this by noting that

$$\frac{i}{\omega^2 - p_z^2 - 2eB\left(n+\frac{1}{2}\right) - m^2 + i\epsilon} = \int_0^\infty ds e^{is[\omega^2 - p_z^2 - 2eB\left(n+\frac{1}{2}\right) - m^2 + i\epsilon]}, \tag{C.20}$$

and using the summation formula for the Laguerre polynomials, namely

$$\sum_{n=0}^{\infty} L_n^\alpha(x) z^n = (1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right). \tag{C.21}$$

We can therefore write the final expression for the translationally invariant part of the scalar propagator in the presence of a constant magnetic field as

$$\begin{aligned}\tilde{D}(\omega, p_x, p_y, p_z) &= 2e^{-\frac{\boldsymbol{p}_\perp^2}{eB}}\int_0^\infty ds\sum_{n=0}^{\infty} e^{is[\omega^2 - p_z^2 - 2eB\left(n+\frac{1}{2}\right) - m^2]} L_n\left(2\frac{\boldsymbol{p}_\perp^2}{eB}\right)\\ &= 2e^{-\frac{\boldsymbol{p}_\perp^2}{eB}}\int_0^\infty ds e^{is[\omega^2 - p_z^2 - m^2]}\frac{e^{-iseB}}{1 + e^{-2iseB}}\exp\left(\frac{2\frac{\boldsymbol{p}_\perp^2}{eB}e^{-2iseB}}{1 + e^{-2iseB}}\right)\\ &= \int_0^\infty ds\frac{e^{-is[-\omega^2 + p_z^2 + m^2 + \frac{\boldsymbol{p}_\perp^2}{eBs}\tan(eBs)]}}{\cos(eBs)},\end{aligned} \tag{C.22}$$

where we in the last line used the relations

$$\frac{e^{-iseB}}{1+e^{-2iseB}} = \frac{1}{2}\frac{1}{\cos(eBs)}, \tag{C.23}$$

and

$$\exp\left(\frac{2\frac{\boldsymbol{p}_\perp^2}{eB}e^{-2iseB}}{1+e^{-2iseB}}\right) = e^{(\boldsymbol{p}_\perp^2 - i\boldsymbol{p}_\perp^2\tan(eBs))/eB}. \tag{C.24}$$

D Calculations of vacuum energy density of a scalar particle in a constant magnetic field: Regularization methods and renormalization

The calculations in this appendix are regarding the vacuum energy density of a scalar particle in a constant magnetic field, Section. 3.5. As we saw, when calculating the vacuum energy density, we obtained an integral which was ultraviolet divergent, namely Eq. (3.80). Thus, to eliminate the divergences we must use appropriate regularization and renormalization schemes. In this appendix, we will regularize the integral using two different regularization schemes, namely dimensional regularization and by introducing an ultraviolet cutoff.

It is important to note that the calculations in this section prove to be useful and are referred to in other sections of this work as well, namely Section. 4.4 and Appendix. E.

D.1 Dimensional regularization

To begin dimensional regularization, we want to evaluate Eq. (3.80) in d dimensions namely

$$\mathcal{I} = \int \frac{d^d p}{(2\pi)^d} e^{-s[\omega^2 + p_z^2 + m^2 + \frac{\boldsymbol{p}_\perp^2}{eBs}\tanh(eBs)]}. \tag{D.1}$$

For simplicity, we shall split the integration elements into its perpendicular and parallel parts

$$d^d p = d^2 p_\perp d^{d-2} p_\parallel, \tag{D.2}$$

where $\boldsymbol{p}_\perp = (p_x, p_y)$ and $\boldsymbol{p}_\parallel = (\omega, p_z)$. Eq. (D.1) then becomes

$$\mathcal{I} = \int \frac{d^2 p_\perp}{(2\pi)^2} e^{-\frac{\boldsymbol{p}_\perp^2}{eB}\tanh(eBs)]} \int \frac{d^{d-2} p_\parallel}{(2\pi)^{d-2}} e^{-s(p_\parallel^2 + m^2)}. \tag{D.3}$$

The $p_\perp$ integral can be easily evaluated by using polar coordinates giving us

$$\mathcal{I}_\perp = \int \frac{d^2 p_\perp}{(2\pi)^2} e^{-\frac{\boldsymbol{p}_\perp^2}{eB}\tanh(eBs)]} = \frac{eB}{4\pi\tanh(eBs)}. \tag{D.4}$$

The $p_\parallel$ integral can be solved by writing the integration element as

$$d^{d-2} p_\parallel = p_\parallel^{d-3} dp_\parallel d\Omega_{d-2} \tag{D.5}$$

where Ω_{d-2} is the solid angle in $(d-2)$-dimensional Euclidean space. It is formally defined as

$$\Omega_{d-2} = 2\frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}, \tag{D.6}$$

where Γ is the Gamma function and has the integral representation

$$\Gamma(a) = \int_0^\infty d\xi \xi^{a-1} e^{-\xi}. \tag{D.7}$$

Thus, the $p_\parallel$ integral can be written as

$$\begin{aligned}\mathcal{I}_\parallel &= \int \frac{d^{d-2}p_\parallel}{(2\pi)^{\frac{d-2}{2}}} e^{-s(p_\parallel^2+m^2)} = \frac{2\pi^{\frac{d-2}{2}}}{(2\pi)^{\frac{d-2}{2}}\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty dp_\parallel p_\parallel^{d-3} e^{-s(p_\parallel^2+m^2)} \\ &= \frac{2}{(4\pi)^{\frac{d-2}{2}}\Gamma\left(\frac{d-2}{2}\right)} e^{-sm^2} \int_0^\infty dp_\parallel p_\parallel^{d-3} e^{-sp_\parallel^2} \\ &= \frac{e^{-sm^2}}{(4\pi)^{\frac{d}{2}-1} s^{\frac{d}{2}-1}},\end{aligned} \tag{D.8}$$

where we in the last step performed a substitution of the form $u = sp_\parallel^2$ and used the integral representation of the Gamma function Eq. (D.7). Thus, the vacuum energy density we obtained in Eq. (3.79) becomes

$$V = -\frac{eB}{(4\pi)^{\frac{d}{2}}} \int_0^\infty ds \frac{e^{-sm^2}}{s^{\frac{d}{2}}\sinh(eBs)}. \tag{D.9}$$

Furthermore, since we are considering 3 + 1-dimensional spacetime, we define $d = 4 - 2\epsilon$ which allows us to isolate the divergences and perform expansions about $\epsilon = 0$. To that end, Eq. (D.9) becomes

$$V = -\frac{eB}{(4\pi)^{2-\epsilon}} \int_0^\infty ds \frac{e^{-sm^2}}{s^{2-\epsilon}\sinh(eBs)}. \tag{D.10}$$

The divergence in Eq. (D.10) becomes clear if we set $\epsilon = 0$ as we obtain the same divergent integral as we saw previously in Eq. (3.81). The divergence is indeed due to the $1/\sinh(eBs)$ factor which blows up at $s = 0$. Thus, to proceed any further we must subtract terms from the integral to eliminate the divergence. The terms we want to subtract can be found from the expansion of $1/\sinh(eBs)$ about $s = 0$, namely

$$\frac{1}{\sinh(eBs)} = \frac{1}{eBs} - \frac{eBs}{6} + \mathcal{O}\big((eBs)^3\big). \tag{D.11}$$

For our purpose, we will only need to subtract terms up to $\mathcal{O}\big((eBs)^3\big)$ from the $1/\sinh(eBs)$ factor in the integrand to remove the divergence. Subtracting a term means we must also add the same term. Thus, we can write Eq. (D.10) as the sum of a converging integral and a diverging one

$$\begin{aligned}V &= -\frac{eB}{(4\pi)^2} \int_0^\infty ds \frac{e^{-sm^2}}{s^2}\left(\frac{1}{\sinh(eBs)} - \frac{1}{eBs} + \frac{eBs}{6}\right) - \frac{eB}{(4\pi)^{2-\epsilon}} \int_0^\infty ds \frac{e^{-sm^2}}{s^{2-\epsilon}}\left(\frac{1}{eBs} - \frac{eBs}{6}\right) \\ &= V_{\text{conv}} + V_{\text{div}},\end{aligned} \tag{D.12}$$

where we have defined

$$V_{\text{conv}} = -\frac{eB}{(4\pi)^2} \int_0^\infty ds \frac{e^{-sm^2}}{s^2}\left(\frac{1}{\sinh(eBs)} - \frac{1}{eBs} + \frac{eBs}{6}\right), \tag{D.13}$$

and

$$V_{\text{div}} = -\frac{eB}{(4\pi)^{2-\epsilon}} \int_0^\infty ds \frac{e^{-sm^2}}{s^{2-\epsilon}}\left(\frac{1}{eBs} - \frac{eBs}{6}\right). \tag{D.14}$$

We also set $\epsilon = 0$ in V_{conv} since it is no longer required as the divergence has been removed. We first calculate V_{div} by making the substitution $u = sm^2$ and using the integral representation of the Gamma function Eq. (D.7). Thus, we have

$$V_{\text{div}} = -\frac{m^{4-2\epsilon}}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon - 2) + \frac{1}{6}\frac{(eB)^2}{(4\pi)^{2-\epsilon}m^{2\epsilon}}\Gamma(\epsilon). \tag{D.15}$$

We can now perform an expansion about $\epsilon = 0$. Before doing so, we must ensure that the prefactor term of the Gamma functions in Eq. (D.15), namely the mass term, is dimensionless such that we can perform expansions on it. To that end, we introduce a mass scale μ where we multiply the mass term by a factor $\mu^{4-d} = \mu^{2\epsilon}$. This procedure is known as the minimal subtraction (MS) scheme. Thus, V_{div} becomes

$$V_{\text{div}} = -\frac{m^4}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon - 2) + \frac{1}{6}\frac{(eB)^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon). \tag{D.16}$$

The Gamma functions can be tackled by using the equation

$$\Gamma(x+1) = x\Gamma(x), \tag{D.17}$$

recursively to get them in the form $\Gamma(\epsilon + 1)$. This allows us to for instance write

$$\Gamma(\epsilon - 2) = \frac{\Gamma(\epsilon + 1)}{\epsilon(\epsilon - 1)(\epsilon - 2)}. \tag{D.18}$$

We can now expand $\Gamma(\epsilon + 1)$ giving us

$$\Gamma(\epsilon + 1) \approx \Gamma(1) + \epsilon\Gamma'(1) + \mathcal{O}\big(\epsilon^2\big) = 1 - \epsilon\gamma + \mathcal{O}\big(\epsilon^2\big), \tag{D.19}$$

where $\Gamma'(1) = -\gamma$ and γ is the Euler-Mascheroni constant. We can also expand the terms in the denominator of Eq. (D.18) where we for example have that

$$(\epsilon - 1)^{-1} \approx -1 + \epsilon + \mathcal{O}\big(\epsilon^2\big). \tag{D.20}$$

Furthermore, the dimensionless mass term can now be expanded as well giving us

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} = e^{\epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right)} \approx 1 + \epsilon\ln\left(\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}\big(\epsilon^2\big). \tag{D.21}$$

Thus, we get the final expression for V_{div} up to $\mathcal{O}(\epsilon)$ namely,

$$V_{\text{div}} = -\frac{m^4}{2(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]. \tag{D.22}$$

To simplify this expression even further, we can introduce the modified minimal substitution ($\overline{\text{MS}}$) scheme. To do this, we bring the Euler-Mascheroni constant into the logarithm factor and make the substitution $\bar{\mu}^2 = \mu^2 4\pi e^{-\gamma}$. Thus, we have

$$V_{\text{div}} = -\frac{m^4}{2(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right]. \tag{D.23}$$

We next turn our focus to V_{conv}, Eq. (D.13). Recall that the individual terms in the integral are still divergent. Thus, we must implement a similar strategy as that in DR, as suggested in [39] and [40], to isolate the divergences. If we temporarily add a small parameter ϵ to the exponent of s in Eq. (D.13), we can evaluate each individual term in a usual manner and then perform an expansion on the resulting terms. Finally, we can set $\epsilon = 0$ in terms where the divergences are non-existent. Thus, we can write Eq (D.13) as

$$V_{\text{conv}} = -\frac{eB}{(4\pi)^2}\int_0^\infty ds \frac{e^{-sm^2}}{s^{2-\epsilon}}\left(\frac{1}{\sinh(eBs)} - \frac{1}{eBs} + \frac{eBs}{6}\right). \tag{D.24}$$

We can now use the table integrals [38]

$$\int_0^\infty dz z^{\mu-1} e^{-az}\frac{1}{\sinh(z)} = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \frac{1}{2}(a+1)\right), \tag{D.25}$$

and

$$\int_0^\infty dz z^{\mu} e^{-az} = a^{-1-\mu}\Gamma(1+\mu), \tag{D.26}$$

where $\zeta(s,x)$ is the Hurwitz zeta-function and can be expressed as

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s}. \tag{D.27}$$

The expansion of the Gamma function follows the same steps as Eq. (D.18)–(D.19). The zeta function can be expanded as

$$\zeta(\epsilon-1,x) \approx \zeta(-1,x) + \epsilon\zeta^{(1,0)}(-1,x) + \mathcal{O}\left(\epsilon^2\right), \tag{D.28}$$

where we can further use that

$$\zeta(-1,x) = -\frac{1}{12} + \frac{x}{2} - \frac{x^2}{2}. \tag{D.29}$$

Thus, V_{conv} becomes

$$V_{\text{conv}} = \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2} - \ln\left(\frac{m^2}{2eB}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[24\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m^2}{2eB}\right) + 1 + \ln\left(\frac{m^2}{2eB}\right)\right] \tag{D.30}$$

Our final expression for the vacuum energy density is therefore

$$\begin{aligned} V = \frac{1}{2}B^2 - \frac{m^4}{2(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] \\ + \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2} - \ln\left(\frac{m^2}{2eB}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[24\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m^2}{2eB}\right) + 1 + \ln\left(\frac{m^2}{2eB}\right)\right], \end{aligned} \tag{D.31}$$

where we also added the tree-level term $\frac{1}{2}B^2$ which is the contribution from the Maxwell Lagrangian.

D.2 Ultraviolet cutoff

We will now regularize the divergent term in Eq. (D.12), namely Eq. (D.14) using an ultraviolet cutoff. Recall that the UV divergence was caused at $s = 0$. Thus, we can avoid this divergence by changing the lower limit of integration to a high enough value with the help of a cutoff value Λ. We therefore set $\epsilon = 0$ in Eq. (D.14) and instead use the cutoff regularization scheme. Thus, we have that

$$V_{\text{div}} = -\frac{eB}{(4\pi)^2}\int_{\frac{1}{\Lambda^2}}^{\infty} ds \frac{e^{-sm^2}}{s^2}\left(\frac{1}{eBs} - \frac{eBs}{6}\right), \tag{D.32}$$

where the integral becomes divergent again in the $\Lambda \to \infty$ limit. We can now use the integral representation of the upper incomplete gamma function namely,

$$\Gamma(a, x) = \int_x^{\infty} dt t^{a-1} e^{-t}, \tag{D.33}$$

to solve the integrals. After making the substitution $u = sm^2$, we obtain

$$V_{\text{div}} = -\frac{m^4}{(4\pi)^2}\Gamma\left(-2, \frac{m^2}{\Lambda^2}\right) + \frac{(eB)^2}{6(4\pi)^2}\Gamma\left(0, \frac{m^2}{\Lambda^2}\right). \tag{D.34}$$

In the $\Lambda \to \infty$ limit, we can use the following expansions for the upper incomplete Gamma function about $\frac{m^2}{\Lambda^2} = 0$, namely

$$\Gamma\left(-2, \frac{m^2}{\Lambda^2}\right) \approx \frac{1}{2}\left(\frac{\Lambda^2}{m^2}\right)^2 - \frac{\Lambda^2}{m^2} + \frac{1}{2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma + \frac{3}{2}\right] + \mathcal{O}\left(\frac{1}{\Lambda^2}\right), \tag{D.35}$$

and

$$\Gamma\left(0, \frac{m^2}{\Lambda^2}\right) \approx \ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma + \mathcal{O}\left(\frac{1}{\Lambda^2}\right). \tag{D.36}$$

Thus, V_{div} becomes

$$V_{\text{div}} = -\frac{1}{2(4\pi)^2}\left\{\Lambda^4 - 2m^2\Lambda^2 + m^4\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma + \frac{3}{2}\right]\right\} + \frac{(eB)^2}{6(4\pi)^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma\right]. \tag{D.37}$$

Combining Eq. (D.37), Eq. (D.30), and the tree-level term $\frac{1}{2}B^2$ the vacuum energy density becomes

$$\begin{aligned} V = {} & \frac{1}{2}B^2\left\{1 + \frac{e^2}{3(4\pi)^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma\right]\right\} - \frac{1}{2(4\pi)^2}\left\{\Lambda^4 - 2m^2\Lambda^2 + m^4\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma + \frac{3}{2}\right]\right\} \\ & + \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2} - \ln\left(\frac{m^2}{2eB}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[24\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m^2}{2eB}\right) + 1 + \ln\left(\frac{m^2}{2eB}\right)\right]. \end{aligned} \tag{D.38}$$

We can see that this expression for the vacuum energy density agrees with that of for instance [17]. Finally, we note that in most cases, the Λ^4 term is disregarded since it does not depend on neither B nor m.

D.3 Renormalization

Having found an expression for the vacuum energy density of a boson coupled to a constant magnetic field, we must ensure that our results are physically reasonable. As it currently stands, we have calculated a diverging integral, Eq. (D.14), using both DR and cutoff scheme. When using DR, we managed to isolate the diverging terms proportional to $1/\epsilon$ as seen in Eq. (D.23). However, since the vacuum energy density cannot be infinite, we cannot have any diverging terms included in its expression. To eliminate these terms, we must therefore renormalize our result. In what follows, we consider the expression we found using DR, Eq. (D.31). Next, we note that we are dealing with two different pole terms. One of these terms is accompanied by a factor of m^4 while the other a factor of $(eB)^2$. We first look at the m^4 term. This term is identical to the diverging term which appeared in the free energy density of a scalar field in Eq. (E.5). Thus, we have located the source of this divergence and can renormalize the m^4 term by adding a suitable counterterm to the free energy density of the theory. We therefore make the substitution $V \to V + \Delta\mathcal{E}$ where

$$\Delta\mathcal{E} = \frac{m^4}{2(4\pi)^2\epsilon}. \tag{D.39}$$

The $(eB)^2$ term can be eliminated by the wave-function renormalization of the $\frac{1}{2}B^2$ term which was the free contribution from the Maxwell field. Thus, in the $\frac{1}{2}B^2$ term, we make the transformation $B^2 \to A^2 B^2$ where

$$A = \sqrt{1 - \frac{e^2}{3(4\pi)^2\epsilon}}, \tag{D.40}$$

is the renormalization term of the wave function. Making these substitutions, we thus obtain our renormalized expression for the vacuum energy density of a scalar particle in the presence of a constant magnetic field, namely

$$\begin{aligned} V = {} & \frac{1}{2}B^2\left[1 + \frac{e^2}{3(4\pi)^2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] - \frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2}\right] \\ & + \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2} - \ln\left(\frac{m^2}{2eB}\right)\right] + \frac{(eB)^2}{6(4\pi)^2}\left[24\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m^2}{2eB}\right) + 1 + \ln\left(\frac{m^2}{2eB}\right)\right]. \end{aligned} \tag{D.41}$$

E Free vacuum energy density of a scalar field

In this section we evaluate the integral in the expression for the free vacuum energy density of a scalar particle which we obtained in Eq. (2.114). We are therefore interested in calculating the integral

$$\rho = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\omega = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}. \tag{E.1}$$

This integral is clearly UV divergent and thus we shall use dimensional regularization scheme to evaluate it. First step is thus to take the integral in $d-1$-dimensions where $d = 4-\epsilon$. Proceeding similarly as Appendix. D.1 we get

$$\rho = \frac{1}{2}\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\sqrt{k^2+m^2} = \frac{1}{(4\pi)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}-\frac{\epsilon}{2}\right)}\left(\frac{\mu}{\sqrt{4\pi}}\right)^{\epsilon}\int_0^\infty dk k^{d-2}\sqrt{k^2+m^2}. \tag{E.2}$$

where we have implemented the MS scheme by introducing the mass scaling factor μ. From here we use Euler's Beta function allowing us to evaluate integrals of the form [21]

$$\int_0^\infty dk\frac{k^{2a-1}}{(k^2+m^2)^b} = \frac{1}{2}m^{2a-2b}\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}. \tag{E.3}$$

Using this, we have

$$\rho = -\frac{m^4}{2(4\pi)^2}\left(\frac{\mu^2}{4\pi}\right)^{\frac{\epsilon}{2}}\Gamma(-2+\epsilon). \tag{E.4}$$

We can now perform the expansions about $\epsilon = 0$ similar to Appendix. D.1. Our final expression for the free vacuum energy density of a scalar particle is therefore

$$\rho = -\frac{m^4}{2(4\pi)^2}\left[\frac{1}{\epsilon}+\frac{3}{4}+\frac{1}{2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right], \tag{E.5}$$

where we simplified the expression further by going from MS scheme to $\overline{\text{MS}}$ scheme by introducing $\bar{\mu}^2 = \mu^2 4\pi e^{-\gamma}$.

F Effective action and effective potential

We want to find the form of the effective action starting from the functional integral representation of the transition amplitude. Once, we have found the effective action, we can then relate this to the effective potential. We will for simplicity look at a single real scalar field theory in the following calculations, however this can be easily generalized to an arbitrary number of real scalar fields. The calculations in this section follows closely to that in [16].

We first start with the functional integral representation of the transition amplitude,

$$Z[J] = e^{-iE[J]} = \int D\phi \exp\Big[i \int d^4x(\mathcal{L}[\phi] + J\phi)\Big], \tag{F.1}$$

where J is an external source and $E[J]$ is the energy functional which is merely the vacuum energy as a function of the external source J. It will now be useful to express the functional derivative of the energy functional with respect to the source, namely

$$\frac{\delta}{\delta J(x)} E[J] = i\frac{\delta}{\delta J(x)} \ln Z = -\frac{\int D\phi \exp\left[i\int d^4x(\mathcal{L}[\phi] + J\phi)\right]\phi(x)}{Z} = -\langle 0|\,\phi(x)\,|0\rangle_J\,. \tag{F.2}$$

Thus, the functional derivative of the energy functional $E[J]$ with respect to the source J simply gives the vacuum expectation value of the field ϕ in the presence of an external source, with an additional minus sign. For brevity we shall define this vacuum expectation value as

$$\langle 0|\,\phi(x)\,|0\rangle_J = \phi_{\text{cl}}(x), \tag{F.3}$$

which we call the classical field. We are now ready to define the effective action via a Legendre transformation of the energy functional, namely

$$\Gamma[\phi_{\text{cl}}] = -E[J] - \int d^4y J(y)\phi_{\text{cl}}(y). \tag{F.4}$$

We can now find the functional derivative of the effective action with respect to the classical field, namely

$$\begin{aligned}
\frac{\delta}{\delta\phi_{\text{cl}}(x)}\Gamma[\phi_{\text{cl}}] &= -\frac{\delta}{\delta\phi_{\text{cl}}(x)}E[J] - \int d^4y\frac{\delta J(y)}{\delta\phi_{\text{cl}}(x)}\phi_{\text{cl}}(y) - J(x)\\
&= -\int d^4y\frac{\delta J(y)}{\delta\phi_{\text{cl}}(x)}\frac{\delta E[J]}{\delta J(y)} - \int d^4y\frac{\delta J(y)}{\delta\phi_{\text{cl}}(x)}\phi_{\text{cl}}(y) - J(x)\\
&= -J(x).
\end{aligned} \tag{F.5}$$

Thus, we see that in the case of no external sources, $J(x) = 0$, the functional derivative of the effective action with respect to the classical field gives the solutions, ϕ_{cl}, that minimize the effective action. In what follows we will simplify the problem by considering only classical fields which are independent of space-time coordinates. This allows us to write the effective action as

$$\Gamma[\phi_{\text{cl}}] = -\Omega V_{\text{eff}} \tag{F.6}$$

where Ω is the four-volume factor of space-time and V_{eff} is the effective potential. Thus, when considering fields independent of space-time coordinates with the absence of an external source, we obtain a function V_{eff} whose derivative with respect to the classical field ϕ_{cl} gives the vacuum state of the quantum field theory, including all effects of quantum fluctuations [16, p. 299]. We shall now compute the effective potential. The method for these computations involves starting with the generating functional defined in Eq. (F.1) and finding a perturbative expansion for it. We can then find the energy functional by taking the logarithm of this function. Continuing, we perform a Legendre transform of the energy functional to find the effective action. Once we have found the effective action, we can find the effective potential by considering the case where the classical field is independent of space-time coordinates. Furthermore, we want to compute the effective action Γ as a function of the classical field ϕ_{cl}. However, since the generating functional $Z[J]$ depends on ϕ_{cl} through its dependence on J, we must first find out how ϕ_{cl} depends on J. At lowest order in perturbation theory, the dependence is simply the classical field equation given by

$$\left.\frac{\delta \mathcal{L}}{\delta \phi}\right|_{\phi=\phi_{\text{cl}}} + J(x) = 0. \tag{F.7}$$

Thus, we start by performing a small shift in the field, namely $\phi(x) = \phi_{\text{cl}}(x) + \eta(x)$, which allows us to perform an expansion in the exponent of Eq. (F.1) about $\phi = \phi_{\text{cl}}$, namely

$$\begin{aligned}\int d^4x(\mathcal{L}[\phi] + J\phi) \approx \int d^4x(\mathcal{L}[\phi_{\text{cl}}] + J\phi_{\text{cl}}) + \int d^4x\left(\frac{\delta \mathcal{L}}{\delta \phi} + J\right)\eta(x) \\ + \frac{1}{2}\int d^4x d^4y \frac{\delta^2 \mathcal{L}}{\delta\phi(x)\delta\phi(y)}\eta(x)\eta(y) + \mathcal{O}(\eta^3).\end{aligned} \tag{F.8}$$

We can see that the second term on the right hand side is zero by using Eq. (F.7). Thus, the lowest order quantum correction to the effective action is given by the term quadratic in η. Consequently, the cubic and higher terms would correspond to the higher order corrections. In what follows, we will only focus on the lowest order quantum corrections, thus only working with terms quadratic in η. The integral over η is a Gaussian integral which can be evaluated in terms of a functional determinant, namely

$$\int D\eta \exp\left[i\left(\int \mathcal{L}[\phi_{\text{cl}}] + J\phi_{\text{cl}} + \frac{1}{2}\int \eta \frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\eta\right)\right] = \exp\left[i\int (\mathcal{L}[\phi_{\text{cl}}] + J\phi_{\text{cl}})\right]\cdot\left(\det\left[-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\right]\right)^{-\frac{1}{2}}. \tag{F.9}$$

Substituting Eq. (F.9) back into Eq. (F.1), taking the natural logarithm of both sides, and multiplying both sides by i we obtain for the energy functional,

$$E[J] = -\int d^4x\,(\mathcal{L}[\phi_{\text{cl}}] + J\phi_{\text{cl}}) - \frac{i}{2}\ln\det\left[-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\right]. \tag{F.10}$$

Performing again the Legendre transform on the energy functional we obtain the effective action

$$\Gamma[\phi_{\text{cl}}] = \int d^4x \mathcal{L}[\phi_{\text{cl}}] + \frac{i}{2}\ln\det\left[-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\right]. \tag{F.11}$$

Finally, assuming that the classical fields are independent of space-time coordinates, we can write the effective potential according to Eq. (F.6) including the first quantum corrections

$$V_{\text{eff}}(\phi_{\text{cl}}) = V(\phi_{\text{cl}}) - \frac{i}{2\Omega}\ln\det\left[-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\right], \tag{F.12}$$

where V is the classical potential belonging to the Lagrangian $\mathcal{L}$. Thus, we can now compute the effective potential to lowest order in perturbation theory given a specific Lagrangian.

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