A Control Engineer's Guide to Sliding Mode Control

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***Abstract*—This paper presents a guide to sliding mode control for practicing control engineers. It offers an accurate assessment of the so-called chattering phenomenon, catalogs implementable sliding mode control design solutions, and provides a frame of reference for future sliding mode control research.**

***Index Terms*— Discrete-time systems, multivariable systems, nonlinear systems, robustness, sampled data systems, singularly perturbed systems, uncertain systems, variable structure systems.**

I. Introduction

During the last two decades since the publication of the survey paper in the IEEE Transactions on Automatic Control in 1977 [1], significant interest on variable structure systems (VSS) and sliding mode control (SMC) has been generated in the control research community worldwide. One of the most intriguing aspects of sliding mode is the discontinuous nature of the control action whose primary function of each of the feedback channels is to switch between two distinctively different system structures (or components) such that a new type of system motion, called sliding mode, exists in a manifold. This peculiar system characteristic is claimed to result in superb system performance which includes insensitivity to parameter variations, and complete rejection of disturbances. The reportedly superb system behavior of VSS and SMC naturally invites criticism and scepticism from within the research community, and from practicing control engineers alike [2]. The sliding mode control research community has risen to respond to some of these critical challenges, while at the same time, contributed to the confusions about the robustness of SMC by offering incomplete analyzes, and design fixes for the so-called chattering phenomenon [3]. Many analytical design methods were proposed to reduce the effects of chattering [4]–[8]—for it remains to be the only obstacle for sliding mode to become one of the most significant discoveries in modern control theory; and its potential seemingly limited by the imaginations of the control researchers [9]–[11].

In contrast to the published works since the 1977 article, which serve as a status overview [12], a tutorial [13] of design methods, or another more recent state of the art assessment [14], or yet another survey of sliding mode research [15], the purpose of this paper is to provide a comprehensive guide to SMC for control engineers. It is our goal to accomplish these objectives:

- provide an accurate assessment of the chattering phenomenon;
- offer a catalog of implementable robust sliding mode control design solutions for real-life engineering applications;
- initiate a dialog with practicing control engineers on sliding mode control by threading the many analytical underpinnings of sliding mode analysis through a series of design exercises on a simple, yet illustrative control problem;
- establish a frame of reference for future sliding mode control research.

The flow of the presentation in this paper follows the chronological order in the development of VSS and SMC: First we introduce issues within continuous-time sliding mode in Section II, then in Section III, we progress to discrete-time sliding mode, (DSM) followed with sampled data SMC design in Section IV.

II. Continuous-Time Sliding Mode

Sliding mode is originally conceived as system motion for dynamic systems whose essential open-loop behavior can be modeled adequately with ordinary differential equations. The discontinuous control action, which is often referred to as variable structure control (VSC), is also defined in the continuous-time domain. The resulting feedback system, the so-called VSS, is also defined in the continuous-time domain, and it is governed by ordinary differential equations with discontinuous right-hand sides. The manifold of the state-space of the system on which sliding mode occurs is the sliding mode manifold, or simply, sliding manifold. For control engineers, the simplest, but vividly perceptible example is a double integrator plant, subject to time optimal control action. Due to imperfections in the implementations of the switching curve, which is derived from the Pontryagin maximum principle, sliding mode may occur. Sliding mode was studied in conjunction with relay control for double integrator plants, a problem motivated by the design of attitude control systems of missiles with jet thrusters in the 1950's [16].

The chattering phenomenon is generally perceived as motion which oscillates about the sliding manifold. There are two possible mechanisms which produce such a motion. First, in the absence of switching nonidealities such as delays, i.e., the switching device is switching ideally at an infinite frequency, the presence of parasitic dynamics in series with the plant causes a small amplitude high-frequency oscillation to appear in the neighborhood of the sliding manifold. These parasitic dynamics represent the fast actuator and sensor dynamics

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which, according to control engineering practice, are often neglected in the open-loop model used for control design if the associated poles are well damped, and outside the desired bandwidth of the feedback control system. Generally, the motion of the real system is close to that of an ideal system in which the parasitic dynamics are neglected, and the difference between the ideal and the real motion, which is on the order of the neglected time constants, decays rapidly. The mathematical basis for the analysis of dynamic systems with fast and slow motion is the theory of singularly perturbed differential equations [17], and its extensions to control theory have been developed and applied in practice [18]. However, the theory is not applicable for VSS since they are governed by differential equations with discontinuous right hand sides. The interactions between the parasitic dynamics and VSC generate a nondecaying oscillatory component of finite amplitude and frequency, and this is generically referred to as chattering.

Second, the switching nonidealities alone can cause such high-frequency oscillations. We shall focus only on the delay type of switching nonidealities since it is most relevant to any electronic implementation of the switching device, including both analog and digital circuits, and microprocessor code executions. Since the cause of the resulting chattering phenomenon is due to time delays, discrete-time control design techniques, such as the design of an extrapolator can be applied to mitigate the switching delays [19]. These design approaches are perhaps more familiar to control engineers.

Unfortunately, in practice, both the parasitic dynamics and switching time delays exist. Since it is necessary to compensate for the switching delays by using a discrete-time control design approach, a practical SMC design may have to be unavoidably approached in discrete time. We shall return to the details of discrete-time SMC after we illustrate our earlier points on continuous-time SMC with a simple design example, and summarize the existing approaches to avoid chattering.

A. Chattering Due to Parasitic Dynamics—A Simple Example

The effects of unmodeled dynamics on sliding mode can be illustrated with an extremely simple relay control system example: Let the nominal plant be an integrator

$$\dot{x} = u, \quad x(t_0) = x_0 \neq 0 \tag{1}$$

and assume that a relay controller has been designed

$$u = -\mathrm{sgn}\,(x). \tag{2}$$

The sliding manifold is the origin of the state-space $x = 0$. Given any nonzero initial condition x_0, the state trajectory $x(t)$ is driven toward the sliding manifold. Ideally, if the relay controller can switch infinitely fast, then $x(t) = 0, t \geq t^*$ where t^* is the first time instant that $x(t^*) = 0$, i.e., once the state trajectory reaches the sliding manifold, it remains on it for good. However, even with such an ideal switching device, unmodeled dynamics can induce oscillations about the sliding manifold. Suppose we have ignored the existence of a "sensor" with second-order dynamics, and the true system dynamics are governed by

$$\dot{x} = -\mathrm{sgn}\,(x_s) \tag{3}$$

$$\tau_s^2 \ddot{x}_s + 2\tau_s \dot{x}_s + x_s = x, \qquad \tau_s \ll 1 \tag{4}$$

where x_s and $\dot{x}_s$ are the states of the sensor dynamics. Clearly, sliding mode cannot occur on $x = 0$ since $\dot{x}$ is continuous, however, since $\dot{x}_s$ is bounded, $x_s(t) - x(t) = O(\tau_s)$ where τ_s is the time constant of the sensor. Furthermore, reaching an $O(\tau_s)$ boundary layer of $x(t) = 0$ is guaranteed since

$$\dot{x}_s = -\mathrm{sgn}\,(x_s + O(\tau_s)). \tag{5}$$

The system behavior inside this $O(\tau_s)$ boundary layer can be analyzed by replacing the infinitely fast switching device with a linear feedback gain approximation whose gain tends to infinity asymptotically

$$u = -g x_s, \quad g \to \infty. \tag{6}$$

The root locus of this system, with g as the scalar gain parameter, has third-order asymptotes as $g \to \infty$. Therefore, the high-frequency oscillation in the boundary layer is unstable.

Instead of having parasitic sensor dynamics, we may have second-order parasitic actuator dynamics in series with the nominal plant, in which case, the closed-loop dynamics are given by

$$\dot{x} = x_a \tag{7}$$

$$\tau_a^2 \ddot{x}_a + 2\tau_a \dot{x}_a + x_a = -\mathrm{sgn}\,(x), \qquad \tau_a \ll 1. \tag{8}$$

The characteristic equation of this system is identical to that of the parasitic sensor case. This is not surprising since the forward transfer function is identical in both cases. Thus, similar instability also occur with infinitely fast switching.

B. Boundary Layer Control

The most commonly cited approach to reduce the effects of chattering has been the so called piecewise linear or smooth approximation of the switching element in a boundary layer of the sliding manifold [20]–[23]. Inside the boundary layer, the switching function is approximated by a linear feedback gain. In order for the system behavior to be close to that of the ideal sliding mode, particularly when an unknown disturbance is to be rejected, sufficiently high gain is needed. Note that in the absence of disturbance, it is possible to enlarge the boundary layer thickness, and at the same time reduce the effective linear gain such that the resulting system no longer exhibits any oscillatory behavior about the sliding manifold. However, this system no longer behaves as dictated by sliding mode, i.e., simply put, in order to reduce chattering, the proposed method of piecewise linear approximation reduces the feedback system to a system with no sliding mode. This proposed method has wide acceptance by many sliding mode researchers, but unfortunately it does not resolve the core problem of the robustness of sliding mode as exhibited in chattering. Many sliding mode researchers cited the work in [3] and [22] as the basis of their optimism that the implementation issues of continuous-time sliding mode are solved with boundary layer control. Unfortunately, the optimism of these researchers was not shared by practicing engineers, and this may be rightly so. The effectiveness of boundary layer control is immediately challenged when realistic parasitic dynamics are considered. An in-depth analysis of the interactions of parasitic actuator and sensor dynamics with the boundary layer control [24] revealed the shortcomings of this approach. Parasitics dynamics must be carefully modeled and considered in the feedback design in order to avoid instability inside the boundary layer which leads to chattering. Without such information of the parasitic dynamics, control engineers must adopt a worst case boundary layer control design in which the disturbance rejection properties of SMC are severely compromised.

A Boundary Layer Controller: We shall continue with the simple relay control example, and consider the design of a boundary layer controller. We assume the same second-order parasitic sensor dynamics as before. The behavior inside the boundary layer is governed by a linear closed-loop system

$$\dot{x} = -gx_s + d(t) \tag{9}$$

$$\tau_s^2 \ddot{x}_s + 2\tau_s \dot{x}_s + x_s = x \tag{10}$$

where $d(t)$ represents a bounded, but unknown exogenous disturbance. Whereas discontinuous control action in VSC can reject bounded disturbances, by replacing the switching control with a boundary layer control, the additional assumption that $\dot{d}$ be bounded is needed since according to singular perturbation analysis, the residue error is proportional to $|\dot{d}|/g$. Given a finite τ_s, we can compute the root locus of this system with respect to the scalar positive gain $g > 0$. An upper bound g_c exists which specifies the crossover point of the root locus on the imaginary axis. Thus, for $0 < g < g_c$, the behavior of this system is asymptotically stable, i.e., for any initial point inside the boundary layer

$$|x_s| \leq 1/g \tag{11}$$

the sliding manifold $x = 0$ is reached asymptotically as $t \to \infty$. The transient response and disturbance rejection of this feedback system are two competing performance measures to be balanced by the choice of an optimum gain value. If we assume $\tau_s = 0.01$ the associated root locus is plotted in Fig. 1 for $0.002 \leq g^{-1} \leq 0.05$ with a step size of 0.001. The critical gain is $g_c = 200$. Thus from the linear analysis, a boundary layer control with $g = 100$ results in a stable sliding mode, whereas with $g = 200$, oscillatory behavior about the sliding manifold is predicted. Fig. 2 shows the simulated error responses of the closed-loop system for these two gain values which agree with the analysis. In this simulation, a unity reference command for the plant state and a constant disturbance $d(t) = 0.5$ is introduced. The tradeoff between chattering reduction and disturbance rejection can be observed from $x(t)$, of which the steady-state value -0.005 (for the stable response), or the average value -0.0025 (for the oscillatory response) is inversely proportional to the gain g. We note that even with $g > g_c$, the resulting response is only oscillatory, but still bounded. This is because the linear analysis is valid only inside the boundary layer, and the VSC always forces the state trajectory back into the boundary layer region. However, as the gain increases, the frequency of oscillation increases as the magnitude of the imaginary parts of the complex root increases. Increasingly smaller amplitude but higher frequency oscillation as gain approaches infinity. This is the chattering behavior observed when the switching feedback control action interacts with the neglected resonant frequencies of the physical plant.

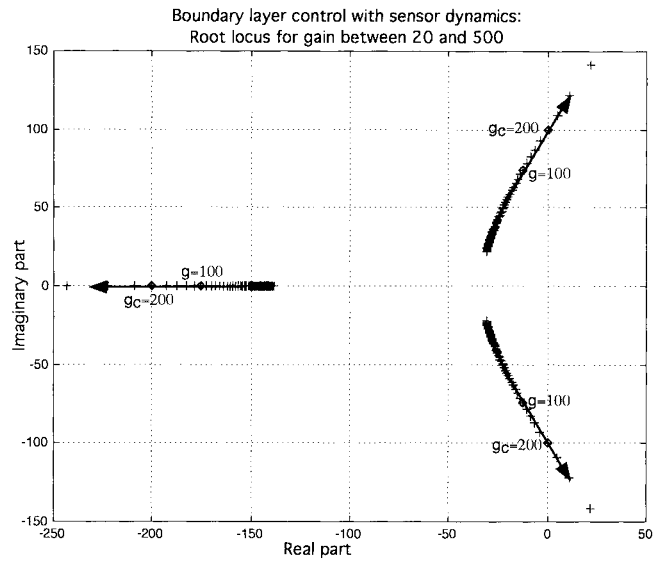

Fig. 1. Boundary layer control with sensor dynamics: root locus.

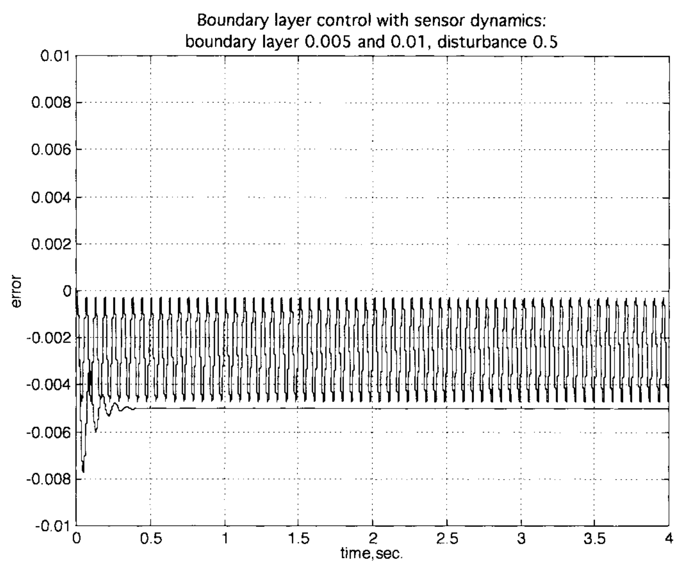

Fig. 2. Boundary layer control with sensor dynamics: Time responses for $g = 200$ (oscillatory), and $g = 100$ (stable).

This example illustrates the advantages of boundary layer control which lie primarily in the availability of familiar linear control design tools to reduce the potentially disastrous chattering. However, it should also be reminded that if the acceptable closed-loop gain has to be reduced sufficiently to avoid instability in the boundary layer, the resulting feedback system performance may be significantly inferior to the nominal system with ideal sliding mode. Furthermore, the precise details of the parasitic dynamics must be known and used properly in the linear design.

C. Observer-Based Sliding Mode Control

Recognizing the essential triggering mechanism for chattering is due to the interactions of the switching action with the parasitic dynamics, an approach which utilizes asymptotic observers to construct a high-frequency by pass loop has been proposed [4]. This design exploits a localization of the high-frequency phenomenon in the feedback loop by introducing a discontinuous feedback control loop which is closed through an asymptotic observer of the plant [25]. Since the model imperfections of the observer are supposedly smaller than those in the plant, and the control is discontinuous only with respect to the observer variables, chattering is localized inside a high-frequency loop which bypasses the plant. However, this approach assumes that an asymptotic observer can indeed be designed such that the observation error converges to zero asymptotically. We shall discuss the various options available in observer based sliding mode control in the following design example.

Design Example of Observer-Based SMC: For the relay control example, we examine the utility of the observer based SMC in localizing the high-frequency phenomenon. For the nominal plant, the following asymptotic observer results from applying conventional state-space linear control design:

$$\dot{\hat{x}} = h(x_s - \hat{x}) + u \tag{12}$$

where $h > 0$ is the observer feedback gain, and x_s is the output of the parasitic sensor dynamics. The SMC and the associated sliding manifold defined on the observer state-space is

$$u = -\text{sgn}\,(\hat{x}). \tag{13}$$

The behavior of the closed-loop system can be deduced from the following fourth-order system:

$$\dot{x} = -\text{sgn}\,(x - e) + d(t), \quad e \triangleq x - \hat{x} \tag{14}$$

$$\dot{e} = -he + h(x - x_s) + d(t) \tag{15}$$

$$\tau_s^2 \ddot{x}_s + 2\tau_s \dot{x}_s + x_s = x. \tag{16}$$

First we consider the case when $d(t) = 0$. Using an infinite gain linear function $g(x - e)$ to approximate the switching function $\text{sgn}\,(x - e)$, and since τ_s is finite, the above system is a singularly perturbed system with g^{-1} being the parasitic parameter. The slow dynamics which are of third-order can be extracted by formally setting $g^{-1} = 0$, and $x = e$,

$$\dot{e} = -hx_s \tag{17}$$

$$\tau_s^2 \ddot{x}_s + 2\tau_s \dot{x}_s + x_s = e. \tag{18}$$

It is possible to further apply a singular perturbation analysis to insure that given τ_s, there exists $h > 0$ such that the asymptotic observer dynamics are of first order, and its eigenvalue is approximately $-h$. Clearly, the adverse effects of the parasitic sensor dynamics are neutralized with an observer-based SMC design. If a switching function is realized in the SMC design, the only remaining concern will be switching time delays, and if the observer is to be implemented in discrete time, the entire feedback design including the compensation of switching time delays may be best carried out in the discrete-time domain. Fig. 3 is a block diagram of this design. Note that the switching element is inside a feedback loop which passes through only the observer, bypassing both blocks of the plant dynamics. This is the so-called high-frequency bypass effects of the observer-based SMC [4], [26].

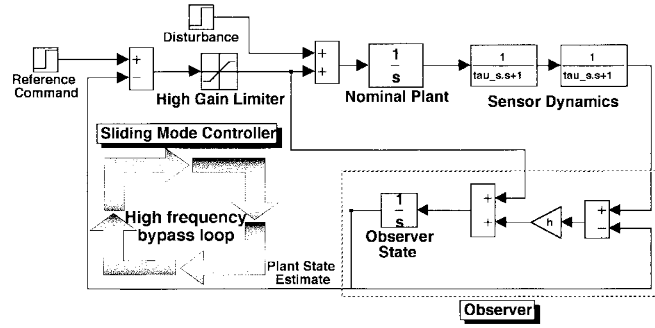

Fig. 3. Block diagram of observer-based sliding mode control.

When $d(t) \neq 0$, its effects on the convergence of asymptotic observers are well known. If $d(t)$ is an unknown constant disturbance, a multivariable servomechanism formulation can be adopted to estimate both the state and exogenous disturbance in a composite asymptotic observer. The resulting feedback system is a variable structure (VS) servomechanism [25], [27]. In general, $d(t)$ can be the output of a linear time-invariant system whose system matrix is known, but the initial conditions are unknown.

For bounded but unknown disturbances with bounded time derivatives, the only known approach to ensure the robustness of the asymptotic observer is to introduce a high-gain loop around the observer itself to reject the unknown disturbance, i.e., by increasing the gain h in the observer such that the effects of $d(t)$ are adequately attenuated. However, the requirements for disturbance attenuation and closed-loop stability must be balanced in the design, and if sliding mode is to be preserved in the manifold $\hat{x} = 0, g$ must be sufficiently larger than h. A switching function implementation of the SMC would seem to ensure the necessary time scale separations, however, the condition $g \ll 1/\tau_s$ should also be imposed to avoid adverse interactions with the parasitic dynamics. Note that if the high-gain loop in the asymptotic observer is implemented with a switching function, it is referred to as a sliding mode observer [28]–[30]. Since two sliding manifolds are employed in the feedback loops, the closed-loop system robustness must be carefully examined when less than infinite switching frequencies are to be expected. In such robustness analysis, the relative time scales of the various motions in the system can be managed with singular perturbation methods, similar to that applied to high-gain observers.

The performance of the observer based SMC can be evaluated by simulation. We let the sensor dynamic time constant be $\tau_s = 0.01$, and assume the same unity reference command

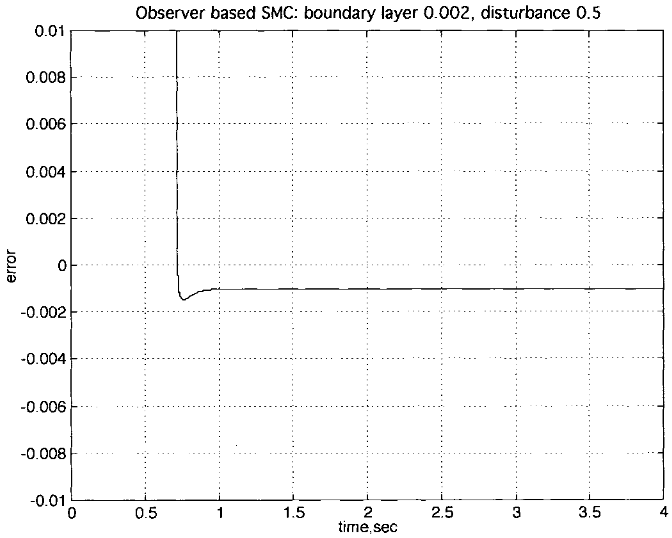

Fig. 4. Observer-based SMC: error between reference command and observer state.

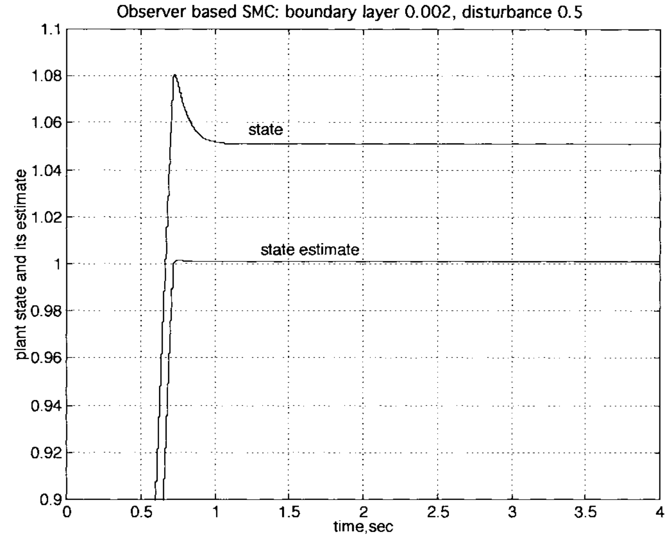

Fig. 5. Observer-based SMC: plant state (upper curve) and its estimate (lower curve).

and constant disturbance as in the boundary layer control example, $d(t) = 0.5$. A linear feedback gain approximation with a boundary layer of 0.002 is used in place of the switching control in the observer based SMC. The closed-loop eigenvalues are at $\{-510$ (due to boundary layer), -12.92 (from observer), $-59.44, -127.62$ (shifted sensor poles)$\}$. Fig. 4 shows the error response between the reference and the observed state. The steady state error of -0.001 reflects the attenuation of the disturbance by the high-gain of 500. Note that sliding mode in the observer state-space can be implemented with high-gain with no adverse interactions with the parasitic dynamics. Fig. 5 shows the observer state's tracking of the unity reference command despite the constant disturbance. The superb rejection of the disturbance by sliding mode in the observer state-space is expected since a large gain value can be chosen freely when the constraints imposed by the parasitic dynamics are no longer present. However, also shown in this figure, the plant state response has a steady-state error of 0.05 which is due to the observation error caused by the relatively low feedback gain of the observer $h = 10$. This error can be reduced by increasing the value of gain h, provided that the time scales and stability of the system are preserved.

D. Disturbance Compensation

In SMC, the main purpose of sliding mode is to reject disturbances and to desensitize against unknown parametric perturbations. Building on the observer based SMC, a sliding mode disturbance estimator which uses sliding mode to estimate the unknown disturbances and parametric uncertainties has also been introduced [8]. In this approach, the control law consists of a conventional continuous feedback control component, and a component derived from the SM disturbance estimator for disturbance compensation. If the disturbance is sufficiently compensated, there is no lneed to evoke a discontinuous feedback control to achieve sliding mode, thus, the remaining control design follows the conventional wisdom, and issues regarding unmodeled dynamics are no longer critical. Also chattering becomes a nonissue since a conventional feedback control instead of SMC is applied. The critical design issues are transferred to the SM disturbance estimator and its associated sliding mode. While there are many engineering issues to be dealt with in this approach, simulation studies and experiment results [31] show that desired objectives are indeed achievable.

An SM Disturbance Estimator: Once again we return to the simple relay example with parasitic sensor dynamics for our design of a disturbance estimator. The plant model is

$$\dot{x} = u + d(t) \tag{19}$$

$$\tau_s^2 \ddot{x}_s + 2\tau_s \dot{x}_s + x_s = x. \tag{20}$$

We shall design a disturbance estimator with sliding mode as follows:

$$\dot{\hat{x}} = u + \mathrm{sgn}\,(x_s - \hat{x}), \quad e \triangleq x_s - \hat{x}. \tag{21}$$

Suppose sliding mode occurs on $e = 0$. Since $u(t)$ is continuous and differentiable, from the error dynamics

$$\dot{e} = -\mathrm{sgn}\,(x_s - \hat{x}) + d(t) + \dot{x}_s - \dot{x}. \tag{22}$$

The "equivalent control" is the control which keeps the trajectories of the system on $e = 0$. It can be solved from $\dot{e} = 0$,

$$[\mathrm{sgn}\,(x_s - \hat{x})]_{\mathrm{eq}} = d(t) + O(\tau_s). \tag{23}$$

Note that from (20), $|\dot{x}_s - \dot{x}| = O(\tau_s)$. Thus, within this estimator, there exists a signal which, under the sliding mode condition, is $O(\tau_s)$ close to the unknown disturbance $d(t)$. This forms the basis of a feedback control design which utilizes this signal to compensate the disturbance to $O(\tau_s)$. The resulting control law has a conventional linear feedback component, and a disturbance compensating component, and for this system

$$u = -k\hat{x} - [\mathrm{sgn}\,(x_s - \hat{x})]_{\mathrm{eq}}. \tag{24}$$

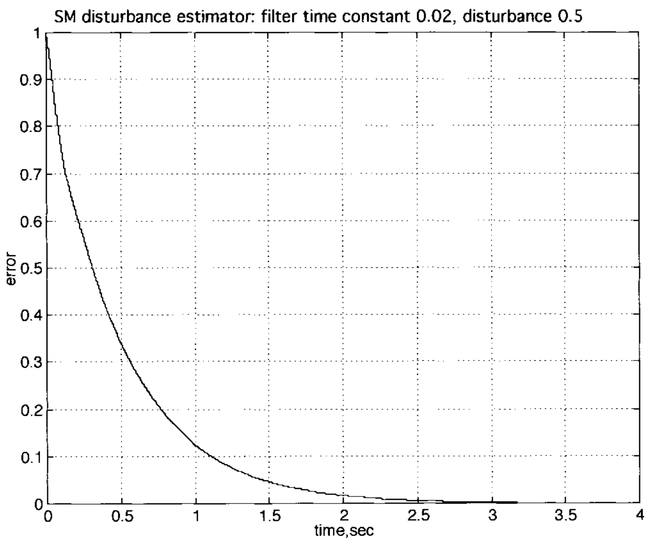

Fig. 6. SM disturbance estimator control: Error between reference command and plant state.

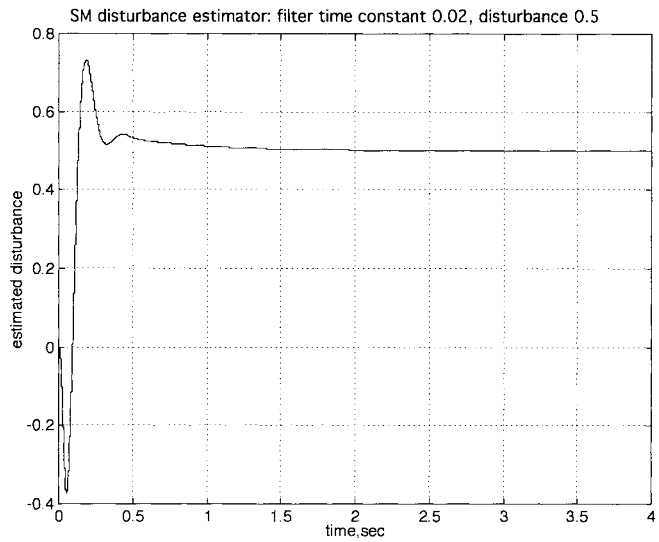

Fig. 7. SM disturbance estimator control: Disturbance estimate.

The extraction of the equivalent control from the sliding model control signal is by low-pass filtering. While theoretically there exists a low-pass filter such that the equivalent control can be found, in practice, the bandwidth of the desired closed-loop system, the spectrum of the disturbance, are all important considerations in the selection of the cutoff frequency of this filter.

For a closer examination of the behavior of this disturbance estimator, we let the sensor time constant be once again $\tau_s = 0.01$, and simulate the system's responses with the same unity reference command, and constant disturbance $d(t) = 0.5$ as before. After canceling the disturbance, we design a closed-loop system with a time constant of one seconds which can be attained with $k = 2$. A boundary layer of 5×10^{-4} replaces the switching function in the estimator. The closed-loop eigenvalues are $\{-2000.$ (from the boundary layer), -1, (the dominant closed-loop pole), $-96.75\pm\jmath 101.83$ (the shifted sensor poles)$\}$. For low-pass filtering, a third-order butterworth filter with a 3 dB corner frequency of 50 rad/s is used to filter the equivalent control. Fig. 6 shows the error between the reference command and the plant state which exhibits the desired one second time constant transient behavior, with the exception of initial minor distortions which are due to the convergence of the disturbance estimate shown in Fig. 7. Despite the constant disturbance, the steady-state error is zero. While standard PID controllers can achieve the same zero steady state error in the presence of unknown constant disturbance, the tracking error is regulated to zero even when $d(t)$ is time varying [8].

E. Actuator Bandwidth Constraints

Despite its desirable properties, VSC is mostly restricted to control engineering problems where the control input of the plant is, by the nature of the control actuator, necessarily discontinuous. Such problems include control of electric drives where pulse-width-modulation is not the exception, but the rule of the game. Space vehicle attitude control is another example where reaction jets operated in an on-off mode are commonly used. The third example, which is closely related to the first one, is power converter and inverter feedback control design. For these classes of applications, the chattering phenomenon still needs to be addressed. However, the arguments against using sliding mode in the feedback design are weakened. The issue in this case is whether VSC should be utilized directly to improve system performance while at the same time produces the required PWM control signal, or a standard PID type controller should first be designed, and then the actual PWM control signal is to be generated by applying standard PWM techniques to approximate the continuous linear control signal. If VSC is to be used, by adopting an observer-based SMC, the high-frequency components of the discontinuous control can be bypassed, and consequently, adverse interactions with the unmodeled dynamics which cause chattering can be avoided.

In plants where control actuators have limited bandwidth, e.g., hydraulic actuators, there are two possibilities: First, the actuator bandwidth is outside the required closed-loop bandwidth. Thus the actuator dynamics become unmodeled dynamics, and our discussions in the previous sections are applicable. While it is possible to ignore the actuator dynamics in linear control design, doing so in VSC requires extreme care. By ignoring actuator dynamics in a classical SMC design, chattering is likely to occur since the switching frequency is limited by the actuator dynamics even in the absence of other parasitic dynamics. Strictly speaking, sliding mode cannot occur, since the control input to the plant is continuous.

Second, the desired closed-loop bandwidth is beyond the actuator bandwidth. In this case, regardless of whether SMC or other control designs are to be used, the actuator dynamics are lumped together with the plant, and the control design model encompasses the actuator-plant in series. With the actuator dynamics no longer negligible, often the matching conditions for disturbance rejection and insensitivity to parameter variations in sliding mode [32] which are satisfied in the nominal plant model are violated. This results from having dominant dynamics inserted between the physical input to the plant, such as force, and the controller output, usually an electrical signal. The design of SMC which incorporates the actuator dynamics as a prefilter for the VSC was proposed in [28]. This design utilizes an expansion of the original state-space by including state derivatives, and formulates an SMC design such that the matching condition is indeed satisfied in the extended space. Another alternative approach is to utilize sliding mode to estimate the disturbance for compensation as discussed earlier. Since sliding mode is not introduced primarily to reject disturbances, the matching conditions are of no significance in this design. Provided that a suitable sliding mode exists such that the disturbance can be estimated from the corresponding equivalent control, this approach resolves the limitations imposed by actuator bandwidth constraints on the design of sliding mode based controllers.

An SMC Design with Prefilter: We shall use the example with a nominal integrator plant and actuator dynamics

$$\dot{x} = x_a + d(t) \tag{25}$$

$$\alpha^2 \dot{x}_a + 2\alpha \ddot{x}_a + x_a = u \tag{26}$$

to illustrate this design. The actuator bandwidth limitation is expressed in the time constant α. Given a discontinuous input $u(t)$, the rate of change of the actuator output $x_a(t)$ is limited by the finite magnitude of α. However, in order for the disturbance $d(t)$ to be rejected, $x_a(t)$ must be an SMC. Also if x_a can be designed as a control input, then the matching condition is clearly satisfied. But since u is the actual input, the matching condition does not hold for finite α. The design begins with an assumption that $d(t)$ has continuous first and second derivatives, and with the introduction of new state variables

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \ddot{x} \tag{27}$$

the control u is designed as an VSC with respect to the sliding manifold

$$s(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + x_3 = 0. \tag{28}$$

With the equivalent control u_{eq} computed from

$$0 = -c_1 c_2 x_1 + (c_1 - c_2^2) x_2 + \left(\ddot{d} - \frac{2}{\alpha} \dot{x}_a - \frac{x_a}{\alpha^2} + \frac{u_{\text{eq}}}{\alpha^2} \right) \tag{29}$$

the resulting sliding mode dynamics are found to be composed of two subsystems in series

$$\ddot{x} + c_2 \dot{x} + c_1 x = 0 \tag{30}$$

$$\ddot{x}_a = (c_2^2 - c_1)\dot{x} + c_1 c_2 x - \ddot{d}. \tag{31}$$

This design shows that although the embedded prefilter in the plant model destroys the matching condition, an SMC can still be designed to reject the unknown disturbance. However, it is necessary to restrict the class of disturbances to those which have bounded derivatives. Furthermore, derivatives of the state, $\dot{x}, \ddot{x}$ are required in the feedback control implementation.

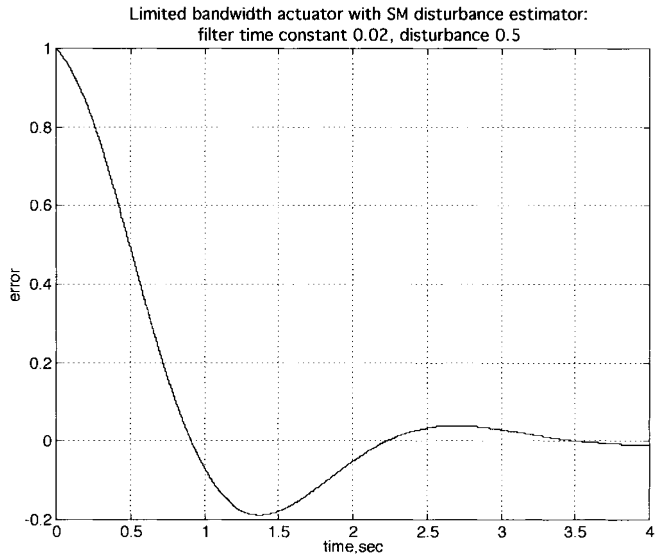

Fig. 8. Limited bandwidth actuator with SM disturbance estimator: Error between reference command and plant state.

2) A Disturbance Estimation Solution: For the nominal integrator plant with limited bandwidth actuator dynamics given by (25), (26), we introduce the same set of sensor dynamics as in (20) and use a disturbance estimator similar to (21), only with x_a replacing u

$$\dot{\hat{x}} = x_a + \operatorname{sgn}(x_s - \hat{x}). \tag{32}$$

With sliding mode occurs on $x - \hat{x} = 0$, the disturbance $d(t)$ is estimated with the equivalent control given by (23) to $O(\tau_s)$. With the disturbance compensated, the remaining task is to design a linear feedback control to achieve the desired transient performance. The resulting feedback control law is given by

$$u = -k_1 \hat{x} - k_2 x_a - [\operatorname{sgn}(x_s - \hat{x})]_{\text{eq}}. \tag{33}$$

With $\tau_s = 0.01$, and $\alpha = 0.2$, the feedback gains $k_1 = 31.25$, and $k_2 = 6.25$ place the poles of third-order system dynamics, which consists of the actuator dynamics and the integrator plant, at $\{-2.5, -2.5, -5\}$. Again, we use the same third-order butterworth low-pass filter with a 50 rad/s bandwidth as before to filter the equivalent control signal. Fig. 8 shows the effects of the constant disturbance $d(t) = 0.5$ are neutralized since the error between the reference command and the plant state is reduced to zero in steady state. The disturbance estimate is shown in Fig. 9 to reach its expected value in steady state.

F. Frequency Shaping

An approach which has been advocated for attenuating the effects of unmodeled parasitic dynamics in sliding mode involves the introduction of frequency shaping in the design of the sliding manifold [5]. Instead of treating the sliding manifold as the intersection of hyperplanes defined in the state-space of the plant, sliding manifolds which are defined as linear operators are introduced to suppress frequency components of the sliding mode response in a designated frequency band. For unmodeled high-frequency dynamics, this approach implants a low-pass filter either as a prefilter, similar to introducing

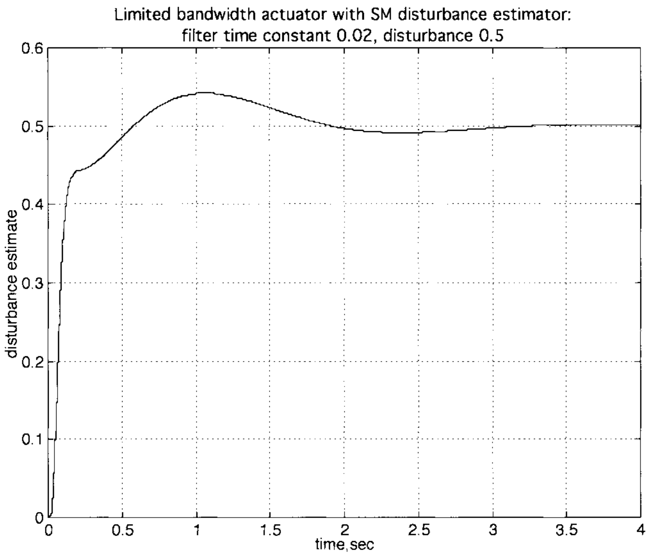

Fig. 9. Limited bandwidth actuator with SM disturbance estimator: Disturbance estimate.

artificial actuator dynamics, or as a postfilter, functioning like sensor dynamics. The premise of this frequency shaped sliding mode design, which was motivated by flexible robotic manipulator control applications [33], is that the effects of parasitic dynamics remain to be critical on the sliding manifold. However, robustness to chattering was only implicitly addressed in this design. By combining frequency shaping sliding mode and the SMC designs introduced earlier, the effects of parasitic dynamics on switching induced oscillations, as well as their interactions with sliding mode dynamics can be dealt with.

A Frequency-Shaped SMC Design: For the nominal integrator plant with parasitic sensor dynamics, we introduce a frequency shaping postfilter

$$\ddot{x}_p + 2\omega_p \dot{x}_p + \omega_p^2 x_p = x_s \quad (34)$$

$$y_p = p_1 \dot{x}_p + p_2 x_p. \quad (35)$$

The sliding manifold is defined as a linear operator, which can be expressed as a linear transfer function

$$\sigma(s) = \left(1 + \frac{p_1 s + p_2}{s^2 + 2\omega_p s + \omega_p^2}\right) x_s(s) = 0. \quad (36)$$

Given an estimate of the lower bound of the bandwidth of parasitic dynamics, the postfilter parameter ω_p can be chosen to impose a frequency dependent weighting function in a linear quadratic optimal design whose solution provides an optimal sliding manifold. The optimal feedback gains are implemented as p_1, p_2 in (36), and they ensure that the sliding mode dynamic response has adequate roll off in the specified frequency band.

G. Robust Control Design Based on the Lyapunov Method

Another nonlinear control design approach for plants whose dynamic models are uncertain is a robust control design which utilizes a Lyapunov function of the nominal plant. The origin of this approach can be traced to the work published in the 1970's by Leitmann and Gutman [34], [35]. Although sliding mode is not explicitly evoked in the Lyapunov control synthesis, nevertheless, the resulting closed-loop system behavior unavoidably includes sliding mode as the system's trajectory approaches the desired equilibrium point.

Given an affine dynamic system

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t), \qquad u \in \mathbb{R}^m \quad (37)$$

where $x \in \mathbb{R}^n$ is the state vector, and $B(x,t) \in \mathbb{R}^{n \times m}$ is the input matrix, and there exists a vector $\lambda(x,t) \in \mathbb{R}^m$ such that

$$h(x,t) = B(x,t)\lambda(x,t). \quad (38)$$

We note that (38) satisfies the Drazenovic matching condition introduced for variable structure systems [32]. The robust feedback control law which stabilizes the above system is given by

$$u = -\rho(x,t)\frac{B^T \nabla V}{||B^T \nabla V||}$$
$$\nabla V \triangleq \frac{\partial V}{\partial x} \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n \quad (39)$$

where $\rho(\cdot,\cdot)$ is a scalar feedback gain satisfying the condition

$$\rho(x,t) > ||\lambda(x,t)|| \quad (40)$$

and $V(x) > 0$ is a Lyapunov function of the nominal plant, i.e., along the trajectories of (37) with $u = 0$ and $h(\cdot,\cdot) = 0, \dot{V}$ is negative definite. For unity feedback gain, $\rho(\cdot,\cdot) = 1$, the norm of the above feedback control is equal to unity for any (x,t), thus it is also referred to as unit control.

Unambiguously, $u(\cdot,\cdot)$ is discontinuous on the manifold

$$s(x) = B^T \nabla V = 0. \quad (41)$$

Moreover, the condition (40) guarantees that sliding mode exists on $s(x) = 0$ inside a domain $||x(t)|| \leq \Omega(x,t)$. For sufficiently large $\rho(\cdot,\cdot)$, sliding mode exists for any x. Since the closed-loop system is asymptotically stable, sliding mode on $s(x) = 0$ is also asymptotically stable, i.e., $x(t) \to 0$ on the manifold $s(x) = 0$ as $t \to \infty$. Moreover, the dynamics of the system in sliding mode are invariant with respect to the unknown disturbance $h(x,t)$.

Since sliding mode is the principal mechanism with which uncertainties and disturbances are rejected in robust control of uncertain systems, the robustness of these feedback controllers with respect to unmodeled dynamics are identical to continuous-time SMC, and the respective engineering design issues can be addressed as outlined in this section.

A Robust Control Stabilization Example: The dynamics of a rolling platform with a rotating eccentric mass [36] are governed by

$$\dot{x}_1 = x_2 \quad (42)$$

$$\dot{x}_2 = \frac{\delta \cos x_1 (d(t) - \delta \sin x_1 x_2^2) - b x_2 - k x_1 + u}{(1 - \delta^2 \cos^2 x_1)} \quad (43)$$

where $\delta < 1$ is a measure of the eccentricity of the rotating inertia, $d(t)$ is the translational displacement of the platform, x_1, x_2 are the angular displacement and velocity of the rotating mass, and u is the control torque. The uncertainty in the dynamics is due to the platfrom's translational motion. The unperturbed system, with $d = 0$ and $u = 0$ in (43), is a conservative mechanical system. However, its total energy

$$E(x_1, x_2) = \tfrac{1}{2}\, kx_1^2 + \tfrac{1}{2}\,(1 - \delta^2 \cos^2 x_1)x_2^2. \tag{44}$$

cannot be used a Lyapunov function for the robust control design because its time derivative is seminegative definite, and not negative definite. The origin of this system is nevertheless asymptotically stable. For this system, the robust control assumes a *de facto* discontinuous control characteristic since the discontinuous manifold is one-dimensional

$$s = \frac{\partial V(x_1, x_2)}{\partial x_2}(1 - \delta^2 \cos^2 x_1)^{-1} = 0 \tag{45}$$

where $V(x_1, x_2)$ is a Lyapunov function which must be computed in the design process. This is a major design issue since finding a suitable Lyapunov function even for this simple nonlinear system is a nontrival task. Given $k > \sqrt{b} > 0$, there exists $k^* > 0$ and $\alpha > 0$ such that, along the trajectories of the unperturbed system, the time derivative of

$$\begin{aligned} V(x_1, x_2) = \tfrac{1}{2}\, k^* x_1^2 + \alpha(1 - \delta^2 \cos^2 x_1)x_1 x_2 \\ + \tfrac{1}{2}\,(1 - \delta^2 \cos^2 x_1)x_2^2 \end{aligned} \tag{46}$$

is negative definite in a domain $\{(x_1, x_2)\colon (x_1^2 + x_2^2)^{1/2} \le r(k^*, \alpha, k, b)\}$. Using this as a Lyapunov function, the resulting robust control is in the form of a sliding mode control which is typically applied to second-order mechanical systems

$$u = -\rho_o(x_1, x_2, t)\,\mathrm{sgn}\,(x_2 + \alpha x_1) \tag{47}$$

$$\rho_o(x_1, x_2, t) > |\delta \cos x_1\, d(t)|. \tag{48}$$

With this example, we have shown that a more effective control design procedure for uncertain dynamic systems is to bypass the detour into the Lyapunov function construction, and to proceed with a sliding mode control design. Fig. 10 shows the phase trajectory of the closed-loop system to which feedback control in the form of (47) has been applied. The system is subjected to a 10 Hz sinusoidal excitation of the platform, $d(t) = \sin(2\pi 10t)$, and the control magnitude is $\rho_o = 1.1\delta$. The stiffness and damping are $k = 50, b = 5$, and the eccentricity parameter $\delta = 0.2$. We solve for the cross product coefficient α in (46) to satisfy the negative definiteness condition. One possible solution is $\alpha = 1$. While it is a fairly straightforward matter in sliding mode control design to change the slope of the switching line, it may require the construction of another Lyapunov function. Such is the case here if it is desirable to speed up the transient process in sliding mode by changing α to 10. For the switching function implementation, we utilize a boundary layer control with a boundary layer thickness of $\epsilon = 0.001$. Due to the finite gain approximation, the effects of the persistingly excitating platform motion on the system's trajectory are only attenuated to $O(\epsilon)$. The residual oscillations in the phase trajectory near the origin are due to the exogenous disturbance. Nevertheless, this trajectory clearly remains inside the boundary layer which indicates that sliding mode would exist on the switching manifold $x_2 + x_1 = 0$ if the control law is implemented with a switching function.

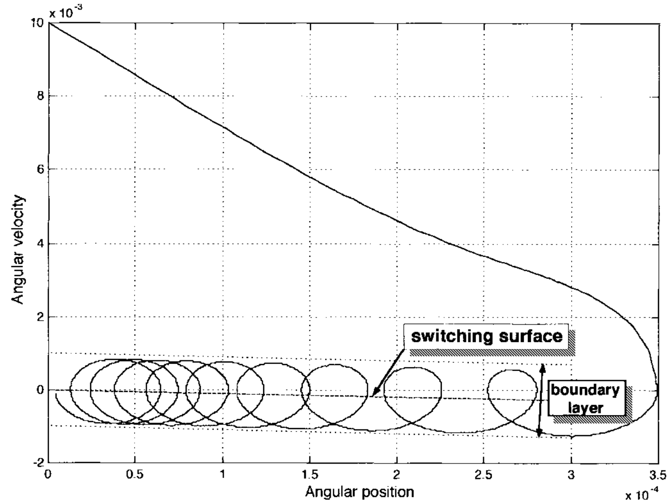

Fig. 10. Phase trajectory of the eccentric rotating mass platform under sinusoidal excitation and robust unit control.

III. Discrete-Time Sliding Mode

While it is an accepted practice for control engineers to consider the design of feedback systems in the continuous-time domain—a practice which is based on the notion that, with sufficiently fast sampling rate, the discrete-time implementation of the feedback loops is merely a matter of convenience due to the increasingly affordable microprocessor hardware. The essential conceptual framework of the feedback design remains to be in the continuous-time domain. For VSS and SMC, the notion of sliding mode subsumes a continuous-time plant, and a continuous-time feedback control, albeit its discontinuous, or switching characteristics. However, SM, with its conceptually continuous-time characteristics, is more difficult to quantify when a discrete-time implementation is adopted. When control engineers approach sampled data control, the choice of sampling rate is an immediate, and extremely critical design decision. Unfortunately, in continuous-time SM, desired closed-loop bandwidth does not provide any useful guidelines for the selection of sampling rate. In the previous section, we indicate that asymptotic observers or sliding mode observers can be constructed to eliminate chattering. Observers are most likely constructed in discrete time for any real life control implementations. However, in order for these observer-based design to work, the sampling rate has to be relatively high since the notion of continuous-time sliding mode is still applied.

For SM, the continuous-time definition and its associated design approaches for sampled data control implementation have been redefined to cope with the finite-time update limitations of sampled data controllers. DSM was introduced [37] for discrete-time plants. The most striking contrast between SM and DSM is that DSM may occur in discrete-time systems with continuous right-hand sides, thus discontinuous control and SM, are finally separable. In discrete time, the notion of

VSS is no longer a necessity in dealing with motion on a sliding manifold.

IV. Sampled Data Sliding Mode Control Design

We shall limit our discussions to plant dynamics which can be adequately modeled by finite dimensional ordinary differential equations, and assume that an *a priori* bandwidth of the closed-loop system has been defined. The feedback controller is assumed to be implemented in discrete-time form. The desired closed-loop behavior includes insensitivity to significant parameter uncertainties and rejection of exogenous disturbances. Without such a demand on the closed-loop performance, it is not worthwhile to evoke DSM in the design. Using conventional design rule of thumb for sampled data control systems, it is reasonable to assume that for the discretization of the continuous-time plant, we include only the dominant modes of the plant whose corresponding corner frequencies are well within the sampling frequency. This is always achievable in practice by antialiasing filters which attenuate the plant outputs at frequencies beyond the sampling frequency before they are sampled. Actuator dynamics are assumed to be of higher frequencies than the sampling frequency. Otherwise, actuator dynamics will have to be handled as part of the dominant plant dynamics. Thus, all the undesirable parasitic dynamics manifest only in the between sampling plant behavior, which is essentially the open-loop behavior of the plant since sampled data feedback control is applied. Clearly, this removes any remote possibilities of chattering due to the interactions of sliding mode control with the parasitic dynamics.

We begin to summarize sampled data sliding mode control designs with the well understood sample and hold process. This may seem to be elementary at first glance, it is however worthwhile since the matching conditions for the continuous-time plant are only satisfied in an approximation sense in the discretized models. We shall restrict our discussions to linear time-invariant plants with uncertainties and exogenous disturbances

$$\dot{x} = Ax + Bu + Ed, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad d \in \mathbb{R}^r \tag{49}$$

where A, B, E are constant matrices, and $d(t)$ is the exogenous disturbance. For the plant (49), we assume that the system matrices are decomposed into nominal and uncertain components

$$A = \overline{A} + \Delta A, \quad B = \overline{B} + \Delta B \tag{50}$$

where $\overline{A}, \overline{B}$ denote the nominal components. Let the admissible parametric uncertainties satisfy the following model matching condition [32]:

$$\operatorname{rank}([\overline{B} \vdots \Delta A \vdots \Delta B \vdots E]) = \operatorname{rank} \overline{B}. \tag{51}$$

The discrete-time model is obtained by applying a sample and hold process to the continuous-time plant with sampling period T, which to $O(T^2)$, is given by

$$x_{k+1} = Fx_k + Gu_k + Dd_k, \quad x_0 = x(t_o) \tag{52}$$

$$x(kT) = x_k, \quad u(kT) = u_k, \quad d(kT) = d_k \tag{53}$$

where F, G, and D result from integrating the solution of (49) over the time interval $t \in [kT, (k+1)T]$ with

$$u(t) = u(kT), \qquad d(t) = d(kT) \tag{54}$$

$$\overline{F} = \exp(\overline{A}T), \qquad \overline{G} = \Gamma \overline{B} \tag{55}$$

$$F = \overline{F} + \Gamma \Delta A, \qquad \Gamma = \int_0^T \exp(\overline{A}\tau)\, d\tau \tag{56}$$

$$G = \overline{G} + \Gamma \Delta B, \qquad D = \Gamma E. \tag{57}$$

This discrete-time model is an $O(T^2)$ approximation of the exact model which is described by the same F and G matrices, but because the exogenous disturbance is a continuous-time function, the sample and hold process yields a D matrix which renders the matching condition for the continuous-time plant to be only a necessary, but not sufficient condition for the exact discrete-time model [40]. However, by adopting the above $O(T^2)$ approximated model, it follows from (57) that, if the continuous-time matching condition (51) is satisfied, the following matching condition for this model holds:

$$\operatorname{rank}[\Gamma\overline{B} \vdots \Gamma\Delta A \vdots \Gamma\Delta B \vdots \Gamma E] = \operatorname{rank}(\Gamma\overline{B}). \tag{58}$$

From an engineering design perspective, the $O(T^2)$ models are adequate since the between sampling behavior of the continuous-time plant is also $O(T^2)$ close to the values at the sampling instants. Let the sliding manifold be defined by

$$s_k = Cx_k = 0, \qquad k = 0, 1, \cdots, \quad s(kT) = s_k \tag{59}$$

Two different definitions of discrete-time sliding mode have been proposed for discrete-time systems. While these definitions share the common base of using the concept of equivalent control, the one proposed in [37] uses a definition of discrete-time equivalent control $u_k^{\mathrm{eq}} = u(kT)$ which is the solution of

$$s_{k+1} = 0, \qquad k = 0, 1, \cdots. \tag{60}$$

On the other hand, u_k^{eq} is defined in [38] as the solution of

$$\Delta_k^s = s_{k+1} - s_k = 0, \qquad k = 0, 1, \cdots. \tag{61}$$

Note that (60) implies (61), however, the converse is not true. Herein, the first definition given by (60) shall be used.

A. DSM Control Design for Nominal Plants

Given the nominal plant with no external disturbance, the DSM design becomes intuitively clear. In DSM, by definition

$$s_{k+1} = Cx_{k+1} = C(\overline{F}x_k + \overline{G}u_k) = 0 \tag{62}$$

and provided that $C\overline{G}$ is invertible, the DSM control which is also the equivalent control, is given by the linear continuous feedback control

$$u_{k_{\mathrm{eq}}} = -[C\overline{G}]^{-1} C\overline{F} x_k. \tag{63}$$

The only other complication is that since $||\overline{G}|| = O(T)$, the required magnitude of this control may be large. If the bounds $\overline{u}$ on u_k are taken into account, the following feedback control has been shown [19] to force the system into DSM:

$$u_k = \begin{cases} u_{k_{eq}}, & \text{if } ||u_{k_{eq}}|| < \overline{u} \\ -\overline{u}\dfrac{u_{k_{eq}}}{||u_{k_{eq}}||}, & \text{if } ||u_{k_{eq}}|| \geq \overline{u}. \end{cases} \tag{64}$$

DSM Control of the Integrator Plant: For the nominal integrator plant with parasitic sensor dynamics (20), we design an DSM controller based on (64). Let the sensor time constant $\tau_s = 0.02$, and the control magnitude $\overline{u} = 1$. The desired closed-loop bandwidth is given to be one Hz. A good choice of the sampling frequency would be 10 Hz $(T = 0.1)$. Since the sensor dynamics are of 50 Hz, and therefore they can be neglected initially in the design. The DSM control takes the form of

$$u_k = \begin{cases} -\dfrac{x_k^s}{T}, & \text{if } |x_k^s| < T \\ -\text{sgn}\,(x_k^s), & \text{if } |x_k^s| \geq T \end{cases} \tag{65}$$

where $x_k^s = x_s(kT)$ is the sampled value of the sensor output $x_s(t)$. Note that due to the control bounds, a linear feedback control law is applied inside a boundary layer of thickness $2T$ about the sliding manifold $x_k^s = 0$. Without sensor dynamics, the behavior inside the boundary layer is that of a deadbeat controller. The sensor dynamics impose a third-order discrete-time system inside this boundary layer, and its eigenvalues are inside the unit circle at $\{-0.002, 0.1 \pm \jmath 0.436\}$. For reference, the discrete model of the open-loop nominal plant and the sensor dynamics has a pair of double real pole almost at the origin $\{4.54 \times 10^{-5})\}$, which result from sampling at a frequency much lower than the sensor's corner frequency, and a pole at unity which is due to the integrator plant. The third-order system response can be seen in Fig. 11 where the sample values of the error between the constant unity reference command and the sensor output is plotted. Note that only the behavior inside the boundary layer is shown, and it agrees well with the predicted third-order behavior. The steady state error magnitude of 0.05 is due to the constant disturbance $d(t) = 0.5$ as applied to this plant as before, and the effective loop gain being $T^{-1} = 10$. Fig. 12 displays the continuous-time error of the plant state and the discrete-time error of the sensor output where the time lag due to the sensor dynamics can be seen during the transient period.

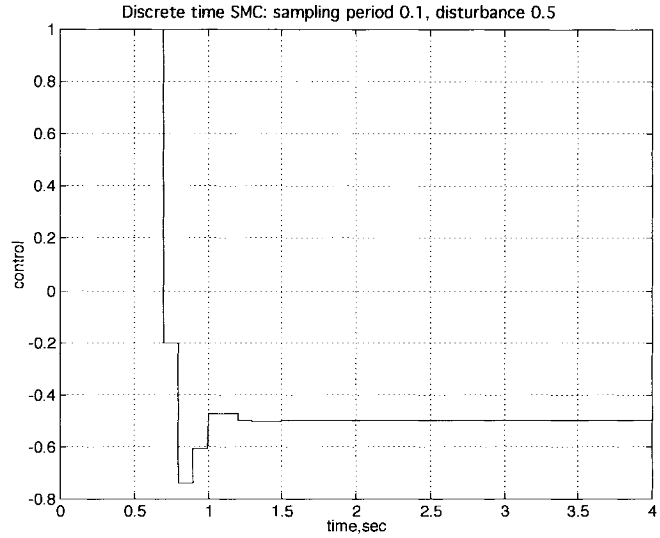

Fig. 11. DSM control for nominal plant: error between reference command and sensor output.

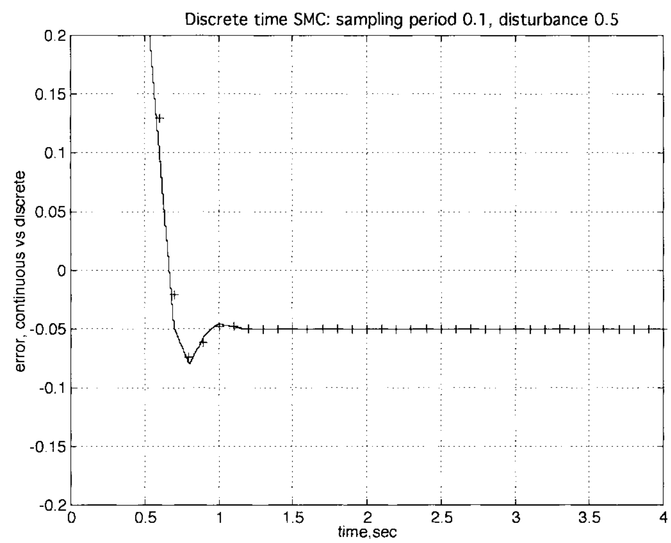

Fig. 12. DSM control for nominal plant: Continuous-time and discrete-time error responses.

B. DSM Control with Delayed Disturbance Compensation

The earlier DSM control design for nominal plants can be modified to compensate for unknown disturbances in the system [39], [40]. From the discrete model in (52), the one step delayed unknown disturbance

$$d_{k-1}^* \triangleq Dd_{k-1} = x_k - \overline{F}x_{k-1} - \overline{G}u_{k-1} \tag{66}$$

can be computed, given the measurements x_k, x_{k-1}, and u_{k-1}, and the nominal system matrices $\overline{F}, \overline{G}$. Let

$$u_{k_{eq}}^* \triangleq -[C\overline{G}]^{-1}C[\overline{F}x_k + d_{k-1}^*]. \tag{67}$$

The feedback controller is of a similar form as (64)

$$u_k = \begin{cases} u_{k_{eq}}^*, & \text{if } ||u_{k_{eq}}^*|| < \overline{u} \\ -\overline{u}\dfrac{u_{k_{eq}}^*}{||u_{k_{eq}}^*||}, & \text{if } ||u_{k_{eq}}^*|| \geq \overline{u}. \end{cases} \tag{68}$$

The effectiveness of this controller is demonstrated by examining the behavior of s_k when the control signal is not saturated

$$s_{k+1} = CD(d_k - d_{k-1}). \tag{69}$$

If the disturbance has bounded first derivatives, i.e., $|\dot{d}| \leq \overline{d} < \infty, d_k - d_{k-1}$ is of $O(T)$, and from the definition given in (57), $||D|| = O(T)$, hence $|s_k| = O(T^2)$, implying that the motion of the system remains within an $O(T^2)$ neighborhood of the sliding manifold. This controller has also been shown

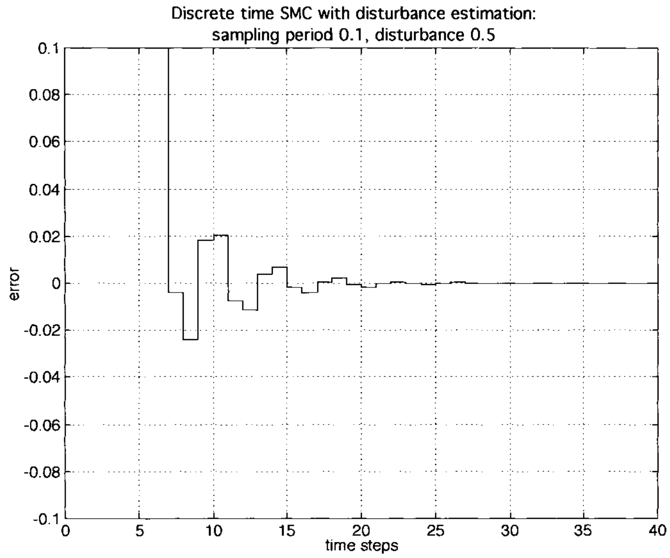

Fig. 13. DSM control with disturbance compensation: Error between reference command and sensor output.

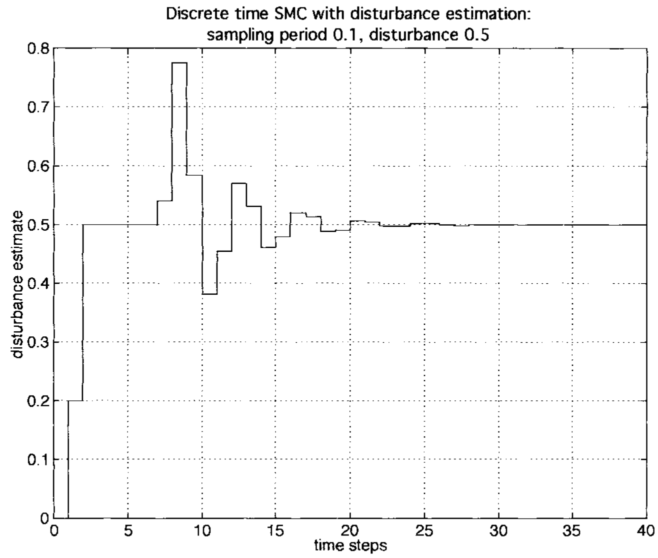

Fig. 14. DSM control with disturbance compensation: One step delayed disturbance estimate.

[19] to force the system into DSM if the control signal is initially saturated.

On the sliding manifold, the system dynamics are, to $O(T^2)$, invariant with respect to the unknown disturbance. Since similar matching conditions exist for the $O(T^2)$ discrete-time models we have adopted, it follows from continuous-time sliding mode [28] that by using a change of state variables, the discrete model can be transformed into

$$x^1_{k+1} = \overline{F}_{11}x^1_k + \overline{F}_{12}x^2_k \tag{70}$$
$$x^2_{k+1} = \overline{F}_{21}x^1_k + \overline{F}_{22}x^2_k + \overline{G}_2u_k + D_2d_k \tag{71}$$

with the sliding manifold given by

$$s_k = C_1x^1_k + C_2x^2_k = 0 \tag{72}$$

and $C_2\overline{G}$ is nonsingular. By eliminating x^2_k, the reduced order sliding mode dynamics are $O(T^2)$ approximated by

$$x^1_{k+1} = (\overline{F}_{11} - \overline{F}_{12}C_2^{-1}C_1)x^1_k. \tag{73}$$

Discrete-Time Disturbance Compensation for the Integrator Plant: We continue with the DSM control design using the same sampling frequency and system parameter values. The controller which takes into account the one step delayed disturbance estimates is given by

$$u_k = \begin{cases} -\dfrac{x^s_k}{T} - \dfrac{x^s_k - x^s_{k-1}}{T} + u_{k-1}, & \text{if } |x^s_k| < T \\ -\operatorname{sgn}(x^s_k), & \text{if } |x^s_k| \geq T. \end{cases} \tag{74}$$

Note the PID controller structure of this controller when the system is inside the boundary layer. Fig. 13 shows the sampled error between the reference command and the sensor output. The practically zero steady-state error is much better than our $O(T^2)$ estimate due to the PID controller structure. The one step delayed disturbance estimate is given in Fig. 14, showing convergence to the expected value. Fig. 15 displays the continuous-time error between the plant state and the reference, and its discrete-time measurements.

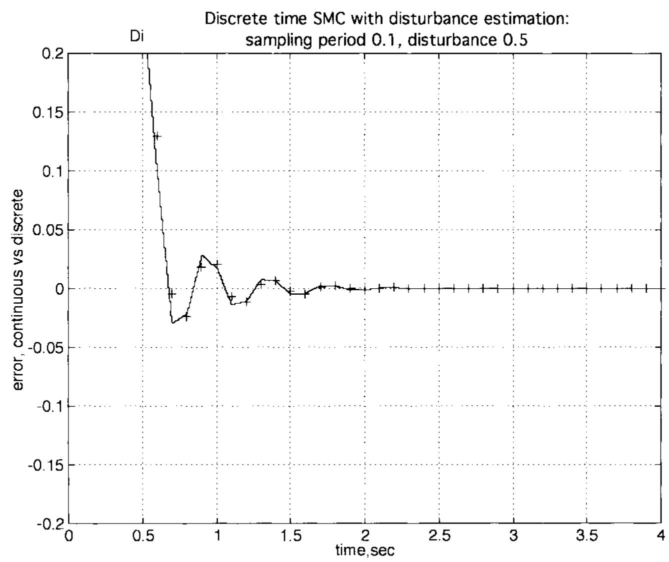

Fig. 15. DSM control with disturbance compensation: Continuous-time and discrete-time error responses.

C. DSM Control with Parameter Uncertainties and Disturbances

With the presence of system parameter uncertainties, the above approach which uses one step delayed disturbance estimates can still be applied. However the one step delayed signal contains both delayed state and control values

$$\begin{aligned} f_{k-1} &\triangleq \Delta F x_{k-1} + \Delta G u_{k-1} + Dd_{k-1} \\ &= x_k - \overline{F}x_{k-1} - \overline{G}u_{k-1} \end{aligned} \tag{75}$$

where $\Delta F = \Gamma\Delta A, \Delta G = \Gamma\Delta B$. The DSM control is of the same form as (68), with d^*_{k-1} replaced by f_{k-1}. The behavior of s_k is prescribed by

$$\begin{aligned} s_{k+1} = & C(f_k - f_{k-1}) = CD(d_k - d_{k-1}) \\ & + C\Delta F(x_k - x_{k-1}) + C\Delta G(u_k - u_{k-1}). \end{aligned} \tag{76}$$

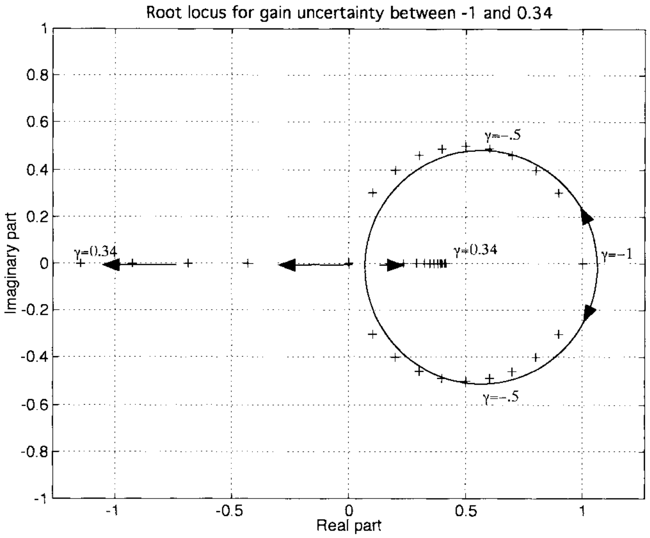

Fig. 16. DSM control with control parameter variations: Root locus for evaluating sliding manifold convergence.

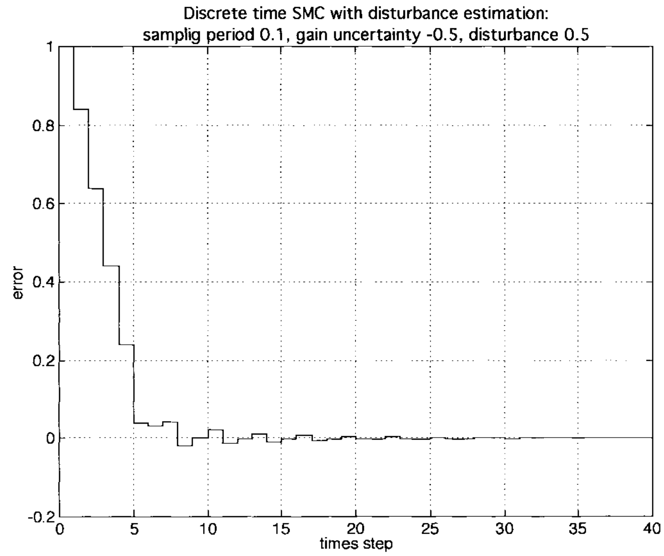

Fig. 17. DSM control with control parameter variations: Error between reference command and sensor output.

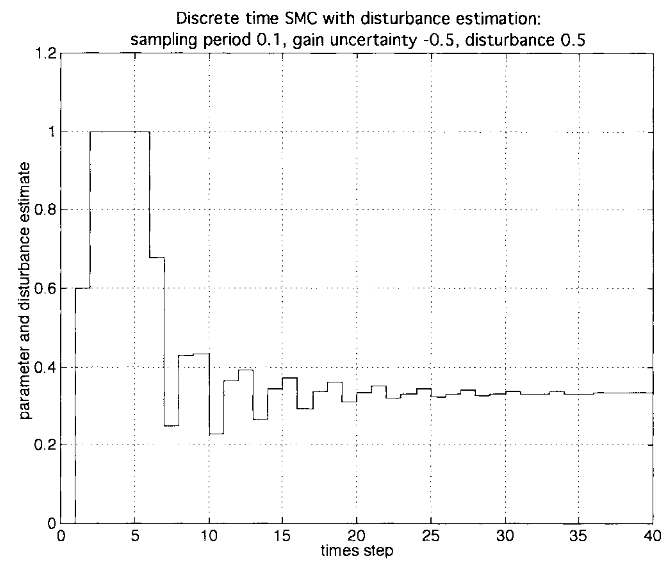

Fig. 18. DSM control with disturbance compensation: One step delayed parameter and disturbance estimate.

Since $|\dot{x}|$ is bounded, $x_k - x_{k-1}$ is of $O(T)$, and since $||\Delta F|| = O(T)$, we have

$$s_{k+1} = C\Delta G(u_k - u_{k-1}) + O(T^2). \tag{77}$$

Due to the coupling between s_k and u_k, it has been shown [40], [41] that the behavior outside the sliding manifold is governed by the following second-order difference equation:

$$s_{k+1} = -C\Delta G(C\overline{G})^{-1}[2s_k - s_{k-1}] + O(T^2) \tag{78}$$

which has poles inside the unit circle for sufficiently small $||\Delta B||$. The permissible control matrix uncertainties are dictated by the above stability condition which determines the convergence on the sliding manifold. Note that provided the parameter uncertainties are in the system matrix, they do not impact the convergence, nor they affect the motion on the manifold.

Compensation for Gain Uncertainties in Integrator Plant: We shall introduce gain uncertainties in the integrator plant to examine their effects on the convergence of the sliding manifold. The actual plant is given by

$$\dot{x} = (1+\gamma)u + d(t) \tag{79}$$

where γ represents the gain uncertainty in the integrator. The DSM controller in (74) can be used again because the right-hand side of the one step delayed signal is the same regardless of the parametric uncertainties. The root locus of the second-order system governing the motion outside the manifold is plotted in Fig. 16 for $-1 \leq \gamma \leq 0.34$. For $\gamma = -1$, there is a pair of double poles at unity, and for $\gamma = 1/3$, one of the poles becomes -1. The case for $\gamma = -0.5$, corresponding to a pole of complex pairs $0.5 \pm \jmath 0.5$, is simulated with the same reference and disturbance as in the previous studies. Fig. 17 shows the convergence of the sampled error between the reference command and the sensor output to zero. Fig. 18 displays the estimates of the exogenous disturbance and the residue control signal due to the gain uncertainty. The continuous-time error of the plant state and the discrete-time error of the sensor output are shown in Fig. 19 for comparison.

V. Conclusions

We have systematically examined SMC designs which are firmly anchored in sliding mode for the continuous-time domain. Most of these designs are focused on guaranteeing the robustness of sliding mode in the presence of practical engineering constraints and realities, such as finite switching frequency, limited bandwidth actuators, and parasitic dynam-

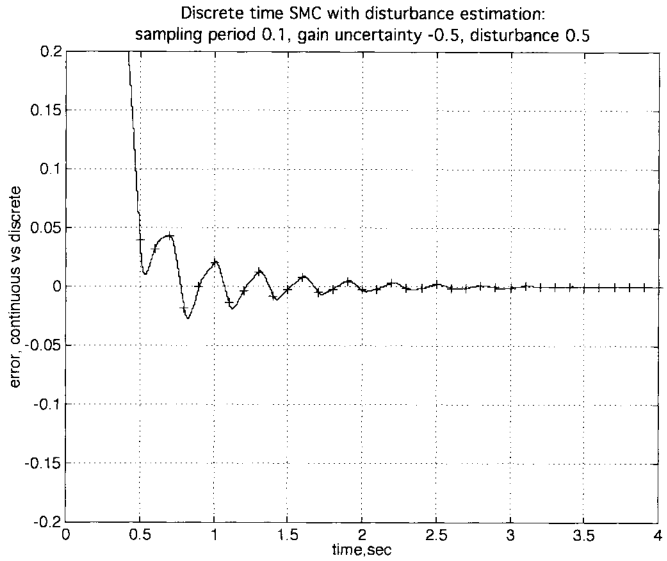

Fig. 19. DSM control with control parameter variations: Continuous-time and discrete-time error responses.

ics. Introducing DSM, and restructuring the SMC design in a sampled data system framework are appropriate, and positive steps in sliding mode control research. It directly addresses the pivotal microprocessor implementation issues; it moves the research in a direction which is more sensitive to the concerns of practicing control engineers who are faced with the dilemma of whether to ignore this whole branch of advanced control methods for fear of the reported implementation difficulties, or to embrace it with caution in order to achieve system performance otherwise unattainable. However, as compared with the ideal continuous-time sliding mode, we should also be realistic about the limitations of DSM control designs in rejecting disturbances, and in its ability to withstand parameter variations. The real test for the sliding mode research community in the near future will be the willingness of control engineers to experiment with these SMC design approaches in their professional practice.

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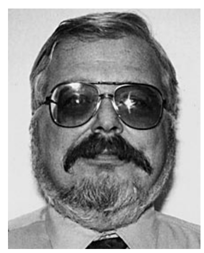

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