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Triangulated Derivators and their Calabi-Yau Dimension

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$$\begin{tikzcd}
 & & x_1 \arrow[rr] \arrow[dl] \arrow[dd] & & x_2 \arrow[rr] \arrow[dl] \arrow[dd] & & x_3 \arrow[dl] \arrow[dd] \\
 & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[dd] & \\
 & & 0_1 \arrow[rr] \arrow[dl] & & 0_3 \arrow[rr] \arrow[dl] & & 0_5 \arrow[dl] \\
 & y_3 \arrow[rr] \arrow[dl] \arrow[dd] & & z_2 \arrow[rr] \arrow[dl] \arrow[dd] & & u_1 \arrow[dl] \arrow[dd] & \\
0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[dd] & & \\
 & 0_4 \arrow[rr] \arrow[dl] & & 0_6 \arrow[rr] \arrow[dl] & & 0_8 \arrow[dl] & \\
v_1 \arrow[rr] & & v_2 \arrow[rr] & & v_3 & &
\end{tikzcd}$$

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Kunnskap for en bedre verden

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Abstract

We introduce the theory of derivators from the perspective of homological algebra. Beginning with motivation from the derived category and Kan extensions, before giving a thorough account of the theory. We extend several results from classical category theory to the setting of derivators, and prove that a triangulated derivator $\mathscr{D}$ induces a canonical triangulation on $\mathscr{D}(\mathbf{J})$, for all small categories $\mathbf{J}$. We also propose a generalized version of the cofiber functor for a derivator, called an n-cofiber functor, and show that this leads to a fractional Calabi-Yau dimension with respect to the suspension.

Sammendrag

Vi introduserer teorien bak derivatorer fra perspektivet til homologisk algebra. Først med motivasjon fra deriverte kategorier og kanutvidelser, deretter en nøye gjennomgang av teorien. Vi utvider flere resultater fra klassisk kategoriteori til derivator perspektivet, og beviser at en triangulert derivator $\mathscr{D}$ induserer en kanonisk triangulering på $\mathscr{D}(\mathbf{J})$, for alle små kategorier $\mathbf{J}$. Vi foreslår også en generalisert versjon av (ko)fiberfunktoren, kalt en n-(ko)fiberfunktor, og viser at dette fører til en brudden Calabi-Yau dimensjon med hensyn på suspensjonen.

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Contents

Introduction

The theory of derivators was developed independently by Grothendieck and Heller in the 80's as an enhancement of triangulated categories. Triangulated categories have been a huge success within several areas of mathematics. However, a major issue for triangulated categories is the lack of a functorial cone construction. Since the cone is not unique up to a unique isomorphism, there is no well-defined functor cone: $\mathcal{T}^{[1]} \to \mathcal{T}$ (see example 6.8). Derivators is a way to remedy this flaw. Let $\mathscr{D}$ be a triangulated derivator. Given any small category $\mathbf{J}$ the underlying category $\mathscr{D}(\mathbf{J})$ carries a canonical triangulated structure. But not only that, in this case we have a functorial cone construction!

In a sense, derivators are a way to extend classical results to a more general homotopy setting. A common slogan for the theory is that it is the 'minimal framework that allows for well-behaved calculus of homotopy (co)limits'. This framework has gained a lot of attention recently, and has been used in a variety of areas such as algebraic geometry, algebraic topology, and representation theory [3, 7, 8, 11, 12, 14, 16, 20]. One thing most recent papers have in common is that they give a rudimentary introduction as to what a derivator is, and then refer to either Grothendieck, Maltsiniotis, or Moritz Rahn (formerly named Moritz Groth) for more details. Unfortunately for the author of this thesis, most of the referenced texts are written in french. This, in addition to the reader friendliness, is the reason for why we largely follow the (at the time unfinished) book of Moritz Rahn, *Introduction to the theory of derivators* [5]. In chapters 1-6 unless otherwise stated, it is safe to assume this as a primary reference for the theory.

The plan for this thesis is to build up the theory of derivators from scratch, beginning with a recollection of abelian categories, localization, and the derived category. In chapter 1 we will show that the derived category of the arrow category $\mathbf{D}(\mathcal{A}^{[\mathbf{1}]})$ has a functorial cone construction (in contrast to $\mathbf{D}(\mathcal{A})^{[\mathbf{1}]}$). This is the motivation behind what we call *coherent diagrams*, and sits at the core of the theory of derivators. Then we introduce *Kan extensions* in chapter 2. This is a way to 'extend' one functor X along another functor u. We give an example to illustrate how this generalizes the notion of (co)limits, and also show how they can be calculated point-wise. Finally, we show that they induce adjoints to precomposition functors, which is a notion we want to demand of our derivators.

In chapter 3 we finally introduce (pre)derivators, and show how this encompasses the previous chapters. We give two examples of derivators, which we will reference throughout the rest of the thesis, and show how to generate new derivators. Then in chapter 4 we introduce *pointed* derivators, which leads to abstract (co)fiber, suspension and loop functors. These functors are motivated by the fact that functorial the cone on $\mathbf{D}(\mathcal{A}^{[\mathbf{1}]})$ is the left derived functor of the cokernel functor (theorem 1.26). After this we introduce *stable* derivators in chapter 5. Here the functors from chapter 4 become equivalences. In particular, we will use this to show that they induce additive categories $\mathscr{D}(\mathbf{J})$, for all small categories $\mathbf{J}$. When we introduce *strong* derivators in chapter 6, we show that derivators that are both stable and strong has even more canonical structure. They induce triangulated categories. For this reason these derivators are called *triangulated*, and we explain how this canonical triangulation amends the issue of functorial cones.

Finally, in chapter 7 we take the (co)fiber functor from chapter 4 and generalize it to composable morphisms of length n. We have coined these generalized functors *n-(co)fiber functors*. The same idea occurs in chapter five of *Abstract representation theory of Dynkin quivers of type A* [7], as compositions of reflection functors and suspensions. However, these functors are all in the context of stable derivators, with only a remark (p. 14) that says there are variants for pointed derivators. The author has found neither name nor construction of generalized (co)fiber functors, hence the proposed name. By generalizing the proof of lemma 5.13 from [6], we show that iterated sequences of n-cofiber functors are naturally isomorphic to powers of the suspension functor. In particular, this leads to a fractional Calabi-Yau dimension for a stable derivator, recovering Theorem 5.19 [7].

1 Derived categories of abelian categories

The derived category of abelian categories are of interest in many areas of mathematics, like representation theory and algebraic geometry. In this chapter we recall the definition and basic properties of abelian categories, and their derived categories. We will explore some properties that the abelian categories enjoy when passing to the derived category, before we highlight the difference between *coherent* and *incoherent* diagrams. The difference between these types of diagrams is important for the motivation of derivators. Finally, we show that the cone of the derived category is a left derived functor of the cokernel. This last result foreshadows the *cofiber* functor, which plays a very important role in the rest of thesis.

1.1 Abelian categories and localization

In this subsection we recall some properties of abelian categories, in particular quasi-isomorphisms and their properties, and introduce the idea behind localization. The localization theory is based on chapter three of *Derived categories, resolutions, and Brown representability* by Henning Krause [9]. We then prove a generalized version of proposition 3.35 in [5] which states that any well-defined localization functor gives rise to an equivalence on functor categories. And finally we define the derived category of an abelian category as a localization with respect to the class $W_{\mathcal{A}}$ of quasi-isomorphisms.

Definition 1.1. Let $\mathcal{A}$ be an abelian category, $\mathbf{Ch}(\mathcal{A})$ its category of chain complexes, and consider
$f\colon X \to Y$ a function in $\mathbf{Ch}(\mathcal{A})$. We say f is a *quasi-isomorphism*, if the induced map in homology, $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.

Example 1.2. Let us consider the two complexes in $\mathbf{Ch}(\operatorname{Mod}\mathbb{Z})$

$$\begin{tikzcd}
a^\bullet\colon & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \cdots \\
b^\bullet\colon & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/_{2\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where the projection morphism π lies in degree 1. Here the morphism is zero for all degrees $n \neq 1$, and so this morphism is clearly not an isomorphism. However one easily sees that

$$H^n(f) = \left\{ \begin{array}{ll} 0, & \text{if } n \neq 1 \\ \mathbb{Z}/_{2\mathbb{Z}}, & \text{if } n = 1 \end{array} \right\}$$

for both $a^\bullet$ and $b^\bullet$. Hence, this is a quasi-isomorphism.

We denote the set of all quasi-isomorphisms in $\mathcal{A}$ by $W_\mathcal{A}$

Proposition 1.3. *Let $\mathcal{A}, \mathcal{B}$ be to two abelian categories, and $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor between them. F is exact if and only if the induced $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{B})$ preserves quasi-isomorphisms.*

Proof. Assume F preserves quasi-isomorphisms, and let $0 \to a_0 \to a_1 \to a_2 \to 0$ be a short exact sequence in $\mathbf{Ch}(\mathcal{A})$. We apply F to this sequence to obtain $F(0) \to F(a_0) \to F(a_1) \to F(a_2) \to F(0)$. Note that the first sequence is short exact if and only if it is quasi-isomorphic to zero (since the homology is zero in each degree). This means that the second sequence is also quasi-isomorphic to zero, and thus a short exact sequence. Hence F is exact if it preservers quasi-isomorphisms.

Conversely, assume that F is exact and let $q\colon a_1^\bullet \to a_2^\bullet$ be a quasi-isomorphism in $\mathbf{Ch}(\mathcal{A})$. Since F is exact, it preserves short exact sequences, which means that we have

(i)
$$\operatorname{Ker}[F(a_1^i) \to F(a_2^i)] = F(\operatorname{Ker}[a_1^i \to a_2^i])$$

(ii)
$$\operatorname{Coker}[F(a_1^i) \to F(a_2^i)] = F(\operatorname{Coker}[a_1^i \to a_2^i])$$

Since images are defined as cokernels of kernels, F preserves images as well. In particular, this means that F preserves homology. Since functors preserve isomorphisms, we thus have $H^i(F(a_1)^\bullet) = F(H^i(a_1^\bullet)) \xrightarrow{\cong} F(H^i(a_2^\bullet)) = H^i(F(a_2)^\bullet =$ which concludes the result. □

So we have a notion of quasi-isomorphisms in a category, and a notion of preserving quasi-isomorphisms for functors. The only natural thing to do is to define quasi-isomorphisms for natural transformations as well.

Definition 1.4. Let $\mathcal{A}, \mathcal{B}$ be two abelian categories, $F, G \colon \mathcal{A} \to \mathcal{B}$ be two functors, and $\alpha \colon F \to G$ a natural transformation between them. If α_a is a quasi-isomorphism for all objects a we say α is a *levelwise quasi-isomorphism.*

Proposition 1.5. *Let $\mathcal{A}$ be an abelian category, and $\mathbf{Ch}(\mathcal{A})$ its category of complexes. If $u \colon \mathbf{J} \to \mathbf{K}$ is a functor between small categories, then the restriction functor*

$$u^* \colon \mathbf{Ch}(\mathcal{A})^{\mathbf{K}} \to \mathbf{Ch}(\mathcal{A})^{\mathbf{J}}, \quad X \mapsto X \circ u$$

preserves levelwise quasi-isomorphisms.

Proof. Let $F, G \colon \mathbf{K} \to \mathbf{Ch}(\mathcal{A})$ be two functors, and $\alpha \colon F \to G$ a levelwise quasi-isomorphism. Write $u^*(F)$ for the precomposition $F \circ u$. Then we get the two induced restriction functors $u^*(F), u^*(G) \colon \mathbf{J} \to \mathbf{Ch}(\mathcal{A})$, along with the natural transformation $u^*(\alpha) \colon u^*(F) \to u^*(G)$. This gives the diagram

$$\begin{tikzcd}
(F \circ u)(j) = F(u(j)) \arrow[r, "\alpha_{u(j)}"] \arrow[d, "F(u(f))"'] & G(u(j)) = (G \circ u)(j) \arrow[d, "G(u(f))"] \\
(F \circ u)(j') = F(u(j')) \arrow[r, "\alpha_{u(j')}"] & G(u(j')) = (G \circ u)(j')
\end{tikzcd}$$

which commutes in $\mathcal{A}$. For every $j \in \mathbf{J}$, $\alpha_{u(j)}$ is a quasi-isomorphism for some $u(j) \in \mathbf{K}$. Hence, this preserves level-wise quasi-isomorphisms. □

When we later consider abstract derivators, we will axiomatize this property.

Proposition 1.6. *Let* $\mathbf{J}$ *be a small category and* $\mathcal{A}$ *an abelian category. Then* $\mathbf{Ch}(\mathcal{A})^{\mathbf{J}} \cong \mathbf{Ch}(\mathcal{A}^{\mathbf{J}})$*.*

Proof. Let $X \colon \mathbf{J} \to \mathbf{Ch}(\mathcal{A})$ be a diagram of chain complexes. Then for any $n \in \mathbb{Z}$ and any object $j \in \mathbf{J}$ we can define $Y(j)^n = X^n(j)$ to be the degree-wise image, so that $Y \in \mathbf{Ch}(\mathcal{A}^{\mathbf{J}})$.

Given any Y in $\mathbf{Ch}(\mathcal{A}^{\mathbf{J}})$ we can define X in the same way. That is, there is a 1-1 correspondence between both objects and morphisms of $\mathbf{Ch}(\mathcal{A})^{\mathbf{J}}$ and $\mathbf{Ch}(\mathcal{A}^{\mathbf{J}})$. $\square$

This is an important isomorphism. It allows us to keep control over (co)limits of the complex categories, and we will use this implicitly throughout this text. However, we are going to see that this is no longer the case for the derived categories $\mathbf{D}(\mathcal{A}^{\mathbf{J}})$ and $\mathbf{D}(\mathcal{A})^{\mathbf{J}}$. In fact, this is not even true for the homotopy category $\mathbf{K}(\mathcal{A})$. This is what motivates the idea of *coherent* and *incoherent* diagrams.

For the rest of this section we go through the basics of localization (based on chapter three [9]), in order to get a good understanding of the derived category.

Definition 1.7. Let $\mathbf{J}$ be a category, and $\mathbf{S}$ a class of morphisms in $\mathbf{J}$. We say $\mathbf{S}$ is a *multiplicative system* if the following hold

MS(1) If $f, g \in \mathbf{S}$ are composable then $(g \circ f) \in \mathbf{S}$, and the identity $\mathrm{id}_j \in \mathbf{S}$ for all $j \in \mathbf{J}$.

MS(2) If $s \colon j_1 \to j_2 \in \mathbf{S}$ then every pair of morphisms $g \colon j_1 \to j'$ and $f \colon j \to j_2$ in $\mathbf{J}$ can be completed to commutative diagrams

$$\begin{tikzcd}
j_1 \arrow[r, "g"] \arrow[d, "s"'] & j' \arrow[d, dashed, "\hat{s_1}"] & & \hat{j} \arrow[r, dashed, "\hat{f}"] \arrow[d, dashed, "\hat{s_2}"'] & j_1 \arrow[d, "s"] \\
j_2 \arrow[r, dashed, "\hat{g}"'] & \hat{j}' & & j \arrow[r, "f"'] & j_2
\end{tikzcd}$$

with $\hat{s_1}, \hat{s_2} \in \mathbf{S}$ and $\hat{f}, \hat{g} \in \mathbf{J}$.

MS(3) If $f, g\colon j \to j'$ are morphisms in $\mathbf{J}$, then there exists a morphism $s_1\colon \hat{j} \to j \in \mathbf{S}$ such that $f \circ s_1 = g \circ s_1$ if and only if there exists a morphism $s_2\colon j' \to \hat{j}' \in \mathbf{S}$ such $s_2 \circ f = s_2 \circ g$.

$$\begin{tikzcd}
\hat{j} \arrow[r, dashed, "s_1"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & j' \arrow[r, dashed, "s_2"] & \hat{j}'
\end{tikzcd}$$

Given a multiplicative system $\mathbf{S}$, we are going to create a category $\mathbf{J}[\mathbf{S^{-1}}]$ where our goal is to formally invert morphisms of $\mathbf{S}$. Therefore we let the objects of $\mathbf{J}[\mathbf{S^{-1}}]$ be the same as the objects of $\mathbf{J}$, but we define the morphisms of $\mathbf{J}[\mathbf{S^{-1}}]$ by

(i) A morphism in $\mathbf{J}[\mathbf{S^{-1}}]$ is a pair (f, s) $\quad j \xrightarrow{f} \hat{j} \xleftarrow{s} j' \quad$ with $s \in \mathbf{S}$. In particular, the identity is given by $(\mathrm{id}_j, \mathrm{id}_j)$ for all $j \in \mathbf{J}[\mathbf{S^{-1}}]$.

(ii) Two morphisms (f_1, s_1) and (f_2, s_2) in $\mathbf{J}[\mathbf{S^{-1}}]$ are equivalent if there exists a third morphism (f_3, s_3) in $\mathbf{J}[\mathbf{S^{-1}}]$, along with two morphisms g, h in $\mathbf{J}$, such that the following diagram commutes

$$\begin{tikzcd}
 & j_1 \arrow[d, "g"'] & \\
j \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & j_3 & j' \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\
 & j_2 \arrow[u, "h"] &
\end{tikzcd}$$

(iii) Composition of two morphisms (f_1, s_1) and (f_2, s_2) in $\mathbf{J}[\mathbf{S^{-1}}]$ is given by $(\hat{f_2} \circ f_1, \hat{s_1} \circ s_2)$, where $\hat{f_2} \in \mathbf{J}$ and $\hat{s_1} \in \mathbf{S}$ comes from $MS(3)$.

$$\begin{tikzcd}
 & & \hat{\hat{j}} & & \\
 & \hat{j_1} \arrow[ur, dashed, "\hat{f_2}"] & & \hat{j_2} \arrow[ul, dashed, "\hat{s_1}"'] & \\
j_1 \arrow[ur, "f_1"] & & j_2 \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & j_3 \arrow[ul, "s_2"']
\end{tikzcd}$$

Remark. This construction is well-defined, and satisfies the properties that makes $\mathbf{J}[\mathbf{S^{-1}}]$ a category. This is rather tedious to show, but the necessary steps can be found in chapter two of Neeman's *Triangulated categories* [17].

Proposition 1.8. *Let* $\mathbf{J}$ *be a category, and* $\mathbf{S}$ *a multiplicative system.*

(i) $\pi\colon \mathbf{J} \to \mathbf{J}[\mathbf{S^{-1}}]$ *defined by* $j \mapsto j$ *on objects and* $f \mapsto (f, \mathrm{id})$ *on morphisms is a well-defined functor.*

(ii) For all $s \in \mathbf{S}$, $\pi(s)$ *is an isomorphism.*

(iii) If $\beta\colon \mathbf{J} \to \mathbf{K}$ *is another functor such that* $\beta(s)$ *is an isomorphism for all* $s \in \mathbf{S}$, *then there exists a unique functor* $\gamma\colon \mathbf{J}[\mathbf{S^{-1}}] \to \mathbf{K}$ *such that* $\beta = \gamma \circ \pi$.

Proof. (i) Since objects are sent to themselves, we only need to check the identity and composition. Clearly, we have $\pi(\mathrm{id}_{\mathbf{J}}) = \mathrm{id}_{\pi(\mathbf{J})}$. Let f and g be composable in $\mathbf{J}$. Then $\pi(g \circ f) = (g \circ f, \mathrm{id}_{g\circ f}) = (g, \mathrm{id}_g) \circ (f, \mathrm{id}_f) = \pi(g) \circ \pi(f)$.

(ii) Let $s\colon j \to j' \in \mathbf{S}$. Then $(s, \mathrm{id}_{j'})$ has $(\mathrm{id}_{j'}, s)$ as an inverse. This can be seen by comparing the composition with the identity

$$\begin{tikzcd}
 & & j' & & \\
 & j' \arrow[ur, "\mathrm{id}"] & & j' \arrow[ul, "\mathrm{id}"'] & \\
j \arrow[ur, "s"] & & j' \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] & & j \arrow[ul, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & j' \arrow[d, "\mathrm{id}"'] & \\
j \arrow[ur, "s"] \arrow[r, "s"] \arrow[dr, "\mathrm{id}"'] & j' & j \arrow[ul, "s"'] \arrow[l, "s"'] \arrow[dl, "\mathrm{id}"] \\
 & j \arrow[u, "s"] &
\end{tikzcd}$$

In the diagram to the left we see the composition, which appears in the upper path of the diagram to the right. Here we also see that it is in the same class as the identity on j.

(iii) We define $\gamma(j)$ to be $\beta(j)$ on objects, as this is our only choice. For morphisms, we define $\gamma(f, s) = \beta(s)^{-1} \circ \beta(f)$ as this is the only way to make compositions well-defined. Since the well-definedness gave us no choice in the definitions, the functor is unique up to isomorphism

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "\beta"] \arrow[d, "\pi"'] & \mathbf{K} \\
\mathbf{J}[\mathbf{S^{-1}}] \arrow[ur, dashed, "\exists!\gamma"'] &
\end{tikzcd}$$

□

The construction we made earlier, along with proposition 1.8 motivates us to make the following definition

Definition 1.9. let $\mathbf{J}$ be a category and $\mathbf{S}$ a class of morphisms in $\mathbf{J}$. The *localization* of $\mathbf{J}$ with respect to $\mathbf{S}$ is a category $\mathbf{J}[\mathbf{S^{-1}}]$ together with a functor $\pi\colon \mathbf{J} \to \mathbf{J}[\mathbf{S^{-1}}]$ such that

(i) $\pi(s)$ is an isomorphism for all $s \in \mathbf{S}$.

(ii) For any other functor $\beta\colon \mathbf{J} \to \mathbf{K}$ such that $\beta(s)$ is an isomorphism for all $s \in \mathbf{S}$, there exists a unique functor $\gamma\colon \mathbf{J}[\mathbf{S^{-1}}] \to \mathbf{K}$ such that $\beta = \gamma \circ \pi$.

Remark. Given two categories $\mathbf{J}, \mathbf{K}$ and a class of morphisms $\mathbf{S}$ in $\mathbf{J}$, we will denote by $\mathrm{Fun}^{\mathbf{S}}(\mathbf{J}, \mathbf{K})$ the collection of functors that maps the morphisms in $\mathbf{S}$ to isomorphisms.

Example 1.10. The homotopy category $\mathbf{K}(\mathcal{A})$ is the localization of $\mathbf{Ch}(\mathcal{A})$ with respect to W the class of chain homotopy equivalences.

Now that we have defined the localization of a category, we are ready to show that they induce equivalences on functor categories.

Theorem 1.11. *Let* $\mathbf{J}, \mathbf{K}$ *be two categories, and* $\mathbf{S}$ *a multiplicative system in* $\mathbf{J}$. *The localization functor* $\pi\colon \mathbf{J} \to \mathbf{J}[\mathbf{S^{-1}}]$ *induces an equivalence on the functor categories* $\pi^*\colon \mathrm{Fun}(\mathbf{J}[\mathbf{S^{-1}}], \mathbf{K}) \to \mathrm{Fun}^{\mathbf{S}}(\mathbf{J}, \mathbf{K})$.

Proof. Let $F, G \in \mathrm{Fun}^{\mathbf{S}}(\mathbf{J}, \mathbf{K})$. Due to the universal property of localization, we know that F and G factors as $F = F' \circ \pi$ and $G = G' \circ \pi$ with $F', G' \in \mathrm{Fun}(\mathbf{J}[\mathbf{S^{-1}}], \mathbf{K})$, which means that π^* is dense. Hence it suffices to show that π^* is also fully faithful by lemma A.7. We will do this by showing that there is a bijection between the respective natural transformations.

Let $\alpha \colon F \to G$ be a natural transformation between F and G. We want to show that there is a natural transformation between F' and G'. The natural transformation α has components for all $f \colon j \to j'$ in $\mathbf{J}$

$$\begin{tikzcd}
F(j) \arrow[r, "\alpha_j"] \arrow[d, "F(f)"'] & G(j) \arrow[d, "G(f)"] \\
F(j') \arrow[r, "\alpha_{j'}"] & G(j')
\end{tikzcd}$$

As the natural transformation α is a morphism in $\mathbf{K}$, we can also think of this as a functor $\beta \colon \mathbf{J} \to \mathbf{K}^{[\mathbf{1}]}$ defined on objects by $\beta(j) = (\alpha_j \colon F(j) \to G(j))$, and $\beta(f)$ defined component-wise through $F(f)$ and $G(f)$.

For any morphism $s \colon j \to j'$ in $\mathbf{S}$, we know that F and G induces vertical isomorphisms

$$\begin{tikzcd}
F(j) \arrow[r, "\alpha_j"] \arrow[d, "F(s)"', "\cong"] & G(j) \arrow[d, "G(s)", "\cong"'] \\
F(j') \arrow[r, "\alpha_{j'}"] & G(j')
\end{tikzcd}$$

in the diagram. This means that β inverts morphisms of $\mathbf{S}$, hence $\beta \in \mathrm{Fun}^{\mathbf{S}}(\mathbf{J}, \mathbf{K}^{[\mathbf{1}]})$. But then the universal property of π implies that there exists a unique γ such that the following diagram

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "\beta"] \arrow[d, "\pi"'] & \mathbf{K}^{[\mathbf{1}]} \\
\mathbf{J}[\mathbf{S^{-1}}] \arrow[ur, "!\exists\gamma"'] &
\end{tikzcd}$$

commutes. Now note that β factoring as $\gamma \circ \pi$ implies that γ corresponds to a natural transformation $\alpha' \colon F' \to G'$ defined level-wise by $\gamma(j) = (\alpha'_j \colon F'((j, \mathrm{id})) \to G'((j, \mathrm{id})))$, for each $(j, \mathrm{id}) \in \mathbf{J}[\mathbf{S^{-1}}]$.

Thus we started with an $\alpha\colon F \to G$ and this induced an $\alpha'\colon F' \to G'$, showing that π^* is full. Uniqueness of γ implies that the α' we found is also unique. Hence, π^* is also faithful and this completes the proof. □

Now that we have some familiarity with localizations, we turn to the localization system we are really interested in.

1.2 The derived category

The derived category of an abelian category $\mathcal{A}$ can give us much information about $\mathcal{A}$. The idea is that we localize with respect to quasi-isomorphisms, which will allow us to identify all projective resolutions of an object $a \in \mathcal{A}$. There are, however, some issues with this type of localization. For instance it can be constructed but in general they are not locally small. Let us begin with a definition

Definition 1.12. Let $\mathcal{A}$ be an abelian category. The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is the localization of $\mathbf{Ch}(\,\mathcal{A}\,)$ at the class $W_{\mathcal{A}}$ of quasi-isomorphisms

$$\mathbf{D}(\mathcal{A}) = \mathbf{Ch}(\,\mathcal{A}\,)[W_{\mathcal{A}}^{-1}]$$

Remark. It is more common to define the derived category as the localization of the *homotopy category* $\mathbf{K}(\,\mathcal{A}\,)$, as is done in [18]. However, it is more appropriate for us to use the category $\mathbf{Ch}(\,\mathcal{A}\,)$ as we want to relate proposition 1.6 to the derived category. In any case, since chain homotopy equivalences are quasi-isomorphisms (proposition 3.2 [2]) the two definitions agree.

Now that we have defined the derived category as a localization at the set of quasi-isomorphisms, we get the immediate corollary.

Corollary 1.13. *For any category* $\mathbf{J}$ *and abelian category* $\mathcal{A}$*, there is an equivalence of categories*

$$\pi^*\colon \mathrm{Fun}(\mathbf{D}(\mathcal{A}), \mathbf{J}) \to \mathrm{Fun}^{\mathbf{W}_{\mathcal{A}}}(\mathbf{Ch}(\,\mathcal{A}\,), \mathbf{J}),$$

where $\mathbf{W}_{\mathcal{A}}$ *is the class of quasi-isomorphisms in* $\mathcal{A}$*.*

Since the derived category is defined as a localization, the objects are rather abstract. This makes it usually hard to describe functors between them. Thankfully this can be avoided using the universal property that follows from the localization.

Proposition 1.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{B})$ a functor between their chain categories. Then F is exact if and only if there exists a functor $D(F)\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\pi_{\mathcal{A}}"'] & \mathbf{Ch}(\mathcal{B}) \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathbf{D}(\mathcal{A}) \arrow[r, "D(F)"] & \mathbf{D}(\mathcal{B})
\end{tikzcd}$$

In this case the functor is unique.

Proof. Assume F is exact, then by proposition 1.3 it preserves quasi-isomorphisms. Thus the composition $\pi_{\mathcal{B}} \circ F\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ maps the quasi-isomorphisms from $\mathbf{Ch}(\mathcal{A})$ to isomorphisms in $\mathbf{D}(\mathcal{B})$. It follows from the universal property of $\pi_{\mathcal{A}}\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ that there exists a unique map $D(F)$ that makes the diagram commutative.

Conversely, assume F is not exact. Then there exists a short exact sequence δ in $\mathbf{Ch}(\mathcal{A})$ such that $F(\delta)$ is not short exact. This means that $\pi_{\mathcal{B}} \circ F$ maps δ to something non-zero, while $D(F) \circ \pi_{\mathcal{A}}$ maps δ to zero. Hence, the square does not commute. □

Note that in proposition 1.14 the functor F could be induced from an exact functor between $\mathcal{A}$ and $\mathcal{B}$, by proposition 1.3. Unfortunately, most functors we come across are not exact and so we can not guarantee the existence of such a functor. For instance, the cokernel functor $\text{cok}\colon \mathbf{Ch}(\mathcal{A})^{[\mathbf{1}]} \to \mathbf{Ch}(\mathcal{A})$ is, in general, not exact. By the above result there can thus be no $D(\text{cok})\colon \mathbf{D}(\mathcal{A})^{[\mathbf{1}]} \to \mathbf{D}(\mathcal{A})$. However, we would still like something that behaves like $D(\text{cok})$. This motivates the following definition.

Definition 1.15. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{B})$ a functor between their chain categories.

(i) A *left derived functor* of F is a pair (LF, ϵ) consisting of a functor $LF \colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ and a natural transformation $\epsilon \colon LF \circ \pi_{\mathcal{A}} \to \pi_{\mathcal{B}} \circ F$ with the universal property that for every other such pair (G, α), there is a unique natural transformation $\beta \colon G \to LF$ such that $\alpha = \epsilon \circ \beta_{\pi_{\mathcal{A}}}$

(ii) A *right derived functor* of F is a pair (RF, η) consisting of a functor $RF \colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ and a natural transformation $\eta \colon \pi_{\mathcal{B}} \circ F \to RF \circ \pi_{\mathcal{A}}$ with the universal property that for every other such pair (G, α), there is a unique natural transformation $\beta \colon RF \to G$ such that $\alpha = \beta_{\pi_{\mathcal{A}}} \circ \eta$

The left and right derived functor can be illustrated as follows

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{A} \arrow[r, "F"] \arrow[d, "\pi_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathbf{D}(\mathcal{A}) \arrow[r, bend left, "LF", ""{name=L, below}] \arrow[r, bend right, "G"', ""{name=G, above}] & \mathbf{D}(\mathcal{B})
\arrow[from=2-1, to=1-2, Rightarrow, shift left=2, "\epsilon"]
\arrow[from=2-1, to=1-2, Rightarrow, shift right=2, "\forall\alpha"']
\arrow[from=G, to=L, Rightarrow, dashed, "!\exists\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{A} \arrow[r, "F"] \arrow[d, "\pi_{A}"'] & B \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathbf{D}(\mathcal{A}) \arrow[r, bend left, "RF", ""{name=R, below}] \arrow[r, bend right, "G"', ""{name=G, above}] & \mathcal{B}
\arrow[from=1-2, to=2-1, Rightarrow, shift right=2, "\eta"']
\arrow[from=1-2, to=2-1, Rightarrow, shift left=2, "\forall\alpha"]
\arrow[from=R, to=G, Rightarrow, dashed, "!\exists\beta"']
\end{tikzcd}$$

Example 1.16. If $F \colon \mathcal{A} \to \mathcal{B}$ is exact, then the pair $(D(F) \colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B}), \mathrm{id} \colon F \circ \pi_{\mathcal{A}} \to \pi_{\mathcal{B}} \circ F)$ is a left derived functor. From proposition 1.14, the identity certainly fulfills the transformation requirement. To see that is also satisfies the universal property, we can consider another pair (G, α). Since $\alpha \colon G \circ \pi_{\mathcal{A}} \to \pi_{\mathcal{B}} \circ F$, and we have $F \circ \pi_{\mathcal{A}} = \pi_{\mathcal{B}} \circ F$, this reads as

$$\alpha \colon G \circ \pi_{\mathcal{A}} \to F \circ \pi_{\mathcal{A}}.$$

Then corollary 1.13 implies that there exists a unique $\beta \colon G \to F$ such that everything commutes. The dual to this example implies that (D(F), $\mathrm{id} \colon \pi_{\mathcal{B}} \circ F \to F \circ \pi_{\mathcal{A}}$) is also a right derived functor.

1.3 Coherent diagrams

We now address the difference between *coherent* and *incoherent* diagrams. For any small category $\mathbf{J}$ and abelian category $\mathcal{A}$, the functor category $\mathcal{A}^{\mathbf{J}}$ is again abelian (proposition A.13) and are frequently referred to as a *diagram of shape* $\mathbf{J}$. As we saw in the last chapter, however, there are some differences when it comes to derived categories of arrow categories and arrow categories of derived categories. When we exchange the arrows [1] with a more general small category $\mathbf{J}$, we get what we call *coherent* and *incoherent* diagrams. They sit at the very core of the theory of derivators, and as such all results has an abstract version which we will get back to in section 3.

Definition 1.17. Let $\mathcal{A}$ be abelian, and $\mathbf{J}$ a small category.

(i) An object X in $\mathbf{D}(\mathcal{A}^{\mathbf{J}})$ is called a *coherent* diagram of shape $\mathbf{J}$.

(ii) An object X in $\mathbf{D}(\mathcal{A})^{\mathbf{J}}$ is called an *incoherent* diagram of shape $\mathbf{J}$.

Some comments on the difference between these types of diagrams. Let $\mathcal{A}$ be an abelian category, and $\mathbf{J}$ a small category. An object of $\mathbf{Ch}(\mathcal{A}^{\mathbf{J}})$ is a chain complex of functors $X\colon \mathbf{J} \to \mathcal{A}$. Recall from proposition 1.6 that $\mathbf{Ch}(\mathcal{A}^{\mathbf{J}}) \cong \mathbf{Ch}(\mathcal{A})^{\mathbf{J}}$, so X actually corresponds to a diagram $X\colon \mathbf{J} \to \mathbf{Ch}(\mathcal{A})$. This diagram commutes 'properly', and since the localization sends objects to objects, so does the derived diagram $\pi(X) \in \mathbf{D}(\mathcal{A}^{\mathbf{J}})$, which is why call this a coherent diagram. An incoherent diagram, on the other hand, is a diagram $X\colon \mathbf{J} \to \mathbf{D}(\mathcal{A})$. By the definition of the derived category (as a localization), this diagram commutes up to an equivalence class.

To really emphasize the difference, consider commutative squares on the form

$$\begin{tikzcd}
j_1 \arrow[r] \arrow[d] & j_2 \arrow[d] \\
j_3 \arrow[r] & j_4
\end{tikzcd}$$

in $\mathbf{J}$. We can either send such a square to $\mathcal{A}$ along X, and then derive it (coherent), or send it straight away to the derived category along $\bar{X}$ (incoherent). In any case morphisms between the resulting squares will look like the two following cubes

$$\begin{tikzcd}
 & X(j_1) \arrow[rr] \arrow[dl, "\simeq"'] \arrow[dd] & & X(j_2) \arrow[dl, "\simeq"'] \arrow[dd] \\
Y(j_1) \arrow[rr, crossing over] \arrow[dd] & & Y(j_2) & \\
 & X(j_3) \arrow[rr] \arrow[dl, "\simeq"'] & & X(j_4) \arrow[dl, "\simeq"'] \\
Y(j_3) \arrow[rr] & & Y(j_4) \arrow[from=uu, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bar{X}(j_1) \arrow[rr, "\simeq"] \arrow[dl, "\simeq"'] \arrow[dd, "\simeq"] & & \bar{X}(j_2) \arrow[dl, "\simeq"'] \arrow[dd, "\simeq"] \\
\bar{Y}(j_1) \arrow[rr, crossing over, "\simeq"] \arrow[dd, "\simeq"] & & \bar{Y}(j_2) & \\
 & \bar{X}(j_3) \arrow[rr, "\simeq"] \arrow[dl, "\simeq"'] & & \bar{X}(j_4) \arrow[dl, "\simeq"'] \\
\bar{Y}(j_3) \arrow[rr, "\simeq"] & & \bar{Y}(j_4) \arrow[from=uu, crossing over, "\simeq"'] &
\end{tikzcd}$$

where the left cube lies in $\mathbf{D}(\mathcal{A}^{\mathbf{J}})$ and the right cube lies in $\mathbf{D}(\mathcal{A})^{\mathbf{J}}$. Each of the morphisms indicated by $\simeq$ represents morphisms on the form

$$X \xrightarrow{f} \tilde{Y} \xleftarrow{q} Y$$

for a morphism $f \in \mathbf{Ch}(\mathcal{A})^{\mathbf{J}}$, and a level-wise quasi-isomorphism q. Now it is easy to see that the back and front face of the left square commutes on the nose, while the remaining sides commute up to chain homotopy. On the square to the right on the other hand, every side of the cube commutes up to chain homotopy.

So there is a big difference between coherent and incoherent diagrams. In general, we cannot replace an incoherent diagram by a coherent diagram. However, the two types are related by a diagram functor the other way $\operatorname{dia}\colon \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \to \mathbf{D}(\mathcal{A})^{\mathbf{J}}$ which we will define later. First we take a look at some nice properties available to coherent diagrams.

Definition 1.18. Let $\mathbb{1}$ denote the category with only one element and the identity morphism. We call this the *terminal category.*

Any object $j \in \mathbf{J}$, gives rise to a corresponding *identification functor* $j\colon \mathbb{1} \to \mathbf{J}$ from the terminal category that simply picks out the object.

For any diagram $X\colon \mathbf{J} \to \mathbf{K}$ we call the induced precomposition functor $j^*\colon \mathbf{K}^{\mathbf{J}} \to \mathbf{K}^{\mathbb{1}} \cong \mathbf{K}$ an *evaluation functor.*

Example 1.19. Let $\mathbf{S}$ be the poset $\{a, b, c, d\}$ with relations $a \leq b, c, d$ and $b, c \leq d$

$$\begin{tikzcd}
& d & \\
b \arrow[ur] & & c \arrow[ul] \\
& a \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Consider the identification functor $a \colon \mathbb{1} \to \mathbf{S}$, along with the presheaf $\mathbf{S}^{\mathrm{op}} \to \mathbf{Ab}$

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & \\
\mathbb{Z} \arrow[dr, "\cdot 2"] & & 0 \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}$$

In this case, the induced evaluation functor $a^* = \mathrm{Presh}_{\mathbf{Ab}}\,\mathbf{S} \circ a$ gives us the abelian group $\mathbb{Z}$ which was in position a.

Similarly to the objects of $\mathbf{J}$, any morphism $f \colon j \to j'$ gives rise to a natural transformation

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "j"{name=U}] \arrow[r, bend right, "j'"'{name=D}] & \mathbf{J} \arrow[from=U, to=D, Rightarrow, "f"]
\end{tikzcd}$$

between the identification functors. For any diagram $X \colon \mathbf{J} \to \mathcal{A}$ we write X_j for the evaluated diagram $j^*(X) \in \mathcal{A}$. Now, every morphism $f \colon j \to j'$ induces a transformation between the evaluation functors $f^* \colon j^* \to j'^*$. We write this as $X_f \colon X_j \to X_{j'}$

Example 1.20. Consider the poset from example 1.19. In this case $X_{a \leq b} \colon \mathbb{Z} \to \mathbb{Z}$ is the map given by multiplication by 2.

For any morphism of diagrams $g \colon X \to Y$, we get an induced morphism evaluated at j. This is denote by $g_j \colon X_j \to Y_j$.

Example 1.21. Consider again the Poset from example 1.19, but this time we include the natural transformation $g\colon X \to Y$

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] \arrow[rrr, bend left, "g_d"] & & & 0 \arrow[dl] \arrow[dr] & \\
\mathbb{Z} \arrow[dr, "\cdot 2"'] \arrow[rrr, bend left, "g_b"] & & 0 \arrow[dl] \arrow[rrr, bend left, "g_c"] & 0 \arrow[dr] & & 0 \arrow[dl] \\
& \mathbb{Z} \arrow[rrr, bend left, "g_a=\pi"] & & & \mathbb{Z}/_{2\,\mathbb{Z}} &
\end{tikzcd}$$

In this case the induced morphism g_a in **Ab** is the projection onto the subgroup $\mathbb{Z}/_{2\,\mathbb{Z}}$.

Lemma 1.22. *Let $\mathcal{A}$ be an abelian category, $\mathbf{J}$ a small category, and $j \in \mathbf{J}$. There is a unique evaluation functor $\bar{j}^*\colon \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \to \mathbf{D}(\mathcal{A})$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{Ch}(\,\mathcal{A}^{\mathbf{J}}\,) \arrow[r, "j^*"] \arrow[d, "\pi"'] & \mathbf{Ch}(\,\mathcal{A}\,) \arrow[d, "\pi"] \\
\mathbf{D}(\mathcal{A}^{\mathbf{J}}) \arrow[r, "\bar{j}^*"'] & \mathbf{D}(\mathcal{A})
\end{tikzcd}$$

Proof. It follows from the discussion of proposition 1.6 that the class of quasi-isomorphisms in $\mathbf{Ch}(\,\mathcal{A}^{\mathbf{J}}\,)$ is the same as levelwise quasi-isomorphisms in $\mathbf{Ch}(\,\mathcal{A}\,)^{\mathbf{J}}$. By proposition 1.5 j^* preserves quasi-isomorphisms. Now the result follows from proposition 1.3 and proposition 1.14 □

By the discussion of morphisms and evaluation above, lemma 1.22 and corollary 1.13 gives us the immediate result.

Corollary 1.23. *Let $\mathcal{A}$ be an abelian category, $\mathbf{J}$ a small category, and $f\colon j \to j'$ a morphism in $\mathbf{J}$. There is a unique natural transformation $\bar{f}^*\colon \bar{j}^* \to \bar{j'}^*\colon \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \to$*

$\mathbf{D}(\mathcal{A})$ *such that* $\pi \circ f^* = \bar{f}^* \circ \pi$

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}^{\mathbf{J}}) \arrow[r, bend left, "j^*"] \arrow[r, bend right, "j'^*"'] \arrow[r, phantom, "f^* \Downarrow"] \arrow[d, "\pi"'] & \mathbf{Ch}(\mathcal{A}) \arrow[d, "\pi"] \\
\mathbf{D}(\mathcal{A}^{\mathbf{J}}) \arrow[r, bend left, "\bar{j}^*"] \arrow[r, bend right, "\bar{j}'^*"'] \arrow[r, phantom, "\bar{f}^* \Downarrow"] & \mathbf{D}(\mathcal{A})
\end{tikzcd}$$

We now turn to the construction of a functor from coherent to incoherent diagrams. To begin with, let us note that for all diagrams $X \in \mathbf{D}(\mathcal{A}^{\mathbf{J}})$ there is an *underlying diagram* $\mathrm{dia}_{\mathbf{J}}(X)\colon \mathbf{J} \to \mathbf{D}(\mathcal{A})$ defined by $j \mapsto X_j$ and $f \mapsto X_f$.

For any morphism of diagrams $g\colon X \to Y \in \mathbf{D}(\mathcal{A}^{\mathbf{J}})$ we also get an induced natural transformation $\mathrm{dia}_{\mathbf{J}}(g)\colon \mathrm{dia}_{\mathbf{J}}(X) \to \mathrm{dia}_{\mathbf{J}}(Y)$. This is done component-wise, and the commutativity of the following diagram

$$\begin{tikzcd}
X_j \arrow[r, "\mathrm{dia}_{\mathbf{J}}(g)_j"] \arrow[d, "X_f"'] & Y_j \arrow[d, "Y_f"] \\
X_{j'} \arrow[r, "\mathrm{dia}_{\mathbf{J}}(g)_{j'}"'] & Y_{j'}
\end{tikzcd}$$

follows from corollary 1.23.

Proposition 1.24. *Let $\mathcal{A}$ be an abelian category and $\mathbf{J}$ a small category. There is a well-defined functor*

$$\mathrm{dia}_{\mathbf{J}}\colon \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \to \mathbf{D}(\mathcal{A})^{\mathbf{J}}$$

given by $X \mapsto \mathrm{dia}_{\mathbf{J}}(X)$ and $g \mapsto \mathrm{dia}_{\mathbf{J}}(g)$.

Proof. We need to check that identity and composition are well-defined.

Consider $\mathrm{id}_X \in \mathbf{D}(\mathcal{A}^{\mathbf{J}})$. This is mapped to $\mathrm{dia}_{\mathbf{J}}(\mathrm{id}_X)$ which is the identity on all components $\mathrm{dia}_{\mathbf{J}}(\mathrm{id}_X)_j$ for all $j \in \mathbf{J}$, hence equals $\mathrm{id}_{\mathrm{dia}_{\mathbf{J}}(X)}$.

Now, let $f\colon X \to Y$ and $g\colon Y \to Z$ be two composable functions in $\mathbf{D}(\mathcal{A}^{\mathbf{J}})$. Their composition $g \circ f$ is sent to $\mathrm{dia}_{\mathbf{J}}(g \circ f)$ which equals the composition $\mathrm{dia}_{\mathbf{J}}(g) \circ \mathrm{dia}_{\mathbf{J}}(f)$ since they are well-defined natural transformations, and hence agree on all components. $\square$

Intuitively speaking this functor takes a coherent diagram X and 'forgets' the strict commutativity. We call this the *underlying diagram functor* since it maps to the underlying homotopy-commutative diagram.

1.4 The derived cone

We end this chapter by combining the results of the previous sections. We give a definition of the mapping cone of a morphism, and using the techniques we have developed so far, show that this induces a left derived functor of the cokernel, as defined in definition 1.15.

Consider a morphism of chain complexes $f\colon a^\bullet \to b^\bullet \in \mathbf{Ch}(\mathcal{A})$. We construct the cone of this map $C(f) \in \mathbf{Ch}(\mathcal{A})$ in the usual sense. We take the direct sum of components $b^{i-1} \oplus a^i$ and define the map $\left(\begin{smallmatrix} d_b^{i-1} & f^i \\ 0 & -d_a^i \end{smallmatrix}\right)$ as the differential. This gives us a nice commutative diagram

$$\begin{tikzcd}[ampersand replacement=\&]
a^\bullet\colon \cdots \arrow[r] \& a^{i-1} \arrow[r, "d_a^{i-1}"] \arrow[d, "i_a"'] \& a^i \arrow[r, "d_a^i"] \arrow[d, "i_a"'] \& a^{i+1} \arrow[r, "d_a^{i+1}"] \arrow[d, "i_a"'] \& \cdots \\
C(f)^\bullet\colon \cdots \arrow[r] \& b^{i-2} \oplus a^{i-1} \arrow[r, "\left(\begin{smallmatrix} d_b^{i-2} \& f^{i-1} \\ 0 \& -d_a^{i-1} \end{smallmatrix}\right)"] \& b^{i-1} \oplus a^i \arrow[r, "\left(\begin{smallmatrix} d_b^{i-1} \& f^{i} \\ 0 \& -d_a^{i} \end{smallmatrix}\right)"] \& b^i \oplus a^{i+1} \arrow[r, "\left(\begin{smallmatrix} d_b^{i} \& f^{i+1} \\ 0 \& -d_a^{i+1} \end{smallmatrix}\right)"] \& \cdots \\
b^\bullet\colon \cdots \arrow[r] \& b^{i-2} \arrow[r, "d_b^{i-2}"] \arrow[u, "i_b"] \& b^{i-1} \arrow[r, "d_b^{i-1}"] \arrow[u, "i_b"] \& b^i \arrow[r, "d_b^i"] \arrow[u, "i_b"] \& \cdots
\end{tikzcd}$$

which in turn induces the nice degree-wise split short exact sequence $b^\bullet \xrightarrow{\binom{1}{0}} C(f)^\bullet \xrightarrow{(0\ 1)} a[1]^\bullet$, where $a[1]^\bullet$ is the shifted complex of $a^\bullet$ given by $a[1]^i = a^{i+1}$ and $d_{a[1]}^i = d_a^{i+1}$. This also induces a long exact sequence of homology. In the case where $f = \mathrm{id}_{a^\bullet}$

we write $C(a)^\bullet$ for the complex instead. It follows from the long exact sequence of homology that $C(a)^\bullet$ is quasi-isomorphic to zero. This construction defines a nice cone functor $C\colon \mathbf{Ch}(\mathcal{A}^{[\mathbf{1}]}) \to \mathbf{Ch}(\mathcal{A})$ under the isomorphism from proposition 1.6.

Lemma 1.25. *The cone functor $C\colon \mathbf{Ch}(\mathcal{A}^{[\mathbf{1}]}) \to \mathbf{Ch}(\mathcal{A})$ preserves levelwise quasi-isomorphisms*

Proof. Let $f^\bullet\colon a^\bullet \to b^\bullet$, and $g^\bullet\colon c^\bullet \to d^\bullet$ be two elements of $\mathbf{Ch}(\mathcal{A}^{[1]})$, and $\phi^\bullet\colon f^\bullet \to g^\bullet$ a level-wise quasi isomorphism

$$\begin{tikzcd}
a^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\phi_1^\bullet"'] & b^\bullet \arrow[d, "\phi_2^\bullet"] \\
c^\bullet \arrow[r, "g^\bullet"'] & d^\bullet
\end{tikzcd}$$

Then we get an induced morphism $\psi^\bullet = \begin{pmatrix} \phi_2^\bullet & 0 \\ 0 & \phi_1^\bullet \end{pmatrix}\colon C(f)^\bullet \to C(g)^\bullet$. Now the long exact sequence of homology becomes

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(a^\bullet) \arrow[r, "H^i(f^\bullet)"] \arrow[d, "\cong"] & H^i(b^\bullet) \arrow[r, "H^i(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))"] \arrow[d, "\cong"] & H^i(C(f)^\bullet) \arrow[r, "H^i((0\ 1))"] \arrow[d, "H^i(\psi^\bullet)"] & H^{i+1}(a^\bullet) \arrow[r, "H^{i+1}(f^\bullet)"] \arrow[d, "\cong"] & H^{i+1}(b^\bullet) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^i(c^\bullet) \arrow[r, "H^i(g^\bullet)"'] & H^i(d^\bullet) \arrow[r, "H^i(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))"'] & H^i(C(g)^\bullet) \arrow[r, "H^i((0\ 1))"'] & H^{i+1}(c^\bullet) \arrow[r, "H^i(g^\bullet)"'] & H^{i+1}(d^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

from which the five-lemma concludes the result. □

Another way to define the cone of a morphism f is as the pushout of the following span $(C(a)^\bullet \leftarrow a^\bullet \xrightarrow{f} b^\bullet)$. It is often useful to look at things from different perspectives. In fact, if we also consider the cokernel of f as the pushout of the span

$(0 \leftarrow a^\bullet \xrightarrow{f} b^\bullet)$, we get diagram

$$\begin{tikzcd}
 & a^\bullet \arrow[rr] \arrow[dl, equal] \arrow[dd] \arrow[ddrr, phantom, "\ulcorner", very near end] & & b^\bullet \arrow[dl, equal] \arrow[dd] \\
a^\bullet \arrow[rr, crossing over] \arrow[dd] \arrow[ddrr, phantom, "\ulcorner", very near end] & & b^\bullet & \\
 & C(a)^\bullet \arrow[rr] \arrow[dl] & & C(f)^\bullet \arrow[dl] \\
0 \arrow[rr] & & \mathrm{cok}(f) \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{1}$$

from which we deduce that there is a unique morphism $\phi\colon C(f)^\bullet \to \mathrm{Cok}(f)$ by the pushout property. Alternatively, since the composition of the cone complex defined in the beginning is zero, we get the same map from the cokernel property.

Now, since the cone functor preserves quasi-isomorphisms, it follows from proposition 1.3 and proposition 1.14 that we get an induced functor $\bar{C}\colon \mathbf{D}(\mathcal{A}^{[\mathbf{1}]}) \to \mathbf{D}(\mathcal{A})$. This functor, together with ϕ above, lets us define a natural transformation

$$\epsilon\colon \bar{C} \circ \pi = \pi \circ C \xrightarrow{\pi\circ\phi} \pi \circ \mathrm{Cok}$$

which we now use in the following theorem.

Theorem 1.26. *The cone functor* $\bar{C}\colon \mathbf{D}(\mathcal{A}^{[\mathbf{1}]}) \to \mathbf{D}(\mathcal{A})$ *together with the natural transformation* $\epsilon\colon \bar{C} \circ \pi \to \pi \circ \mathrm{Cok}$ *is a left derived functor of the cokernel functor.*

Proof. By definition 1.15 we assume that we have another such pair (G, α). corollary 1.13 implies that it is enough to find a unique natural transformation $\bar{\beta}\colon G \circ \pi \to$

$\bar{C} \circ \pi$ such that $\alpha = \epsilon \circ \bar{\beta} \colon G \circ \pi \to \bar{C} \circ \pi \to \mathrm{Cok} \circ \pi$

$$\begin{tikzcd}
\mathcal{A}^{[1]} \arrow[r, "C"] \arrow[d, "\pi"'] & \mathcal{A} \arrow[r, "\phi"] \arrow[d, "\pi_{\mathcal{B}}"] & \mathcal{A} \arrow[d, "\pi"] \\
\mathbf{D}(\mathcal{A}^{[\mathbf{1}]}) \arrow[r, "\bar{C}"] \arrow[rr, bend right, "G"'] \arrow[ur, Rightarrow, shorten=8mm, "\mathrm{id}"] \arrow[urr, Rightarrow, shorten=10mm, "\alpha"] & \mathbf{D}(\mathcal{A}) \arrow[r, "\mathrm{id}_{\mathbf{D}(\mathcal{A})}"] \arrow[ur, Rightarrow, shorten=8mm, "\epsilon"] & \mathbf{D}(\mathcal{A}) \\
& |[alias=Z]| {} \arrow[u, Rightarrow, shorten=2mm, "!\bar{\beta}"] &
\end{tikzcd}$$

as then there is a unique natural transformation $\beta \colon G \to \bar{C}$, which shows that the universal property holds.

Let us consider diagram (1), where we found the ϕ-function. From this we get an induced ‘diagonal’ square

$$\begin{tikzcd}
a^\bullet \arrow[r] \arrow[d, equal] & b^\bullet \oplus C(a)^\bullet \arrow[d] \\
a^\bullet \arrow[r, "f"'] & b^\bullet \oplus 0 \cong b^\bullet
\end{tikzcd}$$

from the direct sum of the corners. Since $C(a)^\bullet$ is quasi-isomorphic to zero, and the identity is certainly a quasi-isomorphism, the vertical arrows are quasi-isomorphisms. Now we apply our functors and natural transformations to get

$$\begin{tikzcd}
(G \circ \pi)[a^\bullet \to b^\bullet \oplus C(a)^\bullet] \arrow[r, "\alpha"] \arrow[d, "\cong"'] & (\pi \circ \mathrm{Cok})[a^\bullet \to b^\bullet \oplus C(a)^\bullet] \arrow[d] & (C \circ \pi)[a^\bullet \to b^\bullet \oplus C(a)^\bullet] \arrow[l, "\epsilon"', "\cong"] \arrow[d, "\cong"] \\
(G \circ \pi)[a^\bullet \to b^\bullet] \arrow[r, "\alpha"] & (\pi \circ \mathrm{Cok})[a^\bullet \to b^\bullet] & (C \circ \pi)[a^\bullet \to b^\bullet] \arrow[l, "\epsilon"']
\end{tikzcd}$$

where the indicated vertical morphisms are isomorphisms due to the above discussion. Since the inclusion $a^\bullet \to b^\bullet \oplus C(a)^\bullet$ is a monomorphism, and the cokernel functor is right exact, it follows that the induced morphism ϕ is a quasi-isomorphism by proposition 1.3. Hence, the upper right arrow is also an isomorphism. By the naturality of the transformations α and ϵ, the above diagram commutes. Hence, we can set $\bar{\beta}$ to be the outer path. The uniqueness then follows from the isomorphisms, so $\alpha = \epsilon \circ \bar{\beta}$ and we are done. □

Remark. This left derived functor of the cokernel is often referred to as the *cofiber functor*. There is the dual version of this construction, which is called the *fiber functor*. The fiber functor induces a right derived functor to the kernel functor in precisely the same way.

Let us do a small recap of what we just did. In ordinary category theory, for any morphism $f\colon a \to b$ the cokernel of f is the pushout of the following diagram

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d] & b \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cok}(f)
\end{tikzcd}$$

This applies especially to a morphism between chain complexes $a^\bullet$ and $b^\bullet$. Now, if we exchange the morphism $a^\bullet \to 0$ with the quasi-isomorphic morphism $a^\bullet \to C(a)^\bullet$, we get different pushout diagram

$$\begin{tikzcd}
a^\bullet \arrow[r, "f"] \arrow[d] & b^\bullet \arrow[d] \\
C(a)^\bullet \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C(f)^\bullet
\end{tikzcd}$$

which induces a left derived functor to the original diagram.

This construction will be generalized to abstract derivators in chapter 4. Before this, though, we introduce the machinery of *Kan extensions*, in order to get left and right adjoints to restriction functors.

2 Kan extensions

When we start our discussion on derivators in chapter 3, we want to use the 'calculus' of Kan extensions. By this we mean that we are interested in the way Kan extensions generalize (co)limits, but in a homotopy setting. In this chapter we therefore give a good introduction, and prove some general results about Kan extensions. We begin by introducing something called *slice categories*, which will also be important for derivators. Then we define Kan extensions, and give an example of how they generalize the (co)limits of a diagram. Finally, we show that there is a point-wise construction for these extensions, and that they induce adjunctions to precomposition functors.

2.1 Slice categories

Slice categories are a special case of what is called *comma categories.* These are interesting in their own right, as they not only relate objects to one another by morphisms, but also allow for morphisms to become objects themselves. In general, you would define a comma category through commutative diagrams in a category $\mathbf{L}$, where the objects came from different categories and are related by functors $u\colon \mathbf{J} \to \mathbf{L}$ and $v\colon \mathbf{K} \to \mathbf{L}$. So in the following diagram the objects are morphisms $(f_j \in \mathbf{J}, f_k \in \mathbf{K})$, and the morphism is the pair $(f_l, f_{l'})$.

$$\begin{tikzcd}
u(j) \arrow[r, "u(f_j)"] \arrow[d, "f_l"'] & u(j') \arrow[d, "f_l'"] \\
v(k) \arrow[r, "v(f_k)"'] & v(k')
\end{tikzcd}$$

We are going to look at the special case of comma categories where u or v is the constant functor $k\colon \mathbb{1} \to \mathbf{K}$ that picks out an object $k \in \mathbf{K}$. In this case we obtain what is called a *slice category.*

Definition 2.1. Let $\mathbf{J}$ and $\mathbf{K}$ be categories, $k \in \mathbf{K}$ and $u\colon \mathbf{J} \to \mathbf{K}$ a functor. A *Slice* *category* is a comma category, denote by $(u \downarrow k)$, where the objects are pairs (j, f) with $j \in \mathbf{J}$ and $f\colon u(j) \to k$, and morphisms are functions $g\colon j \to j'$ such that

$$\begin{tikzcd}
u(j) \arrow[r, "u(g)"] \arrow[d, "f"'] & u(j') \arrow[dl, "f'"] \\
k &
\end{tikzcd}$$

commutes in $\mathbf{K}$.

There is also the dual notion, where the objects are the same but with reversed arrows. They would have diagrams on the form

$$\begin{tikzcd}
u(j) \arrow[r, "u(g)"] & u(j') \\
k \arrow[u, "f"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

and we denote them by $(k \downarrow u)$.

The slice category $(u \downarrow k)$ is often called the 'category of objects over k', while the slice category $(k \downarrow u)$ is often called the 'category of objects under k' [1] [5].

Example 2.2. Let $\mathcal{A}$ be an abelian category, and consider $a \in \mathcal{A}$. The slice category $(\mathrm{id}_{\mathcal{A}} \downarrow a)$ is the subcategory of the arrow category $\mathcal{A}^{[1]}$ where the only target is a.

Example 2.3. Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor from some small category to a cocomplete category, and consider $\lim_{\mathbf{J}} u \in \mathbf{K}$. The slice category $(\lim_{\mathbf{J}} u \downarrow u)$ is the limiting cone.

Lemma 2.4. *Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between two small categories. Then*

(i) For every $k \in \mathbf{K}$, there is a projection functor $\rho_k\colon (u \downarrow k) \to \mathbf{J}$.

(ii) For every morphism $h\colon k \to k'$ in $\mathbf{K}$, there is a functor

$$(u \downarrow h)\colon (u \downarrow k) \to (u \downarrow k')$$

(iii) For every morphism $h\colon k \to k'$ in $\mathbf{K}$, the diagram

$$\begin{tikzcd}
(u \downarrow k) \arrow[r, "(u\downarrow h)"] \arrow[d, "\rho_k"'] & (u \downarrow k') \arrow[dl, "\rho'_k"] \\
\mathbf{J} &
\end{tikzcd}$$

commutes.

Proof. (i) We define ρ_k on objects by $(j, f) \mapsto j$ and on morphisms by $g \mapsto g$. As morphisms are by definition done in $\mathbf{J}$ before passing to $\mathbf{K}$ through the well-defined functor u, it follows that $\rho_k(\mathrm{id}_{\mathbf{J}}) = \mathrm{id}_{\mathbf{J}}$ and $\rho_k(g_1 \circ g_2) = \rho_k(g_1) \circ \rho_k(g_2)$.

(ii) We define $(u \downarrow h)$ on objects by $(j, f) \mapsto (j, h \circ f)$ and on morphisms by $g \mapsto g$. Again, since morphisms are done in $\mathbf{J}$ before passing to $\mathbf{K}$, the result follows.

(iii) Since the functor $(u \downarrow h)$ does nothing to the object j, while the functor ρ_k projects to that object, it the diagram clearly commutes.

□

Note that lemma 2.4 has obvious dual results for the slice category $(k \downarrow u)$. We are going to denote the projection functor from $(k \downarrow u)$ by θ_k so as to keep them separated. Now, we turn to the two squares

$$\begin{tikzcd}
(u \downarrow k) \arrow[r, "\rho_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, shorten=4mm, "f"'] \\
\mathbb{1} \arrow[r, "k"'] & \mathbf{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
(k \downarrow u) \arrow[r, "\theta_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \\
\mathbb{1} \arrow[r, "k"'] \arrow[ur, Rightarrow, shorten=4mm, "f"] & \mathbf{K}
\end{tikzcd}
\tag{2}$$

which we will denote the *slice squares* for further reference.

Proposition 2.5. *The two slice squares define transformations.*

Proof. We focus on the left square as the case for the right square is dual.

The goal is to show that the indicated functors from $(u \downarrow k)$ to $\mathbf{K}$ have a natural transformation f between them. Moving along the upper path we send (j, f) first to

j, then to $u(j)$ in $\mathbf{K}$. Moving along the lower path we project to $\mathbb{1}$, and then identify the element $k \in \mathbf{K}$. We claim that the original $f\colon u(j) \to k$ from $(u \downarrow k)$ is a natural transformation. Indeed, the following diagram

$$\begin{tikzcd}
(u \circ \rho_k)(j,f) \arrow[r, "f"] \arrow[d, "g"] & (k \circ \pi)(j,f) \arrow[d, "\mathrm{id}_b"] \\
(u \circ \rho_k)(j',f') \arrow[r, "f'"'] & (k \circ \pi)(j,f)
\end{tikzcd}$$

commutes by lemma 2.4 (iii). □

Now we have the properties that we need from slice categories, and turn our attention to Kan extensions next.

2.2 Definition of Kan extensions

As the name suggests Kan extensions are a way to extend functors. Given two functors f and g, the Kan extensions of f along g is a functor h which is a 'best approximation' of f by g. In this section we give a proper definition of the Kan extensions of a functor, and as an example we show that it generalizes the notion of limits and colimits. In addition to this, we will also include some easy results for Kan extensions.

Definition 2.6. Let $u\colon \mathbf{J} \to \mathbf{K}$ be functor between small categories, let $\mathbf{C}$ be any category, and $X\colon \mathbf{J} \to \mathbf{C}$ a functor.

(i) A *left Kan extension* of X along u is a functor $\mathrm{LK}_u(X)\colon \mathbf{K} \to \mathbf{C}$ together with a natural transformation $\eta\colon X \to \mathrm{LK}_u(X) \circ u$ satisfying the following universal property; For every pair $(Y\colon \mathbf{K} \to \mathbf{C},\ \alpha\colon X \to Y \circ u)$ there is a unique transformation $\beta\colon\ \mathrm{LK}_u(X) \to Y$ such that $\alpha = \beta_u \circ \eta$.

(ii) A *right Kan extension* of X along u is a functor $\mathrm{RK}_u(X)\colon \mathbf{K} \to \mathbf{C}$ together with a natural transformation $\epsilon\colon\ \mathrm{RK}_u(X) \circ u \to X$ satisfying the following universal property; For every pair $(Y\colon \mathbf{K} \to \mathbf{C}, \alpha\colon Y \circ u \to X)$ there is a unique transformation $\beta\colon Y \to \mathrm{RK}_u(X)$ such that $\alpha = \epsilon \circ \beta_u$

The left and right Kan extension, and the universal property can be illustrated as follows

$$\begin{tikzcd}
\mathbf{J} \arrow[rr, "X"{name=X1}] \arrow[d, "u"'] & & \mathbf{C} \\
\mathbf{K} \arrow[urr, "\mathrm{LK}_u(X)"'{name=L}] \arrow[urr, bend right=60, "Y"'{name=Y1}] & &
\arrow[Rightarrow, from=X1, to=L, "\eta"]
\arrow[Rightarrow, from=X1, to=Y1, "\forall\alpha"']
\arrow[Rightarrow, dashed, from=L, to=Y1, "\exists!\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{J} \arrow[rr, "X"{name=X2}] \arrow[d, "u"'] & & \mathbf{C} \\
\mathbf{K} \arrow[urr, "\mathrm{RK}_u(X)"'{name=R}] \arrow[urr, bend right=60, "Y"'{name=Y2}] & &
\arrow[Rightarrow, from=R, to=X2, "\epsilon"']
\arrow[Rightarrow, from=Y2, to=X2, "\forall\alpha"]
\arrow[Rightarrow, dashed, from=Y2, to=R, "\exists!\beta"']
\end{tikzcd}$$

The notion of a Kan extension might seem similar to that of a derived functor (definition 1.15). The idea is the same. We do not necessarily have a functor that makes the diagram commute, and so we want the 'best approximation' of such a functor. In a way, one might think of the universal property of Kan extensions as a 'pushout and pullback of functors'. As an example let us see that it generalizes the notion of colimits and limits.

Example 2.7. Let $\mathbb{1}$ be the terminal category, $\mathbf{J}$ a small category, and $X\colon \mathbf{J} \to \mathbf{C}$ a functor to some cocomplete category $\mathbf{C}$. There is a canonical functor $\pi\colon \mathbf{J} \to \mathbb{1}$. A left Kan extension of X along π is a functor $\mathrm{LK}_\pi(X)\colon \mathbb{1} \to \mathbf{C}$ and a universal natural transformation $\eta\colon X \to \mathrm{LK}_\pi \circ \pi$. The functor $\mathrm{LK}_\pi(X)$ always maps to the same object $c \in \mathbf{C}$, and the natural transformation η gives us the following commutative diagram for all $f\colon j \to j'$ in $\mathbf{J}$.

$$\begin{tikzcd}
j \arrow[d, "\forall f"'] & X(j) \arrow[r, "\eta_j"] \arrow[d, "X(f)"'] & c \\
j' & X(j') \arrow[ur, "\eta_{j'}"'] &
\end{tikzcd}$$

Then by the universal property of the Kan extension, if $Y\colon \mathbb{1} \to \mathbf{C}$ is another functor that picks out some element $c' \in \mathbf{C}$, and α is a natural transformation of X into

$Y \circ \pi$, we get the following commutative diagram

$$\begin{tikzcd}
X(j) \arrow[r, "X(f)"] \arrow[d, "\eta_j"'] \arrow[dd, bend right, "\alpha_j"'] & X(j') \arrow[dl, "\eta_{j'}"] \arrow[ddl, bend left, "\alpha_{j'}"] \\
c \arrow[d, dashed, "\exists!\beta"] & \\
c' &
\end{tikzcd}.$$

Hence, it follows that the left Kan extension of π along X is simply the colimit of X in $\mathbf{C}$ with η the colimiting cocone.

Dually to example 2.7, the right Kan extension would yield the limit of X in $\mathbf{C}$. This shows that as long as the target category $\mathbf{C}$ is (co)complete we know Kan extensions along the functor $\pi \colon \mathbf{J} \to \mathbb{1}$ exist. In the next section we show that Kan extensions along general functors of small categories exists, and can be calculated by a point-wise formula. Before we do that, however, we want to assert some properties of (co)limits.

Proposition 2.8. *Let $\mathbf{C}$ be a cocomplete category, and $u \colon \mathbf{J} \to \mathbf{K}$ a functor between two small categories. For every diagram $X \colon \mathbf{K} \to \mathbf{C}$ we can precompose with u to get $Y \colon \mathbf{J} \to \mathbf{C}$. This gives two colimits, and a canonical function between them $\phi \colon \operatorname{colim}_{\mathbf{J}} Y \to \operatorname{colim}_{\mathbf{K}} X$.*

Proof. Let $X \colon \mathbf{K} \to \mathbf{C}$ be any diagram. We then have a colimit $\operatorname{colim}_{\mathbf{K}} X \in C$ and a colimiting cocone $\eta \colon X \to \Delta_K(C)$. We can then precompose along u to get $Y = X \circ u \colon \mathbf{J} \to \mathbf{C}$. Precomposing the cocone η, gives a colimiting cocone α on Y given by $\alpha_j \colon Y(j) \to \operatorname{colim}_{\mathbf{K}} X$, $j \in \mathbf{J}$. Hence, for all $f \colon j \to j'$ in $\mathbf{J}$, we get a commutative triangle

$$\begin{tikzcd}
Y(j) \arrow[r, "\alpha_j"] \arrow[d, "Y(f)"'] & \operatorname{colim}_{\mathbf{K}} X \\
Y(j') \arrow[ur, "\alpha_{j'}"] &
\end{tikzcd}$$

In particular, α factors uniquely through the colimit. Since the diagram Y has a colimit of its own, we get a canonical morphism $\phi\colon \operatorname{colim}_{\mathbf{J}} Y \to \operatorname{colim}_{\mathbf{K}} X$ such that

$$\begin{tikzcd}
Y(j) \arrow[r, "\alpha_j"] \arrow[d] & \operatorname{colim}_{\mathbf{K}} X \\
\operatorname{colim}_{\mathbf{J}} Y \arrow[ur, "\phi"] &
\end{tikzcd}$$

commutes for all $j \in \mathbf{J}$. □

Dually, we would get a canonical morphism $\psi\colon \lim_{\mathbf{K}} X \to \lim_{\mathbf{J}} Y$ if $\mathbf{C}$ was a complete category.

Definition 2.9. Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between two small categories.

(i) u is said to be *final* if the induced colimit function $\phi\colon \operatorname{colim}_{\mathbf{J}} Y \to \operatorname{colim}_{\mathbf{K}} X$ is an isomorphism for every cocomplete category $\mathbf{C}$ and every diagram $X\colon \mathbf{K} \to \mathbf{C}$, where Y is the precomposition with u.

(ii) u is said to be *cofinal* if the induced limit function $\psi\colon \lim_{\mathbf{K}} X \to \lim_{\mathbf{J}} Y$ is an isomorphism for every complete category $\mathbf{C}$ and every diagram $X\colon \mathbf{K} \to \mathbf{C}$, where Y is the precomposition with u.

Example 2.10. Any terminal object $t \in \mathbf{C}$, for some category $\mathbf{C}$ induces a final functor $t\colon \mathbb{1} \to \mathbf{C}$.

Lemma 2.11. *Right adjoint functors between small categories are final, and left adjoint functors between small categories are cofinal*

Proof. We will show that right adjoint functors are final. The proof for left adjoint functors is dual

Let $\mathbf{C}$ be a cocomplete category, and $(u, v)\colon \mathbf{J} \leftrightarrows \mathbf{K}$ an adjunction between the small categories $\mathbf{J}$ and $\mathbf{K}$. Then we get an adjunction $(u^*, v^*)\colon \mathbf{C}^{\mathbf{K}} \leftrightarrows \mathbf{C}^{\mathbf{J}}$ by proposition A.11. Since adjoints can be composed to create new adjunctions, the composition with $(\operatorname{colim}_{\mathbf{J}}, \Delta_J)\colon \mathbf{C}^{\mathbf{J}} \leftrightarrows \mathbf{C}$ yields

$$(\operatorname{colim}_{\mathbf{J}} \circ u^*, v^* \circ \Delta_J = \Delta_K)\colon \mathbf{C}^{\mathbf{K}} \leftrightarrows \mathbf{C}$$

Then by the uniqueness of left adjoints, there is a canonical isomorphism $\operatorname{colim}_{\mathbf{K}} \cong (\operatorname{colim}_{\mathbf{J}} \circ u^*)$ □

The above result is really just a more abstract way of saying that right adjoint functors commute with limits, while left adjoint functors commute with colimits. The reason for this formulation becomes more clear when we give the derivator-version in chapter three.

2.3 The point-wise construction

We saw in example 2.7 that Kan extensions generalized the idea of a (co)limit, if the target category was the terminal category $\mathbb{1}$. Now we replace the terminal category by a small category and show that these general Kan extensions exists, and that they can be calculated point-wise. Recall the definition of the slice categories as categories of morphisms into specific objects. We are going to use this to construct a candidate for the left Kan extension of a functor, and then show that this construction satisfies the universal property. For the remainder of this section we will only show that the results are true for left Kan extensions and cocomplete categories, but any result can be dualized.

Let $\mathbf{J}$ and $\mathbf{K}$ be two small categories, $\mathbf{C}$ a cocomplete category, and $u\colon \mathbf{J} \to \mathbf{K}$ and $X\colon \mathbf{J} \to \mathbf{C}$ be two functors. For an object $k \in \mathbf{K}$ consider the slice category $(u \downarrow k)$. From lemma 2.4 (i) we get a projection functor $\rho_k\colon (u \downarrow k) \to \mathbf{J}$. We can compose this with the functor X to get

$$(u \downarrow k) \xrightarrow{\rho_k} \mathbf{J} \xrightarrow{X} \mathbf{C}$$

Since $\mathbf{J}$ is small and $\mathbf{C}$ is cocomplete, there exists a colimit of the composition $\operatorname{colim}_{(\mathbf{u}\downarrow\mathbf{k})}(X \circ \rho_k)$, and we make the following definition on objects

$$L(X)(k) \longmapsto \operatorname{colim}_{(\mathbf{u}\downarrow\mathbf{k})}(X \circ \rho_k)$$

For each $h\colon k \to k'$ in $\mathbf{K}$, we get an induced functor $(u \downarrow h)\colon (u \downarrow k) \to (u \downarrow k')$ by lemma 2.4(ii). Then the canonical function in proposition 2.8 motivates us to define

$$L(X)(h)\colon\ \mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k})}(X \circ \rho_k) \to \mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k'})}(X \circ \rho_{k'})$$

and since these morphisms are unique they are also functorial.

Thus we have shown that we can calculate a functor $L(X)\colon \mathbf{K} \to \mathbf{C}$ point-wise through the slice categories. In order to show that this is a Kan extension, we also need a natural transformation. Consider an object $j \in \mathbf{J}$. The pair $(j, \mathrm{id}_{\mathbf{K}}\colon u(j) \to u(j))$ defines an object in the slice category $(u \downarrow u(j))$, and since $L(X)$ is defined as a colimit, we have a colimiting cocone

$$\eta_j\colon X(j) \to \mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{u(j)})}(X \circ \rho_{u(j)})$$

For a morphism $g\colon j \to j'$ in $\mathbf{J}$ we have the following diagram

$$\begin{tikzcd}
X(j) \arrow[r, "\eta_j"] \arrow[d, "X(g)"'] \arrow[dr, "\eta'_j"] & \mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{u(j)})}(X \circ \rho_{u(j)}) \arrow[d, "L(X)(g)"] \\
X(j') \arrow[r, "\eta_{j'}"] & \mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{u(j')})}(X \circ \rho_{u(j')})
\end{tikzcd}$$

where the lower triangle commutes by colimit properties, and the upper triangle commutes by lemma 2.4 (ii) and proposition 2.8. Hence, we have a natural transformation $\eta\colon X \to L(X) \circ u$.

So now we have a functor and a natural transformation. All that is left, is to tie this up in the following theorem.

Theorem 2.12. *Let* $\mathbf{J}$ *and* $\mathbf{K}$ *be two small categories, and* $\mathbf{C}$ *a cocomplete category. If* $u\colon \mathbf{J} \to \mathbf{K}$*, and* $X\colon \mathbf{J} \to \mathbf{C}$ *are two functors then the pair* $(L(X), \eta)$ *constructed above is a Left Kan extension of* X *along* u*.*

Proof. Due to the above arguments, all we need to prove is the universal property. For each other pair (α, Y), for some $\alpha\colon X \to Y \circ u$, we need a unique $\beta\colon L(X) \to Y$ such that $\alpha = (\beta \circ u) \circ \eta$. In other words, given

$$\phi\colon\ \mathrm{Hom}_{\mathbf{C}^{\mathbf{K}}}(L(X), Y) \to \mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(X, Y \circ u), \quad \beta \mapsto (\beta \circ u) \circ \eta$$

we need to create an inverse, thus proving it's a bijection.

Let $\alpha\colon X \to Y \circ u$, and fix an object $k \in \mathbf{K}$. For every $(j, f) \in (u \downarrow k)$, we define the map $Y(f) \circ \alpha_j : X(j) \to (Y \circ u)(j) \to Y(k)$. So given $g\colon j \to j' \in \mathbf{J}$ the following diagram

$$\begin{tikzcd}
X(j) \arrow[r, "X(g)"] \arrow[d, "\alpha_j"'] & X(j') \arrow[d, "\alpha_{j'}"] \\
(Y \circ u)(j) \arrow[r, "(Y\circ u)(g)"] \arrow[d, "Y(f)"'] & (Y \circ u)(j') \arrow[dl, "Y(f')"] \\
Y(k) &
\end{tikzcd}$$

commute. In other words, this defines a cocone on $(X \circ \rho_k) : (u \downarrow k) \to \mathbf{C}$. Recall the definition of $L(X)$ as $\operatorname{colim}(X \circ \rho_k)$. By proposition 2.8 we get a unique map $\psi(\alpha)_k\colon L(X)(k) \to Y(k)$ such that

$$\begin{tikzcd}
L(X)(k) \arrow[r, "\exists! \psi(\alpha)(k)"] & Y(k) \\
X(j) \arrow[u, "\eta_j"] \arrow[r, "\alpha_j"'] & (Y \circ u)(j) \arrow[u, "Y(f)"']
\end{tikzcd}$$

commutes. It follows that this defines a natural transformation $\psi(\alpha)\colon L(X) \to Y$. Thus, all that is left is to check that ϕ and ψ are indeed inverses to each other.

Let $\alpha = \phi(\beta) = (\beta \circ u) \circ \eta$, for a $\beta\colon L(X) \to Y \circ u$. For every $k \in \mathbf{K}$, and $(j, f) \in (u \downarrow k)$ we get the following diagram

$$\begin{tikzcd}
& L(X)(k) \arrow[r, "\beta_k"] & Y(k) \\
X(j) \arrow[r, "\eta_j"] \arrow[ur, "\eta'_j"] & L(X)(u(j)) \arrow[u, "L(X)(f)"] \arrow[r, "\beta_{u(j)}"'] & (Y \circ u)(j) \arrow[u, "Y(f)"']
\end{tikzcd}$$

where the maps on the left side comes from the cocones and proposition 2.8, and the square is a naturality square. Then the uniqueness of $\psi(\alpha)_k$ implies that $\beta_k = \psi(\alpha)_k = \psi(\phi(\beta))_k$.

Conversely, assume we have $\alpha\colon X \to Y \circ u$. For any $j \in \mathbf{J}$, the corresponding object $(j, \mathrm{id}_{u(j)})$ gives the diagram

$$\begin{tikzcd}
L(X)(k) \arrow[r, "\psi(\alpha)(u(j))"] & Y(u(j)) \\
X(j) \arrow[u, "\eta_j"] \arrow[r, "\alpha_j"'] & (Y \circ u)(j) \arrow[u, "="']
\end{tikzcd}$$

which commutes by definition of $\psi(\alpha)$. Hence we have that $\alpha_j = ((\psi(\alpha) \circ u) \circ \eta)_j = \phi(\psi(\alpha))_j$, concluding the proof. □

Now we show why these extensions are interesting. The following theorem summarizes the properties of Kan extensions that we want for an abstract derivator.

Theorem 2.13. *Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between small categories, and $\mathbf{C}$ be a cocomplete category. For every diagram $X\colon \mathbf{J} \to \mathbf{C}$ we have the following*

(i) There exists a left Kan extension $\mathrm{LK}_u(X)\colon \mathbf{K} \to \mathbf{C}$, which defines a functor $\mathrm{LK}_u\colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}^{\mathbf{K}}$ that is left adjoint to the restriction functor $u^*\colon \mathbf{C}^{\mathbf{K}} \to \mathbf{C}^{\mathbf{J}}$.

(ii) For every object $k \in \mathbf{K}$, there is a canonical isomorphism

$$\mathrm{colim}_{(u\downarrow k)}(X \circ \rho_k) \cong \mathrm{LK}_u(X)_k.$$

Proof. (i) We obtain the restriction functor $u^*\colon \mathbf{C}^{\mathbf{K}} \to \mathbf{C}^{\mathbf{J}}$ by precomposing the diagram X with u. We define $\mathrm{LK}_u\colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}^{\mathbf{K}}$ by $X \mapsto \mathrm{LK}_u(X)$. From theorem 2.12 we know there exists a left Kan extension $\mathrm{LK}_u(X)$ for every diagram $X\colon \mathbf{J} \to \mathbf{C}$. From the same theorem it also follows that

$$\mathrm{Hom}_{\mathbf{C}^{\mathbf{K}}}(\mathrm{LK}_u(X), Y) \cong \mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(X, Y \circ u)$$

hence, there is an adjunction $(\mathrm{LK}_u, u^*)\colon \mathbf{C}^{\mathbf{K}} \leftrightarrows \mathbf{C}^{\mathbf{J}}$.

(ii) Consider the commutative diagram

$$\begin{tikzcd}
X(j) \arrow[r, "X(f)"] \arrow[d, "\eta_j"'] & X(j') \arrow[d, "\eta_{j'}"] \\
(Y \circ u)(j) \arrow[r, "(Y\circ u)(f)"'] & (Y \circ u)(j')
\end{tikzcd}$$

in **C**. We can think of this natural transformation as an element of the comma category $(X \downarrow u^*)$. The universal property of $\mathrm{LK}_u(X)$ makes it an initial object in this category, so $\mathrm{LK}_u(X)_k$ is an initial object for all $k \in \mathbf{K}$. The colimit $\mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k})}(X \circ \rho_k)$ is also initial by the properties of colimits. As any two initial objects are canonically isomorphic (A.2) the result follows.

□

Remark. In a bicomplete category, we can consider the two slice categories $(u \downarrow k)$ and $(k \downarrow u)$, together with the colimits and limits of the projection maps, to canonically form left and right Kan extensions. Then by theorem 2.13 (i) we have both a left and right adjoint to restriction functors.

3 Definition and properties of derivators

In this chapter we define the framework of derivators, and show some of its properties. We begin by introducing prederivators, and becoming familiar with 2-categorical thinking. Then we give a motivation for the definition of a derivator, as well as some examples. Finally, we prove that derivators generalize concepts from chapters 1 and 2, and give motivation for why we want to further investigate derivators.

3.1 Prederivators

Before we discuss derivators, we introduce the notion of *prederivators* in order to become familiar with some of the terminology, and the way we think about higher categories. When we work with categories we are not (usually) considering sets, but classes of objects. As is discussed in [13], Russel's paradox says we can not form the 'set of all sets'. However, we can form something like 'the category of all categories'. In fact, we denote this by **CAT**. We might also consider a subcategory of this category. We denote the category of all small categories (categories whose objects *do* form sets) by **Cat**. The idea of a prederivator is just a functor from the small category of categories to the bigger category of categories.

Definition 3.1. A *prederivator* is a strict 2-functor $\mathscr{D}\colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$.

Let us unravel this definition. A 2-functor is a functor between 2-categories. By this we mean categories consisting of objects, morphisms and morphisms between the morphisms. A short introduction can be found in *Bicategories and 2-categories* [19], but we can think of the functor categories as a standard example. When we say it is strict we mean that the natural transformations, or 2-cells, are strictly associative and not just up to isomorphism. In other words, given three composable natural transformations α, β and γ we have an equality $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ and not just an isomorphism. $\mathbf{Cat}^{\mathrm{op}}$ is the category of all small opposite categories. That is, categories with reversed morphisms. So a prederivator is something that maps small categories (with reversed arrows) to 'bigger' categories. We illustrate this with some examples.

Example 3.2. Let $\mathbf{C}$ be a category, and define $\mathscr{D}_{\text{rep}}(-) := \mathbf{C}^{(-)}$. For any functors $u, v\colon \mathbf{J} \to \mathbf{K}$ between small categories, we define $\mathscr{D}_{\text{rep}}(u)$ and $\mathscr{D}_{\text{rep}}(v)$ to be the precomposition functors. That is, we get

$$\mathscr{D}_{\text{rep}}(u) = u^*\colon \mathbf{C}^{\mathbf{K}} \to \mathbf{C}^{\mathbf{J}}, \quad X \mapsto X \circ u.$$

If $\alpha : u \to v$ is a natural transformation, we also get an induced natural transformation $\alpha^* : u^* \to v^*$ by precomposition

$$\alpha^*_X = X \circ \alpha : X \circ u \mapsto X \circ v, \quad X \in \mathbf{C}^{\mathbf{K}}.$$

This then defines a prederivator. We refer to $\mathscr{D}_{\text{rep}}$ as the *represented prederivator.*

We have already seen ways that the terminal category $\mathbb{1}$ can be useful to generalize concepts. For example Kan extensions along $\mathbb{1}$ become (co)limits (see example 2.7). In this section we will see more uses for it, involving slice categories. We call the category $\mathscr{D}(\mathbb{1})$ the *underlying category* of $\mathscr{D}$. The underlying category of the represented prederivator from example 3.2 is the original category $\mathbf{C}$.

Example 3.3. Let $\mathcal{A}$ be an abelian category, and $\mathbf{Ch}(\mathcal{A})$ be the category of chain complexes. Recall that we denote by $W_{\mathcal{A}}$ the class of quasi-isomorphisms in $\mathbf{Ch}(\mathcal{A})$. Given any small category $\mathbf{J}$, we denote by $W^{\mathbf{J}}_{\mathcal{A}}$ the class of levelwise quasi-isomorphisms in $\mathbf{Ch}(\mathcal{A})^{\mathbf{J}}$. We then define the *homotopy prederivator*

$$\mathscr{D}_{\mathcal{A}}(\mathbf{J}) := \mathbf{Ch}(\mathcal{A})^{\mathbf{J}}[(W^{\mathbf{J}}_{\mathcal{A}})^{-1}], \quad \mathbf{J} \in \mathbf{Cat}$$

For each functor $u\colon \mathbf{J} \to \mathbf{K}$, we get restriction functors $u^*\colon \mathbf{Ch}(\mathcal{A})^{\mathbf{K}} \to \mathbf{Ch}(\mathcal{A})^{\mathbf{J}}$, which preserves levelwise quasi-isomorphisms by proposition 1.5. Then by proposition 1.3 and proposition 1.14, we get canonically induced functors $\mathscr{D}_{\mathcal{A}}(u)\colon \mathscr{D}_{\mathcal{A}}(\mathbf{K}) \to \mathscr{D}_{\mathcal{A}}(\mathbf{J})$. From corollary 1.13 it follows that this is a strict 2-functor from $\mathbf{Cat}^{\mathbf{op}}$ to

CAT. The underlying category of this prederivator is canonically isomorphic to the derived category

$$\mathscr{D}_{\mathcal{A}}(\mathbb{1}) = \mathbf{Ch}(\mathcal{A})^{\mathbb{1}}[(W^{\mathbb{1}})^{-1}] \cong \mathbf{Ch}(\mathcal{A})[(W)^{-1}] = \mathbf{D}(\mathcal{A}).$$

Even more generally, from the discussion of proposition 1.6, we get $\mathscr{D}_{\mathcal{A}}(\mathbf{J}) \cong \mathbf{D}(\mathcal{A}^{\mathbf{J}})$, $\mathbf{J} \in \mathbf{Cat}$.

From this example, we are motivated to introduce a little terminology. The following is an abstract version of subsection 1.3, hence the abuse of notation. Let $\mathscr{D}$ be a prederivator, and $\mathbf{J}$ a small category. A diagram $X \in \mathscr{D}(\mathbf{J})$ is called a *coherent diagram of shape* $\mathbf{J}$, and a diagram $X \in \mathscr{D}(\mathbb{1})^{\mathbf{J}}$ is called an *incoherent diagram of shape* $\mathbf{J}$.

The induced precomposition functor $j^* = \mathscr{D}(j)\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbb{1})$ is called an *evaluation functor*. Similarly to the derived category, given any coherent diagram $X \in \mathscr{D}(\mathbf{J})$, the evaluation functor j^* lets us see what happens to the object j. We also denote this by $j^*(X) = X_j$. Given any morphism of coherent diagrams $g\colon X \to Y$ in $\mathscr{D}(\mathbf{J})$, we get an induced morphism $g_j\colon X_j \to Y_j$ in the underlying category $\mathscr{D}(\mathbb{1})$ by $j^*(X) \mapsto j^*(X \circ g)$.

Similarly, we get the induced map $f^*\colon X_j \to X_{j'}$ from any morphism $f\colon j \to j'$ in $\mathbf{J}$. The following lemma is an abstract version of proposition 1.24, and the proof is practically identical.

Lemma 3.4. *Let $\mathscr{D}$ be a prederivator, $\mathbf{J}$ a small category, and $g\colon X \to X'$ a morphism in $\mathscr{D}(\mathbf{J})$.*

(i) There is a functor $\mathrm{dia}_J(X)\colon \mathbf{J} \to \mathscr{D}(\mathbb{1})$, defined by $j \mapsto X_j$ and $f \mapsto f^*$.

(ii) There is a natural transformation $\mathrm{dia}_J(g)\colon\ \mathrm{dia}_J(X) \to \mathrm{dia}_J(Y)$, defined componentwise by $g_j\colon X_j \to Y_j$, for all $j \in \mathbf{J}$.

(iii) There is a functor $\mathrm{dia}_J\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbb{1})^{\mathbf{J}}$, defined by $X \mapsto \mathrm{dia}_J(X)$ and $g \mapsto \mathrm{dia}_J(g)$

Proof. (i) It follows from the definition that objects are sent to objects, and morphisms to morphisms, so all that is left to check is identity and composition. It is enough to observe that this is evaluating objects along functors. Evaluating an object along the identity changes nothing, so $\mathrm{id}_{\mathbf{J}}^* = \mathrm{id}_{\mathscr{D}(\mathbb{1})}$. Evaluating an object along two maps, is the same as evaluating the object along the composition. So $(g \circ f)(j) = g(f(j)) \mapsto g^*(f^*(j)) = (g^* \circ f^*)(j)$.

(ii) Consider the square

$$\begin{tikzcd}
X_j \arrow[r, "g_j"] \arrow[d, "f^*"'] & Y_j \arrow[d, "f^*"] \\
X_{j'} \arrow[r, "g_{j'}"'] & Y_{j'}
\end{tikzcd}$$

of morphisms in $\mathscr{D}(\mathbb{1})$. As the two maps g_j, and $g_{j'}$ are morphisms in $\mathscr{D}(\mathbb{1})$, and the map f^* comes from natural a transformation $f\colon j \to j'$, the square commutes for all $j \in \mathbf{J}$.

(iii) Again, all that is left to check is identity and composition. $\mathrm{id}_{\mathscr{D}(\mathbf{J})}$ is sent to $\mathrm{dia}_j(\mathrm{id}_{\mathscr{D}(\mathbf{J})})$ which maps X_j to itself for all $j \in \mathbf{J}$. which is just the identity on $\mathscr{D}(\mathbb{1})^{\mathbf{J}}$ from (i). For composition, we can consider $(g_2 \circ g_1)$ in $\mathscr{D}(\mathbf{J})$. This is sent to $\mathrm{dia}_J(g_2 \circ g_1) = \mathrm{dia}_J(g_2) \circ \mathrm{dia}_J(g_1)$, as these are natural transformations from (ii).

□

The functor from lemma 3.4 (iii) is called the *underlying diagram functor*. Note that it takes a coherent diagram and sends it to the underlying incoherent diagram.

Example 3.5. For the represented prederivator the coherent and incoherent diagrams coincide, and so the underlying diagram functor is just the identity functor.

Example 3.6. For the homotopy prederivator the underlying diagram functor is the functor from proposition 1.24.

As we saw in sections 1.3 and 1.4 many constructions available at the level of coherent diagrams, are no longer available when we pass to incoherent diagrams.

This suggests that we generally do not want to use dia_J, as it is usually not an equivalence.

3.2 Derivators

Now that we have seen how prederivators can generalize concepts from the derived category, we want to impose some additional structure so we can apply the machinery of Kan extensions.

Definition 3.7. Let $\mathscr{D}$ be a prederivator, $u\colon \mathbf{J} \to \mathbf{K}$ a functor between small categories, and consider the restriction functor $u^*\colon \mathscr{D}(\mathbf{K}) \to \mathscr{D}(\mathbf{J})$. We say $\mathscr{D}$ admits *left Kan extensions* and *right Kan extensions* along u, if there exists a left and right adjoint, $u_!$ and u_*, to u^*. In other words, if we have,

$$(u_!, u^*)\colon \mathscr{D}(\mathbf{J}) \leftrightarrows \mathscr{D}(\mathbf{K}), \quad (u^*, u_*)\colon \mathscr{D}(\mathbf{J}) \leftrightarrows \mathscr{D}(\mathbf{K}).$$

Remark. In this context, we are talking about abstract objects of $\mathscr{D}$ and so the terminology Kan extension is not really connected to that of chapter 2 yet. However, we can think of this as imposing theorem 2.13 (i) to the prederivator. We will show more similarities between the concepts to justify the terminology.

If we assume that a prederivator $\mathscr{D}$ admits Kan extensions, we can consider the extension along the functor $\pi\colon \mathbf{J} \to \mathbb{1}$. Recall in example 2.7, that Kan extensions along this functor resulted in (co)limits. Motivated by this, we will call the Kan extensions along π for *(co)limits of shape J*. For an abstract derivator, we want to be able to calculate these (co)limits pointwise, as we showed is possible by theorem 2.13 (ii). From there we get the two isomorphic transformations

$$\mathrm{colim}_{(u\downarrow k)}\,(X \circ \rho_k) \cong \mathrm{LK}_u(X)_k, \quad \mathrm{RK}_u(X)_k \cong \lim_{(b\downarrow k)}\,(X \circ \theta_k),$$

where ρ_k and θ_k correspond to the slice categories $(u \downarrow k)$ and $(k \downarrow u)$, respectively.

Recall the two slice squares

$$\begin{tikzcd}
(u\downarrow k) \arrow[r, "\rho_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, shorten=8pt, "f"'] & (k\downarrow u) \arrow[r, "\theta_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \\
\mathbb{1} \arrow[r, "k"] & \mathbf{K} & \mathbb{1} \arrow[r, "k"] \arrow[ur, Rightarrow, shorten=8pt, "f"] & \mathbf{K}
\end{tikzcd}$$

of proposition 2.5. If we apply our prederivator $\mathscr{D}$ along with the left Kan extensions $u_!$ and $\pi_!$ to the left square, we get the following diagram

$$\begin{tikzcd}
\mathscr{D}(\mathbb{1}) & \mathscr{D}((u\downarrow k)) \arrow[l, "\pi_!"'] \arrow[dl, Rightarrow, shorten=10pt, "\epsilon"'] & \mathscr{D}(\mathbf{J}) \arrow[l, "\rho_k^*"'] \arrow[dl, Rightarrow, shorten=10pt, "f_*"'] & {} \arrow[dl, Rightarrow, shorten=10pt, "\eta"'] \\
{} & \mathscr{D}(\mathbb{1}) \arrow[u, "\pi^*"] \arrow[ul, bend left, "\mathrm{id}"] & \mathscr{D}(\mathbf{K}) \arrow[l, "k^*"] \arrow[u, "u^*"] & \mathscr{D}(\mathbf{J}) \arrow[l, "u_!"] \arrow[ul, bend right, "\mathrm{id}"']
\end{tikzcd}$$

where η and ϵ are the unit and counit of the indicated adjunctions. Dually, if apply the right Kan extensions u_* and π_* to the right square we get

$$\begin{tikzcd}
\mathscr{D}(\mathbb{1}) & \mathscr{D}((k\downarrow u)) \arrow[l, "\pi_*"'] & \mathscr{D}(\mathbf{J}) \arrow[l, "\theta_k^*"'] & {} \\
{} \arrow[ur, Rightarrow, shorten=10pt, "\eta"] & \mathscr{D}(\mathbb{1}) \arrow[u, "\pi^*"] \arrow[ul, bend left, "\mathrm{id}"] \arrow[ur, Rightarrow, shorten=10pt, "f_*"] & \mathscr{D}(\mathbf{K}) \arrow[l, "k^*"] \arrow[u, "u^*"] \arrow[ur, Rightarrow, shorten=10pt, "\epsilon"] & \mathscr{D}(\mathbf{J}) \arrow[l, "u_*"] \arrow[ul, bend right, "\mathrm{id}"']
\end{tikzcd}$$

This induces the two following transformations

$$\mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k})} \circ \rho_k^* = \pi_! \circ \rho_k^* \to \pi_! \circ \rho_k^* \circ u^* \circ u_! \to \pi_! \circ \pi^* \circ k^* \circ u_! \to k^* \circ u_!$$

and

$$k^* \circ u_* \to \pi_* \circ \pi^* \circ k^* \circ u_* \to \pi_* \circ \theta_k^* \circ u^* \circ u_* \to \pi_* \circ \theta_k^* = \lim_{(\mathbf{k}\downarrow\mathbf{u})} \circ \theta_k^*$$

We refer to the transformations $\mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k})} \circ \rho_k^* \to k^* \circ u_!$ and $k^* \circ u_* \to \lim_{(\mathbf{k}\downarrow\mathbf{u})} \circ \theta_k^*$ as the *canonical mate transformations*. Now we have what we need to define a derivator.

Definition 3.8. Let $\mathscr{D}$ be a prederivator. We say $\mathscr{D}$ is a derivator if it satisfies the following properties

(Der i) $\mathscr{D}$ sends coproducts to products. That is, $\mathscr{D}(\amalg_{i\in I}\mathbf{J}_i) \cong \Pi_{i\in I}\,\mathscr{D}(\mathbf{J}_i)$. In particular, $\mathscr{D}(\emptyset)$ is equivalent to the terminal category.

(Der ii) For any $\mathbf{J} \in \mathbf{Cat}$, a morphism $f : X \to Y$ in $\mathscr{D}(\mathbf{J})$ is an isomorphism if and only if each $f_j : X_j \to Y_j$ is an isomorphism in $\mathscr{D}(\mathbb{1})$.

(Der iii) For any functor $u\colon \mathbf{J} \to \mathbf{K}$ between small categories, the restriction functor u^* admits both left and right Kan extensions

(Der iv) For any functor $u\colon \mathbf{J} \to \mathbf{K}$ between small categories, and any object $k \in \mathbf{K}$, the canonical mate transformations

$$\mathrm{colim}_{(\mathbf{u}\downarrow\mathbf{k})} \circ \rho_k^* \to k^* \circ u_! \quad \text{and} \quad k^* \circ u_* \to \lim_{(\mathbf{k}\downarrow\mathbf{u})} \circ \theta_k^*$$

are isomorphisms.

Remark. (Der iii) and (Der iv) is the same as imposing theorem 2.13 on abstract categories.

Example 3.9. The represented prederivator $\mathscr{D}_{\mathrm{rep}}$ is a derivator, if the underlying category $\mathbf{C}$ is bicomplete. (Der i) and (Der ii) follow from the same arguments as those given in proposition A.13, and (Der iii) and (Der iv) follows from theorem 2.13.

Recall that an abelian category $\mathcal{A}$ is said to be Grothendieck abelian if it is cocomplete, filtered colimits in $\mathcal{A}$ are exact, and it has a generator.

Example 3.10. If $\mathbf{A}$ is Grothendieck abelian, then the homotopy prederivator

$$\mathscr{D}_{\mathcal{A}}(\mathbf{J}) := \mathbf{Ch}(A)^{\mathbf{J}}[(W^{\mathbf{J}})^{-1}], \quad \mathbf{J} \in \mathbf{Cat}.$$

is a derivator.

(Der i) Let I be some index, $\{\mathbf{J}\}_{i\in I}$ a collection of small categories, and consider $\amalg_{i\in I}\mathbf{J}_i$ a coproduct of them. The derived category preserves products, since $\mathcal{A}$ is

Grothendieck abelian (This follows from the fact that localizing with respect to quasi-isomorphisms is something called *Verdier localization*, see [9] section 3.5). Then we get

$$\mathscr{D}_{\mathcal{A}}(\amalg_{i\in I}\mathbf{J}_i) = \mathbf{D}(\mathcal{A}^{\amalg_{\mathbf{i}\in\mathbf{I}}\mathbf{J}_\mathbf{i}}) = \mathbf{D}(\Pi_{\mathbf{i}\in\mathbf{I}}\,\mathcal{A}^{\mathbf{J}_\mathbf{i}}) = \Pi_{i\in I}\mathbf{D}(\mathcal{A}^{\mathbf{J}_\mathbf{i}}) = \Pi_{i\in I}\,\mathscr{D}_{\mathcal{A}}(\mathbf{J}_i)$$

and in particular $\mathscr{D}_{\mathcal{A}}(\emptyset) = \mathbf{D}(\emptyset) = \mathbb{1}$.

(Der ii) This follows from the uniqueness of corollary 1.23.

(Der iii) Since $\mathcal{A}$ is Grothendieck abelian it is bicomplete, and so is $\mathbf{Ch}(\mathcal{A})$. Hence any restriction functor has a left and right adjoint by theorem 2.13(i), and by Quillen's adjunction theorem for derived functors [15] it follows that the derived functors of these adjunctions are adjunctions.

(Der iv) Consider an object $k \in \mathbf{K}$, and the induced left slice diagram

$$\begin{tikzcd}
\mathbf{D}(\mathcal{A}) & \mathbf{D}(\mathcal{A}^{\mathbf{(u\downarrow k)}}) \arrow[l, "\pi_!"'] & \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \arrow[l, "\rho_k^*"'] & \\
& \mathbf{D}(\mathcal{A}) \arrow[u, "\pi^*"] \arrow[ul, bend left, "\mathrm{id}"] & \mathbf{D}(\mathcal{A}^{\mathbf{K}}) \arrow[l, "k^*"] \arrow[u, "u^*"] & \mathbf{D}(\mathcal{A}^{\mathbf{J}}) \arrow[l, "u_!"] \arrow[ul, bend right, "\mathrm{id}"']
\arrow[from=1-2, to=2-1, Rightarrow, shorten=4mm, "\epsilon"']
\arrow[from=1-3, to=2-2, Rightarrow, shorten=4mm, "f^*"']
\arrow[from=1-4, to=2-3, Rightarrow, shorten=4mm, "\eta"']
\end{tikzcd}$$

The upper path is the derived colimit of ρ_k^* evaluated on any $X \in \mathbf{D}(\mathcal{A}^{\mathbf{J}})$ by example 2.7 and lemma 1.22. By theorem 2.13 (ii) the Kan extension is point-wise isomorphic to the colimit of $(X \circ \rho)$. By the discussion of (Der iii) and the uniqueness of corollary 1.23, this is precisely the lower path of the diagram.

The right mate is dual

The two examples above are the two standard examples to keep in mind for the rest of the thesis. $\mathscr{D}_{\text{rep}}$ is relatively concrete, as the structure of the functor category depends of the target category. $\mathscr{D}_{\mathcal{A}}$ on the other hand, is more abstract as we saw in section 1.3. The nice thing about $\mathscr{D}_{\text{rep}}$ is that this derivator makes it easy to see what properties we generalize from 'regular' category theory. When we get to chapter 5, however, we will see that the theory of stable derivators is an 'abstract theory'. By this we mean that examples where $\mathscr{D}_{\text{rep}}$ is stable are rather rare.

3.3 Properties

In this subsection we explore some simple results which holds for all derivators. The results we go through here are generalized versions from 'ordinary' category theory, and so the reader is encouraged to think of $\mathscr{D}_{\text{rep}}$ as the standard example for this section. Throughout chapter section 2 we used the idea of dualizing to minimize the proofs and make it more readable. Whenever possible, it is often practical to prove a statement and then pass to the opposite category in order to get the dualized result. Thus, in the spirit of not proving too much we introduce the opposite derivator.

Definition 3.11. Let $\mathscr{D}$ be a prederivator. We define the opposite prederivator $\mathscr{D}^{\text{op}}$ by

$$\mathscr{D}^{\text{op}} = \mathscr{D}(\mathbf{J}^{\text{op}})^{\text{op}}, \text{ for all } \mathbf{J} \in \mathbf{Cat}$$

To justify this definition, we have the following proposition.

Proposition 3.12. *$\mathscr{D}$ is a derivator if and only if $\mathscr{D}^{op}$ is a derivator.*

Proof. Let $\{\mathbf{J}\}_i, i \in I$ be a collection of small categories, and consider $\amalg_{i\in I}\mathbf{J}_i$ a coproduct of them. Then we get

$$\mathscr{D}^{\text{op}}(\amalg_{i\in I}\mathbf{J}_i) = \mathscr{D}((\amalg_{i\in I}\mathbf{J}_i)^{\text{op}})^{\text{op}} = \mathscr{D}(\amalg_{i\in I}(\mathbf{J}_i)^{\text{op}})^{\text{op}}$$

$$= (\Pi_{i\in I}\, \mathscr{D}(\mathbf{J}_i^{\text{op}}))^{\text{op}} = \Pi_{i\in I}\, \mathscr{D}(\mathbf{J}_i^{\text{op}})^{\text{op}} = \Pi_{i\in I}\, \mathscr{D}^{\text{op}}(\mathbf{J}_i)$$

which is a product in **CAT**. Similarly, we get

$$\mathscr{D}^{\text{op}}(\emptyset) = \mathscr{D}(\emptyset^{\text{op}})^{\text{op}} = \mathscr{D}(\emptyset)^{\text{op}} = \mathbb{1}^{\text{op}} = \mathbb{1}$$

for the empty set.

Given a morphism $f \in \mathscr{D}^{\text{op}}(\mathbf{J})$, then

$$\begin{aligned} & f \text{ is an isomorphism in } \mathscr{D}(\mathbf{J}^{\text{op}})^{\text{op}} \\ \iff & f^{\text{op}} \text{ is an isomorphism in } \mathscr{D}(\mathbf{J}^{\text{op}}) \\ \iff & f_j^{\text{op}} \text{ is an isomorphism in } \mathscr{D}(\mathbb{1}), \text{ for all } j \in \mathbf{J}^{\text{op}} \\ \iff & f_j \text{ is an isomorphism in } \mathscr{D}(\mathbb{1}), \text{ for all } j \in \mathbf{J}. \end{aligned}$$

For any functor $u\colon \mathbf{J} \to \mathbf{K}$ between small categories, the restriction functor of the opposite functor $(u^{\mathrm{op}})^* \in \mathscr{D}(J^{\mathrm{op}})$ admits left and right Kan extensions, $(u^{\mathrm{op}})_!$ and $(u^{\mathrm{op}})_*$, respectively. By definition these are right and left adjoints, and so we get Kan extensions in $\mathscr{D}^{\mathrm{op}}$ defined by

$$v_! = {(u^{\mathrm{op}})_*}^{\mathrm{op}} \quad \text{and} \quad v_* = {(u^{\mathrm{op}})_!}^{\mathrm{op}}$$

where $v = \mathscr{D}^{\mathrm{op}}(u)$.

Now we show that the canonical mates are isomorphisms. It is enough to show one of them, as the proof for the other is dual. Consider the following slice square, along with its opposite

$$\begin{tikzcd}
(u \downarrow k) \arrow[r, "\rho_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, shorten=10pt, "f"'] \\
\mathbb{1} \arrow[r, "k"] & \mathbf{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
(k \downarrow u^{\mathrm{op}}) \arrow[r, "\rho_k^{\mathrm{op}}"] \arrow[d, "\pi^{\mathrm{op}}"'] & \mathbf{J}^{\mathrm{op}} \arrow[d, "u^{\mathrm{op}}"] \\
\mathbb{1} \arrow[r, "k^{\mathrm{op}}"] \arrow[ur, Rightarrow, shorten=10pt, "f^{\mathrm{op}}"] & \mathbf{K}^{\mathrm{op}}
\end{tikzcd}$$

Applying the derivator $\mathscr{D}$ and then taking the opposite is our definition of $\mathscr{D}^{\mathrm{op}}$ and yields the two diagrams

$$\begin{tikzcd}
\mathscr{D}(\mathbb{1}) & \mathscr{D}((k \downarrow u^{\mathrm{op}})) \arrow[l, "(\pi^{\mathrm{op}})_*"'] & \mathscr{D}(\mathbf{J}^{\mathrm{op}}) \arrow[l, "(\rho_k^{\mathrm{op}})^*"'] & \\
 & \mathscr{D}(\mathbb{1}) \arrow[u, "(\pi^{\mathrm{op}})^*"] \arrow[ul, bend left, "\mathrm{id}"] & \mathscr{D}(\mathbf{K}^{\mathrm{op}}) \arrow[u, "(u^{\mathrm{op}})^*"] \arrow[l, "(k^{\mathrm{op}})^*"] & \mathscr{D}(\mathbf{J}^{\mathrm{op}}) \arrow[l, "(u^{\mathrm{op}})_*"] \arrow[ul, bend right, "\mathrm{id}"']
\arrow[from=2-1, to=1-2, Rightarrow, shorten=10pt, "\eta"]
\arrow[from=2-2, to=1-3, Rightarrow, shorten=20pt, "(f^{\mathrm{op}})^*"]
\arrow[from=2-3, to=1-4, Rightarrow, shorten=10pt, "\epsilon"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{D}^{\mathrm{op}}(\mathbb{1}) & \mathscr{D}^{\mathrm{op}}((u \downarrow k)) \arrow[l, "(\pi^{\mathrm{op}})_*^{\mathrm{op}}"'] & \mathscr{D}^{\mathrm{op}}(\mathbf{J}) \arrow[l, "(\rho_k^{\mathrm{op}})^{*\mathrm{op}}"'] & \\
 & \mathscr{D}^{\mathrm{op}}(\mathbb{1}) \arrow[u, "(\pi^{\mathrm{op}})^{*\mathrm{op}}"] \arrow[ul, bend left, "\mathrm{id}"] & \mathscr{D}^{\mathrm{op}}(\mathbf{K}) \arrow[u, "(u^{\mathrm{op}})^{*\mathrm{op}}"] \arrow[l, "(k^{\mathrm{op}})^{*\mathrm{op}}"] & \mathscr{D}^{\mathrm{op}}(\mathbf{J}) \arrow[l, "{(u^{\mathrm{op}})_*}^{\mathrm{op}}"] \arrow[ul, bend right, "\mathrm{id}"']
\arrow[from=1-2, to=2-1, Rightarrow, shorten=10pt, "\epsilon"']
\arrow[from=1-3, to=2-2, Rightarrow, shorten=20pt, "(f^{\mathrm{op}})^{*\mathrm{op}}"']
\arrow[from=1-4, to=2-3, Rightarrow, shorten=10pt, "\eta"']
\end{tikzcd}$$

where we use that η and ϵ is the unit and counit, respectively. From the lower diagram, we have the mate transformation

$$\operatorname{colim}_{(\mathbf{k}\downarrow\mathbf{u}^{\mathrm{op}})} (\rho_k^{\mathrm{op}})^{*\mathrm{op}} \to {(u^{\mathrm{op}})_*}^{\mathrm{op}} \circ (k^{\mathrm{op}})^{*\mathrm{op}}$$

in $\mathscr{D}^{\mathrm{op}}$. This is an isomorphism if and only if its opposite is an isomorphism. From the upper diagram, the opposite is the mate transformation

$$(u^{\mathrm{op}})_* \circ (k^{\mathrm{op}})^* \to \lim_{(\mathbf{k}\downarrow\mathbf{u}^{\mathrm{op}})} (\rho_k^{\mathrm{op}})^*$$

in $\mathscr{D}$. This is an isomorphism since $\mathscr{D}$ is a derivator. Hence, we have shown that $\mathscr{D}^{\mathrm{op}}$ is derivator

Finally, if $\mathscr{D}^{\mathrm{op}} = \mathscr{D}'$ is a derivator, then by definition

$$\mathscr{D}'^{\mathrm{op}}(\mathbf{J}) = \mathscr{D}'(\mathbf{J}^{\mathrm{op}})^{\mathrm{op}} = (\mathscr{D}^{\mathrm{op}}(\mathbf{J}^{\mathrm{op}}))^{\mathrm{op}} = \mathscr{D}(\mathbf{J}^{\mathrm{opop}})^{\mathrm{opop}} = \mathscr{D}(\mathbf{J})$$

so $\mathscr{D}$ is also a derivator. □

Remark. In this proof we are constantly using the fact that for ordinary categories $\mathbf{C}$ taking opposites preserves isomorphisms, and that $\mathbf{C}^{\mathrm{opop}} = \mathbf{C}$.

When we introduced the canonical mate transformations in the last section, we defined them through the induced natural transformations of the slice squares (proposition 2.5). However, for a derivator $\mathscr{D}$ any diagram with a natural transformation

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "f"] \arrow[d, "h_1"'] & \mathbf{K} \arrow[d, "h_2"] \arrow[dl, Rightarrow, "\phi"'] \\
\mathbf{L} \arrow[r, "g"'] & \mathbf{M}
\end{tikzcd}$$

induce a diagram

$$\begin{tikzcd}
\mathscr{D}(\mathbf{L}) & \mathscr{D}(\mathbf{J}) \arrow[l, "(h_1)_!"'] \arrow[dl, Rightarrow, "\epsilon"'] & \mathscr{D}(\mathbf{K}) \arrow[l, "f^*"'] \arrow[dl, Rightarrow, "\phi^*"'] & {} \arrow[dl, Rightarrow, "\eta"'] \\
{} & \mathscr{D}(\mathbf{L}) \arrow[ul, bend left, "id"] \arrow[u, "h_1^*"] & \mathscr{D}(\mathbf{M}) \arrow[l, "g^*"] \arrow[u, "h_2^*"] & \mathscr{D}(\mathbf{K}) \arrow[l, "(h_2)_!"] \arrow[ul, bend right, "id"']
\end{tikzcd}$$

which yields a more general mate transformation $\phi_! \colon (h_1)_! \circ f^* \to g^* \circ (h_2)_!$. We see that axiom (Der iii) ensures the existence of these mates, while axiom (Der iv) implies that the special case of the slice squares are isomorphisms.

Lemma 3.13. *Given the diagram*

$$\begin{tikzcd}
\mathbf{J_1} \arrow[r, "f_1"] \arrow[d, "h_1"] & \mathbf{J_2} \arrow[r, "f_2"] \arrow[d, "h_2"] \arrow[dl, Rightarrow, shorten=4mm, "\phi_1"'] & \mathbf{J_3} \arrow[d, "h_3"] \arrow[dl, Rightarrow, shorten=4mm, "\phi_2"'] \\
\mathbf{K_1} \arrow[r, "g_1"] & \mathbf{K_2} \arrow[r, "g_2"] & \mathbf{K_3}
\end{tikzcd}$$

the two induced mate transformations

$$(\phi_1)_! \colon (h_1)_! \circ f_1^* \to g_1^* \circ (h_2)_!$$

$$(\phi_2)_! \colon (h_2)_! \circ f_2^* \to g_2^* \circ (h_3)_!$$

are compatible with pasting. That is, we have $(\phi_2)_! \circ (\phi_1)_! = (\phi_3)_!$*, where* ϕ_3 *is the induced natural transformation* $h_1 \circ (f_1 \circ f_2) \to (g_1 \circ g_2) \circ h_3$ *obtained by first applying* ϕ_2 *and then* ϕ_1*.*

Proof. As the precomposition functors have adjoints by the definition of $\mathscr{D}$, the induced unit and counit gives us the following diagram for $(\phi_2)_! \circ (\phi_1)_!$

$$\begin{tikzcd}
\mathscr{D}(\mathbf{K_1}) & \mathscr{D}(\mathbf{J_1}) \arrow[l, "(h_1)_!"'] & \mathscr{D}(\mathbf{J_2}) \arrow[l, "f_1^*"'] & & & \\
& \mathscr{D}(\mathbf{K_1}) \arrow[ul, bend left, "\mathrm{id}"] \arrow[u, "h_1^*"] \arrow[ul, phantom, "\overset{\epsilon_1}{\Leftarrow}" near end] & \mathscr{D}(\mathbf{K_2}) \arrow[l, "g_1^*"] \arrow[u, "h_2^*"] \arrow[ul, phantom, "\overset{\phi_1^*}{\Leftarrow}"] & \mathscr{D}(\mathbf{J_2}) \arrow[l, "(h_2)_!"] \arrow[ul, bend right, "\mathrm{id}"'] \arrow[ul, phantom, "\overset{\eta_2}{\Leftarrow}" near start] & \mathscr{D}(\mathbf{J_3}) \arrow[l, "f_2^*"'] & \\
& & & \mathscr{D}(\mathbf{K_2}) \arrow[ul, bend left, "\mathrm{id}"] \arrow[u, "h_2^*"] \arrow[ul, phantom, "\overset{\epsilon_2}{\Leftarrow}" near end] & \mathscr{D}(\mathbf{K_3}) \arrow[l, "g_2^*"] \arrow[u, "h_3^*"] \arrow[ul, phantom, "\overset{\phi_2^*}{\Leftarrow}"] & \mathscr{D}(\mathbf{J_3}) \arrow[l, "(h_3)_!"] \arrow[ul, bend right, "\mathrm{id}"'] \arrow[ul, phantom, "\overset{\eta_3}{\Leftarrow}" near start]
\end{tikzcd}$$

Note that in the middle we have a triangular identity on h_2^* (eq. (11)), which means that the transformations cancel out. So the diagram reads as

$$(h_1)_! \circ (f_2 \circ f_1)^* = (h_1)_! \circ f_1^* \circ f_2^* \xrightarrow{\eta_3} (h_1)_! \circ f_1^* \circ f_2^* \circ h_3^* \circ (h_3)_! \xrightarrow{\phi_2^*} (h_1)_! \circ f_1^* \circ h_2^* \circ g_2^* \circ (h_3)_!$$

$$\xrightarrow{\phi_1^*} (h_1)_! \circ h_1^* \circ g_1^* \circ g_2^* \circ (h_3)_! \xrightarrow{\epsilon_1} g_1^* \circ g_2^* \circ (h_3)_! = (g_2 \circ g_1)^* \circ h_3!$$

which is $(\phi_3)_!$ by definition. □

This is just an abstract way of saying that given composable functors between small categories, as in the following diagram

$$\begin{tikzcd}
\mathbf{J} \arrow[rr, "X"] \arrow[d, "u"] & & \mathbf{C} \\
\mathbf{K} \arrow[rru, "\mathrm{LK}_u(X)"'] \arrow[d, "v"] & & \\
\mathbf{L} \arrow[rruu, "\mathrm{LK}_v(\mathrm{LK}_u(X))"'] & &
\end{tikzcd}$$

The Kan extensions commute. That is $\mathrm{LK}_{v\circ u}(X) \cong \mathrm{LK}_v(\mathrm{LK}_u(X))$.

Lemma lemma 3.13 is very useful, as it allows us to calculate seemingly complex canonical mates by breaking up the diagrams, and calculating their simpler parts. An example of this is given in the proof of the following lemma.

Lemma 3.14. *Let $\mathscr{D}$ be a derivator, and $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between small categories. If u is fully faithful, then so is the Kan extensions $u_!$ and u_*.*

Proof. It is enough to show that this is true for $u_!$, as the proof for u_* is dual. First, consider the square

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, "\mathrm{id}"', shorten=3mm] \\
\mathbf{J} \arrow[r, "u"'] & \mathbf{K}
\end{tikzcd}$$

and note that the induced mate transformations amounts to applying the unit $\eta\colon \mathrm{id} \to u^* \circ u_!$. Hence, we need to show that this is an isomorphism (lemma A.10). To this end, let us add a slice square to the left of this diagram to get

$$\begin{tikzcd}
(\mathrm{id}_{\mathbf{J}} \downarrow j) \arrow[r, "\rho_k"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dl, Rightarrow, "\mathrm{id}"', shorten=3mm] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, "\mathrm{id}"', shorten=3mm] \\
\mathbb{1} \arrow[r, "j"'] & \mathbf{J} \arrow[r, "u"'] & \mathbf{K}
\end{tikzcd} .$$

Then by lemma 3.13 the mate transformation of the entire diagram factors as

$$\pi_! \circ \rho_k \xrightarrow{\cong} j^* \xrightarrow{\eta_j} j^* \circ u^* \circ u_!.$$

where the first isomorphism comes from the slice square by axiom (Der iv). From this we see that η_j is an isomorphism if and only if the canonical mate $\phi : \pi_! \circ \rho_k \to j^* \circ u^* \circ u_!$ is an isomorphism.

Now we observe that u is fully faithful, so the induced functor $u' : (\mathrm{id}_{\mathbf{J}} \downarrow j) \to (u \downarrow u(j))$ that sends elements (j', f) to $(u(j'), u(f))$ is an equivalence. So now we paste the above diagram together, and add the equivalence to get the following diagram

$$\begin{tikzcd}
(\mathrm{id}_{\mathbf{J}} \downarrow j) \arrow[r, "\cong"] \arrow[d, "\pi"'] & (u \downarrow u(j)) \arrow[r, "\rho_k"] \arrow[d, "\pi"'] \arrow[dl, Rightarrow, "\mathrm{id}"'] & \mathbf{J} \arrow[d, "u"'] \arrow[dl, Rightarrow, "\mathrm{id}"'] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \arrow[r, "u(j)"'] & \mathbf{K}
\end{tikzcd}$$

Note that ϕ is still the mate associated to the outer diagram. Another use of lemma 3.13 implies that ϕ is isomorphic to the mate transformation of a slice square and the transformation of an equivalence. The equivalence implies that the mate transformation is an isomorphism, and the mate of the slice square is an isomorphism by axiom (Der iv). This means that the outer mate transformation is an isomorphism. Thus, ϕ, and in particular η_j, is an isomorphism for all $j \in \mathbf{J}$. By axiom (Der ii) η_j is an isomorphism, which concludes the proof. □

Remark. This lemma partially justifies why we call the adjoints $u_!$ and u_* Kan extensions, since they 'transferred' the fully faithful property from 'regular' category theory to the homotopy setting of derivators.

Now we give a homotopy version of lemma 2.11.

Lemma 3.15. *Let* $(u, v) \colon \mathbf{J} \leftrightarrows \mathbf{K}$ *be an adjunction between small categories. Then* *given the diagrams*

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "v"] \arrow[d, "\pi_{\mathbf{K}}"'] & \mathbf{J} \arrow[d, "\pi_{\mathbf{J}}"] \arrow[dl, Rightarrow, shorten=6mm, "\mathrm{id}"'] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{J} \arrow[r, "u"] \arrow[d, "\pi_{\mathbf{J}}"'] & \mathbf{K} \arrow[d, "\pi_{\mathbf{K}}"] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] \arrow[ur, Rightarrow, shorten=6mm, "\mathrm{id}"] & \mathbb{1}
\end{tikzcd}$$

the induced mate transformations $\phi\colon (\pi_{\mathbf{K}})_! \circ v^* \to (\pi_{\mathbf{J}})_!$ *and* $\theta\colon (\pi_{\mathbf{K}})_* \to (\pi_{\mathbf{J}})_* \circ u^*$ *are isomorphisms.*

Proof. We only show that ϕ is an isomorphism as the proof of θ is dual.

From $(u, v)\colon \mathbf{J} \leftrightarrows \mathbf{K}$ we get the induced adjunction in derivators $(v^*, u^*)\colon \mathscr{D}(\mathbf{J}) \leftrightarrows \mathscr{D}(\mathbf{K})$. Since adjoints are unique up to isomorphism, this implies $v^* \cong u_!$. Now the mate transformation reads as $\phi\colon (\pi_{\mathbf{K}})_! \circ u_! = (\pi_{\mathbf{K}} \circ u)_! \to (\pi_{\mathbf{J}})_!$ which is an isomorphism due to uniqueness of adjoints. $\square$

This result is really just an abstract version of saying that left adjoints preserve colimits. The dual result for right adjoints and limits is also true.

Now, consider the diagonal functor $\Delta_K\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{J})^{\mathbf{K}}$ that just maps any $X \in \mathscr{D}(\mathbf{J})$ to the constant diagram. We would like for this functor to have adjoints as in the usual categorical sense, but for pre-derivators they do not exist in general. We are therefore going to look at a related functor.

Let $\pi_K\colon \mathbf{J} \times \mathbf{K} \to \mathbf{J}$ denote the projection onto $\mathbf{J}$ (or away from $\mathbf{K}$). This induces the restriction functor $\pi_K^*\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{J} \times \mathbf{K})$. By the definition of derivators, this has two adjoints $(\pi_K)_!$ and $(\pi_K)_*$ which we will use later. The two functors, Δ_K and π_K^* have the same domain, but quite different targets. Note that objects in $\mathscr{D}(\mathbf{J} \times \mathbf{K})$ are diagrams that are coherent in both $\mathbf{J}$ and $\mathbf{K}$, while objects in $\mathscr{D}(\mathbf{J})^{\mathbf{K}}$ are only coherent in $\mathbf{J}$. However, they are related by the following.

For any $k \in \mathbf{K}$, we have the natural isomorphism and inclusion functor $\mathrm{id}_{\mathbf{J}} \times k :$ $\mathbf{J} \cong \mathbf{J} \times \mathbb{1} \to \mathbf{J} \times \mathbf{K}$. This induces a restriction functor $(\mathrm{id}_{\mathbf{J}} \times k)^*\colon \mathscr{D}(\mathbf{J} \times \mathbf{K}) \to \mathscr{D}(\mathbf{J})$ which evaluates the object k. Then this gives a generalization of the underlying diagram functor of lemma 3.4 (iii).

Proposition 3.16. *There is a functor, called the partial underlying diagram functor, defined by*

$$\mathrm{dia}_{J,K}\colon \mathscr{D}(\mathbf{J}\times\mathbf{K}) \to \mathscr{D}(\mathbf{J})^{\mathbf{K}}, \quad X \mapsto (\mathrm{id}_{\mathbf{J}} \times k)^*(X).$$

Proof. The proof is analogous to the proof of lemma 3.4. □

If we combine the partial underlying functor $\mathrm{dia}_{J,K}$ with the projection functor π_K^*, we get a functor $\mathrm{dia}_{J,K} \circ \pi_K^*\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{J})^{\mathbf{K}}$ that first maps a diagram X to a diagram X_k that is constant in $\mathbf{K}$, and then evaluates it. So we have the following equality

$$\mathrm{dia}_{J,K} \circ \pi_K^* = \Delta_K$$

Just as with dia_J, in general, the functor $\mathrm{dia}_{J,K}$ is not an equivalence. If we assume for an instance that $\mathrm{dia}_{J,K}$ is an equivalence of categories. Then from the following diagram

$$\begin{tikzcd}
\mathscr{D}(\mathbf{J}\times\mathbf{K}) \arrow[r, "\mathrm{dia}_{J,K}"] \arrow[d, bend right=60, "(\pi_K)_!"'] \arrow[d, bend left=60, "(\pi_K)_*"] & \mathscr{D}(\mathbf{J})^{\mathbf{K}} \arrow[d, bend right=60, dashed] \arrow[d, bend left=60, dashed] \\
\mathscr{D}(\mathbf{J}) \arrow[u, "\pi_K^*"'] \arrow[r, "\mathrm{id}"] & \mathscr{D}(\mathbf{J}) \arrow[u, "\Delta_K"']
\end{tikzcd} \tag{3}$$

we get the existence of adjoints for Δ_K. This implies that the category $\mathscr{D}(\mathbf{J})$ has (co)limits of shape $\mathbf{K}$.

Theorem 3.17. *If $\mathscr{D}$ is derivator, and $\mathbf{J} \in \mathbf{Cat}$, then $\mathscr{D}(\mathbf{J})$ admits (co)products.*

Proof. Let $\mathbf{S}$ be a discrete category. $\mathbf{S}$ can be considered as a set S by removing the identity functions, and S can be considered a discrete category by adding the identity functions. Then there are equivalences $\mathbf{S}\times\mathbf{J} \simeq \amalg_{s\in S}\mathbf{J}$, and $\Pi_{s\in S}J \simeq \mathbf{J}^{\mathbf{S}}$. This gives the following commutative diagram

$$\begin{tikzcd}
\mathscr{D}(\amalg_{s\in S}\mathbf{J}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \Pi_{s\in S}\,\mathscr{D}(\mathbf{J}) \arrow[d, "\simeq"] \\
\mathscr{D}(\mathbf{S}\times\mathbf{J}) \arrow[r, "\mathrm{dia}_{S,J}"] & \mathscr{D}(\mathbf{J})^{S}
\end{tikzcd}$$

where the top equivalence comes from axiom (Der i). Which means $\mathrm{dia}_{S,J}$ is an equivalence. By the discussion of diagram 3, by setting $\mathbf{K} = \mathbf{S}$, we get the existence of (co)products. $\square$

By setting S as the empty set in the above theorem, we get the immediate result.

Corollary 3.18. *If $\mathscr{D}$ is derivator, and $\mathbf{J} \in \mathbf{Cat}$, then $\mathscr{D}(\mathbf{J})$ has initial and terminal objects*

We now introduce the notion of a *shifted* (pre)derivator.

Definition 3.19. Let $\mathscr{D}$ be a prederivator, and $\mathbf{K} \in \mathbf{Cat}$. Then $\mathscr{D}^{\mathbf{K}}$ is defined by

$$\mathscr{D}^{\mathbf{K}}(-) = \mathscr{D}(\mathbf{K} \times (-)),$$

and we call this the *shifted prederivator*.

Note that for any $u\colon \mathbf{J} \to \mathbf{L}$, we get the induced functor $u \times \mathrm{id}_{\mathbf{K}}\colon \mathbf{J} \times \mathbf{K} \to \mathbf{L} \times \mathbf{K}$, which in turn gives a restriction functor on derivators

$$(u \times \mathrm{id}_{\mathbf{K}})^*\colon \mathscr{D}^{\mathbf{K}}(\mathbf{L}) = \mathscr{D}(\mathbf{L} \times \mathbf{K}) \to \mathscr{D}(\mathbf{J} \times \mathbf{K}) = \mathscr{D}^{\mathbf{K}}(\mathbf{J}).$$

As was the case with $\mathscr{D}^{\mathrm{op}}$, we see that the shifted prederivator is defined by an already well-defined derivator. Therefore the next result might not be such a surprise.

Theorem 3.20. *If $\mathscr{D}$ is a prederivator, then $\mathscr{D}$ is a derivator if and only if $\mathscr{D}^{\mathbf{K}}$ is a derivator, for all $\mathbf{K} \in \mathbf{Cat}$.*

Proof. If $\mathscr{D}^{\mathbf{K}}$ is derivator for all $\mathbf{K}$, then certainly for $\mathbf{K} = \mathbb{1}$ we have that $\mathscr{D}^{\mathbb{1}} \cong \mathscr{D}$ is a derivator.

Conversely, let $\mathbf{J}$ be a small category, and consider $\amalg_{i\in I}\mathbf{J}_i$. Then we have

$$\mathscr{D}^{\mathbf{K}}(\amalg_{i\in I}\mathbf{J}_i) = \mathscr{D}(\mathbf{K} \times \amalg_{i\in I}\mathbf{J}_i) = \mathscr{D}(\amalg_{i\in I}\mathbf{K} \times \mathbf{J}_i) = \Pi_{i\in I}\, \mathscr{D}(\mathbf{K} \times \mathbf{J}_i) = \Pi_{i\in I}\, \mathscr{D}^{\mathbf{K}}(\mathbf{J}_i),$$

and

$$\mathscr{D}^{\mathbf{K}}(\emptyset) = \mathscr{D}(\mathbf{K} \times \emptyset) = \mathscr{D}(\emptyset) = \mathbb{1}$$

so the first axiom holds.

Given a morphism $f \in \mathscr{D}^{\mathbf{K}}(\mathbf{J})$. then

$$\begin{aligned}
& f \text{ is an isomorphism in } \mathscr{D}^{\mathbf{K}}(\mathbf{J}) \\
\iff & f \text{ is an isomorphism in } \mathscr{D}(\mathbf{K} \times \mathbf{J}) \\
\iff & f_{j,k} \text{ is an isomorphism in } \mathscr{D}(\mathbb{1}) \text{ for all } (j,k) \in \mathbf{J} \times \mathbf{K} \\
\iff & f_j \text{ is an isomorphism in } \mathscr{D}(\mathbf{K}) \text{ for all } j \in \mathbf{J} \\
\iff & f_j \text{ is an isomorphism in } \mathscr{D}^{\mathbf{K}}(\mathbb{1}) \text{ for all } j \in \mathbf{J},
\end{aligned}$$

where the middle equivalence follows from axiom (Der ii) and the commutative diagram

$$\begin{tikzcd}
\mathscr{D}(\mathbf{J} \times \mathbf{K}) \arrow[r, "X_j"] \arrow[d, "X_{(j,k)}"'] & \mathscr{D}(\mathbf{K}) \arrow[dl, "X_k"] \\
\mathscr{D}(\mathbb{1}) &
\end{tikzcd}$$

For any functor $u \colon \mathbf{J} \to \mathbf{L}$, we get the induced functor on derivators $(u \times \mathrm{id}_{\mathbf{K}})^* \colon \mathscr{D}^{\mathbf{K}}(\mathbf{L}) \to \mathscr{D}^{\mathbf{K}}(\mathbf{J})$. This is by definition a functor between $\mathscr{D}(\mathbf{L} \times \mathbf{K})$ and $\mathscr{D}(\mathbf{J} \times \mathbf{K})$, and so by the property of $\mathscr{D}$, there exists adjoints

$$(u \times \mathrm{id}_{\mathbf{K}})_! \quad \text{and} \quad (u \times \mathrm{id}_{\mathbf{K}})_*$$

For the final axiom, consider the usual slice square, and cross it with $\mathbf{K}$

$$\begin{tikzcd}
(u \downarrow l) \arrow[r, "\rho_l"] \arrow[d, "\pi"'] & \mathbf{J} \arrow[d, "u"] \arrow[dl, Rightarrow, "f"'] & \mathbf{K} \times (u \downarrow l) \arrow[r, "\mathrm{id}_{\mathbf{K}} \times \rho_l"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times \pi"'] & \mathbf{K} \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}} \times u"] \arrow[dl, Rightarrow, "\mathrm{id}_{\mathbf{K}} \times f"'] \\
\mathbb{1} \arrow[r, "l"'] & \mathbf{L} & \mathbf{K} \times \mathbb{1} \arrow[r, "\mathrm{id}_{\mathbf{K}} \times l"'] & \mathbf{K} \times \mathbf{L}
\end{tikzcd}$$

The goal is to show that the mate transformation $(\mathrm{id}_{\mathbf{K}} \times \pi)_! \circ (\mathrm{id}_{\mathbf{K}} \times \rho_l)^* \to (\mathrm{id}_{\mathbf{K}} \times l)^* \circ (\mathrm{id}_{\mathbf{K}} \times u)_!$ of the crossed square is an isomorphism. Note that $\mathrm{id}_{\mathbf{K}} \times \pi$ is a functor that we can build another slice square from, to get

$$\begin{tikzcd}
((\mathrm{id}_{\mathbf{K}} \times \pi) \downarrow k) \arrow[r, "\rho_{k\times 1}"] \arrow[d, "\pi"'] & \mathbf{K} \times (u \downarrow l) \arrow[d, "\mathrm{id}_{\mathbf{K}} \times \pi"] \arrow[dl, Rightarrow, shorten=10pt, "g\times 1"'] \\
\mathbb{1} \arrow[r, "k\times 1"'] & \mathbf{K} \times \mathbb{1}
\end{tikzcd}$$

Now we observe that $((\mathrm{id}_{\mathbf{K}} \times \pi) \downarrow k) \cong (\mathrm{id}_{\mathbf{K}} \downarrow k) \times (u \downarrow l) \cong ((\mathrm{id}_{\mathbf{K}} \times u) \downarrow (k, l))$. This can be seen by the mappings $((k', (j, f)), g) \mapsto ((k', g), (j, f)) \mapsto ((k', j), (g, f))$ on objects. Putting all this together yields the large diagram

$$\begin{tikzcd}
((\mathrm{id}_{\mathbf{K}} \times u) \downarrow (k, l)) \arrow[d, "\pi"] & ((\mathrm{id}_{\mathbf{K}} \times \pi) \downarrow k) \arrow[l, "\cong"'] \arrow[r, "\rho_{k\times 1}"] \arrow[d, "\pi"'] \arrow[dl, Rightarrow, shorten=10pt, "\mathrm{id}"'] & \mathbf{K} \times (u \downarrow l) \arrow[r, "\mathrm{id}_{\mathbf{K}} \times \rho_l"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times \pi"'] \arrow[dl, Rightarrow, shorten=10pt, "g\times 1"'] & \mathbf{K} \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}} \times u"] \arrow[dl, Rightarrow, shorten=10pt, "\mathrm{id}_{\mathbf{K}} \times f"'] \\
\mathbb{1} & \mathbb{1} \arrow[l, "="] \arrow[r, "k\times 1"'] & \mathbf{K} \times \mathbb{1} \arrow[r, "\mathrm{id}_{\mathbf{K}} \times l"'] & \mathbf{K} \times \mathbf{L}
\end{tikzcd}$$

The outer canonical mate of this diagram is a slice square, hence the outer mate is an isomorphism. In addition, it is given by

$$\pi_! \circ \rho^*_{k\times 1} \circ (\mathrm{id}_{\mathbf{K}} \times \rho_l)^* \to (k \times 1)^* \circ (\mathrm{id}_{\mathbf{K}} \times l)^* \circ (\mathrm{id}_{\mathbf{K}} \times u)_!$$

since the leftmost square is an equivalence. By lemma 3.13 this factors as

$$\pi_! \circ \rho^*_{k\times 1} \circ (\mathrm{id}_{\mathbf{K}} \times \rho_l)^* \xrightarrow{\cong} (k \times 1)^* \circ (\mathrm{id}_{\mathbf{K}} \times \pi)_! \circ (\mathrm{id}_{\mathbf{K}} \times \rho_l)^* \to (k \times 1)^* \circ (\mathrm{id}_{\mathbf{K}} \times l)^* \circ (\mathrm{id}_{\mathbf{K}} \times u)_!$$

where the first transformation is an isomorphism by axiom (Der iv) applied to the slice square in the middle. Since the outer transformation is an isomorphism, it follows that the second transformation is an isomorphism for any k. Since axiom (Der ii) lets us calculate isomorphisms pointwise, this is indeed an isomorphism.

The other canonical mate isomorphism is dual. □

So now we have shown that given any derivator $\mathscr{D}$, we can actually generate new derivators by either considering the opposite $\mathscr{D}^{\mathrm{op}}$, or ‘shift’ the derivator by a small category $\mathbf{K}$ through $\mathscr{D}^{\mathbf{K}}$. In addition to this we have shown that a derivator has both initial and terminal objects. So, the next natural step would be to ask what happens if they coincide? This leads to the concept of a *pointed* derivator, which we will discuss in the next chapter.

4 Pointed derivators

As we showed in corollary 3.18, any abstract derivator $\mathscr{D}$ admits initial and terminal objects. In this chapter we discuss the derivators in which these objects are also isomorphic. In which case this is called a *zero object*. In the first section, we show that the zero object allows us to 'extend' coherent morphisms in abstract derivators to coherent diagrams. In the second section we introduce four functors defined through these extensions which are generalisations of the cofiber and fiber functors from theorem 1.26. Finally, we want to show that the shifted derivator preserves properties in a particularly nice way.

4.1 The extensions by zero

In this section we define a pointed derivator, and characterize the essential images of Kan extensions. This will let us extend coherent diagrams by zeroes in a natural way.

Definition 4.1. A derivator $\mathscr{D}$ is said to be *pointed* if the underlying category $\mathscr{D}(\mathbb{1})$ has a zero object. We denote the zero object by $0 \in \mathscr{D}(\mathbb{1})$.

Example 4.2. Let $\mathbf{C}$ be a bicomplete category, with a zero object. Then the represented derivator $\mathscr{D}_{\text{rep}}$ from example 3.2 is pointed.

Example 4.3. The homotopy derivator $\mathscr{D}_{\mathcal{A}}$ from example 3.3 is a pointed derivator. Since $\mathcal{A}$ is abelian, it has a zero object. This object is preserved by the localization, so $\mathscr{D}_{\mathcal{A}}(\mathbb{1})$ has a zero object as well.

In the last section, we showed that given a derivator $\mathscr{D}$ we can generate new derivators by means of opposite or shift. Now we show that these operations preserve the zero object as well.

Lemma 4.4. *Let $\mathscr{D}$ be a pointed derivator. Then we have the following* *(i)* $\mathscr{D}^{\mathrm{op}}$ *is a pointed derivator.*

(ii) For a small category $\mathbf{K}$*, the shifted derivator* $\mathscr{D}^{\mathbf{K}}$ *is a pointed derivator.*

(iii) For any functor $u\colon \mathbf{J} \to \mathbf{K}$*, the restriction functor* $u^*\colon \mathscr{D}(\mathbf{K}) \to \mathscr{D}(\mathbf{J})$*, and the Kan extensions* $u_!, u_*\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{K})$ *preserves zero objects.*

Proof. (i) This follows from the definition. We have that $\mathscr{D}^{\mathrm{op}}(\mathbb{1}) = \mathscr{D}(\mathbb{1}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{D}(\mathbb{1})^{\mathrm{op}}$. Since initial and terminal objects are dual(definition A.1), it follows that $\mathscr{D}^{\mathrm{op}}$ is pointed.

(ii) Consider $\mathscr{D}^{\mathbf{K}}(\mathbb{1}) = \mathscr{D}(\mathbf{K})$. Let $\phi\colon I \to T$ be the unique function between the initial and terminal object in $\mathscr{D}(\mathbf{K})$. For all $k \in \mathbf{K}$, we get the induced evaluation function $\phi_k\colon I_k \to T_k$ in $\mathscr{D}(\mathbb{1})$, which is an isomorphism by definition of pointed derivator. Then axiom (Der ii) implies that ϕ is an isomorphism, hence $\mathscr{D}^{\mathbf{K}}$ is a pointed derivator.

(iii) For all $u(j) \in \mathbf{K}$, the diagram

$$\begin{tikzcd}
\mathscr{D}(\mathbf{K}) \arrow[r, "u(j)^*"] \arrow[dr, "u^*"'] & \mathscr{D}(\mathbb{1}) \\
& \mathscr{D}(\mathbf{J}) \arrow[u, "j^*"']
\end{tikzcd}$$

commutes. From (ii) we know that $k^*(0_{\mathscr{D}(K)}) = 0_{\mathscr{D}(\mathbb{1})}$ for all $k \in \mathbf{K}$, so in particular we have $(j^* \circ u^*)(0_{\mathscr{D}(\mathbf{K})}) = u(j)^*(0_{\mathscr{D}(K)}) = 0_{\mathscr{D}(\mathbb{1})}$ for all $j \in \mathbf{J}$. So the morphism $\phi\colon u^*(0_{\mathscr{D}(K)}) \to 0_{\mathscr{D}(J)}$ is an isomorphism by axiom (Der ii).

Now consider the adjoints $u_!$ and u_*. The result now follows as left adjoint preserves initial objects, and right adjoints preserve terminal objects.

□

We now introduce some definitions that will let us characterize the zero objects, and extensions of it. Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor and consider $u(\mathbf{J})$ in $\mathbf{K}$. This is in general not a category (there can morphisms that compose in $\mathbf{K}$, but not in $\mathbf{J}$).

We therefore introduce the ‘smallest subcategory containing $u(\mathbf{J})$’. This subcategory has some interesting properties.

Definition 4.5. Let $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between categories. We define the *essential image* of u to be all $k \in \mathbf{K}$ such that $k \cong u(j)$ for some $j \in \mathbf{J}$.

Lemma 4.6. *Let* $\mathbf{J}, \mathbf{K} \in \mathbf{Cat}$ *and* $u\colon \mathbf{J} \to \mathbf{K}$ *be fully faithful, with* $X \in \mathscr{D}(\mathbf{K})$.

(i) X is in the essential image of $u_!$ if and only if the counit $\epsilon_k\colon (u_! \circ u^*)(X)_k \to X_k$ is an isomorphism for all $k \in \mathbf{K} - u(\mathbf{J})$.

(ii) X is in the essential image of u_* if and only if the unit $\eta_k\colon X_k \to (u_* \circ u^*)(X)_k$ is an isomorphism for all $k \in \mathbf{K} - u(\mathbf{J})$.

Proof. By duality it is enough to prove this for $u_!$.

Let X be in the essential image of $u_!$. We know from lemma 3.14 that $u_!$ is fully faithful. Then from lemma A.10 we know that X lying in the essential image is the same as saying that the counit $\epsilon(X_k)$ is an isomorphism for all k. But then in particular it is an isomorphism for all $k \in \mathbf{K} - u(\mathbf{J})$.

Conversely, consider the triangular identities from figure (11)

$$\begin{tikzcd}
u^* & u^* \circ u_! \circ u^* \arrow[l, "(u^*\cdot\epsilon)"'] & & u_! & u_! \circ u^* \circ u_! \arrow[l, "(\epsilon\cdot u_!)"'] \\
 & u^* \arrow[u, "(\eta\cdot u^*)"'] \arrow[ul, bend left, "\mathrm{id}"] & & & u_! \arrow[u, "(u_!\cdot\eta)"'] \arrow[ul, bend left, "\mathrm{id}"]
\end{tikzcd}$$

It follows that the identity factors as

$$\mathrm{id}_{u^*(X)} = (u^* \cdot \epsilon)(X) \circ (\eta \cdot u^*)(X) : u^* \xrightarrow{\cong} u^* \circ u_! \circ u^* \to u^*$$

Since $u_!$ is fully faithful we know η is an isomorphism (lemma A.10), and the identity is certainly an isomorphism, so then $u^* \cdot \epsilon$ is also an isomorphism. So then $\epsilon(X_k)$ is an isomorphism for all $k \in u(\mathbf{J})$. Hence, if $\epsilon(X_k)$ is an isomorphism for all $k \in \mathbf{K} - u(\mathbf{J})$ the counit is an isomorphism by axiom (Der ii). Now the result follows from another use of lemma A.10. $\square$

Definition 4.7. Let $u\colon \mathbf{J} \to \mathbf{K}$ be a fully faithful functor.

(i) u is a *sieve*, if for every morphism $k \to u(j)$ it follows that k lies in the image of u.

(ii) u is a *cosieve* if for every morphism $u(j) \to k$ it follows that k lies in the image of u.

This can be interpreted as saying that we don't have any other morphisms in $\mathbf{K}$ going in (sieve) or out (cosieve) of $u(\mathbf{J})$. Clearly if $k \in \mathbf{K}$ is in the essential image of u, then $k \cong u(j)$, and so a functor onto its essential image is both a sieve and a cosieve.

Example 4.8. Let $\bullet \xrightarrow{i_1} \bullet \xrightarrow{i_2} \bullet$ be the linearly oriented quiver A_3. Consider the two inclusions

$$\bullet \xrightarrow{i_0} \bullet \xrightarrow{i_1} \bullet \xrightarrow{i_2} \bullet \qquad\qquad \bullet \xrightarrow{i_1} \bullet \xrightarrow{i_2} \bullet \xrightarrow{i_3} \bullet$$

into A_4. The left inclusion, l, adds an arrow into A_3, while the right inclusion, r, adds an arrow out of A_3. Then l is a cosieve while r is a sieve.

Now we use the sieves and cosieves to characterize the essential images of the Kan extensions.

Proposition 4.9. *Let $\mathscr{D}$ be a derivator, and $u\colon \mathbf{J} \to \mathbf{K}$ be a functor between small categories.*

(i) If u is a cosieve, then $u_!\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{K})$ is fully faithful and induces an equivalence onto the full subcategory of $\mathscr{D}(\mathbf{K})$ spanned by all diagrams X such that $X_k \cong 0$ for all $k \in \mathbf{K} - u(\mathbf{J})$.

(ii) If u is a sieve, then $u_*\colon \mathscr{D}(\mathbf{J}) \to \mathscr{D}(\mathbf{K})$ is fully faithful and induces an equivalence onto the full subcategory of $\mathscr{D}(\mathbf{K})$ spanned by all diagrams X such that $X_k \cong 0$ for all $k \in \mathbf{K} - u(\mathbf{J})$.

Proof. It is enough to show this for a cosieve, as the arguments are dual for a sieve. By definition of cosieve u is fully faithful, and so by lemma 3.14 so is $u_!$. By lemma A.7 all we need to show is that for any diagram X in the essential image of $u_!$ we have $X_k \cong 0$, for all $k \in \mathbf{K} - u(\mathbf{J})$.

By lemma 4.6, a diagram X is in the essential image of $u_!$ if and only if the counit $\epsilon_k \colon (u_! \circ u^*)(X)_k \to X_k$ is an isomorphism for all $k \in \mathbf{K} - u(\mathbf{J})$. By axiom (Der iv) we have canonical isomorphisms

$$(u_! \circ u^*)(X)_k \cong \operatorname{colim}_{(\mathbf{u}\downarrow\mathbf{k})} \rho_k \circ u^*(X), \quad k \in \mathbf{K}.$$

Since u is a cosieve, the slice category $(u \downarrow k)$ is actually empty for all $k \in \mathbf{K} - u(\mathbf{J})$. By axiom (Der i) $\mathscr{D}(\emptyset) = \mathbb{1}$. Since any object therein is a zero object, $\operatorname{colim}_{\emptyset} \rho_k \circ u^*(X)$ is also a zero object, since left adjoints between a pointed derivator preserve zero objects by lemma 4.4(iii). Hence, we have an isomorphism $(X)_k \cong 0$. □

This is referred to as the *left extension by zero* and the *right extension by zero*, depending on if we extend by a left or right adjoint. These extensions are crucial when we begin constructing bigger diagrams.

4.2 Fiber, cofiber, loop and suspension

In this section we introduce fibers and cofibers for pointed derivators. These functors are going to be a generalisation of the kernel and cokernel functors, but set in an abstract setting similar to theorem 1.26. In order to define them properly, we use partially ordered sets, or posets, as categories. Let $[n]$ denote the set of elements $\{0, 1, 2, \cdots, n-1, n\}$, and with the obvious relation $k < m$, if k is a number less than m.

Consider the category $[1] \times [1]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

which we will denote by $\square$. This has two full subcategories

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) & & & (1,0) \arrow[d] \\
(0,1) & & & (0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

which we denote by $\ulcorner$ and $\lrcorner$, respectively. They also come with respective inclusions $i_{\ulcorner} \colon \ulcorner \to \square$ and $i_{\lrcorner} \colon \lrcorner \to \square$. These categories and inclusion functors induces the following definition.

Definition 4.10. Let X be a square in $\mathscr{D}(\square)$.

(i) X is *cocartesian* if it lies in the essential image of the left Kan extension $(i_{\ulcorner})_! \colon \mathscr{D}(\ulcorner) \to \mathscr{D}(\square)$

(ii) X is *cartesian* if it lies in the essential image of the right Kan extension $(i_{\lrcorner})_* \colon \mathscr{D}(\lrcorner) \to \mathscr{D}(\square)$

For both the span $\ulcorner$, and cospan $\lrcorner$, we have fully faithful inclusion functors

$$h \colon [1] \to \ulcorner \qquad v \colon [1] \to \lrcorner$$

that identifies the horizontal and vertical morphisms, respectively. We visualize this as the diagram

$$\begin{tikzcd}
(0,0) \arrow[r] & (1,0) & {} \arrow[r, "h"] & {} & (0,0) \arrow[r] \arrow[d] & (1,0) \\
 & & & & (0,1) &
\end{tikzcd}$$

and the diagram

$$\begin{tikzcd}
(1,0) \arrow[d] & & & & (1,0) \arrow[d] \\
(1,1) & {} \arrow[r, "v"] & {} & (0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

respectively. Note that the only morphism from an object in $\ulcorner$ to an object in the image of h is the horizontal morphism that starts in $(0,0)$. But this is also in the image of h, hence h is a sieve. Dually, v is a cosieve. Similarly, we see that $i_\ulcorner$ and $i_\lrcorner$ are also a sieve and a cosieve. Composing these functors, we obtain

$$h' = i_\ulcorner \circ h \colon [1] \to \Box \qquad\qquad v' = i_\lrcorner \circ v \colon [1] \to \Box$$

which identifies the horizontal and vertical morphism in the square. These functors allows us to make the following definition

Definition 4.11. Let $\mathscr{D}$ be a pointed derivator.

(i) The *cofiber functor* is defined as

$$\mathrm{cof}\colon \mathscr{D}([1]) \xrightarrow{h_*} \mathscr{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathscr{D}(\Box) \xrightarrow{(v')^*} \mathscr{D}([1])$$

(ii) The *fiber functor* is defined as

$$\mathrm{fib}\colon \mathscr{D}([1]) \xrightarrow{v_!} \mathscr{D}(\lrcorner) \xrightarrow{(i_\lrcorner)_*} \mathscr{D}(\Box) \xrightarrow{(h')^*} \mathscr{D}([1])$$

Since h and v are a sieve and a cosieve, it follows from proposition 4.9 that the induced Kan extensions are extensions by zero. Hence, given a morphism $f \in \mathscr{D}([1])$, with the underlying diagram $X \to Y$, there is a cocartesian and cartesian square in $\mathscr{D}(\Box)$ with underlying diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "\mathrm{cof}(f)"] \\
0 \arrow[r] & C(f)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f) \arrow[r, "\mathrm{fib(f)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd}$$

Example 4.12. For $\mathscr{D}_{\mathrm{rep}}$, this is the same as taking the regular cokernel and kernel.

Example 4.13. For $\mathscr{D}_{\mathcal{A}}$ this becomes the cofiber construction for the left derived functor of the cokernel (theorem 1.26).

As one might have noticed, there is a certain similarity between the cofiber and fiber functors. Since they are constructed through adjunctions and restrictions along the same maps, intuitively we might suspect that the functors are adjoint to one another. Indeed this is case, as we will show next.

Proposition 4.14. *The cofiber and fiber functors form an adjoint pair* $(\mathrm{cofib}, \mathrm{fib})\colon \mathscr{D}([\mathbf{1}]) \leftrightarrows \mathscr{D}([\mathbf{1}])$.

Proof. Let $\mathscr{D}^{\mathbf{ex}}(\ulcorner) \subseteq \mathscr{D}(\ulcorner)$, $\mathscr{D}^{\mathbf{ex}}(\lrcorner) \subseteq \mathscr{D}(\lrcorner)$ and $\mathscr{D}^{\mathbf{ex}}(\square) \subseteq \mathscr{D}(\square)$ denote the full subcategories spanned by the coherent diagrams such that they vanish at the lower left corner $(0,1)$. Then by proposition 4.9 we have the following equivalences (h^*, h_*) and $(v_!, v^*)$ between $\mathscr{D}([1])$ and $\mathscr{D}^{\mathbf{ex}}(\ulcorner)$, and between $\mathscr{D}([1])$ and $\mathscr{D}^{\mathbf{ex}}(\lrcorner)$, respectively. Now consider the following sequence of functors

$$\begin{tikzcd}
\text{cofib}\colon \mathscr{D}([1]) \arrow[r, bend right, "h_*"'] & \mathscr{D}^{\mathbf{ex}}(\ulcorner) \arrow[l, bend right, "h^*"'] \arrow[r, bend left, "(i_\ulcorner)_!"] & \mathscr{D}^{\mathbf{ex}}(\square) \arrow[l, bend left, "(i_\ulcorner)^*"] \arrow[r, bend left, "(i_\lrcorner)^*"] & \mathscr{D}^{\mathbf{ex}}(\lrcorner) \arrow[l, bend left, "(i_\lrcorner)_*"] \arrow[r, bend right, "v^*"'] & \mathscr{D}([1])\colon \text{fib} \arrow[l, bend right, "v_!"']
\end{tikzcd}$$

The two outer functor pairs are equivalences, while the two inner functor pairs are adjoints by definition. Hence, we have an adjoint pair. ☐

When we discussed the derived cokernel in section 1.4, we saw that the cone construction was a generalisation of the cokernel. We replaced the zero morphism with a more general morphism. We then showed that the left derived functor of the cokernel is indeed the induced cone functor on the derived category (theorem 1.26). The cofiber and fiber functors above is a generalization of precisely this idea. We now do the reverse, and replace the original morphism of the cofiber functor with a zero morphism. This leads to what we call suspensions and loops. The construction is very similar to the cofiber and fiber, but instead of identifying the horizontal and vertical morphisms, we identify the upper left and lower right corner.

Consider the following fully faithful functors

$$i\colon (0,0) \to \ulcorner \quad \text{and} \quad k\colon (1,1) \to \lrcorner$$

which identifies the middle object of the span and cospan, respectively. We post-compose with $i_\ulcorner$ and $i_\lrcorner$ to get

$$i' = i_\ulcorner \circ i\colon (0,0) \to \square \quad \text{and} \quad k' = i_\lrcorner \circ k\colon (1,1) \to \square$$

which identifies the the two corners we wanted.

Definition 4.15. Let $\mathscr{D}$ be a pointed derivator.

(i) The *suspension functor* is defined as

$$\Sigma\colon \mathscr{D}(\mathbb{1}) \xrightarrow{i_*} \mathscr{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathscr{D}(\square) \xrightarrow{(k')^*} \mathscr{D}(\mathbb{1})$$

(ii) The *loop functor* is defined as

$$\Omega\colon \mathscr{D}(\mathbb{1}) \xrightarrow{k_!} \mathscr{D}(\lrcorner) \xrightarrow{(i_\lrcorner)_*} \mathscr{D}(\square) \xrightarrow{(i')^*} \mathscr{D}(\mathbb{1})$$

Just like with the horizontal and vertical identification, the two functors i and k is a sieve and cosieve, respectively. By proposition 4.9, this means the induced Kan extensions are extensions by zero. Hence, for an $X \in \mathscr{D}(\mathbb{1})$, there is a cocartesian and cartesian square in $\mathscr{D}(\square)$ with underlying diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma X
\end{tikzcd} \qquad \begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}$$

Example 4.16. In $\mathscr{D}_{\text{rep}}$ this is taking the cokernel and kernel of 0, and hence is just 0.

Example 4.17. In $\mathscr{D}_{\mathcal{A}}$ this is the homotopy pushout and homotopy pullback.

As with cofiber and fiber, these two functors are also related by the following result.

Proposition 4.18. *The suspension and loop functors form an adjoint pair* $(\Sigma, \Omega)\colon \mathscr{D}(\mathbb{1}) \leftrightarrows \mathscr{D}(\mathbb{1})$.

Proof. We prove this in exactly the same manner as proposition 4.14.

Let $\mathscr{D}^{\mathbf{ex}}(\ulcorner) \subseteq \mathscr{D}(\ulcorner)$, $\mathscr{D}^{\mathbf{ex}}(\lrcorner) \subseteq \mathscr{D}(\lrcorner)$ and $\mathscr{D}^{\mathbf{ex}}(\square) \subseteq \mathscr{D}(\square)$ denote the full subcategories spanned by the coherent diagrams such that they vanish at both $(0,1)$ and $(1,0)$.

Then again by proposition 4.9 we have equivalences (i^*, i_*) and $(k_!, k^*)$ between $\mathscr{D}(\mathbb{1})$ and $\mathscr{D}^{\mathbf{ex}}(\ulcorner)$, and between $\mathscr{D}(\mathbb{1})$ and $\mathscr{D}^{\mathbf{ex}}(\lrcorner)$, respectively. Now consider the the following sequence of functors

$$\begin{tikzcd}
\Sigma\colon \mathscr{D}(\mathbb{1}) \arrow[r, bend left, "i_*"] & \mathscr{D}^{\mathbf{ex}}(\ulcorner) \arrow[l, bend left, "i^*"] \arrow[r, bend left, "(i_\ulcorner)_!"] & \mathscr{D}^{\mathbf{ex}}(\square) \arrow[l, bend left, "(i_\ulcorner)^*"] \arrow[r, bend left, "(i_\lrcorner)^*"] & \mathscr{D}^{\mathbf{ex}}(\lrcorner) \arrow[l, bend left, "(i_\lrcorner)_*"] \arrow[r, bend left, "k^*"] & \mathscr{D}(\mathbb{1})\colon \Omega \arrow[l, bend left, "k_!"]
\end{tikzcd}$$

where the two outer functor pairs are equivalences, and the two inner pairs are adjunctions by definition. □

4.3 Properties of the shifted derivator

Recall that cocartesian and cartesian squares are defined as squares in the essential image of the left and right Kan extension $(i_\ulcorner)_!$ and $(i_\lrcorner)_*$, respectively. The main results for this section will be to show that a square $X \in \mathscr{D}^{\mathbf{K}}(\square)$ is (co)cartesian if and only if each evaluated square $X_k \in \mathscr{D}(\square)$ are (co)cartesian. This is a rather deep result which requires some technical proofs, however the reward will be worth it. In further studies we only need to consider the underlying diagram $\mathscr{D}(\mathbb{1})$ of a pointed derivator, as we can always shift it afterwards.

Lemma 4.19. *Let $\mathscr{D}$ be a derivator, and consider $0\colon \mathbb{1} \to [1]$ the inclusion functor that identifies the initial object.*

(i) A coherent morphism $X \in \mathscr{D}([1])$ lies in the essential image of $0_!$ if and only if X is an isomorphism.

(ii) $(\mathrm{id}_{[1]} \times 0)_!\colon \mathscr{D}(\ulcorner) \to \mathscr{D}(\square)$ is fully faithful and induces an equivalence onto the full subcategory of $\mathscr{D}(\square)$ such that both vertical morphisms are isomorphisms.

(iii) Let $X \in \mathscr{D}(\square)$*, and let* $X_{(0,0)} \xrightarrow{cong} X_{(0,1)}$ *be an isomorphism. Then* X *is cocartesian if and only if* $X_{(1,0)} \xrightarrow{cong} X_{(1,1)}$ *is an isomorphism.*

Proof. (i) The inclusion of the initial object $0\colon \mathbb{1} \to [1]$ induces a fully faithful functor $0_!\colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}([1])$ by lemma 3.14. Denote by X_0 the category in $\mathscr{D}(\mathbb{1})$, and X_1 the terminal category in $\mathscr{D}([1])$. Then $0^*(X) = X_0$, and by lemma 4.6 $X_1 \cong (0_! \circ 0^*)(X)_1 \cong X_0$ since X lies in the essential image of $0_!$.

(ii) Fully faithfulness follows from the fact that $(\mathrm{id}_{[1]} \times 0)$ is a sieve, and lemma 3.14. The isomorphism is just (i) applied to the shifted derivator $\mathscr{D}^{[\mathbf{1}]}$.

(iii) Since $(\mathrm{id}_{[1]} \times 0)$ factors as $i_{\ulcorner} \circ i\colon [1] \to \ulcorner \to \square$, we get an induced isomorphism of adjoints

$$(\mathrm{id}_{[1]} \times 0)_! \cong i_{\ulcorner !} \circ i_!$$

By (i), we know X lies in the essential image of $(i_1)_!$. By (ii), both morphisms are isomorphisms if and only if X lies in the essential image of $(\mathrm{id}_{[1]} \times 0)_!$. This is if and only if it lies in the essential image of $(i_{\ulcorner})_!$, which is the definition X is being cocartesian.

□

The first thing to notice here is that lemma 4.19 has obvious dual versions. In addition to this, in the proof of lemma 4.19 (i) we could have instead taken any category $\mathbf{J}$ with an initial object i. Then in the essential image of $i_!$, this would induce an isomorphism $X_j \cong X_i$, for all $j \in \mathbf{J}$. In particular, the initial object $X_i \cong \lim_{\mathbf{J}} X$, which gives the corollaries.

Corollary 4.20. *Let* $u\colon \mathbf{J} \to \mathbf{K}$ *be a functor between small categories.*

(i) In the slice category $(k \downarrow u)$*, the initial object* k *induces an isomorphism* $X_k \cong \lim_{(\mathbf{k}\downarrow\mathbf{u})} X$*, for* $X \in \mathscr{D}((k \downarrow u))$*.*

(ii) In the slice category $(u \downarrow k)$*, the terminal object* k *induces an isomorphism* $X_k \cong \operatorname{colim}_{(\mathbf{u}\downarrow\mathbf{k})} X$*, for* $X \in \mathscr{D}((u \downarrow k))$*.*

Lemma 4.21. *Let* $\mathbf{J}, \mathbf{J}', \mathbf{K}$ *and* $\mathbf{K}'$ *be small categories,* $u\colon \mathbf{J} \to \mathbf{J}'$ *and* $v\colon \mathbf{K} \to \mathbf{K}'$ *functors between them, and consider the square*

$$\begin{tikzcd}
\mathbf{K} \times \mathbf{J} \arrow[r, "v \times \mathrm{id}_{\mathbf{J}}"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times u"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'} \times u"] \arrow[dl, Rightarrow, shorten=15pt, "\mathrm{id}_{\mathbf{K}' \times \mathbf{J}'}"'] \\
\mathbf{K} \times \mathbf{J}' \arrow[r, "v \times \mathrm{id}_{\mathbf{J}'}"'] & \mathbf{K}' \times \mathbf{J}'
\end{tikzcd} .$$

Then the two mate transformations

$$\phi\colon (\mathrm{id}_{\mathbf{K}} \times u)_! \circ (v \times \mathrm{id}_{\mathbf{J}})^* \to (v \times \mathrm{id}_{\mathbf{J}'})^* \circ (\mathrm{id}_{\mathbf{K}'} \times u)_!$$

$$\theta\colon (v \times \mathrm{id}_{\mathbf{J}'})^* \circ (\mathrm{id}_{\mathbf{K}'} \times u)_* \to (\mathrm{id}_{\mathbf{K}} \times u)_* \circ (v \times \mathrm{id}_{\mathbf{J}})^*$$

are isomorphisms.

Proof. We only prove that ϕ is an isomorphism, as the proof for θ is dual. The proof will be done in three steps. First, we reduce to the case where $u = \pi_{\mathbf{J}}\colon \mathbf{J} \to \mathbb{1}$. Second, we reduce yet again to the case of evaluation functors as opposed to restriction functors. Third, we prove the result for the special case.

Step one. Consider the following pasting

$$\begin{tikzcd}
\mathbf{K} \times (u \downarrow j') \arrow[r, "\mathrm{id}_{\mathbf{K}} \times \rho_{j'}"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times \pi"'] & \mathbf{K} \times \mathbf{J} \arrow[r, "v \times \mathrm{id}_{\mathbf{J}}"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times u"'] \arrow[dl, Rightarrow, shorten=15pt, "\mathrm{id}_{\mathbf{K}} \times f"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'} \times u"] \arrow[dl, Rightarrow, shorten=15pt, "\mathrm{id}_{\mathbf{K}' \times \mathbf{J}'}"'] \\
\mathbf{K} \arrow[r, "\mathrm{id}_{\mathbf{K}} \times j'"'] & \mathbf{K} \times \mathbf{J}' \arrow[r, "v \times \mathrm{id}_{\mathbf{J}'}"'] & \mathbf{K}' \times \mathbf{J}'
\end{tikzcd}$$

Here we have 'glued' together a slice category to the left of our original diagram. Then by theorem 3.20 the square to the left in the diagram induces a slice square in a shifted derivator, hence the induced mate of that diagram will be an isomorphism. Thus, by lemma 3.13, if we can show that the the induced mate of the outer square is an isomorphism, then the induced mate of the rightmost square is an isomorphism. To this end, we introduce another pasting

$$\begin{tikzcd}
\mathbf{K} \times (u \downarrow j') \arrow[r, "v \times \mathrm{id}_{(u \downarrow j')}"] \arrow[d, "\mathrm{id}_{\mathbf{K}} \times \pi"'] & \mathbf{K}' \times (u \downarrow j') \arrow[r, "\mathrm{id}_{\mathbf{K}'} \times \rho_{j'}"] \arrow[d, "\mathrm{id}_{\mathbf{K}'} \times \pi"'] \arrow[dl, Rightarrow, shorten=15pt, "\mathrm{id}_{\mathbf{K}'}"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'} \times u"] \arrow[dl, Rightarrow, shorten=15pt, "\mathrm{id}_{\mathbf{K}'} \times f"'] \\
\mathbf{K} \arrow[r, "v"'] & \mathbf{K}' \arrow[r, "\mathrm{id}_{\mathbf{K}'} \times j'"'] & \mathbf{K}' \times \mathbf{J}'
\end{tikzcd}$$

Tracing the two diagrams, we see that the outer squares agree. This means that we need to show that the induced mate of the outer square of this diagram is an isomorphism. Again, by theorem 3.20 and lemma 3.13, since the right square is a shifted slice square, this reduces to show that the the induced mate of the left diagram is an isomorphism. This is precisely the case of $u = \pi_{\mathbf{J}}$.

Step two. Consider the following pasting

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "(k{,}\mathrm{id})\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] & (\mathrm{id}_{\mathbf{K}} \downarrow k) \times \mathbf{J} \arrow[r, "\rho_k\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] \arrow[dl, Rightarrow, shorten=12pt, "\mathrm{id}"'] & \mathbf{K} \times \mathbf{J} \arrow[r, "v\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\mathrm{id}_{\mathbf{K}}\times\pi"'] \arrow[dl, Rightarrow, shorten=12pt, "f"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'}\times\pi"] \arrow[dl, Rightarrow, shorten=12pt, "\mathrm{id}"'] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \arrow[r, "k"'] & \mathbf{K} \arrow[r, "v"'] & \mathbf{K}'
\end{tikzcd}$$

The middle square is a slice square, hence the induced mate is an isomorphism. The leftmost square is induced by the functor that maps to the terminal object $k \in (\mathrm{id}_{\mathbf{K}} \downarrow k)$. Since the terminal object in a slice category is a right adjoint (corollary 4.20), it follows from lemma 3.15 that the mate transformation in this square is an isomorphism as well. Hence, lemma 3.13 implies that the rightmost square induces an isomorphism if and only if the outer square does it. The outer square can also be described as the following diagram

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "v(k)\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'}\times\pi"] \arrow[dl, Rightarrow, shorten=12pt, "\mathrm{id}_{\mathbf{K}'}"'] \\
\mathbb{1} \arrow[r, "v(k)"'] & \mathbf{K}'
\end{tikzcd}$$

Which, when we pass to derivators, is just the case of evaluation functors.

Step three. Finally, let us consider the square

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "k'\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'}\times\pi"] \arrow[dl, Rightarrow, shorten=12pt, "\mathrm{id}_{\mathbf{K}'}"'] \\
\mathbb{1} \arrow[r, "k'"'] & \mathbf{K}'
\end{tikzcd}$$

If we can show that the induced mate is an isomorphism, then we are done. As with the other two cases, we consider a pasting in which case the individual squares induce mates that are easier to compute. The square above can be extended to the following pasting

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "(k',\mathrm{id}_{\mathbf{K}'})\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] & (\mathrm{id}_{\mathbf{K}'} \downarrow k') \times \mathbf{J} \arrow[r, "\rho_{k'}\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "\pi"'] \arrow[dl, Rightarrow, shorten=10pt, "\mathrm{id}"'] & \mathbf{K}' \times \mathbf{J} \arrow[d, "\mathrm{id}_{\mathbf{K}'}\times\pi"] \arrow[dl, Rightarrow, shorten=10pt, "f"'] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \arrow[r, "k'"'] & \mathbf{K}'
\end{tikzcd}$$

Here we see that the square to the right becomes a slice square in a shifted derivator, hence the induced mate is an isomorphism. Similarly to step two, the square to the left is induced by the functor k', which maps to the terminal object of a slice category, hence is a right adjoint. Then by lemma 3.15 this square induces an isomorphism as well. Then the result follows from lemma 3.13. □

Now we use this result to gain some insight in the essential image of the adjoints to the shifted functors $(\mathrm{id}_{\mathbf{K}} \times u)$.

Lemma 4.22. *Let* $\mathscr{D}$ *be a derivator,* $\mathbf{J}, \mathbf{J}', \mathbf{K} \in \mathbf{Cat}$*, and* $u\colon \mathbf{J} \to \mathbf{J}'$ *a fully faithful functor.*

(i) $X \in \mathscr{D}(\mathbf{K}\times\mathbf{J}')$ *lies in the essential image of* $(\mathrm{id}_{\mathbf{K}} \times u)_!$ *if and only if* $X_k \in \mathscr{D}(J')$ *is in the essential image of* $u_!$*, for all* $k \in K$*.*

(ii) $X \in \mathscr{D}(\mathbf{K}\times\mathbf{J}')$ *lies in the essential image of* $(\mathrm{id}_{\mathbf{K}} \times u)_*$ *if and only if* $X_k \in \mathscr{D}(J')$ *is in the essential image of* u_**, for all* $k \in K$*.*

Proof. We prove the first case, the second is dual.

Consider the square

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "k\times\mathrm{id}_{\mathbf{J}}"] \arrow[d, "u"'] & \mathbf{K}\times\mathbf{J} \arrow[r, "\mathrm{id}_{\mathbf{K}}\times u"] \arrow[d, "\mathrm{id}_{\mathbf{K}}\times u"'] \arrow[dl, Rightarrow, shorten=10pt, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}'}"'] & \mathbf{K}\times\mathbf{J}' \arrow[d, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}}"] \arrow[dl, Rightarrow, shorten=10pt, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}'}"'] \\
\mathbf{J}' \arrow[r, "k\times\mathrm{id}_{\mathbf{J}'}"'] & \mathbf{K}\times\mathbf{J}' \arrow[r, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}}"'] & \mathbf{K}\times\mathbf{J}'
\end{tikzcd}$$

Since u is fully faithful, so is $(\mathrm{id}_{\mathbf{K}} \times u)$. Then by lemma 4.6 we know that X lies in the essential image of $(\mathrm{id}_{\mathbf{K}} \times u)_!$ if and only if the counit $\epsilon\colon (\mathrm{id}_{\mathbf{K}} \times u)_! \circ (\mathrm{id}_{\mathbf{K}} \times u)^*(X)_{k\times j'} \to$

$X_{k\times j'}$ is an isomorphism on X, for all $k \times j' \in \mathbf{K} \times \mathbf{J}' - (\mathrm{id}_{\mathbf{K}} \times u)(\mathbf{K} \times \mathbf{J})$, which is the same as for all $j' \in \mathbf{J}' - u(\mathbf{J})$. Note that the induced mate transformation of the square to the right is precisely this counit. The square to the left also induces an isomorphism by lemma 4.21. Thus, by lemma 3.13 the induced mate of the entire diagram is an isomorphism. This diagram can also be described by

$$\begin{tikzcd}
\mathbf{J} \arrow[r, "u"] \arrow[d, "u"'] & \mathbf{J}' \arrow[r, "k\times \mathrm{id}_{\mathbf{J}'}"] \arrow[d, "\mathrm{id}_{\mathbf{J}'}"'] \arrow[dl, Rightarrow, shorten=4mm, "\mathrm{id}_{\mathbf{J}'}"'] & \mathbf{K}\times\mathbf{J}' \arrow[d, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}}"] \arrow[dl, Rightarrow, shorten=4mm, "\mathrm{id}_{\mathbf{K}\times\mathbf{J}'}"'] \\
\mathbf{J}' \arrow[r, "\mathrm{id}_{\mathbf{J}'}"'] & \mathbf{J}' \arrow[r, "k\times \mathrm{id}_{\mathbf{J}'}"'] & \mathbf{K}\times\mathbf{J}'
\end{tikzcd}$$

Hence, the outer diagram induces an isomorphism of the mate transformation. The right diagram also induces an isomorphism by lemma 4.21, which means that the induced mate transformation of the left diagram is an isomorphism by lemma 3.13. As with the first diagram, this transformation is simply the counit $\epsilon\colon u_! \circ u^* \to \mathrm{id}_{\mathbf{J}'}$. As u is fully faithful, lemma 4.6 together with axiom (Der ii), implies that this is the case if and only if X_k lies in the essential image of $u_!$ for all $k \in \mathbf{K}$. $\square$

Now finally, by applying lemma 4.22 above to the fully faithful functors $i_{\ulcorner}\colon \ulcorner \to \square$ and $i_{\lrcorner}\colon \lrcorner \to \square$ we immediately get the main result

Proposition 4.23. *let $\mathscr{D}$ be a derivator.*

(i) A square $X \in \mathscr{D}^{\mathbf{K}}(\square)$ is cocartesian if and only if $X_k \in \mathscr{D}(\square)$ is cocartesian for every $k \in \mathbf{K}$.

(ii) A square $X \in \mathscr{D}^{\mathbf{K}}(\square)$ is cartesian if and only if $X_k \in \mathscr{D}(\square)$ is cartesian for every $k \in \mathbf{K}$.

This proposition is a very important result. For instance, if we want to calculate the homotopy pushout of a complex in $\mathbf{D}(\mathcal{A}^{\mathbf{J}})$ we see that this is the homotopy pushout of $\mathbf{D}(\mathcal{A})$, evaluated in every $j \in \mathbf{J}$. This also specializes to the kernel and cokernel functor in proposition A.13 for $\mathscr{D}_{\mathrm{rep}}(-) = \mathcal{A}^{(-)}$.

So cocatersian and cartesian squares are interesting, as they generalize cokernels, kernels, pushouts, and pullbacks. In addition to this proposition 4.23 lets us calculate them pointwise, which is very convenient. As we did with initial and terminal objects for a pointed derivator, the next natural step is ask what happens when the cocartesian and cartesian squares coincide? This is what we will explore in the next chapter.

5 Stable derivators

In this chapter we introduce the notion of a *stable* derivator. Stable derivators are of interest as they unite cocartesian and cartesian squares. As we will see in the first section, this gives us a lot of control over our diagrams. In the second section we will show that the stable derivators gives rise to pre-additive categories, and in the last section that they are even additive. The group action on the set of morphisms is very technical, which is why we have dedicated an entire section to properly develop the necessary techniques.

5.1 Properties of stable derivators

In this section we begin by defining the stable derivator, and prove some easy properties. After that we give a classification result for stable derivators.

Definition 5.1. Let $\mathscr{D}$ be a derivator. We say $\mathscr{D}$ is *stable* if $\mathscr{D}$ is pointed, and a square $X \in \mathscr{D}(\square)$ is cartesian if and only if it is cocartesian. We call such squares *bicartesian.*

Since a stable derivator is by definition pointed, all of the results from the previous chapter apply. Similarly to the previous chapter, we begin by showing that stability is a sensible property.

Lemma 5.2. *Let $\mathscr{D}$ be a derivator. Then $\mathscr{D}$ is stable if and only if $\mathscr{D}^{op}$ is stable.*

Proof. We know that $\mathscr{D}$ is pointed if and only $\mathscr{D}^{\mathrm{op}}$ is pointed by lemma 4.4 (i), so we only have to consider cocartesian squares.

Let X be a cocartesian square in $\mathscr{D}(\square)$. The underlying diagram looks like

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & Cf
\end{tikzcd}$$

The corresponding X^{op} in $\mathscr{D}^{\mathrm{op}}$ looks like

$$\begin{tikzcd}
X^{\mathrm{op}} & Y^{\mathrm{op}} \arrow[l] \\
0 \arrow[u] & Cf^{\mathrm{op}} \arrow[l] \arrow[u]
\end{tikzcd}$$

and it follows from the definition of $\mathscr{D}^{\mathrm{op}}$ that X is cocartesian if and only if X^{op} is cartesian. Similarly, X is cartesian if and only if X^{op} is cocartesian. Hence $\mathscr{D}$ is stable if and only if $\mathscr{D}^{\mathrm{op}}$ is stable. □

Lemma 5.3. *Let $\mathscr{D}$ be a derivator and $\mathbf{K} \in \mathbf{Cat}$. Then $\mathscr{D}$ is stable if and only if $\mathscr{D}^{\mathbf{K}}$ is stable.*

Proof. Pointedness is taken care of by lemma 4.4 (ii), so we only have to show that cocartesian and cartesian squares coincide.

let X be a square in $\mathscr{D}^{\mathbf{K}}(\square)$. Then by proposition 4.23 (i) X is cocartesian if and only if X_k is cocartesian for all $k \in \mathbf{K}$. Since $\mathscr{D}$ is stable this is if and only if X_k is cartesian for all $k \in \mathbf{K}$. Then by proposition 4.23 (ii) this is if and only if X is cocartesian.

Consequently, if $\mathscr{D}^{\mathbf{K}}$ is stable for all $\mathbf{K} \in \mathbf{Cat}$, then certainly it is stable for $\mathscr{D}^{\mathbb{1}} \cong \mathscr{D}$. □

Example 5.4. The represented derivator $\mathscr{D}_{\mathrm{rep}}$ from example 3.2 is stable if and only if $\mathbf{C} = \mathbb{1}$. This follows from the fact that $\Sigma x \cong 0$ for all $x \in \mathbf{C}$, and since this is a pullback diagram as well as a pushout diagram, $x \cong 0$ for all 0.

Example 5.5. The homotopy derivator $\mathscr{D}_{\mathcal{A}}$ from example 3.3 is always stable. This follows from the fact that $\mathbf{D}(\mathcal{A})$ is a triangulated category for any abelian category $\mathcal{A}$, and so in particular for any category $\mathcal{A}^{\mathbf{J}}$ (proposition A.13). In triangulated categories, fiber and cofiber sequences are the same.

As we see from these examples, the stable property is more a homotopy property. Whenever we see stable derivator, we therefore think of $\mathscr{D}_{\mathcal{A}}$ as our standard example.

Now, consider a commutative diagram of the form

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[d] \\
x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

In classical category theory it is known that if the left square is a pushout diagram, the right square is a pushout diagram if and only if the outer square is a pushout diagram. The dual result is true if the right square is a pullback square. We want this handy calculus of squares in the abstract setting as well.

To this end, denote the following diagram

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1) \arrow[r] & (2,1)
\end{tikzcd}$$

by $\square\!\square$. Let us call the the left square $\square_l$, the right square $\square_r$ and the outer square $\square_o$

Proposition 5.6. *let $\mathscr{D}$ be a pointed derivator, and consider the diagram $\square\!\square$. If $\square_l$ is cocartesian, then $\square_r$ is cocartesian if and only if $\square_o$ is cocartesian.*

Proof. Consider the two full subcategories of $\square\!\square$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\
(0,1) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \\
(0,1) \arrow[r] & (1,1) &
\end{tikzcd}$$

which we will denote by D_1 and D_2, respectively. The inclusion $i\colon D_1 \to \square\!\square$ factors as two inclusions $i_1\colon D_1 \to D_2$ and $i_2\colon D_2 \to \square\!\square$ which is obtained by adding the two points $(1,1)$ and $(1,2)$, respectively. In particular all three inclusions are sieves. From lemma 4.6 we know that the essential image of i_* is characterized by diagrams where the unit $\eta\colon \mathrm{id}_{\mathscr{D}(\square\!\square)} \to (i_* \circ i^*)$ is an isomorphism. This unit factors as

$$\eta\colon \mathrm{id}_{\mathscr{D}(\square\!\square)} \to i_* \circ i^* = (i_2 \circ i_1)_* \circ (i_2 \circ i_1)^* \cong i_{2*} \circ i_{1*} \circ i_1^* \circ i_2^* \xrightarrow[\cong]{\bar{\eta}^{-1}} i_{2*} \circ \mathrm{id}_{\mathscr{D}(D_1)} \circ i_2^* \cong i_{2*} \circ i_2^*$$

where the middle isomorphism comes from the fact that $\square_l$ is an isomorphism. From this it is clear that $\eta\colon \mathrm{id}_{\mathscr{D}(\square\square)} \to (i_* \circ i^*)$ is an isomorphism precisely when $\eta\colon \mathrm{id}_{\mathscr{D}(\square\square)} \to (i_{2*} \circ i_2^*)$ is an isomorphism. □

As a direct consequence of this result, along with the dual result for cartesian squares, we have the following corollary

Corollary 5.7. *Let $\mathscr{D}$ be a stable derivator. If any two of the three squares $\square_l$, $\square_r$ or $\square_o$ are bicartesian then so is the third.*

As we saw in proposition 4.18 there is an adjunction between the loop and the suspension functors. However, when we are in the stable setting we can do a lot better. If we combine our results with the proof of the adjunctions in proposition 4.14 and proposition 4.18, we get the immediate result.

Corollary 5.8. *Let $\mathscr{D}$ be a stable derivator. The cofiber and fiber functors are equivalences on $\mathscr{D}([1])$, and the loop and suspension functors are equivalences on $\mathscr{D}(\mathbb{1})$.*

So in a stable derivator, we have good control over our squares. We end this section with a nice characterization of stable derivators.

Proposition 5.9. *Let $\mathscr{D}$ be a derivator. The following are equivalent*

(i) $\mathscr{D}$ is a stable derivator

(ii) Squares of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

are cartesian if and only if they are cocartesian

Proof. It is clear that (i) implies both (ii) from the adjunction of cofiber and fiber functors (proposition 4.14).

Conversely, consider the square

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\ z \arrow[r] & v \end{tikzcd}$$

and assume it is cartesian. The goal is to show that it is also cocartesian. Consider the inclusion $i_1\colon \square \to D_1$, where the diagram D_1 is obtained by adding the arrow $g\colon (-1,1) \to (0,1)$ into the lower left corner. Then we include this diagram into the double square $i_2\colon D_1 \to \square\square$. Note that both inclusions are cosieves, so by proposition 4.9 the resulting compositions $\mathscr{D}(\square) \xrightarrow{i_{2*}} \mathscr{D}(D_1) \xrightarrow{i_{1!}} \mathscr{D}(\square\square)$ has an underlying diagram that looks like the following

$$\begin{tikzcd} Ff \arrow[r, "\mathrm{fib}(f)"] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\ 0 \arrow[r] & z \arrow[r] & v \end{tikzcd}$$

Since the left square is also cartesian, it follows from the dual result of proposition 5.6 that the outer square is cartesian. By our assumption, squares of this form are cartesian if and only if they are cocartesian. Thus, the left and outer square are also cocartesian and so proposition 5.6 concludes our result. $\square$

As a special case we also see that in a stable derivator squares of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x' \end{tikzcd}$$

are cartesian if and only if they are cocartesian.

5.2 The pre-additivity of a stable derivator

In this section we are going to prove that a stable derivator $\mathscr{D}$ induces an pre-additive category $\mathscr{D}(\mathbf{J})$ for all small categories $\mathbf{J}$. To begin with we give some arguments for a way to characterize coproducts and products within $\mathscr{D}(\mathbf{J})$. Consider the pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & y \arrow[d] \\
x \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & z
\end{tikzcd}$$

in an ordinary cocomplete category $\mathbf{J}$. In this case we know that $z \cong x \amalg y$. We want the same to be true in a derivator setting. We therefore consider the functor $((1,0)(0,1))\colon \mathbb{1} \amalg \mathbb{1} \to \square$ that identifies the upper right and lower left corner of the square. This functor factors in two ways

$$\mathbb{1} \amalg \mathbb{1} \xrightarrow{i} \ulcorner \xrightarrow{i_\ulcorner} \square \qquad\qquad \mathbb{1} \amalg \mathbb{1} \xrightarrow{j} \lrcorner \xrightarrow{i_\lrcorner} \square$$

This is what motivates the following definition.

Definition 5.10. Let $\mathscr{D}$ be a derivator.

(i) A coherent cospan $X \in \mathscr{D}(\lrcorner)$ is a *coproduct cocone* if it lies in the essential image of $j_!\colon \mathscr{D}(\mathbb{1} \amalg \mathbb{1}) \to \mathscr{D}(\lrcorner)$

(ii) A coherent span $X \in \mathscr{D}(\ulcorner)$ is a *product cone* if it lies in the essential image of $i_*\colon \mathscr{D}(\mathbb{1} \amalg \mathbb{1}) \to \mathscr{D}(\ulcorner)$

These coproduct cocones and product cones behave in the way we want them to.

Lemma 5.11. *For every derivator $\mathscr{D}$ the category $\mathscr{D}(\mathbb{1} \amalg \mathbb{1})$ is equivalent to the full subcategory $\mathscr{D}(\square)^{copr} \subseteq \mathscr{D}(\square)$ spanned by the cocartesian squares X such that $X_{(0,0)}$ is an initial object and the restriction $i_\lrcorner^*(X) \in \mathscr{D}(\lrcorner)$ is a coproduct cocone.*

Proof. Since the functor $k = ((0,1),(1,0))\colon \mathbb{1} \amalg \mathbb{1} \to \square$ is fully faithful, so is $k_!$ by lemma 3.14. By the factorization of k above, we get induced natural isomorphisms

$$k_! \cong (i_\ulcorner)_! \circ i_! \cong (i_\lrcorner)_! \circ j_!$$

which induce equivalences onto the essential image. We will use the two ways to describe the essential image of $k_!$ to deduce our result. Consider first the composition

$(i_{\ulcorner})_! \circ i_!$. By lemma 4.19(i) and (iii) we know that X lies in the essential image of k if and only if X is cocartesian and $X_{(0,0)}$ is isomorphic to the initial object. And using the factorization $(i_{\lrcorner})_! \circ j_!$ along with the dual of lemma 4.19(i) and (iii) implies that the essential image of k is precisely those cocartesian X such that $X_{(0,0)}$ is isomorphic to an initial object, and such that $(i_{\lrcorner})^*(X)$ are coproduct cocones. □

Now we are ready to prove this sections main result

Theorem 5.12. *Let* $\mathscr{D}$ *be a stable derivator, and* $\mathbf{J} \in \mathbf{Cat}$*. The category* $\mathscr{D}(\mathbf{J})$ *is pre-additive.*

Proof. It is enough to show that the underlying category $\mathscr{D}(\mathbb{1})$ is pre-additive, by lemma 5.3. We will go through the steps of pre-additive categories as described in section 2.1 of [5].

(Add 1) By definition the stable derivator is pointed, and hence $\mathscr{D}(\mathbb{1})$ has a zero element.

(Add 2) It follows from theorem 3.17 that $\mathscr{D}(\mathbb{1})$ has both products and coproducts.

(Add 3) Finally, we show that the products and coproducts coincide. Or said with other words, we want the induced map

$$x \amalg y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} x \Pi y$$

to be an isomorphsim. By (Der i) we have that $\mathscr{D}(\mathbb{1}) \Pi \mathscr{D}(\mathbb{1}) \cong \mathscr{D}(\mathbb{1} \amalg \mathbb{1})$, so for two objects $(x, y) \in \mathscr{D}(\mathbb{1}) \Pi \mathscr{D}(\mathbb{1})$ we identify them with objects of $\mathscr{D}(\mathbb{1} \amalg \mathbb{1})$.

For two objects $(x, y) \in \mathscr{D}(\mathbb{1} \amalg \mathbb{1})$, we are going to construct a coherent diagram of shape $[2] \times [2]$. Consider the following diagram

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
(0,1) \arrow[r] \arrow[d] & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] \\
(0,2) \arrow[r] & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

which we denote by Q, and let $D_1 \subseteq Q$ be the full subcategory consisting of the objects $(0,2), (1,2), (2,0)$ and $(2,1)$. We define $i_1\colon \mathbb{1} \amalg \mathbb{1} \to D_1$ as the inclusion which identifies the two points $(1,2)$ and $(2,1)$. Note that this is a cosieve. Then we define $i_2\colon D_1 \to D_2$ as the inclusion into the full subcategory $D_2 \subseteq Q$ defined by adding the cornerpoint $(2,2)$

$$\begin{tikzcd}
 & (2,0) \arrow[d] & & & & (2,0) \arrow[d] \\
 & (2,1) & \arrow[r, "i_2"] & & & (2,1) \arrow[d] \\
(0,2) \arrow[r] & (1,2) & & (0,2) \arrow[r] & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

Note that this a sieve. Finally, we let $i_3\colon D_2 \to Q$ be the inclusion into the full diagram Q. This last inclusion is done step-wise ($i_3 = k_4 \circ k_3 \circ k_2 \circ k_1$) by adding points one by one backwards. Similarly to the proof of proposition 5.6 this amounts to adding a new cartesian square. These four inclusions induce fully faithful Kan extensions

$$\mathscr{D}(\mathbb{1} \amalg \mathbb{1}) \xrightarrow{i_{1!}} \mathscr{D}(D_1) \xrightarrow{i_{2*}} \mathscr{D}(D_2) \xrightarrow{k_*} \mathscr{D}(Q)$$

If we denote $\mathscr{D}(Q)^{\mathsf{ex}}$ the full subcategory spanned by all diagrams which is cartesian in all four squares, and vanishes at the corner points $(0,2), (2,0)$ and $(2,2)$, then by proposition 4.9 there is an equivalence of categories $\mathscr{D}(\mathbb{1} \amalg \mathbb{1}) \cong \mathscr{D}(Q)^{\mathsf{ex}}$. Furthermore by proposition 5.6 and proposition 5.9 the four squares are bicartesian. Hence, for objects $(x, y) \in \mathscr{D}(\mathbb{1} \amalg \mathbb{1})$ the induced underlying diagram in $\mathscr{D}(Q)^{\mathbf{ex}}$ looks like

$$\begin{tikzcd}
z \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & x' \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & 0 \arrow[d] \\
y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{3}"] & b \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{4}"] & y \arrow[d] \\
0 \arrow[r] & x \arrow[r] & 0
\end{tikzcd}$$

for some objects x', y', z and $b \in \mathscr{D}(\mathbb{1})$. Corollary 5.7 implies that any combination of squares are also bicartesian.

Let us compare some of the squares. If we consider the entire outer square, then lemma 4.19 (iii) implies that $z \cong 0$. Similarly, inspecting the square $2 + 4$ implies that $x' \cong x$, and the square $3 + 4$ implies that $y' \cong y$. Now we apply lemma 5.11 to square 1 and its dual to square 4, which implies that the object in the middle, b, is both the product and the coproduct of x and y.

□

5.3 The additivity of a stable derivator

In this section we want to show that stable derivators induce more than just preadditive categories. They are in fact additive. As mentioned, a lot of the theory is based on the writings of Mortiz Rahn [5]. The fact that stable derivators induce additive categories is no different, however *Introduction to the theory of derivators* does not give a very satisfying explanation. Instead we will focus on the theory from the related article *Derivators, pointed derivators and stable derivators* [4]. This is a very thorough treatment of the additivity, and provides good insight to stable derivators.

So the main goal is to show that for all $x, y \in \mathscr{D}(\mathbf{J})$, where $\mathbf{J}$ is a small category, the morphism set $\mathscr{D}(\mathbf{J})(x, y)$ is actually an abelian group. Given two morphisms

$f, g\colon x \to y$, we define $f+g$ in the usual way

$$f+g\colon x \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} x \oplus x \xrightarrow{\begin{pmatrix}f & 0\\0 & g\end{pmatrix}} y \oplus y \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} y$$

Let us justify the use of this operation on morphisms

Lemma 5.13. *The operation defined on morphisms turns the morphism set of $\mathscr{D}(\mathbf{J})$ into a commutative monoid.*

Proof. To see that the $(\mathscr{D}(\mathbf{J})(x,y),+)$ is commutative, it is enough to verify that the following diagram commutes

$$\begin{tikzcd}
x \arrow[r, "\begin{pmatrix}1\\1\end{pmatrix}"] \arrow[d, equal] & x \oplus x \arrow[r, "\begin{pmatrix}f & 0\\0 & g\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}"] & y \oplus y \arrow[r, "\begin{pmatrix}1 & 1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}"] & y \arrow[d, equal] \\
x \arrow[r, "\begin{pmatrix}1\\1\end{pmatrix}"'] & x \oplus x \arrow[r, "\begin{pmatrix}g & 0\\0 & f\end{pmatrix}"'] & y \oplus y \arrow[r, "\begin{pmatrix}1 & 1\end{pmatrix}"'] & y
\end{tikzcd}$$

Similarly to check that it is associative, we only need to verify that the following diagram commutes

$$\begin{tikzcd}
x \arrow[r, "{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}"] \arrow[d, equal] & x \oplus x \arrow[r, "{\begin{pmatrix} f+g & 0 \\ 0 & h \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}"'] & y \oplus y \arrow[r, "{\begin{pmatrix} 1 & 1 \end{pmatrix}}"] & y \arrow[d, equal] \\
x \arrow[r, "{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}"] \arrow[d, equal] & x \oplus x \oplus x \arrow[r, "{\begin{pmatrix} f & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & h \end{pmatrix}}"] & y \oplus y \oplus y \arrow[r, "{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}"] \arrow[u, "{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}"'] \arrow[d, "{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}"'] & y \arrow[d, equal] \\
x \arrow[r, "{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}"'] & x \oplus x \arrow[r, "{\begin{pmatrix} f & 0 \\ 0 & g+h \end{pmatrix}}"'] \arrow[u, "{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}}"] & y \oplus y \arrow[r, "{\begin{pmatrix} 1 & 1 \end{pmatrix}}"'] & y
\end{tikzcd}$$

□

In light of this result, all we really need to show is that the +-operation also has an inverse. This is where things gets complicated. We begin with a discussion on the loop space Ωx.

Consider $n+1$ elements $\{e_0, e_1, \cdots, e_n\}$ and some other element t. We define the poset $\lrcorner_n$ as the set consisting of the above elements generated by the relations $e_i \leq t$. So $\lrcorner_1$ and $\lrcorner_2$ can be illustrated as the following two diagrams

$$\begin{tikzcd}
 & e_1 \arrow[d] & & e_1 \arrow[dr] & e_2 \arrow[d] \\
e_0 \arrow[r] & t & & e_0 \arrow[r] & t
\end{tikzcd}$$

We write $\langle n \rangle$ for the set $\{0, 1, \cdots, n\}$. Now any set-theoretic map $f\colon \langle n \rangle \to \langle m \rangle$ induces a functor from **Set** to **Cat**, $f_\lrcorner\colon \lrcorner_n \to \lrcorner_m$ by setting $f_\lrcorner(e_i) = e_{f(i)}$ and $f_\lrcorner(t) = t$.

So for instance $f\colon \langle 2\rangle \to \langle 1\rangle$ defined by $f(0) = f(1) = 1$ and $f(2) = 0$ induces the functor in the following picture

$$\begin{tikzcd}
e_1 \arrow[rd] & e_2 \arrow[d] & & & & e_0 \arrow[d] \\
e_0 \arrow[r] & t & {} \arrow[r, "f_\lrcorner"] & {} & e_1 \arrow[r] & t
\end{tikzcd}$$

Furthermore, let t denote the functor $t\colon \mathbb{1} \to \lrcorner_n$ that identifies the terminal object t. Note that this is a cosieve, so the induced functor $t_!\colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\lrcorner_n)$ is an extension by zero. We define P_n as the composite functor

$$P_n\colon \mathscr{D}(\mathbb{1}) \xrightarrow{t_!} \mathscr{D}(\lrcorner_n) \xrightarrow{\pi_*} \mathscr{D}(\mathbb{1})$$

where we take the homotopy limit of the resulting diagram. Note that this gives an isomorphism $P_1x \cong \Omega x$

$$\begin{tikzcd}
P_1x \arrow[r] \arrow[d] & 0 \arrow[d] & & \Omega x \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & x \arrow[r, phantom, ""] & {} & 0 \arrow[r] & x
\end{tikzcd}
\quad \xrightarrow{\cong} \quad$$

The nice thing about this construction is that it extends to a functor which lets us keep track of the homotopy limits.

Lemma 5.14. *Let $\mathscr{D}$ be a stable derivator, and let* **Fin** *be the category of finite sets. Then there is a well-defined functor*

$$P\colon \mathbf{Fin}^{op} \times \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1}), \quad (\langle n\rangle, x) \mapsto P_nx$$

Proof. The mapping in the second degree, which simply sends x to itself is clearly a well-defined functor. It remains to see that the first degree is also well behaved.

Consider a morphism $f\colon \langle n\rangle \to \langle m\rangle$. By the above discussion, we get the following diagram along with the induced diagram of derivators

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathrm{id}"] \arrow[d, "t"'] & \mathbb{1} \arrow[d, "t"] \\
\lrcorner_n \arrow[r, "f_\lrcorner"] \arrow[d, "\pi"'] & \lrcorner_m \arrow[d, "\pi"] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{D}(\mathbb{1}) \arrow[d, "t_!"'] \arrow[dr, Rightarrow, shorten=4mm, "\phi"] & \mathscr{D}(\mathbb{1}) \arrow[l, "\mathrm{id}"'] \arrow[d, "t_!"] \\
\mathscr{D}(\lrcorner_n) \arrow[d, "\pi_*"'] & \mathscr{D}(\lrcorner_m) \arrow[l, "f_\lrcorner^*"'] \arrow[d, "\pi_*"] \arrow[dl, phantom, ""] \\
\mathscr{D}(\mathbb{1}) & \mathscr{D}(\mathbb{1}) \arrow[l, "\mathrm{id}"] \arrow[from=3-2, to=2-1, Rightarrow, shorten=4mm, "\theta"']
\end{tikzcd}$$

where ϕ and θ are the induced mate transformations. Since t is a right adjoint to $\pi_{\lrcorner_n}$ it follows from lemma 3.15 that ϕ is an isomorphism, and hence invertible. We can therefore define P_f as

$$P_f\colon P_m = \pi_* \circ t_! \xrightarrow{\theta} \pi_* \circ f_\lrcorner \circ t_! \xrightarrow{\phi^{-1}} \pi_* \circ t_! = P_n$$

and then functorality follows from the properties of mates and pasting of mates. □

Let us see how we can relate this to the loop space. If we consider the $n+1$ elements in $\langle n\rangle$, for $n \geq 1$, then we can pick out any two elements $\{k-1, k\}$. Let $(k-1,k)$ denote the morphism $\langle 1\rangle \to \langle n\rangle$ that maps $\{0,1\} \in \langle 1\rangle$ to the two elements

$$\{0,1\} \xrightarrow{i=(3,4)} \{0,1,2,i(0),i(1),5,\cdots,n\}$$

So by lemma 5.14 above, we get the existence of functors

$$(k-1,k)^* = P((k-1,k),\mathrm{id}_x)\colon P_n x \longrightarrow P_1 x \cong \Omega x$$

Now we show that these functors induce important isomorphisms for the loop space.

Proposition 5.15. *Let $\mathscr{D}$ be a stable derivator and $x \in \mathscr{D}(\mathbb{1})$. For any $n \geq 1$ and $1 \leq k \leq n$, the functors $(k-1,k)^*$ define a natural isomorphism in $\mathscr{D}(\mathbb{1})$*

$$s_n\colon P_n x \xrightarrow{\cong} \prod_{i=1}^{n} P_1 x \xrightarrow{\cong} \prod_{i=1}^{n} \Omega x$$

Proof. We will prove this by induction. For the case where n is equal to one, s_1 becomes the identity and so this is trivially true.

Assume the statement holds true for some $n > 1$. Define D as the poset obtained from $\lrcorner_n$ by adding ω_i such that $\omega_i \leq e_i, e_n$ for all $0 \leq i \leq n-1$. So the picture for $n = 2$ looks like

$$\begin{tikzcd}
 & \omega_1 \arrow[d] \arrow[dr] & \\
\omega_0 \arrow[r] \arrow[dr] & e_2 \arrow[dr] & e_1 \arrow[d] \\
 & e_0 \arrow[r] & t
\end{tikzcd}$$

Let us denote $j\colon \lrcorner_n \to D$ as the inclusion. Note that e_n has the same relation for each ω_i as t has for each e_i, so there is actually an isomorphism $D \cong [1]\times\lrcorner_{n-1}$. This means that we can consider the adjunction $((\delta^1 \times \mathrm{id}), (\sigma^0 \times \mathrm{id}))\colon \lrcorner_{\mathbf{n-1}} \leftrightarrows [\mathbf{1}]\times\lrcorner_{\mathbf{n-1}}$, where δ and σ are the standard coface and codegeneracy maps, as really an adjunction $(L, R)\colon \lrcorner_{n-1} \leftrightarrows D$. This means that the restriction functor L^* is a right adjoint. Consider the diagram induced by the functors

$$\mathscr{D}(\mathbb{1}) \xrightarrow{t_!} \mathscr{D}(\lrcorner_n) \xrightarrow{j_*} \mathscr{D}(D) \xrightarrow{L^*} \mathscr{D}(\lrcorner_{n-1}) \xrightarrow{\pi_*} \mathscr{D}(\mathbb{1}) \tag{4}$$

The first functor is the same as for P_n, while the next two functors are right adjoints. Comparing the diagram

$$\begin{tikzcd}
\lrcorner_{n-1} \arrow[r, "(j^*\circ L)"] \arrow[d, "\pi"'] & \lrcorner_n \arrow[d, "\pi"] \\
\mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1}
\end{tikzcd}$$

to lemma 3.15 where j^* is the restriction to $\lrcorner_n$, gives a natural isomorphism between P_n and the diagram induced by the functors (4). By proposition 4.9 and proposition 5.9 it follows that the diagram induced by the first two functors is a diagram which is isomorphic to 0 at each e_i and Ωx at each ω_i. Now L^* restricts to the diagram consisting of just the Ωx's and one terminal object 0 located at the position of e_n. Now, the result follows from the induction hypothesis combined with lemma 5.11. □

Let us illustrate what this construction looks like for $n = 2$

$$x \xrightarrow{t_!} \begin{tikzcd} 0 \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & x \end{tikzcd} \xrightarrow{j_*} \begin{tikzcd} & \Omega x \arrow[d] \arrow[dr] & \\ \Omega x \arrow[r] \arrow[dr] & 0 \arrow[dr] & 0 \arrow[d] \\ & 0 \arrow[r] & x \end{tikzcd}$$

$$\xrightarrow{L^*} \begin{tikzcd} & \Omega x \arrow[d] \\ \Omega x \arrow[r] & 0 \end{tikzcd} \xrightarrow{\pi_*} \Omega x \oplus \Omega x$$

So now that we have a nice functorial construction for direct sums of loop objects, we can define an operation $\star$ on $\Omega x \oplus \Omega x$. Motivated by topology, this is called the *concatenation* map. By proposition 5.15 we can invert the maps s_n, so we define $\star$ by the composition

$$\star \colon \Omega x \oplus \Omega x \xleftarrow{\cong} P_2 x \xrightarrow{(0,2)^*} P_1 x \cong \Omega x$$

This mapping will be used to prove that $\mathscr{D}(\mathbf{J})(x, y)$ is indeed an abelian group. Before we do that, we need to assure ourselves that this is an associative mapping.

Lemma 5.16. *let $\mathscr{D}$ be a stabel derivator, and $x \in \mathscr{D}(\mathbb{1})$. The concatenation map is associative.*

Proof. Similarly to the what we did when we showed that the +-operation was associative, this will be showed through a diagram. Let $y \in \mathscr{D}(\mathbb{1})$ be some other object and consider three maps $f, g, h\colon y \to \Omega x$. In the following diagram

$$\begin{tikzcd}
& \Omega x & & P_2x & \\
y \arrow[ur, "f\star(g\star h)"] \arrow[rr, "{f,g\star h}"] \arrow[dr, "{f,g,h}"'] & & \Omega x \bigoplus \Omega x & & P_3x \arrow[ullll, bend right, "{(0,3)^*}"'] \arrow[ul, "(013)^*"'] \arrow[dl, "s"] \arrow[dllll, bend left, "s_3"] \\
& \Omega x \bigoplus \Omega x \bigoplus \Omega x & & \Omega x \bigoplus P_2x \arrow[ul, "\mathrm{id}\times(0{,}2)^*"] \arrow[ll, bend right, "\mathrm{id}\times s_2"'] &
\arrow[from=1-4, to=1-2, bend left, "{(0,2)^*}"]
\arrow[from=1-4, to=2-3, "s_2"']
\end{tikzcd}$$

all maps labeled s are the isomorphisms from proposition 5.15. The two middle quadrilaterals commute as they are simply the definition of $\star$, and the quadrilateral to the right commutes by lemma 5.14. Define $m(f, g, h)\colon y \to P_3x$ as the unique map such that $s \circ m(f, g, h) = f, g, h$. Now we have a nice description of $f \star (g \star h)$ as the composite $(0, 3)^* \circ m(f, g, h)$. We could draw a similar diagram for $(f \star g) \star h$, and then uniqueness of $m(f, g, h)$, $(0, 3)^*$ and the Yoneda lemma together imply that $\star$ is associative. □

We now prove this chapters main theorem

Theorem 5.17. *Let $\mathscr{D}$ be a stable derivator, and $\mathbf{J} \in \mathbf{Cat}$. The category $\mathscr{D}(\mathbf{J})$ is additive.*

Proof. By the discussions of the section, all that is left to prove is that the +-operation has an inverse. The strategy is to relate the concatenation map to the additive map. Inspired by group theory, for $i, j \in \langle 2\rangle$, let $\tau_{i,j}$ denote the transposition of two elements. In particular, we consider τ the non-identity on $\langle 1\rangle$. Then the following diagram commutes

$$\begin{tikzcd}
P_2x \arrow[r, "\tau_{0,2}"] \arrow[d, "s_2"'] & p_2x \arrow[d, "s_2"] \\
\Omega x \bigoplus \Omega x \arrow[r, "\left(\begin{smallmatrix} 0 & \tau^* \\ \tau^* & 0 \end{smallmatrix}\right)"'] & \Omega X \bigoplus \Omega X
\end{tikzcd}$$

by the definition of s_2. In a similar way, we deduce the matrix corresponding to $\tau_{0,1}$. From the relation

$$\tau_{0,1} \circ (0,1) = (0,1) \circ \tau \colon \langle 1\rangle \to \langle 2\rangle$$

we gather that the induced morphism on $\Omega x \oplus \Omega x$ is similar to an inclusion on the first degree. That is, we have a lower triangular matrix as in the following diagram

$$\begin{tikzcd}
P_2x \arrow[r, "\tau_{0,1}"] \arrow[d, "s_2"'] & p_2x \arrow[d, "s_2"] \\
\Omega x \bigoplus \Omega x \arrow[r, "\left(\begin{smallmatrix} \tau^* & 0 \\ \alpha & \beta \end{smallmatrix}\right)"'] & \Omega X \bigoplus \Omega X
\end{tikzcd}$$

for some maps $\alpha, \beta \colon \Omega x \to \Omega x$. Since $\tau_{0,1}$ is a transposition, we have that $(\tau_{0,1})^2 = \mathrm{id}_{P_2x}$. The same is true for the corresponding matrix, which gives us

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tau^* & 0 \\ \alpha & \beta \end{pmatrix}^2 = \begin{pmatrix} \tau^* & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \tau^* & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha \circ \tau^* + \beta \circ \alpha & \beta^2 \end{pmatrix}$$

from which we get the two equations

$$\alpha \circ \tau^* + \beta \circ \alpha = 0, \quad \beta^2 = 1$$

The goal for the rest of the proof is now to show that these to morphisms, α and β, are identities. The two equations above will then imply that $\tau^* + \mathrm{id}_{\Omega x} = 0$, hence the identity has an additive inverse. This in turn will imply that for any morphism

f, we get that $f^{-1} = \tau^* \circ f$ is an additive inverse, and since Ω is an equivalence on $\mathscr{D}(\mathbb{1})$, we are done.

So for step one, we consider the relation $(0,2) = \tau_{0,1} \circ (1,2)$. This induces the equality $(0,2)^* = (1,2)^* \circ \tau_{0,1} \colon P_2x \to \Omega x$. So for any $f, g \colon y \to \Omega x$ we get

$$y \oplus y \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)} \Omega x \oplus \Omega x \xrightarrow{s_2} P_2x \xrightarrow{(1,2)^* \circ \left(\begin{smallmatrix} \tau^* & 0 \\ \alpha & \beta \end{smallmatrix}\right)} P_1x \xrightarrow{\cong} \Omega X$$

which by the definition of the concatenation map gives us $f \star g = \alpha \circ f + \beta \circ g$. By the associativity of the concatenation, we have the equation $0 \star (0 \star \mathrm{id}_{\Omega x}) = (0 \star 0) \star \mathrm{id}_{\Omega x}$. The left hand side equals β^2, which is the identity by our two equations. The right hand side equals β. Hence, we have one of our identities.

For the second step, we consider another relation of $(0,2)$. This time, we consider $(0,2) = \tau_{1,2} \circ (0,1)$. Since we have the equality $\tau_{1,2} = \tau_{0,2} \circ \tau_{0,1} \circ \tau_{0,2}$, matrix multiplication gives the matrix corresponding to $\tau_{1,2}$ as

$$\begin{pmatrix} \tau^* \circ \beta \circ \tau^* & \tau^* \circ \alpha \circ \tau^* \\ 0 & \tau^* \end{pmatrix} \colon \Omega x \oplus \Omega x \to \Omega x \oplus \Omega x$$

Using the same trick, we get an induced alternative description of the concatenation map as
$f \star g = \tau^* \circ \beta \circ \tau^* \circ f + \tau^* \circ \alpha \circ \tau^* \circ g$. Since these maps has to agree on f, we combine this with the equations above to get $\alpha = \tau^* \circ \beta \circ \tau^* = (\tau^*)^2 = \mathrm{id}_{\Omega x}$, completing the proof. $\square$

Given a functor between small categories $u \colon \mathbf{J} \to \mathbf{K}$, we know that $\mathscr{D}(\mathbf{J})$ and $\mathscr{D}(\mathbf{K})$ are additive categories. So the next question is what can we say about the induced functors $u^*, u_!$ and u_*?

Proposition 5.18. *Let $u \colon \mathbf{J} \to \mathbf{K}$ be a functor between small categories. The restriction functor u^* and the Kan extensions $u_!, u_*$ are additive functors.*

Proof. We already know that the functors preserve the zero object by lemma 4.4(iii). Now the result follows from the two adjunctions $(u_!, u^*) \colon \mathscr{D}(\mathbf{J}) \leftrightarrows \mathscr{D}(\mathbf{K})$ and $(u^*, u_*) \colon \mathscr{D}(\mathbf{J}) \leftrightarrows \mathscr{D}(\mathbf{K})$, and lemma 3.15 $\square$

Thus, we have shown that stable derivators induce additive categories. However, as the next section will show, we can say even more about the abstract derivators.

6 Canonical triangulations in triangulated derivators

The last chapter showed that given a stable derivator $\mathscr{D}$, and any small category $\mathbf{J}$, the category $\mathscr{D}(\mathbf{J})$ is an additive category. In this chapter we are going to show that we only need to impose one more property on the derivator to get a triangulated category. We begin with a recollection of triangulated categories, and then give a lengthy proof that the categories $\mathscr{D}(\mathbf{J})$ are triangulated for suitable derivators.

6.1 Triangulated categories

Triangulated categories have made a great impact on several areas of mathematics. They were introduced independently by Dold & Puppe, and by Verdier. The difference being that Verider also included the octahedron axiom. The motivation for triangulated categories was to axiomatize the structure of the derived category of an abelian category. One slogan for triangles appearing in a triangulated category is that they are 'shadows' of short exact sequences.

Definition 6.1. Let $\mathcal{T}$ be an additive category, with an additive auto-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$, and a class of morphisms $x \to y \to z \to \Sigma x$ called *distinguished triangles*, which we denote by $\triangle$. We denote by $x[n]$ the autoequivalence to some power $\Sigma^n x$, where n is an integer. The pair $(\Sigma, \triangle)$ defines a *triangulated* structure on $\mathcal{T}$ if the following axioms are satisfied.

(T1) (a) Any morphism $x \xrightarrow{f} y \in \mathcal{T}$ can be completed to a triangle $x \to y \to z \to x[1] \in \triangle$

(b) the trivial triangle $x \xrightarrow{\text{id}} x \to 0 \to x[1]$ lies in $\triangle$.

(c) The class of distinguished triangles $\triangle$ is closed under isomorphisms.

(T2) If $x \to y \to z \to x[1]$ lies in $\triangle$, then so does $z[-1] \to x \to y \to z$ and $y \to z \to x[1] \to y[1]$

(T3) Given the solid part of the following commutative diagram, where the rows lie in $\triangle$

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d, dotted, "h"] & x[1] \arrow[d] \\
x' \arrow[r] & y' \arrow[r] & z' \arrow[r] & x'[1]
\end{tikzcd}$$

we can always find a morphism h such that the diagram commutes.

(T4) Given the solid part of the following diagram, where the top rows and left column lie in $\triangle$

$$\begin{tikzcd}
x \arrow[r] \arrow[d, equal] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d, dotted] & x[1] \arrow[d, equal] \\
x \arrow[r] & u \arrow[r] \arrow[d] & v \arrow[r] \arrow[d, dotted] & x[1] \\
 & w \arrow[r, equal] \arrow[d] & w \arrow[d, dotted] & \\
 & y[1] \arrow[r] & z[1] &
\end{tikzcd}$$

the indicated morphisms exists such that the diagram commutes, and they define a triangle $z \to v \to w \to z[1] \in \triangle$.

We say the $\mathcal{T}$ is a *triangulated category* if it has a triangulated structure.

Remark. In the first axiom we can always find an object that completes a morphism to a triangle. That object is often called the 'cone' of the triangle.

The second axiom is often called the 'rotation' axiom. Applying this to the third axiom, lets us shift the dotted arrow to the position we want. This is why the third axiom is often called the 'two-out-of-three' axiom, since given two out of any three morphisms f, g, h we can always find the third.

Finally, the fourth axiom is called the 'octahedron' axiom, because if you think of the triangles as actual triangles, and fold them together you get an octahedron.

Let us consider an example

Example 6.2. Let $\mathcal{A}$ be an abelian category, then the homotopy category $K(\mathcal{A})$ is triangulated. The triangles are of the form $x \xrightarrow{f} y \to \operatorname{cone}(f) \to x[1]$, where $\operatorname{cone}(f)$ is the mapping cone.

As with groups, rings, and generally most things in mathematics, once we have introduced the concept there should also be morphisms of some kind that preserve structure. Triangulated categories are no different.

Definition 6.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories. An *exact* functor is an additive functor $F\colon \mathcal{T} \to \mathcal{T}'$ together with a natural isomorphism $\varepsilon_x\colon F(x[1]) \to F(x)[1]$ such that for any triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} x[1]$$

in $\triangle_{\mathcal{T}}$, the resulting triangle

$$F(x) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z) \xrightarrow{\varepsilon_x(F(h))} F(x)[1]$$

lies in $\triangle_{\mathcal{T}'}$

Example 6.4. Let $\mathcal{A}$ be an abelian category, then the derived category $D(\mathcal{A})$ is a triangulated category, and the localization functor $\gamma\colon K(\mathcal{A}) \to D(\mathcal{A})$ is an exact functor

Remark. This is actually a consequence of a more general result known as *Verdier localization* (see the proposition in section 3.5 [9] for more details).

We introduce some useful properties of triangulated categories

Lemma 6.5. *Let $x \to y \to z \to x[1]$ be a distinguished triangle in a triangulated category $\mathcal{T}$. For any $t \in \mathcal{T}$, we get an exact sequence*

$$\operatorname{Hom}_{\mathcal{T}}(t,x) \to \operatorname{Hom}_{\mathcal{T}}(t,y) \to \operatorname{Hom}_{\mathcal{T}}(t,z)$$

Proof. Let $g \in \operatorname{Hom}_{\mathcal{T}}(t, y)$. Comparing the standard triangle to the trivial triangle associated to t, yields the following diagram

$$\begin{tikzcd}
t \arrow[r, "\mathrm{id}"] \arrow[d, dotted, "f"] & t \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, dotted] & t[1] \\
x \arrow[r] & y \arrow[r] & z \arrow[r] & x[1]
\end{tikzcd}$$

By the rotation axiom, and the two-out-of-three axiom, the existence of f is equivalent to g factoring through the zero morphism such that everything commutes. In other words, the Hom-sequence is exact. □

Any functor H from a triangulated category $\mathcal{T}$ into an abelian category $\mathcal{A}$ such that the induced sequence $H(x) \to H(y) \to H(z)$ is exact is called *homological.* We will not use this general type of functors, but the above lemma is needed to show the following strong result.

Proposition 6.6. *Let $\phi\colon \mathcal{T} \to \mathcal{T}'$ be a functor between triangulated categories. If any two out of the three morphisms are isomorphisms, then so is the third.*

Proof. Consider the following diagram for two distinct triangles

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "\phi_1"'] & y \arrow[r] \arrow[d, "\phi_2"'] & z \arrow[r] \arrow[d, "\phi_3"'] & x[1] \arrow[d, "\phi_1[1]"] \\
x' \arrow[r] & y' \arrow[r] & z' \arrow[r] & x'[1]
\end{tikzcd}$$

By the two-out-of three axiom, it is enough to show that the statement is true for ϕ_2. So assume ϕ_1 and ϕ_3 are isomorphisms, and let $t \in \mathcal{T}$ be any other object. By lemma 6.5 we get an induced diagram of abelian groups

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{T}}(t, z[-1]) \arrow[r] \arrow[d, "\cong"', "\phi_3[-1]\circ(-)"] & \operatorname{Hom}_{\mathcal{T}}(t, x) \arrow[r] \arrow[d, "\cong"', "\phi_1\circ(-)"] & \operatorname{Hom}_{\mathcal{T}}(t, y) \arrow[r] \arrow[d, "\phi_2\circ(-)"] & \operatorname{Hom}_{\mathcal{T}}(t, z) \arrow[r] \arrow[d, "\cong"', "\phi_3\circ(-)"] & \operatorname{Hom}_{\mathcal{T}}(t, x[1]) \arrow[d, "\cong"', "\phi_1[1]\circ(-)"] \\
\operatorname{Hom}_{\mathcal{T}}(t, z'[-1]) \arrow[r] & \operatorname{Hom}_{\mathcal{T}}(t, x') \arrow[r] & \operatorname{Hom}_{\mathcal{T}}(t, y') \arrow[r] & \operatorname{Hom}_{\mathcal{T}}(t, z') \arrow[r] & \operatorname{Hom}_{\mathcal{T}}(t, x'[1])
\end{tikzcd}$$

from which the five-lemma for abelian groups implies that $\mathrm{Hom}_{\mathcal{T}}(t,y) \xrightarrow{\phi_2\circ(-)} \mathrm{Hom}_{\mathcal{T}}(t,y')$ is a natural isomorphism. Then Yoneda's lemma implies that ϕ_2 is an isomorphism. □

Let f be any morphism in a triangulated category, and consider C the cone of f. If we had any other candidate for the cone, say C', then the two-out-of-three axiom implies that there exists a morphism $h\colon C \to C'$. By the above results, we then get the immediate corollary.

Corollary 6.7. *In a triangulated category $\mathcal{T}$ the cone is unique up to isomorphism.*

One thing to notice here is that the induced isomorphism is *not* unique. That is, if we had $\psi_3 \neq \phi_3$ fit into the diagram such that it commutes, then ψ_3 is also an isomorphism.

Example 6.8. Consider the group homomorphism $\cdot 5\colon {}^{\mathbb{Z}}/_{2\mathbb{Z}} \to {}^{\mathbb{Z}}/_{5\mathbb{Z}}$ in $\mathrm{Mod}(\mathbb{Z})$. This gives rise to the complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & {}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r, "\mathrm{id}"] \arrow[d, "\cdot 5"] & {}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r] \arrow[d, "\cdot 5"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \arrow[r, "\mathrm{id}"] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in $\mathbf{K}(\mathrm{Mod}(\mathbb{Z}))$. Then we get induced isomorphisms between triangles of the form

$$\begin{tikzcd}
{}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r, "\cdot 5"] \arrow[d, "\mathrm{id}"] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \arrow[r, "\binom{1}{0}"] \arrow[d, "\mathrm{id}"] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix}1 & q\\ 0 & 1\end{smallmatrix}\right)"] & {}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[d, "\mathrm{id}"] \\
{}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r, "\cdot 5"] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \arrow[r, "\binom{1}{0}"] & {}^{\mathbb{Z}}/_{5\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{2\mathbb{Z}} \arrow[r, "(0\ 1)"] & {}^{\mathbb{Z}}/_{2\mathbb{Z}}
\end{tikzcd}$$

for any integer q.

In the example above we see that there are infinitely many choices for q, but they are all isomorphic (they are homotopic). Hence, there is no functorial cone construction. We turn now to the triangulated construction of a derivator

6.2 Triangulated derivators

In this subsection we show that given any *strong* and stable derivator, and small category $\mathbf{J}$, the category $\mathscr{D}(\mathbf{J})$ is canonically triangulated. Recall from proposition 3.16 that we have a partial underlying diagram functor

$$\mathrm{dia}_{\mathbf{J},\mathbf{K}} \colon \mathscr{D}(\mathbf{J} \times \mathbf{K}) \to \mathscr{D}(\mathbf{J})^{\mathbf{K}}, \quad X \mapsto (\mathrm{id}_{\mathbf{J}} \times k)^*(X).$$

which makes the coherent diagram incoherent in the $\mathbf{K}$-direction.

Definition 6.9. A derivator $\mathscr{D}$ is *strong* if $\mathrm{dia}_{\mathbf{J}\times[1]} \colon \mathscr{D}(\mathbf{J} \times [1]) \to \mathscr{D}(\mathbf{J})^{[1]}$ is full and essentially surjective for every $\mathbf{J} \in \mathbf{Cat}$.

Remark. Many authors define strong by a different notion, where they require $\mathrm{dia}_{\mathbf{J}\times\mathbf{K}}$ to be essentially surjective for $\mathbf{K}$ a finite free category.

It is worth noting here that being strong does not ask that $\mathrm{dia}_{\mathbf{J},\mathbf{K}}$ is faithful, and so this is not an equivalence.

Example 6.10. $\mathscr{D}_{rep}$ is strong, and this can be easily seen as $\mathrm{dia}_{\mathbf{J},\mathbf{K}}$ is actually an equivalence.

Example 6.11. $\mathscr{D}_{\mathcal{A}}$ is strong, for a Grothendieck abelian category $\mathcal{A}$. Consider $q^{-1} \circ f \in \mathbf{D}(\mathcal{A})^{[\mathbf{1}]}$. By the discussion of localizations (above proposition 1.8) we can simply recover the morphism $f \in \mathbf{D}(\mathcal{A}^{[\mathbf{1}]})$.

Similar to pointedness and stability, we show that strongness is preserved by the operations on derivators.

Lemma 6.12. *Let $\mathscr{D}$ be a derivator.*

(i) $\mathscr{D}$ is strong if and only if $\mathscr{D}^{op}$ is strong.

(ii) $\mathscr{D}$ is strong if and only if $\mathscr{D}^{\mathbf{K}}$ is strong, for all $\mathbf{K} \in \mathbf{Cat}$.

Proof. (i) Let the following morphism between diagrams

$$Y^{\mathrm{op}} \xrightarrow{f^{*\mathrm{op}}} X^{\mathrm{op}}$$

be an element in $\mathscr{D}^{\mathrm{op}}(\mathbf{J})^{[1]}$. Then the opposite morphism

$$Y^{\mathrm{op}} \xleftarrow{f^{*}} X^{\mathrm{op}}$$

can be considered as an element in $\mathscr{D}(\mathbf{J}^{\mathrm{op}})^{[1]}$. This can be lifted to a coherent morphism

$$Y^{\mathrm{op}} \xleftarrow{f^{*}} X^{\mathrm{op}}$$

in $\mathscr{D}(\mathbf{J}^{\mathrm{op}} \times [1])$, which has an opposite morphism

$$Y^{\mathrm{op}} \xrightarrow{f^{*\mathrm{op}}} X^{\mathrm{op}}$$

in $\mathscr{D}^{\mathrm{op}}(\mathbf{J} \times [1])$.

(ii) This follows from the commutative diagram

$$\begin{tikzcd}
\mathscr{D}^{\mathbf{K}}(\mathbf{J} \times [1]) \arrow[r, "\mathrm{dia}_{\mathbf{J},[1]}"] \arrow[d, equal] & \mathscr{D}^{\mathbf{K}}(\mathbf{J})^{[1]} \arrow[d, equal] \\
\mathscr{D}(\mathbf{J} \times \mathbf{K} \times [1]) \arrow[r, "\mathrm{dia}_{\mathbf{J}\times\mathbf{K},[1]}"] & \mathscr{D}(\mathbf{J} \times \mathbf{K})^{[1]}
\end{tikzcd}$$

where the two partial underlying diagram functors are the same functor

□

Being strong is a common property for stable derivators, although there are examples of stable derivators which do not satisfy it (see corollary 3.6.11 [11]). For this reason, it is usual to refer to strong and stable derivators as *triangulated derivators*.

Definition 6.13. A derivator $\mathscr{D}$ is said to be *triangulated* if it is both stable and strong.

We justify this name with the main result of this chapter.

Theorem 6.14. *Let $\mathscr{D}$ be a triangulated derivator, and $\mathbf{J} \in \mathbf{Cat}$. Then $\mathscr{D}(\mathbf{J})$ is a triangulated category.*

Proof. By lemma 5.3 and lemma 6.12(ii) above, it is enough to show that the underlying category $\mathscr{D}(\mathbb{1})$ is triangulated.

Before we start going through the axioms, following the definition of a triangulated category we need an additive autoequivalence and a natural isomorphism. For the additive autoequivalence we will use the suspension functor $\Sigma\colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1})$ from definition 4.15.

For the class of triangles $\triangle$ let us once more consider the cofiber functor from definition 4.11. Given a morphism $f\colon x \to y$ in $\mathscr{D}([1])$, we apply the cofiber functor to f, yielding a coherent square. Then, if we apply the cofiber functor to the resulting morphism $\mathrm{cof}(f)$, something interesting happens. The process is illustrated as follows

$$\begin{tikzcd}
 & & & & x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[d, "\mathrm{cof}(f)"] & & & x \arrow[r, "f"] \arrow[d] & y \arrow[r] \arrow[d, "\mathrm{cof}(f)"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
x \arrow[r, "f"] & y & {} \arrow[r] & {} & 0 \arrow[r] & z & {} \arrow[r] & {} & 0 \arrow[r] & z \arrow[r, "\mathrm{cof}^2(f)"'] & x'
\end{tikzcd}$$

Now note that by proposition 5.9 and corollary 5.7 the outer square is bicartesian, so there is an isomorphism $\phi\colon x' \xrightarrow{\phi} \Sigma x$. This gives us a sequence

$$x \xrightarrow{f} y \xrightarrow{\mathrm{cof}(f)} z \xrightarrow{\phi\circ\mathrm{cof}^2(f)} \Sigma x$$

by restricting to the indicated zig-zag. Applying the dia functor to this gives us an element of $\mathscr{D}(\mathbb{1})^{[3]}$, and we define $\triangle$ to be the class of morphisms that are isomorphic to such triangles. We are now ready to prove the axioms.

(T1) The first axiom is not to difficult. By definition $\triangle$ is closed under isomorphisms. For any morphism f, we define the functor $\mathrm{tria}(f)$ as the composition

$$\mathrm{tria}(f)\colon\ \mathscr{D}(\mathbb{1})^{[1]} \xrightarrow{\mathrm{lift}} \mathscr{D}([1]) \to \mathscr{D}(\square\square) \to \mathscr{D}([3]) \xrightarrow{\mathrm{dia}_{[3]}} \mathscr{D}(\mathbb{1})^{[3]}$$

where the first arrow uses strongness to lift a diagram, the second arrow comes from the above discussion, and the third arrow is just restriction. From this it is clear that any f can be completed to a triangle in $\triangle$. In the case where $f = \mathrm{id}_x$, it follows from lemma 4.19(iii) that we get the sequence

$$x \xrightarrow{\mathrm{id}_x} x \to 0 \to \Sigma x$$

so the trivial triangle is also in $\triangle$.

(T2) Let us consider a triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$$

in $\triangle$. The goal is to show that the corresponding shifts

$$y \xrightarrow{g} z \xrightarrow{h} \Sigma x \xrightarrow{-\Sigma f} \Sigma y$$

and

$$\Omega z \xrightarrow{-\Omega(\phi \circ h)} x \xrightarrow{f} y \xrightarrow{g} z$$

are also in $\triangle$. We show that this is true for the former, and the latter shift is dual.

We might assume WLOG that the triangle came from a sequence $\mathrm{tria}(f)$. Then writing $h = \phi \circ h'$ for the corresponding isomorphism $x' \cong \Sigma x$ and $h' \colon z \to x'$, we apply the cofiber functor to h'. This is illustrated by the following commutative diagram

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & z \arrow[r, "h'"] \arrow[d] \arrow[dr, phantom, "\square"] & x' \arrow[d, "\mathrm{cof}(h') = f'"] \\
& 0 \arrow[r] & y'
\end{tikzcd}$$

Note that we have an isomorphism $\psi \colon y' \xrightarrow{\cong} \Sigma y$. This means that the we have a triangle $y \xrightarrow{g} z \xrightarrow{h'} x' \xrightarrow{\psi \circ f'} \Sigma y$ in $\triangle$. Our goal is now to show that the shifted triangle is isomorphic to this triangle. Or in other words, to show that the following diagram commutes

$$\begin{tikzcd}
y \arrow[r, "g"] \arrow[d, equal] & z \arrow[r, "h'"] \arrow[d, equal] & x' \arrow[r, "\psi\circ f'"] \arrow[d, "\phi"'] & \Sigma y \arrow[d, equal] \\
y \arrow[r, "g"'] & z \arrow[r, "h"'] & \Sigma x \arrow[r, "-\Sigma f"'] & \Sigma y
\end{tikzcd}$$

Since the leftmost square obviously commute, and the middle commutes by the definition of h, all we need to show is that the right hand side commutes, as then by similar arguments as the proof of proposition 6.6 the two triangles are isomorphic. Consider the two triangle constructions for f and g, respectively

$$\begin{tikzcd}
\mathrm{tria}(f)\colon & x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & 0_2 \arrow[d] \\
& 0_1 \arrow[r] & z \arrow[r, "h"'] & \Sigma x
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{tria}(g)\colon & y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & z \arrow[r] \arrow[d, "h'"] \arrow[dr, phantom, "\square"] & 0_3 \arrow[d] \\
& 0_2 \arrow[r] & x' \arrow[r, "\psi\circ f'"'] & \Sigma y
\end{tikzcd}$$

We included the numeration on the zeroes to keep track of them. Recall that flipping the positions of the zeroes is the additive inverse in $\mathscr{D}(\mathbb{1})(x, y)$ from theorem 5.17. We can extend to the following diagram, where most of the sides on the front and back are bicartesian squares

$$\begin{tikzcd}
& x \arrow[rr, "f"] \arrow[dl, "f"] \arrow[dd] & & y \arrow[rr] \arrow[dl] \arrow[dd, "g"] & & 0_2 \arrow[dl, equal] \arrow[dd] \\
y \arrow[rr] \arrow[dd, "g"] & & 0_2 \arrow[rr, equal] \arrow[dd] & & 0_2 \arrow[dd] & \\
& 0_1 \arrow[rr] \arrow[dl] \arrow[dd, equal] & & z \arrow[rr, "h'"] \arrow[dl, "h'"] \arrow[dd] & & x' \arrow[dl, equal] \arrow[dd, "f'"] \\
z \arrow[rr, "g"] \arrow[dd] & & x' \arrow[rr, equal] \arrow[dd, "f'"] & & x' \arrow[dd, "f'"] & \\
& 0_1 \arrow[rr] \arrow[dl] & & 0_3 \arrow[rr] \arrow[dl] & & y' \arrow[dl, equal] \\
0_3 \arrow[rr] & & y' \arrow[rr, equal] & & y' &
\end{tikzcd}$$

This can be pasted to the simple cube

$$\begin{tikzcd}
& x \arrow[rr] \arrow[dl, "f"] \arrow[dd] & & 0_2 \arrow[dl] \arrow[dd] \\
y \arrow[rr] \arrow[dd] & & 0_2 \arrow[dd] & \\
& 0_1 \arrow[rr] \arrow[dl] & & x' \arrow[dl, "f'"] \\
0_3 \arrow[rr] & & y' &
\end{tikzcd}$$

in which the front and back face are bicartesian by corollary 5.7 and proposition 5.9. From this cube we get the natural isomorphism $f' \cong \Sigma f$. However, comparing the two squares

$$\begin{tikzcd}
y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0_2 \arrow[d] \\
0_3 \arrow[r] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0_3 \arrow[d] \\
0_2 \arrow[r] & y'
\end{tikzcd}$$

from the cube and from the pasting of tria(g), we see that there is a reversing of zeroes, which induces a minus sign in $-\Sigma f$.

Extending the original diagram to include the isomorphisms ϕ and ψ

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & 0_2 \arrow[r] \arrow[d] & 0_2 \arrow[d] \\
0_1 \arrow[r] & z \arrow[r, "h'"'] \arrow[d] \arrow[dr, phantom, "\square"] & x' \arrow[r, "\phi"'] \arrow[d, "f'"] & \Sigma x \arrow[d, "-\Sigma f"] \\
& 0_3 \arrow[r] & y' \arrow[r, "\psi"'] & \Sigma y
\end{tikzcd}$$

it follows from lemma 4.19(iii) that the lower right square is bicartesian, and commutes.

(T3) Using the fact that $\mathscr{D}$ is strong, we know that the composite functor

$$\mathscr{D}(\square) \xrightarrow{\mathrm{dia}_{[1]}} \mathscr{D}([1])^{[1]} \xrightarrow{\mathrm{dia}_{[1]}} (\mathscr{D}(\mathbb{1})^{[1]})^{[1]} \cong \mathscr{D}(\mathbb{1})^{\square}$$

is essentially surjective. This means that we can lift a commutative square from $\mathscr{D}(\mathbb{1})^{\square}$ to a coherent square in $\mathscr{D}(\square)$. Consider the morphisms between two triangles

$$\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"'] & y_1 \arrow[r, "g_1"] \arrow[d, "\phi_2"] & z_1 \arrow[r, "h_1"] \arrow[d, dotted, "\phi_3"] & \Sigma x_1 \arrow[d, "\Sigma\phi"] \\
x' \arrow[r, "f_2"] & y' \arrow[r, "g_2"] & z' \arrow[r, "h_2"] & \Sigma x_2[1]
\end{tikzcd}$$

Since $\mathscr{D}$ is strong we can lift the first square

$$\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & y_1 \arrow[d, "\phi_2"] \\
x_2 \arrow[r, "f_2"] & y_2
\end{tikzcd}$$

which lies in $\mathscr{D}(\mathbb{1})^{\square}$, to a coherent square in $\mathscr{D}(\square)$. If we now apply the triafunctor to f_1 and f_2 we first get the following induced diagram

$$\begin{tikzcd}
 & x_1 \arrow[rr, "f_1"] \arrow[dl, "\phi_1"] \arrow[dd] & & y_1 \arrow[rr] \arrow[dl, "\phi_2"] \arrow[dd, "g_1"] & & 0 \arrow[dl] \arrow[dd] \\
x_2 \arrow[rr, "f_2"] \arrow[dd] & & y_2 \arrow[rr] \arrow[dd, "g_2"] & & 0 \arrow[dd] & \\
 & 0 \arrow[rr] \arrow[dl] & & z_1 \arrow[rr, "h_1"] \arrow[dl, "\phi_3"] & & \Sigma x_1 \arrow[dl, "\Sigma\phi_1"] \\
0 \arrow[rr] & & z_2 \arrow[rr, "h_2"] & & \Sigma z_2 &
\end{tikzcd} \tag{5}$$

which we then restrict to a morphism between the two triangles.

(T4) The octahedron axiom asks that three composable morphisms $x \xrightarrow{f_1} y \xrightarrow{f_2} z \in \mathscr{D}(\mathbb{1})$ can be completed to an octahedron. These morphisms are not coherent, so we first have to argue that we can lift this to a coherent object in $\mathscr{D}([2])$. We can think of our sequence as a square

$$\begin{tikzcd}
x \arrow[r, "f_1"] \arrow[d, equal] & y \arrow[d, "f_2"] \\
x \arrow[r, "f_3"] & z
\end{tikzcd}$$

where $f_3 = f_2 \circ f_1$. By the strongness property we can lift f_1 to some $F_1 \in \mathscr{D}([1])$ with underlying diagram $x \to y$. Now there are natural isomorphisms

$$\mathscr{D}(\mathbb{1})(y, z) \xrightarrow{f_1^*} \mathscr{D}(\mathbb{1})([x \to y], [x \to z]) \xrightarrow{\text{lift}} \mathscr{D}([1])(F_1, 1_* z)$$

where the image of $1_* z$ is naturally isomorphic to $\pi^* z \colon z \xrightarrow{\text{id}} z$, for $\pi \colon [1] \to \mathbb{1}$, by the dual of lemma 4.19 (i). Let $\phi \colon F_1 \to \pi^* z$ be the image of f_2 under this isomorphism. Another use of the strongness property gives us the square

$$\begin{tikzcd}
x \arrow[r, "f_1"] \arrow[d, "\phi_0"] & y \arrow[d, "\phi_1"] \\
z \arrow[r, equal] & z
\end{tikzcd}$$

which we denote by D. Then defining $i \colon [2] \to \square$ by the functor that classifies all arrows that passes through the upper right corner $(1, 0)$(without being the identity), we can now set $F = i^* D \in \mathscr{D}([2])$ to get the wanted coherent sequence.

Let us now assume we have three (coherent) triangles in the following diagram

$$\begin{tikzcd}
x \arrow[r, "f_1"] \arrow[d, equal] & y \arrow[r, "f_2"] \arrow[d, "h_1"] & z \arrow[r, "f_3"] \arrow[d, dotted] & \Sigma x \arrow[d, equal] \\
x \arrow[r, "g_1"] & u \arrow[r, "g_2"] \arrow[d, "h_2"] & v \arrow[r, "g_3"] \arrow[d, dotted] & \Sigma x \\
 & w \arrow[r, equal] \arrow[d, "h_3"] & w \arrow[d, dotted] & \\
 & \Sigma y \arrow[r, "\Sigma f_2"] & \Sigma z &
\end{tikzcd} \tag{6}$$

and show that we can always find the triangle that fits. First, the morphism $\phi_1 \colon z \to v$ comes from the two-out-of-three axiom. However, if we consider diagram (5) above, we see that in the left cube both the back face, and front face are bicartesian. Since the cube commutes, we can paste together the 'upper' and 'lower' part to get the two diagrams

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[d] & x_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
y_1 \arrow[r] & y_2 \arrow[r] & z_2
\end{tikzcd}
\qquad
\begin{tikzcd}
x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
y_1 \arrow[r] & z_1 \arrow[r] & z_2
\end{tikzcd}$$

From this, it follows from corollary 5.7 and lemma 4.19(iii) that the leftmost square is bicartesian if and only if the outer left square is bicartesian if and only if the outer right square is bicartesian if and only if the rightmost square is bicartesian if and only if $z_1 \to z_2$ is an isomorphism. Shifting the two upper morphisms in diagram 6, and applying the above arguments, we end up with the following new diagram

$$\begin{tikzcd}
x \arrow[r, "f_1"] \arrow[d, equal] & y \arrow[r, "f_2"] \arrow[d, "h_1"] \arrow[dr, phantom, "\square"] & z \arrow[r, "f_3"] \arrow[d, "j_1"] & \Sigma x \arrow[d, equal] \\
x \arrow[r, "g_1"] & u \arrow[r, "g_2"] \arrow[d, "h_2"] & v \arrow[r, "g_3"] \arrow[d, dotted, "j_2"] & \Sigma x \\
& w \arrow[r, equal] \arrow[d, "h_3"] & w \arrow[d, dotted, "j_3"] & \\
& \Sigma y \arrow[r, "\Sigma f_2"] & \Sigma z &
\end{tikzcd} \tag{7}$$

in which the indicated square is bicartesian. Now we can just complete j_1 to a triangle with (T_1), and apply the same arguments as above. Since the indicated square is bicartesian, we get an isomorphism between the cones, thus finishing the diagram.

All of the above diagrams are equivalent to $\mathscr{D}([2])$ by proposition 4.9, proposition 5.9, and the tria functor, which concludes the proof.

□

Now we know that $\mathscr{D}(\mathbf{J})$ is a triangulated category, but what about the induced functors $u^*, u_!, u_*$? It turns out that for abstract derivators, these functors behave in a very nice way.

Proposition 6.15. *Let $\mathscr{D}$ be a triangulated derivator, and $u\colon \mathbf{J} \to \mathbf{K}$ a functor between small categories. The induced functors $u^*, u_!$ and u_* are exact functors between triangulated categories*

Proof. By proposition 5.18 these are additive functors, so we only need to show that triangles are sent to triangles.

Cofiber and fiber sequences are preserved by u^*, since u^* is an additive functor. So consider $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ a triangle in $\mathscr{D}(\mathbf{K})^{[2]}$. By strongness of $\mathscr{D}$, we can lift this to a cofiber sequence of the form

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & z \arrow[r, "h"] & \Sigma x
\end{tikzcd}$$

in $\mathscr{D}^{\square\square}(\mathbf{K})$. This is then sent to the cofiber sequence

$$\begin{tikzcd}
u^*(x) \arrow[r, "u^*(f)"] \arrow[d] \arrow[dr, phantom, "\square"] & u^*(y) \arrow[r] \arrow[d, "u^*(g)"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & u^*(z) \arrow[r, "u^*(h)"] & u^*(\Sigma x)
\end{tikzcd}$$

in $\mathscr{D}^{\square\square}(\mathbf{J})$. By proposition 5.9 there is an isomorphism $\phi\colon u^*(\Sigma x) \cong \Sigma u^*(x)$, which induces a natural isomorphism. This diagram can now be restricted to a triangle in $\mathscr{D}(\mathbf{J})^{[2]}$.

Since adjoints of exact functors are exact this also proves that $u_!$ and u_* are exact functors, which concludes the proof. □

We end this chapter with a brief discussion on the functorial cone construction of derivators. As advertised in the previous chapter, for a general triangulated category $\mathcal{T}$ there is no functorial cone construction cone: $\mathcal{T}^{[1]} \to \mathcal{T}$. However, as we saw in section 1.4, the were a construction for $\mathbf{D}(\mathcal{A}^{[\mathbf{1}]})$. The same is true for derivators. For a triangulated derivator $\mathscr{D}$, there is no functorial cone $\mathscr{D}(\mathbb{1})^{[1]} \to \mathscr{D}(\mathbb{1})$. The big difference is that strongness allows us to essentially surjectively lift any morphism to a coherent morphism, which HAS a functorial cone

$$\mathscr{D}(\mathbb{1})^{[1]} \xrightarrow{\text{lift}} \mathscr{D}([1]) \xrightarrow{1^* \circ \text{cof}} \mathscr{D}(\mathbb{1})$$

thus solving the issue.

7 The Calabi-Yau dimension of an abstract derivator

In this chapter we introduce the notion of *n-cofiber functors*, which is a generalized version of the cofiber functor definition 4.11. By adapting the proof of Lemma 5.13 from [6], we prove that repeated sequences of n-cofiber functors has a natural equivalence with powers of the suspension, thus leading to an alternative proof of the fractional Calabi-Yau dimension of a stable derivator.

7.1 2-cofiber sequences

recall from section 4 how we defined the cofiber functor. We take a coherent morphism $f \in \mathscr{D}([1])$, create a diagram of the form

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \arrow[d, "\mathrm{cof}(f)"] \\
0 \arrow[r] & C(f)
\end{tikzcd}$$

and then restrict to the morphism $\mathrm{cof}(f)$. In this section we are going to extend this to a coherent composition $g \circ f \in \mathscr{D}([2])$.

Consider the following inclusions of diagrams

$$\begin{tikzcd}
(0,0) \arrow[r] & (1,0) \arrow[r] & (2,0) & & & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\
& & & {} \arrow[r, "i_1"] & {} & (0,1) & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) & & & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
{} \arrow[r, "i_2"] & {} & (0,1) \arrow[r] & (1,1) & & {} \arrow[r, "i_3"] & {} & (0,1) \arrow[r] & (1,1) \arrow[r] & (2,1)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] & & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
\xrightarrow{i_4} & (0,1) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (2,1) & \xrightarrow{i_5} & (0,1) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] \\
 & & (1,2) & & & & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

where we denote the final diagram by Q, the diagrams induced by the inclusions i_j for D_j and denote by v the inclusion of the full subcategory $(2,0) \to (2,1) \to (2,2)$ of the vertical arrows.

Definition 7.1. Let $\mathscr{D}$ be a pointed derivator. We define the functor $2\text{-cof}\colon \mathscr{D}([2]) \to \mathscr{D}([2])$ by the composition of functors

$$\mathscr{D}([2]) \xrightarrow{(i_1)_!} \mathscr{D}(D_1) \xrightarrow{(i_2)_*} \mathscr{D}(D_2) \xrightarrow{(i_3)_*} \mathscr{D}(D_3) \xrightarrow{(i_4)_!} \mathscr{D}(D_4) \xrightarrow{(i_5)_*} \mathscr{D}(Q) \xrightarrow{v^*} \mathscr{D}([2])$$

Since all the inclusions are sieves, the functors $(i_1)_!$ and $(i_4)_!$ are extensions by zero. So for a given pair of composable morphisms $g \circ f \in \mathscr{D}([2])$ with underlying diagram $x_1 \xrightarrow{f} x_2 \xrightarrow{g} x_3$, we get induced diagrams of the form

$$\begin{tikzcd}
x_1 \arrow[r, "f"] \arrow[d] & x_2 \arrow[r, "g"] \arrow[d] & x_3 \arrow[d] \\
0 \arrow[r] & C_f \arrow[r] \arrow[d] & C_{g\circ f} \arrow[d] \\
 & 0 \arrow[r] & C_g
\end{tikzcd}$$

which gives us the sequence $2\text{-cof}(g \circ f)\colon x_3 \xrightarrow{\text{cof}(g\circ f)} C_{g\circ f} \xrightarrow{\text{cof}(g)} C_g$.

There is also the dual functor, defined by the sequence of inclusions

$$\begin{tikzcd}
(2,0) \arrow[d] & & & (2,0) \arrow[d] \\
(2,1) \arrow[d] & \xrightarrow{j_1} & & (2,1) \arrow[d] \\
(2,2) & & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & (2,0) \arrow[d] & & & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
{} \arrow[r, "J_2"] & {} & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] & {} \arrow[r, "j_3"] & {} & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] \\
 & & (1,2) \arrow[r] & (2,2) & & & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] & & & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[d] \\
{} \arrow[r, "j_4"] & {} & (0,1) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] & {} \arrow[r, "j_5"] & {} & (0,1) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[d] \\
 & & & (1,2) \arrow[r] & (2,2) & & & & (1,2) \arrow[r] & (2,2)
\end{tikzcd}$$

with each j_i an inclusion into a diagram E_i, and h the inclusion of the full subcategory $(0,0) \to (1,0) \to (2,0)$ of the horizontal arrows.

Definition 7.2. Let $\mathscr{D}$ be a pointed derivator. We define the functor $2\text{-fib}\colon \mathscr{D}([2]) \to \mathscr{D}([2])$ by the sequence of functors

$$\mathscr{D}([2]) \xrightarrow{(j_1)_*} \mathscr{D}(E_1) \xrightarrow{(j_2)_!} \mathscr{D}(E_2) \xrightarrow{(j_3)_!} \mathscr{D}(E_3) \xrightarrow{(j_4)_*} \mathscr{D}(E_4) \xrightarrow{(j_5)_!} \mathscr{D}(Q) \xrightarrow{h^*} \mathscr{D}([2])$$

In a similar fashion this functors takes the coherent composable morphisms $x_1 \xrightarrow{f} x_2 \xrightarrow{g} x_3$, makes a diagram of the form

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d] & F_{g \circ f} \arrow[r] \arrow[d] & x_1 \arrow[d, "f"] \\
0 \arrow[r] & F_g \arrow[r] \arrow[d] & x_2 \arrow[d, "g"] \\
 & 0 \arrow[r] & x_3
\end{tikzcd}$$

and restricts to $2\text{-fib}(g \circ f)\colon F_f \xrightarrow{\text{fib}(f)} F_{g \circ f} \xrightarrow{\text{fib}(g \circ f)} x_1$.

Example 7.3. Let $\mathscr{D}_{\mathrm{rep}}$ be a pointed represented derivator, with underlying category $\mathbf{C}$. Then $\mathscr{D}_{\mathrm{rep}}([2]) = \mathbf{C}^{[2]}$ is the category of composable morphisms in $\mathbf{C}$, and 2-cof maps the morphisms $(c_1 \xrightarrow{f_1} c_2 \xrightarrow{f_2} c_3)$ to the cokernels $(c_3 \xrightarrow{k_1} \mathrm{Cok}(f_2 \circ f_1) \xrightarrow{k_2} \mathrm{Cok}\, f_2)$.

Example 7.4. In the homotopy derivator $\mathscr{D}_{\mathcal{A}}$, we can denote by DCok the left derived cokernel functor. By theorem 1.26 this is the induced cone functor, and 2-cof gives us the following composition of derived cokernels

$$(x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3) \xrightarrow{\text{2-cof}} (x_3 \to \mathrm{DCok}(f_2 \circ f_1) \to \mathrm{DCok}(f_2))$$

As with the cofiber and fiber functors, there is an obvious relation between 2-cof and 2-fib.

Proposition 7.5. *The two functors* $(\text{2-cof}, \text{2-fib})\colon \mathscr{D}([\mathbf{2}]) \leftrightarrows \mathscr{D}([\mathbf{2}])$ *are an adjoint pair.*

Proof. This is done in exactly the same way as for proposition 4.14. Consider the pair of adjunctions

$$\begin{tikzcd}
\text{2-cof}\colon \mathscr{D}([2]) \arrow[r, bend right, "(i_1)_!"'] & \mathscr{D}^{\mathrm{ex}}(D_1) \arrow[l, bend right, "(i_1)^*"'] \arrow[r, bend left, "(i_2)_*"] & \mathscr{D}^{\mathrm{ex}}(D_2) \arrow[l, bend left, "(i_2)^*"] \arrow[r, bend left, "(i_3)_*"] & \mathscr{D}^{\mathrm{ex}}(D_3) \arrow[l, bend left, "(i_3)^*"] \arrow[r, bend right, "(i_4)_!"'] & \mathscr{D}^{\mathrm{ex}}(D_4) \arrow[l, bend right, "(i_4)^*"'] \arrow[r, bend left, "(i_5)_*"] & \mathscr{D}^{\mathrm{ex}}(Q) \arrow[l, bend left, "(i_5)^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{D}^{\mathrm{ex}}(Q) \arrow[r, bend left, "(j_5)^*"] & \mathscr{D}^{\mathrm{ex}}(E_4) \arrow[l, bend left, "(j_5)_!"] \arrow[r, bend right, "(j_4)^*"'] & \mathscr{D}^{\mathrm{ex}}(E_3) \arrow[l, bend right, "(j_4)_*"'] \arrow[r, bend left, "(j_3)^*"] & \mathscr{D}^{\mathrm{ex}}(E_2) \arrow[l, bend left, "(j_3)_!"] \arrow[r, bend left, "(j_2)^*"] & \mathscr{D}^{\mathrm{ex}}(E_1) \arrow[l, bend left, "(j_2)_!"] \arrow[r, bend right, "(j_1)^*"'] & \mathscr{D}([2])\colon \text{2-fib} \arrow[l, bend right, "(j_1)_*"']
\end{tikzcd}$$

where $\mathscr{D}^{\mathrm{ex}}$ is the restriction to the full subcategories induced by diagrams that vanish at the the positions $(0,1)$ and $(1,2)$. The extensions by zero are equivalences, while the rest are all adjoints, so the composition is clearly an adjunction. □

Note that in the stable setting all squares are bicartesian, and in particular the cofibers and fibers agree, so we get the immediate corollary.

Corollary 7.6. *Let* $\mathscr{D}$ *be a stable derivator. The two functors* $(\text{2-cof}, \text{2-fib})$ *are equivalences.*

In *Mayer-Vietoris sequences in stable derivators* Groth, Ponto and Shulman show that there is a natural equivalence between cof^3 and Σ for a stable derivator (Lemma 5.13 [6]). As a precursor to the main result, we now extend this to the functor 2-cof, by slightly modifying the proof.

Lemma 7.7. *Let $\mathscr{D}$ be a pointed derivator. There is a natural equivalence between the functors $(2\text{-cof})^4$ and $\Sigma^2\colon \mathscr{D}([2]) \to \mathscr{D}([2])$.*

Proof. Consider $D_1 \subseteq [6] \times [4]$ the full subcategory spanned by elements

$$D_1 = \{(i,j) \in [6] \times [4] \quad | \quad j-1 \leq i \leq j+3\}$$

By using combinations of extensions by zero (proposition 4.9) and left Kan extensions, we get a functor $\mathscr{D}([2]) \to \mathscr{D}(D_1)$, which sends a coherent diagram $(x_1 \to x_2 \to x_3)$ to a diagram Q of shape D_1

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[d] & x_2 \arrow[r] \arrow[d] & x_3 \arrow[r] \arrow[d] & 0_2 \arrow[d] & & & \\
0_1 \arrow[r] & y_1 \arrow[r] \arrow[d] & y_2 \arrow[r] \arrow[d] & y_3 \arrow[r] \arrow[d] & 0_5 \arrow[d] & & \\
& 0_3 \arrow[r] & z_1 \arrow[r] \arrow[d] & z_2 \arrow[r] \arrow[d] & z_3 \arrow[r] \arrow[d] & 0_6 \arrow[d] & \\
& & 0_4 \arrow[r] & u_1 \arrow[r] \arrow[d] & u_2 \arrow[r] \arrow[d] & u_3 \arrow[r] \arrow[d] & 0_8 \arrow[d] \\
& & & 0_7 \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & v_3
\end{tikzcd} \tag{8}$$

All the squares and rectangles of diagram (8) are cocartesian by construction, so we have the following canonical identifications

$$\begin{aligned}
2\text{-cof}(x_1 \to x_2 \to x_3) &\cong (x_3 \to y_2 \to z_1)\\
2\text{-cof}^2(x_1 \to x_2 \to x_3) &\cong (z_1 \to z_2 \to z_3)\\
2\text{-cof}^3(x_1 \to x_2 \to x_3) &\cong (z_3 \to u_2 \to v_1)\\
2\text{-cof}^4(x_1 \to x_2 \to x_3) &\cong (v_1 \to v_2 \to v_3)
\end{aligned}$$

Now let Y be the set of elements

$$Y = \{(0,0,2),(1,0,2),(2,0,2),(0,2,0),(1,2,0),(2,2,0)\}$$

and $D_2 \subseteq [2]^3$ the full subcategory of the cube omitting Y. Let $q\colon D_2 \to D_1$ be the functor such that $q^*(Q)$ has the form

$$\begin{tikzcd}
 & & x_1 \arrow[rr] \arrow[dl] \arrow[dd] & & x_2 \arrow[rr] \arrow[dl] \arrow[dd] & & x_3 \arrow[dl] \arrow[dd] \\
 & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[dd] & \\
 & & 0_1 \arrow[rr] \arrow[dl] & & 0_3 \arrow[rr] \arrow[dl] & & 0_5 \arrow[dl] \\
 & y_3 \arrow[rr] \arrow[dl] \arrow[dd] & & z_2 \arrow[rr] \arrow[dl] \arrow[dd] & & u_1 \arrow[dl] \arrow[dd] & \\
0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[dd] & & \\
 & 0_4 \arrow[rr] \arrow[dl] & & 0_6 \arrow[rr] \arrow[dl] & & 0_8 \arrow[dl] & \\
v_1 \arrow[rr] & & v_2 \arrow[rr] & & v_3 & &
\end{tikzcd}$$

The subscripts of the zero-elements match those of diagram (8), to indicate the definition of q. Since all the left, middle and right sides are cocartesian in $\mathscr{D}$, and the inclusion $\ulcorner \to \square$ is fully faithful, the two cubes

$$\begin{tikzcd}
 & x_1 \arrow[rr] \arrow[dl] \arrow[dd] & & x_2 \arrow[rr] \arrow[dl] \arrow[dd] & & x_3 \arrow[dl] \arrow[dd] \\
0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[dd] & \\
 & 0_1 \arrow[rr] \arrow[dl] & & 0_3 \arrow[rr] \arrow[dl] & & 0_5 \arrow[dl] \\
y_3 \arrow[rr] & & z_2 \arrow[rr] & & u_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_3 \arrow[rr] \arrow[dl] \arrow[dd] & & z_2 \arrow[rr] \arrow[dl] \arrow[dd] & & u_1 \arrow[dl] \arrow[dd] \\
0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[rr] \arrow[dd] & & 0_7 \arrow[dd] & \\
 & 0_4 \arrow[rr] \arrow[dl] & & 0_6 \arrow[rr] \arrow[dl] & & 0_8 \arrow[dl] \\
v_1 \arrow[rr] & & v_2 \arrow[rr] & & v_3 &
\end{tikzcd}$$

are cocartesian in $\mathscr{D}^{[\mathbf{2}]}$ by proposition 4.23. Hence, this gives the identification $2\text{-cof}^4(x_1 \to x_2 \to x_3) \cong \Sigma^2(x_1 \to x_2 \to x_3)$. □

Remark. Note that 2-cof^2 is not equivalent to Σ.

There is also the dual equivalence for 2-fib^4 and Ω^2. Now we extend the above discussions to a more general setting.

7.2 N-cofiber sequences

In this section we generalize the above arguments, resulting in the fractional Calabi-Yau dimension of $\mathscr{D}$. The original result (theorem 5.19 [7]) involves using a relation between the Auslander-Reiten translation and the suspension of a stable derivator. There is a remark at page 14 where the authors explains that there are corresponding results for a pointed derivator. However, it

The proof at the end of this section is essentially the same idea, but more focused on the particular generalization.

Let $n \in \mathbf{N}$ be a natural number, and consider $Q \subset \mathbb{Z}^2$ the subposet of the form

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \arrow[r] \arrow[d] & \cdots \arrow[r] & (n,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (2,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (n,1) \arrow[d] \\
 & (1,2) \arrow[r] & (2,2) \arrow[r] \arrow[d] & \cdots \arrow[r] & (n,2) \arrow[d] \\
 & & (2,3) \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[d] \\
 & & & (n-1,n) \arrow[r] & (n,n)
\end{tikzcd}$$

For a subposet $D_{(k,j)}$ we write $i_{(k,j+1)}$ and $i_{(k+1,j)}$ for the inclusions into the subposets $D_{(k,j+1)}$ and $D_{(k+1,j)}$ that adds the indicated object and morphisms.

Definition 7.8. Let $\mathscr{D}$ be a pointed derivator, and $x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \to \cdots \to x_n \xrightarrow{f_n} x_{n+1}$ be a coherent sequence in $\mathscr{D}([n])$. we define the functor n-cof as the composition

$$n\text{-cof}\colon \mathscr{D}([n]) \xrightarrow{(i_{(0,1)})_!} \mathscr{D}(D_{(0,1)}) \xrightarrow{(i_{(1,1)})_*} \mathscr{D}(D_{(1,1)}) \xrightarrow{(i_{(2,1)})_*} \cdots$$

$$\xrightarrow{(i_{(n,1)})_*} \mathscr{D}(D_{(n,1)}) \xrightarrow{(i_{(1,2)})_!} \mathscr{D}(D_{(1,2)}) \xrightarrow{(i_{(2,2)})_*} \cdots \xrightarrow{(i_{(n,n)})_*} \mathscr{D}(Q) \xrightarrow{v^*} \mathscr{D}([n])$$

where v denotes the inclusion of the full subcategory of the vertical morphism $[(n,0) \to (n,1) \to \cdots \to (n,n)] \to Q$.

Note that the functor n-cof has an underlying diagram of the form

$$\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d] & x_2 \arrow[r, "f_2"] \arrow[d] & x_3 \arrow[r, "f_3"] \arrow[d] & \cdots \arrow[r, "f_n"] & x_{n+1} \arrow[d, "\alpha_1"] \\
0 \arrow[r] & C(f_1) \arrow[r] \arrow[d] & C(f_2 \circ f_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & C(f_n \circ \cdots \circ f_1) \arrow[d, "\alpha_2"] \\
 & 0 \arrow[r] & C(f_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & C(f_n \circ \cdots \circ f_2) \arrow[d] \\
 & & 0 \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[d, "\alpha_n"] \\
 & & & 0 \arrow[r] & C(f_n)
\end{tikzcd}$$

where $\alpha_1 = \mathrm{cof}(f_n \circ \cdots \circ f_1)$, and $(\alpha_i \circ \alpha_{i-1} \circ \cdots \circ \alpha_1) = \mathrm{cof}(f_n \circ \cdots \circ f_i)$.

In a completely dual manner, we define the functor n-fib: $\mathscr{D}([n]) \to \mathscr{D}([n])$ by the composition

$$n\text{-fib}\colon \mathscr{D}([n]) \xrightarrow{(i_{(n-1,n)})_*} \mathscr{D}(D_{(n-1,n)}) \xrightarrow{(i_{(n-1,n-1)})_!} \mathscr{D}(D_{(n-1,n-1)}) \xrightarrow{(i_{(n-1,n-2)})_!} \cdots$$

$$\xrightarrow{(i_{(n-1,0)})_!} \mathscr{D}(D_{(n-1,0)}) \xrightarrow{(i_{(n-2,n-1)})_*} \mathscr{D}(D_{(n-2,n-1)}) \xrightarrow{(i_{(n-2,n-2)})_!} \cdots \xrightarrow{(i_{(0,0)})_*} \mathscr{D}(Q) \xrightarrow{h^*} \mathscr{D}([n])$$

where h denotes the inclusion of the full subcategory of the horizontal morphism $[(0,0) \to (1,0) \to \cdots \to (n,0)] \to Q$.

Lemma 7.9. *The two functors* $(n\text{-cof}, n\text{-fib})\colon \mathscr{D}([\mathbf{n}]) \leftrightarrows \mathscr{D}([\mathbf{n}])$ *are an adjoint pair.*

Proof. This is the same as the proof for 2-cof and 2-fib, we only restrict to the diagrams which vanish on $(i-1, i)$ for $1 \leq i \leq n$. Then the extensions by zero are equivalences, while the rest are adjunctions. □

And of course, there is the corresponding result for the stable setting.

Corollary 7.10. *Let* $\mathscr{D}$ *be a stable derivator. Then* $(n\text{-cof}, n\text{-fib})\colon \mathscr{D}([\mathbf{n}]) \leftrightarrows \mathscr{D}([\mathbf{n}])$ *is an equivalence.*

Example 7.11. For the represented derivator $\mathscr{D}_{\text{rep}}$ we get the induced sequence of cokernels

$$n\text{-cof}(x_1 \xrightarrow{f_1} x_2 \to \cdots \xrightarrow{f_n} x_n) \to (x_n \to \operatorname{Cok}(f_n \circ \cdots \circ f_1) \to \cdots \to \operatorname{Cok}(f_n))$$

We now turn to the main result of this chapter. The fact that the functors n-cof have an inherit connection to Σ. More precisely, we have the following result (theorem 5.19 in [7]).

Theorem 7.12. *Let* $\mathscr{D}$ *be a pointed derivator. For all* $n \geq 1$ *there is a natural equivalence between the functors*

$$(n\text{-cof})^{n+2} \cong \Sigma^n$$

Proof. The proof is very similar to that of lemma 7.7, we only differentiate between whether n is even or odd. If n is an even number, we let $m_1 = \dfrac{n^2+4n}{2}$ and $m_2 = \dfrac{n^2+2n}{2}$, and if n is an odd number we let $m_1 = m_2 = \dfrac{n^2+3n}{2}$. All arguments are the same regardless of whether n is odd or even, this is just to make the dimensions agree. In any case, we consider $D_1 \subseteq [m_1] \times [m_2]$ the full subcategory spanned by the set of elements (i, j) defined by

$$D_1 = \{(i,j) \in [m_1] \times [m_2] \quad | \quad j-1 \leq i \leq j+n+1\}$$

By combinations of extensions by zero (proposition 4.9) and left Kan extensions, we get a functor $\mathscr{D}([n]) \to \mathscr{D}(D_1)$, which sends a coherent diagram $(x_1 \to x_2 \to \cdots \to x_n)$ to a coherent diagram Q of shape D_1. For an even n this looks like the following diagram

$$\begin{tikzcd}
x_{(0,0)} \arrow[r] \arrow[d] & x_{(1,0)} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & x_{(n,0)} \arrow[r] \arrow[d] & 0_2 \arrow[d] & & & \\
0_1 \arrow[r] & x_{(1,1)} \arrow[r] \arrow[d] & x_{(2,1)} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & x_{(n+1,1)} \arrow[r] \arrow[d] & 0_4 \arrow[d] & & \\
& 0_3 \arrow[r] & x_{(2,2)} \arrow[r] \arrow[d] & x_{(3,2)} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & x_{(n+2,2)} \arrow[r] \arrow[d] & 0_6 \arrow[d] & \\
& & \cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[d] & \\
& & & 0_{2\cdot m_2+1} \arrow[r] & x_{(m_1-n,m_2)} \arrow[r] & x_{(m_1-n+1,m_2)} \arrow[r] & \cdots \arrow[r] & x_{(m_1,m_2)}
\end{tikzcd} \tag{9}$$

where the zero elements are indexed for future purposes. Now consider the map $f\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$f\colon (i,j) \mapsto (j+n+1, i+1)$$

and let $D_2 \subseteq [n]^3$ be the full subcategory of the cube generated by the elements (i,j,k) such that the difference between j and k lie in $\{0,1\}$. Furthermore, we let

$q\colon D_1 \to D_2$ be the functor such that the restriction $q^*(Q)$ has the form

$$\begin{tikzcd}
 & & & x_{(0,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & x_{(1,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & \cdots \arrow[rr] \arrow[dl] \arrow[dd] & & x_{(n,0)} \arrow[dl] \arrow[dd] \\
 & & 0_2 \arrow[rr] \arrow[dd] & & 0_2 \arrow[rr] \arrow[dd] & & \cdots \arrow[rr] \arrow[dd] & & 0_2 \arrow[dd] & \\
 & & & 0_1 \arrow[rr] \arrow[dl] & & 0_3 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & 0_{2\cdot n+1} \arrow[dl] \\
 & & x_{f(0,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f(0,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & \cdots \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f(0,0)} \arrow[dl] \arrow[dd] & \\
 & 0_{2\cdot n+3} \arrow[rr] \arrow[dd] & & 0_{2\cdot n+3} \arrow[rr] \arrow[dd] & & \cdots \arrow[rr] \arrow[dd] & & 0_{2\cdot n+3} \arrow[dd] & & \\
 & & 0_4 \arrow[rr] \arrow[dl] & & 0_6 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & 0_{2\cdot(n+2)} \arrow[dl] & \\
 & x_{f^2(0,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f^2(1,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & \cdots \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f^2(n,0)} \arrow[dl] \arrow[dd] & & \\
\cdots \arrow[rr] \arrow[dd] & & \cdots \arrow[rr] \arrow[dd] & & \cdots \arrow[rr] \arrow[dd] & & \cdots \arrow[dd] & & & \\
 & \cdots \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & \cdots \arrow[dl] & & \\
x_{f^n(0,0)} \arrow[rr] & & x_{f^n(1,0)} \arrow[rr] & & \cdots \arrow[rr] & & x_{f^n(n,0)} & & &
\end{tikzcd}$$

where $f(i,j)$ and the indexed zeroes match up with those of diagram (9) to indicate the definition of q. All the sides are cocartesian by construction so each cube of the form

$$\begin{tikzcd}
 & x_{f^i(0,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f^i(1,0)} \arrow[rr] \arrow[dl] \arrow[dd] & & \cdots \arrow[rr] \arrow[dl] \arrow[dd] & & x_{f^i(n,0)} \arrow[dl] \arrow[dd] \\
0 \arrow[rr] \arrow[dd] & & 0 \arrow[rr] \arrow[dd] & & \cdots \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\
 & 0 \arrow[rr] \arrow[dl] & & 0 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & 0 \arrow[dl] \\
x_{f^{i+1}(0,0)} \arrow[rr] & & x_{f^{i+1}(1,0)} \arrow[rr] & & \cdots \arrow[rr] & & x_{f^{i+1}(n,0)} &
\end{tikzcd}$$

for some $0 \leq i \leq n-1$, are cocartesian in $\mathscr{D}^{[\mathbf{n}]}$ by proposition 4.23. By the definition of the suspension functor (definition 4.15), f induces a functor that maps an element $x_{(i,j)}$ to its suspended element $\Sigma x_{(i,j)}$. In a similar matter the functor g defined by

$$g\colon (i,j) \mapsto (j+n, i)$$

induces a functor which identifies n-cof when applied to a coherent sequence of length n. This gives two canonical identifications

$$\Sigma^i(x_{(0,0)} \to x_{(1,0)} \to \cdots \to x_{(n,0)}) \cong (x_{f^i(0,0)} \to x_{f^i(1,0)} \to \cdots \to x_{f^i(n,0)})$$

$$n\text{-cof}^i(x_{(0,0)} \to x_{(1,0)} \to \cdots \to x_{(n,0)}) \cong (x_{g^i(0,0)} \to x_{g^i(1,0)} \to \cdots \to x_{g^i(n,0)})$$

So the question is for what degrees do the functors agree on a sequence of length n. The general formulas for iterated uses are given by

$$f^k(i,j) = \left\{ \begin{array}{lll} (i+m\cdot(n+2), & j+m\cdot(n+2)), & \text{if } k=2m \\ (j+m\cdot(n+2)-1, & i+(m-1)\cdot(n+2)+1), & \text{if } k=2m-1 \end{array} \right\}$$

$$g^k(i,j) = \left\{ \begin{array}{lll} (i+m\cdot n, & j+m\cdot n), & \text{if } k=2m \\ (j+m\cdot n, & i+(m-1)\cdot n), & \text{if } k=2m-1 \end{array} \right\}$$

Now let k_1 and k_2 be the smallest natural numbers such that $f^{k_1} = g^{k_2}$. This equation is satisfied if and only if the above formulas agree for all (i,j), if and only if they agree for $(i,j) = (0,0)$, so we restrict ourselves to this case. If either k_1 is even and k_2 is odd, or the other way around, we get an immediate contradiction from the above equalities. Assume therefore that k_1 and k_2 are both even (the same arguments apply to the case when they are both odd). We then have the following equality

$$m_1 \cdot (n+2) = m_2 \cdot n \tag{10}$$

where m_1 and m_2 are the smallest natural numbers satisfying the above equation. Let d denote the greatest common divisor of n and $n+2$. In this case, we can write

$$n+2 = d\cdot b_1, \quad b_1 = \frac{n+2}{d}$$

$$n = d\cdot b_2, \quad b_2 = \frac{n}{d}$$

and in particular $\gcd(b_1, b_2) = 1$. Or in other words, they are coprime. factoring out d gives us the new equation

$$m_1 \cdot b_1 = m_2 \cdot b_2$$

which means that there exists an integer a such that $m_1 = a \cdot b_2$ and $m_2 = a \cdot b_1$. Since d has to divide the difference of $n+2$ and n, this means that $d \in \{1,2\}$. If $d = 2$, this contradicts lemma 7.7, so $d = 1$. If we now set $a = 1$ (in order to get the smallest possible numbers that solve the equation), we see that $m_1 = n$ and $m_2 = n+2$. □

We end the chapter by explaining what we mean by a fractional Calabi-Yau dimension (as described in [10]), and why this is an interesting result. Let $\mathrm{Hom}(x,y)^\vee$ denote the dual of $\mathrm{Hom}(x,y)$. Given a triangulated category $\mathcal{T}$, we define a *Serre functor* as an auto-equivalence $S\colon \mathcal{T} \to \mathcal{T}$ with a bifunctorial isomorphism

$$\mathrm{Hom}(x,y)^\vee \cong \mathrm{Hom}(y, S(x))$$

for all $x, y \in \mathcal{T}$. We say $\mathcal{T}$ is a *d-Calabi-Yau category* if $S \cong \Sigma^d$, for some integer d. This integer is then referred to as the *CY-dimension* of $\mathcal{T}$. Related to this we have the weaker structure where $S^{d_1} \cong \Sigma^{d_2}$ for two integers d_1 and d_2. In this case, we say $\mathcal{T}$ is a *fractional Calabi-Yau categori*, and has a CY-dimension ${}^{d_2}/_{d_1}$. Note that if a category has a CY-dimension of ${}^{4}/_{2}$, this does not imply that it has a CY-dimension of 2. For this reason, the CY-dimension is never simplified. The functor n-cof satisfies the properties of a Serre functor on a triangulated derivator (see chapter five in [7]), and so theorem 7.12 implies that $\mathscr{D}([n])$ has CY-dimension ${}^{n}/_{n+2}$.

A Appendix

This section is for recalling some of the basic properties of homological algebra. We mention the most necessary results from *Homological Algebra* [18] along with some extra properties from *Abstract and concrete categories. The joy of cats* [1]. For further detail the reader is referred to this literature.

We begin by defining the initial and terminal object, and justify their uniqueness (up to isomorphism).

Definition A.1. Let $\mathbf{J}$ be a category and j an object in $\mathbf{J}$.

(i) j is *initial* if for each object $j' \in \mathbf{J}$ there is exactly one morphism $f\colon j \to j'$.

(ii) j is called *terminal* if for each object j' there is exactly one morphism $f\colon j' \to j$.

Remark. An initial object in a category $\mathbf{J}$, is a terminal object in $\mathbf{J}^{\mathrm{op}}$, and vice versa.

The following proposition is a justification of why we only refer to *the* initial and *the* terminal object.

Proposition A.2. *Let* $\mathbf{J}$ *be a category with initial and terminal objects.*

(i) Any two initial objects are canonically isomorphic

(ii) Any two terminal objects are canonically isomorphic

Proof. We prove the first statement, the proof for terminal objects is dual. Let j and j' be two initial objects. Then there exists exactly two morphisms $f\colon j \to j'$ and $g\colon j' \to j$. The composition $g \circ f$ then defines a map from j to itself. Since j is initial, this map has to be the identity. The same argument shows that $f \circ g\colon j' \to j'$ is also the identity. Hence, the two objects are canonically isomorphic. □

Example A.3. In the poset $[n] = 0 \leq 1 \leq \cdots \leq n$, we have the initial object 0 and the terminal object n.

Example A.4. Any additive category admits a zero object. i.e an object that is both initial and terminal.

An important notion in homological algebra is the idea of equivalences and adjunctions. In the spirit of category theory it is often too much to ask that something is exactly equivalent. Usually one asks that two categories are equivalent if they are 'practically the same'. By this we mean that there are functors between them that makes the compositions isomorphic to the identities. Adjunctions is another example of this as they give a relation between two functors that makes the categories 'almost' equivalent.

Since the theory of derivators heavily relies on their calculus, we give some attention to adjunctions and equivalences here.

Definition A.5. A functor $F\colon \mathbf{J} \to \mathbf{K}$ is called an *equivalence* if there exists a functor $G\colon \mathbf{K} \to \mathbf{J}$ such that $F \circ G \cong \mathrm{id}_{\mathbf{K}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{J}}$.

Example A.6. Let k be a field, and $\mathbf{Vec}_k$ the category of finite dimensional vector spaces over k. Denote by $\mathbf{Mat}_k$ the category whose objects are natural numbers, and morphisms are $n \times m$ matrices with entries in k. Then $F\colon \mathbf{Mat}_k \to \mathbf{Vec}_k$ which maps n to k^n is an equivalence.

One might ask what are the necessary conditions for a functor to be an equivalence. The following lemma is from *Homological Algebra* [18], and lets us recognize equivalences through other functor properties.

Lemma A.7. *Given two categories* $\mathbf{J}, \mathbf{K}$ *and a functor* $F\colon \mathbf{J} \to \mathbf{K}$*, then* F *is an equivalence if it is full, faithful and dense*

Proof. Assume F is fully faithful and dense. We are then going to create a functor in the other direction which satisfies the definition above.

Since F is dense we may, by a strong version of the axiom of choice, for any $k \in \mathbf{K}$ fix an object $G(k)$ and an isomorphism $\phi_k\colon (F \circ G)(k) \to k$. For a morphism

$f\colon k \to k'$ we get an induced bijection

$$\mathrm{Hom}_{\mathbf{J}}(G(k), G(k')) \to \mathrm{Hom}_{\mathbf{K}}((F \circ G)(k), (F \circ G)(k'))$$

from F being fully faithful. Define $G(f)$ to be the preimage of $\phi_{k'}^{-1} \circ f \circ \phi_k$. Then we have that

$$G(\mathrm{id}_{\mathbf{K}}) = F^{-1}(\phi_{\mathbf{K}}^{-1} \circ \mathrm{id}_{\mathbf{K}} \circ \phi_{\mathbf{K}}) = F^{-1}(\phi_{\mathbf{K}}^{-1} \circ \phi_{\mathbf{K}}) = F^{-1}(\mathrm{id}_{(F\circ G)(\mathbf{K})}) = F^{-1}F(\mathrm{id}_{G(\mathbf{K})}) = \mathrm{id}_{G(\mathbf{K})}$$

and for morphisms $f\colon k \to k'$ and $g\colon k' \to k''$ we get

$$G(g \circ f) = F^{-1}(\phi_{k''}^{-1} \circ (g \circ f) \circ \phi_k) = F^{-1}(\phi_{k''}^{-1} \circ g \circ (\phi_{k'} \circ \phi_{k'}^{-1}) \circ f \circ \phi_k)$$

$$= F^{-1}(\phi_{k''}^{-1} \circ g \circ \phi_{k'}) \circ F^{-1}(\phi_{k'}^{-1} \circ f \circ \phi_k) = G(g) \circ G(f)$$

which shows that G defines a functor $G\colon \mathbf{K} \to \mathbf{J}$.

Now, let $f\colon k \to k'$ be a morphism in $\mathbf{K}$. Then we have

$$\phi_{k'} \circ (F \circ G)(f) = \phi_{k'} \circ (F \circ F^{-1})(\phi_{k'}^{-1} \circ f \circ \phi_k) = \phi_{k'} \circ \phi_{k'}^{-1} \circ f \circ \phi_k = f \circ \phi_k$$

which shows that ϕ is a natural isomorphism so that $(F \circ G) \cong \mathrm{id}_{\mathbf{K}}$.

Finally, since F is fully faithful, by considering $(\phi \circ F)\colon \mathbf{F} \circ \mathbf{G} \circ \mathbf{F} \to \mathbf{F}$, we can find a unique morphism $\eta_j\colon (G \circ F)(j) \to j$ and $\bar{\eta}_j\colon j \to (G \circ F)(j)$ such that $F(\eta_j) = \phi_{F(j)}$ and $F(\bar{\eta}_j) = \phi_{F(j)}^{-1}$. Thus, η is a natural transformation, with an inverse, and hence $(G \circ F) \cong \mathrm{id}_{\mathbf{J}}$ and we are done. □

Remark. This lemma is really an if and only if statement, however we are only going to use the one implication in this text and as the proof takes up quite a bit of space, it has been left out. The reader is referred to [18] for further details.

Definition A.8. Let $\mathbf{J}, \mathbf{K}$ be two categories, and $F\colon \mathbf{J} \to \mathbf{K}$ and $G\colon \mathbf{K} \to \mathbf{J}$ be two functors between them. We say F and G is an *adjoint pair* $(F, G)\colon \mathbf{J} \leftrightarrows \mathbf{K}$ if

$$\mathrm{Hom}_{\mathbf{B}}(F(-), -) \text{ and } \mathrm{Hom}_{\mathbf{A}}(-, G(-))$$

are naturally isomorphic. In this case F is a left adjoint, and G is a right adjoint.

There is an equivalent statement for adjoint pairs. Namely that $(F, G)\colon \mathbf{J} \leftrightarrows \mathbf{K}$ if and only if there exists natural transformations $\eta\colon \mathrm{id}_{\mathbf{J}} \to G \circ F$ and $\epsilon\colon F \circ G \to \mathrm{id}_{\mathbf{K}}$ such that

$$\mathrm{id}_F = \epsilon_{F(-)} \circ F(\eta(-)) \quad \text{and} \quad \mathrm{id}_G = G(\epsilon(-)) \circ \eta_{G(-))}.$$

See [18] for details. This can be visualized as the two diagrams

$$\begin{tikzcd}
F & F \circ G \circ F \arrow[l, "(\epsilon\cdot F)"'] & G & G \circ F \circ G \arrow[l, "(G\cdot\epsilon)"'] \\
& F \arrow[u, "(F\cdot\eta)"'] \arrow[ul, bend left, "\mathrm{id}"] & & G \arrow[u, "(\eta\cdot G)"'] \arrow[ul, bend left, "\mathrm{id}"]
\end{tikzcd} \tag{11}$$

known as the *triangular identities.*

Example A.9. Let **Set** be the category of sets and **Grp** the category of groups. Define Forget: **Grp** → **Set** as the functor that maps a group to the underlying set, and Free: **Set** → **Grp** the functor that assigns to each set the free group generated by the elements of that set. Then we have an adjunction $(\mathrm{Free}, \mathrm{Forget})\colon \mathbf{Set} \leftrightarrows \mathbf{Grp}$.

Lemma A.10. *Let* $\mathbf{J}, \mathbf{K}$ *be two categories, and* $(F, G)\colon \mathbf{J} \leftrightarrows \mathbf{K}$ *an adjunction between them.*

(i) G *is fully faithful if and only if the counit* $\epsilon\colon F \circ G \to \mathrm{id}_{\mathbf{K}}$ *is a natural isomorphism.*

(ii) F *is fully faithful if and only if the unit* $\eta\colon \mathrm{id}_{\mathbf{J}} \to G \circ F$ *is a natural isomorphism.*

Proof. We prove the first statement, the second one is dual.

The counit is by definition a natural transformation $\epsilon\colon F \circ G \to \mathrm{id}_{\mathbf{K}}$. Now note that

$$\mathrm{Hom}_{\mathbf{K}}(k, k') \xrightarrow{G} \mathrm{Hom}_{\mathbf{J}}(G(k), G(k')) \xrightarrow[\cong]{\phi} \mathrm{Hom}_{\mathbf{K}}((F \circ G)(k), k')$$

implies that G is fully faithful if and only if $\epsilon_k \cong \mathrm{id}_k$ for all $k \in \mathbf{K}$ if and only if ϵ is a natural isomorphism. □

For two categories $\mathbf{J}, \mathbf{K}$, we can consider the category of functors between them. Denote by $\mathbf{J}^{\mathbf{K}}$ the category of functors from $\mathbf{K}$ to $\mathbf{J}$. Then objects are functors $F\colon \mathbf{K} \to \mathbf{J}$, and morphisms are natural transformations $\eta\colon F \to G\colon \mathbf{K} \to \mathbf{J}$. Often when we have certain properties of categories, they can be transferred to functor categories.

Proposition A.11. *Let* $\mathbf{C}$ *be a category. If* $(F \leftrightarrows G)\colon \mathbf{J} \to \mathbf{K}$ *is an adjunction between two small categories, then we get an induced adjunction* $(F^* \leftrightarrows G^*)\colon \mathbf{C}^{\mathbf{K}} \to \mathbf{C}^{\mathbf{J}}$ *by precomposition.*

Proof. This follows from the fact that for all $X \in \mathbf{C}^{\mathbf{K}}$ we get $(X \circ F) \in \mathbf{C}^{\mathbf{J}}$ and similarly for all $Y \in \mathbf{C}^{\mathbf{J}}$ we can precompose with G. So then $X \circ (F \circ G) \in \mathbf{C}^{\mathbf{K}}$, and the counit $\epsilon\colon (F \circ G) \to \mathrm{id}_{\mathbf{K}}$ induces a unit through $\epsilon^*\colon \mathrm{id}_{\mathbf{C}^{\mathbf{K}}} \to (G^* \circ F^*) = (F \circ G)^*$ defined by $\epsilon^*_X = X \circ \epsilon$. Similarly, we obtain $\eta^*\colon \mathrm{id}_{\mathbf{C}^{\mathbf{J}}} \to (G \circ F)^*$. □

Definition A.12. If $F\colon \mathbf{J} \to \mathbf{K}$ is a functor between categories, with $\mathbf{J}$ a small category, then the *colimit* of the functor F is an object in $\mathbf{K}$, denoted $\operatorname{colim}_{\mathbf{J}} F$, together with a natural isomorphism such that

$$\mathrm{Hom}_{\mathbf{K}}(\operatorname{colim}_{\mathbf{J}} F, -) \simeq \mathrm{Hom}_{\mathbf{K}^{\mathbf{J}}}(\Delta -, F)$$

We define the *limit* dually.

Remark. It follows from the definitions that (co)limits are really just adjunctions $(\operatorname{colim}_{\mathbf{B}} \leftrightarrows \Delta_B)$, and $(\Delta_B \leftrightarrows \lim_{\mathbf{B}})$

Recall the definition of an abelian category as an additive category where the natural morphism $\phi\colon \mathrm{Coim} \to \mathrm{Im}$ is an isomorphism. It turns out that functor categories 'inherit' this property.

Proposition A.13. *Let* $\mathcal{A}$ *be an abelian category, and* $\mathbf{J}$ *be a small category. Then the category* $\mathcal{A}^{\mathbf{J}}$ *of functors from* $\mathbf{J}$ *to* $\mathcal{A}$ *is also abelian.*

Proof. First we show that $\mathcal{A}^{\mathbf{J}}$ is additive. Let j be any object in $\mathbf{J}$, and $F, G\colon \mathbf{J} \to \mathcal{A}$ be two functors. Since all natural transformations between F and G are also morphisms in $\mathcal{A}$, they satisfy the definition of a pre-additive category. We can then define the biproduct component-wise by $(F \bigoplus G)(j) = F(j) \bigoplus G(j)$. This is well-defined since $\mathcal{A}$ is abelian. Because there exists a zero object $0 \in \mathcal{A}$, and all objects have morphisms to 0, there is a zero functor $0 \colon \mathbf{J} \to \mathcal{A}$ that maps all objects and morphisms to 0. We define $\mathbf{0} \colon F \to 0$ as the natural transformation that maps any $F(j)$ to 0, and similarly 0 to any $F(j)$. This then defines the zero object.

We now show that this is also an abelian category. First, since each $\eta_j \colon F(j) \to G(j)$ has a kernel $\ker_j \in \mathcal{A}$, we can define the kernel functor $\ker \colon \mathbf{J} \to \mathcal{A}$ by $\ker(j) = \ker_j$. Dually, we define the cokernel functor. Since this construction was made component-wise in $\mathcal{A}$, it follows that the induced natural transformation $\bar{f}_j \colon \mathrm{Im}_j \to \mathrm{Coim}_j$ is a natural isomorphism for all j. Hence $\mathcal{A}^{\mathbf{J}}$ is abelian. □

Remark. That the functor category $\mathbf{C}^{\mathbf{J}}$ 'inherited' the underlying properties of $\mathbf{C}$ is not unique to the abelian categories. In fact, it is rather common that we can pass on properties from the target category to the functor category.

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