Master's thesis

NTNU
Norwegian University of Science and Technology
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Department of Mathematical Sciences

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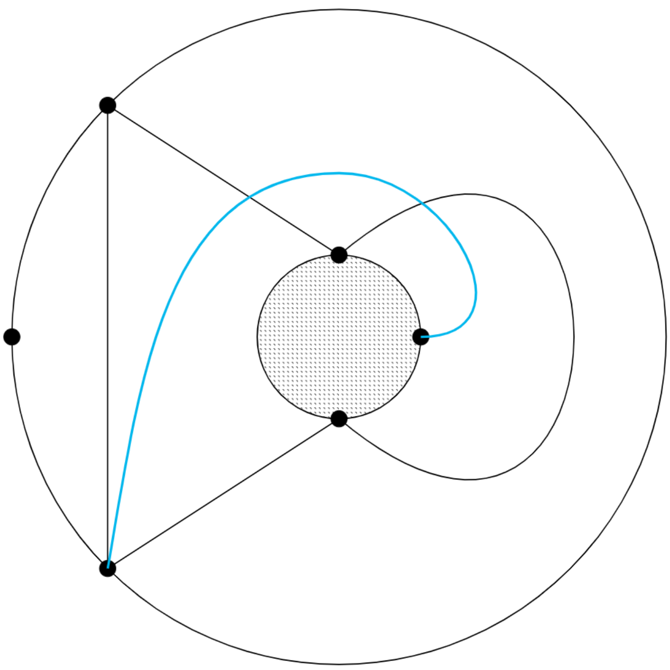
Gentle Algebras and a Geometric Model for the Module Category

Master's thesis in Mathematical Sciences
Supervisor: Steffen Oppermann
June 2021



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Abstract

In this thesis we show that gentle algebras are isomorphic to tiling algebras. Tiling algebras can be translated into lamination algebras, which are used to restore the orginal gentle algebra. These isomorphisms define the foundation for the construction of a geometric model for the module category over a gentle algebra.

Sammendrag

I denne oppgaven viser vi at milde algebraer er isomorfe til flisalgebraer. Flisalgebraer kan visulaiseres som en lamineringsalgebra, som brukes til å gjenopprette den milde algebraen. Disse isomorfiene danner grunnlaget for en konstruksjon av en geometrisk modell for modulkategorien over en mild algebra.

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Christina Dønvold Sjøborg
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Chapter 1

Introduction

Representing a complex mathematical object in a simpler way is a core part of representation theory. For instance, we can represent a linear transformation on a finite dimensional vector space by its reduction or its Jordan normal form. When we study representations of an algebra today, this is done by classifying the indecomposable modules over that algebra and the homomorphisms between them. In other words, we want to describe the algebra through how it effects the structures connected to it.

The study of gentle algebras started with Assem and Skowrónski in 1987 [1], and has been an active area of research in representaion theory since then. Gentle algebras are considered to be a nice class of algebras because of the simple combinatorial definition and the many connections to other classes of algebras. For instance, they are connected to string algebras, biserial algebras, special biserial algebras, tame algebras and so on.

Every gentle algebra is a string algebra. This allows us to describe the indecomposable modules as string and band modules, which is done in Chapter 4. Using this description we can find the irreducible morphisms between string modules. Before this we give some preliminaries in Chapter 2 and a brief introduction to Auslander-Reiten theory in Chapter 3. In Chapter 4 we consider Auslander-Reiten theory for gentle algebras specifically. In Chapter 5 we introduce the notion of a tiling algebra and prove that a tiling algebra is gentle. We then want to recover the gentle algebra. This is done by using the notion of ribbon graphs and ribbon surfaces to define the lamination algebra of a gentle algebra in Chapter 6. Finally, we use these isomorphisms to construct a geometric model for the module category over a gentle algebra. We do so in terms of tilings in Chapter 7. We can summarize the main theorems in this thesis as follows.

Theorem. *Let A be a finite dimensional algebra. Then the following are equivalent.*

(1) A is gentle
(2) A is a tiling algebra
(3) A is a lamination algebra.

Theorem. *Let A_{P} be the tiling algebra of a surface S with a set of marked points M on the boundary and a partial triangulation P. Then there are bijections*

(1) between the equivalence classes of non-trivial premissible arcs of (S, M) and non-zero strings of A.
(2) between pivot elementary move of permissible arcs and irreducible morhisms in $\operatorname{mod} A$.

The geometric model for the module category is constucted in the article of Baur and Simões [2]. A similar geometric model is given by Opper, Plamondon and Schroll in their article [3]. They described geometrically the bounded derived category of a gentle algebra up to shift. This thesis is mainly based upon Refs. [4, 5, 2, 3], where the reader will find the main part of the proofs and results given in this thesis.

This thesis is written to be understandable for someone who has completed an introductionary course in representation theory. However, some basic representation theory will be introduced in Chapter 2. The reader should also be familiar with some homological algebra, at the level of basic definitions and terminology. Any necessary theory beyond this will be introduced.

Chapter 2

Preliminaries

In this chapter we recall some defintions and results that gives the necessary background information. The information can be found in any introductionary book in representaion theory, for instance we refer to [4] and [6]. We will assume that some basic homological algebra is known and we refer to [7] or any introductionary book on the topic, for more information. Throughout this thesis K is a field and any algebras over K are considered to be finite dimensional, unless otherwise is stated. Whenever we write $\operatorname{mod} A$ we mean the category of finitely generated right A-modules and when we write $\operatorname{Mod} A$ we mean the category of all right A-modules, where A is a K-algebra.

2.1 Algebras and Modules

Definition 2.1. Let K be a field. We define a K**-algebra** A to be a vector space over K with a multiplication as follows,

$$\lambda(aa') = (\lambda a)a' = a(\lambda a') \ \ \forall \lambda \in K \ \ and \ \ \forall a, a' \in A.$$

We say that A is **finite dimensional** if there is a finite set $\{e_1, \dots, e_n\}$ in A such that $A = \sum_{i=1}^{n} K_i e_i$.

In a sense, we can think of an algebra as both a ring and a vector space at the same time. An algebra is said to be **local** if it has a unique maximal right ideal.

Definition 2.2. The (Jacobsen) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all the maximal right ideals in A. We define the right $(A/\operatorname{rad} A)$-module **top of** M as

$$\operatorname{top} M = M / \operatorname{rad} M.$$

The set $\operatorname{Hom}_A(M,N)$ of all A-module homomorphisms from M to N is a K-vector space with respect to the scalar multiplication $(f,\lambda) \mapsto f\lambda$ given by $(f\lambda)(m) = f(m\lambda)$ for $f \in \operatorname{Hom}_A(M,N)$, $\lambda \in K$ and $m \in M$. If the modules M and N are finite dimensional, then the K-vector space $\operatorname{Hom}_A(M,N)$ is finite dimensional. Additionally, the K-vectorspace $\operatorname{End}(M) = \operatorname{Hom}_A(M,M)$ of all A-module endomorphisms of a right A-module M is an associative K-algebra with respect to the composistion of morphisms.

Recall that whenever we talk about module-homomorphisms, a monomorphism and an epimorphism are the same as an injective and surjvective A-module homomorphism, respectively. In particular a split monomorphism is an injective A-module homomorphism, and a split epimorphism is a surjective A-module homomorphism.

Definition 2.3. A right A-module P is **projective** if for any epimorphism $h : M \to N$, and for any $f \in \operatorname{Hom}_A(P,M)$ there is an $f' \in \operatorname{Hom}_A(P,N)$ such that the following diagram is commutative.

$$\begin{tikzcd} & P \arrow[dl, "f'"'] \arrow[d, "f"] & \\ M \arrow[r, "h"'] & N \arrow[r] & 0 \end{tikzcd}$$

Similarly, we say that an A-module I is **injective** if for any monomorphism $u : L \to M$ and any $g \in \operatorname{Hom}_A(L,I)$, there is a $g' \in \operatorname{Hom}_A(M,I)$ such that the following diagram is commutative.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "g"'] & M \arrow[dl, "g'"] \\ & I & \end{tikzcd}$$

An A-module homomorphism $f : M \to N$ is called an **essential epimorphism** if f is an epimorphism and if $g : X \to N$ is such that $f \circ g : X \to N$ is onto, then g is onto.

Let $f : P \to M$ be an A-module homomorphism. Then f is a **projective cover of** M if P is projective and f is an essential epimorphism.

Definition 2.4. An exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

in $\operatorname{mod} A$ is called a **minimal projective presentation** of an A-module M if p_0 and p_1 are projective covers.

Let A be a ring and let M and N be A-modules, where M is a submodule of N. We say that M is an **essential submodule of** N if for each non-zero submodule L of N we have that $M \cap L \neq (0)$. A monomorphism $i : M \to N$ is an **essential monomorphism** if $i(M)$ is an essential submodule of N.

A monomorphism $i : M \to I$ is an **injective envelope** if I is injective and i is an essential monomorphism.

Definition 2.5. An exact sequence

$$0 \to N \xrightarrow{u^0} I^0 \xrightarrow{u^1} I^1$$

is a **minimal injective presentation** of an A-module N if the monomorphisms u^0 and u^1 are injective envelopes.

Definition 2.6. Let A be a finite dimensional K-algebra. We define the functor

$$D = \operatorname{Hom}_K(-, K) : \operatorname{mod} A \to \operatorname{mod} A^{op}$$

by assigning each right module M in $\operatorname{mod} A$ to the dual K-vector space

$$DM = \operatorname{Hom}_K(M, K) = M^*.$$

DM is endowed with a left A-module structure. The functor D is often called a **standard duality**.

Note that if we consider left A-modules we will have a standard duality

$$D : \operatorname{mod} A^{op} \to \operatorname{mod} A.$$

This will in fact be the inverse of the previous duality. Remark that we here denote the left finitely generated A-modules by $\operatorname{mod} A^{op}$. The standard duality will be important in defining the Auslander-Reiten translation of a module.

The following theorem states some facts about the standard duality D, the proof can be found in [8].

Theorem 2.7. *Let A be a finite dimensional K-algebra and let D be the standard duality. Then the following hold.*

(a) A sequence $0 \to L \xrightarrow{u} N \xrightarrow{h} M \to 0$ in $\operatorname{mod} A$ is exact if and only if the induced sequence $0 \to D(M) \xrightarrow{D(h)} D(N) \xrightarrow{D(u)} D(L) \to 0$ is exact in $\operatorname{mod} A^{op}$.

(b) A module I in $\operatorname{mod} A$ is injective if and only if the module $D(I)$ is projective in $\operatorname{mod} A^{op}$. A module P in $\operatorname{mod} A$ is projective if and only if $D(P)$ is injective in $\operatorname{mod} A^{op}$.

(c) A module S in $\operatorname{mod} A$ is simple if and only if the module $D(S)$ is simple in $\operatorname{mod} A^{op}$

(d) A monomorphism $u : M \to I$ in $\operatorname{mod} A$ is an injective envelope if and only if the morphism $D(u) : D(I) \to D(M)$ is a projective cover in $\operatorname{mod} A^{op}$. An epimorphism $h : P \to M$ in $\operatorname{mod} A$ is a projective cover if and only if the morphism $D(h) : D(M) \to D(P)$ is an injective envelope.

Suppose that A is a K-algebra that can be written as $A = P_1 \oplus \cdots \oplus P_N$, where each P_i are indecomposable right ideals of A and $P_i = e_i A$, where all the e_i's are primitive orthogonal idempotents of A such that $1 = \sum_{i=1}^{n} e_i$. We then say that the set $\{e_1, \dots, e_n\}$ is a **complete set of primitive orthogonal idempotents**. If in addition the following is satisfied; $e_i A \ncong e_j A \ \ \forall i \neq j$, we define A to be **basic**.

Proposition 2.8. Let A be a finite dimensional K-algebra. Then P is a projective A-module if and only if there exists a free A-module F and an A-module Q such that $F \cong P \oplus Q$.

Proof. Suppose P is generated by $\{m_j | j \in J\}$. If F is free, we can write $F = \bigoplus_{j \in J} x_j A$, where the set $\{x_j\}_{j \in J}$ is the set of generators. Now, we define a morphism $f : F \to P$ such that $f(x_j) = m_j$. Then f is an epimorphism, and by the projectivity of P there exists a morphism $g : P \to F$ such that $f g = 1_P$. This implies that $F \cong P \oplus \operatorname{Ker} f$.

Assume ϕ is the isomorphism $\phi : F \xrightarrow{\cong} P \oplus Q$, where F is free. Suppose that $g : B \to C$ is an epimorphism and let $f : P \to C$ be an A-homomorphism, illustrated as the solid arrows in the diagram below.

$$\begin{tikzcd}
F \cong P \oplus Q \arrow[r, two heads, "\pi"] \arrow[d, dotted, "h'"'] & P \arrow[d, "f"] \arrow[l, dotted, bend left, "i"] & \\
B \arrow[r, two heads, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

Here we have that π is the canonical projection and i is the canonical inclusion, i.e. $\pi(p,q) = p$ and $i(p) = (p,0)$. We observe that $\pi i(p) = p$, and hence $\pi i = 1_P$. By the fact that a free module is projective, there is a morphism $h' : F \to B$ such that

$$\begin{aligned}
& gh' = f\pi\phi \\
\rightsquigarrow\ & gh'\phi^{-1} = f\pi \\
\rightsquigarrow\ & gh'\phi^{-1}i = f\pi i = f
\end{aligned}$$

Hence, P is projective. ■

Let $\{e_1, \ldots, e_n\}$ be the set of primitive orthogonal idempotents of a finite dimensional K-algebra A. Then we know the right A-module A_A can be written as $A_A = e_1A \oplus \cdots \oplus e_nA$. Recall that if an A-module M is isomorphic to e_iA, then it is indecomposable.

Proposition 2.9. Let A be a finite dimensional K-algebra, and let $\{e_1, \ldots, e_n\}$ be the set of primitive orthogonal idempotents such that $A_A = e_1A \oplus \cdots \oplus e_nA$. If P is a projective A-module, then $P = P_1 \oplus \cdots \oplus P_s$, for some $s \leq n$, where every P_i is indecomposable and isomorphic to some e_iA.

Proof. Let P be projective. By the Proposition 2.8 there is a free A-mod F and an A-module Q such that $F \cong P \oplus Q$. By assumption F is a direct sum of indecomposable modules $e_1A, \ldots, e_mA$. Since e_iA are indecomposable we know that $\mathrm{End}(e_iA)$ is local and that the decomposistion is unique, by the unique decomposition theorem. Hence;

$$\begin{aligned}
\rightsquigarrow\ & F \cong e_1A \oplus \cdots \oplus e_mA \cong P \oplus Q \\
\rightsquigarrow\ & P \cong e_1A \oplus \cdots \oplus e_sA, \text{ for some } s.
\end{aligned}$$

■

Corollary 2.10. Let A be a finite dimensional K-algebra, and let $\{e_1, \dots, e_n\}$ be the set of primitive orthogonal idempotents. Then every indecomposable projective right A-module is isomorphic to e_iA for some $i \in \{1, \dots, n\}$.

2.2 Quivers and Representations

Definition 2.11. A **quiver** $Q = (Q_0, Q_1)$ is an oriented graph which consists of two sets:

$$Q_0 = \{vertices\}, \quad Q_1 = \{arrows\}.$$

Definition 2.12. A **path** in a quiver Q is an ordered sequence of arrows $p = \alpha_n \alpha_{n-1} \dots \alpha_1$ where the endpoint of α_i is equal to the startpoint of α_{i+1}, in other words $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, n-1$. Additionally, there is a trivial path e_i for each i in Q_0. The trivial path is stationary at the vertex i.

Note that $s(\alpha_i)$ denotes the source or the startpoint of α_i and $e(\alpha_i)$ denotes the target or the endpoint of α_i. We will also use the notation $t(\alpha)$ for the endpoint of α. If the startpoint and endpoint of a path coincide we call it an oriented cycle or a cycle for short.

Definition 2.13. Let Q be a quiver and let K be a field. We define KQ to be the **path algebra**, which is the vector space with the paths in Q as basis-elements. The elements in KQ are of the form

$$a_1 p_1 + a_2 p_2 + \dots + a_t p_t$$

where $a_i \in K$ and p_i are paths in Q. The identity element of KQ is $e_1 + e_2 + \dots + e_t$, where e_i is the trivial path at the vertex i, for all $i \in Q_0$.

Definition 2.14. Let Q be a quiver and let $\rho = \langle \text{arrows} \rangle$ be the ideal in KQ generated by the arrows in Q. We say that an ideal $I \subseteq KQ$ is **admissible** if $\rho^m \subseteq I \subseteq \rho^2$ for $m \geq 2$. If I is an admissible ideal of KQ, then we say that (Q, I) is a **bounded quiver**.

Note that if I is an admissible ideal, we define $A = KQ/I$ to be the path algebra in a similar way as above.

Definition 2.15. Let Q be a finite quiver. A K**-linear representation**, or briefly a **representation**, M of Q is defined by the following:

- For each vertex v in Q_0 there is associated a K-vector space M_v.
- For each arrow $\alpha : v_1 \to v_2$ in Q_1 there is a K-linear map $\phi_\alpha : M_{v_1} \to M_{v_2}$.

Recall that each K-representation of Q corresponds to a KQ-module. In particular a finite representation corresponds to a finitly generated module. The simple indecomposable KQ-modules (or representations) is denoted by S_v and can be described as follows: for each vertex $w \in Q_0$, $(S_v)_w = 0$ if $w \neq v$ and $(S_v)_w = K$ if $w = v$. For each arrow $\alpha \in Q_1$ define $\phi_\alpha = 0$.

Now, let A denote the path algebra KQ. We can find all projective indecomposable A-modules by letting each vector space M_v be generated by all the paths from a fixed vertex i to the vertex v. We denote this projective module by $P(i)$. Similarly, we can find all injective indecomposable A-modules by letting each vector space M_v be generated by all the paths from v to a fixed vertex i. We denote the indecomposable injective modules by $I(i)$ for each vertex $i \in Q_0$.

Recall that the radical of a representation M is a representation where each vertex is the sum of images of the linear maps ending at that vertex. Also, recall that we can find the socle of a representation. The socle is again a representation, where each vector space is the the intersection of the kernels of morphisms starting at that vertex. The linear maps in both the radical and the socle of a representation are the linear maps from the representation restiricted to the new vector spaces.

We finish this chapter by considering an example.

Example 2.16. Let K be a field, and let Q be the quiver

$$Q = \quad \begin{tikzcd} 1 \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & 2 \\ & 3 \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad I = \langle \gamma\beta \rangle$$

We want to find all the indecomposable projective and injective modules of the path algebra $A = KQ/I$, and we start by finding the projective module at vertex 1. There is precisely one path from the vertex 1 to vertex 1. This means that the vector space at vertex 1 is K. Now, there are two paths from vertex 1 to vertex 2, the paths α and $\gamma\beta$. However, the latter path is in the ideal, and thus, we will have the vector space K at vertex 2 as well. There is precisely one path from vertex 1 to vertex 3, namely β. Thus we have the following temporary representation:

$$M = \quad \begin{tikzcd} K \arrow[rr] \arrow[dr] & & K \\ & K \arrow[ur] & \end{tikzcd}$$

Now, we need to find the linear maps. The morphisms corresponding to α and β will be the identities. For the criteria defined by the ideal we need to have the composition $\gamma\beta$ to be zero. Thus, we let the morphism corresponding to γ to be the zero morphism. Finally, we have the following projective module.

$$P(1) = \quad \begin{tikzcd} K \arrow[rr, "1"] \arrow[dr, "1"'] & & K \\ & K \arrow[ur, "0"'] & \end{tikzcd}$$

In a similar way we find the projective indecomposable modules $P(2)$ and $P(3)$.

$$P(2) \quad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & K \\ & 0 \arrow[ur] & \end{tikzcd} \qquad P(3) = \quad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd}$$

We remark that $P(2)$ is a simple projective module.

By dual consideration we find the indecomposable injective modules. We start by finding the indecomposable injective module at vertex 1. There is precisely one path from vertex 1 to vertex 1. Thus the vectorspace at vertex 1 is K. Now there are no paths from vertex 2 to vertex 1, neither from vertex 3 to vertex 1. Hence the vector spaces at vertex 2 and 3 are both 0. Thus we have the following.

$$I(1) = \quad \begin{tikzcd} K \arrow[rr] \arrow[dr] & & 0 \\ & 0 \arrow[ur] & \end{tikzcd}$$

In a similar way we find the indecomposable injective modules for vertex 2 and 3.

$$I(2) = \quad \begin{tikzcd} K \arrow[rr, "1"] \arrow[dr, "0"'] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd} \qquad I(3) = \quad \begin{tikzcd} K \arrow[rr] \arrow[dr, "1"'] & & 0 \\ & K \arrow[ur] & \end{tikzcd}$$

We will now find the radical of $P(1)$ and the socle of $I(2)$. To find the radical we consider the images of the linear maps. There are no map ending at the vertex one. Thus, the vector space will be 0. There is precisely one map ending at vertex 3, which has image equal K, and therefor the vector space at vertex 3 is K. For the vertex two, there are two maps ending here, one with image equal 0 and one with image equal K. The sum of those is K. Hence, we have the following representation.

$$\operatorname{rad} P(1) = \quad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd}$$

To find the socle of the $I(2)$ we consider the kernels of the linear maps. First of, there are no maps starting at vertex 2, thus the vector space is still K. Now for the other vertices, they are both the domain of a map with kernel 0. Thus, regardless of the other maps the vector spaces will be the zero vector sapce. This gives us the following representation.

$$\operatorname{soc} I(2) = \quad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & K \\ & 0 \arrow[ur] & \end{tikzcd}$$

Chapter 3

Auslander-Reiten Theory

In this chapter we will give a brief introduction to Auslander-Reiten theory. This will be usefull to characterize the module category of a gentle algebra. Throughout this chapter A denotes a finite dimensional K-algebra, where K is an algebraically closed field. All A-modules are right finite dimensional A-modules. We will state some results without proofs. However, the proofs can be found in [4, ch.4], which is also the main source for this chapter.

3.1 Irreducible Morphisms and Almost Split Sequences

In this section we define irreducible, minimal and almost split morphisms in the category $\operatorname{mod} A$. Recall that any module in $\operatorname{mod} A$ can be written as a direct sum of indecomposable modules, and that such a decomposistion is unique up to isomorphism and permutation of the summands. Thus, to describe the objects and morphisms of $\operatorname{mod} A$ it suffices to describe the indecomposable summands and the homomorphisms between them. These morphisms turn out to be the irreducible.

Definition 3.1. Let L, M, N be modules in $\operatorname{mod} A$, and let $f : L \to M$ and $g : M \to N$ be A-homomorphisms.

(1) We define f to be **left minimal** if every $h \in \operatorname{End}(M)$ satisfying $hf = f$, is an isomorphism.

(2) We define g to be **right minimal** if every $k \in \operatorname{End}(M)$ satisfying $gk = g$, is an isomorphism.
(3) The homomorphism f is said to be **left almost split** if f is not a split monomorphism, and for every $u \in \operatorname{Hom}_A(L,U)$ which is not a split monomorphism there exists a $u' \in \operatorname{Hom}_A(M,U)$ such that the following diagram is commutative:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[dl, "u'"] \\ U & \end{tikzcd}$$

(4) The homomorphism g is said to be **right almost split** if g is not a split epimorphism, and $\forall v \in \operatorname{Hom}_A(V,N)$ that is not a split epimorphism there exists $v' \in \operatorname{Hom}_A(V,M)$ such that the following diagram is commutative

$$\begin{tikzcd} V \arrow[d, "v'"'] \arrow[dr, "v"] & \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

(5) The homomorphism f is defined to be **left minimal almost split** if it is both left minimal and left almost split.
(6) The homomorphism g is defined to be **right minimal almost split** if it is both right minimal and right almost split.

Note that each "right" morphism is the dual of the corresponding "left" morphism. We will now prove that left minimal almost split morphisms uniquely determine their targets. In a similar way we can prove that right minimal almost split morphisms uniquely determine their sources.

Proposition 3.2. (1) Let $f \in \operatorname{Hom}_A(L,M)$ and $f' \in \operatorname{Hom}_A(L,M')$ be left minimal almost split. Then there is an isomorphism $h \in \operatorname{Hom}_A(M,M')$ such that $f' = hf$.
(2) Let $g \in \operatorname{Hom}_A(M,N)$ and $g' \in \operatorname{Hom}_A(M',N)$ be right minimal almost split. Then there is an isomorphism $k \in \operatorname{Hom}_A(M,M')$ such that $g = g'k$.

Proof. We will only prove the first statement, the second statment is proven similarly. Since f and f' are almost split, we know there is a $m \in \mathrm{Hom}_A(M, M')$ and $m' \in \mathrm{Hom}_A(M', M)$ such that $f' = mf$ and $f = m'f'$. This implies that $f = m'mf$ and $f' = mm'f'$. Since f and f' are both minimal, we have by definition that mm' and $m'm$ are both isomorphisms. Hence, m is an isomorphism. ■

Lemma 3.3. *(1) If $f \in \mathrm{Hom}_A(L, M)$ is left almost split, then L is indecomposable.*
(2) If $g \in \mathrm{Hom}_A(M, N)$ is right almost split, then N is indecomposable.

Proof. We will only prove the first statement, the second statement is proven similarly. Assume for contradiction that L can be written as $L = L_1 \oplus L_2$, where L_1 and L_2 are non-zero. Let p_i be the projections $p_i : L \to L_i$ for $i = 1, 2$. By definition of left almost split morphisms, the projections p_i cannot be split monomorphisms. Since p_i is not split monomorphisms there is an $u_i : M \to L_i$ such that $u_i f = p_i$. However, we then have $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : M \to L = L_1 \oplus L_2$, which satisfies $uf = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 1_L$. This contradicts the fact that f cannot be a split monomorphism. ■

We wil now define irreducible morphisms in $\operatorname{mod} A$. These play a crucial role in Auslander-Reiten theory and will be important when we give a geometric model for the module category of a gentle algebra in Chapter 7.

Definition 3.4. A homomorphism $f : X \to Y$ in $\operatorname{mod} A$ is **irreducible** if f is not a split monomorphism nor a split epimorphism, and if $f = f_2 f_1$, then either f_2 is a split epimorphism or f_1 is a split monomorphism.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y \\ & Z \arrow[ur, "f_2"'] & \end{tikzcd}$$

Proposition 3.5. (1) If $f : L \to M$ is an irreducible monomorphism, then the cokernel of f, $\operatorname{Cok} f$ is indecomposable.
(2) If $g : M \to N$ is an irreducible epimorphism, then the kernel of g, $\operatorname{Ker} g$ is indecomposable.

Proof. We will only prove the first statement, the second one is proven similarly. Let $g : M \to N$ be the cokernel of f and suppose that $N = N_1 \oplus N_2$ with $N_i \neq 0$, for $i = 1, 2$. Define $q_i : N_i \to N$ be the corresponding inclusions. If there is a homomorphism $u_i : M \to N_i$ such that $g = q_i u_i$, then q_i is an epimorphism, because g is an epimorphism. Thus, q_i is an isomorphism, which contradicts that $N_i \neq 0$. Hence, there is a homomorphism $v_i : N_i \to M$ for all i, such that $g v_i = 1_N$, and thus, g is a split epimorphism. Then f needs to be a split monomorphism, and this contradicts the fact that f is irreducible. ■

We finish this section by defining what an Auslander-Reiten sequence is and give some results regarding Auslander-Reiten sequences.

Definition 3.6. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\operatorname{mod} A$. The short exact sequence is an **almost split sequence** or an **Auslander-Reiten sequence** if f is left minimal almost split and g is right minimal almost split.

Note that the existence of such a sequence is not obvious and is written out in detail in [4, Ch.4]. However, given that there exists such a sequence, we have several results. We write some them in the following remark.

Remark 3.7. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an almost split sequence.

- Then L and N are indecomposable modules.
- Since f is not a split monomorphism and g is not a split epimorphism, an Auslander-Reiten sequence is never split.
- L is never injective and N is never projective.
- If $0 \to L' \xrightarrow{f} M' \xrightarrow{g} N' \to 0$ is another almost split sequence in $\operatorname{mod} A$, then the sequences are isomorphic if and only if $L \cong L'$ as A-modules if and only if $N \cong N'$ as A-modules.

The next theorem can be useful to determine whether a sequence is almost split or not. The proof is written out in [4, Ch.4, Thm. 1.13].

Theorem 3.8. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence in* $\operatorname{mod} A$*. Then the following are equivalent.*

(1) The sequence is almost split

(2) *L is indecomposable, and g is right almost split.*
(3) *N is indecomposable, and f is left almost split.*
(4) *f is left minimal almost split.*
(5) *g is right minimal almost split.*
(6) *L and N are indecomposable, and f and g are irreducible.*

3.2 Auslander-Reiten Translations

In this section we will define the Auslander-Reiten translations and give some results. But before giving a precise definition we need some terminology. Let A be a finite dimensional K-algebra, and let $\operatorname{mod} A$ represent the category of finite dimensional A-modules. We define the following A-dual functor

$$(-)^t = \operatorname{Hom}_A(-, A)\colon \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{op}}.$$

This functor sends each right A-module to a left A-module. So, if P is a projective right A-module, then $P^t = \operatorname{Hom}_A(P, A)$ is a projective left A-module. Furthermore, if $P \cong eA$ with $e \in A$ a primitive idempotent, then $P^t = \operatorname{Hom}_A(eA, A) \cong Ae$. The functor $(-)^t$ induces a duality between the category proj A, of projective right A-modules, and the category proj A^{op} of projective left A-modules. We define the **transposition** to be this new duality of $\operatorname{mod} A$.

Thus, let

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective resolution of M. If we now apply the duality $(-)^t$, we get an exact sequence of left A-modules

$$0 \to M^t \xrightarrow{p_0'} P_0^t \xrightarrow{p_1'} P_1^t \to \operatorname{Cok} p_1^t \to 0$$

We denote the cokernel $\operatorname{Cok} p_1^t$ by $\operatorname{Tr} M$ and we call it the **transpose** of M. We see that the transpose of M is uniquely determined up to isomorphism, because of the uniqueness of minimal projective resolutions. The following result is an overview of the main properties of Tr , see [4, Ch. 4.2] for the proof.

Proposition 3.9. Let M be an indecomposable module in $\operatorname{mod} A$.

(a) The transpose $\operatorname{Tr} M$ has no nonzero projective direct summands.
(b) If M is not projective, then the sequence $P_0^t \xrightarrow{p_1^t} P_1^t \to \operatorname{Tr} M \to O$ induced from the minimal projective resolution $P_1 \to P_0 \to M \to 0$ of M is a minimal projective resolution of the module $\operatorname{Tr} M$.
(c) M is projective if and only if $\operatorname{Tr} M = 0$. If M is not projective, then $\operatorname{Tr} M$ is indecomposable and $\operatorname{Tr} \operatorname{Tr} M \cong M$
(d) If M and N are indecomposable nonprojective modules then $M \cong N$ if and only if $\operatorname{Tr} M \cong \operatorname{Tr} N$

The duality Tr transforms right A-modules into left A-modules and conversely. If we want to define an endofunctor of $\operatorname{mod} A$, we need to compose Tr *w*ith another duality between right and left A-modules. We will use the standard duality D defined in Chapter 2. Then we are finally ready to define the Auslander-Reiten translations.

Definition 3.10. The **Auslander-Reiten translations** are defined to be the compositions of D and Tr , that is

$$\tau = D\mathrm{Tr} \qquad \text{and} \qquad \tau^{-1} = \mathrm{Tr}\, D$$

The following gives an overview of some properties of τ and τ^{-1}.

Proposition 3.11. Let M and N be indecomposable modules in $\operatorname{mod} A$.

(a) M is projective if and only if τM is zero.
(b) N is injective if and only if $\tau^{-1} N$ is zero.
(c) If M is non-projective, then τM is indecomposable non-injective and $\tau^{-1}\tau M \cong M$.
(d) If N is non-injective, then $\tau^{-1} N$ is indecomposable non-projective and $\tau\tau^{-1} N \cong N$.
(e) If M and N are non-projective, then $M \cong N$ if and only if $\tau M \cong \tau N$.
(f) If M and N are non-injective, then $M \cong N$ if and only if $\tau^{-1} M \cong \tau^{-1} N$.

The following result is the main existence theorem for almost split sequences, and the proof can be found in [4].

Proposition 3.12. For any indecomposable non-projective right A-module M, there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \to \tau M \to E \to M \to 0.$$

For any indecomposable non-injective right A-module N, there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \to N \to F \to \tau^{-1} N \to 0.$$

We will now state some results regarding how to find irreducible morphisms and how to construct Auslander-Reiten sequences.
In Chapter 4 we will give precise methods of finding all irreducible morphisms when A is a gentle algebra. For now, we will show some useful results for finding irreducible morphisms when A is an algebraically closed finite dimesional K-algebra.

Proposition 3.13. (1) Let P be an indecomposable projective module in $\operatorname{mod} A$. Then an A-homomorphism $g : M \to P$ is right almost split if and only if g is a monomorphism with image equal to $\operatorname{rad} P$.
(2) Let I be an indecomposable injective module in $\operatorname{mod} A$. Then an A-homomorphism $f : I \to M$ is left almost split if and only if f is an epimorphism with kernel equal to $\operatorname{soc} I$.

Proof. We will only prove the first statement, the second one is proven similarly. It is enough to show that the inclusion $i : \operatorname{rad} P \to P$ is right minimal almost split, by Proposition 3.2. The morphism i is a monomorphism, and thus it is right minimal. Let $v : V \to P$ be a homomorphism that is not a split epimorphism. The morphism i is not a split epimorphism, since it is the inclusion. The radical of P is the unique maximal submodule of P, since P is projective. Since v is not an epimorphism, we know that we then have $\operatorname{Im}(v) \subseteq \operatorname{rad} P$. Thus, v factors through i. ∎

We then have the following immediate consequence.

Corollary 3.14. Let M be indecomposable in $\operatorname{mod} A$.

(a) Then there exists a right minimal almost split morphism $g : N \to M$. Furthermore, M is simple projective if and only if $N = 0$.

(b) Then there exists a left minimal almost split morphism $f : M \to N$. Furthermore, M is simple injective if and only if $N = 0$.

Proposition 3.15. (a) If M is an indecomposable non-projective module in $\operatorname{mod} A$, then there is an irreducible morphism $f : L \to M$ if and only if there is an irreducible morphism $f' : \tau M \to L$.

(b) If N is an indecomposable non-injective module in $\operatorname{mod} A$, then there is an irreducible morphism $g : N \to L$ if and only if there is an irreducible morphism $g' : L \to \tau^{-1}N$.

Proof. We will only prove the first statement, the second statement is proven similarly. By Theorem 1.10 in [4, Ch. 4.1], there is a morphism $h : L' \to M$ such that $\begin{bmatrix} f & h \end{bmatrix} : L \oplus L' \to M$ is right almost split. Since M is not projective, this implies that $\begin{bmatrix} f & h \end{bmatrix}$ is an epimorphism. By Proposition 3.5, we then have that $I = \operatorname{Ker}\begin{bmatrix} f & h \end{bmatrix}$ is indecomposable. Thus, the short exact sequence

$$0 \to I \xrightarrow{\begin{bmatrix} f' \\ h' \end{bmatrix}} L \oplus L' \xrightarrow{\begin{bmatrix} f & h \end{bmatrix}} M \to 0$$

is almost split. This implies that there is an isomorphism $g : \tau M \to I$ and the homomorphism $f'g : \tau M \to L$ is irreducible. The proof of the converse is similar. ■

Corollary 3.16. (a) If S is simple projective non-injective in $\operatorname{mod} A$, and if $f : S \to M$ is irreducible, then M is projective.

(b) If S is simple injective non-projective in $\operatorname{mod} A$, and if $g : N \to S$ is irreducible, then N is injective.

Proof. We will only prove the first statement, the second statment can be proven similarly. Assume that M is indecomposable. If M is not projective, then there is, by Proposition 3.15, an irreducible morphism $\tau M \to S$. However, this contradicts Corollary 3.14. ■

This corollary allows us to construct some almost split sequences. If we let S be simple projective non-injective and let $f : S \to P$ be irreducible, then P is projective. By Proposition 3.13 we have that for each indecomposable summand P_i of P, the corresponding component of f

$f_i : S \to P_i$ is a monomorphism with image equal to $\operatorname{rad} P_i$. This implies that P is a direct sum of all such indecomposable projectives P_i, and thus the sequence $0 \to S \xrightarrow{f} P \to \operatorname{Cok} f \to 0$ is almost split.

Proposition 3.17. Let P be a non-simple indecomposable projective-injective module. Then the sequence

$$0 \to \operatorname{rad} P \xrightarrow{\begin{bmatrix} q \\ i \end{bmatrix}} (\operatorname{rad} P/\operatorname{soc} P) \oplus P \xrightarrow{\begin{bmatrix} -j & p \end{bmatrix}} P/\operatorname{soc} P \to 0$$

is almost split, where i and j are inclusions and p and q are projections.

Proof. The module $\operatorname{rad} P$ is indecomposable, since it has a simple socle. By Proposition 3.13 $i : \operatorname{rad} P \to P$ is, up to isomorphism, the unique non-tirvial irreducible morphism ending in P. Dually, the module $P/\operatorname{soc} P$ is indecomposable, and the morphism $p : P \to P/\operatorname{soc} P$ is up to isomorphism, the unique non-trivial morphism from P. By Proposition 3.15, we have that $\operatorname{rad} P \cong \tau(\operatorname{rad} P/\operatorname{soc} P)$. Since the given sequence is non-split, we now only need to show that $\begin{bmatrix} q \\ i \end{bmatrix}$ is left almost split. Suppose that $u : \operatorname{rad} P \to U$ is not a split monomorphism. If u is a monomorphism, then u factors through P, since P is injective. This finishes the proof for this case.

If u is not a monomorphism, then there is a factorization $u = u'u''$, with $u'' : \operatorname{rad} P \to U'$ a proper epimorphism and $u' : U' \to U$ a monomorphism. Since $\operatorname{Ker} u \neq 0$, the simple socle of $\operatorname{rad} P$ is contained in $\operatorname{Ker} u = \operatorname{Ker} u''$. Thus, u'' factors through $\operatorname{rad} P/\operatorname{soc} P$. This means that there is a morphism $u_1 : \operatorname{rad} P/\operatorname{soc} P \to U'$ such that $u'' = u_1 q$. Hence, the morphism $\begin{bmatrix} u'u_1 & 0 \end{bmatrix}$ satifies

$$\begin{bmatrix} u'u_1 & 0 \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} = u'u'' = u$$

Consequently, the morphism $\begin{bmatrix} q \\ i \end{bmatrix}$ is left almost split. ∎

3.3 The Auslander-Reiten Quiver

In this section we give the definition of an Auslander-Reiten quiver and we construct it in an example. The Auslander-Reiten quiver is a quiver which contains information about $\operatorname{mod} A$, in terms of indecomposable modules, irreducible morphisms and Auslander-Reiten sequences. From these we can build modules in general, morphisms and short exact sequences, respectively [9].

Definition 3.18. Let A be a basic, connected and finite dimensional K-algebra. The quiver $\Gamma(\operatorname{mod} A)$ is then defined as:

(a) The vertices corresponds to isomorphism classes of indecomposable modules in $\operatorname{mod} A$.
(b) The arrows correspond to irreducible morphisms between between the indecomposable modules.

The quiver $\Gamma(\operatorname{mod} A)$ is called the **Auslander-Reiten quiver** of A.

If we have an Auslander-Reiten quiver, we then know all the indecomposable modules and how the homomorphisms between them look like. To construct such a quiver we can use some of the results presented in earlier sections. Below is an example of constructing the Auslander-Reiten quiver.

Example 3.19. Let A be the K-algebra defined as

$$A = KQ = K(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$$

We can then find all the projective or injective indecomposable modules.

$$\begin{aligned}
P(1) &= K \xrightarrow{1} K \xrightarrow{1} K \\
P(2) &= 0 \longrightarrow K \xrightarrow{1} K \\
P(3) &= 0 \longrightarrow 0 \longrightarrow K = S(3) \\
I(1) &= K \longrightarrow 0 \longrightarrow 0 = S(1) \\
I(2) &= K \xrightarrow{1} K \longrightarrow 0 \\
I(3) &= K \xrightarrow{1} K \xrightarrow{1} K
\end{aligned}$$

Note that $S(2)$ is also an indecomposable module, but is neither projective nor injective. We then observe the following equalities.

$$\begin{aligned} P(1) &= I(3) & I(1) &= I(2)/S(2) \\ P(3) &= \operatorname{rad} P(2) & I(2) &= I(3)/S(3) \\ P(2) &= \operatorname{rad} P(1) & I(2) &= P(1)/S(3) \end{aligned}$$

Since $P(3)$ is simple projective and non-injective, we have by Corollary 3.16 that the target of each irreducible morphism starting at $P(3)$ is projective. The equality $P(3) = \operatorname{rad} P(2)$, combined with the fact that $P(3)$ is not a summand of $\operatorname{rad} P(1)$, implies that the only such morphism is the inclusion

$$i : P(3) \to P(2).$$

This is actually the only right minimal almost split morphisms ending at $P(2)$. Hence, we have an exact sequence

$$0 \to P(3) \xrightarrow{i} P(2) \longrightarrow \operatorname{Cok} i \to 0,$$

where $\operatorname{Cok} i = P(2)/P(3) = S(2)$.

Since $P(1) = I(3)$ is projective-injective we have the following sequence by Proposition 3.17

$$0 \to \operatorname{rad} P(1) \longrightarrow (\operatorname{rad} P(1)/\operatorname{soc} I(3)) \oplus P(1) \longrightarrow P(1)/\operatorname{soc} I(3) \to 0,$$

which equals the following sequence

$$0 \to P(2) \longrightarrow S(2) \oplus P(1) \longrightarrow I(2) \to 0.$$

Furthermore, the homomorphism $I(2) \to I(2)/S(2) = S(1)$ is left almost split with kernel $S(2)$. Thus, we obtain the following Auslander-Reiten quiver when we combine everything.

$$\begin{tikzcd}
 & & P(1) \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I(2) \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dotted, no head] & & S(2) \arrow[ur] \arrow[rr, dotted, no head] & & S(1)
\end{tikzcd}$$

The dotted lines indicate what the modules are sent to under τ.

It is a usual notation to write the sequences such that M and τM are on the same horisontal line. The next remark can be useful when constructing the Auslander-Reiten quiver. It allows us to not have all the information before constructing the next steps of the quiver.

Remark 3.20. Whenever we have an almost split sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, the dimension vector of L and N is equal to the dimension vector of M, i.e.

$$\mathbf{dim}\, L + \mathbf{dim}\, N = \mathbf{dim}\, M.$$

Chapter 4

Gentle Algebras and String Modules

In this chapter we will give an introduction to gentle algebras and certain modules over such, called string modules. We finish this chapter by finding all irreducible morphisms between string modules and by giving a characterization of all Auslander-Reiten sequences containing string modules. In later chapters we will give a geometric model for a gentle algebra, and modules over such. The main sources in this chapter are [10, 2, 5].

Definition 4.1. Let K be a field, Q a quiver and I an admissible ideal. A finite dimensional K-algebra A is a **string algebra** if it is isomorphic to $A = KQ/I$ and satisfies the following:

(G1) Each vertex $i \in Q_0$ is the source of maximal two arrows and is the target of maximal two arrows.
(G2) For arrows $\alpha, \beta, \gamma \in Q_1$ with $t(\alpha) = t(\beta) = s(\gamma)$ and $\alpha \neq \beta$ we either have $\alpha\gamma \in I$ or $\beta\gamma \in I$.
(G3) For arrows $\alpha, \beta, \gamma \in Q_1$ with $s(\alpha) = s(\beta) = t(\gamma)$ and $\alpha \neq \beta$ we either have $\gamma\alpha \in I$ or $\gamma\beta \in I$.
(G4) The ideal I can be generated by zero relations.

Definition 4.2. Let $A = KQ/I$ be a string algebra. We define A to a **gentle algebra** if in addition the following is satisfied:

(G5) For arrows $\alpha, \beta, \gamma \in Q_1$ with $t(\alpha) = t(\beta) = s(\gamma)$ and $\alpha \neq \beta$ we either have $\alpha\gamma \notin I$ or $\beta\gamma \notin I$.

(G6) For arrows $\alpha, \beta, \gamma \in Q_1$ with $s(\alpha) = s(\beta) = t(\gamma)$ and $\alpha \neq \beta$ we either have $\gamma\alpha \notin I$ or $\gamma\beta \notin I$.

(G7) The ideal I can be generated by paths of length 2.

To simplify this we can think of a gentle algebra as a quiver where there is only one way to move forward from each vertex. Note that every gentle algebra is a string algebra.

Example 4.3. We will now give two examples of gentle algebras. Immediately, we see that the path algebra defined by

$$Q = 1 \longrightarrow 2$$

is a gentle algebra.

The algebra defined by

$$Q = \begin{tikzcd} 1 \arrow[dr, "\alpha"] & & 4 \\ & 3 \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & \\ 2 \arrow[ur, "\beta"'] & & 5 \end{tikzcd} \qquad I = \langle \gamma\alpha, \delta\beta \rangle$$

is also gentle. We see that there are maximum two arrows starting at each vertex and maximum two arrows ending at each vertex. The ideal is generated by paths of length 2, and there is only one way to move forward from a vertex.

4.1 Strings and String Modules

For the rest of this thesis let $A = KQ/I$ be a gentle algebra, unless otherwise stated. Note that the definitions and results given in this chapter also holds for string algebras. However, we will state them for gentle algebras, since this is what the thesis is about.

Given an arrow α of Q let α^{-1} denote the formal inverse of α, with $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. Similarly, we define the inverse of a path $p = \alpha_1\alpha_2 \ldots \alpha_n \in Q$ to be $p^{-1} = \alpha_n^{-1} \ldots \alpha_2^{-1}\alpha_1^{-1}$, where $s(p^{-1}) = t(p)$ and $t(p^{-1}) = s(p)$. The set of inverse paths is denoted by Q^{-1}.

Definition 4.4. Let Q be a quiver and let $A = KQ/I$ be a gentle algebra. A **string** $w = \alpha_1\alpha_2\ldots\alpha_n$, $\alpha_i \in Q_1 \bigcup Q_1^{-1}$, is a reduced walk in the quiver such that there are no subwalks of the form $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for $\alpha \in Q_1$, nor sub-walks $\alpha\beta$ such that $\alpha\beta \in I$ or $(\alpha\beta)^{-1} \in I$. The **length of the string** $w = \alpha_1\alpha_2\ldots\alpha_n$ is n.

The next natural step is to define composition of strings. Before doing so we need to define trivial strings. Each vertex will have two trivial strings. In practice, we may neglect one of these. However, for technical reasons when defining composition of strings, we need to have two.

Definition 4.5. For each vertex $v \in Q_0$ we have **trivial strings** e_v^+ and e_v^-. The vertex v is both the startpoint and endpoint of the trivial strings. The inverse of these trivial strings will act by swapping the sign, i.e. $(e_v^{\pm})^{-1} = e_v^{\mp}$.

For technical reasons, we also consider the **empty string** or the **zero string**, denoted as $w = 0$. We say that a string $w = \alpha_1\ldots\alpha_n$ is **direct** if $\alpha_i \in Q_1$ for all $i = 1\ldots n$. Similarly, we define w to be **inverse** if every α_i are in Q^{-1}. A string is **cyclic** if its startpoint and endpoint coincide.

Now, we are ready to define a composition of two strings. This is a technical definition, but for a gentle algebra the intuition is quite nice, because there are maximally two arrows going into a vertex and maximally two arrows out of the vertex. We let one of these be a forbidden walk, i.e. there is only one way to pass through the vertex such that the path is direct. Hence, we only have one way going through the vertex, and the other way is zero.

For the more technical definition we first let σ and ϵ be two sign functions from Q_1 to $\{-1, 1\}$ satisfying:

(1) If $\beta_1 \neq \beta_2$ are two arrows such that $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$,
(2) If $\alpha_1 \neq \alpha_2$ are two arrows such that $t(\alpha_1) = t(\alpha_2)$, then $\epsilon(\alpha_1) = -\epsilon(\alpha_2)$,
(3) If $\alpha\beta$ is not a relation, then $\sigma(\alpha) = -\epsilon(\beta)$.

To choose these functions we proceed as follows. Choose some vertex v and define the σ-value for the arrows β which start at v. Similarly, define ϵ-value for the arrows α ending at v. If there are arrows β_0 and

α_0 such that $\beta_0\alpha_0 \notin I$, then choose $\sigma(\beta_0) = 1$ and $\epsilon(\alpha_0) = -1$. We then use conditions (1) and (2) in the definition above, to define $\sigma(\beta)$ and $\epsilon(\alpha)$ for the remaining arrows around v. Note that the third condition is automatically satisfied. If there are no composable arrows β_0 and α_0 at the vertex v, then we only need to consider the first and the second condtion.

We extend these functions to strings in general in the following way: if β is an arrow in Q_1 let

$$\sigma(\beta^{-1}) = \epsilon(\beta) \text{ and } \epsilon(\beta^{-1}) = \sigma(\beta)$$

if $w = w_1 \dots w_n$ is a string of length $n \geq 1$, let

$$\sigma(w) = \sigma(w_n) \text{ and } \epsilon(w) = \epsilon(w_1)$$

finally, define

$$\sigma(e_v^{\pm}) = \mp 1 \text{ and } \epsilon(e_v^{\pm}) = \pm 1.$$

Definition 4.6. Let $w' = \alpha_1 \dots \alpha_n$ and $w = \beta_1 \dots \beta_m$ be two strings of length greater or equal to one. Then the **composition of** w' **and** w, $w'w$, is defined provided that $\sigma(w') = -\epsilon(w)$, i.e. $\sigma(\alpha_n) = -\epsilon(\beta_1)$.

We say that the composition of $e_v^{\pm}$ and w is defined if $t(w) = v$ and $\epsilon(w) = \pm 1$. In this case let $e_v^{\pm} w = w$. Similarly, we say that the composition of w' and $e_v^{\pm}$ is defined provided $s(w') = v$ and $\sigma(w') = \mp 1$. In this case let $w' e_v^{\pm} = w'$.

For any string $w = \alpha_1 \dots \alpha_n$ or $w = e_v^{\pm}$, define $u : \{0, 1, \dots, n\} \to Q_0$ to be the map such that:

$$u(i) = t(\alpha_i) \quad \text{for} \quad i \neq 0 \quad \text{and} \quad u(n) = s(w) \quad \text{for} \quad i = 0.$$

For a vertex v in Q_0, define $I_v = \{i \mid u(i) = v\}$. Now, we will define the representation $M(w)$ of Q.

Definition 4.7. Let $M(w)$ be the representation such that

(1) for each vertex $v \in Q_0$ let $M(w)_v = \bigoplus_{i \in I_v} K$

(2) For each arrow $\alpha \in Q$ with $s(\alpha) = v$ and $t(\alpha) = v'$, let

$$M_\alpha : \bigoplus_{i \in I_v} K \to \bigoplus_{j \in I_{v'}} K$$

with entries

$$(M_\alpha)_{i,j} = \begin{cases} \mathrm{id}_K & \text{if } j = i+1 \text{ and } \alpha_j \text{ is direct} \\ \mathrm{id}_K & \text{if } i = j+1 \text{ and } \alpha_i \text{ is inverse} \\ 0 & \text{otherwise} \end{cases}$$

We see that $M(w)$ is a representation of Q which satisfies the relations in I, and we define $M(w)$ to be a the **string module** over A.

We can think of this definition as: each vertex in the representation, in which the string w passes through, is replaced by a copy of K and the action of the arrows in w is just the identity morphism. For vertices and arrows that are not included in the string, let the vertices be zero and the morphisms be zero-morphisms. Note that $M(w)$ and $M(w^{-1})$ always are isomorphic. Also note that $M(e_v^{\pm})$ is the simple representation corresponding to the vertex v.

Example 4.8. Let A be the gentle algebra defined by

$$Q = \begin{tikzcd} 1 \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & 2 \\ & 3 \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad I = \langle \gamma\beta \rangle$$

The arrow α is a string and corresponds to the following string module.

$$\begin{tikzcd} K \arrow[rr, "\mathrm{id}"] \arrow[dr, "0"'] & & K \\ & 0 \arrow[ur, "0"'] & \end{tikzcd}$$

The string module of the trivial string at vertex 3 will be of the form

$$\begin{tikzcd} 0 \arrow[rr, "0"] \arrow[dr, "0"'] & & 0 \\ & K \arrow[ur, "0"'] & \end{tikzcd}$$

From this point on, whenever we talk about modules we will mainly consider string modules. This is because they are in fact indecomposable. There are another type of indecomposable modules induced by so called bands, which we think of as cyclic strings.

A **band** is a cyclic string b for which each power b^n is a string, but b itself is not a proper power of any string. In a similar way as we defined a string module, we also define a **band module**. Each band $b = \alpha_1 \dots \alpha_n$ defines a family of band modules $M(b, n, \phi)$ where ϕ is an indecomposable automorphism of K^n, for some $n \in \mathbb{N}$. Here each vertex of b is replaced by a copy of K^n and the action of an arrow α on $M(b, n, \phi)$ is induced by identity morphism if $\alpha = \alpha_i$ for $i = 1, \dots, n-1$ and is induced by ϕ if $\alpha = \alpha_n$. We refer to [5, p. 160] for a more detailed definition. Note that in this thesis we have chosen to focus mainly on strings and not bands.

The next theorem gives a description of indecomposable modules whenever A is a gentle algebra. We will use this in later chapters whenever we work with indecomposable modules. The proof is quite technical, and we will refer to [11] for a detailed proof.

Theorem 4.9. *All string- and band modules are indecomposable, and every indecomposable module is either a string- or band module.*

4.2 Irreducible Morphisms between String Modules

In this section we will give a characterization of irreducible morphisms between string modules. This allows us to find all irreducible morphisms, and from this we can construct the Auslander-Reiten quiver whenever A is a finite gentle algebra.

Definition 4.10. Let w be a string of Q. We define the string w to **start on a peak** if there is no arrow α such that $w\alpha$ is a string. Similarly, we define w to **start in a deep** if there is no arrow α such that $w\alpha^{-1}$ is a string.

Let w be a string of length n, w' be a string of length m and let α be an arrow such that the composition $w\alpha w'$ is defined. Then there is a canonical embedding $M(w) \to M(w\alpha w')$ which sends the basevectors of $M(w)$ to the submodule $M(w)$ of $M(w\alpha w')$. The proof of the following lemma can be found in [5, p. 167].

Lemma 4.11. *Let w be a string that does not start on a peak. In other words, there is an arrow $\alpha \in Q_1$ such that $w\alpha$ is a string. Let $w' = \beta_1 \dots \beta_n$ be the maximal direct string starting at $s(\alpha)$, that does not start with α. Define w_l to be string starting in a deep such that $w_l = w\alpha w'^{-1}$. Then the canonical embedding $M(w) \to M(w_l)$ is irreducible.*

This process is also called **adding a hook on** $s(w)$, and can be illustrated as in Figure 4.1.

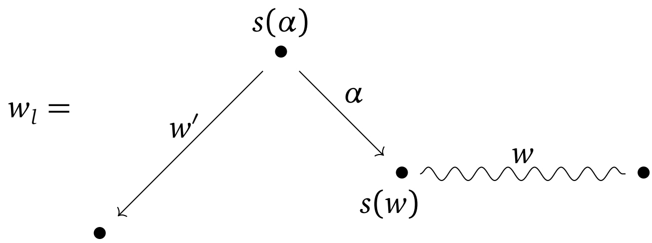


Figure 4.1: The string w with an added hook at $s(w)$. The string w' is direct and α is an arrow.

We will now define the dual concept of starting on a peak and starting in a deep. This allows us to obtain new irreducible morphisms.

Definition 4.12. Let w be a string. We define w to **end on a peak** if there is no arrow α such that $\alpha^{-1}w$ is a string. Dually, we define w to **end in a deep** if there is no arrow α such that αw is a string.

Observe that the string w ends on a peak if and only if w^{-1} starts on a peak. Similarly, w ends in a deep if and only if w^{-1} starts in a deep. If we have two strings w and w' and an arrow α such that $w'\alpha^{-1}w$ is a string, then there is a canonical embedding $M(w) \to M(w'\alpha^{-1}w)$. The previous lemma may then be reformulated as the following.

Lemma 4.13. *Let w be a string that does not end on a peak. In other words there is an arrow α such that $\alpha^{-1}w$ is a string. Let $w' = \beta_1 \dots \beta_n$ be the maximal direct string starting at $s(\alpha)$ such that it does not end with α^{-1}. Define w_r to be the string ending in a deep such that $w_r = w'\alpha^{-1}w$. Then the canonical embedding $M(w) \to M(w_r)$ is irreducible.*

This process is called **adding a hook on** $t(w)$ and is similar to adding a hook on $s(w)$. This can be illustrated as in Figure 4.2

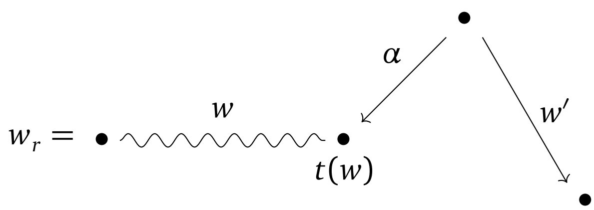


Figure 4.2: The string w with an added hook at $t(w)$. The string w' is direct and α is an arrow.

We have now established two irreducible embeddings $M(w) \to M(w_l) = M(w\alpha(w')^{-1})$ and $M(w) \to M(w_r) = M(w'\alpha^{-1}w)$, for approriate strings w. By duality, we obtain two canonical projections.

Lemma 4.14. *Let w be a string that does not start in a deep. In other words there is an arrow α such that $w\alpha^{-1}$ is a string. Let $w' = \beta_1 \ldots \beta_n$ be the maximal direct string starting at $t(\alpha)$ such that it does not end with α^{-1}. Define ${}_lw$ to be the string starting on a peak such that ${}_lw = w\alpha^{-1}\beta_1 \ldots \beta_n$. Then the canonoical projection $M({}_lw) \to M(w)$ is irreducible.*

We call this process **removing a hook on** $s(w)$ and can be pictured as in Figure 4.3

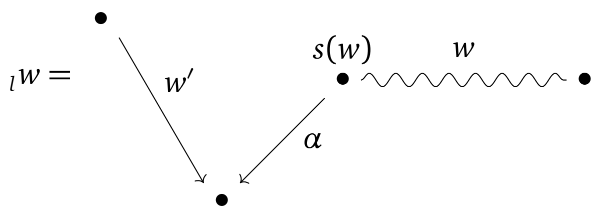


Figure 4.3: The string w with a removed hook at $s(w)$. The string w' is direct and α is an arrow.

Lemma 4.15. *Let w be a string that does not end in a deep. In other words there is an arrow α such that αw is a string. Let $w' = \beta_1 \ldots \beta_n$ be the maximal direct string starting at $t(\alpha)$ such that it does not start with α. Define ${}_rw$ to be the string ending on a peak, as ${}_rw = (w')^{-1}\alpha w$. Then the canonical projection $M({}_rw) \longrightarrow M(w)$ is irreducible.*

We call this process **removing a hook on** $t(w)$, and can be pictured as in Figure 4.4.

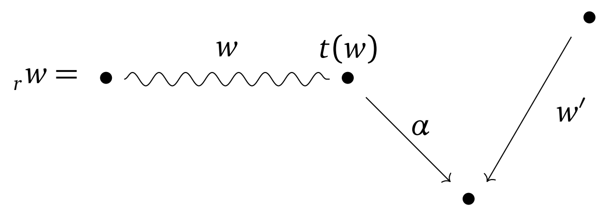


Figure 4.4: The string w with a removed hook at $s(w)$. The string w' is direct and α is an arrow.

Lemma 4.16. *Let w be a string.*

(1) If w does not end on a peak and does not start in a deep (or start on a peak), then the composition $_r(-)$ with $_l(-)$ (or $(-)_l$) is commutative.
(2) Similarly, if w does not end in a deep and does not start in a deep (or start on a peak), then the composition of $(-)_r$ with $_l(-)$ (or $(-)_l$) is commutative.

Proof. We only prove the first statement, the second one is proven similarly. Assume that w is a string that does not end on a peak and does not start on a peak. Then there are arrows α and β such that $w\alpha$ is a string and $\beta^{-1}w$ is a string. Since A is a gentle algebra, there is only one way to continue a string, in a given direction. Thus, if we have constructed w_l the arrow β will be the arrow such that $\beta^{-1}w_l$ is defined. This implies that:

$$\begin{aligned}(w_l)_r &= (w\alpha w')_r \\ &= w''\beta^{-1}(w\alpha w'^{-1}) \\ &= (w''\beta^{-1}w)\alpha w'^{-1} \\ &= (w_r)_l\end{aligned}$$

where w' and w'' are maximal strings which does not contain w. Similarly, assume that w does not start in a deep and does not end on a peak. Then there are arrows α and β such that $w\alpha^{-1}$ and $\beta^{-1}w$ are defined.

Thus, by a similar argument we have that:

$$\begin{aligned}({}_l w)_r &= (w\alpha^{-1}w')_r \\ &= w''\beta^{-1}(w\alpha^{-1}w') \\ &= (w''\beta^{-1}w)\alpha^{-1}w' \\ &={}_l (w_r)\end{aligned}$$

■

We have now obtained a number of irreducible morphisms between string modules, and we will prove that we have obtained all of them. Recall that a string $w = \beta_1 \dots \beta_n$ is direct if all β_i are arrows, and that w is inverse if all β_i are inverses of arrows. Hence by definition the strings of length zero are both direct and inverse.

Proposition 4.17. Let w be a string of a gentle algebra A and $M(w)$ be the corresponding string module. Then the canonical projections $M({}_-w) \to M(w)$ and the canonical embeddings $M(w) \to M(w_-)$ are all of the irreducible morphisms between string modules.

Proof. We know that irreducible morphisms occur in three different scenarios. In an Auslander-Reiten sequence, with a projetive module as target or an injective module as the source. Thus, we prove this proposition by proving Proposition 4.18 and Proposition 4.19. ■

Proposition 4.18. Let $M(w)$ be a string module over a gentle algebra A.

(1) Let $M(w)$ be projective non-simple module. Then we have $(\operatorname{rad} M(w))_l = M(w)$ or $(\operatorname{rad} M(w))_r = M(w)$. Additionally, the morphism $\operatorname{rad} M(w) \to M(w)$ is irreducible.
(2) Let $M(w)$ be injective non-simple module. Then we have ${}_l(M(w)/\operatorname{soc} M(w)) = M(w)$ or ${}_r(M(w)/\operatorname{soc} M(w)) = M(w)$. Additionally, the morphism $M(w) \to M(w)/\operatorname{soc} M(w)$ is irreducible.

Proof. First, let v be a vertex of Q, and let $P(v)$ be the indecomposable projective module corresponding to v. Observe that $P(v) = M(w_1 w_2)$, where w_1 is direct, w_2 is inverse and both strings start and end in a deep. The string w_1 needs to start at the vertex v, and let v have ϵ-value ± 1. Note that w_1 and w_2 might be of zero length.

If the length of both w_1 and w_2 is zero, then $P(v)$ is a simple projective module. Thus, there are no irreducible morphisms ending at $P(v)$.

Suppose now, that $w_1 = \beta_n \dots \beta_1$ with $n \geq 1$. If $n = 1$ and β_1 ends at the vertex v', then w_1 is just the trivial string at vertex v'. In either case, the string module $M(\beta_n \dots \beta_2)$ is a direct summand of $\operatorname{rad} P(v)$, and the inclusion $M(\beta_n \dots \beta_2) \hookrightarrow M(w_1 w_2)$ is the irreducible embedding

$$M(\beta_n \dots \beta_2) \hookrightarrow M(\beta_n \dots \beta_2)_l = M(\beta_n \dots \beta_2 \beta_1 w_2) = M(w_1 w_2).$$

Similarly, for the string $w_2 = \gamma_1^{-1} \dots \gamma_m^{-1}$ with $m \geq 1$, we have that the string module $M(\gamma_2^{-1} \dots \gamma_m^{-1})$ is a direct summand of $\operatorname{rad} P(v)$. The corresponding inclusion map is the canonical embedding

$$M(\gamma_2^{-1} \dots \gamma_m^{-1}) \hookrightarrow M(\gamma_2^{-1} \dots \gamma_m^{-1})_r = M(w_1 \gamma_1^{-1} \gamma_2^{-1} \dots \gamma_m^{-1}) = M(w_1 w_2).$$

Now, we consider the indecomposable injective module $I(v)$ at vertex v. Similarly as before $I(v)$ is a string module of the form $M(z_1 z_2)$, where z_1 is inverse, z_2 is direct and both strings start and end on a peak. The string z_1 starts at the vertex v. By duality of the above argument, we have that we can write $M(z_1 z_2)/\operatorname{soc} M(z_1 z_2)$ as the direct sum of at most two string modules of direct strings. Additionally, the projections are the irreducible projections. ■

To finish the proof of Proposition 4.17 we need to find the Auslander-Reiten sequences containing string modules. First consider the Auslander sequences with only one middle term.

$$0 \to M(A) \longrightarrow M(B) \longrightarrow M(C) \to 0$$

where A, B and C are strings of Q. By [5] we know that B starts in a deep and ends on a peak, and that we can write B as $B = \gamma_m^{-1} \dots \gamma_1^{-1} \beta \delta_1^{-1} \dots \delta_n^{-1}$ for a suitable arrow β and arrows γ_i, δ_j. Additionally, we can write $A = \gamma_m^{-1} \dots \gamma_1^{-1}$ and $C = \delta_1^{-1} \dots \delta_n^{-1}$. The morphisms in the exact sequnce is then the canonical embedding and the canonical projection. Note that $B = A_l =_r C$, and that A ends on a peak, but does not start on a peak. Similarly, C starts in a deep, but does not end in a deep. We will call this Auslander-Reiten sequence a canonical exact sequence.

Let w be a string of a gentle algebra A such that the corresponding string module $M(w)$ is not injective and that w is not inverse. We will have several cases to consider. The first one is when w neither starts nor ends on a peak, thus both w_l and w_r exist, and also $(w_r)_l = w_{r,l}$. Then the sequence

$$0 \to M(w) \xrightarrow{\begin{bmatrix} i & i \end{bmatrix}} M(w_l) \oplus M(w_r) \xrightarrow{\begin{bmatrix} i \\ -i \end{bmatrix}} M(w_{r,l}) \to 0$$

will be exact, where i is the canonical embedding. We will call this the canonical exact sequence.

The second case is when w does not start on a peak, but ends on peak, thus w_l is defined. Since w is not inverse we can write w as $w = \gamma_t^{-1} \dots \gamma_1^{-1} \gamma_0 u$, where u is a string not starting on a peak, γ_i are arrows for all i and $t \geq 0$. Thus, $w =_r u$. Since u does not start on a peak, u_l is defined, and there is the following exact sequence.

$$0 \to M(w) \xrightarrow{\begin{bmatrix} p & i \end{bmatrix}} M(u) \oplus M(w_l) \xrightarrow{\begin{bmatrix} i \\ -p \end{bmatrix}} M(u_l) \to 0$$

where i is the canonical inclusion and p is the canonical projection. We will call this the canonical exact sequence.

Similarly we obtian the following two canonical exact sequences.

$$0 \to M(w) \xrightarrow{\begin{bmatrix} i & p \end{bmatrix}} M(w_r) \oplus M(u) \xrightarrow{\begin{bmatrix} p \\ -i \end{bmatrix}} M(u_r) \to 0$$

$$0 \to M(w) \xrightarrow{\begin{bmatrix} p & p \end{bmatrix}} M({}_r u) \oplus M({}_l u) \xrightarrow{\begin{bmatrix} p \\ p \end{bmatrix}} M(u) \to 0$$

Proposition 4.19. The canonical exact sequences defined above are all Auslander-Reiten sequences containing string modules.

Proof. To prove this, we only need to verify that the given morphisms are irreducible. If the middle term is indecomposable, the sequence only contains indecomposable modules. Thus, it is clear that the exact sequnce is an Auslander-Reiten sequnce.

Now, let the the middle term be of the form $Y_1 \oplus Y_2$, where the two modules are indecomposable and non-isomorphic. If $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$, where X is an indecomposable module, then the morphism $\begin{bmatrix} f_1 & f_2 \end{bmatrix} : X \to Y_1 \oplus Y_2$ is irreducible. This finishes the proof for this case.

Hence, we only need to consider the case when w neither starts nor ends on a peak and $M(w_l) \cong M(w_r)$. The case where w neither starts nor ends in a deep and $M({}_l w) \cong M({}_r w)$ follows by duality. We prove the former by proving the following two claims.

Claim 1: Let α, β_1 be two arrows in Q_0 such that

$$\begin{gathered} \alpha \neq \beta_1, \;\; t(\alpha) = t(\beta_1), \;\; s(\alpha) = s(\beta_1), \\ \text{and } \gamma\beta, \; \gamma\alpha \in I \;\; \forall \gamma \in Q_1 \end{gathered} \tag{4.1}$$

Let $w = (\alpha\beta_1)^s$ for some $s \geq 0$. We then have that
$w_r = (\alpha\beta_1)^{s+1} = w_l$, and the space of irreducible morphisms from $M(w)$ to $M(w_l)$ is two-dimensional and is given by the residue classes of the embeddings $M(w) \to M(w_l)$ and $M(w) \to M(w_r)$. Note that, if $s = 0$, we let $(\alpha\beta_1)^0 = e_u^{\pm}$ provided that α is composable with $e_u^{\pm}$.

Claim 2: Let w be a string which neither starts nor ends on a peak, and assume $M(w_r) \cong M(w_l)$. Then there are arrows α and β_1 satisfying equation 4.1 such that $w = (\alpha\beta_1^{-1})^s$ for some $s \geq 0$.

■

Proof of Claim 1. Let α and β_1 be arrows satisfying equation 4.1, and let $w = (\alpha\beta_1^{-1})^s$, for some $s \geq 0$. We want to show that the two embeddings $M(w) \to M(w_l)$ and $M(w) \to M(w_r)$ are linearly independent in the space of irreducible morphisms from $M(w)$ to $M(w_l)$. To do so, we restrict the situation to only consider the subquiver given by the arrows α, β_1 and the vertices $s(\alpha)$ and $t(\alpha)$. Thus, we have two different scenarios.

If $s(\alpha) = t(\alpha)$, then we have a quiver with one vertex and two arrows, which does not give rise to a gentle algebra.
Now, if $s(\alpha) \neq t(\alpha)$, then the subquiver is the Kronecker quiver.

$$\begin{tikzcd} i \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta_1"'] & j \end{tikzcd}$$

Note that $M(w)$ will be the representation with K^s in i-th position and K^{s+1} in the j-th position. Now, $w_l = (\alpha\beta_1^{-1})^s\alpha\beta_1^{-1}$. This implies that $M(w_l)$ is equal to $M(w)$, but with one more K at each vertex. Since w_l adds a hook on the start of w, we consider the additional K to be the first K in the direct sum. Similarly, $w_r = \alpha\beta_1^{-1}(\alpha\beta_1^{-1})^s$. This implies that $M(w_r) = M(w_l)$, but since w_r adds a hook at the end of w, we consider the additional K to be the last K in the direct sum. Thus, the morphism $M(w) \to M(w_l)$ is the embedding of the last coordinates and the morphism $M(w) \to M(w_r)$ is the embedding of the first coordinates. Hence, the morphisms are linearly independent. ∎

Proof of Claim 2. Let w be a string which neither starts nor ends on a peak, and assume that $M(w_r) \cong M(w_l)$, where $w_r = \gamma_m \dots \gamma_1\alpha'^{-1}w$ and $w_l = w\alpha\beta_1^{-1}\dots\beta_n^{-1}$.

First, we observe that we must have $w_l = w_r$. If not, we would have $w_l = (w_r)^{-1} = (w^{-1})_l$, and thus $w = w^{-1}$. If the length w is odd, then the middle arrows of w and w^{-1} would have different exponents, which contradicts $w = w^{-1}$. If the length w is even, then the middle part would be of the form $\beta\beta^{-1}$ or $\beta^{-1}\beta$, which contradicts the definition of a string. If w is of length zero, our definitions imply that $w \neq w^{-1}$. Thus, we must have $w_l = w_r$.

Second, observe that since $w_l = w_r$, we need $n = m$, where $n \geq 1$. If not, w needs to start with α'^{-1}, since w_l does, but then w starts with $\alpha'^{-1}\alpha'^{-1}$, and so on, giving a contradiction.

Now, we will prove by induction that $w = (\alpha\beta_1^{-1})^s$, $s \geq 1$, where α and β_1 satisfying equation 4.1. The proof will be inductive on the length of the string w.

First, let w be of length 0. Then since $w_l = w_r$ we need to have $m = n = s = 1$, thus the arrows α and β_1 satisfy equation 4.1.

Now, for the induction hypothesis, assume that there are arrows α, β_1 satisfying equation 4.1 such that $w = (\alpha\beta_1^{-1})^s$ for all lenghts of w smaller than k.

Let w be of length k. Since $w_r = w_l$, the string w must contain both direct and inverse arrows. Thus, we can write w as both

$$w = \gamma_n \dots \gamma_1\alpha'^{-1}\tilde{w}$$
$$w = \tilde{\tilde{w}}\alpha\beta_1^{-1}\dots\beta_n^{-1}$$

where $\tilde{w}$ and $\tilde{\tilde{w}}$ are two strings. When inserting this in the equation $w_r = w_l$ we get that $\tilde{w} = \tilde{\tilde{w}}$. Now, $\tilde{w}$ neither starts nor ends on a peak, and by the above equation we have that

$$\tilde{w}_l = \tilde{w}_r = w.$$

The length of $\tilde{w}$ is properly smaller than the length w. Hence, by the induction hypothesis we know that

$$\tilde{w} = (\delta_0 \delta_1^{-1})^s,$$

where δ_0 and δ_1 satisfy equation 4.1. Thus, we have

$$\gamma_n \dots \gamma_1 \alpha'^{-1} (\delta_0 \delta_1^{-1})^s = (\delta_0 \delta_1^{-1})^s \alpha \beta_1^{-1} \beta_n^{-1}.$$

For this to hold, we need to have $n = 1$ and

$$\gamma_n = \gamma_1 = \delta_0 = \alpha \text{ and } \alpha' = \delta_1 = \beta_1 = \beta_n.$$

Hence,

$$w = \tilde{w}_l = \tilde{w}_r = (\alpha\beta_1)^{s+1}.$$

■

Remark 4.20. Note that we can reformulate this proposition such that we only have two cases of Auslander-Reiten sequences starting with $M(w)$, where $M(w)$ is not injective. In Lemma 4.14 we gave the definition of ${}_l w = w\alpha^{-1} w'$ for a string w, and obtained an irreducible morphim $M({}_l w) \to M(w)$. Define u to be the string equal to ${}_l w$. We still have an irreducible morphism from $M({}_l w) = M(u)$ to $M(w)$. If we now define $w = u_l$, we then have an irreducible morphism $M(u) \to M(u_l)$. Now u_l is obtained from u by removing a hook, and we have the following diagram.

$$\begin{tikzcd}
M({}_l w) \arrow[r] \arrow[d, equal] & M(w) \arrow[d, equal] \\
M(u) \arrow[r] & M(u_l)
\end{tikzcd}$$

Similarly we can reformulate Lemma 4.15 such that we have an irreducible morphism from $M(u) \to M(u_r)$. If we use these reformulations we have the following exact sequence.

$$0 \to M(w) \to M(w_l) \oplus M(w_r) \to M(w_{r,l}) \to 0$$

In particular, $\tau^{-1}(M(w)) = M(w_{r,l})$. This reformulation will be useful in later chapters.

We finish of this chapter with an example of constructing an Auslander-Reiten quiver of a gentle algebra.

Example 4.21. Let A be the gentle algebra defined by

$$Q = \begin{tikzcd} 1 \arrow[dr, "\alpha"] & & \\ & 2 \arrow[r, "\beta"] & 4 \\ 3 \arrow[ur, "\gamma"'] & & \end{tikzcd}$$

with admissible ideal $I = \langle \beta\gamma \rangle$. We will construct the Auslander-Reiten quiver by finding all irreducible morphisms and using Remark 3.20. Before we can find the irreducible morphisms, we need to consider the possible compositions in Q. Let us start with at the vertex 2 and consider the arrows starting at 2. The arrow β is the only arrow starting here. Thus, let $\sigma(\beta) = 1$. Now, since $\beta\gamma$ is in the ideal I, we immediately have $\epsilon(\gamma) = 1$. Since the composition $\beta\alpha$ is not in the ideal, we need to have $\epsilon(\alpha) = -1$. We then have the following possible strings

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha, \alpha^{-1}\gamma\}$$

where each give rise to an isomorphism class of an indecomposable module. Note that since we have two trivial paths with different epsilon values, we can always compose a string or an arrow with a trivial path. We will therefor denote the trivial strings by e_v regardless of the epsilon-value.

Now we are ready to find the irreducible morphisms. Let us first consider the strings that do not start on a peak. In other words, for a string w there is an arrow μ such that $w\mu$ is defined. Then we have the following irreducible morphisms:

$$\begin{aligned} M(e_2) &\hookrightarrow M(e_2\alpha) = M(\alpha) \qquad & M(e_4) &\hookrightarrow M(e_4\beta) = M(\beta) \\ M(e_2) &\hookrightarrow M(e_2\gamma) = M(\gamma) \qquad & M(\beta) &\hookrightarrow M(\beta\alpha) \end{aligned}$$

For the strings not ending in a deep, we have the following irreducible morphisms.

$$\begin{array}{rcl} M(e_2) \hookrightarrow M(\alpha^{-1}e_2) = M(\alpha) & \qquad & M(\gamma) \hookrightarrow M(\alpha^{-1}\gamma) \\ M(e_2) \hookrightarrow M(\gamma^{-1}e_2) = M(\gamma) & & M(\alpha) \hookrightarrow M(\gamma^{-1}\alpha) \\ M(e_4) \hookrightarrow M(\beta^{-1}e_4) = M(\beta) & & \end{array}$$

Similarly, for the strings not starting or not ending in a deep we have the following irreducible morphisms.

$$\begin{array}{rcl} M(e_1\alpha^{-1}\gamma) \twoheadrightarrow M(e_1) & \qquad & M(e_2\beta^{-1}) \twoheadrightarrow M(e_2) \\ M(e_3\gamma^{-1}\alpha) \twoheadrightarrow M(e_3) & & M(\gamma^{-1}\alpha e_1) \twoheadrightarrow M(e_1) \\ M(\beta e_2) \twoheadrightarrow M(e_2) & & M(\alpha^{-1}\gamma e_3) \twoheadrightarrow M(e_3) \\ M(\beta\alpha) \twoheadrightarrow M(\alpha) & & \end{array}$$

Note that some morphisms are repeated. This is because we have ignored the ϵ-value for the trivial strings at each vertex.

We are now ready to construct the Auslander-Reiten quiver. Let us start with the module $M(e_4)$. This module is only involved with one morphism, namely $M(e_4) \hookrightarrow M(\beta)$. This morphism is injective, which means that the kernel is equal to zero. We know that it is a morphism in an Auslander-Reiten sequence, $0 \to M(e_4) \longrightarrow M(\beta) \longrightarrow N \to 0$. Thus, we can use the equation

$$\mathbf{dim}\, N = \mathbf{dim}\, M(\beta) - \mathbf{dim}\, M(e_4)$$

and we get that $\mathbf{dim}\, N = \begin{smallmatrix} 0 & & \\ & 1 & 0 \\ 0 & & \end{smallmatrix}$. This implies that N is the module $M(e_2)$, and we then have the following Auslader-Reiten sequence.

$$0 \to M(e_4) \longrightarrow M(\beta) \longrightarrow M(e_2) \to 0$$

Now, we consider the module $M(\beta)$. There are two irreducible morphisms from $M(\beta)$, $M(\beta) \twoheadrightarrow M(e_2)$ and $M(\beta) \hookrightarrow M(\beta\alpha)$. Hence the middle term of the Auslander-Reiten sequence starting with $M(\beta)$ contains a direct sum of two indecomposable modules. We then use the same equality as above to get that

$$\mathbf{dim}\, M(e_2) + \mathbf{dim}\, M(\beta\alpha) - \mathbf{dim}\, M(\beta) = \begin{array}{ccc} 1 & & \\ & 1 & 0 \\ 0 & & \end{array} = \mathbf{dim}\, M(\alpha)$$

Thus, we have the following Auslander-Reiten sequence.

$$0 \to M(\beta) \longrightarrow M(e_2) \oplus M(\beta\alpha) \longrightarrow M(\alpha) \to 0$$

We continue this process until there are no irreducible morphisms starting at the last vertex. When combining everything, we have the following quiver.

$$\begin{tikzcd}
 & & & M(\gamma) \arrow[dr] \arrow[rr, dashed, no head] & & M(e_1) \\
M(e_4) \arrow[dr] \arrow[rr, dashed, no head] & & M(e_2) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & M(\alpha^{-1}\gamma) \arrow[ur] \arrow[dr] & \\
 & M(\beta) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & M(\alpha) \arrow[ur] \arrow[rr, dashed, no head] & & M(e_3) \\
 & & M(\beta\alpha) \arrow[ur] & & &
\end{tikzcd}$$

Chapter 5

Tiling Algebras as Gentle Algebras

In this chapter we will prove that a surface with a tiling gives rise to a gentle algebra. Throughout this chapther let S be an unpunctured connected oriented Riemann surface with a non-empty boundary. This means that S is a surface in $\mathbb{R}^n$, in which we have a way of measuring angles and distances and we have a clockwise (counterclockwise) orientation. Additionally, let M denote a finite set of marked points on the boundary of S. We denote such a surface with marked points on the boundary by (S, M), and call it a marked surface. In this thesis, we will always imagine the marked surface as a disc or an annulus, i.e. S is a surface in $\mathbb{R}^2$. If S is a disc, we will consider M to consist of at least four marked points. In addition we allow boundary components of S without any marked points and define these as unmarked boundary components. An **arc** in (S, M) is a curve v in S satisfying the following properties:

- The endpoints of v are in M.
- v intersects the boundary of S only in its endpoints
- v does not cut out a monogon or a digon.

We always consider arcs up to homotopy relative to their endpoints. A collection P of arcs that do not intersect themselves or each other in the interior of S is defined to be a **partial triangulation**. Moreover a triple $(\mathsf{S}, M, \mathsf{P})$ is a **tiling** if it satisfies the conditions above.

5.1 Tiling Algebras

To define an algebra of a tiling we need a way to remember the arcs in a nice order. In order to do so we introduce some terminology. An arc $\mathtt{v}$ in the triple $(\mathsf{S}, M, \mathsf{P})$ has two **end segments**, defined by cutting the arc in three pieces and deleting the middle part. So, an end segment can be looked as a piece of the arc containing one of the endpoints. Let $\overline{\mathtt{v}}$ and $\overline{\overline{\mathtt{v}}}$ denote the two end segments.

Let p be a marked point in M, and let m', m'' be two points in the same boundary component as p such that m' and m'' are not in M, and p is the only marked point lying in the boundary segment δ between m' and m''. Let c be a curve homotopic to δ which lies in the interior of S except for its endpoints m' and m''. We define the **complete fan at** p to be the sequence of arcs $\mathtt{v}_1, \ldots, \mathtt{v}_k$ in P which c crosses in clockwise order. A **fan at** p is a subsequence $\mathtt{v}_i, \mathtt{v}_{i+1}, \ldots, \mathtt{v}_j$ of the complete fan at p. The arc $\mathtt{v}_i$ is said to be the **start** of this fan, and similarly, $\mathtt{v}_j$ is called the **end** of the fan. If a marked point q is not the endpoint of any arc in P, then the (complete) fan at q is the **empty fan**.

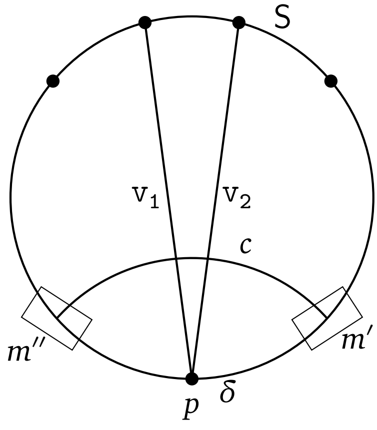


Figure 5.1: S is a disc and the set of marked points M consists of five points, including p. δ is the boundary segment between the two points m' and m'', and c is a curve homtopic to δ. The complete fan at p is $\mathtt{v}_1$, $\mathtt{v}_2$.

Definition 5.1. Let the bound quiver $(Q_{\mathsf{P}}, I_{\mathsf{P}})$ associated to the partial triangulation P of (S, M) be defined as follows.

(1) The vertices in $(Q_{\mathsf{P}})_0$ are in one-to-one correspondence with the arcs in P.
(2) Let $\alpha : s(\alpha) \to e(\alpha)$ be an arrow in $(Q_{\mathsf{P}})_1$. The arrow α exists if the arcs a_s and a_e corresponding to the vertices $s(\alpha)$ and $e(\alpha)$ share an endpoint $p_\alpha \in M$, and the arc a_e is the immediate successor of a_s in the complete fan at p_α.
(3) The ideal I_{P} is generated by paths $\beta\alpha$ of length 2, where the paths satisfy one of the following conditions:
    a. if $\alpha = \beta$.
    b. if $\beta \neq \alpha$ and $p_\alpha \neq p_\beta$.
    c. if $\beta \neq \alpha$, $p_\alpha = p_\beta$ and a is a loop, we are in the situation described in Figure 5.2.

The bound quiver algebra $A_{\mathsf{P}} = KQ_{\mathsf{P}}/I_{\mathsf{P}}$ is called a **tiling algebra**.

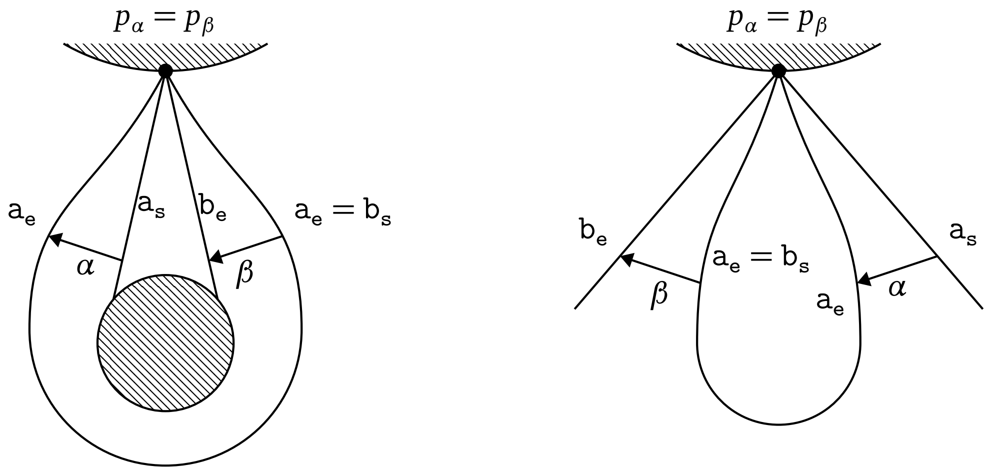


Figure 5.2: The two different cases when $\beta\alpha = 0$ in Q_{P} with $p_\alpha = p_\beta$.

Remark 5.2. If $\alpha, \beta \in (Q_{\mathsf{P}})_1$ are such that $\alpha\beta \in I_{\mathsf{P}}$, then the arcs corresponding to $s(\alpha), e(\beta) = s(\alpha)$ and $e(\beta)$ bound the same region in $(\mathsf{S}, M, \mathsf{P})$. However, the converse might not be true in general.

Intuitively, if we cut along the arcs of the surface, then we obtain smaller surfaces that fit together as the pieces of a puzzle. Every relation will be in the same “puzzle-piece”.

Lemma 5.3. *Every oriented cycle in the tiling algebra A_{P} of length ≥ 2 has a zero relation.*

Proof. Assume that $c = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{k-1}} v_k \xrightarrow{\alpha_k} v_1$ is an oriented cycle of length $k \geq 2$, and let p_i be the marked point associated to arrow α_i for $i = 1, \dots, k, k+1$, where $k+1 = 1$. If we have $\alpha_i = \alpha_{i+1}$, then by definition we would have $\alpha_{i+1}\alpha_i \in I_{\mathsf{P}}$. Moreover, if $p_i \neq p_{i+1}$ for some i, we would agian have by definition that $\alpha_{i+1}\alpha_i \in I_{\mathsf{P}}$. Therefore, suppose that $\alpha_i \neq \alpha_{i+1}$ and $p_i = p_{i+1} = p$ for all i. Now, the cycle c corresponds to a fan at p with v_1 as the start and endpoint. This implies that v_1 is a loop. Hence, by definition $\alpha_1\alpha_k$ is in the ideal I_{P}, and consequently the entire cycle c is. ■

Corollary 5.4. Every tiling algebra is finite dimensional.

This follows directly from the definition. For an algebra to be of infinite dimension, we need to have infinite number of paths, which only happens when there is a cycle. Since we have just proven that cycles have a zero relation, every tiling algebra is finite dimensional.

We say that a **triangulation of** (S, M) is a partial triangulation that also cuts the surface S into triangles. In other words a triangulation T of (S, M) is defined as a maximal collection of arcs that do not intersect in the interior of S.

Let $(\mathsf{S}, M, \mathsf{P})$ be a marked surface, and add a marked point to each unmarked boundary component. We have now obtained a new marked surface (S, M'). The additional marked points in (S, M') do not affect the definition of the tiling algebra, since the partial triangulation P is equal in the two surfaces, and the definition of the tiling algebra only depends on P. Hence the tiling algebras associated with $(\mathsf{S}, M, \mathsf{P})$ and $(\mathsf{S}, M', \mathsf{P})$ are isomorphic. A partial triangulation P of (S, M') can be completed to a triangulation T by adding arcs.

Previously when we introduced the notion of tiling, we wanted to define an algebra of this tiling. Similarly, we now want to define an algebra of a triangulation T of a surface (S, M). We will follow the definitions from [12].

Definition 5.5. We say that a triangle is an **internal triangle** if none of the edges are homotopic to a boundary segment. The **medial quiver** Q_{T}

is the quiver where the vertices are in one-to-one correspondence with the arcs of T. The arrows are defined if there is a triangle containing two arcs a and b, where a is the predecessor of b with respect to clockwise orientation at the common vertex of a and b. If this is satisfied there is an arrow from a to b in Q_T.

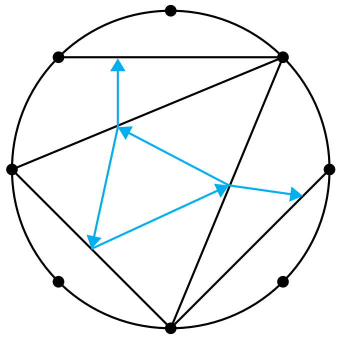


Figure 5.3: Example of a medial quiver. (S, M) is a disc with eigth marked points and a triangulation. The blue arrows indicate the arrows of the quiver.

Definition 5.6. We define the **Jacobian algebra** A_T as the quotient

$$A_\mathsf{T} = KQ_\mathsf{T}/I_\mathsf{T}$$

where I_T is the ideal generated by all paths of internal triangles. In other words, the relations of Q_T are compositions of any two arrows in an oriented three cycle in Q_T.

Remark 5.7. Let T be triangulation of (S, M). Then the algebra A_T is finite-dimensional.

In the same way as before, this follows directly from the definition of A_T, since we cannot have a cycle, we cannot have an infinite algebra.

Definition 5.8. Let $A_{\mathsf{P},\mathsf{T}}$ be the algebra obtained from $(Q_{\mathsf{P},\mathsf{T}}, I_{\mathsf{P},\mathsf{T}})$ in the following way. The vertices are in one-to-one correspondence with the arcs in P of $(\mathsf{S}, M, \mathsf{P})$.

$$(Q_{\mathsf{P},\mathsf{T}})_0 = (Q_\mathsf{T})_0 \setminus \{v | \mathsf{v} \in \mathsf{T} \setminus \mathsf{P}\}$$

For each direct string $v \to u_1 \to u_2 \to \cdots \to u_k \to v'$ in Q_{T}, where $\mathrm{v}, \mathrm{v}' \in \mathsf{P}$ and $\mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{k} \in \mathsf{T} \setminus \mathsf{P}$, we have an arrow $v \to v'$ in $(Q_{\mathsf{P},\mathsf{T}})$. All the arrows in $(Q_{\mathsf{P},\mathsf{T}})$ are obtained in the same way.

Let $v \xrightarrow{\alpha} v' \xrightarrow{\beta} v''$ be a path in $(Q_{\mathsf{P},\mathsf{T}})$. This means we have two direct strings in Q_{T}: $v \to u_1 \to \cdots \to u_k \xrightarrow{\alpha'} v'$ and $v' \xrightarrow{\beta'} u'_1 \to \cdots \to u'_l \to v''$, where $\mathrm{u}_i, \mathrm{u}'_j \in \mathsf{T} \setminus \mathsf{P}$ for all i, j and $\mathrm{v}, \mathrm{v}', \mathrm{v}'' \in \mathsf{P}$. If $\beta'\alpha' \in I_{\mathsf{T}}$, then we have $\beta\alpha \in I_{\mathsf{P},\mathsf{T}}$.

Lemma 5.9. *The tiling algebra A_{P} is isomorphic to $A_{\mathsf{P},\mathsf{T}}$ for any triangulation T completing the partial triangulation P in (S, M').*

Proof. We will prove that the quivers are equal, then we will prove that the set of relations are equal. By definition we have that the set of vertices $(Q_{\mathsf{P}})_0$ and the set $(Q_{\mathsf{P},\mathsf{T}})_0$ coincide, since they both correspond to the arcs in P. An arrow $\alpha : v \to v'$ is an element of $(Q_{\mathsf{P},\mathsf{T}})_1$ if and only if there is a direct string $v \to u_1 \to u_2 \to \cdots \to u_k \to v'$ in Q_{T}, where all u_i correspond to an arc in $\mathsf{T} \setminus \mathsf{P}$ for all i. Equivalently, we can say that the sequence $\mathrm{v}, \mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{k}, \mathrm{v}'$ is a fan at p of $(\mathsf{S}, M', \mathsf{T})$. This is again equivalent to saying that the sequence v, v' is a fan at p of $(\mathsf{S}, M, \mathsf{P})$. Thus, the arrow $\alpha : v \to v'$ belongs to $(Q_{\mathsf{P}})_1$ and hence, $Q_{\mathsf{P}} = Q_{\mathsf{P},\mathsf{T}}$.

We will now prove that the set of relations of the two algebras are isomorphic. Let $\beta\alpha \in I_{\mathsf{P},\mathsf{T}}$. By definition of $I_{\mathsf{P},\mathsf{T}}$ we have the following subquiver of Q_{T}:

$$\begin{tikzcd} s(\alpha) \arrow[r] & u_1 \arrow[r, squiggly] & u_k \arrow[r, "\alpha'"] & e(\alpha) \arrow[r, "\beta'"] \arrow[l, bend left, "\gamma'"] & u'_1 \arrow[r, squiggly] & u'_s \arrow[r] & e(\beta) \end{tikzcd}$$

where composition of any two arrows of the 3-cycle is zero. This is equivalent to say that β' and α' are in the same internal triangle of Q_{T}, which is equivalent to say that $\beta'\alpha' \in I_{\mathsf{T}}$. This is again equivalent to the whole string $\beta\alpha$ being in the ideal I_{T}. Which again is equivalent to β and α sharing the same polygon of Q_{P}. By the previous argument, we know that β and α are composable arrows in Q_{P}. Since β starts where α ends we have two different scenarios.

Case 1: If $p_\alpha \neq p_\beta$ and $\beta \neq \alpha$, then by definition of Q_{P} we have that the composition is in the ideal I_{P}, see Figure 5.4.

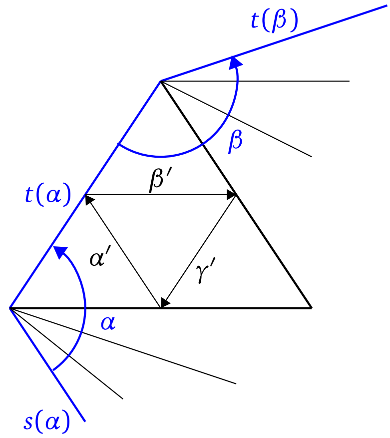


Figure 5.4: Case 1: The blue lines and arrows indicate the part that is in Q_{P} and in Q_{T}. The black lines and arrows indicate what is only in Q_{T}.

Case 2: If $p_\alpha = p_\beta$ and $\beta \neq \alpha$, then we need the endpoint of α to be a loop. This means that we are in the situation described in Figure 5.5. By Defnition 5.1 the composition is again in the ideal I_{P}.

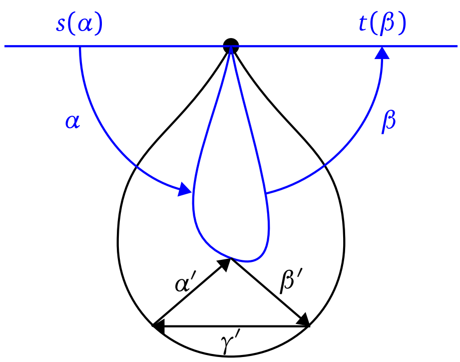


Figure 5.5: Case 2: The blue lines and arrows indicate the part that is in Q_{P} and Q_{T}. The black lines and arrows indicate what is only in Q_{T}.

The case where the arrows are equal is trivial. Hence, $\beta\alpha$ is in the ideal I_{P}. Since the quivers are equal, we have now constructed a isomorphism between the set of ideals. ■

5.2 Cluster Categories

We will now prove some results, which follow from Lemma 5.9. To prove the next result we need to introduce the concept of cluster categories. Our main sources in this section are [13] and [2].

The definition of a cluster category is quite technical and abstract, which requires details that are unecessary to acheive the goal of this thesis. We will therefore give a simplified definition. The cluster category $\mathcal{C}_{(\mathsf{S},M)}$ provides a categorifictaion of the algebra defined on the surface. So, we will use the Jacobian algebra to define the cluster category. This category does not depend on the triangulation T of (S, M).

Definition 5.10. A **cluster category** $\mathcal{C}_{(\mathsf{S},M)}$ is a triangulated category where;

1. We will only consider the objects called **string objects**, which are indexed by the homotopy classes of non-contratible curves in (S, M). The curves are not homotopic to a boundary segment of (S, M) and we consider objects to be homotopic to their inverse, i.e. $\gamma \simeq \gamma^{-1}$.
2. There is a morphism between two string objects if the objects intersect in a point not on the boundary of (S, M).
3. The shift functor $[-]$ is defined as counterclockwise rotation of start- and end-point of a curve to the next marked point at the boundary. In fact the shift of an object will be the same as the Auslander-Reiten translation of the same object.

This is a simplified definition, and to simplify it even more we consider the objects as all possible arcs, without arcs along the boundary and trivial arcs at a single point. We do not in general consider specific morphisms between these arcs. However, we know there exists morphism between two arcs if they intersect in a point not on the boundary of (S, M). Let γ, δ be two arcs in (S, M). Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{(\mathsf{S},M)}}(\gamma, \delta) = \mathrm{Ext}^1(\gamma[1], \delta) = \text{vector space generated by the} \\ \text{intersection points of } \gamma[1] \text{ and } \delta\end{aligned}$$

To illustrate the objects and a morphisms in the cluster category, we will consider a disc as the surface S, with marked points M. Let γ and δ

be two arcs in (S, M). To find $\gamma[1]$ we rotate its endpoints counterclockwise, pictured as a red arc below, Figure 5.6. We find $\delta[1]$ in a similar way.

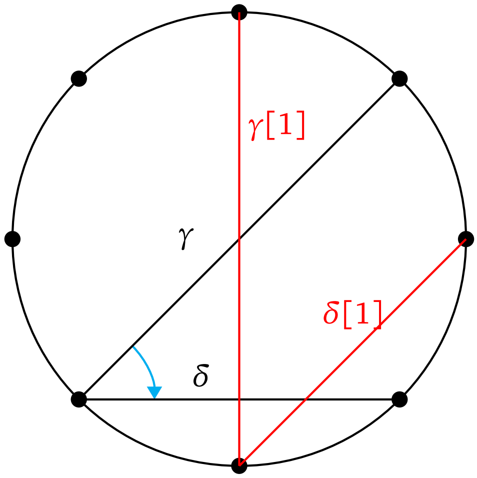


Figure 5.6: γ, δ are two objects of the cluster category $\mathcal{C}_{(\mathsf{S},M)}$. The blue arrow indicates a morphism from γ to δ.

Now if we want to know whether or not there is a morphism between for example γ and γ we can use the above result.

$$\begin{aligned} \mathrm{Hom}(\gamma,\gamma) &= \mathrm{Ext}^1(\gamma[1],\gamma) \\ &= \text{V.S. generated by number of intersectionpoints } = K \end{aligned}$$

Hence there is one morphism between γ and γ. This is reasonable, since there is always an identity morphism for an object in a category. If we now want to check if there are any morphisms between γ and δ in Figure 5.6, we do the same thing and see that there is one intersection point between $\gamma[1]$ and δ. Therefore, there is one morphism from γ to δ, which we symbolize as a blue arrow. We see that there are no intersection points between $\delta[1]$ and γ, which means that there are no morphisms from δ to γ.

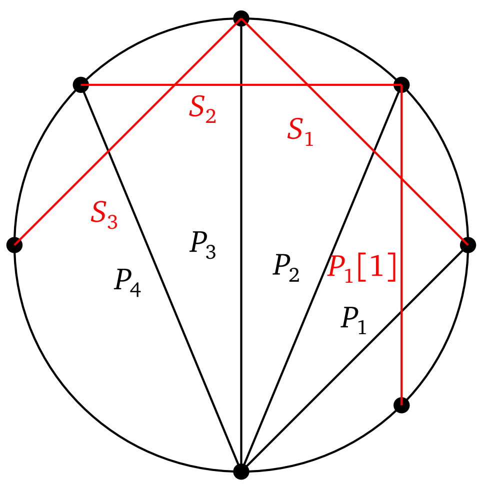
We define **partial cluster tilting objects** as objects that do not have a self-extencion. In other words a partial cluster tilting object is a collection or a direct sum of arcs without any intersection points. So, if we want to find the dimension of Hom-sets we look at each component seperatly. We can think of this as taking the direct sum outside the Hom, and consider the Hom-sets as we did earlier. In Figure 5.6 the object $\gamma \oplus \delta$ is a partial cluster tilting object.

In the next example we will see the connection to Auslander-Reiten quivers and in this way we can picture the morphisms between arcs. This does not work in general, but in this case where S is a disc, it does.

Example 5.11. Let $Q = A_4$ be the quiver $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$. We can then look at the derived category, and consider this part of the the Auslander-Reiten quiver:

$$\begin{tikzcd}
 & & & P_1 \arrow[dr] & & P_4[1] \arrow[dr] & & & \\
 & & P_2 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & P_3[1] \arrow[dr] & & \\
 & P_3 \arrow[ur] \arrow[dr] & & N \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[dr] & \\
P_4 \arrow[ur] & & S_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1]
\end{tikzcd}$$

Here N is an indecomposable module. The last diagonal column is the first one in opposite order with the shift functor. This continues infinitely in both directions. Since the Auslander-Reiten translation is equal to the shift functor we have $S_3[1] = P_4$, $S_2[1] = S_3$ and so on. We would like to put this in a picture, as we have before, and we consider the disc once more.



If we had continued drawing in the arcs, we would see that, every diagonal column in the Auslander-Reiten quiver has it own starting point, and every Auslander-Reiten translation of a module is also the module shifted by one. In the picture above we have drawn a few shifts of P_4, and we could continue this process until we had the starting arrow P_4 again. Every object of the Auslander-Reiten quiver is indecomposable, and hence every object of the cluster category is indecomposable.

Now we are ready to give the final results which lead up to the main result of this chapter, namely a tiling algebra is gentle. Assume that (S, M) is such that each boundary component of S contains a marked point in M. We can then associate the cluster category $\mathcal{C}_{(\mathsf{S},M)}$ to the surface (S, M). As we have seen in the example above, every indecomposable object can be described in terms of arcs in the surface. In particular, there is a bijection between (partial) triangulations P of (S, M) and the partial cluster tilting objects $\mathcal{P}$ in $\mathcal{C}_{(\mathsf{S},M)}$. Given a triangulation T of (S, M), the endomorphism algebra $B_{\mathcal{T}}$ of the corresponding cluster-tilting object $\mathcal{T}$ is isomorphic to the Jacobian algebra A_{T}.

Proposition 5.12. Let A be a finite dimensional algebra. The following are equivalent

(1) A is a tiling algebra.
(2) A is isomorphic to the endomorphism algebra of a partial cluster-tilting object in a generalized cluster category.

Proof. By Lemma 5.9, A is a tiling algebra if and only if $A = A_{\mathsf{P},\mathsf{T}}$, for some triangulation T completing the partial triangulation P. We know that partial triangulations P of a marked surface, which admits triangulations, are in one-to-one correspondence with partial cluster tilting objects $\mathcal{P}$ in the cluster category $\mathcal{C}_{(\mathsf{S},M)}$. Let $B_{\mathcal{P}}$ denote the corresponding endomorphism algebra. Now, we only need to prove that $A_{\mathsf{P},\mathsf{T}} \cong B_{\mathcal{P}}$.

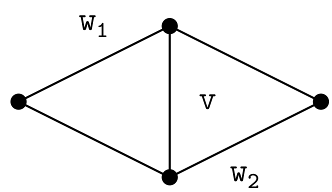
We know that the Jacobian algebra A_{T} is isomorphic to the endomorphism algebra $B_{\mathcal{T}}$ of the cluster tilting object $\mathcal{T}$. This implies that the vertices of the quiver of A_{T} are in one-to-one correspondence with the indecomposable summands of $\mathcal{T}$. The non-zero paths of A_{T} correspond to non-zero morphisms in $\mathcal{C}_{(\mathsf{S},M)}$ between indecomposable summands associated to the endpoints of the path. We will use the same notation for vertices in A_{T} and indecomposable objects in $\mathcal{C}_{(\mathsf{S},M)}$. Similarly we use the same notation for arrows in A_{T} and the corresponding morphisms in $\mathcal{C}_{(\mathsf{S},M)}$.

Let v, v' be two vertices of A_{T} corresponding to two indecomposable summands of $\mathcal{P}$. Each arrow $v \to v'$ in $B_{\mathcal{P}}$ corresponds to a non-zero morphism from v to v' which does not factor through any other summand of $\mathcal{P}$. Since $\mathcal{P}$ can be completed into $\mathcal{T}$, the non-zero morphism from v to v' will correspond to a path in Q_T. In other words, it will correspond to a direct string $v \to u_1 \to \cdots \to u_r \to v'$ in Q_{T}, where $\mathtt{u_i}$ lies in $\mathsf{T} \setminus \mathsf{P}$. Hence the set of arrows of $B_{\mathcal{P}}$ coincides with the arrows of $Q_{\mathsf{P},\mathsf{T}}$.

Finally, we need to prove that ideals for the two algebras coincide. A path $v_1 \to v_2 \to \cdots \to v_k$ in $B_{\mathcal{P}}$ is sent to a zero-morphism in $\mathcal{C}_{(\mathsf{S},M)}$ if and only if there is a direct string $v_i \to u_{i1} \to u_{i2} \to \cdots \to u_{ii_r} \to v_{i+1}$, for each i such that $\mathtt{u_{ij}} \in \mathsf{T} \setminus \mathsf{P}$, and the composition of these strings give rise to a relation. This is equivalent to saying there is a relation in I_{T} of the form $u_{ii_r} \to v_{i+1} \to u_{i+1,1}$. Thus, the ideal of $B_{\mathcal{P}}$ coincides with the ideal of $A_{\mathsf{P},T}$. ■

Lemma 5.13. *Let* T *be a triangulation of* (S, M). *Then* A_{T} *is a gentle algebra.*

Proof. By previous results, we know that the algebra A_{T} is finite-dimensional, so we only need to verify the axioms for the quiver $(Q_{\mathsf{T}}, I_{\mathsf{T}})$ of A_{T}. By definition the ideal I_{T} is generated by paths of length two. Now, let us consider the condition that there are at most two arrows starting in each vertex and ending at each vertex. Let v be a point of Q_{T} corresponding to an internal arc $\mathtt{v}$ of T. Since T is a triangulation of a surface, the arc $\mathtt{v}$ is contained in at most two triangles.



Thus, by the definition of Q_{T} there are at most two arrows ending in $\mathtt{v}$, $\alpha_1 : \mathtt{w}_1 \to \mathtt{v}$ and $\alpha_2 : \mathtt{w}_2 \to \mathtt{v}$. The same argument holds for arrows starting at $\mathtt{v}$.

For the criteria that there is one way to proceed a walk going through a vertex, we assume that Q_{T} contains the following:

$$\begin{tikzcd} \mathsf{b} \arrow[dr, "\alpha"] & & \\ & \mathsf{a} \arrow[r, "\gamma"] & \mathsf{c} \\ \mathsf{d} \arrow[ur, "\beta"'] & & \end{tikzcd}$$

We need to show that precisely one of $\gamma\alpha$ and $\gamma\beta$ belongs to I_{T}. The internal arcs b, d, a in T belong to two triangles as considered above. Since there is an arrow γ we know that the arc c is the successor of a in one of these triangles. Assume that it is the one formed by a, b and c. This gives rise to a relation $\gamma\alpha$. Thus, $\gamma\beta$ does not belong to I_{T} since β and γ arise from different triangles. ■

Proposition 5.14. Every tiling algebra A_{P} is gentle.

Proof. To prove this we will prove that the algebra $A_{\mathsf{P},\mathsf{T}}$ is gentle. We know by Lemma 5.9 that $A_{\mathsf{P}} \cong A_{\mathsf{P},\mathsf{T}}$, and by Lemma 5.13 that A_{T} is gentle.

Suppose that v be a vertex in $A_{\mathsf{P},\mathsf{T}}$, and assume for contradiction that there are three distinct arrows starting at v. These direct strings are subpaths of three distinct non-trivial maximal direct strings in A_{T} which contain the vertex v. Since A_{T} is gentle, every vertex lies in at most two non-trivial maximal direct strings. Hence we have a contradiction. Similarly, we can prove that every vertex in $A_{\mathsf{P},\mathsf{T}}$ is the target of at most two arrows. If we consider the following subquiver of $Q_{\mathsf{P},\mathsf{T}}$,

$$\begin{tikzcd} & & v_3 \\ v_1 \arrow[r, "\alpha"] & v_2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & v_4 \end{tikzcd}$$

then we have the following subquiver in Q_{T}.

$$
\begin{tikzcd}
 & & & & & v_3 \\
 & & & & u' \arrow[ur, rightsquigarrow] & \\
v_1 \arrow[r] & \cdots \arrow[r] & u \arrow[r, "\alpha'"] & v_2 \arrow[ur, "\beta'"] \arrow[dr, "\gamma'"'] & & \\
 & & & & u'' \arrow[dr, rightsquigarrow] & \\
 & & & & & v_4
\end{tikzcd}
$$

Since A_{T} is gentle, then either $\beta'\alpha' = 0$ or $\gamma'\alpha' = 0$, which implies that either $\beta\alpha = 0$ or $\gamma\alpha = 0$. A similar argument shows the dual condition. Hence, every tiling algebra is gentle. ■

Chapter 6

Surfaces with Boundaries of Gentle Algebras

In the previous chapter we proved that a tiling algebra gives rise to a gentle algebra. In this chapter we will prove the dual, in other words that a gentle algebra can be transformed into a tiling algebra. We will construct a surface with boundary associated with a gentle algebra, and then prove that the gentle algebra is isomorphic to the algebra constructed from this surface. Before this we need some graph-theory. Our main refrences are [3] and [14].

6.1 Ribbon Graphs and Ribbon Surfaces

In this section we will define ribbon graphs and ribbon surfaces. To do so precisely, we will first define a graph as a collection of vertices and half-edges, where each half-edge is connected to a vertex and another half-edge.

Definition 6.1. A **graph** is a tuple $\gamma = (V, E, s, i)$, where

- V is a finite set consisting of vertices
- E is a finite set consisting of half-edges.
- $s : E \to V$ is a function sending each half-edge to the vertex it is attached to.
- $\iota : E \to E$ is a function sending each half-edge to the other half-edge it is glued to. It has no fixed-points and is such that $\iota^2 = 1$.

The **degree** of a vertex is the number of half-edges that are incident to the vertex. We will denote the degree of a vertex v by $d(v)$.

Definition 6.2. We say that a graph Γ is a **ribbon graph** if it has a cyclic ordering of the half-edges around each vertex. In other words Γ is defined with a permutation $\sigma : E \to E$, where the orbits correspond to $\{s^{-1}(v) | v \in V\}$.

We can embedd a ribbon graph into the interior of an oriented surface with boundary. A surface like this is called a **ribbon surface**. The embedding is such that the orientation of the surface is given by the cyclic ordering of the ribbon graph.

Definition 6.3. Let Γ be a connected ribbon graph, where E contains at least one half-edge. Define S_Γ to be a surface constructed by gluing polygons together in the following way.

- Define P_v to be an oriented $2d(v)$-gon, for all vertices v in V.
- Every other side of P_v is labelled with the half-edges $e \in E$ attched to v, respecting the cyclic ordering.
- Let e be a half-edge attached to v and let $\iota(e)$ be a half-edge attached to v'. We glue the side of P_v labelled e with the side of P'_v labelled $\iota(e)$ respecting the orientations of the polygon.

The surface S_Γ is called a **ribbon surface**.

Note that, this definition excludes the trivial case where the graph only has one vertex and has no half-edges.

Definition 6.4. Let Γ be a ribbon graph. We define a **face** of Γ to be an equivalence class, up to cyclic rotations, of $n-$tuples of half-edges $(e_1, \ldots, e_n)$. The elements of the tuples are defined by $e_{i+1} = f_i(e_i)$ with f_i alternating between σ and ι, where the indices are taken modulo n. Additionally, the elements of a face does not reoccur in the sense that if $i \neq j$ and $e_i = e_j$, then $e_{i+1} \neq e_{j+1}$.

We can think of a face as drawing closed curves following the edges without crossing any half-edges or vertices. Each of these closed curves will correspond to a face.

The embedding of a ribbon graph Γ into the ribbon surface S_Γ can be described as follows: the vertices of Γ are sent to the centers of the corresponding $2d(v)$-gon P_v, and the half-edges are sent to arcs joining the polygons with the same label. This embedding makes a unique (up to homeomorphism) oriented surface which preserves the cyclic ordering of the half-edges around each vertex of Γ. In addition, S_Γ is the only surface such that the complement of Γ in S_Γ is a disjoint union of discs.

Moreover, the number of faces will correspond to the number of boundary components of S_Γ. We find the faces of a graph by drawing continuely around the graph without crossing any vertices or edges. This can also be thought of as making the graph thicker. The thickend graph can be considered as a surface, and each hole including the outer boundary of the surface will correspond to a face. Thus we might suspect that this surface is the ribbon surface.

In the following example we will illustrate how to find the faces of a ribbon graph and how to construct the corresponding ribbon surface.

Example 6.5. Let Γ be the ribbon graph in Figure 6.1. If we want to find

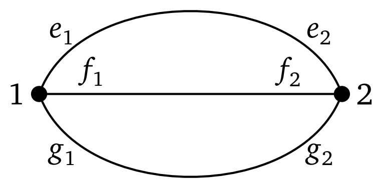


Figure 6.1: The ribbon graph Γ.

the faces of the graph, we start at a half-edge, say e_1. We have a choice in whether to apply σ or ι first. Let us start by applying ι, and we get the half-edge $\iota(e_1) = e_2$. We then have to use σ to obtain the next half-edge in the face, $\sigma(e_2) = f_2$. To find the next half-edge we apply ι to f_2, and obtain f_1. Now we have to use σ on f_1, and we get the half-edge e_1. If continue this process, we would have a repeating sequnce. Thus, we have a face (e_1, e_2, f_2, f_1). In a similar way we find the faces (f_1, f_2, g_2, g_1) and (g_1, g_2, e_2, e_1). These three faces are unique up to permutation. Below is a picture where the faces are illustrated as closed circles not crossing any of the edges or vertices.

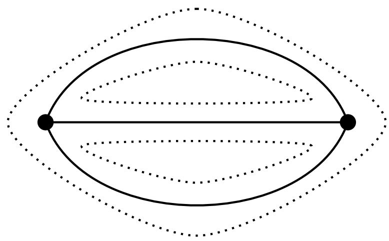


Figure 6.2: The dotted curves indicate the faces of the ribbon graph Γ.

The orientation of the ribbon graph is given by planar embedding. Now we would like to construct the ribbon surface, and we first consider the vertex 1. The vertex 1 has degree $d(1) = 3$. Thus, P_1 is a 6-gon. We label every other side of P_1 with the half-edges connected to 1, such that the orientation is preserved. In a similar way, we obtain the polygon P_2. Following the Definition 6.3, we glue the side of P_1 labelled e_1 to the side of P_2 labelled e_2. We do the same for f_i and g_i and obtain the following surface.

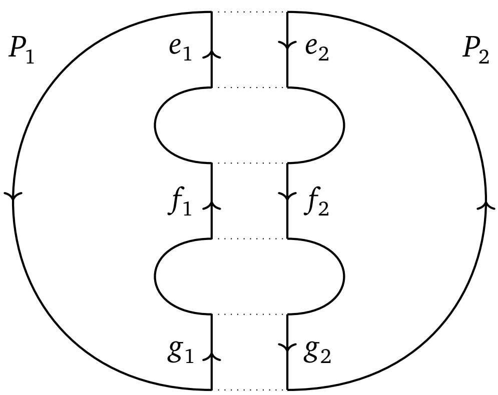


Figure 6.3: The ribbon surface S_Γ of the ribbon graph Γ of Example 6.5. It is obtained by gluing the polygons P_1 and P_2 together as indicated by the dotted lines.

The ribbon surface has three boundary components, which coincide with the number of faces of the graph.

6.2 Marked Ribbon Graphs

In this section we will define marked ribbon graphs and marked ribbon surfaces. This is the next step towards studying ribbon surfaces of gentle algebras.

Definition 6.6. Let Γ be a ribbon graph, and let m be a function from V to E such that $m(v) \in s^{-1}(v)$ for all vertices $v \in V$. A **marked ribbon graph** is a ribbon graph with such a function m. Equivalently, we can say that a marked ribbon graph is a ribbon graph where we have chosen one half-edge $m(v)$ around each vertex v.

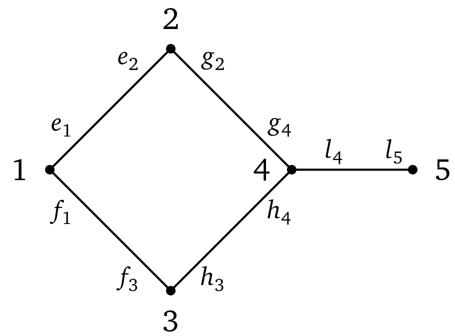
The construction of the ribbon surface for a marked ribbon graph is equivalent to the one in Section 6.1. The information given by m gives the following result.

Construction 6.7. Let Γ be a marked ribbon graph and let S_Γ be the ribbon surface. Then there exists an embedding of Γ into S_Γ, which sends each vertex in V to a point on the boundary segment between $m(v)$ and $\sigma(m(v))$ following the orientation. This embedding presevers the orientation and is unique up to homotopy relative to the boundary of S_Γ.

Proof. We start with proving the existence of such an embedding. The ribbon surface is unique up to homeomorphism, and by Definition 6.3, we only need to move the center of P_v to the side of P_v between the sides labelled $m(v)$ and $\sigma(m(v))$ following the orientation. Thus, there exists such a surface. The construction of S_Γ and the restrictions of the embedding leave us with no choice in how to embed. Hence, the embedding is unique. ■

We define an embedding as in Construction 6.7 a **marked embedding of** Γ **in** S_Γ. We will denote the set of marked points on the boundary of S_Γ by M.

Example 6.8. Let Γ be the following ribbon graph with counterclockwise orientation around each vertex.



By Definition 6.6 we choose the values of $m(v)$ for all vertices v. For instance, choose:

$$m(1) = e_1 \qquad m(2) = e_2 \qquad m(3) = h_3$$
$$m(4) = h_4 \qquad m(5) = l_5$$

For the vertex 5 we only have one choice, since there is only one half-edge connected to the vertex. We construct the ribbon surface, and obtain Figure 6.4.

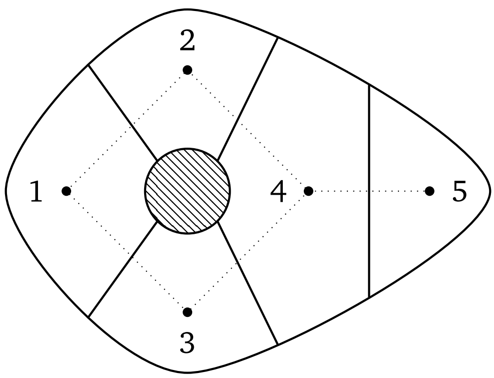
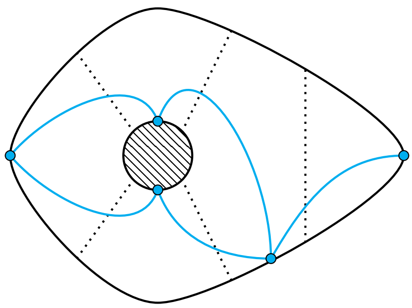


Figure 6.4: The ribbon surface of Γ, where the dotted lines represent Γ. The black lines indicate the polygons defined by the vertices of Γ.

In Construction 6.7 we imagine moving the centers of each polygon out to the boundary of S_Γ. We do so by first applying the m-function to each vertex and then moving it to the next unmarked boundary component of the corresponding polygon. Let us start with the vertex 1. When we apply m to 1 we get the half-edge e_1. Now, following the orientation, we move the point to the unmarked boundary component between e_1 and $\sigma(e_1) = f_1$. Thus, we have a point on the outer boundary of S_Γ. We continue this for the remaining points, and connect the points with non-intersecting arcs, such that the arcs correspond to the edges of the ribbon graph. We then have the following ribbon surface, where the arcs and marked points on the boundary are the blue lines and points.



6.3 The Marked Ribbon Graph of a Gentle Algebra

We are now ready to define a marked ribbon graph of a gentle algebra.

Definition 6.9. Let $A = KQ/I$ be a gentle algebra and let

- $\mathcal{M}$ be the set of maximal paths in (Q, I). The maximal paths $w \in \mathcal{M}$ are not in I and for any arrows composable with w, the composition is in the ideal.
- $\mathcal{M}_0$ be the set of trivial paths e_v, where e_v satifies either
  1) v is the source or target of exactly one arrow, or
  2) v is the target of exactly one arrow α and the start of exactly one arrow β and $\beta\alpha \notin I$.
- $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}_0$.

We then define the **marked ribbon graph** Γ_A **of** A as:

(1) $V(\Gamma_A) = \overline{\mathcal{M}}$, i.e. the set of vertices is $\overline{\mathcal{M}}$.
(2) For every vertex v of Γ_A there are half-edges attached to v which correspond to the vertices that the path v passes through. The half-

edges are labelled by the vertices of Q accordingly. This includes the vertices where the path v starts and ends. If v passes through i two times, then there are one half-edge labelled by i for each such passage.

(3) There are precisely two half-edges labelled by i for all vertices $i \in Q_0$, where ι sends the one to the other.

(4) The half-edges around each vertex are ordered from starting point to ending point. The permutation σ sends the half-edges to the next in this ordering, and sends the ending half-edge to the starting one.

(5) The function m sends every vertex v of Γ_A to the half-edge labelled by its ending point.

Note that the third condition follows from the fact that A is gentle and that we excluded the case where Q is disconnected. If we assume there is at least three half-edges labelled with i, where i is a vertex of Q, then there are at least three paths in $\overline{\mathcal{M}}$ that passes through i. But, for A to be gentle there is a maximum of two maximal paths passing through i. If there are two maximal paths passing through i, then the trivial path at i cannot be contained in $\overline{\mathcal{M}}$. Thus, there are two maximal paths passing trough i, and thus 2 half-edges labelled with i. The case when there is precisely one half-edge labelled with i, implies that the vertex i is disconnected from the rest of the quiver.

By Definition 6.9, the edges of Γ_A are in bijection to the vertices of Q. The edges around each vertex of Γ_A are ordered such that we can see the vertex (or the path) going through the edges, as in the quiver. Thus, we let the angles between the edges of Γ_A correspond to the arrows in Q, such that the angles (arrows in Γ_A) have the same startpoint and endpoint as in Q. Using the previous sections we can now define a surface with boundary and marked points for every gentle algebra.

Definition 6.10. Let $A = KQ/I$ be a gentle algebra. The **ribbon surface** of A is defined to be the ribbon surface of Γ_A, and is denoted S_A. S_A is oriented with marked points on the boundary, denoted (S_A, M). The set of marked points M corresponds to the vertices $(\Gamma_A)_0$ and the embedding of Γ_A in (S_A, M) is as in Construction 6.7.

By definition, we see that the vertices of Q are in bijection with the edges of Γ_A which again are in bijection with the arcs joining the marked points in S_A. Similarly, the arrows of Q are in bijection to the angles of the marked ribbon graph, which again are in bijection to the angles between the arcs of S_A.

Example 6.11. In this example we will illustrate what a marked ribbon graph of a gentle algebra is and how to construct the corresponding marked ribbon surface. Let $A = KQ/I$ be the gentle algebra defined by

$$Q = \qquad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\gamma"'] & 3 \end{tikzcd}$$

Let I be the ideal generated by $\beta\alpha$. By Definition 6.9 we first need to find the set of maximal paths $\mathcal{M}$ and the set of trivial paths $\mathcal{M}_0$. The path $\gamma\alpha$ is a maximal path. We see that the composition with $\beta\alpha$ is in the ideal. Thus, β is a maximal path.

$$\mathcal{M} = \{\gamma\alpha, \beta\}$$

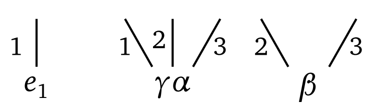
The vertex 2 is the target of one arrow and the source of two, thus the trivial path at 2 is not in $\mathcal{M}_0$. Neither is the trival path at vertex 3. However, the trivial path at 1 is the source of only one arrow, so it is included in the set $\mathcal{M}_0$.

$$\mathcal{M}_0 = \{e_1\}$$

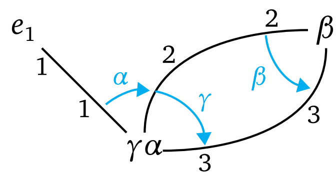
The vertices of the marked ribbon graph of A is then the set

$$(\Gamma_A)_0 = \overline{\mathcal{M}} = \{\gamma\alpha, \beta, e_1\}.$$

We would now like to find the half-edges of the graph. Let us start with the half-edges attached to the vertex e_1. The path e_1 passes only through the vertex 1. Thus, the vertex e_1 is only attached to one half-edge labelled 1. The path $\gamma\alpha$ passes through the vertices 1, 2 and 3. Hence, the vertex will be connected to three half-edges, labelled 1, 2, and 3, ordered from startpoint to endpoint of the corresponding path. Similarly we find the half-edges attached to vertex β. We can then draw the following picture.



Now we connect the half-edges labelled with the same number such that the orientation is preserved, and we get the following picture



This is the ribbon graph of A. The conditions (1) and (2) in Definition 6.9 are already satisfied. There are exactly two half-edges labelled by the vertices of Q. Hence, condition (3) is also satisfied. Since we placed the half-edges in order from startpoint to endpoint of each path the fourth condition is satisfied.

To find the ribbon surface S_A we follow Construction 6.7, and recall that m sends every vertex to its endpoint. We connect the points on the boundary with non-intersecting arcs, and obtain Figure 6.5.

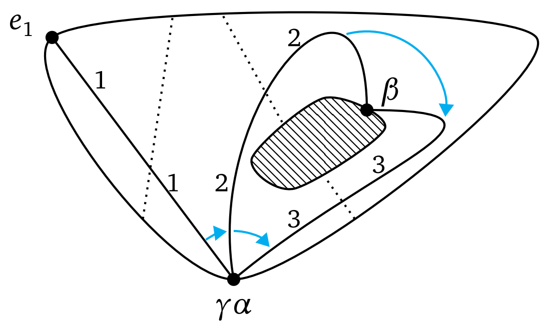


Figure 6.5: The marked ribbon surface (S_A, M) of Example 6.11. The dashed lines indicate the polygons defined by the vertices of the ribbon graph.

Note that the part close to β changed when we constructed the ribbon surface. This is due to the uniqueness of the embedding, and this surface is homotopic to the following surface in Figure 6.6.

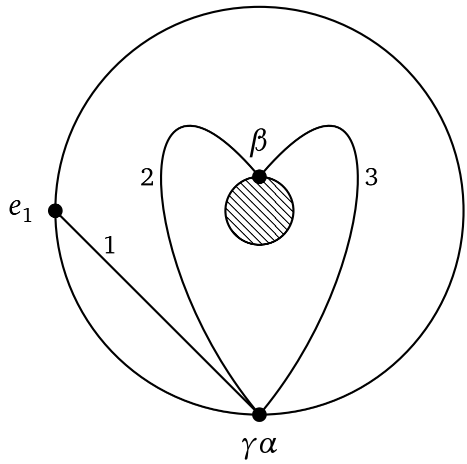


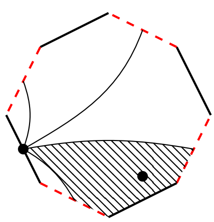
Figure 6.6: The marked ribbon surface (S_A, M) of the algebra A in Example 6.11. M is the set of marked points, these are joined by arcs.

Proposition 6.12. Let $A = KQ/I$ be a gentle algebra, and Γ_A and S_A be the corresponding ribbon graph and ribbon surface, respectively. Then S_A is glued together by two types of tiles.

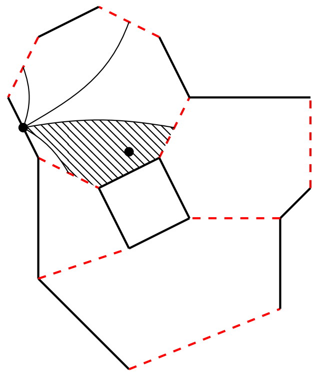
(1) Polygons where the edges are edges of Γ_A, except for one which is a boundary segment of S_A. There are no boundary compoenents in the interior of these polygons.

(2) Polygons where the edges are edges of Γ_A. There is precisely one unmarked boundary component of S_A in the interior of these polygons.

Proof. Let x be any point in the interior of S_A which does not belong to any edge of Γ_A. This point belongs to a polygon P_v with $2d(v)$ sides, as in Definition 6.3. There is exactly one marked point on one of the boundary segments of P_v. This marked point is the endpoint of $d(v)$ edges of Γ_A. Below is a picture of P_v as an octagon.

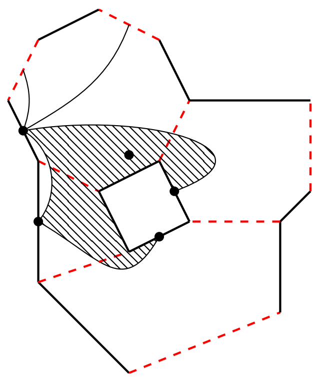


The region around x is partly bounded by a boundary segment of S_A. This segment is part of a boundary component B of S_A. Other segments of B bound other polygons $P_{v_1}, P_{v_2}, \dots, P_{v_r}$. Each of these polygons has a marked point on one of its boundary segments. Below is a picture of how the polygons around B might look like.



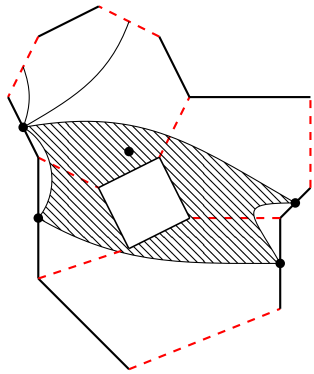
The arcs that bound the region in which x belongs to needs to end on a marked point on S_A. This gives rise to two different cases.

Case 1: There is at least one arc ending on B. Then x belongs to a region (or polygon) where the edges are edges of Γ_A and exactly one is a boundary edge of B.



Note that the two boundary points on B might be connected by an arc. If so, we will have two polygons, and the same holds.

Case 2: There are no arcs ending on B. Then x belongs to a region, where the edges are edges of Γ_A and the interior contains all of B.



■

6.4 A Lamination on the Surface of a Gentle Algebra

In this section we will define a lamination on a ribbon surface. The lamination will give rise to an algebra, and in Section 6.5 we will prove it is isomorphic to a gentle algebra. We will first give a definition for a general surface, before defining the lamination for a marked ribbon surface.

Definition 6.13. Let S be a surface and let M be a finite set of marked points on the boundary of S. A **lamination on** S is a finite collection of curves such that they do not intersect themselves or each other on S. The curves are considered up to isotopy relative to M, and is one of the following:

- a closed curve, not homotopic to a point
- a curve from a point on the boundary $x \notin M$ to a point on the boundary $y \notin M$. We exclude curves that are isotopic to a boundary segment without any marked points.

A curve in a lamination is called a **laminate**.

We will now define a lamination of a ribbon surface of a gentle algebra. This lamination is in fact unique.

Construction 6.14. Let $A = KQ/I$ be a gentle algebra, and let S_A be its ribbon surface. The lamination L_A of S_A is the unique lamination such that

(1) for every $\gamma \in L_A$, γ is not a loop
(2) for every arc i connecting two marked points in S_A there is a unique curve $\gamma_i \in L_A$ such that γ_i crosses the arc i once and crosses no other arcs.
(3) L_A contains no other curves than those described (2).

and we define it to be **the lamination of** A.

Remark 6.15. Recall that the arcs of S_A are in bijection to the edges of Γ_A, which again are in bijection to the vertices of Q.

Proof. Every edge i of Γ_A defines two, not necessarily distinct, faces. These faces gives us the boundary components of S_A. Hence, if a curve γ in L crosses i, it either starts or ends on these two boundary compoenents or it has to cross two edges. Moreover, the curve γ is then unique up to isotopy relative to M. ■

Example 6.16. In the previous example we considered the gentle algebra defined by

$$Q = \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\gamma"'] & 3 \end{tikzcd}$$

with the ideal $I = \langle \beta\alpha \rangle$. This had the corresponding ribbon surface, Figure 6.7.

Shown in Construction 6.14 this ribbon surface has a unique lamination L_A, where every curve of L_A crosses the arcs once and does not intersect each other or themselves. Hence we have the lamination marked by red lines in Figure 6.8.

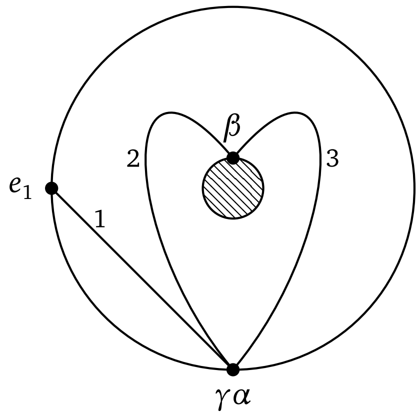


Figure 6.7: The ribbon surface S_A of Example 6.16.

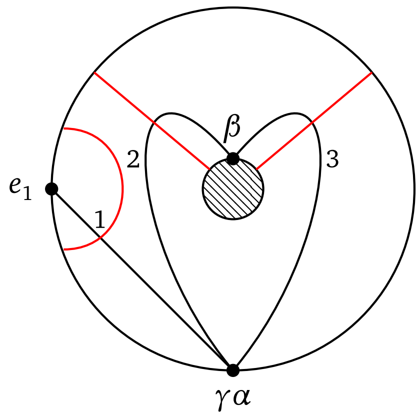


Figure 6.8: The ribbon surface S_A of the gentle algebra A, with the laminates of Example 6.16 as red curves.

6.5 Recovering the Gentle Algebra from its Lamination

The ribbon surface S_A and its lamination L_A contain, by construction, enough information to recover the gentle algebra A. We do so by defining an algebra, where the vertices correspond to the curves of L_A and define the arrows whenever two laminates end on the same boundary segment. Then we prove that this algebra is isomorphic to the original gentle algebra.

Proposition 6.17. Let $A = KQ/I$ be a gentle algebra, and let L_A be the associated lamination. We define Q_L as follows:

- $(Q_L)_0$ is the set corresponding to the curves in L_A.
- There is an arrow from i to j, where $i, j \in (Q_L)_0$ if both have an endpoint on the same boundary segment, the endpoint of j follows the endpoint of i in clockwise order, and there is no other curve ending in between.

The ideal I_L of KQ_L is defined by the following relations: if $i, j, k \in (Q_L)_0$ end on the same boundary segment of S_A, such that there are arrows $\alpha : i \to j$ and $\beta : j \to k$, then $\beta\alpha \in I_L$. Then, $A \cong KQ_L/I_L$.

Proof. We first prove that KQ_L is isomorphic to KQ, then we prove that $I_L \cong I$.

The arcs of S_A correspond to the vertices of Γ_A which again correspond to the vertices of Q, by definition. Each curve in L_A cross an arc of S_A exactly once. Thus, the set of vertices of Q are in bijection to the set of vertices of Q_L.

The angles between the edges of the ribbon graph Γ_A correspond to the arrows of Q. When embedding Γ_A into S_A, we get equivalent angles between the arcs. Now, when constructing the lamination, the angles between the arcs will give dual angles between the laminates. Hence $KQ_L \cong KQ$.

To prove that the ideals are isomorphic, we consider an algebra homomorphism from KQ_L to A. Then we prove that the kernel is equal to I_L. Let $f : KQ_L \to A$ be the homomorphism, which sends one path to the corresponding one. If $f(x) = 0$, for some x in KQ_L, then $f(x)$ is in the ideal I. Since A is a gentle algebra, $f(x)$ is a composition of two non-trivial arrows, $f(x) = \beta\alpha$ for $\beta, \alpha \in Q_1$. Since, $\beta\alpha$ is in the ideal I, $\beta\alpha$ is not a maximal path, and thus $\beta\alpha \notin \overline{\mathcal{M}}$. This implies that $\beta\alpha \notin (\Gamma_A)_0$. The arrows α and β are then subpaths of two maximal paths, denoted by w and w', respectively. Note that these maximal paths might be the arrows itselves. The maximal paths w and w' are not trivial, thus there is at least two half-edges connected to each of them in Γ_A. Since the source of β equals the target of α, the two maximal paths w and w' are connected by an edge in Γ_A (or an arc in S_A). This edge is the side of two polygons, P_1 and P_2, in S_A, where P_1 and P_2 are polygons as in Proposition 6.12. Recall that in Proposition 6.12 we proved that S_A is glued together by two types of polygons. We denote them by 1) and 2).

Assume without loss of generality that α and β are arrows (angles) in P_1. Note that if they are angles in different polygons, they are subpaths of the same maximal path, which is a contradiction. If P_1 is of type 1), then P_1 has at least four edges, where one of these is a boundary segment of S_A. If we now construct the lamination L_A, then all the laminates that cross the sides of P_1 must end on that boundary segment. Thus, we have at least three laminates ending on that boundary segment, which implies that $\beta\alpha \in I_L$.

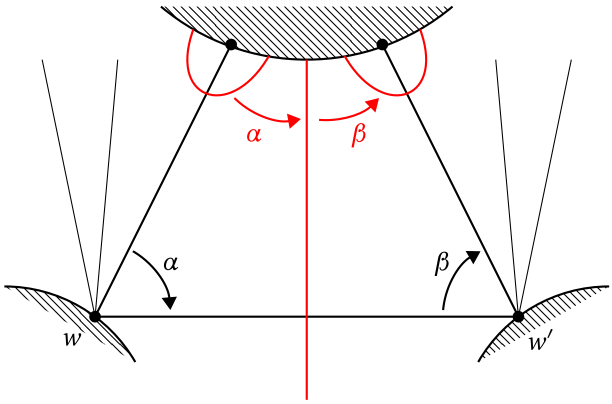


Figure 6.9: The polygon P_1 is of type 1) and the laminates are indicated as red lines with the dual angles.

If P_1 is of type 2) it has at least two edges, because of the number of half-edges connected to w and w'. If P_1 has $n > 2$ edges, then there are n laminates ending on the unmarked boundary component in the interior of P_1. This imply that $\beta\alpha \in I_L$. If P_1 has two edges, there are exactly two laminates ending on the unmarked boundary component in the interior of P_1. Then the laminate corresponding to the source of α is also the target of β. Thus, we have three laminates ending on the unmarked boundary component (where one is counted twice). This implies that $\beta\alpha \in I_L$.

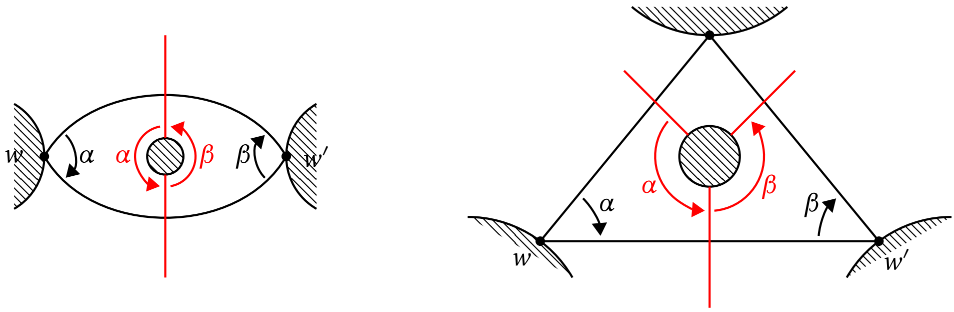


Figure 6.10: The left side shows the case when P_1 is of type 2) with two sides. The right side illustrates the case when P_1 is of type 2) with at least three sides. On both sides the red lines and arrows indicate the laminates and the laminate arrows.

Hence, we have $\mathrm{Ker}(f) \subseteq I_L$.

We now want to prove that $I_L \subseteq \mathrm{Ker}\, f$. Suppose that $\beta\alpha \in I_L$, where $\alpha, \beta \in (Q_L)_1$. By previous arguments we know that α, β correspond to composable arrows. Thus, let $f(\beta\alpha) = \beta'\alpha'$. We want to show that $\beta'\alpha' \in I$, so assume for contradiction that $\beta'\alpha' \notin I$. This implies that $\beta'\alpha'$ is either a maximal path or a subpath of a maximal path. Either way, let w denote the maximal path containing $\beta'\alpha'$. The maximal path w is a vertex of Γ_A connected to at least three edges. This implies that the arrows α' and β' are arrows in different polygons. See figure 6.11 for the different possiblities.

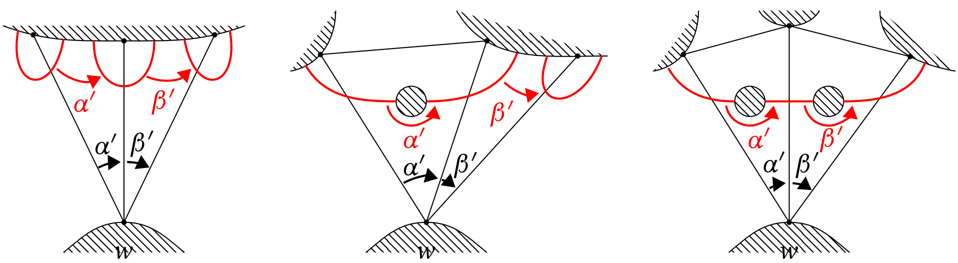


Figure 6.11: The left figure illustrates the case when α and β are arrows of polygons of type 1). The middle figure shows the case when α is an arrow in a polygon of type 1) and β is an arrow in a polygon of type 2). The right figure shows the case when α and β are arrows of two polygons of type 2).

If we now construct the lamination, we see that in either case of the polygons the composition $\beta'\alpha'$ is not in the ideal I_L. This contradicts the fact that we started with a composition in the ideal. Thus, we have $\beta'\alpha' \in I$. This implies that $f(\beta\alpha) = \beta'\alpha' = 0$, and thus, $\beta\alpha \in \mathrm{Ker}(f)$.

Hence, $I_L \subseteq \mathrm{Ker}(f)$, and we have $\mathrm{Ker}(f) = I_L$. By the first isomorphism theorem this implies that:

$$KQ_L/I_L = KQ_L/(\mathrm{Ker}(f)) \cong \mathrm{Im}(f) = KQ/I = A.$$

■

We have in this chapter established a way of going from a gentle algebra to a surface. If we add a marked point to each boundary segment that is homotopic to an arc of S_A, then Γ_A can be realised as a partial triangulation of (S_A, M). Thus the tiling algebra A_{Γ_A} is isomorphic to the algebra obtained by the lamination A_L. Hence, we have the following for a gentle algebra A:

$$A \cong A_L \cong A_{\mathsf{P}} = A_{\Gamma_A}.$$

Chapter 7

A Geometric Model of the Module Category of a Tiling Algebra

We have proven that every gentle algebra is a tiling algebra and the converse. In this chapter we will characterize the module category of a gentle algebra geometrically by using the corresponding tiling algebra. To do so, we consider tilings $(\mathsf{S}, M, \mathsf{P})$ where P divides the surface S into regions or tiles. We will only consider the tiles listed below.

- n-gons, with $n \geq 3$. The edges of the n-gons are arcs of P or boundary segments of S, and the interior contains no unmarked boundary component of S.
- one-gons, with precisely one marked point and precisely one unmarked boundary component in the interior. These will be refered to as loops or **tiles of type I**.
- two-gons, with exactly one unmarked boundary component in the interior and two marked points. We call these **tiles of type II**.

In the last chapter we saw that the marked ribbon graph can be realised as a partial triangulation of the marked ribbon surface. In Proposition 6.12 we gave a description of the regions of S_A. If we remove any unmarked boundary components from n-gons, where $n \geq 3$, as in Proposition 6.12, then the polygons coincide with the tiles above. The removal of unmarked boundary components does not affect the partial

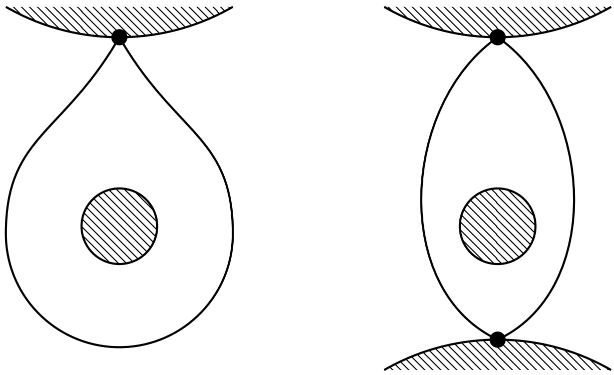


Figure 7.1: Tiles of type I and II

triangulation.

7.1 Permissible Arcs and Closed Curves

In Section 5.2 arcs in (S, M) corresponded to indecomposable objects in the cluster category. In this section we would like to characterize indecomposable objects in the module category of a tiling algebra A_{P}. It turns out that non-homotopic curves in (S, M) can correspond to the same indecomposable module. Therefore we introduce the notion of permissible curves and equivalence between these curves.

Definition 7.1. (1) Let γ be a curve in $(\mathsf{S}, M, \mathsf{P})$ such that γ crosses the arcs a_1 and $\mathsf{a}_2 \in \mathsf{P}$ at the points p_1 and p_2. Let $\overline{\gamma}$ be the segment of γ between p_1 and p_2. We say that γ **consecutively cross** a_1 and a_2 if $\overline{\gamma}$ does not intersect any other arc in P.

(2) Let B be an unmarked boundary component of S and let γ be an arc in (S, M). We write γ as $\gamma = \gamma_1\gamma'\gamma_2$, where γ_1 is the segment of γ between the startpoint of γ and the first crossing with P, and similarly, γ_2 is the segment of γ between the endpoint of γ and the last crossing P. Denote the homotopy class of γ_i for $i = 1, 2$ by β. We say that the **winding number of** γ_i **around** B is the minimum number of times β travels around B in either direction.

(3) Let γ be an arc in $(\mathsf{S}, M, \mathsf{P})$. We define γ to be a **permissible arc** if it satisfies the following:

(a) The winding number of γ_i around an unmarked boundary component is either zero or one.

(b) Whenever γ crosses two arcs x and y in P the crossing is consecutively, the arcs x and y share an endpoint $p \in M$, and locally we a triangle, i.e γ satifies the situation shown in Figure 7.2.

(4) If γ is a closed curve, we define γ to be a **permissible closed curve** if it satisfies (3)(b).

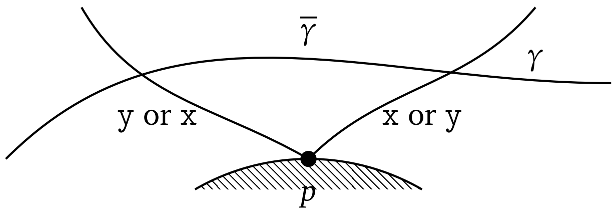


Figure 7.2: γ is a permissble arc

We remark that boundary segments and arcs of P are considered as permissbible arcs.

Remark 7.2. From the definition we have that each consecutive crossing of a permissible arc with an arc in P is associated to an arrow of the tiling algebra.

Example 7.3. The Figures 7.3, 7.4 and 7.5 show some examples of permissible and non-permissible arcs.

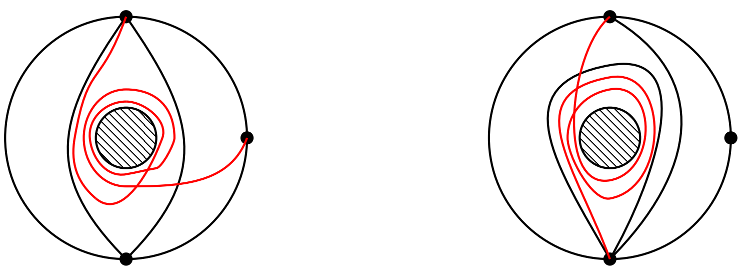


Figure 7.3: Examples of arcs not satisfying (3)(a)

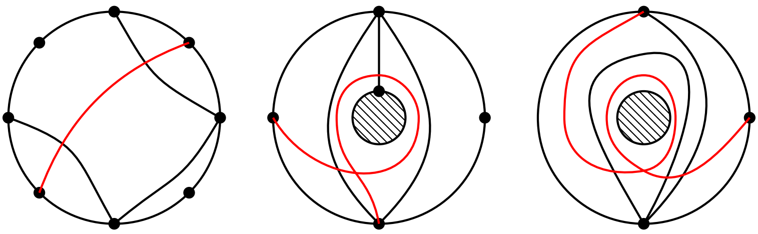


Figure 7.4: Examples of arcs not satisfying (3)(b)

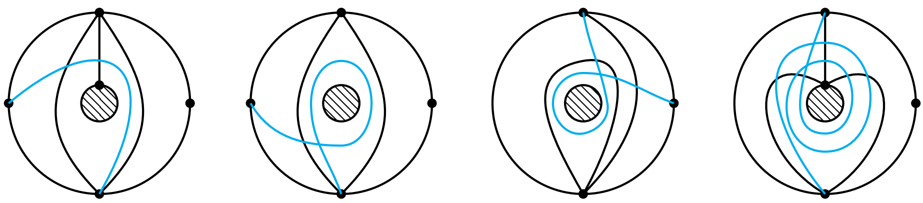


Figure 7.5: Examples of permissible arcs

Remark 7.4. If P is a triangulation of the surface, then every arc in P is permissible.

This remark follows directly from the definitions. If P is a triangulation we cannot have loops or arcs intersecting themselvs. Additionally, a triangulation does not have any unmarked boundary components. Thus, the first condition of the defnition is automatically satisfied. Since the arcs does not intersect each other, except at the marked points, there are no arcs that consecutively cross other arcs.

Definition 7.5. Let γ and γ' be two permissible arcs in (S, M). We say that the arcs are **equivalent**, $\gamma \simeq \gamma'$, if one of the following conditions is satisfied.

(1) Let Δ be a tile in $(\mathsf{S}, M, \mathsf{P})$, not of type I, and let $\delta_1, \ldots, \delta_k$ be a consecutive sides of Δ such that:

   (a) γ is homotopic to the concatenation of γ' and $\delta_1, \ldots, \delta_k$ and,
   (b) γ starts at an endpoint of δ_1, γ' starts at an endpoint of δ_k and their first crossing with P is with the same side of Δ. Or equivalently, γ starts at an endpoint of δ_k, γ' starts at an endpoint of δ_1 and their first crossing with P is with the same side of Δ.

(2) γ and γ' both start at a marked point of a tile Δ of type I or II. Their first crossing with P is on the same side of Δ. The segments of γ and γ' between p and their endpoints are homotopic.

The equivalence class of a permissible arc γ will be denoted $[\gamma]$. Intuitively, two permissible arcs are equivalent as long as they cross the same arcs in P.

Example 7.6. Figure 7.6 and Figure 7.7 show examples of equivalent permissible arcs.

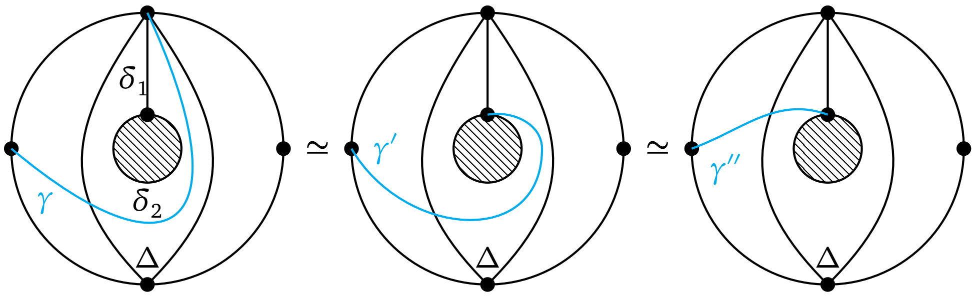


Figure 7.6: Example of equivalent permissible arcs.

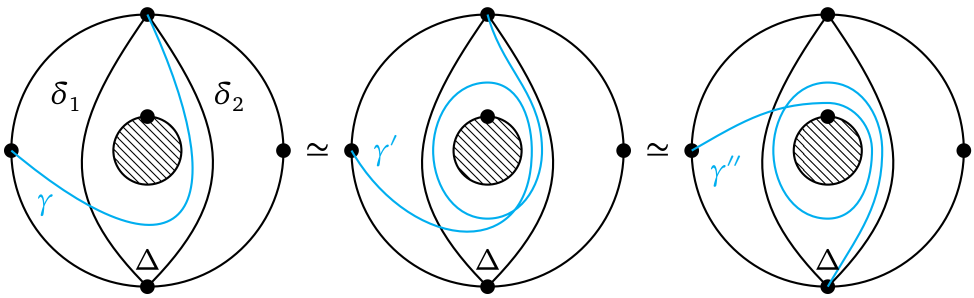


Figure 7.7: Example of equivalent permissible arcs when Δ is of type II.

Remark 7.7. Note that when the tile Δ is of type I, curves that wind around the unmarked boundary component σ in counterclockwise direction are considered equivalent to the one winding in clockwise direction. Similarly, if Δ is of type II, curves winding around in counterclockwise direction or in clockwise direction are considered equivalent. See Figure 7.7.

Remark 7.8. If P is a triangulation, then the equivalence classes of permissible arcs coincide with the homotopy classes. The homotopy classes correspond to classes in the cluster category and the equivalence classes correspond to classes in the module category.

This remark follows directly form the definitions. Since P is a triangulation, we have that every arc is permissible, and it follows that the definitions coincide.

Let γ and γ' be two curves in (S, M). We define $I(\gamma, \gamma')$ to be the minimal number of transversal intersections of representatives of the homotopy classes of γ and γ'. We remark that we only consider crossings in the interior of S. This implies that we might have $I(\gamma, \gamma') = 0$ for two curves sharing an endpoint. In particular, if γ and γ' lies in P, we always have intersection number zero. Let $I_{\mathsf{P}}(\gamma)$ denote the **intersection vector** of a curve $\gamma \in (\mathsf{S}, M, \mathsf{P})$ with respect to P. This means that $I_{\mathsf{P}}(\gamma)$ is equal to the vector of the form $(I(\gamma, \mathsf{a}_i))_{\mathsf{a}_i \in \mathsf{P}}$. Define the **intersection number** $|I_{\mathsf{P}}(\gamma)|$ **of** γ **with respect to** P to be $\Sigma_{\mathsf{a} \in \mathsf{P}}(I(\gamma, \mathsf{a}))$. A permissible arc is said to be **trivial** if $|I_{\mathsf{P}}(\gamma)| = 0$. Thus, a zero string is associated to any trivial permissible arc. This includes boundary segments and arcs in P.

Theorem 7.9. *If* $(\mathsf{S}, M, \mathsf{P})$ *is a tiling, then there is a bijection between the equivalence classes of non-trivial permissible arcs in* (S, M) *and the non-zero strings of* A_{P}*. This bijection sends the intersection vector to the dimension vector of the corresponding string module.*

Proof. We will first show a method of constructing a permissible arc from a string. This gives us three cases for the possible different tiles. Then we prove the converse, i.e. a method of constructing a string from a permissible arc.

Suppose that $w = v_1 \overset{\alpha_1}{\longleftrightarrow} v_2 \overset{\alpha_2}{\longleftrightarrow} \dots \overset{\alpha_{k-1}}{\longleftrightarrow} v_k$ is a non-zero string in A_{P}. The double headed arrows indicate a fixed, but arbitrary orientation of the arrows α_i. Let $\mathtt{v_i}$ denote the arc in P corresponding to the vertex v_i of Q. We will construct a curve $\gamma(w)$ in (S, M), which is the concatenation of $\gamma_0, \gamma_1, \dots, \gamma_k$. The curves γ_i are constructed in the following way.

Since there is an arrow α_1 between the vertices v_1 and v_2, the corresponding arcs $\mathtt{v}_1$ and $\mathtt{v}_2$ share an endpoint $p \in M$. Additionally, there are no other arcs between $\mathtt{v}_1$ and $\mathtt{v}_2$. This implies that $\mathtt{v}_1$ and $\mathtt{v}_2$ are sides of a unique tile Δ_1. Let q_1 be a point on $\mathtt{v}_1$ and q_2 be a point on $\mathtt{v}_2$. Define

γ_1 to be a curve that connects the two points q_1 and q_2, such that locally we have a triangle. See Figure 7.8.

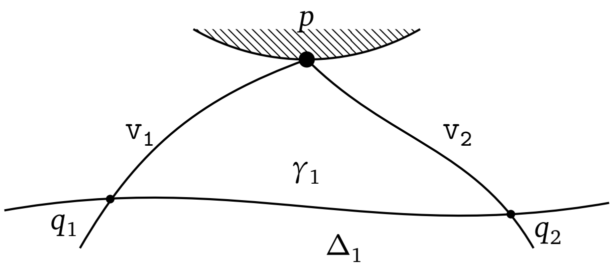


Figure 7.8: The curve γ_1 and the tile Δ_1.

In the same way we construct curves $\gamma_2, \ldots \gamma_{k-1}$ from the arcs $v_2 \ldots v_k$. The arc v_1 is incident with two tiles, Δ_1 and Δ_0. Note that these tiles may coincide. Moreover, the possible number of marked points on Δ_0 gives us three different scenarios for the curve γ_0.

Case 1: Δ_0 has only one marked point p.

If Δ_0 only has one marked point, then v_1 is a loop and Δ_0 is of type I. Let σ_0 be the unmarked boundary component in the interior of Δ_0 and let γ_0 be a curve in the interior of Δ_0 with endpoints p and q_1 and winding number 1 around σ_0. If the winding number is zero, then the concatenation of γ_0 with γ_1 would be homotopic to a curve which does not intersect v_1.

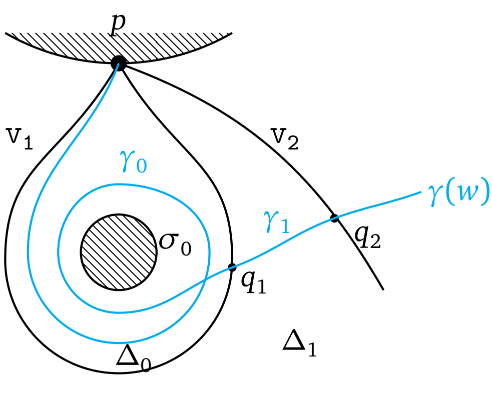


Figure 7.9: Case 1: v_1 is a loop and Δ_0 is of type I.

Case 2: Δ_0 has two marked points, p and p'.

If Δ_0 has two marked points, then the marked points must be the endpoints of v_1. The arc v_1 is not homotopic to a boundary segment, and thus, Δ_0 must be of type II. Now let γ_0 be a curve in the interior of Δ_0 with endpoints q_1 and p, such that we locally have a triangle with vertices p, q_1 and p'. See figure 7.10.

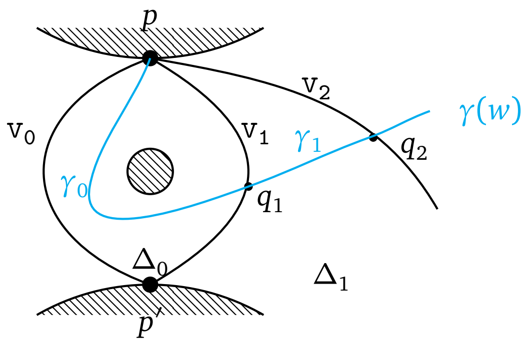


Figure 7.10: Case 2: Δ_0 is of type II.

Case 3: Δ_0 has at least three marked points.

If Δ_0 has at least three marked points, we choose γ_0 to be a curve in the interior of Δ_0 with endpoints q_1 and q, where q is a marked point in Δ_0, which is not an endpoint of v_1.

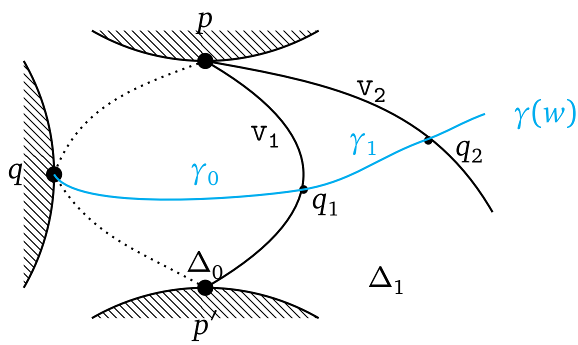


Figure 7.11: Case 3: Δ_0 has at least three marked points.

We have now constructed a curve γ_0 for the different possiblities for the tile Δ_0. In the same way we construct a curve γ_k on the other end of the string. Now, define $\gamma(w)$ to be the concantentaion of $\gamma_0, \ldots \gamma_k$. We then have the following:

- $v_i \neq v_{i+1}$, since w is a string, except for case when $\mathtt{v_i}$ is a loop. In either case, the curves $\gamma_i \forall i$ are not homotopic to a piece of an arc in P.
- The intersections between $\gamma(w)$ and P are transversal, and the intersection points are indexed by vertices of w.
- $\gamma(w)$ is a permissible arc.

Hence the intersection numbers are minimal and $I_{\mathsf{P}}(\gamma(w)) = \mathbf{dim}\, M(w)$, where $\mathbf{dim}\, M(w)$ denotes the dimension vector of the string module $M(w)$.

Note that we have a choice in the form of the curves γ_0 and γ_k. However, if we keep the winding number around unmarked boundary components at zero or one, in order to obtain a permissible arc, every possible choice gives a permissible arc equivalent to the one we built above. Hence, we have constructed a permissible arc from a string.

Now, we construct the converse. The idea is that we consider an arc that is not in P and represent this as a string. We do so by considering the intersections with P and order the intersecting arcs (from P) from start to finish to construct a string.

Let γ be a permissible arc in (S, M) with $|I_{\mathsf{P}}(\gamma)| \neq 0$ and endpoints $s(\gamma)$ and $t(\gamma)$. Suppose that γ is chosen such that the intersections with P are transversal and minimal.

Let γ be oriented from $s(\gamma) \in M$ to $t(\gamma) \in M$, and let $\mathtt{v_1}$ denote the first arc of P that intersects γ, $\mathtt{v_2}$ denote the second arc and so on. Thus, we have a sequence $\mathtt{v_1}, \dots, \mathtt{v_k}$ of arcs in P. Since the intersections are minimal, we have $\mathtt{v_i} \neq \mathtt{v_{i+1}}$, except for the case when $\mathtt{v_i}$ is a loop. In both cases, there are arrows $\alpha_i : v_i \to v_{i+1}$ or $\alpha_i : v_{i+1} \to v_i$ in $Q(A_{\mathsf{P}})$. Since A_{P} is isomorphic to A we have a walk $w(\gamma) = v_1 \overset{\alpha_1}{\longleftrightarrow} v_2 \overset{\alpha_2}{\longleftrightarrow} \dots \overset{\alpha_{k-1}}{\longleftrightarrow} v_k$ in Q_{P}. This walk avoids relations and the arrows are such that $\alpha_{i+1} \neq \alpha_i^{-1}$. Hence, $w(\gamma)$ is a non-zero string in A_{P}.

By defnition of equivalent permissible arcs, we have that if $\gamma \simeq \gamma'$, then necessarily $w(\gamma') = w(\gamma)$. Thus, the map is also well-defined for equivalence classes of permissible arcs, and by construction, the two constructions are inverses. ■

In the next example we will illustrate the construction in the proof of Theorem 7.9.

Example 7.10. Let A be the following gentle algebra without any relations.

$$A = K(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4).$$

The tiling algebra A_{P} then corresponds to the following picture, Figure 7.12.

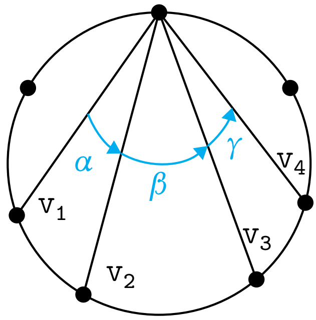
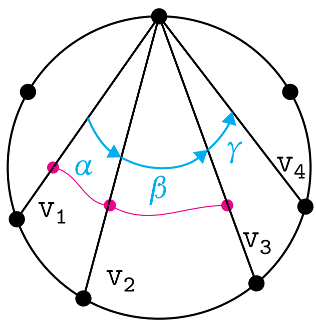


Figure 7.12: The tiling algebra of $A_{\mathrm{P}} = K(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4)$

We will consider the string $w = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. There is an arrow between the vertices 1 and 2. Thus, the arcs v_1 and v_2 share an endpoint and are sides of a unique tile Δ_1. Now, we choose a point q_1 of v_1, a point q_2 of v_2 and a point q_3 of v_3, visualised as pink dots. Then we connect points with a curve and obtain the following picture.



The arc v_1 is incident with two tiles, the tile defined by v_1, v_2 and a boundary segment, and the tile defined by v_1 and just a boundary segment. We call the latter tile Δ_0. The tile Δ_0 has three marked points, where two of these are endpoints of v_1. Let q be the marked point that is not an endpoint of v_1. Now, we choose a curve starting at q that ends at the point q_1 of v_1. We construct a similar curve from q_3 to the marked point q_4, that is not an endpoint of v_3, in the tile defined by v_3 and v_4. Hence, we get Figure 7.13.

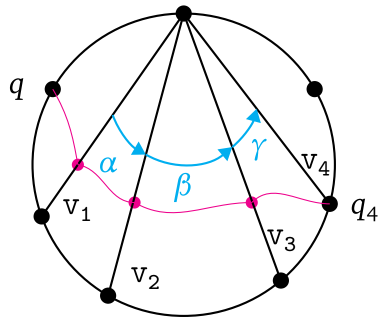


Figure 7.13: The permissible arc $\gamma(w)$.

We have now constructed an arc from a string, and we denote it by $\gamma(w)$. The arc $\gamma(w)$ will in fact be a permissible arc; it does not wind around any unmarked boundary comoponent, and whenever it crosses two arcs it crosses consecutively. The curve is defined such that it only crosses the arcs in P corresponding to the string $w = \beta\alpha$.

Conversely, let γ be a permissible arc. We order the arcs that γ crosses from start to finish, and obtain a sequence of arcs. By previous results and definitions we know that there are arrows between the arcs in Q_{P}. Thus, we can translate these arcs to vertices of a quiver, and we constructed a string.

In this thesis the main focus when talking about modules is string modules. However, to complete the characterization of indecomposable modules we have the following results regarding band modules.

Proposition 7.11. Let $(\mathsf{S}, M, \mathsf{P})$ be a tiling. Then the homotopy classes of permissible closed curves c in (S, M) with $|I_{\mathsf{P}}(c)| \geq 2$ is in one-to-one correspondence to powers of bands of A_{P}. Additionally, the Auslander-Reiten translate of $M(c)$, is equal $M(c)$ itself.

Sketch of the proof. The proof of the first statement is similar to the proof of Theorem 7.9. However, we now consider bands and not strings. This means that the pink curve in the previous example will not start nor end at a marked point. It will be a closed curve around a boundary component.

The second statement follows from the fact that the Auslander-Reiten translate acts on band modules as the identity. ■

Example 7.12. In this example we will illustrate the bijection of Proposition 7.11. Let A be the gentle algebra defined by the following quiver Q.

$$\begin{tikzcd} 1 \arrow[r, "a"] & 4 \\ 3 \arrow[u, "b"] \arrow[r, "d"'] & 2 \arrow[u, "c"'] \end{tikzcd}$$

A corresponds to the following tiling algebra, Figure 7.14.

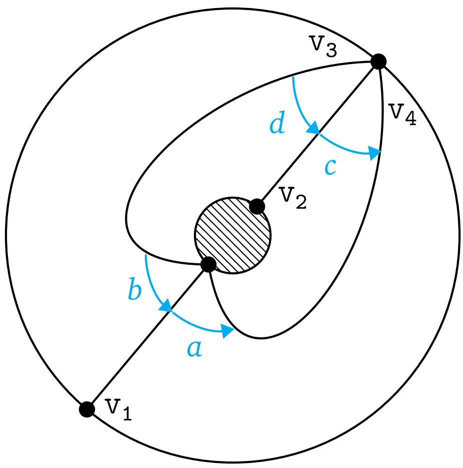


Figure 7.14: The tiling algebra A_{P}

Consider the band $w = ac^{-1}d^{-1}b$ in A. The vertices of the band correspond to arcs and are labelled accordingly. If we want to construct a curve in A_{P} corresponding to the band, we follow the arrows listed in the band and draw it as a closed curve, as in Figure 7.15.

If the band had a power of n, for $n \geq 2$ the pink curve would wind n times around the boundary component. We see that the pink curve is a permissible closed curve, since whenever it crosses two arcs we locally have a triangle.

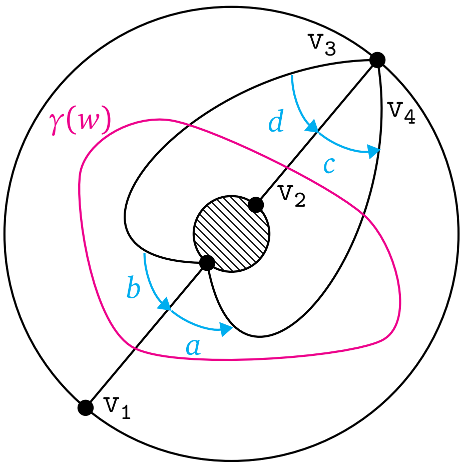


Figure 7.15: The band $bd^{-1}c^{-1}a$ visualised geometrically as the closed pink curve.

Corollary 7.13. Let A_{P} be a tiling algebra. Every permissible simple closed curve c has intersection number $|I_{\mathsf{P}}(c)| \leq 1$ if and only if A_{P} is of finite representation type.

Proof. This follows directly from Proposition 7.11. If we have a closed curve with intersection number greater than 1, it will correspond to a band. A band generates infinite representation type algebras, thus it is not finite dimensional. ∎

7.2 Pivot Elementary Moves

In this section we will give a description of irreducible morphisms between string modules in terms of pivot elementary moves. Let γ be a permissible arc in the marked surface and let $w(\gamma)$ be the associated string in the gentle algebra. We will use $M([\gamma])$ as notation for the string module $M(w(\gamma))$.

Definition 7.14. Let γ be a permissible arc corresponding to a non-empty string $w(\gamma)$. Let s and t be the startpoint and endpoint of γ, respectively. Define Δ to be the tile in (S, M) which contains the segment of γ between s and the first crossing with P in the interior of S. Let γ_s be a permissible arc equivalent to γ with startpoint s', which we find as described below. We define $f_t(\gamma_s)$ to be the permissible arc obtained from γ, such that $f_t(\gamma_s)$ is the concatenation of γ_s with the boundary segment connecting s' to its counterclockwise neighbour. We have different ways of constructing the arc γ_s, depending on the tile Δ.

Case 1: Δ is not of type I or II.

If Δ is not of type I or II we move s in counterclockwise direction around Δ to the vertex s'. We obtain a new arc γ_s with start point s' and endpoint t. It is constructed such that $\gamma_s \simeq \gamma$ and such that γ is not equivalent to the concatenation of γ_s and the side of Δ connecting the point s' and its counterclocwise neighbour. Note that γ and γ_s might coincide in some cases, and if so, we have $s = s'$.

Case 2: Δ is of type I.

If Δ is of type I, we define γ_s to be the permissible arc equivalent to γ with starting point $s = s'$. The arc γ_s wraps around the unmarked boundary component in the interior of Δ in counterclockwise direction.

Case 3: Δ is of type II.

If Δ is of type II we define γ_s to be the permissible arc equivalent to γ with the other marked point of Δ as the startpoint. The arc γ_s will have winding number around the unmarked boundary comoponent equal to zero, and the unmarked boundary component is to the left of γ_s.

We define **pivot elementary move** $f_t([\gamma])$ to be the the equivalence class of the permissible arc $f_t(\gamma_s)$. The pivot elementary move $f_s([\gamma]) = [f_s(\gamma_t)]$ is defined similarly.

We can think of pivot elementary move as moving the end of γ connected to s as far as we can in counterclockwise direction, without moving outside of the equivalence class. We then obatin a new cruve γ_s with startpoint s'. To construct the pivot elementary move curve $f_t(\gamma_s)$, we combine γ_s with the boundary segment between s' and the next marked point in counterclockwise direction. Then choose the curve homotopic to this one, without unecessary crossings of P to be the pivot elementary move of γ_s.

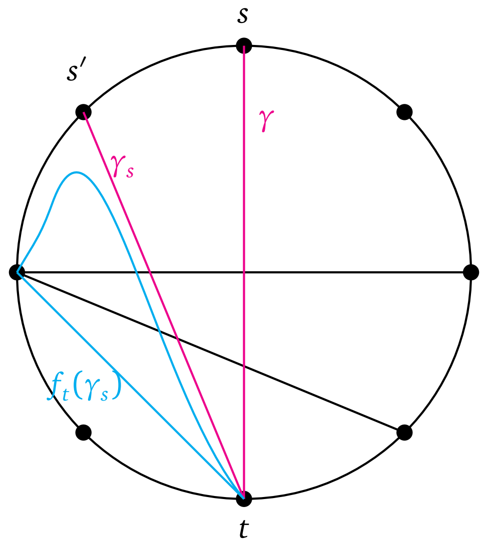


Figure 7.16: Example of pivot elementary move - *Case 1*. γ is equivalent to γ_s. The curved blue line is how we first think of $f_t(\gamma_s)$ before we draw it without unecessary crossings of the triangulations.

Theorem 7.15. *Let w be a string in the gentle algebra A with corresponding tiling algebra A_{P}. Let $M(w)$ be the corresponding string module over A_{P}.*

(1) The irreducible morphisms starting at $M(w)$ are obtained by pivot elementary moves on either endpoint of the corresponding permissible arc $\gamma(w)$.
(2) Let γ be a permissible arc. If $M(\gamma)$ is not injective, then the Auslander-Reiten sequence of string modules in mod A_{P} *are of the form* $0 \to M([\gamma]) \longrightarrow M(f_s(\gamma)) \oplus M(f_t(\gamma)) \longrightarrow M(f_s(f_t(\gamma))) \to 0$ *for all permissible arcs γ.*

Before proving the theorem, recall that we can obatin all irreducible morphisms between string modules by adding or removing a hook. If w is a string, we defined ${}_l w = w\alpha^{-1}v$ (see Chapter 4) and obtained an irreducible morphism from $M({}_l w) \to M(w)$. Recall the reformulation of Lemma 4.14 and Lemma 4.15, where we considered ${}_l w = u$ to be the string, and obtained an irreducible morphism $M(u) \to M(u_l)$.

$$\begin{tikzcd} M({}_l w) \arrow[r] \arrow[d, equal] & M(w) \arrow[d, equal] \\ M(u) \arrow[r] & M(u_l) \end{tikzcd}$$

This reformulation will be useful in the next proof.

Proof. For simplicity we will denote the permissible arc $\gamma(w)$ by γ throughout this proof.

By the fact we can find all irreducible morphisms between string modules, we only need to prove that $w_l = w(f_t(\gamma_s))$ and $w_r = w(f_s(\gamma_t))$. We will only prove the former, since the proof of the latter is similar.

If w is a trivial string, let $w = e_v^{\pm}$, for some vertex v. Recall that, whenever w is a trivial string it correponds to a permissible arc, which only intersects with P once. If there is no arrow ending at v, then $M(w)$ is an injective module, and $w_l = 0 = w_r$. Regardless of the orientation of γ, we see that the pivot elementary moves at either endpoint of γ yields a permissible arc with no intersections with P. This finishes the proof for this case.

If w is a non-trivial string, let $w = v_1 \overset{\alpha_1}{\longleftrightarrow} v_2 \overset{\alpha_2}{\longleftrightarrow} \dots \overset{\alpha_{k-1}}{\longleftrightarrow} v_k$, with $k \geq 2$. We orient the arc γ, corresponding to w, such that s and t is the startpoint and endpoint of γ, respectively. We then orient γ such that it crosses the arcs $\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{k}$ in this order.

Assume that there is at least one arrow in Q with target v. The arc v is the side of two different tiles Δ_0 and Δ_1. Let α be an arrow with target v such that $w\alpha$ is a string. Assume that the arc corresponding to $s(\alpha)$ is a side of Δ_0. We then let γ be oriented such that s is a marked point of Δ_0 and t is a marked point of Δ_1. If there is no arrow with target v such that $w\beta$ is defined, we then orient γ such that s lies in the tile which does not contain the source of β. If the two tiles Δ_0 and Δ_1 coincide, we can use a similar argument to orient the arc γ.

Let $\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{k}$ be the arcs of P that γ crosses in this order. Define Δ to be the tile determined by the marked point s and the arc v_1. Assume that there is an arrow α in Q_{A_P} such that $w\alpha$ is a string. Note that this arrow is unique by the properties of string-composition in a gentle algebra. To find the corresponding arc for w_l, we use the pivot elementary move on

γ However, the pivot elementary move depends on the tile Δ. Thus, we have three different cases.

Case 1: Δ is an n-gon with $n \geq 3$.

If Δ is an n-gon, it cannot be of type I or II. Thus, there is one side $\mathtt{v}_0$ of Δ that is an arc and has an endpoint in common with $\mathtt{v}_1$. This marked point corresponds to α, i.e. α is the arrow going from $\mathtt{v}_0$ to $\mathtt{v}_1$. Let s' be the other endpoint of $\mathtt{v}_0$. The arc γ_s is then the permissible arc obtained from the concatenation of γ with the sides of Δ when travelling counterclockwise from s to s'. Clearly, $\gamma_s \simeq \gamma$, see Figure 7.17.

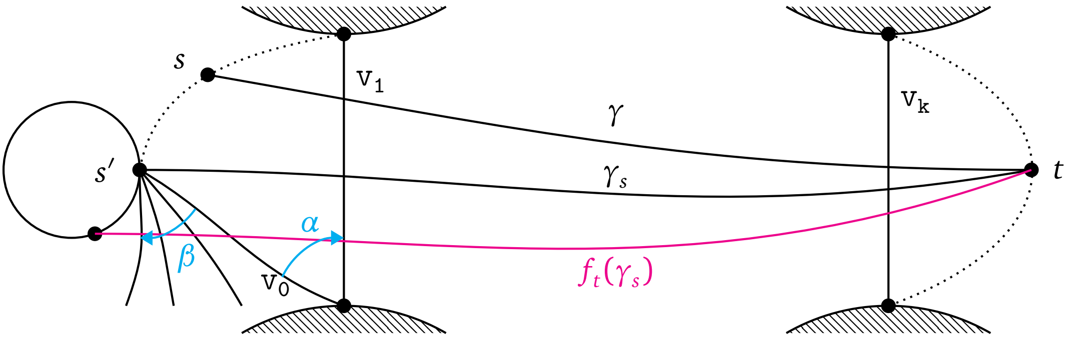


Figure 7.17: *Case 1:* Pivot elementary move which adds a hook.

Now, $f_t(\gamma_s)$ is the concatenation of γ_s and the boundary segment connecting s' and its counterclockwise neighbour. The pivot elementary move $f_t(\gamma_s)$ will then cross the fan at s' starting at $\mathtt{v}_0$, together with $\mathtt{v}_1, \ldots, \mathtt{v}_\mathtt{k}$, as in Figure 7.17. Hence $f_t(\gamma_s)$ is permissible, and the corresponding string $w(f_t(\gamma_s)) = w\alpha\beta^{-1}$, where β corresponds to the fan at s' starting at $\mathtt{v}_0$. Hence, $w(f_t(\gamma_s)) = w_l$.

Case 2: Δ is of type I.

If Δ is of type I the arrow starting and ending at v_1 cannot be the first arrow in the string w. Hence, α is the loop arrow at v_1. Let γ_s be the curve starting at $s = s'$ and wraps around the unmarked boundary component in the tile Δ, as in Defnintion 7.14. Now, the path β corrreponds to the fan at $s' = s$ starting with $\mathtt{v}_1$. The pivot elementary move $f_t(\gamma_s)$ will then cross the fan, then wind around the boundary component in the interior of Δ and then follow the curve γ. Hence we have, $w(f_t(\gamma_s)) = w\alpha\beta^{-1} = w_l$. See the Figure 7.18.

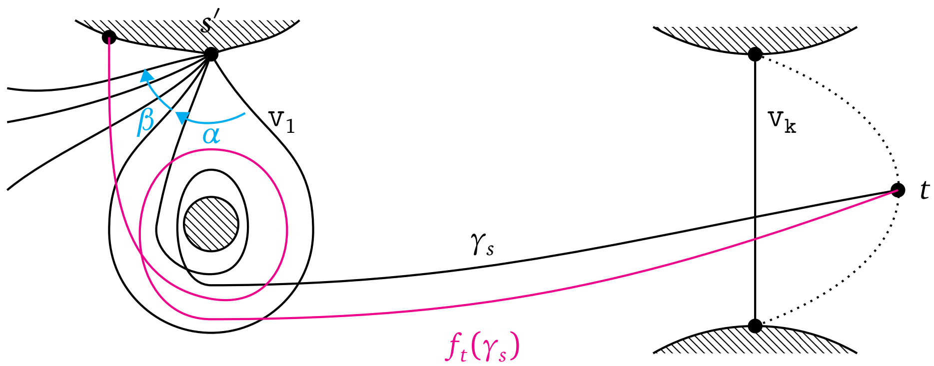


Figure 7.18: *Case 2:* Pivot elementary move which adds a hook.

Case 3: Δ is of type II.

If Δ is of type II, then $\gamma \simeq \gamma_s$, where γ_s is as in Definition 7.14. Let the other arc of Δ be denoted by v_0 and let $p_1 = s'$ be the start point of γ_s and p_2 be the other marked point of Δ. The arrow α then starts at v_0 and ends at v_1 at the marked point p_2. Let β be the path associated to the fan at p_1 starting at v_0. Then, $w(f_t(\gamma_s))$ is again w_l.

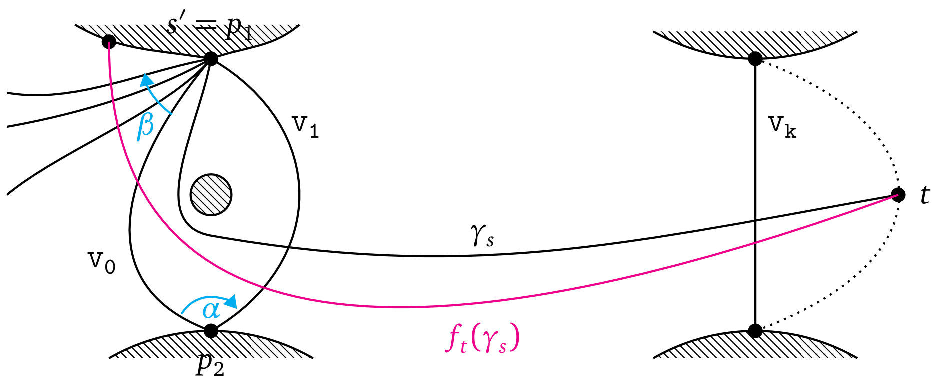


Figure 7.19: Case 3: Pivot elementary move which adds a hook.

For the final step, assume that there is no arrow α such that $w\alpha$ is a string. Then, Δ cannot be of type I nor of type II. If we would like to construct a module such that we have an irreducible morphism, we need to use the reformulation above the proof and remove a hook. Let

p_1 and p_2 be the endpoints of the arc $\mathtt{v}_1$ such that p_1, p_2, s and s' are in counterclockwise order around the boundary of Δ. Thus, the tile Δ has a side that is a boundary segment bound by s' and p_1. This implies that $f_t(\gamma_s)$ has endpoints p_1 and t, and it only crosses $\mathtt{v}_1$, $\mathtt{v}_{1+1}$ $\ldots$, $\mathtt{v}_\mathtt{k}$. The arc $\mathtt{v}_1$ is the first arc of the sequence $\mathtt{v}_1, \ldots, \mathtt{v}_\mathtt{k}$ such that the substring $v_1 \overset{\alpha_1}{\longleftrightarrow} v_2 \overset{\alpha_2}{\longleftrightarrow} \ldots \overset{\alpha_{l-2}}{\longleftrightarrow} v_{l-1}$ is direct and $\alpha_{l-1} : v_l \to v_{l-1}$. In particular, if w is direct, then $f_t(\gamma_s)$ does not intersect with P. Hence, $w(f_t(\gamma_s)) = w_l$.

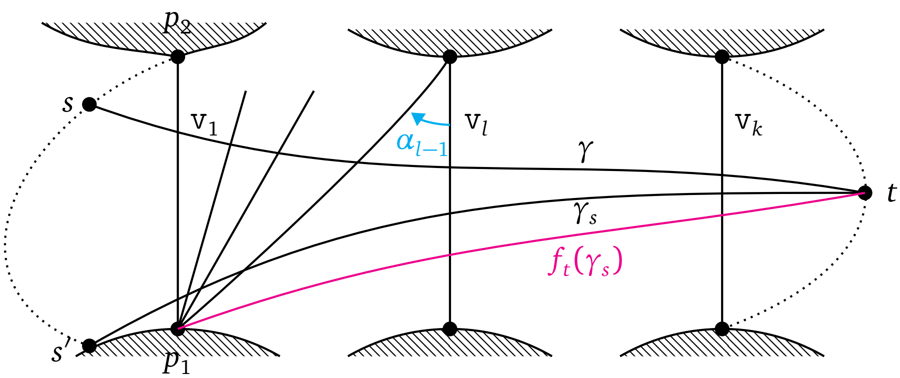


Figure 7.20: Pivot elementary move which removes a hook.

Now, we need to prove that $w(f_s(f_t(\gamma_s))_t) = (w_l)_r$. However, by previous parts of the proof, and the dual, we have $w(f_t(\gamma_s)) = w_l$ and $w(f_s(f_t(\gamma_s))_t) = (w(f_t(\gamma_s))_t)_r = (w_l)_r$. Hence the proof is finished. ■

Remark 7.16. Let γ be a permissible arc. The construction of γ_s does not affect the ending point of γ. Dually, γ_t does not change the startpoint of γ. Thus, we have $(\gamma_t)_s = (\gamma_s)_t$, and we will denote it by $\gamma_{s,t}$. In addition, the ending points of $f_t(\gamma_s)$ and γ_s coincide. Similarly, the staring points of $f_s(\gamma_t)$ and γ_t coincide. We can then conclude that $f_t(\gamma_s) = f_t(\gamma_{s,t})$, and $f_s(\gamma_t) = f_s(\gamma_{s,t})$.

Let γ be a permissible arc. We define $\tau^{-1}(\gamma)$ to be the equivalence class of the arc obtained from $\gamma_{s,t}$ by moving its endpoints to the counterclockwise neighbours. This is denoted as $\tau^{-1}([\gamma])$ or $\tau^{-1}(\gamma_{s,t})$. Remark that $\tau^{-1}(\gamma_{s,t})$ is again a permissible arc.

Corollary 7.17. If $M([\gamma])$ is a string module, then $M([\gamma])$ is non-injective if and only if $|I_\mathsf{P}(\tau^{-1}(\gamma_{s,t}))| \neq 0$. If this holds, we have that $\tau^{-1}(M([\gamma])) = M(\tau^{-1}([\gamma]))$.

Proof. By the proof of Theorem 7.15 it follows that $\tau^{-1}([\gamma]) = [f_s f_t(\gamma_{s,t})] = [f_t f_s(\gamma_{s,t})]$. By the second point of Theorem 7.15, we have $M([f_t f_s(\gamma_{s,t})]) = M(\tau^{-1}(M([\gamma])))$, in the case where $M([\gamma])$ is not injective. If $M([\gamma])$ is injective, then the corresponding string w is such that there are no arrows α, β in A_{P}, such that either $w\alpha$ or βw are strings. Hence $w_{r,l}$ is the empty string and $|I_{\mathsf{P}}(\tau^{-1}(\gamma_{s,t}))| = 0$. ∎

Example 7.18. In this example we will illustrate how to the Auslander-Reiten translate of a geometric model of a string module. We finish this example by illustrating the Auslander-Reiten quiver in terms of the geometric model. Let Q be the following quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \\
3 \arrow[u, "\beta"] \arrow[r, "\delta"'] & 4 \arrow[u, "\gamma"']
\end{tikzcd}$$

with admissible ideal $I = \langle \alpha\beta, \gamma\delta \rangle$. This corresponds to the following tiling algebra.

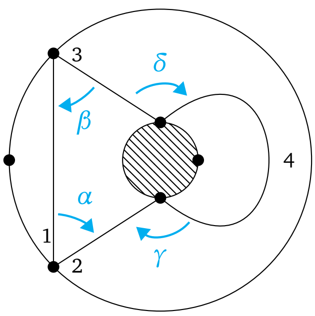


Figure 7.21: The tiling algebra of Example 7.18.

Let us consider the string e_1 in A. Note that we will write e_1 for the corresponding permissible arc in A_{P} as well as for the string in A. We will now construct τ^{-1} of the module $M(e_1)$ over A_{P}. The string module $M(e_1)$ can be visualised as in the following Figure 7.22.

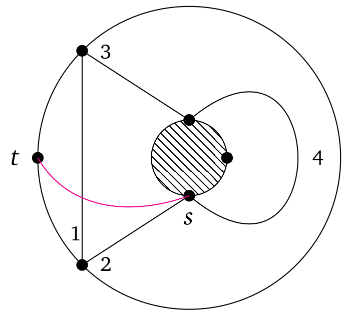


Figure 7.22: The string module $M(e_1)$ of Example 7.18 as a A_{P}-module.

We choose the startpoint s and endpoint t as described in the proof of Theorem 7.9. In practice, when finding τ^{-1}, the orientation of the curve does not matter, since we do the same steps for both endpoints.

Now, we find $(e_1)_s$ by moving the point s as far as we can in the counterclockwise direction along the edges of the tile. Note that the orientation around the boundary component in the interior is the opposite of the orientation of the outer boundary. We then get the following curve, Figure 7.23.

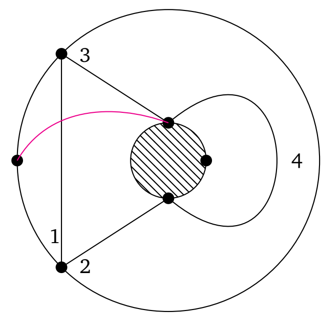


Figure 7.23: The permissible arc $(e_1)_s$ of Example 7.18.

Now when constructing $((\gamma(e_1))_s)_t$ we do the same thing for the endpoint t. However, we cannot move t to any marked point inside the tile and still have an equivalent curve to $(e_1)_s$. Thus, $(e_1)_{s,t}$ is equal to $(e_1)_s$.

For the final step we move both endpoints in the counterclockwise direction. Hence, we have Figue 7.24.

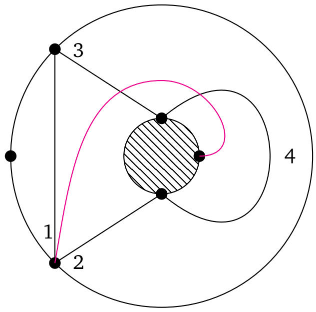


Figure 7.24: $\tau^{-1}(M(e_1))$ in Example 7.18.

We see that this corresponds to the sting module $M(\delta)$, which in fact is equal to $\tau^{-1}(M(e_1))$.

Lastly, we give an example of an Auslander-Reiten quiver of a tiling algebra. The gentle algebra above will have the following Auslander-Reiten quiver:

$$\begin{tikzcd}
& M(\alpha) \arrow[dr] \arrow[rr, dashed, no head] & & M(e_4) \arrow[dr] \arrow[rr, dashed, no head] & & M(\beta) \arrow[dr] & \\
M(e_2) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & M(\gamma^{-1}\alpha) \arrow[ur] \arrow[dr] & & M(\beta\delta^{-1}) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & M(e_3) \\
& M(\gamma) \arrow[ur] \arrow[rr, dashed, no head] & & M(e_1) \arrow[ur] \arrow[rr, dashed, no head] & & M(\delta) \arrow[ur] &
\end{tikzcd}$$

The Auslander-Reiten quiver of the tiling algebra A_P is as shown in Figure 7.25, where we have chosen a representative for each equivalence class $[\gamma]$.

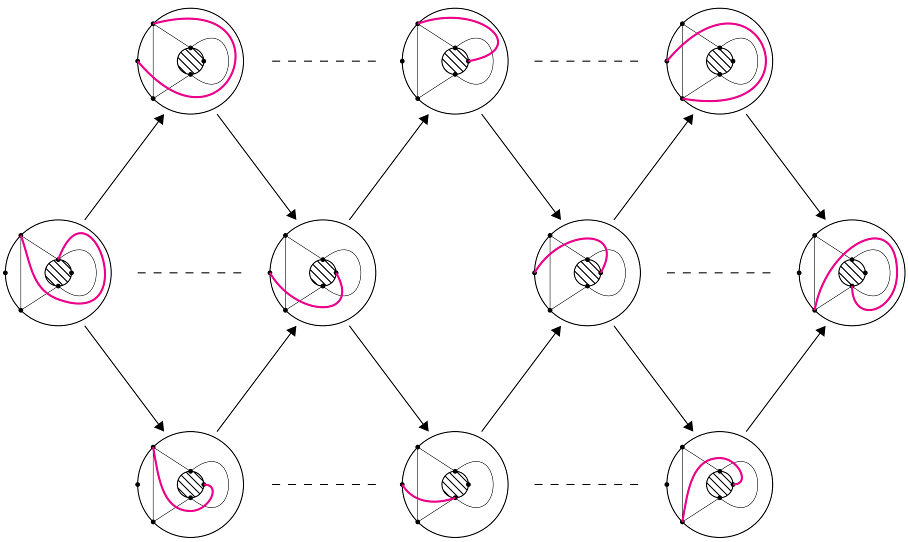


Figure 7.25: The Auslander-Reiten quiver of Example 7.18 in terms of the geometric model.

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Gentle Algebras and a Geometric Model for the Module Category

Kunnskap for en bedre verden