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Realization functors and HRS-tilting

Master's thesis in Mathematical Sciences
Supervisor: Steffen Oppermann
June 2021

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Acknowledgement

I would like to express my sincere thanks to my supervisor Professor Steffen Oppermann, who has dedicated hours of his highly valuable time to make this thesis come to fruition. I would also like thank everyone at my study room, who has helped me through this last year with invaluable feedback and support. Lastly I owe a great debt to my wonderful girlfriend Nan Amalie, who has supported and helped me through my entire five-year master program. None of this would be possible without any of you.

Abstract

Let $\mathcal{A}$ be an abelian category, and $\mathcal{H}$ be the abelian heart of a t-structure over $D^b(\mathcal{A})$. We show that we can always construct a realization functor $D^b(\mathcal{H}) \to D^b(\mathcal{A})$, whose restriction on $\mathcal{H}$ is equal to the identity functor. We will further give sufficient conditions for this functor to be a derived equivalence. Lastly, we will use the realization functor to construct the derived equivalence from the HRS-tilting $D^b(\mathcal{B})$ to $D^b(\mathcal{A})$

Sammendrag

La $\mathcal{A}$ vær en abelsk kategori, og $\mathcal{H}$ vær det abelske hjertet til en t-struktur over $D^b(\mathcal{A})$. Vi vil da vise at vi alltid kan konstruere en “realization functor” $D^b(\mathcal{H}) \to D^b(\mathcal{A})$, som restriktert til $\mathcal{H}$ vil være lik identitets-funktoren. Videre vil vi gi tilstrekkelige kriterier for at funktoren vil bli en derivert ekvivalens. Til slutt vil vi bruke denne funktoren til å konstruere en derivert ekvivalens fra HRS-tiltingen $D^b(\mathcal{B})$ til $D^b(\mathcal{A})$

Contents

Chapter 1

Introduction

In the study of triangulated categories, the notion of t-structures sometimes arises. First introduced by Beilinson, Bernstein and Deligne [BBD83], a t-structure is a certain pair of full subcategories of a triangulated category $\mathcal{T}$, whose intersection (often named the *heart*, $\mathcal{H}$, of the given t-structure) is an abelian subcategory of $\mathcal{T}$. A natural question then arises: *Can the category $\mathcal{T}$ be recovered from the heart $\mathcal{H}$ of a given t-structure?*

As often in mathematics the answer is: sometimes. We will show that if the t-structure is bounded, the heart generates the category $\mathcal{T}$. Then another natural question to ask is how does the bounded derived category $D^b(\mathcal{H})$ over the heart compare to the category $\mathcal{T}$? We will construct a functor $F : D^b(\mathcal{H}) \to \mathcal{T}$ given some conditions on $\mathcal{T}$, and prove that this becomes an equivalence under relatively mild conditions. The functor is called a *realization functor*, and was also first introduced in [BBD83], and then generalized by Beilinson in [Bei87]. We will study how this functor can be used to create derived equivalences. An important example of the realization functor is the HRS-tilting, first introduced by Happel, Reiten and Smalø [HRS96], which induces a certain derived equivalence.

The thesis will start by introducing the Yoneda extension groups in Chapter 2; an important tool that will be used throughout the whole thesis. In particular we use the fact that in an abelian category a short exact sequence induces a long exact sequence of Yoneda extension groups. In Chapter 3 we will introduce the notion of t-structures on triangulated categories, and investigate properties of the heart and its connection to the Yoneda extension. Another important tool introduced in this chapter will be a cohomological functor from the triangulated category to the heart of the t-structure. The culmination of these two chapters will be the first section of Chapter 4; the construction of a special realization functor and one of the main theorems of the thesis:

Theorem (Theorem 4.10)**.** *Let $\mathcal{A}$ be an abelian category, and let $D^b(\mathcal{A})$ be the bounded derived category equipped with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with heart $\mathcal{H}$. There exist a t-exact functor*

$$\mathrm{real} : D^b(\mathcal{H}) \longrightarrow D^b(\mathcal{A})$$

such that $\mathrm{real}|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$

Section 4.3 consist of quite surprising, immediate consequences of the realization functor. It is in general not known if realization functors are unique, however we get an especially surprising result:

Theorem (Corollary 4.10.3 and Theorem 4.11)**.** *Let $F : D^b(\mathcal{H}) \to \mathcal{T}$ be a realization functor. Then the following are equivalent*

(1) F is full
(2) F is dense
(3) F is an equivalence
(4) Any other realization functor $G : D^b(\mathcal{H}) \to \mathcal{T}$ is an equivalence

Lastly, in Chapter 5, we will introduce the notion of torsion pairs, and observe how torsion pairs induce t-structures. We will look at an important example of a realization functor, namely the HRS-tilting [HRS96]. We will arrive at a derived equivalence $F : D^b(\mathcal{B}) \xrightarrow{\cong} D^b(\mathcal{A})$ between the bounded derived category over an abelian category $\mathcal{A}$ and the bounded derived category over a certain full abelian subcategory $B \subseteq D^b(\mathcal{A})$.

It is assumed the reader has prior knowledge in homological algebra, in particular in the study of triangulated categories and localization. For a recap of the notation used, and what is assumed known one is advised to read the lecture notes on Homological Algebra by Steffen Oppermann [Opp16] and the lecture notes on Derived categories, resolutions, and Brown representability by Henning Krause [Kra07]. We will only look at bounded derived categories in this thesis.

It is worth noting that in order to keep the thesis as self-contained as possible no proofs have been omitted. Thus, wherever a theorem and proof is similar or identical to one in a previous paper there is a reference to the original proof in the title of the theorem. At the end, there is also an appendix that gives some basic results found in neither of the above mentioned lecture notes, but that still is assumed known, and will be used throughout the text.

Chapter 2

Yoneda extensions

Recall that if an abelian category $\mathcal{A}$ has enough injectives and projectives the functor $\mathrm{Ext}^n(-,-)$ can be defined. Yoneda gave an alternative definition of the functor using extensions without assuming enough projectives and injectives [Yon54]. The new functor, $\mathrm{YExt}^n(-,-)$, sometimes called the Yoneda extension, does not require enough projectives and injectives, however when $\mathrm{Ext}^n(-,-)$ are defined the two functors coincide. In this chapter we will define the Yoneda extension, and prove that it induces a long exact sequence applied to a short exact sequence.

2.1 Extensions

We will begin by defining extensions in an abelian category, and proving some basic properties of the extensions.

Definition 1. Let $\mathcal{A}$ be an abelian category, and $X, Y \in \mathcal{A}$. Then an n**-fold extension**, $\mathbb{E}$, of Y by X is an exact sequence

$$\mathbb{E} : 0 \to X \hookrightarrow E_1 \to \cdots \to E_n \twoheadrightarrow Y \to 0$$

where $E_i \in \mathcal{A}$
If $\mathbb{E}, \mathbb{F}$ are two n-fold extensions of X by Y, then a map $f : \mathbb{E} \to \mathbb{F}$ is a sequence of maps f_i, $i = 1, \cdots, n$

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & X & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow & & \downarrow{\scriptstyle f_n} & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

such that the diagram commutes.

Lemma 2.1. *Let $\mathbb{E}$ and $\mathbb{F}$ be two n-fold extensions of X by Y. If there is a map $\mathbb{E} \to \mathbb{F}$ then there exist a n-fold extension $\mathbb{G}$ and maps $\mathbb{E} \leftarrow \mathbb{G} \twoheadrightarrow \mathbb{F}$ where $\mathbb{G} \twoheadrightarrow \mathbb{F}$ is an epimorphism.*

Proof. Let $\mathbb{E} \to \mathbb{F}$ be given by

$$\begin{array}{cccccccccccccccc}
\mathbb{E}: & 0 & \longrightarrow & X & \xrightarrow{e_0} & E_1 & \xrightarrow{e_1} & E_2 & \xrightarrow{e_2} & \cdots & \xrightarrow{e_{n-1}} & E_n & \xrightarrow{e_n} & Y & \longrightarrow & 0 \\
\downarrow & & & \| & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow & & \downarrow{\scriptstyle \alpha_n} & & \| & & \\
\mathbb{F}: & 0 & \longrightarrow & X & \xrightarrow{f_0} & F_1 & \xrightarrow{f_1} & F_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & F_n & \xrightarrow{f_n} & Y & \longrightarrow & 0
\end{array}$$

We construct the exact sequence $\tilde{\mathbb{F}}$

$$0 \to F_1 \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} F_1 \oplus F_2 \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} F_2 \oplus F_3 \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} \cdots \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} F_{n-2} \oplus F_{n-1} \xrightarrow{(0\ 1)} F_{n-1} \to 0$$

Now let $\mathbb{G} = \mathbb{E} \oplus \tilde{\mathbb{F}}$

$$0 \longrightarrow X \xrightarrow{g_0} E_1 \oplus F_1 \xrightarrow{g_1} E_2 \oplus F_1 \oplus F_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} E_n \oplus F_{n-1} \xrightarrow{g_n} Y \longrightarrow 0$$

where

$$g_0 = \left(\begin{smallmatrix} e_0 \\ 0 \end{smallmatrix}\right), \ g_1 = \begin{pmatrix} e_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \ g_{n-1} = \left(\begin{smallmatrix} e_{n-1} & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right), \ g_n = \left(\begin{smallmatrix} e_n & 0 \end{smallmatrix}\right)$$

and

$$g_i = \begin{pmatrix} e_i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \text{for } i = 2, \cdots, n-2$$

Clearly the direct sum of the exact sequences, with the maps defined above is again exact. Thus $\mathbb{G}$ is an n-fold extension of X by Y.

Denote $E_i \oplus F_{i-1} \oplus F_i$ in $\mathbb{G}$ by G_i. Then we have a diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\left(\begin{smallmatrix} e_0 \\ 0 \end{smallmatrix}\right)} & G_1 & \xrightarrow{\left(\begin{smallmatrix} e_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} & G_2 & \xrightarrow{\left(\begin{smallmatrix} e_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)} & \cdots & \xrightarrow{\left(\begin{smallmatrix} e_{n-1} & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} & G_n & \xrightarrow{(e_n\ 0)} & Y & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle (\alpha_1\ 1)} & & \downarrow{\scriptstyle (\alpha_2\ f_1\ 1)} & & \downarrow & & \downarrow{\scriptstyle (\alpha_n\ f_{n-1})} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow[f_0]{} & F_1 & \xrightarrow[f_1]{} & F_2 & \xrightarrow[f_2]{} & \cdots & \xrightarrow[f_{n-1}]{} & F_n & \xrightarrow[f_n]{} & Y & \longrightarrow & 0
\end{array}$$

where the vertical maps $G_i \to F_i$ are $\left(\begin{smallmatrix} \alpha_i & f_{i-1} & 1 \end{smallmatrix}\right)$ for $i = 2, \cdots, n-1$. Note that all maps are epimorphisms (to see the rightmost epimorphism apply the five lemma), and since $f_i \alpha_i = \alpha_{i+1} e_i$ and $f_{i+1} f i = 0$ the diagram commutes.
By taking the projection $\mathbb{G} \to \mathbb{E}$ we have have maps $\mathbb{E} \leftarrow \mathbb{G} \twoheadrightarrow \mathbb{F}$. $\square$

The next lemma is a horizontal extension of the regular 3×3 lemma, which will be useful when studying at extensions of length bigger than 1.

Lemma 2.2. *Let*

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & X_1 & & X_2 & & \cdots & & X_{n-1} & & X_n & & \\
 & & \hookdownarrow & & \hookdownarrow & & & & \hookdownarrow & & \hookdownarrow & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & \cdots & \xrightarrow{y_{n-2}} & Y_{n-1} & \xrightarrow{y_{n-1}} & Y_n & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & Z_1 & \xrightarrow{z_1} & Z_2 & \xrightarrow{z_2} & \cdots & \xrightarrow{z_{n-2}} & Z_{n-1} & \xrightarrow{z_{n-1}} & Z_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

be a commutative diagram in an abelian category, where the two bottom rows are exact and all the columns are short exact sequences. Then there exist maps

$$0 \to X_1 \hookrightarrow X_2 \to \cdots \to X_{n-1} \twoheadrightarrow X_n \to 0$$

and the sequence is exact.

Proof. We have for each $i = 1, \cdots, n$ short exact sequences

$$0 \to \operatorname{Im}(y_{i-1}) \to Y_i \to \operatorname{Im}(y_i) \to 0$$

since $\operatorname{Im}(y_{i-1}) = \ker(y_i)$ and $\operatorname{Im}(y_i) = Y_i / \ker(y_i) = Y_i / \operatorname{Im}(y_{i-1})$.
By the universal property of images we get commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}(y_{i-1}) & \longrightarrow & Y_i & \longrightarrow & \operatorname{Im}(y_i) & \longrightarrow & 0 \\
 & & \vdots\!\twoheaddownarrow & & \twoheaddownarrow & & \vdots\!\twoheaddownarrow & & \\
0 & \longrightarrow & \operatorname{Im}(z_{i-1}) & \longrightarrow & Z_i & \longrightarrow & \operatorname{Im}(z_i) & \longrightarrow & 0
\end{array}$$

where the dashed arrows are epimorphisms. We then have, for each i a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker & \longrightarrow & X_i & \longrightarrow & \ker & \longrightarrow & 0 \\
 & & \hookdownarrow & & \hookdownarrow & & \hookdownarrow & & \\
0 & \longrightarrow & \operatorname{Im}(y_{i-1}) & \longrightarrow & Y_i & \longrightarrow & \operatorname{Im}(y_i) & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & \operatorname{Im}(z_{i-1}) & \longrightarrow & Z_i & \longrightarrow & \operatorname{Im}(z_i) & \longrightarrow & 0
\end{array}$$

where the top row exist and is exact by the 3×3 lemma. We obtain the exact sequence

$$0 \to X_1 \hookrightarrow X_2 \to \cdots \to X_{n-1} \twoheadrightarrow X_n \to 0$$

by the splicing of all the short exact sequences in the top row of the diagrams above. $\square$

The following two lemmas concerns the construction of new extensions from existing ones by taking pullbacks and pushouts.

Lemma 2.3. *Given the following commutative diagram with exact rows*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & 0 \\
 & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \\
0 & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & 0
\end{array}$$

If $Z_i \to Y_i$ is an epimorphism for every i then we have an exact sequence

$$0 \to PB_1 \to PB_2 \to \cdots \to PB_n \to 0$$

where PB_i denotes the pullback $X_i \coprod_{Y_i} Z_i$

Proof. Note that, from [Opp16, Proposition 13.4], given a pullback diagram

$$\begin{array}{ccc}
PB_i & \longrightarrow & Z_i \\
\twoheaddownarrow & & \twoheaddownarrow \\
X_i & \longrightarrow & Y_i
\end{array}$$

where $Z_i \twoheadrightarrow Y_i$ is an epimorphism, we have a short exact sequence

$$0 \to PB_i \hookrightarrow X_i \oplus Z_i \twoheadrightarrow Y_i \to 0$$

We then have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & PB_1 & & PB_2 & & \cdots & & PB_n & & \\
 & & \hookdownarrow & & \hookdownarrow & & \hookdownarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 \oplus Z_1 & \longrightarrow & X_2 \oplus Z_2 & \longrightarrow & \cdots & \longrightarrow & X_n \oplus Z_n & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & 0
\end{array}$$

Thus by Lemma 2.2 we have an exact sequence

$$0 \to PB_1 \to PB_2 \to \cdots \to PB_n \to 0$$

□

Lemma 2.4. *Given a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_1 & \hookrightarrow & Y_1 & \longrightarrow & Z_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_2 & \hookrightarrow & Y_2 & \longrightarrow & Z_2
\end{array}$$

with exact rows. Then, by taking the pushout of the left square, there exist a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_1 & \hookrightarrow & Y_1 & \longrightarrow & Z_1 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & X_2 & \hookrightarrow & X_2 \coprod_{X_1} Y_1 & \longrightarrow & Z_1 \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_2 & \hookrightarrow & Y_2 & \longrightarrow & Z_2
\end{array}$$

Dually, given the commutative diagram

$$\begin{array}{ccccccc}
X_1 & \longrightarrow & Y_1 & \twoheadrightarrow & Z_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
X_2 & \longrightarrow & Y_2 & \twoheadrightarrow & Z_2 & \longrightarrow & 0
\end{array}$$

with exact rows. Then, by taking the pullback of the right square, there exist a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
X_1 & \longrightarrow & Y_1 & \twoheadrightarrow & Z_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
X_2 & \longrightarrow & Y_2 \prod_{Z_2} Z_1 & \twoheadrightarrow & Z_1 & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
X_2 & \longrightarrow & Y_2 & \twoheadrightarrow & Z_2 & \longrightarrow & 0
\end{array}$$

Proof. We only prove the first part; the second is dual. Pushout preserves monomorphisms, and the cokernel of $X_1 \to Y_1$ is equal to the cokernel of $X_2 \to X_2 \coprod_{X_1} Y_1$. Since $Y_1 \to Z_1$ factors through this cokernel the

middle row is exact and the top two squares commute.
By the pushout property the map $X_2 \coprod_{X_1} Y_1 \to Y_2$ exist and the bottom left square commutes. What is left to show is that the bottom right square commutes. Since the outer square commutes we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & Y_1 & & \\ \downarrow & & \downarrow & \searrow & \\ X_2 & \longrightarrow & X_2 \coprod_{X_1} Y_1 & & Z_1 \\ & \searrow & & \searrow & \downarrow \\ & & Y_2 & \longrightarrow & Z_2 \end{array}$$

Showing that there is a unique map $X_2 \coprod_{X_1} Y_1 \to Z_2$ and the lower right square commutes. □

2.2 The Yoneda extension group

We are now ready to define the Yoneda extension groups, and showing the abelian group structure.

Definition 2 (Yoneda extension)**.** Let $\mathcal{A}$ be an abelian category. For two objects $X, Y \in \mathcal{A}$ and $n \geq 1$, let $\mathscr{E}$ be the collection of all n-fold extensions of Y by X.

$$\mathbb{E} : 0 \to Y \to E_1 \to \cdots \to E_n \to X \to 0$$

We consider two exact sequences, $\mathbb{E}$ and $\mathbb{F}$ to be *similar* if there is a map from $\mathbb{E}$ to $\mathbb{F}$

$$\begin{array}{lccccccccccccc} \mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \mathbb{F}: & 0 & \longrightarrow & Y & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

We consider two exact sequences $\mathbb{E}$ and $\mathbb{F}$ *equivalent* if there exists a commutative diagram

$$\begin{array}{lccccccccccccc} \mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\ \tilde{\mathbb{E}}: & 0 & \longrightarrow & Y & \longrightarrow & \tilde{E}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{E}_n & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \mathbb{F}: & 0 & \longrightarrow & Y & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

where $\tilde{\mathbb{E}}$ is also exact. That this forms an equivalence relation will be proved in Lemma 2.5
The **(n-th) Yoneda extension group**, $\mathrm{YExt}^n_{\mathcal{A}}(X,Y)$, of X and Y is the group whose elements are the equivalence classes $[\mathbb{E}] \in \mathscr{E}/\sim$.

Lemma 2.5. *The equivalence in the definition of the Yoneda extension group is an equivalence relation.*

Proof. The reflexive- and symmetric property is trivial, so what needs to be proven is the transitive property. Assume $\mathbb{E}$ is equivalent to $\mathbb{F}$, and $\mathbb{F}$ is equivalent to $\mathbb{G}$. We then have a commutative diagram

$$
\begin{array}{lccccccccccccc}
\mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
\tilde{\mathbb{E}}: & 0 & \longrightarrow & Y & \longrightarrow & \tilde{E}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{E}_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathbb{F}: & 0 & \longrightarrow & Y & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
\tilde{\mathbb{F}}: & 0 & \longrightarrow & Y & \longrightarrow & \tilde{F}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{F}_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathbb{G}: & 0 & \longrightarrow & Y & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_n & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

and from Lemma 2.1 there exist an n-fold extension $\mathbb{H}$ and maps $\tilde{\mathbb{E}} \leftarrow \mathbb{H} \twoheadrightarrow \mathbb{F}$. By using Lemma 2.3 on the maps $\mathbb{H} \twoheadrightarrow \mathbb{F} \leftarrow \tilde{\mathbb{F}}$ we get an exact sequence of pullbacks, $\mathbb{PB}$. Since, given $A \in \mathcal{A}$, the pullback of

$$
\begin{array}{ccc}
 & & A \\
 & & \| \\
A & = & A
\end{array}
$$

is again A, $\mathbb{PB}$ is an n-fold extension of Y by X. We have the maps

$$\mathbb{E} \leftarrow \tilde{\mathbb{E}} \leftarrow \mathbb{H} \leftarrow \mathbb{PB} \rightarrow \tilde{\mathbb{F}} \rightarrow \mathbb{G}$$

and thus a commutative diagram

$$\begin{array}{lccccccccccccc}
\mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
\mathbb{PB}: & 0 & \longrightarrow & Y & \longrightarrow & PB_1 & \longrightarrow & \cdots & \longrightarrow & PB_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathbb{G}: & 0 & \longrightarrow & Y & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

We conclude that $\mathbb{E}$ is equivalent to $\mathbb{G}$ and the transitive property holds. □

Example 1. In $\mathrm{YExt}^1_{\mathcal{A}}(X,Y)$ if $\mathbb{E}$ is equivalent to $\mathbb{F}$ we have a commutative diagram

$$\begin{array}{lccccccccc}
\mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \| & & \\
\tilde{\mathbb{E}}: & 0 & \longrightarrow & Y & \longrightarrow & \tilde{E} & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \| & & \\
\mathbb{F}: & 0 & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

and the five lemma forces $E \cong \tilde{E} \cong F$.

The Yoneda extension group is indeed an abelian group. To see this we first need to define what the Yoneda extension does on maps in $\mathcal{A}$

Definition 3. Given a function $f : Y_1 \to Y_2$ we define

$$\mathrm{YExt}^n_{\mathcal{A}}(X,f) : \mathrm{YExt}^n_{\mathcal{A}}(X,Y_1) \to \mathrm{YExt}^n_{\mathcal{A}}(X,Y_2)$$

by taking the pushout as shown in the following diagram:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y_1 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & Y_2 \coprod_{Y_1} E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

where the first row is an element in $\mathrm{YExt}^n_{\mathcal{A}}(X,Y_1)$ represented by $\mathbb{E}$ and the second row is the element represented by $\mathrm{YExt}^n_{\mathcal{A}}(X,f)(\mathbb{E})$. It is an easy observation to see that the bottom row is exact and the diagram commutes. Thus the class $[\mathrm{YExt}^n_{\mathcal{A}}(X,f)(\mathbb{E})] \in \mathrm{YExt}^n_{\mathcal{A}}(X,Y_2)$ and the definition makes sense. One usually writes $f \cdot -$ for $\mathrm{YExt}^n_{\mathcal{A}}(X,f)(-)$

Similarly, for a map $g : X_1 \to X_2$, we define the map

$$\mathrm{YExt}^n_{\mathcal{A}}(g, Y) : \mathrm{YExt}^n_{\mathcal{A}}(X_2, Y) \to \mathrm{YExt}^n_{\mathcal{A}}(X_1, Y)$$

by taking the pullback as shown in the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n \prod X_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n & \longrightarrow & X_2 & \longrightarrow & 0
\end{array}$$

One usually writes $- \cdot g$ for $\mathrm{YExt}^n_{\mathcal{A}}(g, Y)$

Lemma 2.6. *The maps* $\mathrm{YExt}^n_{\mathcal{A}}(X, f)$ *and* $\mathrm{YExt}^n_{\mathcal{A}}(g, Y)$ *in the definition above are well-defined.*

Proof. Let $\mathbb{E}$ represent an element in $\mathrm{YExt}^n_{\mathcal{A}}(X, Y_1)$ and let $\mathbb{F}$ be equivalent to $\mathbb{E}$ as shown in the diagram

$$\begin{array}{cccccccccccccc}
\mathbb{E}: & 0 & \longrightarrow & Y_1 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
\mathbb{G}: & 0 & \longrightarrow & Y_1 & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathbb{F}: & 0 & \longrightarrow & Y_1 & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Then, given $f : Y_1 \to Y_2$ we have a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y_2 & \longrightarrow & Y_2 \coprod_{Y_1} E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & Y_2 \coprod_{Y_1} G_1 & \longrightarrow & G_2 & \longrightarrow & \cdots & \longrightarrow & G_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & Y_2 \coprod_{Y_1} F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

showing that $f\mathbb{E}$ is equivalent to $f\mathbb{F}$. □

Lemma 2.7. *Let* $\mathbb{E}$ *represent an element in* $\mathrm{YExt}^n_{\mathcal{A}}(X, Y)$, *and let* $f : Y \to \tilde{Y}$ *and* $g : \tilde{X} \to X$ *be two maps. Then* $(f\mathbb{E})g = f(\mathbb{E}g)$

Proof. The statement is trivial for all $n > 1$, so we only need to check for a sequence $\mathbb{E} \in \mathrm{YExt}^1_{\mathcal{A}}(X, Y)$. We have a commutative diagram

$$\begin{array}{lccccccccc}
\mathbb{E}g: & 0 & \longrightarrow & Y & \hookrightarrow & E\prod_X \tilde{X} & \twoheadrightarrow & \tilde{X} & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow g & & \\
\mathbb{E}: & 0 & \longrightarrow & Y & \hookrightarrow & E & \twoheadrightarrow & X & \longrightarrow & 0 \\
& & & f\downarrow & & \downarrow & & \| & & \\
f\mathbb{E}: & 0 & \longrightarrow & \tilde{Y} & \hookrightarrow & \tilde{Y}\coprod_Y E & \twoheadrightarrow & X & \longrightarrow & 0
\end{array}$$

Look at the composite $\mathbb{E}g \to f\mathbb{E}$ and from the pullback variant of Lemma 2.4 we get a commutative diagram

$$\begin{array}{lccccccccc}
\mathbb{E}g: & 0 & \longrightarrow & Y & \longrightarrow & E\prod_X \tilde{X} & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\
& & & f\downarrow & & \downarrow & & \| & & \\
(f\mathbb{E})g: & 0 & \longrightarrow & \tilde{Y} & \longrightarrow & (\tilde{Y}\coprod_Y E)\prod_X \tilde{X} & \longrightarrow & \tilde{X} & \longrightarrow & 0
\end{array}$$

From the pushout variant of Lemma 2.4 we get the commutative diagram

$$\begin{array}{lccccccccc}
\mathbb{E}g: & 0 & \longrightarrow & Y & \longrightarrow & E\prod_X \tilde{X} & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\
& & & \downarrow f & & \downarrow & & \| & & \\
f(\mathbb{E}g): & 0 & \longrightarrow & \tilde{Y} & \longrightarrow & \tilde{Y}\coprod_Y (E\prod_X \tilde{X}) & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \| & & \\
(f\mathbb{E})g: & 0 & \longrightarrow & \tilde{Y} & \longrightarrow & (\tilde{Y}\coprod_Y E)\prod_X \tilde{X} & \longrightarrow & \tilde{X} & \longrightarrow & 0
\end{array}$$

and we conclude that

$$(g\mathbb{E})f = f(\mathbb{E}g)$$

We therefore can omit the parentheses without ambiguity. □

Lemma 2.8. *Let* $Y_1 \xrightarrow{\alpha} Y_2 \xrightarrow{\beta} Y_3$ *and* $X_2 \xrightarrow{\gamma} X_1 \xrightarrow{\delta} X$ *be maps in* $\mathcal{A}$*. We then have* $(\beta\alpha)\mathbb{E} = \beta(\alpha\mathbb{E})$ *and* $(\mathbb{E}\delta)\gamma = \mathbb{E}(\delta\gamma)$

Proof. We have a commutative diagram

$$\begin{array}{ccccccccccc}
Y_1 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X \\
\alpha\downarrow & & \downarrow & & \| & & \| & & \| & & \| \\
Y_2 & \longrightarrow & Y_2\coprod_{Y_1} E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X \\
\beta\downarrow & & \downarrow & & \| & & \| & & \| & & \| \\
Y_3 & \longrightarrow & Y_3\coprod_{Y_2}(Y_2\coprod_{Y_1} E_1) & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X
\end{array}$$

and from [Opp16, Exercise I.14] we know that the "iterated pushout"

$$Y_3 \coprod_{Y_2} (Y_2 \coprod_{Y_1} E_1) \cong Y_3 \coprod_{Y_1} E_1$$

proving $(\beta\alpha)\mathbb{E} = \beta(\alpha\mathbb{E})$. The proof for $(\mathbb{E}\delta)\gamma = \mathbb{E}(\delta\gamma)$ is similar. □

What is left to show that the Yoneda extensions have a group structure is to define the addition operator.

Definition 4 (Baer-sum)**.** Let $\mathbb{E}$ and $\mathbb{F}$ represent two elements in $\mathrm{YExt}^n_{\mathcal{A}}(X, Y)$. Then we denote by $\mathbb{E} \oplus \mathbb{F}$ the exact sequence

$$0 \to Y \oplus Y \to E_1 \oplus F_1 \to E_2 \oplus F_2 \to \cdots \to E_n \oplus F_n \to X \oplus X \to 0$$

with the canonical diagonal maps. Now the addition (*Baer-sum*) in $\mathrm{YExt}^n_{\mathcal{A}}(X, Y)$ is defined to be

$$\mathbb{E} + \mathbb{F} = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix} \right) \cdot \mathbb{E} \oplus \mathbb{F} \cdot \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) \in \mathrm{YExt}^n_{\mathcal{A}}(X, Y)$$

Theorem 2.9. *Yoneda extension* $\mathrm{YExt}^n_{\mathcal{A}}(X, Y)$ *with Baer sum is a abelian group. The zero element in* $\mathrm{YExt}^n_{\mathcal{A}}(X, Y)$ *for* $n = 1$ *is*

$$0 \to Y \to Y \oplus X \to X \to 0$$

For $n > 1$ *the zero element is defined to be*

$$0 \longrightarrow Y \xrightarrow{\cong} Y \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{\cong} X \longrightarrow 0$$

This group structure defines an additive functor

$$\mathrm{YExt}^n_{\mathcal{A}}(-, -) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \boldsymbol{Ab}$$

to the category of abelian groups.

Proof. Since direct sum is commutative, the Baer-sum is commutative. Further, we have, given $\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \in \mathrm{YExt}^n_{\mathcal{A}}$, that

$$\mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3 = \left(\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix} \right) (\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is independent of brackets so the Baer-sum is associative.

We have a commutative diagram

$$
\begin{array}{lccccccccccccc}
\mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \binom{1}{1}} & & \downarrow & & \| & & \| & & \| & & \| & & \\
\binom{1}{1}\mathbb{E}: & 0 & \longrightarrow & Y \oplus Y & \longrightarrow & PO & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle \binom{1}{1}} & & \downarrow & & \downarrow & & \| & & \\
(\mathbb{E} \oplus \mathbb{E})\binom{1}{1}: & 0 & \longrightarrow & Y \oplus Y & \longrightarrow & E_1 \oplus E_1 & \longrightarrow & E_2 \oplus E_2 & \longrightarrow & \cdots & \longrightarrow & PB & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \| & & \| & & \| & & \| & & \downarrow & & \downarrow{\scriptstyle \binom{1}{1}} & & \\
\mathbb{E} \oplus \mathbb{E}: & 0 & \longrightarrow & Y \oplus Y & \longrightarrow & E_1 \oplus E_1 & \longrightarrow & E_2 \oplus E_2 & \longrightarrow & \cdots & \longrightarrow & E_n \oplus E_n & \longrightarrow & X \oplus X & \longrightarrow & 0
\end{array}
$$

where the dashed arrows exist by the pushout- and pullback propery. Thus

$$\binom{1}{1}\mathbb{E} = (\mathbb{E} \oplus \mathbb{E})\binom{1}{1}$$

Then

$$
\begin{aligned}
f\mathbb{E} + g\mathbb{E} &= \begin{pmatrix} 1 & 1 \end{pmatrix}(f\mathbb{E} \oplus g\mathbb{E})\binom{1}{1} \\
&= \begin{pmatrix} f & g \end{pmatrix}(\mathbb{E} \oplus \mathbb{E})\binom{1}{1} \\
&= \begin{pmatrix} f & g \end{pmatrix}\binom{1}{1}\mathbb{E} \\
&= (f+g)\mathbb{E}
\end{aligned}
$$

It is an easy observation that $0 \cdot \mathbb{E} = 0$ and $\mathbb{E} \cdot 0 = 0$ so

$$\mathbb{E} + 0 = 1 \cdot \mathbb{E} + 0 \cdot \mathbb{E} = (1+0) \cdot \mathbb{E} = \mathbb{E}$$

and

$$\mathbb{E} + (-1) \cdot \mathbb{E} = (1-1) \cdot \mathbb{E} = 0 \cdot \mathbb{E} = 0$$

□

2.3 The long exact sequence

The last tool we need to prove the long exact sequence of Yoneda extension groups is the following definition.

Definition 5. Let $[\mathbb{F}] \in \mathrm{YExt}^n_{\mathcal{A}}(X, Y)$ and $[\mathbb{E}] \in \mathrm{YExt}^m_{\mathcal{A}}(Y, Z)$. Then the **Yoneda product** or **cup product** $[\mathbb{E}] \cup [\mathbb{F}] \in \mathrm{YExt}^{n+m}_{\mathcal{A}}(X, Z)$ is defined to be the class represented by the splicing of $\mathbb{E}$ and $\mathbb{F}$.

$$\begin{array}{llcccccccccccccccc}
\mathbb{E}: & 0 \to Z \to E_1 \to \cdots \to E_m \to & Y & \longrightarrow 0 & & & & \\
 & & \| & & & & & \\
\mathbb{F}: & 0 \longrightarrow & Y & \to F_1 \to \cdots \to F_n \to X \to 0 \\
 & \nearrow & & \searrow & \\
\mathbb{E} \cup \mathbb{F}: & 0 \to Z \to E_1 \to \cdots \to E_m & \dashrightarrow & F_1 \to \cdots \to F_n \to X \to 0
\end{array}$$

We are now ready for the first half of the proof of the long exact sequence.

Proposition 2.1. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be a short exact sequence in an abelian category* $\mathcal{A}$*. Then given* $A \in \mathcal{A}$*, there is an exact sequence*

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A, X) \to \operatorname{Hom}_{\mathcal{A}}(A, Y) \to \operatorname{Hom}_{\mathcal{A}}(A, Z)$$

$$\xrightarrow{\alpha} \operatorname{YExt}^1_{\mathcal{A}}(A, X) \xrightarrow{f \cdot -} \operatorname{YExt}^1_{\mathcal{A}}(A, Y) \xrightarrow{g \cdot -} \operatorname{YExt}^1_{\mathcal{A}}(A, Z)$$

where, given $\varphi \in \operatorname{Hom}_{\mathcal{A}}(A, Z)$*,* $\alpha(\varphi)$ *is defined to be the top exact row in the following commutative diagram.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \hookrightarrow & A \prod_Z Y & \twoheadrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0
\end{array}$$

To show the exactness in $\operatorname{Hom}_{\mathcal{A}}(A, Z)$ in the proposition we use the following lemma

Lemma 2.10. *Given an element in* $\operatorname{YExt}^1_{\mathcal{A}}(Z, X)$ *represented by a sequence* $\mathbb{E}$*, and a diagram on the form*

$$\begin{array}{lccccccccc}
 & & & & & & A & & \\
 & & & & & & \downarrow a & & \\
\mathbb{E}: & 0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow 0
\end{array}$$

Then $\mathbb{E} \cdot a = 0$ *if and only if* a *can be factored through* $Y \to Z$*.*
Dually, given a map diagram on the form

$$\begin{array}{lcccccccc}
\mathbb{E}: & 0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow 0 \\
 & & & \downarrow b & & & & & \\
 & & & B & & & & &
\end{array}$$

Then $b \cdot \mathbb{E} = 0$ if and only if b can be factored through $X \to Y$

Proof. We only prove the first part, the second is dual.

" $\Rightarrow$ " Assume $\mathbb{E} \cdot a = 0$ then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \hookrightarrow & X \oplus A & \overset{\hookleftarrow}{\twoheadrightarrow} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0
\end{array}$$

where $X \oplus A \to A$ is split epi, so we have $A \to X \oplus A \to Y \to Z$ is equivalent to $A \to Z$. Thus A factors through Y

" $\Leftarrow$ " Assume $h : A \overset{\varphi}{\to} Y \twoheadrightarrow Z$ we can then form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & X \oplus A & \twoheadrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & \swarrow_{\varphi} & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0
\end{array}$$

Then from Lemma 2.4 we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & X \oplus A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & Y \prod_Z A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0
\end{array}$$

Showing that $\mathbb{E} \cdot a = 0$

□

Proof of proposition 2.1. We know $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is left exact, and from Lemma 2.10 we have that the sequence is exact in $\mathrm{Hom}_{\mathcal{A}}(A, Z)$. Therefore we only need to show exactness in $\mathrm{YExt}^1_{\mathcal{A}}(A, X)$ and $\mathrm{YExt}^1_{\mathcal{A}}(A, Y)$ in the sequence

$$\mathrm{Hom}_{\mathcal{A}}(A, Z) \overset{\alpha}{\to} \mathrm{YExt}^1_{\mathcal{A}}(A, X) \xrightarrow{f \cdot -} \mathrm{YExt}^1_{\mathcal{A}}(A, Y) \xrightarrow{g \cdot -} \mathrm{YExt}^1_{\mathcal{A}}(A, Z) \tag{2.1}$$

To show exactness in $\mathrm{YExt}^1_{\mathcal{A}}(A, X)$ let $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, Z)$ Then we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Y\coprod_A(Y\prod_Z A) & \longrightarrow & A & \longrightarrow & 0\\
 & & {\scriptstyle f}\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & X & \hookrightarrow & Y\prod_Z A & \twoheadrightarrow & A & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0
\end{array}$$

We see that $X \to Y$ factors through $Y \prod A$, and from Lemma 2.10 we have that $f \cdot \alpha(\varphi) = 0$ and $\mathrm{Im}(\alpha) \subseteq \ker(f \cdot -)$.
Now let

$$0 \to X \to E \to A \to 0$$

be in the kernel of $f \cdot -$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \hookrightarrow & Y & \twoheadrightarrow & Z & \longrightarrow & 0\\
 & & \| & & \uparrow & & & & \\
0 & \longrightarrow & X & \hookrightarrow & E & \twoheadrightarrow & A & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \rightleftarrows & Y \oplus A & \twoheadrightarrow & A & \longrightarrow & 0
\end{array}$$

Where $E \to Y$ is the composition $E \to Y \oplus A \to Y$ making the top left square commute. Since A is the cokernel of the map $X \to E$ by the cokernel property there exist a map $A \to Z$ making the top right square commute. We have then shown that $\ker(f \cdot -) = \mathrm{Im}(\alpha)$ and the sequence 2.1 is exact in $\mathrm{YExt}^1_{\mathcal{A}}(A, X)$.

Lastly we need to check exactness in $\mathrm{YExt}^1_{\mathcal{A}}(A, Y)$. Let $\mathbb{E}$ be a short exact sequence in $\mathrm{YExt}^1_{\mathcal{A}}(A, X)$ We have $gf\mathbb{E} = (gf)\mathbb{E} = 0 \cdot \mathbb{E}$, and thus $\mathrm{Im}(f \cdot -) \subseteq \ker(g \cdot -)$.
Let

$$0 \to Y \to E \to A \to 0$$

be in $\ker(g \cdot -)$. We then have a commutative diagram

$$\begin{array}{ccccccccc}
 & & X & & & & A & & \\
 & & \downarrow & & & & \| & & \\
0 & \longrightarrow & Y & \hookrightarrow & E & \twoheadrightarrow & A & \longrightarrow & 0\\
 & & \twoheaddownarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Z & \rightleftarrows & Z \oplus A & \twoheadrightarrow & A & \longrightarrow & 0
\end{array}$$

We want to show there exist an element B and a map $B \to E$ such that we get a short exact sequence $0 \to X \to B \to A \to 0$. Let $\varphi : E \to$

$Z \oplus A \to Z$. Note that by the five lemma $E \to Z \oplus A$ is an epimorphism, so φ is an epimorphism. We then get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \dashrightarrow & B = \ker(\varphi) & \dashrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \hookrightarrow & E & \twoheadrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \varphi & & \downarrow & & \\
0 & \longrightarrow & Z & = & Z & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Where all the columns, and the two bottom rows, are exact. By the 3×3 lemma the top exact row exist making the diagram commute. Thus we have shown that $\ker(g \cdot -) = \operatorname{Im}(f \cdot -)$ and the sequence in the proposition is exact. □

Because of this proposition we define $\operatorname{YExt}^0_{\mathcal{A}}(A, -) := \operatorname{Hom}_{\mathcal{A}}(A, -)$ and $\operatorname{YExt}^n_{\mathcal{A}}(A, -) = 0$ for $n < 0$. We are now ready to prove the general result that using the Yoneda extension functor on a short exact sequence leads to a long exact sequence of Yoneda groups. First we need a useful lemma.

Lemma 2.11. *Let*

$$0 \to Y \to E_1 \to \cdots \to E_n \to X \to 0$$

be a exact sequence in $\operatorname{YExt}^n_{\mathcal{A}}(X, Y)$ *then the following are equivalent*

(1) The sequence is equivalent to zero
(2) There exist a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
 & & 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

with the bottom row exact

(3) There exist a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & 0 & &
\end{array}$$

with the bottom row exact

Proof. We only prove that 1 is equivalent to 2. The proof for 1 equivalent to 3 is similar.
(1) ⇒ (2) : Assume the sequence is equivalent to zero. Then there exist a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & \tilde{H}_1 & \longrightarrow & H_1 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\cong} & Y & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & X & \xrightarrow{\cong} & X & \longrightarrow & 0
\end{array}$$

and we note that $Y \to \tilde{H}_1$ is a split monomorphism. We can then write the middle row as

$$0 \to Y \hookrightarrow Y \oplus H_1 \to H_2 \to \cdots \to H_n \to X \to 0$$

We then have the commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Y \oplus H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
 & & 0 & \longrightarrow & H_1 & \hookrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & \tilde{H}_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Since Y is isomorphic to the kernel of the map $Y \oplus H_1 \to H_2$ the map $H_1 \to H_2$ is a monomorphisms.
(2) ⇒ (1) : Assume we have a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\alpha} & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \beta\uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
 & & 0 & \longrightarrow & H_1 & \xrightarrow{\delta} & H_2 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

We can then extend the diagram to the commutative diagram

$$
\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\alpha} & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle(\beta\ \alpha)} & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\binom{0}{1}} & H_1 \oplus Y & \xrightarrow{(\delta\ 0)} & H_2 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\cong} & Y & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & X & \xrightarrow{\cong} & X & \longrightarrow & 0
\end{array}
$$

showing that the sequence is equivalent to zero. □

Theorem 2.12. *Given a short exact sequence* $\mathbb{X} : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *in* $\mathcal{A}$ *we get a long exact sequence*

$$\cdots \to \mathrm{YExt}^n_{\mathcal{A}}(A,X) \xrightarrow{f\cdot -} \mathrm{YExt}^n_{\mathcal{A}}(A,Y) \xrightarrow{g\cdot -} \mathrm{YExt}^n_{\mathcal{A}}(A,Z) \xrightarrow{[\mathbb{X}]\cup[-]} \mathrm{YExt}^{n+1}_{\mathcal{A}}(A,X) \to \cdots$$

Proof. The proof is similar to the proof of the proposition above. We need to check for exactness in $\mathrm{YExt}^n_{\mathcal{A}}(A,Y)$, $\mathrm{YExt}^n_{\mathcal{A}}(A,Z)$ and $\mathrm{YExt}^{n+1}_{\mathcal{A}}(A,X)$.

To show exactness in $\mathrm{YExt}^n_{\mathcal{A}}(A,Y)$, let $\mathbb{E}$ represent an element in $\mathrm{YExt}^n_{\mathcal{A}}(A,X)$. We then have $gf\mathbb{E} = (gf)\mathbb{E} = 0$ and $\mathrm{Im}(f\cdot -) \subseteq \ker(g\cdot -)$

Now assume

$$0 \to Y \to E_1 \to \cdots \to E_n \to A \to 0$$

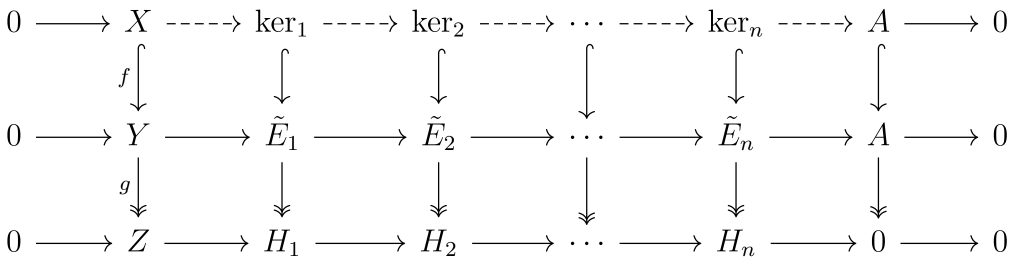
represents an element in $\ker(g\cdot -)$ we then, from Lemma 2.11, get a commutative diagram

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & Z & \longrightarrow & PO & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

where PO denotes the pushout. Now by using Lemma 2.1 we can construct an n-fold extension $\tilde{\mathbb{E}}$ such that we have the following commutative diagram where every vertical map is an epimorphism

$$
\begin{array}{cccccccccccccc}
\tilde{\mathbb{E}}: & 0 & \longrightarrow & Y & \longrightarrow & \tilde{E}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{E}_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
 & 0 & \longrightarrow & Z & \longrightarrow & H_1 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & 0 & &
\end{array}
$$

Then we have a commutative diagram

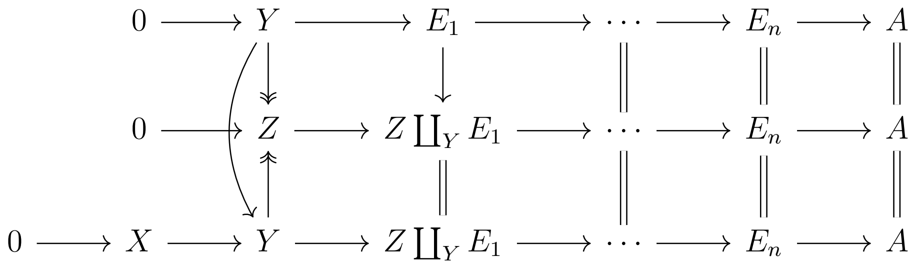


From Lemma 2.2 the top row is exact and represents an element in $\mathrm{YExt}^n_{\mathcal{A}}(A, X)$, and we have proved $\mathrm{Im}(f \cdot -) = \ker(g \cdot -)$

To show exactness in $\mathrm{YExt}^n_{\mathcal{A}}(A, Z)$ Let

$$0 \to Y \to E_1 \to \cdots \to E_n \to A \to 0$$

represent an element in $\mathrm{YExt}^n_{\mathcal{A}}(A, Y)$. Then we have a commutative diagram

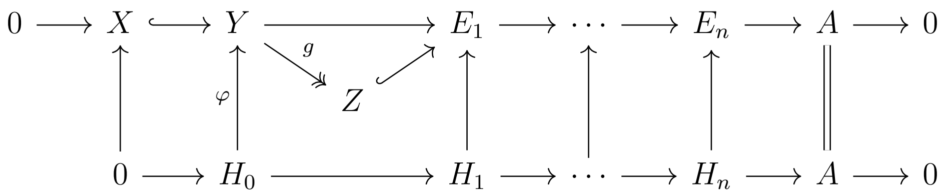


and from Lemma 2.11 the bottom row is equivalent to zero showing that $\mathrm{Im}(g \cdot -) \subseteq \ker([\mathbb{X}] \cup [-])$

Now assume a sequence

$$\mathbb{E} : 0 \to Z \to F_1 \to \cdots \to F_n \to A \to 0$$

represents an element in the kernel of $[\mathbb{X}] \cup [-]$. We then have a commutative diagram



where the bottom row can be seen as an element $\mathbb{F}$ in $\mathrm{YExt}^n_{\mathcal{A}}(A, H_0)$. We then have $g\varphi\mathbb{F} = g \cdot (\varphi\mathbb{F}) = \mathbb{E}$, where $[\varphi\mathbb{F}] \in \mathrm{YExt}^n_{\mathcal{A}}(A, Y)$, and we have proved $\ker([\mathbb{X}] \cup [-]) = \mathrm{Im}(g \cdot -)$

To show exactness in $\mathrm{YExt}^{n+1}_{\mathcal{A}}(A, X)$, first let

$$0 \to Z \to E_1 \to \cdots \to E_n \to A \to 0$$

represent an element in $\mathrm{YExt}^n_{\mathcal{A}}(A, Z)$. We then have a commutative diagram

$$\begin{array}{ccccccccccccccc}
 & & 0 & \longrightarrow & Z & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \twoheaduparrow & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Y \coprod_X Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \vdots & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\cong} & Y & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & A & \xrightarrow{\cong} & A & \longrightarrow & 0
\end{array}$$

where the dashed arrow exist by the pushout property, making $\mathrm{Im}(f \cdot -) \subseteq \ker([\mathbb{X}] \cup [-])$

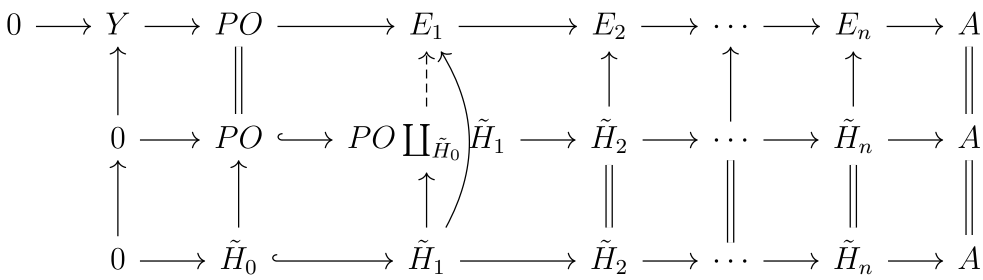
Now let

$$\mathbb{E} : 0 \to X \to E_0 \to E_1 \to \cdots \to E_n \to A \to 0$$

represent an element in $\ker(f \cdot -)$ We then have a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & X & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & PO & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \| & & \\
 & & 0 & \longrightarrow & \tilde{H}_0 & \longrightarrow & \tilde{H}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{H}_n & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

where PO denotes the pushout $Y \coprod_X E_0$. Note that this diagram can be extended in the following way:



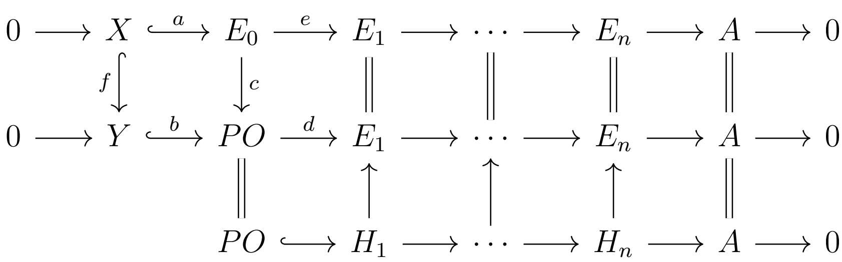
showing that we can replace the sequence

$$0 \to \tilde{H}_0 \to \tilde{H}_1 \to \cdots \to \tilde{H}_n \to A \to 0$$

with

$$0 \to PO \to H_1 \to H_2 \to \cdots \to H_n \to A \to 0$$

We then end up with the following diagram

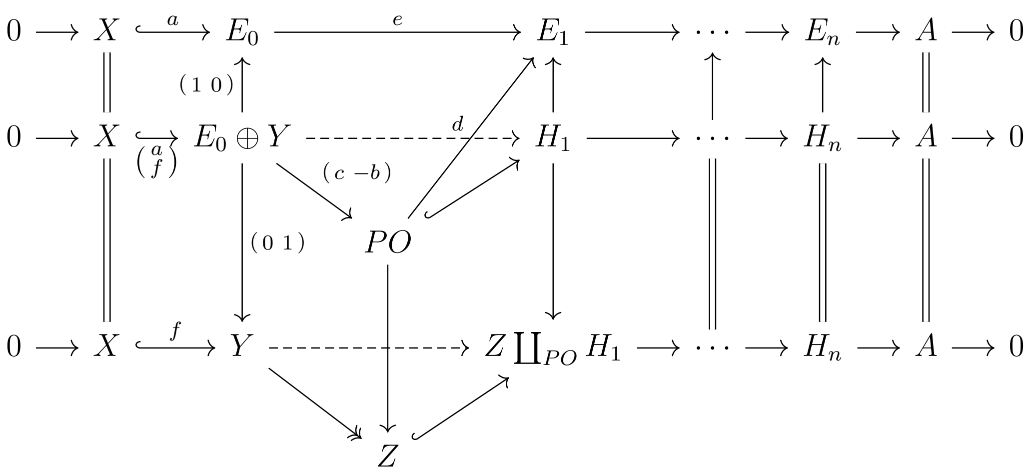


(2.2)

Since the top left square is a pushout, and the maps $X \to E_0$ and $X \to Y$ are monomorphisms, we have from [Opp16, Proposition 13.4] that the sequence

$$0 \longrightarrow X \xrightarrow{\binom{a}{f}} E_0 \oplus Y \xrightarrow{(c\ -b)} PO \longrightarrow 0$$

is short exact. We have a new diagram



where the only square we need to check commute is

$$\begin{array}{ccc} E_0 & \xrightarrow{e} & E_1 \\ {\scriptstyle (1\ 0)}\uparrow & & \uparrow{\scriptstyle d} \\ E_0 \oplus Y & \xrightarrow{(c\ -b)} & PO \end{array}$$

We have

$$d \circ \left(c\ -b \right) = dc - db$$

And from diagram 2.2 we have that $db = 0$ and $dc = e$ so the diagram commutes and we have proved

$$[\mathbb{E}] = [X \to Y \to Z] \cup [Z \to Z \coprod_{PO} H_1 \to H_2 \to \cdots \to H_n \to A]$$

and $\ker(f \cdot -) = \operatorname{Im}([X] \cup [-])$ □

Remark. A similar proof shows that

$$\cdots \to \mathrm{YExt}^n_{\mathcal{A}}(Z, A) \xrightarrow{-\cdot g} \mathrm{YExt}^n_{\mathcal{A}}(Y, A) \xrightarrow{-\cdot f} \mathrm{YExt}^n_{\mathcal{A}}(X, A) \xrightarrow{[-]\cup[\mathbb{X}]} \mathrm{YExt}^{n+1}_{\mathcal{A}}(Z, A) \to \cdots$$

is a long exact sequence, where $\mathrm{YExt}^0_{\mathcal{A}}(-, A) := \mathrm{Hom}_{\mathcal{A}}(-, A)$ and $\mathrm{YExt}^n_{\mathcal{A}}(-, A) = 0$ for $n < 0$

Chapter 3

t-structures

An open question in the study of triangulated categories is if and when two derived categories can be equivalent as triangulated categories. Rickard developed a morita theory to give a sufficient and necessary condition for two derived categories over module categories to be equivalent [Ric89]. However the genereal question of when two derived categories over different abelian categories are triangle equivalent stands unanswered. A natural approach to begin finding derived equivalences is to find abelian subcategories of triangulated categories, and then building the derived categories. In [BBD83] the notion of t-structure was introduced to recover various abelian subcategories of triangulated categories. In this chapter we will introduce t-structures, and study relationships between the triangulated categories and the underlying abelian subcategories.

3.1 t-structures

Definition 6. Let $\mathscr{C}$ be a triangulated category. A ***t*-structure** $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ on $\mathscr{C}$ is a pair of full subcategories of $\mathscr{C}$ such that the following conditions holds for
$C^{\geq n} := C^{\geq 0}[-n]$, $\mathscr{C}^{\leq n} := \mathscr{C}^{\leq 0}[-n]$, $n \in \mathbb{N}$.

(1) $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathscr{C}^{\leq 0}$ and $Y \in \mathscr{C}^{\geq 1}$.
(2) $\mathscr{C}^{\leq 0} \subseteq \mathscr{C}^{\leq 1}$ and $\mathscr{C}^{\geq 1} \subseteq \mathscr{C}^{\geq 0}$. (i.e. if $X \in \mathscr{C}^{\leq 0}$ then $X[1] \in \mathscr{C}^{\leq 0}$, and if $Y \in \mathscr{C}^{\geq 0}$ then $Y[-1] \in \mathscr{C}^{\geq 0}$)
(3) For all $X \in \mathscr{C}$ there is a triangle

$$X' \to X \to X'' \to X'[1]$$

such that $X' \in \mathscr{C}^{\leq 0}$ and $X'' \in \mathscr{C}^{\geq 1}$

For a t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ we denote by $\mathcal{H}$ the full subcategory $\mathscr{C}^{\leq 0} \cap \mathscr{C}^{\geq 0}$ of $\mathscr{C}$. $\mathcal{H}$ is called the **heart** of the t-structure

Theorem 3.1. *The heart, $\mathcal{H} = \mathscr{C}^{\leq 0} \cap \mathscr{C}^{\geq 0}$, of a t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ is an abelian category.*

In order to prove the theorem, we will need a very useful lemma

Lemma 3.2. *A t-structure is closed under extensions. I.e. let $\mathscr{C}$ be a triangulared category, and $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ be a t-structure. If*

$$X \to Y \to Z \to X[1]$$

is an triangle in $\mathscr{C}$ with $X, Z \in \mathscr{C}^{\leq 0}$ (resp. in $\mathscr{C}^{\geq 0}$) then $Y \in \mathscr{C}^{\leq 0}$ (resp. in $\mathscr{C}^{\geq 0}$).

Proof. Assume $X, Z \in \mathscr{C}^{\leq 0}$ From the third condition of a t-structure there exist a triangle

$$A \to Y \to B \to A[1]$$

where $A \in \mathscr{C}^{\leq 0}$ and $B \in \mathscr{C}^{\geq 1}$. Thus we have a diagram

$$\begin{array}{ccccccc} & & A & & & & \\ & & \downarrow & & & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ & & \downarrow & & & & \\ & & B & & & & \\ & & \downarrow & & & & \\ & & A[1] & & & & \end{array}$$

We then have a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathscr{C}}(Z, B) \to \mathrm{Hom}_{\mathscr{C}}(Y, B) \to \mathrm{Hom}_{\mathscr{C}}(X, B) \to \cdots$$

and from condition 1 of t-structures we have $\mathrm{Hom}_{\mathscr{C}}(Z, B) = \mathrm{Hom}_{\mathscr{C}}(X, B) = 0$ forcing $\mathrm{Hom}_{\mathscr{C}}(Y, B) = 0$. We have another long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathscr{C}}(A[1], B) \to \mathrm{Hom}_{\mathscr{C}}(B, B) \to \mathrm{Hom}_{\mathscr{C}}(Y, B) \to \mathrm{Hom}_{\mathscr{C}}(A, B) \to \cdots$$

and from the same condition $\mathrm{Hom}_{\mathscr{C}}(A[1], B) = \mathrm{Hom}_{\mathscr{C}}(A, B) = 0$, forcing

$$\mathrm{Hom}_{\mathscr{C}}(B, B) \cong \mathrm{Hom}_{\mathscr{C}}(Y, B) = 0$$

We conclude that $Y \cong A$. In particular $Y \in \mathscr{C}^{\leq 0}$. The proof for $X, Z \in \mathscr{C}^{\geq 0}$ is similar. □

Proof of Theorem 3.1. First note that

$$A_1 \to A_1 \oplus A_2 \to A_2 \to A_1[1]$$

is a triangle and by Lemma 3.2, since $A_1, A_2 \in \mathcal{H}$, $A_1 \oplus A_2$ is in $\mathcal{H}$, the subcategory is additive.

To show that $\mathcal{H}$ is abelian we let $f : A_1 \to A_2$ be a map between A_1 and A_2 in $\mathcal{H}$. We want to show that the map $A_1 \xrightarrow{f} A_2$ has a kernel and cokernel in $\mathcal{H}$ and that $\mathrm{Im}(f) = \mathrm{Coim}(f)$.
Look at the triangle $A_1 \to A_2 \to \mathrm{Cone}(f) \to A_1[1]$. From condition 3 of a t-structure there is a triangle

$$X \to \mathrm{Cone}(f)[-1] \to Y \to X[1]$$

where $X \in \mathscr{C}^{\leq 0}$ and $Y \in \mathscr{C}^{\geq 1}$. Thus we obtain a diagram

$$\begin{array}{ccccccccc}
X & & & & & & X[1] & & \\
\downarrow & & & & & & \downarrow & & \\
\mathrm{Cone}(f)[-1] & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & A_1[1] \\
\downarrow & & & & & & \downarrow & & \\
Y & & & & & & Y[1] & & \\
\downarrow & & & & & & \downarrow & & \\
X[1] & & & & & & X[2] & &
\end{array}$$

Since $A_2 \in \mathcal{H}$ we have that $A_2 \in \mathscr{C}^{\geq 0} \subseteq \mathscr{C}^{\geq -1}$ by condition 2 of a t-structure. Then since $A_1[1] \in \mathscr{C}^{\geq -1}$ we have by Lemma 3.2 that $\mathrm{Cone}(f) \in \mathscr{C}^{\geq -1}$. Further, since $A_2 \in \mathscr{C}^{\leq 0}$ and $A_1[1] \in \mathscr{C}^{\leq -1} \subseteq \mathscr{C}^{\leq 0}$, we have $\mathrm{Cone}(f) \in \mathscr{C}^{\leq 0}$. Thus $\mathrm{Cone}(f) \in \mathscr{C}^{\leq 0} \cap \mathscr{C}^{\geq -1}$.

Since $\mathrm{Cone}(f) \in \mathscr{C}^{\leq 0}$ and $X[2] \in \mathscr{C}^{\leq -2} \subseteq \mathscr{C}^{\leq 0}$ we get that $Y[1] \in \mathscr{C}^{\leq 0}$. We also have that $Y[1] \in \mathscr{C}^{\geq 0}$, and thus $Y[1] \in \mathcal{H}$. We want to show that $Y[1]$ is a cokernel of the map f. First we show that $Y[1]$ is a weak cokernel:

Let $T \in \mathcal{H}$. We then have an exact sequence

$$\mathrm{Hom}_{\mathcal{H}}(X[2], T) \to \mathrm{Hom}_{\mathcal{H}}(Y[1], T) \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{Cone}(f), T) \to \mathrm{Hom}_{\mathcal{H}}(X[1], T)$$

Since $X[1] \in \mathscr{C}^{\leq -1}$, $X[2] \in \mathscr{C}^{\leq -2}$ and $T \in \mathscr{C}^{\geq 0}$ we have from the first condition of a t-structure that

$$\mathrm{Hom}_{\mathcal{H}}(X[1], T) = \mathrm{Hom}_{\mathcal{H}}(X[2], T) = 0$$

and we get $\mathrm{Hom}_{\mathcal{H}}(\mathrm{Cone}(f), T) \cong \mathrm{Hom}_{\mathcal{H}}(Y[1], T)$.
Since $\mathrm{Cone}(f)$ is a weak cokernel we then get that $Y[1]$ is a weak cokernel. To show the uniqueness in the universal property of the cokernel we look at the following exact sequence

$$\mathrm{Hom}_{\mathcal{H}}(A[1], T) \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{Cone}(f), T) \to \mathrm{Hom}_{\mathcal{H}}(A_2, T) \to \mathrm{Hom}_{\mathcal{H}}(A_1, T)$$

Since $A[1] \in \mathscr{C}^{\leq -1}$ we have that $\mathrm{Hom}_{\mathcal{H}}(A[1], T) = 0$. This, together with the isomorphism above, gives us the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(Y[1], T) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(A_2, T) \to \mathrm{Hom}_{\mathcal{H}}(A_1, T)$$

showing that $Y[1]$ is indeed a cokernel of the morphism f.
Dually one can verify that X is the kernel of the map f

Then what is left is to show $\mathrm{Im}(f) \cong \mathrm{Coim}(f)$. We can embed $A_2 \to \mathrm{coker}(f)$ into the triangle

$$A_2 \to \mathrm{coker}(f) \to I[1] \to A_2[1]$$

where, from Lemma 3.2 and the second condition of a t-structure, $I[1] \in \mathscr{C}^{\geq -1}$. Especially we have $I \in \mathscr{C}^{\geq 0}$. We then use the octahedral axiom to get a commutative diagram

$$\begin{array}{ccccccc}
A_2[-1] & \longrightarrow & \mathrm{Cone}(f)[-1] & \longrightarrow & A_1 & \longrightarrow & A_2 \\
\| & & \downarrow & & \downarrow & & \| \\
A_2[-1] & \longrightarrow & \mathrm{coker}(f)[-1] & \longrightarrow & I & \longrightarrow & A_2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \ker(f)[1] & = & \ker(f)[1] & \longrightarrow & \mathrm{Cone}(f) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{Cone}(f) & \longrightarrow & A_1[1] & &
\end{array}$$

We then have the triangle

$$\ker(f) \to A_1 \to I \to \ker(f)[1]$$

and by Lemma 3.2 $I \in \mathscr{C}^{\leq 0}$. Thus $I \in \mathcal{H}$.
We can define triangles

$$\ker(f) \xrightarrow{\varphi} A_1 \to \mathrm{coker}(\varphi) \to \ker(f)[1]$$

$$\ker(\psi) \to A_2 \xrightarrow{\psi} \mathrm{coker}(f) \to \ker(\psi)[1]$$

Thus we have the commutative diagrams

$$\begin{array}{ccccccc}
\ker(f) & \longrightarrow & A_1 & \longrightarrow & I & \longrightarrow & \ker(f)[1] \\
\| & & \| & & \downarrow \cong & & \| \\
\ker(f) & \xrightarrow{\varphi} & A_1 & \longrightarrow & \operatorname{coker}(\varphi) & \longrightarrow & \ker(f)[1]
\end{array}$$

$$\begin{array}{ccccccc}
\operatorname{coker}(f)[-1] & \longrightarrow & I & \longrightarrow & A_2 & \longrightarrow & \operatorname{coker}(f) \\
\| & & \downarrow \cong & & \| & & \| \\
\operatorname{coker}(f)[-1] & \longrightarrow & \ker(\psi) & \longrightarrow & A_2 & \xrightarrow{\psi} & \operatorname{coker}(f)
\end{array}$$

and

$$\operatorname{Im}(f) := \ker(A_2 \to \operatorname{coker}(f)) \cong I \cong \operatorname{coker}(\ker(f) \to A_1) =: \operatorname{Coim}(f)$$

Thus we conclude that $\operatorname{Im}(f) \cong \operatorname{Coim}(f) \in \mathcal{H}$ □

An immediate consequence is the connection between short exact sequences in the abelian heart and triangles, as explained by the following corollary.

Corollary 3.2.1. *A sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in the heart $\mathcal{H}$ of a t-structure over $\mathscr{C}$ is a short exact sequence if and only if it gives rise to a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$$

in $\mathscr{C}$.

Proof. "⇒" Embed $X \xrightarrow{f} Y$ into the triangle

$$X \to Y \to \operatorname{Cone}(f) \to X[1]$$

Then from the triangle

$$\ker(f)[1] \to \operatorname{Cone}(f) \to \operatorname{coker}(f) \to \ker(f)[2]$$

from the proof above we get, since $\ker(f) = 0$ and $\operatorname{coker}(f) \cong Z$, that $\operatorname{Cone}(f) \cong Z$

"$\Leftarrow$" Assume

$$Z[-1] \to X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$$

is a triangle in $\mathscr{C}$ with $X, Y, Z \in \mathcal{H}$. Given $T \in \mathcal{H}$, using $\operatorname{Hom}(T, -)$ on the triangle we get an exact sequence

$$\operatorname{Hom}(T, Z[-1]) \to \operatorname{Hom}(T, Z) \to \operatorname{Hom}(T, Y) \to \operatorname{Hom}(T, Z) \to \operatorname{Hom}(T, X[1])$$

where from the axioms of a t-structure $\operatorname{Hom}(T, Z[-1]) = \operatorname{Hom}(T, X[1]) = 0$ making

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

an exact sequence in $\mathcal{H}$.

□

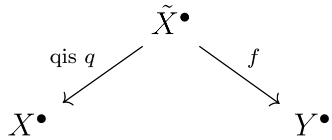
Given the bounded derived category $D^b(\mathcal{A})$, the next example illustrates how we can recover the underlying abelian subcategory $\mathcal{A}$ through the notion of t-structures.

Example 2. Let $\mathcal{A}$ be an abelian category, and $D^b(\mathcal{A})$ be its bounded derived category. Then the pair

$$D^b(\mathcal{A})^{\leq n} = \{X^\bullet | H^i(X^\bullet) = 0, \text{ for } i > n\}$$
$$D^b(\mathcal{A})^{\geq n} = \{X^\bullet | H^i(X^\bullet) = 0, \text{ for } i < n\}$$

defines a t-structure on $D^b(\mathcal{A})$. We need to check the three conditions for a t-structure:

1. Let $X^\bullet \in D^b(\mathcal{A})^{\leq 0}$, $Y^\bullet \in D^b(\mathcal{A})^{\geq 1}$ and $\varphi \in \operatorname{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet)$. Then φ can be represented by the roof



Since $Y^\bullet \in D^b(\mathcal{A})^{\geq 1}$, $Y^\bullet$ is quasi-isomorphic to a complex on the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots$$

we can assume $Y^\bullet$ is of this form. Now since $X^\bullet$ is quasi-isomorphic to $\tilde{X}^\bullet$ we have that $\tilde{X}^\bullet \in D^b(\mathcal{A})^{\leq 0}$. We therefore have that $\tilde{X}^\bullet$ is quasi-isomorphic to a complex $Z^\bullet$, on the form

$$\cdots \longrightarrow Z^{-2} \longrightarrow Z^{-1} \longrightarrow Z^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

and we can represent φ by the roof

$$X^\bullet \xleftarrow{\text{qis } q} \tilde{X}^\bullet \xleftarrow{\text{qis } r} Z^\bullet \xrightarrow{\text{qis } r} \tilde{X}^\bullet \xrightarrow{f} Y^\bullet$$

and we see that $f \circ r$ must be zero.

2. Clearly $D^b(\mathcal{A})^{\leq 0} \subseteq D^b(\mathcal{A})^{\leq 1}$ and $D^b(\mathcal{A})^{\geq 1} \subseteq D^b(\mathcal{A})^{\geq 0}$
3. Given $X^\bullet = (X^i, d^i_X) \in D^b(\mathcal{A})$ we have a complexes

$$Y^\bullet = \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \xrightarrow{d} \ker(d^0_X) \longrightarrow 0 \longrightarrow \cdots$$

$$X^\bullet/Y^\bullet = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^0/\ker(d^0_X) \xhookrightarrow{\tilde{d}} X^1 \longrightarrow \cdots$$

where d is the map in the canonical factorization

$$\begin{array}{ccc} X^{-1} & \longrightarrow & X^0 \\ & {\scriptstyle d}\searrow \quad \nearrow & \\ & \ker(d^0_X) & \end{array}$$

and $\tilde{d}$ is the canonical composite

$$X^0/\ker(d^0_X) \xrightarrow{\cong} \operatorname{Im}(d^0_X) \hookrightarrow X^1$$

It is clear that $Y^\bullet \in D^b(\mathcal{A})^{\leq 0}$, and since $\tilde{d}$ is injective $X^\bullet/Y^\bullet \in D^b(\mathcal{A})^{\geq 1}$ and that we have a short exact sequence in the complex category over $\mathcal{A}$

$$0 \to Y^\bullet \to X^\bullet \to X^\bullet/Y^\bullet \to 0$$

and from [Opp16, Example 34.9] this induces a triangle

$$Y^\bullet \to X^\bullet \to X^\bullet/Y^\bullet \to Y^\bullet[1]$$

in $D^b(\mathcal{A})$

Thus $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$ forms a t-structure on $D^b(\mathcal{A})$. We also have that the heart is $D^b(\mathcal{A})^{\leq 0} \cap D^b(\mathcal{A})^{\geq 0} = \{X^\bullet | H^i(X^\bullet) = 0, \text{ for } i \neq 0\} \cong \mathcal{A}$. This is called the **Canonical t-structure** on the derived category $D^b(\mathcal{A})$

There is a very important connection between the hom-functor in a triangulated category and exact sequences in an abelian heart. To see this we use the Yoneda extension from Chapter 2. The rest of this chapter is dedicated to explain this connection.

Construction 1. Let $\mathbb{E} : 0 \to Y \xrightarrow{f} E \xrightarrow{g} X \to 0$ be a short exact sequence in $\mathcal{H}$. From 3.2.1 this fits uniquely into a triangle

$$Y \xrightarrow{f} E \xrightarrow{g} X \xrightarrow{\theta^1_{X,Y}(\mathbb{E})} Y[1]$$

and we can define a map

$$\theta^1_{X,Y} : \mathrm{YExt}^1_{\mathcal{H}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y[1])$$

This can be generalized in the following way: Let $\mathbb{E} \in \mathrm{YExt}^{n+1}_{\mathcal{H}}(X, Y)$.

$$\mathbb{E} : 0 \to Y \to E_{n+1} \to \cdots \to E_1 \to X \to 0$$

Then we can let $[\mathbb{E}] = [\mathbb{E}_1] \cup [\mathbb{E}_2]$ with $[\mathbb{E}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(Z, Y)$ and $[\mathbb{E}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, Z)$.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_{n+1} & \longrightarrow & Z & \longrightarrow & 0 & & & & & & \\
 & & & & & & \| & & & & & & & & \\
 & & & & 0 & \longrightarrow & Z & \longrightarrow & E_n & \longrightarrow \cdots \longrightarrow & E_1 & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

From here we define $\theta^{n+1}_{X,Y}(\mathbb{E}) = (\theta^1_{Z,Y}(\mathbb{E}_1))[n] \circ \theta^n_{X,Z}(\mathbb{E}_2)$.

$$[Z[n] \to Y[n+1]] \circ [X \to Z[n]] = [X \to Y[n+1]]$$

Thus we arrive at the following definition

Definition 7. Let $\mathscr{C}$ be a triangulated category with t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ with heart $\mathcal{H}$. For $X, Y \in \mathcal{H}$ and $n \geq 1$, let the *canonical map* θ^n be defined by

$$\begin{aligned}\theta^n = \theta^n_{X,Y} : \mathrm{YExt}^n_{\mathcal{H}}(X, Y) &\to \mathrm{Hom}_{\mathscr{C}}(X, Y[n]) \\ [\mathbb{E}] &\mapsto \theta^n(\mathbb{E})\end{aligned}$$

Note that we will just write θ^n instead of $\theta^n_{X,Y}$ in the cases when it is clear which Yoneda extensions we are working over.

Lemma 3.3. *The canonical map θ^n is well-defined.*

Proof. For $n = 1$: Assume $[\mathbb{E}] \in \mathrm{YExt}^1_{\mathcal{H}}(X, Y)$ is represented by the extensions $\mathbb{E}_1$ and $\mathbb{E}_2$. Then $\theta^1(\mathbb{E}_1)$ and $\theta^1(\mathbb{E}_2)$ can be represented by the diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & E_1 & \longrightarrow & X & \xrightarrow{\theta^1(\mathbb{E}_1)} & Y[1] \\ \| & & \downarrow \cong & & \| & & \| \\ Y & \longrightarrow & E_2 & \longrightarrow & X & \xrightarrow{\theta^1(\mathbb{E}_2)} & Y[1] \end{array}$$

And we see that $\theta^1(\mathbb{E}_1) = \theta^1(\mathbb{E}_2)$
For $n > 1$ let $[\mathbb{E}] \in \mathrm{YExt}^n_{\mathcal{H}}(X, Y)$ be represented by the extensions

$$\mathbb{E} : 0 \to Y \to E_1 \to \cdots \to E_n \to X \to 0$$

$$\mathbb{F} : 0 \to Y \to F_1 \to \cdots \to F_n \to X \to 0$$

We then know that there exist an extension $\mathbb{G}$ such that there exist a commutative diagram

$$\begin{array}{cccccccccccccc} \mathbb{E}: & 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\ \mathbb{G}: & 0 & \longrightarrow & Y & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_n & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \mathbb{F}: & 0 & \longrightarrow & Y & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

As in the construction above let

- $[\mathbb{E}] = [\mathbb{E}_1] \cup [\mathbb{E}_2]$ with $[\mathbb{E}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(Z_1, Y)$ and $[\mathbb{E}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, Z_1)$
- $[\mathbb{G}] = [\mathbb{G}_1] \cup [\mathbb{G}_2]$ with $[\mathbb{G}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(V_1, Y)$ and $[\mathbb{G}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, V_1)$
- $[\mathbb{F}] = [\mathbb{F}_1] \cup [\mathbb{F}_2]$ with $[\mathbb{F}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(W_1, Y)$ and $[\mathbb{F}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, W_1)$.

From Corollary 3.2.1 $\mathbb{E}_1$, $\mathbb{G}_1$ and $\mathbb{F}_1$ corresponds to the triangles

$$\begin{array}{ccccccc} Y & \longrightarrow & E_1 & \longrightarrow & Z_1 & \longrightarrow & Y[1] \\ \| & & \uparrow & & \uparrow & & \| \\ Y & \longrightarrow & G_1 & \longrightarrow & V_1 & \longrightarrow & Y[1] \\ \| & & \downarrow & & \downarrow & & \| \\ Y & \longrightarrow & F_1 & \longrightarrow & W_1 & \longrightarrow & Y[1] \end{array}$$

where the dotted arrows exist by the axioms of triangulated categories. In particular, by shifting, we have a commutative diagram

$$\begin{array}{ccc} Z_1[n-1] & \longrightarrow & Y[n] \\ \uparrow & & \| \\ V_1[n-1] & \longrightarrow & Y[n] \\ \downarrow & & \| \\ W_1[n-1] & \longrightarrow & Y[n] \end{array}$$

Now look at $[\mathbb{E}_2]$, $[\mathbb{G}_2]$ and $[\mathbb{F}_2]$. We have a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & Z_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & V_1 & \longrightarrow & U_2 & \longrightarrow & \cdots & \longrightarrow & U_n & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & W_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

again as in the construction let

- $[\mathbb{E}_2] = [\tilde{\mathbb{E}}_2] \cup [\mathbb{E}_3]$ with $[\tilde{\mathbb{E}}_2] \in \mathrm{YExt}^1_{\mathcal{H}}(Z_2, Z_1)$ and $[\mathbb{E}_3] \in \mathrm{YExt}^n_{\mathcal{H}}(X, Z_2)$
- $[\mathbb{G}_2] = [\tilde{\mathbb{G}}_2] \cup [\mathbb{G}_3]$ with $[\tilde{\mathbb{G}}_2] \in \mathrm{YExt}^1_{\mathcal{H}}(V_2, V_1)$ and $[\mathbb{G}_3] \in \mathrm{YExt}^n_{\mathcal{H}}(X, V_2)$
- $[\mathbb{F}_2] = [\tilde{\mathbb{F}}_2] \cup [\mathbb{F}_3]$ with $[\tilde{\mathbb{F}}_2] \in \mathrm{YExt}^1_{\mathcal{H}}(W_2, W_1)$ and $[\mathbb{F}_3] \in \mathrm{YExt}^n_{\mathcal{H}}(X, W_2)$.

and by the same argument as above we have a commutative diagram

$$\begin{array}{ccc} Z_2[n-2] & \longrightarrow & Z_1[n-1] \\ \uparrow & & \uparrow \\ V_2[n-2] & \longrightarrow & V_1[n-1] \\ \downarrow & & \downarrow \\ W_2[n-2] & \longrightarrow & W_1[n-1] \end{array}$$

By continuing this process we compose and end up with a commutative diagram

$$\begin{array}{ccccccccc} X & \longrightarrow & Z_{n-1}[1] & \longrightarrow & \cdots & \longrightarrow & Z_1[n-1] & \longrightarrow & Y \\ \| & & \uparrow & & \uparrow & & \uparrow & & \| \\ X & \longrightarrow & V_{n-1}[1] & \longrightarrow & \cdots & \longrightarrow & V_1[n-1] & \longrightarrow & Y \\ \| & & \downarrow & & \downarrow & & \downarrow & & \| \\ X & \longrightarrow & W_{n-1}[1] & \longrightarrow & \cdots & \longrightarrow & W_1[n-1] & \longrightarrow & Y \end{array}$$

and we conclude that $\theta^n(\mathbb{E}) = \theta^n(\mathbb{F}) = \theta^n(\mathbb{G})$. Thus $\theta^n(\mathbb{E})$ is well defined and is independent of the choice of $\mathbb{E}_1$ and $\mathbb{E}_2$ in the construction. □

Lemma 3.4 ([CHZ18] Lemma 2.1)**.** *Let $\mathscr{C}$ be a triangulated category with t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ with heart $\mathcal{H}$. Then for the canonical map θ^n the following holds:*

(1) θ^1 is an isomorphism, and θ^2 is injective.
(2) Assume that $\theta^n_{X,Y}$ are isomorphisms for all objects $X, Y \in \mathcal{H}$. Then θ^{n+1} is injective
(3) For $n \geq 2$, a morphism $f : X \to Y[n]$ lies in the image $\theta^n_{X,Y}$ if and only if f admits a factorization $X \to X_1[1] \to \cdots \to X_{n-1}[n-1] \to Y[n]$ with each $X_i \in \mathcal{H}$

Proof.

(1) By the previous lemma the homomorphism θ^1 is well defined and we can then define $\Psi : \mathrm{Hom}_{\mathscr{C}}(X, Y[1]) \to \mathrm{YExt}^1_{\mathcal{H}}(X, Y)$ which sends $\psi \in \mathrm{Hom}_{\mathscr{C}}(X, Y[1])$ to the unique extension class corresponding to the unique triangle associated with ψ. Thus Ψ is an inverse of θ^1 and θ^1 is an isomorphism. θ^2 being injective is a special case of (2).

(2) Assume $[\mathbb{E}] \in \ker(\theta^{n+1})$ for $[\mathbb{E}] \in \mathrm{YExt}^{n+1}_{\mathcal{H}}(X, Y)$. Let

$$\mathbb{E}_1 = 0 \to Y \to E \xrightarrow{g} Z \to 0$$

such that $[\mathbb{E}] = [\mathbb{E}_1] \cup [\mathbb{E}_2]$ where $[\mathbb{E}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, Z)$. We then have a commutative diagram.

$$\begin{array}{ccccc}
\mathrm{YExt}^n_{\mathcal{H}}(X,E) & \xrightarrow{\mathrm{YExt}^n_{\mathcal{H}}(X,g)} & \mathrm{YExt}^n_{\mathcal{H}}(X,Z) & \xrightarrow{[\mathbb{E}_1]\cup[-]} & \mathrm{YExt}^{n+1}_{\mathcal{H}}(X,Y) \\
\downarrow{\scriptstyle \theta^n_{X,E}} & & \downarrow{\scriptstyle \theta^n_{X,Z}} & & \downarrow{\scriptstyle \theta^{n+1}_{X,Y}} \\
\mathrm{Hom}_{\mathscr{C}}(X,E[n]) & \xrightarrow{-\circ g[n]} & \mathrm{Hom}_{\mathscr{C}}(X,Z[n]) & \xrightarrow{-\circ\theta^1(\mathbb{E}_1))[1]} & \mathrm{Hom}_{\mathscr{C}}(X,Y[n+1])
\end{array}$$

We know the bottom row is exact, and from Lemma 2.12 the top row is exact. By diagram chasing we see that since $\theta^{n+1}_{X,Y}([\mathbb{E}_1] \cup [\mathbb{E}_2]) = 0$ and $\theta^n_{X,Z}$ is an isomorphism there is an element $\varphi \in \mathrm{Hom}_{\mathscr{C}}(X, Z[n])$ such that $\theta^n_{X,Z}(\varphi) = \mathbb{E}_2$ and φ is sent to zero in $\mathrm{Hom}_{\mathscr{C}}(X, Y[n+1])$. Since the bottom row is exact we can lift φ to an element $\psi \in \mathrm{Hom}_{\mathscr{C}}(X, E[n])$ which can be lifted further to an element $\mathbb{E}_0 \in \mathrm{YExt}^n_{\mathcal{H}}(X, E)$ that is sent to $\mathbb{E}_2$. Since the top row is exact $\mathbb{E}_2$ must be in the kernel of $[\mathbb{E}_1] \cup -$ and $\mathbb{E} = 0$. Thus $\ker(\theta^{n+1}) = 0$ and $\theta^{n+1}_{X,Y}$ is injective.

(3) We use induction to prove the statement.
For $i = 2$

"⇒" Assume $f : X \to Y[2]$ lies in the image of $\theta^2_{X,Y}$. Then there exist short exact sequences

$$\mathbb{E}_1 : 0 \to Y \to E_1 \to X_1 \to 0 \text{ and } \mathbb{E}_2 : 0 \to X_1 \to E_2 \to X \to 0$$

such that $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$ and $f = \theta^2_{X,Y}(\mathbb{E})$ Then from the definition of θ^n we have

$$\begin{aligned} \theta^2(\mathbb{E}) &= (\theta^1(\mathbb{E}_1))[1] \circ \theta^1(\mathbb{E}_2) \\ &= [X_1[1] \to Y[2]] \circ [X \to X_1[1]] \\ &= [X \to X_1[1] \to Y[2]] \end{aligned}$$

"⇐" Now assume $f : X \xrightarrow{\alpha_1} X_1[1] \xrightarrow{\alpha_2} Y[2]$, with $X, X_1, Y \in \mathcal{H}$. Then there exist short exact sequences $\mathbb{E}_2$, where $[\mathbb{E}_2] \in \mathrm{YExt}^1_{\mathcal{H}}(X, X_1)$ and $\mathbb{E}_1$, where $[\mathbb{E}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(X_1, Y)$. From (1) we know $\theta^1_{X,Y}$ is an isomorphism; in particular it is surjective and we can assume

$$(\theta^1(\mathbb{E}_1))[1] = [X_1[1] \xrightarrow{\alpha_2} Y[2]]$$

and

$$\theta^1(\mathbb{E}_2) = [X \xrightarrow{\alpha_1} X_1[1]]$$

Then let $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$ and

$$\theta^2(\mathbb{E}) = (\theta^1(\mathbb{E}_1))[1] \circ \theta^1(\mathbb{E}_2) = [X \xrightarrow{\alpha_1} X_1[1] \xrightarrow{\alpha_2} Y[2]]$$

Now assume the statement hold for $i = n$

"⇒" Assume $f : X \to Y[n+1]$ lies in the image of $\theta^{n+1}_{X,Y}$. Then there exist extensions

$$\mathbb{E}_1 : 0 \to Y \to E_{n+1} \to X_n \to 0$$

and

$$\mathbb{E}_2 : 0 \to X_n \to E_1 \to \cdots \to E_n \to X \to 0$$

such that $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$ and $f = \theta^{n+1}_{X,Y}(\mathbb{E})$ Then from the definition of θ^{n+1} we have

$$\theta^{n+1}(\mathbb{E}) = (\theta^1(\mathbb{E}_1))[n] \circ \theta^n(\mathbb{E}_2)$$

and from the induction assumption we have

$$\theta^n(\mathbb{E}_2) = [X \to X_1[1] \to \cdots \to X_n[n]]$$

and

$$\begin{aligned} \theta^{n+1}(\mathbb{E}) &= (\theta^1(\mathbb{E}_1))[n] \circ \theta^n(\mathbb{E}_2) \\ &= [X_n[n] \to Y[n+1]] \circ [X \to X_1[1] \to \cdots \to X_n[n]] \\ &= [X \to X_1[1] \to \cdots \to X_n[n] \to Y[n+1]] \end{aligned}$$

"$\Leftarrow$" Now assume $f : X \xrightarrow{\alpha_1} X_1[1] \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n[n] \xrightarrow{\alpha_{n+1}} Y[n+1]$, with $X, X_i, Y \in \mathcal{H}$ for $i = 1, \cdots, n$. Then there exist short exact sequences $\mathbb{E}_2$, where $[\mathbb{E}_2] \in \mathrm{YExt}^n_{\mathcal{H}}(X, X_n)$ and $\mathbb{E}_1$, where $[\mathbb{E}_1] \in \mathrm{YExt}^1_{\mathcal{H}}(X_n, Y)$. From the induction assumption we have

$$\theta^n(\mathbb{E}_2) = [X \xrightarrow{\alpha_1} X_1[1] \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n[n]]$$

and from the surjectivity of θ^1 we have

$$(\theta^1(\mathbb{E}_1))[n] = [X_n[n] \xrightarrow{\alpha_{n+1}} Y[n+1]]$$

Then let $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$ and

$$\begin{aligned} \theta^{n+1}(\mathbb{E}) &= (\theta^1(\mathbb{E}_1))[n] \circ \theta^n(\mathbb{E}_2) \\ &= [X \xrightarrow{\alpha_1} X_1[1] \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n[n] \xrightarrow{\alpha_{n+1}} Y[n+1]] \end{aligned}$$

□

3.2 Cohomological functor

Given the derived category $D^b(\mathcal{A})$ a triangle induces a long exact sequence of homologies in $\mathcal{A}$. The object of this subsection is to generalize this to get a functor that maps triangles in a triangulated category to a long exact sequence of objects in the abelian heart. First we need an important proposition about maps of triangles

Proposition 3.1. *Let $\mathscr{C}$ be a triangulated category. Given the solid part of the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \xrightarrow{b} & Z' & \longrightarrow & X'[1] \end{array}$$

where the rows are distinguished triangles, the following are equivalent:

(1) $bga = 0$
(2) There is a morphism f such that the left square commutes
(3) There is a morphism h such that the middle square commutes
(4) The diagram is a morphism of triangles

If any of these conditions are satisfied, and in addition $\mathrm{Hom}_{\mathscr{C}}(X[1], Z') = 0$ *then f and g in condition* (2) *and* (3) *are unique.*

Proof. (1) $\Rightarrow$ (2) If $bga = 0$ then $X \to Y'$ factors through $X' \to Y'$ and there exist a $f : X \to X'$ such that the left square commutes.
(2) $\Rightarrow$ (3) If f and g exist, and the diagram is commutative, we can from the axioms of triangulated categories find a $h : Z \to Z'$ such that the diagram commutes.
(3) $\Rightarrow$ (4) By shifting the triangles, and using the same argument as in (2) $\Rightarrow$ (3) the maps in the diagram exists and form a triangle morphism.
(4) $\Rightarrow$ (1) If the diagram is a triangle morphism, by the commutativity $bga = 0$.

We have an exact sequences

$$\mathrm{Hom}_{\mathscr{C}}(X[1], Z') \to \mathrm{Hom}_{\mathscr{C}}(Z, Z') \to \mathrm{Hom}_{\mathscr{C}}(Y, Z') \to \mathrm{Hom}_{\mathscr{C}}(X, Z')$$

$$\mathrm{Hom}_{\mathscr{C}}(X[1], Z') \to \mathrm{Hom}_{\mathscr{C}}(X[1], X'[1]) \to \mathrm{Hom}_{\mathscr{C}}(X[1], Y'[1]) \to \mathrm{Hom}_{\mathscr{C}}(X[1], Z'[1])$$

We have that $\mathrm{Hom}_{\mathscr{C}}(X, Z') = 0$. If we assume $\mathrm{Hom}_{\mathscr{C}}(X[1], Z') = 0$ then

$$\mathrm{Hom}_{\mathscr{C}}(Z, Z') \cong \mathrm{Hom}_{\mathscr{C}}(Y, Z')$$

$$\mathrm{Hom}_{\mathscr{C}}(X[1], X'[1]) \cong \mathrm{Hom}_{\mathscr{C}}(X[1], Y'[1])$$

showing that the maps f and h are unique. □

We draw two immediate corollaries that shows the uniqueness of the triangle in the third axiom of t-structures.

Corollary 3.4.1. *Let $\mathscr{C}$ be a triangulared category, and $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ be a t-structure. Let $X, Y \in \mathscr{C}$ and $f : X \to Y$ then there exist a commutative diagram of triangles*

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow f & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & Y'[1] \end{array}$$

With $X', Y' \in \mathscr{C}^{\leq 0}$ and $X'', Y'' \in \mathscr{C}^{\geq 1}$ where the maps $X' \to Y'$ and $X'' \to Y''$ are unique.

Proof. This follows immediately from the previous proposition □

Corollary 3.4.2. *Given $X \in \mathscr{C}$ the triangle*

$$X' \to X \to X'' \to X'[1]$$

with $X' \in \mathscr{C}^{\leq 0}$ and $X'' \in \mathscr{C}^{\geq 1}$ is unique up to unique isomorphism.

Proof. Assume

$$X' \to X \to X'' \to X'[1]$$
$$Y' \to X \to Y'' \to Y'[1]$$

both are triangles from the third condition of t-structures. We then have from the previous lemma a commutative diagram

$$\begin{array}{ccccccc}
X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle g_1} & & \downarrow \\
Y' & \longrightarrow & X & \longrightarrow & Y'' & \longrightarrow & Y'[1] \\
\downarrow{\scriptstyle f_2} & & \| & & \downarrow{\scriptstyle g_2} & & \downarrow \\
X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle g_1} & & \downarrow \\
Y' & \longrightarrow & X & \longrightarrow & Y'' & \longrightarrow & Y'[1]
\end{array}$$

In particular we have two commutative diagrams

$$\begin{array}{ccccccc}
X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\
\downarrow{\scriptstyle f_2\circ f_1} & & \| & & \downarrow{\scriptstyle g_2\circ g_1} & & \downarrow \\
X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1]
\end{array}
\quad
\begin{array}{ccccccc}
Y' & \longrightarrow & X & \longrightarrow & Y'' & \longrightarrow & Y'[1] \\
\downarrow{\scriptstyle f_1\circ f_2} & & \| & & \downarrow{\scriptstyle g_1\circ g_2} & & \downarrow \\
Y' & \longrightarrow & X & \longrightarrow & Y'' & \longrightarrow & Y'[1]
\end{array}$$

By the uniqueness of the maps $f_2 \circ f_1 = id_{X'}$ and $f_1 \circ f_2 = id_{Y'}$. Therefore $X' \cong Y'$. Similarly $X'' \cong Y''$ □

We can then formalize the triangle by the following definition

Definition 8. Let $\mathscr{C}$ be a triangulated category, and $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ be a t-structure. Let $X \in \mathscr{C}$, and

$$X' \to X \to X'' \to Y[1]$$

be the unique triangle from the previous corollary. We define $X^{\leq 0} := X'$
$X^{\geq 1} := X''$
and

$$X^{\leq n} := X[n]^{\leq 0}[-n] \text{ and } X^{\geq n} := X[n-1]^{\geq 1}[1-n]$$

Indeed this definition induces a functor, commonly called the **truncation functor** on a triangulated category with a t-structure, which is explained by the following lemma and corollary

Lemma 3.5. *$(-)^{\geq n} : \mathscr{C} \to \mathscr{C}^{\geq n}$ and $(-)^{\leq n} : \mathscr{C} \to \mathscr{C}^{\leq n}$ define functors.*

Proof. We only prove that $(-)^{\geq n}$ defines a functor. The proof for $(-)^{\leq n}$ is similar. Note that $X^{\geq n} = X[n-1]^{\geq 1}[1-n] \in \mathscr{C}^{\geq 1}[1-n] = \mathscr{C}^{\geq n}$. So the functor is defined on objects. We need to check what $(-)^{\geq n}$ does on morphisms.
Without loss of generality we can let $n = 1$. Let $X \to X'$ be a map in $\mathscr{C}$. From Corollary 3.4.1 We have a diagram between triangles

$$\begin{array}{ccccccc} X^{\leq 0} & \longrightarrow & X & \longrightarrow & X^{\geq 1} & \longrightarrow & X^{\leq 0}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'^{\leq 0} & \longrightarrow & X' & \longrightarrow & X'^{\geq 1} & \longrightarrow & X'^{\leq 0}[1] \end{array}$$

where in particular $X^{\geq 1} \to X'^{\geq 1}$ is unique and $(-)^{\geq 1}$ defines a functor. By shifting we get the general result. □

Corollary 3.5.1. *Given $X \in \mathscr{C}$ we have a triangle*

$$X^{\leq n} \to X \to X^{\geq n+1} \to X^{\leq n}[1]$$

Proof. Look at $X[n]$ we then have a triangle

$$X[n]^{\leq 0} \to X[n] \to X[n]^{\geq 1} \to X[n]^{\leq 0}[1]$$

Now if we shift the triangle by $[-n]$ we get

$$X^{\leq n} \to X \to X^{\geq n+1} \to X^{\leq n}[1]$$

□

To understand where the name *truncation functor* comes from we look at the following example in the derived category with the canonical t-structure.

Example 3. Let $D^b(\mathcal{A})$ be the derived category, over an abelian category $\mathcal{A}$, with the canonical t-structure $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$. Let

$$X^\bullet = \cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots$$

Be an element in $D^b(\mathcal{A})$. From Example 2 we know that $X^\bullet$ fits in a triangle

$$Y^\bullet \to X^\bullet \to X^\bullet/Y^\bullet \to Y^\bullet[1]$$

where

$$Y^\bullet = \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow \ker(d_X^0) \longrightarrow 0 \longrightarrow \cdots$$

$$X^\bullet/Y^\bullet = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^0/\ker(d_X^0) \hookrightarrow X_1 \longrightarrow \cdots$$

From the uniqueness of the triangle we have $(X^\bullet)^{\leq 0} \cong Y^\bullet$ and $(X^\bullet)^{\geq 1} \cong X^\bullet/Y^\bullet \cong X^\bullet/(X^\bullet)^{\leq 0}$. We then have $(X^\bullet)^{\leq n} = (X^\bullet)[n]^{\leq 0}[-n]$

$$(X^\bullet)^{\leq n} = \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow \ker(d_X^n) \longrightarrow 0 \longrightarrow \cdots$$

with $\ker(d_X^n)$ in the n-th degree, and $(X^\bullet)^{\geq n} = (X^\bullet)[n-1]^{\geq 1}[1-n] \cong \frac{X^\bullet[n-1]}{(X^\bullet[n-1])^{\leq 0}}[1-n]$

$$(X^\bullet)^{\geq n} = \cdots \longrightarrow 0 \longrightarrow \frac{X^{n-1}}{\ker(d_X^{n-1})} \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots$$

with X^n in the n-th degree. The above complexes $(X^\bullet)^{\leq n}$ and $(X^\bullet)^{\geq n}$ are called the *(good) truncations*, above and below n respectively, of a chain complex $X^\bullet$. The trucations are often denoted as $_{\tau_{\leq n}}X^\bullet$ and $_{\tau_{\geq n}}X^\bullet$ respectively. [Wei95, Truncation 1.2.7]

We actually get an adjoint pair of the inclusion functor and truncation functor

Lemma 3.6. *Let $i : \mathscr{C}^{\leq n} \to \mathscr{C}$ be the inclusion from $\mathscr{C}^{\leq n}$ to $\mathscr{C}$, and $j : \mathscr{C}^{\geq n} \to \mathscr{C}$ be the inclusion from $\mathscr{C}^{\geq n}$ to $\mathscr{C}$. Then $(-)^{\leq n}$ and i, and $(-)^{\geq n}$ and j are adjoint pairs respectively:*

$$\mathrm{Hom}_{\mathscr{C}^{\leq n}}(-, (-)^{\leq n}) \cong \mathrm{Hom}_{\mathscr{C}}(i(-), -)$$

$$\mathrm{Hom}_{\mathscr{C}^{\geq n}}((-)^{\geq n}, -) \cong \mathrm{Hom}_{\mathscr{C}}(-, j(-))$$

Proof. We only prove the first isomorphism, the second is similar. Given $X \in \mathscr{C}$, and $Y \in \mathscr{C}^{\leq n}$ we want to show

$$\mathrm{Hom}_{\mathscr{C}^{\leq n}}(Y, (X)^{\leq n}) \cong \mathrm{Hom}_{\mathscr{C}}(i(Y), X)$$

Note that $Y[n] \in \mathscr{C}^{\leq n}[n] = \mathscr{C}^{\leq 0}$. Let $Y' = Y[n]$ and $X' = X[n]$. Given the triangle

$$X'^{\leq 0} \to X' \to X'^{\geq 1} \to X'^{\leq 0}[1]$$

look at the exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(Y', X'^{\geq 1}[-1]) \to \mathrm{Hom}_{\mathscr{C}}(Y', X'^{\leq 0}) \to \mathrm{Hom}_{\mathscr{C}}(Y', X') \to \mathrm{Hom}_{\mathscr{C}}(Y', X'^{\geq 1})$$

And note that from the third condition of t-structures that, since $X'^{\geq 1}[-1] \in \mathscr{C}^{\geq 1}$, we get $\mathrm{Hom}_{\mathscr{C}}(Y', X'^{\geq 1}[-1]) = \mathrm{Hom}_{\mathscr{C}}(Y', X'^{\geq 1}) = 0$ and

$$\mathrm{Hom}_{\mathscr{C}}(Y', X'^{\leq 0}) \cong \mathrm{Hom}_{\mathscr{C}}(Y', X')$$

Shift X' and Y' back and we get

$$\mathrm{Hom}_{\mathscr{C}^{\leq n}}(Y, X[n]^{\leq 0}[-n]) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

□

We actually get the following general relationships, which are obvious in the case of the canonical t-structure over the derived category.

Corollary 3.6.1. *If $m \leq n$ then:*

(i) $((-)^{\leq n})^{\leq m} = ((-)^{\leq m})^{\leq n} = (-)^{\leq m}$

(ii) $((-)^{\geq m})^{\geq n} = ((-)^{\geq n})^{\geq m} = (-)^{\geq n}$

Proof. We will only prove the first part, the second is similar. Given any $Y \in \mathscr{C}^{\leq m}$ we then have from Lemma 3.6

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}^{\leq m}}(Y, (X^{\leq n})^{\leq m}) &\cong \mathrm{Hom}_{\mathscr{C}}(Y, X^{\leq n}) \cong \mathrm{Hom}_{\mathscr{C}^{\leq n}}(Y, X^{\leq n}) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(Y, X) \cong \mathrm{Hom}_{\mathscr{C}^{\leq m}}(Y, X^{\leq m})\end{aligned}$$

Therefore we have $X^{\leq m} \cong (X^{\leq n})^{\leq m}$. Let $Y \in \mathscr{C}^{\leq n}$ then we have

$$\mathrm{Hom}_{\mathscr{C}^{\leq n}}(Y, (X^{\leq m})^{\leq n}) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X^{\leq m}) \cong \mathrm{Hom}_{\mathscr{C}^{\leq n}}(Y, X^{\leq m})$$

where the last isomorphism comes from the second axiom of t-structures. Thus $X^{\leq m} \cong (X^{\leq m})^{\leq n}$. □

Lemma 3.7. $((-)^{\leq n})^{\geq n} = ((-)^{\geq n})^{\leq n}$

Proof. We can construct the diagram

$$\begin{array}{ccccccc}
(X^{\leq n})^{\leq n-1} & \longrightarrow & X^{\leq n} & \longrightarrow & (X^{\leq n})^{\geq n} & \longrightarrow & (X^{\leq n})^{\leq n-1}[1] \\
\| & & \downarrow & & \vdots\downarrow & & \| \\
X^{\leq n-1} & \longrightarrow & X & \longrightarrow & X^{\geq n} & \longrightarrow & X^{\leq n-1}[1] \\
 & & \downarrow & & \vdots\downarrow & & \downarrow \\
 & & X^{\geq n+1} & = & X^{\geq n+1} & \longrightarrow & X^{\leq n}[1] \\
 & & \downarrow & & \vdots\downarrow & & \\
 & & X^{\leq n}[1] & \longrightarrow & (X^{\leq n})^{\geq n}[1] & &
\end{array}$$

and observe from the previous corollary that $(X^{\leq n})^{\leq n-1} \cong X^{\leq n-1}$ so the vertical equalities makes sense. Therefore, by the octahedral axiom, the dashed arrows form a triangle. We then have triangles

$$\begin{array}{ccccccc} (X^{\leq n})^{\geq n} & \longrightarrow & X^{\geq n} & \longrightarrow & X^{\geq n+1} & \longrightarrow & (X^{\leq n})^{\geq n}[1] \\ \downarrow & & \| & & \downarrow \cong & & \downarrow \\ (X^{\geq n})^{\leq n} & \longrightarrow & X^{\geq n} & \longrightarrow & (X^{\geq n})^{\geq n+1} & \longrightarrow & (X^{\geq n})^{\leq n}[1] \end{array}$$

Where, from the last corollary, the third arrow is an isomorphism. Thus the dashed arrows exist and are isomorphisms. □

Observe that if $n = 0$ we have $((-)^{\leq 0})^{\geq 0} = ((-)^{\geq 0})^{\leq 0} \in \mathscr{C}^{\leq 0} \cap \mathscr{C}^{\geq 0} = \mathcal{H}$. We can therefore define a functor from $\mathscr{C}$ to $\mathcal{H}$

Definition 9. We define a functor

$$H^0_{\mathcal{H}} : \mathscr{C} \to \mathcal{H}$$

from a triangulated category $\mathscr{C}$ to the heart $\mathcal{H}$ of a t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ by

$$H^0_{\mathcal{H}}(X) = (X^{\leq 0})^{\geq 0} = (X^{\geq 0})^{\leq 0}$$

We let $H^n_{\mathcal{H}}(X) = H^0_{\mathcal{H}}(X[n])$

We get the following immediate lemma

Lemma 3.8. *Let $\mathscr{C}$ be a triangulated category with t-structure with heart $\mathcal{H}$. Let $X \in \mathcal{H}$ then $X \cong H^0_{\mathcal{H}}(X)$.*

Proof. For $X \in \mathcal{H}$ there exist a triangle

$$X^{\leq 0} \to X \to X^{\geq 1} \to X^{\leq 0}[1]$$

Since $X \in \mathcal{H}$ the map $X \to X^{\geq 1}$ is zero. Thus we have the isomorphism $X^{\leq 0} \cong X$. Similarly $X^{\geq 0} \cong X$ and we have the isomorphism

$$H^0_{\mathcal{H}}(X) = X^{\leq 0 \geq 0} \cong X$$

□

This is indeed the functor we are looking for, what is left to show is that any triangle in a triangulated category induces a long exact sequence through this functor. The following two results shows this.

Lemma 3.9. *Let $\mathscr{C}$ be a triangulated category with t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ and heart $\mathcal{H}$. Given $A, B, C \in \mathscr{C}$ we have*

(i) If there exist a triangle $A \to B \to C^{\geq 0} \to A[1]$ then we have an exact sequence

$$0 \to H^0_{\mathcal{H}}(A) \to H^0_{\mathcal{H}}(B) \to H^0_{\mathcal{H}}(C)$$

(ii) If there exist a triangle $A^{\leq 0} \to B \to C \to A^{\leq 0}[1]$ then we have an exact sequence

$$H^0_{\mathcal{H}}(A) \to H^0_{\mathcal{H}}(B) \to H^0_{\mathcal{H}}(C) \to 0$$

Proof. The proof for the second part is similar to the first, so only a sketch will be provided for (ii)

(i) Let $T \in \mathscr{C}^{\leq -1}$. We have an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(T, C^{\geq 0}[-1]) \to \mathrm{Hom}_{\mathscr{C}}(T, A) \to \mathrm{Hom}_{\mathscr{C}}(T, B) \to \mathrm{Hom}_{\mathscr{C}}(T, C^{\geq 0})$$

and we see that $\mathrm{Hom}_{\mathscr{C}}(T, C^{\geq 0}[-1]) = \mathrm{Hom}_{\mathscr{C}}(T, C^{\geq 0}) = 0$ and

$$\mathrm{Hom}_{\mathscr{C}}(T, A) \cong \mathrm{Hom}_{\mathscr{C}}(T, B)$$

From Lemma 3.6 we then have

$$\mathrm{Hom}_{\mathscr{C}^{\leq -1}}(T, A^{\leq -1}) \cong \mathrm{Hom}_{\mathscr{C}}(T, A) \cong \mathrm{Hom}_{\mathscr{C}}(T, B) \cong \mathrm{Hom}_{\mathscr{C}^{\leq -1}}(T, B^{\leq -1})$$

in particular

$$A^{\leq -1} \cong B^{\leq -1}$$

We then have the commutative diagram

$$\begin{array}{ccccccc}
A^{\leq -1} & \longrightarrow & A & \longrightarrow & A^{\geq 0} & \longrightarrow & A^{\leq -1}[1] \\
\| & & \downarrow & & \downarrow & & \| \\
B^{\leq -1} & \longrightarrow & B & \longrightarrow & B^{\geq 0} & \longrightarrow & B^{\leq -1}[1] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C^{\geq 0} & = & C^{\geq 0} & \longrightarrow & A[1] \\
 & & \downarrow & & \downarrow & & \\
 & & A[1] & \longrightarrow & A^{\geq 0}[1] & &
\end{array}$$

where the dashed arrows form a triangle by the octahedral axiom. Let $M \in \mathcal{H}$. We then have an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(M, C^{\geq 0}[-1]) \to \mathrm{Hom}_{\mathscr{C}}(M, A^{\geq 0}) \to \mathrm{Hom}_{\mathscr{C}}(M, B^{\geq 0}) \to \mathrm{Hom}_{\mathscr{C}}(M, C^{\geq 0})$$

By the first condition for a t-structure we have $\mathrm{Hom}_{\mathscr{C}}(M, C^{\geq 0}[-1]) = 0$, and from Lemma 3.6 it follows that

$$\mathrm{Hom}_{\mathscr{C}}(M, A^{\geq 0}) \cong \mathrm{Hom}_{\mathscr{C}^{\leq 0}}(M, (A^{\geq 0})^{\leq 0}) = \mathrm{Hom}_{\mathcal{H}}(M, H^0_{\mathcal{H}}(A))$$

By the same argument on B and C we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(M, H^0_{\mathcal{H}}(A)) \to \mathrm{Hom}_{\mathcal{H}}(M, H^0_{\mathcal{H}}(B)) \to \mathrm{Hom}_{\mathcal{H}}(M, H^0_{\mathcal{H}}(C))$$

and in particular

$$0 \to H^0_{\mathcal{H}}(A) \to H^0_{\mathcal{H}}(B) \to H^0_{\mathcal{H}}(C)$$

is exact.

(ii) Similar to the first proof we can show that given a triangle

$$A^{\leq 0} \to B \to C \to A^{\leq 0}[1]$$

we have that

$$B^{\geq 1} \cong C^{\geq 1}$$

Then we have the diagram

$$\begin{array}{ccccccc}
B & \longrightarrow & C & \longrightarrow & A^{\leq 0}[1] & \longrightarrow & B[1] \\
\| & & \downarrow & & \downarrow & & \| \\
B & \longrightarrow & B^{\geq 1} & \longrightarrow & B^{\leq 0}[1] & \longrightarrow & B[1] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C^{\leq 0}[1] & = & C^{\leq 0}[1] & \longrightarrow & C[1] \\
 & & \downarrow & & \downarrow & & \\
 & & C[1] & \longrightarrow & A^{\leq 0}[2] & &
\end{array}$$

and from the octahedral axiom there exist a triangle

$$A^{\leq 0} \to B^{\leq 0} \to X^{\leq 0} \to X^{\leq 0}[1]$$

Given $N \in \mathcal{H}$ we have an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(A^{\leq 0}, N) \to \mathrm{Hom}_{\mathscr{C}}(B^{\leq 0}, N) \to \mathrm{Hom}_{\mathscr{C}}(C^{\geq 0}, N) \to \mathrm{Hom}_{\mathscr{C}}(A^{\leq 0}[1], N)$$

where $\mathrm{Hom}_{\mathscr{C}}(A^{\leq 0}[1], N) = 0$. Similarly to (i) we then get an exact sequence

$$H^0_{\mathcal{H}}(A) \to H^0_{\mathcal{H}}(B) \to H^0_{\mathcal{H}}(C) \to 0$$

□

Theorem 3.10. *The functor $H^0_{\mathcal{H}}$ is cohomological, i.e. given a triangle*

$$X \to Y \to Z \to X[1]$$

in $\mathscr{C}$, we have a long exact sequence

$$H^0_{\mathcal{H}}(X) \to H^0_{\mathcal{H}}(Y) \to H^0_{\mathcal{H}}(Z)$$

in $\mathcal{H}$.

Proof. We have a composite map $Y \to Z \to Z^{\geq 0}$. Embed this into the triangle

$$W \to Y \to Z^{\geq 0} \to W[1]$$

and from the previous lemma we get an exact sequence

$$0 \to H^0_{\mathcal{H}}(W) \to H^0_{\mathcal{H}}(Y) \to H^0_{\mathcal{H}}(Z)$$

Now we have the following diagram

$$\begin{array}{ccccccc}
Y & \longrightarrow & Z & \longrightarrow & X[1] & \longrightarrow & Y[1] \\
\| & & \downarrow & & \downarrow & & \| \\
Y & \longrightarrow & Z^{\geq 0} & \longrightarrow & W[1] & \longrightarrow & Y[1] \\
 & & \downarrow & & \downarrow & & \\
 & & Z^{\leq -1}[1] & = & Z^{\leq -1}[1] & & \\
 & & \downarrow & & \downarrow & & \\
 & & Z[1] & \longrightarrow & X[2] & &
\end{array}$$

and from the octahedral axiom we have a triangle

$$Z^{\leq -1}[-1] \to X \to W \to Z^{\leq -1}$$

Since $Z^{\leq -1}[-1] \in \mathscr{C}^{\leq 0}$ we have from the previous lemma we have that

$$H^0_{\mathcal{H}}(X) \to H^0_{\mathcal{H}}(W) \to 0$$

is exact and by splicing the sequences

$$\begin{array}{ccccccc} H^0_{\mathcal{H}}(X) & \twoheadrightarrow & H^0_{\mathcal{H}}(W) & \longrightarrow & 0 & & \\ & & \| & & & & \\ 0 & \longrightarrow & H^0_{\mathcal{H}}(W) & \hookrightarrow & H^0_{\mathcal{H}}(Y) & \longrightarrow & H^0_{\mathcal{H}}(Z) \end{array}$$

we get that the sequence

$$H^0_{\mathcal{H}}(X) \to H^0_{\mathcal{H}}(Y) \to H^0_{\mathcal{H}}(Z)$$

is exact. □

To justify that this is a generalization of the induced long exact sequence of homology on triangles in the derived category, we note the following example

Example 4. Let $D^b(\mathcal{A})$ be the derived category, over an abelian category $\mathcal{A}$, with the canonical t-structure $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$. The heart of the t-structure is $\mathcal{A}$. Let $X^\bullet \in D^b(\mathcal{A})$.
Look at $H^n_{\mathcal{A}}(X^\bullet) = H^0_{\mathcal{A}}(X^\bullet[n]) = ((X^\bullet[n])^{\geq 0})^{\leq 0} = ((X^\bullet[n-1])^{\geq 1}[1])^{\leq 0} = (\frac{X^\bullet[n-1]}{(X^\bullet[n-1])^{\leq 0}}[1])^{\leq 0}$ We then have

$$\begin{aligned} H^n_{\mathcal{A}}(X^\bullet) &= \left(\cdots \to 0 \to \frac{X^{n-1}}{\ker(d_X^{n-1})} \hookrightarrow X^n \to X^{n+1} \to \cdots\right)^{\leq 0} \\ &= \left(\cdots \to 0 \to \frac{X^{n-1}}{\ker(d_X^{n-1})} \hookrightarrow \ker(d^n_X) \to 0 \to \cdots\right) \end{aligned}$$

with $\ker(d^n_X)$ in degree 0. Note that $\frac{X^{n-1}}{\ker(d_X^{n-1})} \cong \mathrm{Im}(d_X^{n-1}) \subseteq \ker(d^n_X)$ so the image of the non-trivial map in the bottom complex is still $\mathrm{Im}(d_X^{n-1})$. Therefore $H^n_{\mathcal{A}}(X^\bullet)$ is quasi-isomorphic to the stalk complex of the regular homology $H^n(X^\bullet)$.

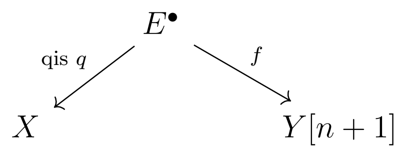
We can now prove the following well known result

Theorem 3.11. *Let $\mathcal{A}$ be an abelian category, and $D^b(\mathcal{A})$ be its bounded derived category. Let $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$ be the canonical t-structure on $D^b(\mathcal{A})$, with heart $\mathcal{A}$. Let $X, Y \in \mathcal{A}$ Then the map*

$$\theta^n : \mathrm{YExt}^n_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[n])$$

is an isomorphism for $n \geq 1$

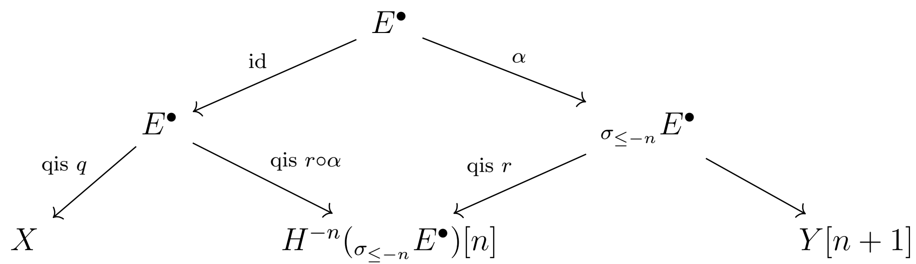
Proof. We use induction to prove that θ^n are isomorphisms for all $n \geq 1$. We know from Lemma 3.4 that θ^1 is an isomorphism, so assume θ^i are isomorhpisms for all $i = 1, \cdots, n$. From the second part of Lemma 3.4 we then know θ^{n+1} is injective, so we need to show the surjectivity.
Let



denote an element in $\operatorname{Hom}_{D^b(\mathcal{A})}(X, Y[n+1])$, With $E^\bullet = (E_i, d_E^i)$. Note that since $E^\bullet$ is quasi-isomorphic to X, $E^\bullet$ is exact in every non-zero degree. Let ${}_{\sigma_{\leq -n}}E^\bullet$ denote the *brutal truncation*

$$ {}_{\sigma_{\leq -n}}E^\bullet = \cdots \longrightarrow E_{-(n+1)} \longrightarrow E_{-n} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots $$

where $({}_{\sigma_{\leq -n}}E^\bullet)^i = (E^\bullet)^i$ for $i \leq -n$ and $({}_{\sigma_{\leq -n}}E^\bullet)^i = 0$ for $i > -n$ [Wei95, Truncation 1.2.7]. Note that $H^n({}_{\sigma_{\leq -n}}E^\bullet) = E^{-n}/\operatorname{Im}(d_E^{-(n+1)}) \cong E^{-n}/\ker(d_E^{-n}) \in \mathcal{A}$. We then have a quasi-isomorphism $r : {}_{\sigma_{\leq -n}}E^\bullet \to H^{-n}({}_{\sigma_{\leq -n}}E^\bullet)[n]$ since ${}_{\sigma_{\leq -n}}E^\bullet$ is exact in every degree $\neq -n$. We then have roof equal to the roof $f \cdot q^{-1}$:



Since θ^n is assumed to an isomorphism, we have from Lemma 3.4 that $X \to H^{-n}({}_{\sigma_{\leq -n}}E^\bullet)[n]$ admits a factorization

$$ X \to X_1[1] \to \cdots \to X_{n-1}[n-1] \to H^{-n}({}_{\sigma_{\leq -n}}E^\bullet)[n] $$

with $X_i \in \mathcal{A}$. Thus $X \to Y[n+1]$ admits the factorization

$$ X \to X_1[1] \to \cdots \to X_{n-1}[n-1] \to H^{-n}({}_{\sigma_{\leq -n}}E^\bullet)[n] \to Y[n+1] $$

and again from the third part of Lemma 3.4 θ^{n+1} is an isomorphism. $\square$

Observation 3.11.1. In the derived category $D^b(\mathcal{A})$, given $X, Y \in D^b(\mathcal{A})$, the extension groups (given enough injectives and projectives) are defined as $\operatorname{Ext}^n_{\mathcal{A}}(X, Y) := \operatorname{Hom}_{D^b(\mathcal{A})}(X, Y[n])$. (Observe that $\operatorname{Ext}^n_{\mathcal{A}}(X, Y) = 0$ for $n < 0$). A consequence of the previous theorem is therefore $\operatorname{YExt}^n_{\mathcal{A}}(X, Y) \cong \operatorname{Ext}^n_{\mathcal{A}}(X, Y)$ for $n \geq 1$ in the bounded derived categories.

3.3 t-exact functors

In our quest to understand a bit more about triangulated equivalences between derived categories, we need a way of going between the abelian subcategories of different triangulated categories. To do this we introduce the concept of t-exact functors.

Definition 10. Let $\mathscr{C}$ and $\mathscr{D}$ be triangulated categories. Then a **triangulated functor** from $\mathscr{C}$ to $\mathscr{D}$ is a pair (F, η) consisting of an additive functor $F : \mathscr{C} \to \mathscr{D}$ together with natural isomorhpism

$$\eta : F(-[1]) \to F(-)[1]$$

such that for every triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathscr{C}$, then

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\eta_X \circ F(h)} F(X)[1]$$

is a triangle in $\mathscr{D}$. Now if $\mathscr{C}$ and $\mathscr{D}$ have t-structures, then a triangulated functor (F, η) from $\mathscr{C}$ to $\mathscr{D}$ is called a ***t*-exact functor** if $F(\mathscr{C}^{\leq 0}) \subseteq \mathscr{D}^{\leq 0}$ and $F(\mathscr{C}^{\geq 0}) \subseteq \mathscr{D}^{\geq 0}$.

We have the following two immediate consequences of this definition

Lemma 3.12. *$F(\mathscr{C}^{\leq n}) \subseteq \mathscr{D}^{\leq n}$ and $F(\mathscr{C}^{\geq n}) \subseteq \mathscr{D}^{\geq n}$*

Proof. Let $X \in \mathscr{C}^{\leq n}$ then $X[n] \in \mathscr{C}^{\leq n}[n] = \mathscr{C}^{\leq 0}$. Since (F, η) is triangulated we have

$$F(X) \cong F(X[n][-n]) \cong F(X[n])[-n] \in \mathscr{D}^{\leq 0}[-n] = \mathscr{D}^{\leq n}$$

Similarly $F(\mathscr{C}^{\geq n}) \subseteq \mathscr{D}^{\geq n}$. □

Lemma 3.13. *$F(X^{\leq n}) \cong F(X)^{\leq n}$ and $F(X^{\geq n}) \cong F(X)^{\geq n}$*

Proof. WLOG we can assume $n = 0$. We have a triangle

$$F(X^{\leq 0}) \to F(X) \to F(X^{\geq 1}) \to F(X^{\leq 0})[1]$$

Where from the t-exactness $F(X^{\leq 0}) \in \mathscr{D}^{\leq 0}$ and $F(X^{\geq 1}) \in \mathscr{D}^{\geq 1}$. We also have a triangle

$$F(X)^{\leq 0} \to F(X) \to F(X)^{\geq 1} \to F(X)^{\leq 0}[1]$$

with $F(X)^{\leq 0} \in \mathscr{D}^{\leq 0}$ and $F(X)^{\geq 1} \in \mathscr{D}^{\geq 1}$. Thus by Corollary 3.4.2

$$F(X^{\leq 0}) \cong F(X)^{\leq 0} \text{ and } F(X^{\geq 1}) \cong F(X)^{\geq 1}$$

□

The following lemma gives us a relationship between the hearts of two triangulated categories through a t-exact functor

Lemma 3.14 ([CHZ18, Lemma 2.3])**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a t-exact functor as in the definition. Then:*

(1) The restriction $F|_{\mathcal{H}} : \mathcal{H} \to \mathcal{G}$ is exact.

(2) Given $X \in \mathscr{C}$ we have isomorphisms

$$H^n_{\mathcal{H}}(F(X)) \cong F|_{\mathcal{H}}(H^n_{\mathcal{H}}(X))$$

for $n \in \mathbb{Z}$

(3) If F is an equivalence then so is the restriction $F|_{\mathcal{H}}$

Proof. (1) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathcal{H}$. Then from Corollary 3.2.1 the sequence gives rise to a triangle

$$X \to Y \to Z \to X[1]$$

in $\mathscr{C}$. Then since F is t-exact we have a triangle

$$F|_{\mathcal{H}}(X) \to F|_{\mathcal{H}}(Y) \to F|_{\mathcal{H}}(Z) \to F|_{\mathcal{H}}(X)[1]$$

with $F|_{\mathcal{H}}(X), F|_{\mathcal{H}}(Y), F|_{\mathcal{H}}(Z) \in \mathcal{G}$. Then again from Corollary 3.2.1

$$0 \to F|_{\mathcal{H}}(X) \to F|_{\mathcal{H}}(Y) \to F|_{\mathcal{H}}(Z) \to 0$$

is a short exact sequence.

(2) We have isomorphisms

$$\begin{aligned} H^n_{\mathcal{G}}(F(X)) &\cong H^0_{\mathcal{G}}(F(X)[n]) = (F(X)[n])^{\leq 0 \geq 0} && (3.1)\\ &\cong F(X[n])^{\leq 0 \geq 0} && (3.2)\\ &\cong F|_{\mathcal{H}}(X[n]^{\leq 0 \geq 0}) && (3.3)\\ &\cong F|_{\mathcal{H}}(H^n_{\mathcal{H}}(X)) && (3.4) \end{aligned}$$

where the isomorphism (3.2) comes from the fact that F is a t-exact functor. The isomorphism in (3.3) comes from Lemma 3.13. Note further that since $X[n]^{\leq 0 \geq 0} = H^n_{\mathcal{H}}(X) \in \mathcal{H}$ we can restrict the functor to $\mathcal{H}$ without changing anything. The last isomorphism, (3.4), comes from Lemma 3.8.

(3) Since F is an equivalence $F|_{\mathcal{H}}$ is clearly fully faithful. What is left is to show the denseness. Assume $Y \in \mathcal{G}$. Then, since F is an equivalence, we have $Y \cong F(\tilde{X})$ for some $\tilde{X} \in \mathscr{C}$. Then we use the isomorphism from (2) and Lemma 3.8 to get

$$Y \cong F(\tilde{X}) \cong H^0_{\mathcal{G}}(F(\tilde{X})) \cong F|_{\mathcal{H}}(H^0_{\mathcal{H}}(\tilde{X}))$$

Thus for every $Y \in \mathcal{G}$ there exist a $X := H^0_{\mathcal{H}}(\tilde{X}) \in \mathcal{H}$ such that $Y \cong F|_{\mathcal{H}}(X)$

□

Again there is a connection between the Yoneda extension of the hearts of two different triangulated categories and the Hom-sets

Proposition 3.2 ([CHZ18, Proposition 2.4]). *Let $(F, \eta) : \mathscr{C} \to \mathscr{D}$ be a t-exact functor as in the definition. Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{YExt}^n_{\mathcal{H}}(X, Y) & \xrightarrow{F|_{\mathcal{H}}} & \mathrm{YExt}^n_{\mathcal{G}}(F(X), F(Y)) \\ \downarrow{\theta^n_1} & & \downarrow{\theta^n_2} \\ \mathrm{Hom}_{\mathscr{C}}(X, Y[n]) & \xrightarrow{(F,\eta)} & \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)[n]) \end{array}$$

For any $X, Y \in \mathcal{H}$ and $n \geq 1$.

In the diagram above θ^n_1 and θ^n_2 are the canonincal maps associated with the t-structures on $\mathscr{C}$ and $\mathscr{D}$. $F|_{\mathcal{H}}$ on the top row is defined to map an exact sequence

$$\mathbb{E} : 0 \to Y \to E_1 \to \cdots \to E_n \to X \to 0$$

to the exact sequence

$$F|_{\mathcal{H}} : 0 \to F(Y) \to F(E_1) \to \cdots \to F(E_n) \to F(X) \to 0$$

(F, η) on the bottom row is defined to send a morphism $f : X \to Y[n]$ to $\eta_y[n] \circ F(f) : F(X) \to F(Y)[n]$

Proof. We prove the statement for $n = 1$, the general case is shown by induction similar to the proof of Lemma 3.4.
For $n = 1$: First look at $\theta^1_2 \circ F|_{\mathcal{H}}$
Let $[\mathbb{E}] \in \mathrm{YExt}^1_{\mathcal{H}}(X, Y)$ be represented by the sequence

$$0 \to Y \xrightarrow{f} E \xrightarrow{g} X \to 0$$

then we have the sequence

$$0 \to F(Y) \xrightarrow{F(f)} F(E) \xrightarrow{F(g)} F(X) \to 0$$

in $\mathrm{YExt}^1_{\mathcal{G}}(F(X), F(Y))$. By embedding this into the unique triangle

$$F(Y) \xrightarrow{F(f)} F(E) \xrightarrow{F(g)} F(X) \xrightarrow{\theta^1_2(F(\mathbb{E}))} F(Y)[1]$$

we obtain the map $(\theta_2^1 \circ F|_{\mathcal{H}})(\mathbb{E})$
Next look at $(F, \eta) \circ \theta_1^1$. Embed $\mathbb{E}$ into the triangle

$$Y \xrightarrow{f} E \xrightarrow{g} X \xrightarrow{\theta_1^1(\mathbb{E})} Y[1]$$

to get map $\theta_1^1(\mathbb{E})$ Use (F, η) to get

$$F(Y) \xrightarrow{F(f)} F(E) \xrightarrow{F(g)} F(X) \xrightarrow{\eta_Y \circ F(\theta_1^1(\mathbb{E}))} F(Y)[1]$$

and we see that $\eta_Y \circ F(\theta_1^1(\mathbb{E})) = \theta_2^1 \circ F(\mathbb{E})$ making the diagram commute. □

3.4 Bounded t-structures

A question we have not yet asked is if we can recover a triangulated category from the abelian heart of a t-structure. It is well known that the bounded derived category $D^b(\mathcal{A})$ is generated by the abelian subcategory $\mathcal{A}$. We will see that with an extra assumption on t-structures, we can generalize this result to see that the heart will generate the triangulated category.

Definition 11. Let $\mathscr{C}$ be a triangulated category. We say the t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ is **bounded**, if

$$\mathscr{C} = \bigcup_{i,j \in \mathbb{Z}} (\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j})$$

Lemma 3.15. *Let $\mathscr{C}$ be a triangulated category with bounded t-structure. Then the heart $\mathcal{H}$ generates $\mathscr{C}$, and*

$$\mathscr{C} = \bigcup_{i \geq j} \mathcal{H}[-j] * \cdots * \mathcal{H}[-i]$$

for $i, j \in \mathbb{Z}$

Proof. Since $\mathscr{C}$ is bounded, $\mathscr{C} = \bigcup_{i,j \in \mathbb{Z}} (\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j})$. Fix $i, j \in \mathbb{Z}$, and look at $\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j}$. If $i < j$ then it is clear that $\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j} = 0$, and if $i = j$ then $\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j} = \mathcal{H}[-i]$. Let $i = j + 1$. We then show that $\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j} = \mathcal{H}[-j] * \mathcal{H}[-i]$. We show this for $j = 0$ and $i = 1$, the general result is analogous. Given $X \in \mathcal{H} * \mathcal{H}[-1]$, we have a triangle

$$H \to X \to \tilde{H}[-1] \to H[1]$$

for some $H, \tilde{H} \in \mathcal{H}$. From the definition of t-structure, we see that $H \in \mathscr{C}^{\leq 0} \subseteq \mathscr{C}^{\leq 1}$, and $\tilde{H}[-1] \in \mathscr{C}^{\geq 1} \subseteq \mathscr{C}^{\geq 0}$. Thus we have $H, \tilde{H}[-1] \in \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$, and since t-structures are closed under extensions we see that $X \in \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$ and $\mathcal{H} * \mathcal{H}[-1] \subseteq \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$.

Now let $X \in \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$. We then have from the definition of t-structure that X fits in a triangle

$$X^{\geq 1}[-1] \to X^{\leq 0} \to X \to X^{\geq 1} \to X^{\leq 0}[1]$$

We have $X^{\geq 1}[-1] \in \mathscr{C}^{\geq 2} \subseteq \mathscr{C}^{\geq 0}$, and $X^{\leq 0}[1] \in \mathscr{C}^{\leq -1} \subseteq \mathscr{C}^{\leq 1}$. Thus, since $X \in \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$, we see that $X^{\leq 0} \in \mathcal{H}$ and $X^{\geq 1} \in \mathcal{H}[-1]$, and we conclude that $\mathcal{H} * \mathcal{H}[-1] = \mathscr{C}^{\leq 1} \cap \mathscr{C}^{\geq 0}$.

An easy induction proves that for $i \geq j$ we have $\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j} = \mathcal{H}[-j] * \mathcal{H}[-(j+1)] * \cdots * \mathcal{H}[-i]$. Thus in conclusion

$$\mathscr{C} = \bigcup_{i,j \in \mathbb{Z}} (\mathscr{C}^{\leq i} \cap \mathscr{C}^{\geq j}) = \bigcup_{i \geq j} \mathcal{H}[-j] * \cdots * \mathcal{H}[-i]$$

□

The next example illustrates why we can view this as a generalization of the special case with the bounded derived category and canonical t-structure.

Example 5. Given the canonical t-structure on $D^b(\mathcal{A})$ we see that since $D^b(\mathcal{A})$ is bounded, we have $H^i(X^\bullet) = 0$ for small- and large enough i, and the t-structure is bounded. Thus, by the previous lemma, $D^b(\mathcal{A})$ is generated by the heart $\mathcal{A}$.

The next lemma shows that if we assume that t-structure are bounded, a t-exact functor can be an equivalence under certain restrictions.

Lemma 3.16. *Let $\mathscr{C}$ and $\mathscr{D}$ be triangulated categories with bounded t-structure, and let $(F, \eta) : \mathscr{C} \to \mathscr{D}$ be a triangulated functor. Let $\mathcal{H}$ be the heart of the t-structure on $\mathscr{C}$.*

(1) If $\mathrm{Hom}_{\mathscr{C}}(X, Y[n]) \cong \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)[n])$ for all $X, Y \in \mathcal{H}$ and all $n \in \mathbb{Z}$. Then F is fully faithful.
(2) If further $F(\mathcal{H})$ is equal to the heart of the t-structure on $\mathscr{D}$, then F is also dense.

Proof. First observe that since $\mathscr{C}$ has a bounded t-structure, we know $\mathscr{C} = \bigcup_{i \geq j} \mathcal{H}[-j] * \cdots * \mathcal{H}[-i]$. Given an object $A \in \mathcal{H}[-j] * \cdots * \mathcal{H}[-i]$ we define $l(A) = i - j$.

To prove the first part, we must show that there is an isomorphism $\mathrm{Hom}_{\mathscr{C}}(X,Y) \cong \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ for all $X,Y \in \mathscr{C}$. We do this by double induction on $l(X)$ and $l(Y)$.

If $l(X) = 0 = l(Y)$, we can assume $X \in \mathcal{H}[-i]$ and $Y \in \mathcal{H}[-j]$. Since the shift functor is an autoequivalence, $([1],[-1])$ forms an adjoint pair and we have isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \cong \mathrm{Hom}_{\mathscr{C}}(X,Y[i][-i]) \cong \mathrm{Hom}_{\mathscr{C}}(X[i],Y[i])$$

We observe that $X[i] \in \mathcal{H}$. We further observe that $Y[j][-j+i] \cong Y[i]$ with $Y[j] \in \mathcal{H}$ and from the assumption we have the isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X[i],Y[i]) \cong \mathrm{Hom}_{\mathscr{D}}(F(X)[i],F(Y[j])[-j+i]) \cong \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)[i][-i])$$

Now assume the assertion is true for all X with $l(X) = n$, and Y with $l(Y) = 0$. Given $X \in \mathcal{H}[-j] * \cdots * \mathcal{H}[-i]$, with $l(X) = j - i = n+1$ there is a triangle

$$H \to X \to \tilde{H} \to H[1]$$

with $H \in \mathcal{H}[-j] * \cdots * \mathcal{H}[-i-1]$ and $\tilde{H} \in \mathcal{H}[-i]$. Thus we have that $l(H), l(\tilde{H}), l(H[1]), l(\tilde{H}[-1]) \leq n$. Applying the cohomological functors $\mathrm{Hom}_{\mathscr{C}}(-,Y)$ and $\mathrm{Hom}_{\mathscr{D}}(-,F(Y))$ we get a diagram with exact rows (we write $\mathscr{C}(-,-)$ for $\mathrm{Hom}_{\mathscr{C}}(-,-)$)

$$\begin{array}{ccccccccc}
\mathscr{C}(\tilde{H}[-1],Y) & \longrightarrow & \mathscr{C}(H,Y) & \longrightarrow & \mathscr{C}(X,Y) & \longrightarrow & \mathscr{C}(\tilde{H},Y) & \longrightarrow & \mathscr{C}(H[1],Y) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathscr{D}(F(\tilde{H})[-1],F(Y)) & \to & \mathscr{D}(F(H),F(Y)) & \to & \mathscr{D}(F(X),F(Y)) & \to & \mathscr{D}(F(\tilde{H}),FY)) & \to & \mathscr{D}(F(H)[1],F(Y))
\end{array}$$

and we see from the induction hypothesis that by the five lemma the middle vertical row must be an isomorphism. To complete the double induction a dual argument is used on $l(Y)$.

To prove (2) we let $X \in \mathcal{D}$, and proceed by induction on $l(X)$. If $l(X) = 0$, then we can assume $X \in F(\mathcal{H})[-i] \cong F(\mathcal{H}[-i])$ and we are done. Now assume the assertion is true for $l(X) \leq n$, and let $Y \in F(\mathcal{H})[-j] * \cdots * F(\mathcal{H})[-i]$ with $l(Y) = i - j = n+1$. Then there exist a triangle

$$H \to Y \to \tilde{H} \to H[1]$$

with $H \in F(\mathcal{H})[-j] * \cdots * F(\mathcal{H})[-i-1]$ and $\tilde{H} \in F(\mathcal{H})[-i]$. We also observe that $l(\tilde{H}[-1]) \leq n$, and let $f : \tilde{H}[-1] \to H$. From the induction hypothesis there exist a triangle $X_1 \overset{\alpha}{\to} X_2 \to Z \to Z_1[1]$ where $F(X_1) \cong (\tilde{H}[-1])$, $F(X_2) \cong H$ and $F(\alpha) \cong f$. We then get a map of triangles

$$\begin{array}{ccccccc}
F(X_1) & \xrightarrow{F(\alpha)\cong f} & F(X_2) & \longrightarrow & F(Z) & \longrightarrow & F(X_1)[1] \\
\downarrow{\cong} & & \downarrow{\cong} & & \vdots & & \downarrow{\cong} \\
\tilde{H}[-1] & \xrightarrow{f} & H & \longrightarrow & Y & \longrightarrow & \tilde{H}
\end{array}$$

and we see that $F(Z) \cong Y$, and we conclude that F is dense. $\square$

Chapter 4

Realization functors

In our quest to construct derived equivalences to a derived category $D^b(\mathcal{A})$, we will look at the heart of t-structures, and study the relationship between the derived categories over the hearts and $D^b(\mathcal{A})$. It turns out that it is always possible to construct a t-exact functor from the bounded derived category over the heart of a t-structure back to $D^b(\mathcal{A})$, where the restriction to the heart becomes the identity functor. This construction is called a *realization functor*, and was first introduced in [BBD83]. We will first start by introducing filtered derived categories, and then show how we can construct a realization functor as a composition through the filtered derived category. Lastly we will give sufficient criteria for the realization functor to become an equivalence.

4.1 Filtered derived categories

Definition 12. Let $\mathcal{A}$ be an abelian category. The **category of finite filtered objects** $F(\mathcal{A})$ is the category of objects $X \in \mathcal{A}$ together with a finite filtration F. We denote an object in $F(\mathcal{A})$ by (X, F). Note that $F(\mathcal{A})$ is an additive category. The **bounded category of filtered chain complexes** $C^bF(\mathcal{A})$ is the category of bounded complexes of objects in $F(\mathcal{A})$. Pictorially a complex $(X^\bullet, F) \in C^bF(\mathcal{A})$ looks like

$$
\begin{array}{ccccccccccc}
\vdots & & & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
X^{-1}: & \cdots & \supseteq & & F_{-1}X^{-1} & \supseteq & F_0X^{-1} & \supseteq & F_1X^{-1} & \supseteq & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
X^{0}: & \cdots & \supseteq & & F_{-1}X^{0} & \supseteq & F_0X^{0} & \supseteq & F_1X^{0} & \supseteq & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
X^{1}: & \cdots & \supseteq & & F_{-1}X^{1} & \supseteq & F_0X^{1} & \supseteq & F_1X^{1} & \supseteq & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where the *finite* filtration implies that for every X^i there exist an $a \in \mathbb{Z}$ such that $F_nX^i = X^i$ for all $n \leq a$, and there exist a $b \in \mathbb{Z}$ such that $F_mX^i = 0$ for all $m \geq b$. Unless it is necessary to describe the filtration we will just denote an object in $C^bF(\mathcal{A})$ as $X^\bullet$. Note that $C^bF(\mathcal{A})$ is an additive category.

There are certain canonical maps that will be useful in the study of filtered derived categories.

Definition 13. For each $i \in \mathbb{Z}$ we define a functor

$$\mathrm{gr}^i : C^bF(\mathcal{A}) \to C^b(\mathcal{A})$$

by $\mathrm{gr}^i(X^\bullet) = F_iX^\bullet / F_{i+1}X^\bullet$, and a forgetful functor

$$\omega : C^bF(\mathcal{A}) \to C^b(\mathcal{A})$$

by $\omega(X^\bullet) = X^\bullet$, forgetting the filtration on $X^\bullet$.

Lemma 4.1. *Given an exact sequence*

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

in $C^bF(\mathcal{A})$. Then the sequences

$$0 \to \mathrm{gr}(X^\bullet) \to \mathrm{gr}(Y^\bullet) \to \mathrm{gr}(Z^\bullet) \to 0$$

$$0 \to \omega(X^\bullet) \to \omega(Y^\bullet) \to \omega(Z^\bullet) \to 0$$

are exact

Proof. ω is exact by definition. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_nX^\bullet & \longrightarrow & F_nY^\bullet & \longrightarrow & F_nZ^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{n+1}X^\bullet & \longrightarrow & F_{n+1}Y^\bullet & \longrightarrow & F_{n+1}Z^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \frac{F_nX^\bullet}{F_{n+1}X^\bullet} \cong \mathrm{gr}^n(X^\bullet) & & \frac{F_nZ^\bullet}{F_{n+1}Y^\bullet} \cong \mathrm{gr}^n(Y^\bullet) & & \frac{F_nZ^\bullet}{F_{n+1}Z^\bullet} \cong \mathrm{gr}^n(Z^\bullet) & &
\end{array}$$

with exact columns and rows. Thus by the 3×3 lemma the bottom row is exact. □

As the derived category is defined as the localization of the chain complex category with respect to quasi-isomorphism, we want some sort of analogous definition of a "quasi-isomorphism" in the filtered chain complex category. The following definition and lemma gives us a definition that makes sense.

Definition 14.
(1) We say a map $f : X^\bullet \to Y^\bullet$ in $C^bF(\mathcal{A})$ is a **filtered quasi-isomorphism** if $F_n(f) : F_n(X^\bullet) \to F_n(Y^\bullet)$ is a quasi-isomorphism for all $n \in \mathbb{Z}$.
(2) If $\mathrm{gr}^n(X^\bullet)$ is acyclic for all $n \in \mathbb{Z}$ we say $X^\bullet$ is **filtered acyclic**.

Lemma 4.2. *Let $f : X^\bullet \to Y^\bullet$ in $C^bF(\mathcal{A})$ The following are equivalent:*

(1) $F_n(f) : F_n(X^\bullet) \to F_n(Y^\bullet)$ *is a quasi-isomorphism for all* $n \in \mathbb{Z}$
(2) $\mathrm{gr}^n(f) : \mathrm{gr}^n(X^\bullet) \to \mathrm{gr}^n(Y^\bullet)$ *is a quasi-isomorphism for all* $n \in \mathbb{Z}$
(3) $\mathrm{Cone}(f)$ *is filtered acyclic*

Proof. (1) $\Rightarrow$ (2) Note that one can construct the commutative diagram with short exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{n+1}X^\bullet & \longrightarrow & F_nX^\bullet & \longrightarrow & \frac{F_nX^\bullet}{F_{n+1}X^\bullet} \cong \mathrm{gr}^n(X^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{n+1}Y^\bullet & \longrightarrow & F_nY^\bullet & \longrightarrow & \frac{F_nY^\bullet}{F_{n+1}Y^\bullet} \cong \mathrm{gr}^n(Y^\bullet) & \longrightarrow & 0
\end{array}$$

we then take homology and get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H^i(F_{n+1}X^\bullet) & \longrightarrow & H^i(F_nX^\bullet) & \longrightarrow & H^i(\mathrm{gr}^n(X^\bullet)) & \longrightarrow & H^{i+1}(F_{n+1}X^\bullet) & \longrightarrow & H^{i+1}(F_nX^\bullet) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
H^i(F_{n+1}Y^\bullet) & \longrightarrow & H^i(F_nY^\bullet) & \longrightarrow & H^i(\mathrm{gr}^n(Y^\bullet)) & \longrightarrow & H^{i+1}(F_{n+1}Y^\bullet) & \longrightarrow & H^{i+1}(F_nY^\bullet)
\end{array}$$

and from the five lemma we get that $\mathrm{gr}^n(f)$ is a quasi-isomorphism for all $n \in \mathbb{Z}$
(2) $\Leftrightarrow$ (3) We have a short exact sequence in $C^bF(\mathcal{A})$

$$0 \to Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1] \to 0$$

use the the gr^n functor to get an exact sequence

$$0 \to \mathrm{gr}^n(Y^\bullet) \to \mathrm{gr}^n(\mathrm{Cone}(f)) \to \mathrm{gr}^n(X^\bullet[1]) \to 0$$

Taking homology we get a long exact sequence

$$H^i(\mathrm{gr}^n(X^\bullet)) \longrightarrow H^i(\mathrm{gr}^n(Y^\bullet)) \longrightarrow H^i(\mathrm{gr}^n(\mathrm{Cone}(f))) \longrightarrow H^{i+1}(\mathrm{gr}^n(X^\bullet)) \longrightarrow H^{i+1}(\mathrm{gr}^n(Y^\bullet))$$

Thus we have that $\mathrm{gr}^n(\mathrm{Cone}(f))$ is acyclic, i.e. $\mathrm{Cone}(f)$ is filtered acyclic, if and only if $\mathrm{gr}^n(f)$ is a quasi-isomorphism for all $n \in \mathbb{Z}$.
(2) $\Rightarrow$ (1) Assume $\mathrm{gr}^n(f)$ is a quasi-isomorphism for all $n \in \mathbb{Z}$. Since $C^bF(\mathcal{A})$ has finite filtration, there exist a N such that $F_nX^\bullet = 0 = F_nY^\bullet$ for all $n > N$. Then we have $\mathrm{gr}^N(X^\bullet) = F_N(X^\bullet)$, $\mathrm{gr}^N(Y^\bullet) = F_N(Y^\bullet)$ and $H^i(F_NX^\bullet) \cong H^i(F_NY^\bullet)$. We then have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_NX^\bullet & \longrightarrow & F_{N-1}X^\bullet & \longrightarrow & \mathrm{gr}^{N-1}(X^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_NY^\bullet & \longrightarrow & F_{N-1}Y^\bullet & \longrightarrow & \mathrm{gr}^{N-1}(Y^\bullet) & \longrightarrow & 0 \end{array}$$

and as above we take homology to get a diagram

$$\begin{array}{ccccccccc} H^{i-1}(\mathrm{gr}^{N-1}(X^\bullet)) & \to & H^i(F_NX^\bullet) & \longrightarrow & H^i(F_{N-1}X^\bullet) & \longrightarrow & H^i(\mathrm{gr}^{N-1}(X^\bullet)) & \to & H^{i+1}(F_NX^\bullet) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ H^{i-1}(\mathrm{gr}^{N-1}(Y^\bullet)) & \to & H^i(F_NY^\bullet) & \to & H^i(F_{N-1}(Y^\bullet)) & \longrightarrow & H^i(\mathrm{gr}^{N-1}Y^\bullet) & \longrightarrow & H^{i+1}(F_NY^\bullet) \end{array}$$

and $F_{N-1}(f)$ is a quasi-isomorphism. Inducting this process we get that $F_{N-n}(f)$ is a quasi-isomorphism for all $n \in \mathbb{Z}$ □

Observation 4.2.1. Since $F(\mathcal{A})$ is an additive category, we can lift $C^bF(\mathcal{A})$ to the homotopy category, which will be aptly named **the bounded filtered homotopy category** $K^bF(\mathcal{A})$. Since the homotopy category is triangulated, and since gr^i and ω are additive functors they will be lifted to triangulated functors in the homotopy category [Opp16, Observation 33.7]

Lemma 4.3. *The class of filtered quasi-isomorphisms form a multiplicative system compatible with the triangulation on* $K^bF(\mathcal{A})$

Proof. Since gr^i preserves triangles the functor $H^n \circ gr^i$ is homological for all $n, i \in \mathbb{Z}$. Then from Lemma 4.2 we have that $H^n(\mathrm{gr}^i(q))$ is an isomorphism if and only if q is a filtered quasi-isomorphism. From [Kra07, Lemma 3.1] this is a sufficient condition for a multiplicative system compatible with the triangulation. □

Now we are ready to define the bounded filtered derived category analogously to the definition of the standard derived category

Definition 15. The **bounded filtered derived category**, $D^bF(\mathcal{A})$, is the localization of the bounded filtered homotopy category, $K^bF(\mathcal{A})$, with respect to the class of all filtered quasi-isomorphisms.
We will denote objects $X^\bullet$ in $D^bF(\mathcal{A})$ by a bullet-notation when it is necessary to remember the chain complex structure of the given object, otherwise we will just denote the object as X.

Observation 4.3.1. Observe that

$$K^bF(\mathcal{A}) \xrightarrow{\mathrm{gr}^i} K^b(\mathcal{A}) \xrightarrow{\textit{localization}} D^b(\mathcal{A})$$

sends every quasi-isomorphism to an isomorphism. Further, since $F^nX^\bullet$ eventually stabilize to $X^\bullet$ the composition

$$K^bF(\mathcal{A}) \xrightarrow{\omega} K^b(\mathcal{A}) \xrightarrow{\textit{localization}} D^b(\mathcal{A})$$

also sends every quasi-isomorphism to an isomorphism. From the definition of localization (see [Kra07]) each of the above compositions factors uniquely through two maps $D^bF(\mathcal{A}) \to D^b(\mathcal{A})$. By abuse of notation we denote the two unique maps as gr^i and ω. Since gr^i, ω and the localization functor are triangulated, the induced maps will also be triangulated.

The rest of this section is dedicated to introducing necessary tools to study the filtered derived categories.

Definition 16. Let $(X^\bullet, F)$ be an object in $C^bF(\mathcal{A})$ Define

$$a(X^\bullet) = \sup\{a \in \mathbb{Z} | F_aX^\bullet = X^\bullet\}$$

$$b(X^\bullet) = \inf\{b \in \mathbb{Z} | F_bX^\bullet = 0\}$$

We can then define the **length of the filtration** as.

$$l(X^\bullet) = |b(X^\bullet) - a(X^\bullet)|$$

If $X^\bullet = 0$ we say the length is undefined. We say $(X^\bullet, F)$ has **trivial filtration** if $l(X) = 1$. We can define **truncation on the filtration**, $(X^\bullet)_F^{\geq i}$ and $(X^\bullet)_F^{\leq i-1}$ where the filtration on $(X^\bullet)_F^{\geq i}$ is defined by

$$F_n((X^\bullet)_F^{\geq i}) = \begin{cases} F_i X^\bullet, & \text{for all } n \leq i \\ F_n X^\bullet, & \text{for all } n > i \end{cases}$$

We define the filtration on $(X^\bullet)_F^{\leq i-1}$ to be

$$F_n((X^\bullet)_F^{\leq i-1}) = F_n(X^\bullet)/F_n((X^\bullet)_F^{\geq i})$$

Observe that $a((X^\bullet)_F^{\geq i}) = i$, $b((X^\bullet)_F^{\geq i}) = b(X^\bullet)$ and $a((X^\bullet)_F^{\leq i-1}) = a(X^\bullet)$, $b((X^\bullet)_F^{\leq i-1}) = i$

Example 6. Let X be an object with filtration

$$\cdots = X = F_a X \supseteq F_{a+1} X \supseteq \cdots \supseteq F_0 X \supseteq \cdots \supseteq F_{b-1} X \supseteq F_b X = 0 = \cdots$$

such that $a(X) = a$ and $b(X) = b$ The filtration on $(X)_F^{\geq 1}$ is

$$\cdots = F_1 X = F_1 X \supseteq F_2 X \supseteq \cdots \supseteq F_{b-1} X \supseteq F_b X = 0 = \cdots$$

We see that $a((X)_F^{\geq 1}) = 1$ and $b((X)_F^{\geq 1}) = b(X) = b$ The filtration on $(X)_F^{\leq 0}$ is

$$\cdots = \frac{X}{F_1 X} = \frac{F_a X}{F_1 X} \supseteq \frac{F_{a+1} X}{F_1 X} \supseteq \cdots \supseteq \frac{F_0 X}{F_1 X} \supseteq \frac{F_1 X}{F_1 X} = 0 = \cdots$$

We see that $a((X)_F^{\leq 0}) = a(X) = a$ and $b((X)_F^{\leq 0}) = 1$ Given a complex $X^\bullet$ observe that in $D^b F(\mathcal{A})$ we get a triangle

$$(X^\bullet)_F^{\geq 1} \to X^\bullet \to (X^\bullet)_F^{\leq 0} \to (X^\bullet)_F^{\geq 1}[1]$$

Indeed for each $n \in \mathbb{Z}$ we get a triangle

$$(X^\bullet)_F^{\geq n} \to X^\bullet \to (X^\bullet)_F^{\leq n-1} \to (X^\bullet)_F^{\geq n}[1]$$

Lemma 4.4. *If* $\mathrm{Hom}_{D^b(\mathcal{A})}(\omega(X^\bullet), \omega(Y^\bullet)) = 0$ *then* $\mathrm{Hom}_{D^b F(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$

Proof. Let $f \in \mathrm{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet)$. Then since $\mathrm{Hom}_{D^b(\mathcal{A})}(\omega(X^\bullet), \omega(Y^\bullet)) = 0$ we will eventually have that $F_n f = 0$ for all $n \leq a$ for some $a \in \mathbb{Z}$. We have the map of triangles

$$\begin{array}{ccccccc}
F_{a+1}X^\bullet & \hookrightarrow & F_aX^\bullet & \longrightarrow & \mathrm{gr}^a(X^\bullet) & \longrightarrow & F_{a+1}X^\bullet[1] \\
\downarrow{\scriptstyle F_{a+1}f} & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\
F_{a+1}Y^\bullet & \hookrightarrow & F_aY^\bullet & \longrightarrow & \mathrm{gr}^a(Y^\bullet) & \longrightarrow & F_{a+1}Y^\bullet[1]
\end{array}$$

and we see that the commutativity of the diagram forces $F_{a+1}f = 0$. By induction we see that $F_nf = 0$ for all $n \in \mathbb{Z}$ and $\mathrm{Hom}_{D^bF(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$ □

Lemma 4.5. *Given $X^\bullet, Y^\bullet \in D^bF(\mathcal{A})$ with trivial filtration, if $a(Y^\bullet) < a(X^\bullet)$ then* $\mathrm{Hom}_{D^bF(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$.

Proof. Let $f : X^\bullet \to Y^\bullet$ be represented by the roof $X^\bullet \xleftarrow{\text{qis } q} Z^\bullet \xrightarrow{r} Y^\bullet$ where q is a filtered quasi-isomorphism. Let $a(Y^\bullet) = a$, then pictorially the map looks like

$$\begin{array}{ccccccccccc}
\cdots & = & X^\bullet & = & X^\bullet & = & X^\bullet & = & X^\bullet & = & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \supseteq & F_{a-1}Z^\bullet & \supseteq & F_aZ^\bullet & \supseteq & F_{a+1}Z^\bullet & \supseteq & F_{a+1}Z^\bullet & \supseteq & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & = & Y^\bullet & = & Y^\bullet & \supseteq & 0 & = & 0 & = & \cdots
\end{array}$$

Since q is a filtered quasi-isomorphism, F_nq is a quasi-isomorphism for all $n \in \mathbb{Z}$. We can then construct a $\widetilde{Z}^\bullet$ with filtration

$$F_n\widetilde{Z}^\bullet = \begin{cases} F_{a(X)}Z^\bullet & \text{for } n \leq a(X) \\ F_nZ^\bullet & \text{for } n > a(X) \end{cases}$$

such that we have a composite quasi-isomorphisms $\widetilde{Z}^\bullet \xrightarrow{s} Z^\bullet \xrightarrow{q} X^\bullet$. Then f is also represented by the roof $X^\bullet \xleftarrow{qs} \widetilde{Z}^\bullet \xrightarrow{rs} Y^\bullet$ with qs and rs being maps in $C^bF(\mathcal{A})$. Pictorially the maps looks like

$$\begin{array}{ccccccccccc}
\cdots & = & X^\bullet & = & X^\bullet & = & X^\bullet & = & X^\bullet & = & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & = & F_{a(X)}Z^\bullet & = & F_{a(X)}Z^\bullet & = & F_{a(X)}Z^\bullet & = & F_{a(X)}Z^\bullet & = & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & = & Y^\bullet & = & Y^\bullet & \supseteq & 0 & = & 0 & = & \cdots
\end{array}$$

and from the commutativity of the diagrams we see that the map rs must be zero. □

Lemma 4.6. *Let $\mathcal{T}$ be a triangulated category, and let*

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{a'} & Y' \end{array}$$

be a commutative diagram in $\mathcal{T}$. Then this can be extended to a 3×3 diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\ X' & \xrightarrow{a'} & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & Z''[1] \\ \downarrow & & \downarrow & & \downarrow & \ominus & \downarrow \\ X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2] \end{array}$$

where the bottom right square is anti-commutative (indicated by $\ominus$). Each row and column are distinguished triangles.

Proof. Complete each map in the square to the triangles

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{a'} & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & & & \\ X'' & & Y'' & & & & \\ \downarrow & & \downarrow & & & & \\ X[1] & & Y[1] & & & & \end{array}$$

also complete the map $X \xrightarrow{a'f=ga} Y'$ to the triangle

$$X \longrightarrow Y' \longrightarrow A \longrightarrow X[1]$$

Now from the octahedral axiom we get commutative diagrams and two new distinguished triangles:

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle a} & & \| \\
X & \longrightarrow & Y' & \longrightarrow & A & \longrightarrow & X[1] \\
 & & \downarrow & & \downarrow{\scriptstyle b} & & \downarrow \\
 & & Y'' & = & Y'' & \longrightarrow & Y[1] \\
 & & \downarrow & & \downarrow{\scriptstyle c} & & \\
 & & Y[1] & \longrightarrow & Z[1] & &
\end{array}
\qquad
\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & X'' & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle a'} & & \| \\
X & \longrightarrow & Y' & \longrightarrow & A & \longrightarrow & X[1] \\
 & & \downarrow & & \downarrow{\scriptstyle b'} & & \downarrow \\
 & & Z' & = & Z' & \longrightarrow & X'[1] \\
 & & \downarrow & & \downarrow{\scriptstyle c'} & & \\
 & & X'[1] & \longrightarrow & X''[1] & &
\end{array}
$$

Observe that if we combine the diagrams above we get commutative diagrams

$$
\begin{array}{ccccc}
X' & \longrightarrow & X'' & \longrightarrow & X[1] \\
\downarrow & & \downarrow{\scriptstyle a'} & & \| \\
Y' & \longrightarrow & A & \longrightarrow & X[1] \\
\downarrow & & \downarrow{\scriptstyle b} & & \downarrow \\
Y'' & = & Y'' & \longrightarrow & Y[1]
\end{array}
\qquad (4.1)
\qquad
\begin{array}{ccccc}
Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow{\scriptstyle a} & & \| \\
Y' & \longrightarrow & A & \longrightarrow & X[1] \\
\downarrow & & \downarrow{\scriptstyle b'} & & \downarrow \\
Z' & = & Z' & \longrightarrow & X'[1]
\end{array}
\qquad (4.2)
$$

Now we can complete the map $X'' \xrightarrow{ba'} Y''$ into the triangle

$$X'' \longrightarrow Y'' \longrightarrow Z'' \longrightarrow X''[1]$$

Again use the octahedral axiom on the new triangles to get the commutative diagram and the new triangle

$$
\begin{array}{ccccccc}
X'' & \longrightarrow & A & \longrightarrow & Z' & \longrightarrow & X''[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle h} & & \| \\
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
 & & \downarrow & & \downarrow{\scriptstyle h'} & & \downarrow \\
 & & Z[1] & = & Z[1] & \longrightarrow & A[1] \\
 & & \downarrow & & \downarrow{\scriptstyle -b'[1]a[1]} & & \\
 & & A[1] & \longrightarrow & Z'[1] & &
\end{array}
$$

Combining all the triangles we get a diagram

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & (1) & \downarrow{\scriptstyle b'a} & (2) & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\
\downarrow & (3) & \downarrow & & \downarrow{\scriptstyle h} & \overset{c'}{\searrow} & \downarrow \\
X'' & \xrightarrow{ba'} & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
\downarrow & (4) & \downarrow & \overset{c}{\searrow} & \downarrow{\scriptstyle h'} & & \downarrow \\
X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2]
\end{array}
$$

We see that the squares (1) and (2) commutes from diagram 4.2, and the squares (3) and (4) commutes from diagram (4.1). Since both $Y \to Z'$ and $Y' \to Y''$ factors through A, the middle square commutes. From the octahedral diagrams above we see that the map c' factorizes as both $Z' \to X'[1] \to X''[1]$ and $Z' \to Z'' \to X''[1]$, and the map c factorizes as both $Y'' \to Z'' \to Z[1]$ and $Y'' \to Y[1] \to Z[1]$. Thus the middle right and bottom squares commute. Observe that we have a map of triangles

$$
\begin{array}{ccccccc}
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle a'[1]} \\
A & \longrightarrow & Y'' & \longrightarrow & Z[1] & \xrightarrow{-a[1]} & A[1]
\end{array}
$$

From the commutativity of the last square and the fact that $X''[1] \to X[2]$ and $Z[1] \to X[2]$ both factor through $A[1]$ via the maps a and a' we conclude that the bottom right square in the diagram is anti-commutative. □

4.2 Filtered derived t-structure

As we have seen, the filtered derived category is closely connected to the derived category. A natural question to ask is whether a t-structure on the derived category induces a t-structure on the filtered derived category, and if so, is there a relation between the abelian hearts of the t-structures? As we shall see, the answer is all that we could hope for.

Definition 17. Let $D^b(\mathcal{A})$ be the bounded derived category with t-structure $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$, and $D^bF(\mathcal{A})$ the filtered derived category. We define two subcategories of $D^bF(\mathcal{A})$ by

$$D^bF(\mathcal{A})^{\leq 0} = \{X \in D^bF(\mathcal{A}) | \operatorname{gr}^n(X) \in D^b(\mathcal{A})^{\leq n} \textit{ for all } n \in \mathbb{Z}\}$$

$$D^bF(\mathcal{A})^{\geq 0} = \{X \in D^bF(\mathcal{A}) | \operatorname{gr}^n(X) \in D^b(\mathcal{A})^{\geq n} \textit{ for all } n \in \mathbb{Z}\}$$

We can further define the subcategories

$$D^bF(\mathcal{A})^{\leq i} := D^bF(\mathcal{A})^{\leq 0}[-i] = \{X \in D^bF(\mathcal{A}) |\, \mathrm{gr}^n(X) \in D^b(\mathcal{A})^{\leq n+i} \text{ for all } n \in \mathbb{Z}\}$$

$$D^bF(\mathcal{A})^{\geq i} := D^bF(\mathcal{A})^{\geq 0}[-i] = \{X \in D^bF(\mathcal{A}) |\, \mathrm{gr}^n(X) \in D^b(\mathcal{A})^{\geq n+i} \text{ for all } n \in \mathbb{Z}\}$$

Proposition 4.1. *Let $D^b(\mathcal{A})$ be the bounded derived category with t-structure $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$, and $D^bF(\mathcal{A})$ the filtered derived category. Then $(D^bF(\mathcal{A})^{\leq 0}, D^bF(\mathcal{A})^{\geq 0})$ defines a t-structure on $D^bF(\mathcal{A})$.*

The proof of the proposition is divided into the following three lemmas, proving each of the three axioms of t-structures

Lemma 4.7. *Let $D^bF(\mathcal{A})^{\leq 0}$ and $D^bF(\mathcal{A})^{\geq 1}$ be defined as above, and let $X \in D^bF(\mathcal{A})^{\leq 0}$ and $Y \in D^bF(\mathcal{A})^{\geq 1}$. Then* $\mathrm{Hom}_{D^bF(\mathcal{A})}(X, Y) = 0$

Proof. Let $f \in \mathrm{Hom}_{D^bF(\mathcal{A})}(X, Y)$. Assume X has filtration

$$\cdots = X = F_aX \supseteq F_{a+1}X \supseteq \cdots \supseteq F_{b-1}X \supseteq F_bX = 0 = \cdots$$

We truncate on the filtration to get a triangle

$$(X)_F^{\geq b} \to X \to (X)_F^{\leq b-1} \to (X)_F^{\geq b}[1]$$

where $l((X)_F^{\geq b}) = 1$ and $l((X)_F^{\leq b-1}) = l(X) - 1$. Observe that $(X)_F^{\geq b}, (X)_F^{\leq b-1} \in D^bF(\mathcal{A})^{\leq 0}$. Using $\mathrm{Hom}_{D^bF(\mathcal{A})}(-, Y)$ we get a long exact sequence

$$\cdots \to \mathrm{Hom}((X)_F^{\leq b-1}, Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}((X)_F^{\geq b}, Y) \to \cdots$$

and we get $\mathrm{Hom}(X, Y) = 0$ provided that

$$\mathrm{Hom}((X)_F^{\geq b}, Y) = 0 = \mathrm{Hom}((X)_F^{\leq b-1}, Y)$$

Truncating again on $(X)_F^{\leq n-1}$ we can by induction on the length of the filtration show that $\mathrm{Hom}(X, Y) = 0$ provided that $\mathrm{Hom}(\tilde{X}, Y) = 0$ for every $\tilde{X} \in D^bF(\mathcal{A})^{\leq 0}$ where $\tilde{X}$ have trivial filtration. A similar induction argument on Y shows that we can reduce the problem to showing $\mathrm{Hom}(\tilde{X}, \tilde{Y}) = 0$ for all $\tilde{Y} \in D^bF(\mathcal{A})^{\geq 1}$, $\tilde{X} \in D^bF(\mathcal{A})^{\leq 0}$ where $\tilde{Y}$ and $\tilde{X}$ has trivial filtration.
Without loss of generality, we can assume that $a(\tilde{X}) = 0$. Then

$$\mathrm{gr}^n(\tilde{X}) = \begin{cases} \tilde{X}, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \qquad \mathrm{gr}^n(\tilde{Y}) = \begin{cases} \tilde{Y}, & \text{if } n = a(\tilde{Y}) \\ 0, & \text{if } n \neq a(\tilde{Y}) \end{cases}$$

Now we have two cases

- If $a(\tilde{Y}) \geq 0$ then $\mathrm{Hom}_{D^bF(\mathcal{A})}(\tilde{X}, \tilde{Y}) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(\mathrm{gr}^0(\tilde{X}), \mathrm{gr}^{a(\tilde{Y})}(\tilde{Y})) = 0$ since $\mathrm{gr}^0(\tilde{X}) \in D^b(\mathcal{A})^{\leq 0}$ and $\mathrm{gr}^{a(\tilde{Y})}(\tilde{Y}) \in D^b(\mathcal{A})^{\geq a(\tilde{Y})} \subseteq D^b(\mathcal{A})^{\geq 1}$
- If $a(\tilde{Y}) < 0$ we have from Lemma 4.5 that $\mathrm{Hom}_{D^bF(\mathcal{A})}(\tilde{X}, \tilde{Y}) = 0$

We conclude that $f = 0$, and therefore $\mathrm{Hom}_{D^bF(\mathcal{A})}(X, Y) = 0$ □

Lemma 4.8. *Let $D^bF(\mathcal{A})$ and $D^b(\mathcal{A})$ be defined as above. Then $D^bF(\mathcal{A})^{\leq 0} \subseteq D^bF(\mathcal{A})^{\leq 1}$ and $D^bF(\mathcal{A})^{\geq 1} \subseteq D^bF(\mathcal{A})^{\geq 0}$*

Proof. If $X \in D^bF(\mathcal{A})^{\leq 0}$ then, for all $n \in \mathbb{Z}$, $gr^n(X) \in D^b(\mathcal{A})^{\leq n} \subseteq D^b(\mathcal{A})^{\leq n+1}$ and $X \in D^bF(\mathcal{A})^{\leq 1}$. Thus $D^bF(\mathcal{A})^{\leq 0} \subseteq D^bF(\mathcal{A})^{\leq 1}$. Similarly one shows $D^b(\mathcal{A})^{\geq 1} \subseteq D^b(\mathcal{A})^{\geq 0}$ □

Lemma 4.9. *Let $D^bF(\mathcal{A})$ and $D^b(\mathcal{A})$ be defined as above. Then given any $X \in D^bF(\mathcal{A})$ there exist a triangle*

$$X' \to X \to X'' \to X'[1]$$

with $X' \in D^bF(\mathcal{A})^{\leq 0}$ and $X'' \in D^bF(\mathcal{A})^{\geq 1}$

Proof. We prove the statement by induction on $l(X)$.
If $l(X) = 1$ we can without loss of generality assume $a(X) = 0$, $b(X) = 1$. Given $n \leq 0$, for each $F_nX = X$ we can find a triangle in $D^b(\mathcal{A})$

$$(X)^{\leq 0} \to X \to (X)^{\geq 1} \to (X)^{\leq 0}[1]$$

Now equip $(X)^{\leq 0}$ with the trivial filtration and $a((X)^{\leq 0}) = 0$, $b((X)^{\leq 0}) = 1$ and similar for $(X)^{\geq 1}$. Then the triangle above becomes a triangle in $D^bF(\mathcal{A})$ and we have for all $n \in \mathbb{Z}$

$$\mathrm{gr}^n((X)^{\leq 0}) = \begin{cases} (X)^{\leq 0}, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases} \qquad \mathrm{gr}^n((X)^{\geq 1}) = \begin{cases} (X)^{\geq 1}, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases}$$

Thus $(X)^{\leq 0} \in D^bF(\mathcal{A})^{\leq 0}$ and $(X)^{\geq 1} \in D^bF(\mathcal{A})^{\geq 1}$. We also see that $\omega((X)^{\leq 0}) \in D^b(\mathcal{A})^{\leq 0} = D^b(\mathcal{A})^{\leq b(X)-1}$ and $\omega((X)^{\geq 1}) \in D^b(\mathcal{A})^{\geq 1} = D^b(\mathcal{A})^{\geq a(X)+1}$.

Now assume for all X such that $l(X) \leq n$ we can find triangles

$$(X)^{\leq 0} \to X \to (X)^{\geq 1} \to (X)^{\leq 0}[1]$$

in $D^bF(\mathcal{A})$ where $\omega((X)^{\leq 0}) \in D^b(\mathcal{A})^{\leq b(X)-1}$ and $\omega((X)^{\geq 1}) \in D^b(\mathcal{A})^{\geq a(X)+1}$. Let X be such that $l(X) = n + 1$. Without loss of generality we can assume $a(X) < 0$ and $b(X) > 0$. By using the truncation on filtration there is a triangle

$$(X)_F^{\geq 1} \to X \to (X)_F^{\leq 0} \to (X)_F^{\geq 1}[1]$$

where $l((X)_F^{\geq 1}), l((X)_F^{\leq 0}) \leq n$. Note that $b((X)_F^{\leq 0}) = 1 = a((X)_F^{\geq 1})$. We can then, from the induction hypothesis, find the diagram

$$\begin{array}{ccccccc}
((X)_F^{\leq 0})^{\leq 0}[-1] & \longrightarrow & (X)_F^{\leq 0}[-1] & \longrightarrow & ((X)_F^{\leq 0})^{\geq 1}[-1] & \longrightarrow & ((X)_F^{\leq 0})^{\leq 0} \\
 & & \big\downarrow & & & & \\
((X)_F^{\geq 1})^{\leq 0} & \longrightarrow & (X)_F^{\geq 1} & \longrightarrow & ((X)_F^{\geq 1})^{\geq 1} & \longrightarrow & ((X)_F^{\geq 1})^{\leq 0}[1]
\end{array}$$

where the rows are triangles. By the assumption, and from the fact that ω is triangulated, we have that

$$\omega(((X)_F^{\leq 0})^{\leq 0}) \in D^b(\mathcal{A})^{\leq b((X)_F^{\leq 0})-1} = D^b(\mathcal{A})^{\leq 0}$$

$$\omega(((X)_F^{\leq 0})^{\leq 0}[-1]) \cong \omega(((X)_F^{\leq 0})^{\leq 0})[-1] \in D^b(\mathcal{A})^{\leq b((X)_F^{\leq 0})-1}[-1] = D^b(\mathcal{A})^{\leq 1}$$

$$\omega(((X)_F^{\geq 1})^{\geq 1}) \in D^b(\mathcal{A})^{\geq a((X)_F^{\geq 1})+1} = D^b(\mathcal{A})^{\geq 2}$$

From Lemma 4.4 we then have that

$$\mathrm{Hom}_{D^bF(\mathcal{A})}\left(((X)_F^{\leq 0})^{\leq 0}[-1], ((X)_F^{\geq 1})^{\geq 1}\right) = 0$$

$$\mathrm{Hom}_{D^bF(\mathcal{A})}\left(((X)_F^{\leq 0})^{\leq 0}, ((X)_F^{\geq 1})^{\geq 1}\right) = 0$$

Thus, from Proposition 3.1 there exist a unique map $((X)_F^{\leq 0})^{\leq 0}[-1] \longrightarrow ((X)_F^{\geq 1})^{\leq 0}$ such that the induced left square in the diagram above commutes. Then from Lemma 4.6 we can find a unique commutative diagram where the rows and columns are triangles

$$\begin{array}{ccccccc}
((X)_F^{\leq 0})^{\leq 0}[-1] & \longrightarrow & ((X)_F^{\geq 1})^{\leq 0} & \longrightarrow & A & \longrightarrow & ((X)_F^{\leq 0})^{\leq 0} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
(X)_F^{\leq 0}[-1] & \longrightarrow & (X)_F^{\geq 1} & \longrightarrow & X & \longrightarrow & (X)_F^{\leq 0} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
((X)_F^{\leq 0})^{\geq 1}[-1] & \longrightarrow & ((X)_F^{\geq 1})^{\geq 1} & \longrightarrow & B & \longrightarrow & ((X)_F^{\leq 0})^{\geq 1} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & \ominus & \big\downarrow \\
((X)_F^{\leq 0})^{\leq 0} & \longrightarrow & ((X)_F^{\geq 1})^{\leq 0} & \longrightarrow & A[1] & \longrightarrow & ((X)_F^{\leq 0})^{\leq 0}[1]
\end{array}$$

We claim that $A = (X)^{\leq 0}$ and $B = (X)^{\geq 1}$. Since gr^n is a triangulated functor we see that gr^n applied to each row is a triangle in $D^b(\mathcal{A})$. Since t-structures are closed under extensions by Lemma 3.2, we see that $\mathrm{gr}^n(A) \in D^b(\mathcal{A})^{\leq 0}$ for all $n \in \mathbb{Z}$ and therefore $A \in D^bF(\mathcal{A})^{\leq 0}$. Similarly $B \in D^bF(\mathcal{A})^{\geq 1}$. What is left to show is that $\omega(A) \in D^b(\mathcal{A})^{\leq b(X)-1}$ and $\omega(B) \in D^b(\mathcal{A})^{\geq a(X)+1}$.

We have $\omega((X)_F^{\geq 1})^{\leq 0}) \in D^b(\mathcal{A})^{\leq b((X)_F^{\geq 1})-1} = D^b(\mathcal{A})^{\leq b(X)-1}$ and $\omega((X)_F^{\leq 0})^{\leq 0} \in D^b(\mathcal{A})^{\leq b((X)_F^{\leq 0}-1} = D^b(\mathcal{A})^{\leq 1-1} \subseteq D^b(\mathcal{A})^{\leq b(X)-1}$ where the last inclusion comes from the fact that $b(X) > 0$ and the second axiom of t-structures.

Thus, since ω is a triangulated functor, $\omega(A) \in D^b(\mathcal{A})^{\leq b(X)-1}$. Similarly one shows the statement about $\omega(B)$. □

The next proposition shows the relationship between the hearts of the two t-structures.

Proposition 4.2. *Let $D^b(\mathcal{A})$ and $D^bF(\mathcal{A})$ be equipped with t-structures as above. Denote by $\mathcal{H}$ and $\mathcal{H}_F$ the respective hearts. Then there is an equivalence of categories*

$$E : \mathcal{H}_F \xrightarrow{\cong} C^b(\mathcal{H})$$

Proof. We construct E as follows: Let $E(X)^\bullet$ be the complex defined by $(E(X))^n = \mathrm{gr}^n(X)[n] \in \mathcal{H}$. Given the two triangles

$$F_{n+1}X \to F_n \to \mathrm{gr}^n(X) \to F_{n+1}X[1]$$

$$F_{n+2}X \to F_{n+1} \to \mathrm{gr}^{n+1}(X) \to F_{n+2}X[1]$$

we define the differential d^n to be the composition

$$\mathrm{gr}^n X[n] \to F_{n+1}X[n+1] \to \mathrm{gr}^{n+1}[n+1]$$

Observe that $d^{n+1} \circ d^n$ is the composition.

$$\mathrm{gr}^n X[n] \to F_{n+1}X[n+1] \to \mathrm{gr}^{n+1}[n+1] \to F_{n+2}[n+2] \to \mathrm{gr}^{n+2}[n+2]$$

and since the two middle map are two consecutive maps in a triangle, the composition is equal to the zero map. Thus $d^{n+1} \circ d^n = 0$ and we have indeed a chain complex over $\mathcal{H}$

$$E(X)^\bullet = \cdots \longrightarrow \mathrm{gr}^{n-1}(X)[n-1] \xrightarrow{d^{n-1}} \mathrm{gr}^n(X)[n] \xrightarrow{d^n} \mathrm{gr}^{n+1}(X)[n+1] \longrightarrow \cdots$$

Since X has finite filtration $\mathrm{gr}^n = 0$ for $n < a(X)$ and $n \geq b(X)$, and $E(X)^\bullet$ is a bounded complex. We have to check E is fully faithful and dense:

(faithful) If we can show that E is an exact functor, and reflects 0-objects, then we can use Lemma A.4 to show that E is faithful.
Consider a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{H}_F$. From Corollary 3.2.1 this gives rise to a triangle $X \to Y \to Z \to X[1]$ in $D^bF(\mathcal{A})$. Since gr^n is a triangulated functor we have a triangle

$$\mathrm{gr}^n(X)[n] \to \mathrm{gr}^n(Y)[n] \to \mathrm{gr}^n(Z)[n] \to \mathrm{gr}^n(X)[n+1]$$

in $D^b(\mathcal{A})$ for all $n \in \mathbb{Z}$. Again from Corollary 3.2.1 we have that

$$0 \to \operatorname{gr}^n(X)[n] \to \operatorname{gr}^n(Y)[n] \to \operatorname{gr}^n(Z)[n] \to 0$$

is a short exact sequence in $\mathcal{H}$ for all $n \in \mathbb{Z}$. We conclude that $0 \to E(X) \to E(Y) \to E(Z) \to 0$ is a short exact sequence, and E is exact.

Let X be such that $E(X)^\bullet = 0$, i.e. $\operatorname{gr}^n(X) = 0$ for all $n \in \mathbb{Z}$. Assume $X \neq 0$, and look at $0 = \operatorname{gr}^{b(X)-1}(X) = \frac{F_{b(X)-1}X}{F_{b(X)}X} = \frac{F_{b(X)-1}X}{0} = F_{b(X)-1}X$ which contradicts the definition of $b(X)$. Thus we conclude that $X = 0$. We then have from Lemma A.4 that E is a faithful functor.

(full) Let $X, Y \in \mathcal{H}_F$, and $\varphi^\bullet \in \operatorname{Hom}_{C^b(\mathcal{H})}(E(X)^\bullet, E(Y^\bullet))$. Visually $\varphi^\bullet$ looks like

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \operatorname{gr}^{n-1}(X)[n-1] & \longrightarrow & \operatorname{gr}^n(X)[n] & \longrightarrow & \operatorname{gr}^{n+1}(X)[n+1] & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \varphi^{n-1}} & & \big\downarrow{\scriptstyle \varphi^{n}} & & \big\downarrow{\scriptstyle \varphi^{n+1}} & & \\
\cdots & \longrightarrow & \operatorname{gr}^{n-1}(Y)[n-1] & \longrightarrow & \operatorname{gr}^n(Y)[n] & \longrightarrow & \operatorname{gr}^{n+1}(Y)[n+1] & \longrightarrow & \cdots
\end{array}$$

Now we want to construct a map ψ such that $E(\psi)^\bullet = \varphi^\bullet$. We do this by constructing $F_n\psi$ for smaller and smaller n. Let $b = \max\{b(X), b(Y)\}$, and define $F_n\psi = 0$ for $n \geq b$. We then have a commutative diagram of triangles

$$\begin{array}{ccccccc}
F_bX & \longrightarrow & F_{b-1}X & \longrightarrow & \operatorname{gr}^{b-1}(X) & \xrightarrow{0} & F_bX[1] \\
\big\downarrow{\scriptstyle 0} & & \vdots & & \big\downarrow{\scriptstyle \varphi^{b-1}[-b+1]} & & \big\downarrow{\scriptstyle 0} \\
F_bY & \longrightarrow & F_{b-1}Y & \longrightarrow & \operatorname{gr}^{b-1}(Y) & \xrightarrow{0} & F_bY[1]
\end{array}$$

And we complete this to a map of triangles and define the dashed arrow as $F_{b-1}\psi$. Using this map we get a new map of triangles

$$\begin{array}{ccccccc}
F_{b-1}X & \longrightarrow & F_{b-2}X & \longrightarrow & \operatorname{gr}^{b-2}(X) & \longrightarrow & F_{b-1}X[1] \\
\big\downarrow{\scriptstyle F_{b-1}\psi} & & \vdots & & \big\downarrow{\scriptstyle \varphi^{b-2}[-b+2]} & & \big\downarrow{\scriptstyle F_{b-1}\psi[1]} \\
F_{b-1}Y & \longrightarrow & F_{b-2}Y & \longrightarrow & \operatorname{gr}^{b-2}(Y) & \longrightarrow & F_{b-1}Y[1]
\end{array}$$

and we define the dashed arrow to be $F_{b-2}\psi$. Inducting this process we get maps $F_n\psi$ for all $n \in \mathbb{Z}$ that forms commutative diagrams

$$\begin{array}{ccc}
F_nX & \longrightarrow & F_{n-1}X \\
\big\downarrow{\scriptstyle F_n\psi} & & \big\downarrow{\scriptstyle F_{n-1}\psi} \\
F_nY & \longrightarrow & F_{n-1}Y
\end{array}$$

By gluing these maps we get a map $\psi : X \to Y$ such that $E(\psi)^\bullet = \varphi^\bullet$ and E is full.

(dense) Given a complex $X^\bullet \in C^b(\mathcal{H})$ we need to find an object $\tilde{X} \in \mathcal{H}_F$ such that $E(\tilde{X}) \cong X^\bullet$. Let $X^\bullet$ be a bounded chain complex in $C^b(\mathcal{H})$. Without loss of generality we can assume $X^n = 0$ for all $n < 0$. $X^\bullet$ is then of the form

$$\cdots \to 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{b-1}} X^b \to 0 \to \cdots$$

We induct on the brutal truncation, ${}_{\sigma_{\leq n}}(X^\bullet)$ (See Theorem 3.11).

- We let $\tilde{X}_0^\bullet$ be the complex with X^0 in degree 0 equipped with trivial filtration such that $a(\tilde{X}_0^\bullet) = 0$.

  Then we have $\mathrm{gr}^n(\tilde{X}_0^\bullet) = \begin{cases} X^0, & n = 0 \\ 0, & n \neq 0 \end{cases}$. Since $X^0 \in \mathcal{H}$, we see that $\tilde{X}_0^\bullet \in \mathcal{H}_F$, and

  $$E(X^\bullet) = [\cdots \to 0 \to X^0 \to 0 \to \cdots] = {}_{\sigma_{\leq 0}}(X^\bullet)$$

- Let X_F^1 be the complex with X^1 in degree 0 equipped with trivial filtration such that $a(X_F^1) = 1$. Then, by abuse of notation, there is a map $d^0 : \tilde{X}_0^\bullet \to X_F^1$ where

  $$F_n(d^0) = \begin{cases} d^0, & n \leq 0 \\ 0, & n > 0 \end{cases}$$

  .

  We can then complete this map to a triangle in $D^bF(\mathcal{A})$

  $$\tilde{X}_0^\bullet \xrightarrow{d^0} X_F^1 \to A \to \tilde{X}_0^\bullet[1]$$

  define $\tilde{X}_1^\bullet := A[-1]$. Note that $X_F^1[-1], \tilde{X}_0^\bullet \in \mathcal{H}_F$ so $\tilde{X}_1^\bullet \in \mathcal{H}_F$. Now we have the following observations:

  - $\mathrm{gr}^0(X_F^1) = 0$ so $\mathrm{gr}^0(\tilde{X}_0^\bullet) \cong \mathrm{gr}^0(\tilde{X}_1^\bullet) = X^0$
  - $\mathrm{gr}^1(\tilde{X}_0^\bullet) = 0$ so $\mathrm{gr}^1(\tilde{X}_1^\bullet) \cong \mathrm{gr}^1(X_F^1)[-1] = X^1[-1]$
  - $F_1\tilde{X}_0^\bullet = 0$ so $F_1\tilde{X}_1^\bullet \cong F_1(X_F^1)[-1] = X^1[-1]$
  - $F_n\tilde{X}_0^\bullet = 0 = F_nX_F^1[-1]$ for all $n \geq 2$, so $F_n\tilde{X}_1^\bullet = 0$ for all $n \geq 2$

  We have two triangles

  $$F_2\tilde{X}_1^\bullet \to F_1\tilde{X}_1^\bullet \xrightarrow{\beta} \mathrm{gr}^1(\tilde{X}_1^\bullet) \to F_2\tilde{X}_1^\bullet[1]$$
  $$F_1\tilde{X}_1^\bullet \to F_0\tilde{X}_1^\bullet \to \mathrm{gr}^0(\tilde{X}_1^\bullet) \xrightarrow{\alpha} F_1\tilde{X}_1^\bullet[1]$$

  and by definition $d^0_{E(\tilde{X}_1^\bullet)} := \beta[1] \circ \alpha$. Since $F_2\tilde{X}_1^\bullet = 0$ we have $\beta[1] \cong \mathrm{id}_{X_F^1}$. From the map $\tilde{X}_1^\bullet \to \tilde{X}_0^\bullet$ we get a map of triangles

$$\begin{array}{ccccccc}
F_1\tilde{X}_1^\bullet & \longrightarrow & F_0\tilde{X}_1^\bullet & \longrightarrow & \mathrm{gr}^0(\tilde{X}_1^\bullet) & \longrightarrow & F_1\tilde{X}_1^\bullet[1] \\
\downarrow & & \downarrow & & f\downarrow\cong & & \downarrow \\
F_1\tilde{X}_0^\bullet & \longrightarrow & F_0\tilde{X}_0^\bullet & \xrightarrow[g]{\cong} & \mathrm{gr}^0(\tilde{X}_0^\bullet) & \longrightarrow & F_1\tilde{X}_0^\bullet[1]
\end{array}$$

Since, in particular, the middle square commutes we get a commutative diagram

$$\begin{array}{ccccccc}
F_0\tilde{X}_1^\bullet & \longrightarrow & F_0\tilde{X}_0^\bullet, & \xrightarrow{d^0} & F_0X_F^1 & \longrightarrow & F_0\tilde{X}_1^\bullet[1] \\
\| & & g\circ f^{-1}\downarrow\cong & & & & \| \\
F_0\tilde{X}_1^\bullet & \longrightarrow & \mathrm{gr}^0(\tilde{X}_1^\bullet) & \xrightarrow{\alpha} & F_1\tilde{X}_1^\bullet[1] & \longrightarrow & F_0\tilde{X}_1^\bullet[1]
\end{array}$$

which we can complete to a map of triangles, and in particular we see that $\alpha \cong d^0$. Thus

$$E(\tilde{X}_1^\bullet) \cong [\cdots \to 0 \to X^0 \xrightarrow{d^0} X^1 \to 0 \to \cdots] =_{\sigma_{\leq 0}} (X^\bullet)$$

- ∘ Now assume we can find $\tilde{X}_n^\bullet$ such that $E(\tilde{X}_n^\bullet) = {}_{\sigma_{\leq n}}(X^\bullet)$ then we have the following observations
  - $\mathrm{gr}^i(\tilde{X}_n^\bullet) = \begin{cases} X^i[-i], & \text{for } 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$
  - $F_n\tilde{X}_n^\bullet = X_n[-n]$
  - $F_{n+i}\tilde{X}_n^\bullet = 0$ for all $i > 0$

  Let Y_F^{n+1} be the complex with X^{n+1} in degree 0 equipped with trivial filtration such that $a(Y_F^{n+1}) = n$. Now given $E(\tilde{X}_n^\bullet)$ and $E(Y_F^{n+1}[-n]))$ there exist a map

$$\begin{array}{ccccccccc}
E(\tilde{X}_n^\bullet) : \cdots & \xrightarrow{d^{n-2}} & X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow d^n & & \downarrow & & \\
E(Y_F^{n+1}[-n])) : \cdots & \longrightarrow & 0 & \longrightarrow & X^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

  From the construction in the proof of E being full, there exist a map $\varphi : \tilde{X}_n^\bullet \to Y_F^{n+1}[-n]$ such that $F_n\varphi = d^n[-n]$. Now let $X_F^{n+1}[-n]$ be the complex with $X^{n+1}[-n]$ in degree 0 equipped with trivial filtration such that $a(X_F^{n+1}[-n]) = n+1$ then there exist a map $\iota : Y_F^{n+1}[-n] \to X_F^{n+1}[-n]$ where

$$F_i(\iota) = \begin{cases} \mathrm{id}_{X^{n+1}[-n]}, & i \leq n \\ 0, & i > 0 \end{cases}$$

By abuse of notation define $d^n[-n] := \iota \circ \varphi : \tilde{X}^\bullet_n \to X^{n+1}_F[-n]$, and observe that in particular $F_n(d^n[-n]) = d^n[-n]$. We complete this map to a triangle

$$\tilde{X}^\bullet_n \xrightarrow{d^n[-n]} X^{n+1}_F[-n] \to A \to \tilde{X}^\bullet_n[1]$$

and define $\tilde{X}^\bullet_{n+1} = A[-1]$. Observe that since $\tilde{X}^\bullet_n, X^{n+1}_F[-n-1] \in \mathcal{H}_F$ we have that $\tilde{X}^\bullet_{n+1} \in \mathcal{H}_F$. By a similar argument as before we conclude that $d^n_{E(\tilde{X}^\bullet_{n+1})}[-n] \cong d^n[-n]$, and $E(\tilde{X}^\bullet_{n+1}) \cong {}_{\sigma_{\leq n+1}}(X^\bullet)$. Thus given our original complex $X^\bullet$ we see that $E(\tilde{X}^\bullet_b) \cong {}_{\sigma_{\leq b}}(X^\bullet) \cong X^\bullet$ and E is dense.

□

4.3 The realization functor

We are now ready for the main results of the thesis. We have defined the tools to construct the realization functor, and to prove when this functor becomes an equivalence.

Theorem 4.10. *Let $\mathcal{A}$ be an abelian category, and let $D^b(\mathcal{A})$ be the bounded derived category equipped with t-structure $(D^b(\mathcal{A})^{\leq 0}, D^b(\mathcal{A})^{\geq 0})$ with heart $\mathcal{H}$. There exist a t-exact functor*

$$\mathrm{real} : D^b(\mathcal{H}) \longrightarrow D^b(\mathcal{A})$$

such that $\mathrm{real}|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$

Proof. Let E be the equivalence from the lemma above, then we have the composition

$$\widetilde{\mathrm{real}} : C^b(\mathcal{H}) \xrightarrow{E^{-1}} \mathcal{H}_F \hookrightarrow D^bF(\mathcal{A}) \xrightarrow{\omega} D^b(\mathcal{A})$$

Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism in $C^b(\mathcal{H})$. Then we have a short exact sequence

$$0 \to B^\bullet \to \mathrm{Cone}(f^\bullet) \to A^\bullet[1] \to 0$$

where $\mathrm{Cone}(f^\bullet)$ is acyclic. From Corollary 3.2.1, that E^{-1} is exact, and the fact that ω is triangulated, the short exact sequence will be sent to the triangle

$$\widetilde{\mathrm{real}}(A^\bullet) \to \widetilde{\mathrm{real}}(B^\bullet) \to \widetilde{\mathrm{real}}(\mathrm{Cone}(f^\bullet)) \to \widetilde{\mathrm{real}}(A^\bullet)[1]$$

Further, given the cohomological functor $H^n_{\mathcal{H}}$ from Theorem 3.10, we see that $H^n_{\mathcal{H}} \circ \widetilde{\text{real}}$ is equal to the canonical homology functor. In particular

$$H^n_{\mathcal{H}}(\widetilde{\text{real}}(\text{Cone}(f^\bullet))) \cong H^n(\text{Cone}(f^\bullet)) = 0$$

for all $n \in \mathbb{Z}$. Thus every quasi-isomorphism is sent to an isomorphism. And from the definition of localization we get a commutative diagram

$$\begin{array}{ccc} C^b(\mathcal{H}) & \xrightarrow{\quad\widetilde{\text{real}}\quad} & D^b(\mathcal{A}) \\ & {\scriptstyle \textit{localization}} \searrow \qquad \nearrow {\scriptstyle \exists !} & \\ & D^b(\mathcal{H}) & \end{array}$$

where the unique map is the map we are looking for. We name the new functor the **realization functor**, or just real. Since the diagram commutes we see that real is triangulated. Given an object $X \in \mathcal{H}$, X is isomorphic to the stalk complex in $C^b(\mathcal{H})$ with X in degree 0. In particular $H^n(X) = \begin{cases} X, & \text{if } n = 0 \\ 0, & \text{else} \end{cases}$.

Thus, since we have $H^n_{\mathcal{H}}(\widetilde{\text{real}}(X)) \cong H^n(X) = X$ we conclude that $\text{real}|_{\mathcal{H}} \cong \text{id}_{\mathcal{H}}$. Remember the the canonical t-structure on $D^b(\mathcal{H})$ is given by

$$D^b(\mathcal{H})^{\leq 0} = \{X^\bullet | H^i(X^\bullet) = 0, \text{ for } i < 0\}$$
$$D^b(\mathcal{H})^{\geq 0} = \{X^\bullet | H^i(X^\bullet) = 0, \text{ for } i > 0\}$$

It is clear that real is t-exact with respect to the canonical t-structure. □

Remark. Given a triangulated category $\mathcal{T}$ with t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ with heart $\mathcal{H}$, it is possible to generalize this construction to get a realization functor real : $D^b(\mathcal{H}) \to \mathcal{T}$ provided that $\mathcal{T}$ can be lifted to a **filtered triangulated category**; See [Bei87, Appendix] and [PV17]. It is further possible to show that $\mathcal{T}$ can be lifted to a filtered triangulated category provided that $\mathcal{T}$ is algebraic; see [Han20, Appendix]. The uniqueness of such a realization functor is in general not known.

Now that we know the existence of such a functor, we can combine everything previously shown in the thesis to arrive at a few very important results, particularly we will show sufficient conditions for the realization functor to be an equivalence.

Corollary 4.10.1. *Let $F : D^b(\mathcal{H}) \to \mathcal{T}$ be a realization functor to the triangulated category $\mathcal{T}$. Then given $X \in D^b(\mathcal{H})$ we have*

(1) isomorphisms

$$H^n_{\mathcal{H}}(F(X)) \cong H^n(X)$$

for $n \in \mathbb{Z}$

(2) If X is such that $F(X) \in \mathcal{H}$ then $X \in \mathcal{H}$ and consequently $F(X) \cong X$

Proof. (1) Since the realization functor is t-exact we have from Lemma 3.14 the isomorphisms

$$H^n_{\mathcal{H}}(F(X)) \cong F|_{\mathcal{H}}(H^n_{\mathcal{H}}(X))$$

By definition of the realization functor we have that $F|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$ and from Example 4 we see that $H^n_{\mathcal{H}}(X) \cong H^n(X)$. Thus we have

$$H^n_{\mathcal{H}}(F(X)) \cong H^n(X)$$

(2) If $X \in \mathcal{H}$ then $H^n_{\mathcal{H}}(X) = 0$ for $n \neq 0$. Combining this with the first part we conclude that $F(X) \in \mathcal{H}$ and $F(X) \cong \mathrm{id}_{\mathcal{H}}(X) \cong X$

□

Corollary 4.10.2. *Let $F : D^b(\mathcal{H}) \to \mathcal{T}$ be a realization functor to the triangle category $\mathcal{T}$, and let $X, Y \in \mathcal{H}$ then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Ext}^n_{\mathcal{H}}(X,Y) & \xrightarrow{\quad F \quad} & \mathrm{Hom}_{\mathcal{T}}(X, Y[n]) \\ {\scriptstyle \theta^n_1}\nwarrow{\scriptstyle \cong} & & \nearrow{\scriptstyle \theta^n_2} \\ & \mathrm{YExt}^n_{\mathcal{H}}(X,Y) & \end{array}$$

for $n \geq 1$ and $F : \mathrm{Ext}^i_{\mathcal{H}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Y[i])$ is an isomorphism for $i = 1$ and an injection for $i = 2$.

Proof. From Proposition 3.2 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{YExt}^n_{\mathcal{H}}(X,Y) & \xrightarrow{\quad F|_{\mathcal{H}} \quad} & \mathrm{YExt}^n_{\mathcal{G}}(F(X), F(Y)) \\ \downarrow{\scriptstyle \theta^n_1} & & \downarrow{\scriptstyle \theta^n_2} \\ \mathrm{Hom}_{D^b(\mathcal{H})}(X, Y[n]) & \xrightarrow{\quad F \quad} & \mathrm{Hom}_{\mathcal{T}}(F(X), F(Y)[n]) \end{array}$$

Since F is a realization functor we have $F|_{\mathcal{H}} \cong \mathrm{id}_{\mathcal{H}}$. This fact combined with Theorem 3.11 gives us the diagram. Then from Lemma 3.4 part (1) $F : \mathrm{Ext}^i_{\mathcal{H}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Y[i])$ is an isomorphism for $i = 1$ and an injection for $i = 2$. □

Corollary 4.10.3. *Let $\mathcal{T}$ be a triangulated category with bounded t-structure with heart $\mathcal{H}$. Let $F : D^b(\mathcal{H}) \to \mathcal{T}$ be a realization functor to the triangle category $\mathcal{T}$, and let $\theta^n : \mathrm{YExt}^n_{\mathcal{H}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}}(X,Y[1])$ be the canonical map from Definition 7. Then the following are equivalent*

1. *The realization functor F is full,*
2. *The realization functor F is an equivalence,*
3. *The canonical maps θ^n are isomorphisms for all $n \geq 1$*
4. *The canonical maps θ^n are surjective for all $n \geq 1$*

Proof. "(1) $\Rightarrow$ (2)" Assume F is full. Let $X \in \ker F$, then $F(X) \in \mathcal{H}$, and from Corollary 4.10.1 part (2) we get that $X \cong 0$. Thus from Lemma A.5 we have that F is faithful. Further, since F is a realization functor, $F(\mathcal{H}) \cong \mathcal{H}$. Thus, by the boundedness of the t-structure on $\mathcal{T}$ we have by Lemma 3.16 part (2) that F is dense.
"(2) $\Rightarrow$ (3)" If F is an equivalence then from the commutative diagram from the previous corollary we see that θ^n must be an isomorphism for all $n \geq n$
"(3) $\Leftrightarrow$ (4)" If θ^n is an isomorphism then it is clearly surjective. If θ^n is surjective for all $n \geq 1$, then inductively using Lemma 3.4 part (1) and (2) it is clear that θ^n is an isomorphism for all $n \geq 1$.
"(3) $\Rightarrow$ (1)" Observe that F in the diagram an isomorphism for $n = 0$, and $\mathrm{Ext}^n_{\mathcal{H}}(X,Y) = 0 = \mathrm{Hom}_{\mathcal{T}}(X,Y[n])$ for $n < 0$. Since $\mathcal{T}$ has bounded t-structure we have from Lemma 3.16 part (1) that F is fully faithful. □

Observation 4.10.1. Since θ^n is constucted independently of the realization functor, we see that if part (3) is satisfied then every realization functor is an equivalence. In other words if one realization functor is an equivalence, then all realization functors are equivalences.

Theorem 4.11 ([CHZ18, Theorem 2.9])**.** *Let $\mathcal{T}$ be a triangulated category with bounded t-structure with heart $\mathcal{H}$, and let $F : D^b(\mathcal{H}) \to \mathcal{T}$ be a realization functor. If F is dense, then F is an equivalence.*

Proof. From Corollary 4.10.3 part (4) it suffices to show that the map $\theta^n : \mathrm{YExt}^n_{\mathcal{H}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}}(X,Y[n])$ is surjective for all $X,Y \in \mathcal{H}$ and $n \geq 1$. We prove this by induction on n. Observe that by Lemma 3.4 part (1) the assertion is true for $n = 1$. Now assume it is true for $i \leq n-1$. Then by Lemma 3.4 part (3) it is enough to show that any $f \in \mathrm{Hom}_{\mathcal{T}}(X,Y[n])$ with $X,Y \in \mathcal{H}$ admits a factorization

$$X \to X_{n-1}[n-1] \to Y[n]$$

for some $X_{n-1} \in \mathcal{H}$.
First note that f can be embedded in a triangle

$$X \xrightarrow{f} Y[n] \xrightarrow{g} \tilde{Z} \to X[1]$$

and since F is dense there exist a complex $Z^\bullet \in D^b(\mathcal{H})$ such that $F(Z^\bullet) \cong \tilde{Z}$ Applying the cohomological functor $H^0_{\mathcal{H}}$ we get a long exact sequence

$$\cdots \to H^{-1}_{\mathcal{H}}(Y[n]) \to H^{-1}_{\mathcal{H}}(F(Z^\bullet)) \to H^0_{\mathcal{H}}(X) \to H^0_{\mathcal{H}}(Y[n]) \to \cdots$$

Since $H^i_{\mathcal{H}}(X) = 0$ for all $i \neq 0$, and $H^i_{\mathcal{H}}(Y[n]) = 0$ for all $i \neq -n$ we get $H^i_{\mathcal{H}}(F(Z^\bullet)) = 0$ for all $i \neq -n, -1$. In particular we get that $H^{-n}_{\mathcal{H}}(g)$ is an isomorphism. From Corollary 4.10.1 part (1) we have $H^i(Z^\bullet) \cong H^i_{\mathcal{H}}(F(Z^\bullet)) = 0$ for $i \neq -n, -1$, and $Z^\bullet$ is quasi-isomorphic to a complex of the form

$$\cdots \to 0 \to Z^{-n} \to Z^{-n+1} \to \cdots \to Z^{-1} \to 0 \to \cdots$$

Therefore we may assume $Z^\bullet$ is of this form in $D^b(\mathcal{H})$. Let $\pi : Z^\bullet \to Z^{-n}[n]$ be the canonical projection. Note that since $Z^{-n} \in \mathcal{H}$ we have from Lemma 4.10.1 part (2) that $F(Z^{-n}) \cong Z^{-n}$. We then have a map $h[n] : Y[n] \to Z^{-n}[n]$ given by the composition

$$Y[n] \xrightarrow{g} F(Z^\bullet) \xrightarrow{F(\pi)} F(Z^{-n}[n]) \xrightarrow{\cong} F(Z^{-n})[n] \xrightarrow{\cong} Z^{-n}[n]$$

Now $H^{-n}(\pi)$ is a monomorphism, therefore by Corollary 4.10.1 part (1) the map $H^{-n}_{\mathcal{H}}(F(\pi))$ is mono. Thus we conclude that, since $H^{-n}_{\mathcal{H}}(g)$ is an isomorphism, the map

$$Y \cong H^{-n}_{\mathcal{H}}(Y[n]) \xrightarrow{H^{-n}_{\mathcal{H}}(h[n]) \cong h} H^{-n}_{\mathcal{H}}(Z^{-n}[n]) \cong Z^{-n}$$

is a monomorphism. We then have a short exact sequence in $\mathcal{H}$

$$0 \to Y \xrightarrow{h} Z^{-n} \to \operatorname{coker}(h) \to 0$$

This embeds uniquely into a triangle in $\mathcal{T}$ by Corollary 3.2.1 and we have a diagram of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y[n] & \longrightarrow & F(Z) & \longrightarrow & X[1] \\ \downarrow & & \| & & \downarrow & & \downarrow \\ \operatorname{coker}(h)[n-1] & \longrightarrow & Y[n] & \xrightarrow{h[n]} & Z^{-n}[n] & \longrightarrow & \operatorname{coker}(h)[n] \end{array}$$

Since $g \circ f = 0$, and from the definition of $h[n]$ we see that $h \circ f = 0$ and the dashed arrows exist by Lemma 3.1 making the diagram into a map of triangles. Thus we see that $f : X \to Y[n]$ admits a factorization

$$X \to \operatorname{coker}(h)[n-1] \to Y[n]$$

with $\operatorname{coker}(h) \in \mathcal{H}$ and we are done. $\square$

Chapter 5

HRS-tilting

An important example of the application of the realization functor is the *HRS-tilt*. This was developed by Happel, Reiten and Smalø in 1996 [HRS96], and is an important tool in representation theory of quasi-tilted algebras, and in the study of derived equivalence [Huy06]. We will first define *torsion pairs*, and then describe how this induces a certain t-structure on the derived category. Lastly we will show that if the torsion pair is *tilting*, the realization functor on the induced t-structure becomes an equivalence.

Definition 18. Let $\mathcal{A}$ be an abelian category, and $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories in $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a **torsion pair** in $\mathcal{A}$ if the following conditions are satisfied:

(1) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$
(2) For all $X \in \mathcal{A}$ there exist a short exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0$$

such that $t(x) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then

- $\mathcal{T}$ is called the **torsion class**
- $T \in \mathcal{T}$ is called a **torsion object**
- $\mathcal{F}$ is called the **torsion free class**
- $F \in \mathcal{F}$ is called a **torsion free object**

Observation 5.0.1. It is clear that $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions, $\mathcal{T}$ is closed under taking quotients and $\mathcal{F}$ is closed under subobjects.

Lemma 5.1. *Let $\mathcal{A}$ be an abelian category with torsion pair $(\mathcal{T}, \mathcal{F})$*

(1) If $X \in \mathcal{A}$, and $\operatorname{Hom}_{\mathcal{A}}(X, F) = 0$ for all $F \in \mathcal{F}$, then $X \in \mathcal{T}$.
(2) If $X \in \mathcal{A}$, and $\operatorname{Hom}_{\mathcal{A}}(T, X) = 0$ for all $T \in \mathcal{T}$, then $X \in \mathcal{F}$

Proof. We prove part (1), the second is dual. Given $X \in \mathcal{A}$ there is a triangle

$$0 \to t(X) \to X \to X/t(X) \to 0$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$. If $\operatorname{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0$, then in the above short exact sequence we see that $t(X) \cong X$ and $X \in \mathcal{T}$ □

We can now show that a torsion pair induces a t-structure on the derived category

Proposition 5.1 ([HRS96, Proposition 2.1]). *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$.*
Let $\mathcal{D}^{\leq 0} = \{X^\bullet \in D^b(\mathcal{A}) | H^i(X^\bullet) = 0,\ i > 0,\ H^0(X^\bullet) \in \mathcal{T}\}$
and $\mathcal{D}^{\geq 0} = \{X^\bullet \in D^b(\mathcal{A}) | H^i(X^\bullet) = 0,\ i < -1,\ H^{-1}(X^\bullet) \in \mathcal{F}\}$
Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded t-structure on $D^b(\mathcal{A})$.

Proof. We verify condition (1), (2) and (3) of the definition for a t-structure.

(1) Let $X^\bullet \in \mathcal{D}^{\leq 0}$ and $Y^\bullet \in \mathcal{D}^{\geq 1} = \{X^\bullet \in \mathcal{D}^b(\mathcal{A}) | H^i(X^\bullet) = 0$ for $i < 0$ and $H^0(X^\bullet) \in \mathcal{F}\}$. Assume there exist $0 \neq f \in \operatorname{Hom}_{D^b(\mathscr{A})}(X^\bullet, Y^\bullet)$. So f can be represented by the roof

$$\begin{array}{ccccc} & & \tilde{X}^\bullet & & \\ & \overset{q}{\swarrow} & & \overset{f}{\searrow} & \\ X^\bullet & & & & Y\bullet \end{array}$$

where $f^\bullet$ is given by a morphism of complexes not homotopic to zero. Using the truncation, and Proposition 3.1 we obtain the following map of triangles in $\mathcal{D}^b(\mathcal{A})$

$$\begin{array}{ccccccc} (\tilde{X}^\bullet)^{\leq 0} & \xrightarrow{a^\bullet} & \tilde{X}^\bullet & \longrightarrow & (\tilde{X}^\bullet)^{\geq 1} & \longrightarrow & (\tilde{X}^\bullet)^{\leq 0}[1] \\ \downarrow{\scriptstyle (f^\bullet)^{\leq 0}} & & \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle (f^\bullet)^{\geq 1}} & & \downarrow \\ (Y^\bullet)^{\leq 0} & \longrightarrow & Y^\bullet & \longrightarrow & (Y^\bullet)^{\geq 1} & \longrightarrow & (Y^\bullet)^{\leq 0}[1] \end{array}$$

Following Example 3, since $\tilde{X}^\bullet \in \mathcal{D}^{\leq 0}$, we have that $H^i(\tilde{X}^\bullet) = 0$ for $i > 0$. Then $H^i((\tilde{X}^\bullet)^{\geq 1}) = 0$ for all $i \in \mathbb{Z}$. Thus $(\tilde{X}^\bullet)^{\geq 1} \cong 0$ in $D^b(\mathcal{A})$, and $a^\bullet$ is an isomorphism in $\mathcal{D}^b(\mathcal{A})$. In particular, this makes $(f^\bullet)^{\leq 0} \neq 0$.
Again using trucation we get the map of triangles

$$\begin{array}{ccccccc}
((\tilde{X}^\bullet)^{\leq 0})^{\leq -1} & \longrightarrow & (\tilde{X}^\bullet)^{\leq 0} & \longrightarrow & ((\tilde{X}^\bullet)^{\leq 0})^{\geq 0} & \longrightarrow & ((\tilde{X}^\bullet)^{\leq 0})^{\leq -1}[1] \\
\downarrow{\scriptstyle ((f^\bullet)^{\leq 0})^{\leq -1}} & & \downarrow{\scriptstyle (f^\bullet)^{\leq 0}} & & \downarrow{\scriptstyle ((f^\bullet)^{\leq 0})^{\geq 0}} & & \downarrow \\
((Y^\bullet)^{\leq 0})^{\leq -1} & \longrightarrow & (Y^\bullet)^{\leq 0} & \xrightarrow{b^\bullet} & ((Y^\bullet)^{\leq 0})^{\geq 0} & \longrightarrow & ((Y^\bullet)^{\leq 0})^{\leq -1}[1]
\end{array}$$

Since $Y^\bullet \in \mathcal{D}^{\geq 0}$ we have that $H^i(Y^\bullet) = 0$ for $i < -1$, and $((Y^\bullet)^{\leq 0})^{\leq -1} = 0$ in $D^b(\mathcal{A})$. We then get that $b^\bullet$ is an isomorphism. Since $((\tilde{X}^\bullet)^{\leq 0})^{\geq 0} = H^0(\tilde{X}^\bullet) \cong H^0(X^\bullet) \in \mathcal{T}$, $((Y^\bullet)^{\leq 0})^{\geq 0} = H^0(Y^\bullet) \in \mathcal{F}$ and since $(\mathcal{T}, \mathcal{F})$ is a torsion pair this forces $((f^\bullet)^{\leq 0})^{\geq 0} = 0$ which again forces $(f^\bullet)^{\leq 0} = 0$ which is a contradiction. Thus $f^\bullet = 0$.

(2) We have

$$\mathcal{D}^{\geq 0} = \{X^\bullet \in \mathcal{D}^b(\mathscr{A}) | H^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } H^{-1}(X^\bullet) \in \mathcal{F}\}$$
$$\mathcal{D}^{\geq 1} = \{X^\bullet \in \mathcal{D}^b(\mathscr{A}) | H^i(X^\bullet) = 0 \text{ for } i < 0 \text{ and } H^{0}(X^\bullet) \in \mathcal{F}\}$$

$$\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathcal{D}^b(\mathscr{A}) | H^i(X^\bullet) = 0 \text{ for } i > 0 \text{ and } H^{0}(X^\bullet) \in \mathcal{T}\}$$
$$\mathcal{D}^{\leq 1} = \{X^\bullet \in \mathcal{D}^b(\mathscr{A}) | H^i(X^\bullet) = 0 \text{ for } i > 1 \text{ and } H^{1}(X^\bullet) \in \mathcal{T}\}$$

In $\mathcal{D}^{\leq 0}$, we have $H^1(X^\bullet) = 0$ we have that $H^1(X^\bullet) \in \mathcal{T}$. Thus $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$. A similar argument shows $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$

(3) Let $X^\bullet = (X^i, d^i) \in D^b(\mathscr{A})$. Since $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ we have an exact sequence

$$0 \to t(H^0(X^\bullet)) \xrightarrow{\iota} H^0(X^\bullet) \xrightarrow{\pi} H^0(X^\bullet)/t(H^0(X^\bullet)) \to 0$$

where $t(H^0(X^\bullet)) \in \mathcal{T}$ and $H^0(X^\bullet)/t(H^0(X^\bullet)) \in \mathcal{F}$.
We also have a short exact sequence

$$0 \to \mathrm{Im}(d^{-1}) \to \ker(d^0) \to H^0(X^\bullet) \to 0$$

Taking the pullback, PB, of the diagram

$$\begin{array}{ccc}
 & & t(H^0(X^\bullet)) \\
 & & \downarrow{\scriptstyle \iota} \\
\ker(d^0) & \longrightarrow & H^0(X^\bullet)
\end{array}$$

we get from [Opp16, Proposition 13.7] a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Im}(d^{-1}) & \xrightarrow{\alpha} & \mathrm{PB} & \longrightarrow & t(H^0(X^\bullet)) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\iota} & & \\
0 & \to & \operatorname{Im}(d^{-1}) & \longrightarrow & \ker(d^0) & \longrightarrow & H^0(X^\bullet) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle\pi} & & \\
 & & & & H^0(X^\bullet)/t(H^0(X^\bullet)) & = & H^0(X^\bullet)/t(H^0(X^\bullet)) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Let d^{-1} be the composite $X^{-1} \xrightarrow{\rho} \operatorname{Im}(d^{-1}) \xrightarrow{i} X^0$. Let $\tilde{d}^{-1} = \alpha\rho : X^{-1} \to \mathrm{PB}$. We can then construct $\tilde{X}^\bullet$, as the following subcomplex of $X^\bullet$:

$$\cdots \xrightarrow{d^{-3}} X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{\tilde{d}^{-1}} \mathrm{PB} \xrightarrow{0} 0 \longrightarrow 0 \longrightarrow \cdots$$

with PB in degree 0. Then $H^0(\tilde{X}) = PB/\operatorname{Im}(\alpha\rho) \cong PB/\operatorname{Im}(d^{-1}) \cong t(H^0(X^\bullet)) \in \mathcal{T}$ and $\tilde{X}^\bullet \in D^{\leq 0}$.
Let $X''^\bullet$ be the quotient complex $X^\bullet/X'^\bullet$. Thus we obtain a triangle

$$X'^\bullet \to X^\bullet \to X''^\bullet \to X'^\bullet[1]$$

in $D^b(\mathscr{A})$. We have to show that $X''^\bullet \in D^{\geq 1}$.
Note that $H^i(X''^\bullet) = 0$ for $i < 0$. Now $X''^0 = X^0/\mathrm{PB}$ and $X''^1 = X^1$. We get a commutative diagram with exact columns:

$$\begin{array}{ccccccccccc}
X'^\bullet : \cdots & \longrightarrow & X^{-1} & \longrightarrow & PB & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
X^\bullet : \cdots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \| & & \| & & \\
X''^\bullet : \cdots & \longrightarrow & 0 & \longrightarrow & X^0/PB & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots
\end{array}$$

In particular we have a diagram with short exact rows and columns

$$\begin{array}{ccccccccc} & & PB & & \ker(d^0) & & \ker(\tilde{d^0}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & PB & \longrightarrow & X^0 & \longrightarrow & X^0/PB & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & X^1 & \longrightarrow & X^1 & \longrightarrow & 0 \end{array}$$

and from the 3×3 lemma we get $H^0(X''^\bullet) = \ker(\tilde{d^0}) \cong \ker(d^0)/\mathrm{PB}$. From the first diagram we know that $\ker(d^0)/PB \cong H^0(X^\bullet)/t(H^0(X^\bullet)) \in \mathcal{F}$ and $X''^\bullet \in \mathcal{D}^{\geq 1}$. Hence the assertion holds.

□

In order to construct a derived equivalence, we need one more definition

Definition 19. Let $D^b(\mathcal{A})$ be the bounded derived category over an abelian category $\mathcal{A}$, and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair over $\mathcal{A}$. The abelian heart,

$$\mathcal{B} = \{X^\bullet \in D^b(\mathcal{A}) | H^{-1}(X^\bullet) \in \mathcal{F}, H^0(X^\bullet) \in \mathcal{T}, H^i(X^\bullet) = 0 \text{ for } i \neq 0, 1\}$$

of the t-structure defined in the previous proposition is called **the HRS-tilt of $\mathcal{A}$ with respect to the torsion pair** $(\mathcal{T}, \mathcal{F})$.

Definition 20. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$.

(1) We say a torsion class $\mathcal{T}$ is a **tilting torsion class** if $\mathcal{T}$ is a cogenerator for $\mathcal{A}$ (I.e. if for all $X \in \mathcal{A}$, there exist an object $T_X \in \mathcal{T}$ together with a monomorphism $\mu_X : X \hookrightarrow T_X$)
(2) Dually we say a torsion free class $\mathcal{F}$ is a **cotilting torsion free class** if $\mathcal{F}$ is a generator for $\mathcal{A}$.

Lemma 5.2. *Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. If $\mathcal{T}$ is tilting then every $X^\bullet \in D^b(\mathcal{A})$ is quasi-isomorphic to a complex $T^\bullet$ where every $T^i \in \mathcal{T}$*

Proof. Let $X^\bullet \in D^b(\mathcal{A})$. Since $D^b(\mathcal{A})$ is bounded we can without loss of generality assume $X^i = 0$ for $i < 0$ and $i > n$ for some $n \geq 1$. Since $\mathcal{T}$ is tilting there exist a T_X^0 and a monomorphism $X^0 \hookrightarrow T_X^0$. Taking the pushout we then construct a complex as the bottom in the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & T_X^0 & \longrightarrow & T_X^0 \coprod X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \end{array}$$

Observe that since the map $X^0 \to T_X^0$ is mono, the pushout square is also a pullback square. Thus by [Opp16, Proposition 13.7] the complexes are quasi-isomorphic. Again taking the pushout and using the fact that $\mathcal{T}$ is tilting we construct a further complex as the bottom row in the diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & T_X^0 & \longrightarrow & T_X \coprod X^1 & \longrightarrow & X^2 & \longrightarrow & X^3 & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & T_X^0 & \longrightarrow & T_X^1 & \longrightarrow & T_X^1 \coprod X^2 & \longrightarrow & X^3 & \longrightarrow & \cdots
\end{array}$$

The complexes are again quasi-isomorphic. Continuing this process we get a quasi-isomorphism $X^\bullet \hookrightarrow T_X^\bullet$ where every $T_X^i \in \mathcal{T}$ □

Now we are ready to construct a derived equivalence and state the main theorem of this section:

Theorem 5.3. *Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$, and let $\mathcal{B}$ be its corresponding HRS-tilt. If $\mathcal{T}$ is a tilting torsion class then there exist a triangle equivalence $F : D^b(\mathcal{B}) \xrightarrow{\cong} D^b(\mathcal{A})$, where $F|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$*

Proof. From Theorem 4.10 there exist a realization functor $F : D^b(\mathcal{B}) \to D^b(\mathcal{A})$ where $F|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$. To show that F is an equivalence it is enough by Theorem 4.11 to show that F is dense. Let $X^\bullet \in D^b(\mathcal{A})$. Since $\mathcal{T}$ is tilting, we get from the previous lemma a quasi-isomorphism $X^\bullet \to T^\bullet$, where $T^i \in \mathcal{T}$. $T^\bullet$ is on the form

$$\cdots \xrightarrow{d_T^{n-2}} T^{n-1} \xrightarrow{d_T^{n-1}} T^n \xrightarrow{d_T^n} T^{n+1} \xrightarrow{d_T^{n+1}} \cdots$$

Observe that each T^i is in $\mathcal{B}$, thus $F(T^i) = F|_{\mathcal{B}}(T^i) = \mathrm{id}_{\mathcal{B}}(T^i) = T^i$ and $F(d_T^i) = F|_{\mathcal{B}}(d_T^i) = \mathrm{id}_{\mathcal{B}}(d_T^i) = d_T^i$.
We conclude that $F(T^\bullet) = T^\bullet \cong X^\bullet$ and F is dense. □

Remark. In the original construction and proof of the derived equivalence of the HRS-tilt, [HRS96, Theorem 3.3], it is assumed enough injectives and projectives, however as we have seen this is unnecessary.

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Appendix A

Basic results

Lemma A.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category, let and $C \in \mathscr{C}$, and let $F : \mathscr{C} \to$ **Set** be a covariant functor. Then the map*

$$\{\textit{natural transformations } \mathrm{Hom}_{\mathscr{C}}(C, -) \to F)\} \to F(C)$$
$$\alpha \mapsto \alpha_C(1_C)$$

is a bijection.

Corollary A.1.1. *Let*

$$\alpha : \mathrm{Hom}_{\mathscr{C}}(A, -) \to \mathrm{Hom}_{\mathscr{C}}(B, -)$$

is a natural transformation. Then, for all $C \in \mathscr{C}$

$$\alpha_C = [- \circ f] : \mathrm{Hom}_{\mathscr{C}}(A, C) \to \mathrm{Hom}_{\mathscr{C}}(B, C)$$

with a unique $f : B \to A$.

Proof. Let $\varphi \in \mathrm{Hom}_{\mathscr{C}}(A, C)$ Then from the naturality of α the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(A, A) & \xrightarrow{\alpha_A} & \mathrm{Hom}_{\mathscr{C}}(B, A) \\ \downarrow{\scriptstyle \varphi\circ -} & & \downarrow{\scriptstyle \varphi\circ -} \\ \mathrm{Hom}_{\mathscr{C}}(A, C) & \xrightarrow{\alpha_C} & \mathrm{Hom}_{\mathscr{C}}(B, C) \end{array}$$

In particular $\varphi \circ (\alpha_A(1_A)) = \alpha_C(\varphi)$ From the Yoneda lemma we have the bijection

$$\mathrm{Hom}_{\mathrm{fun}}(\mathrm{Hom}_{\mathscr{C}}(A, -), \mathrm{Hom}_{\mathscr{C}}(B, -)) \to \mathrm{Hom}_{\mathscr{C}}(B, A)$$
$$\alpha \mapsto \alpha_A(1_A)$$

So define $f := \alpha_A(1_A) \in \mathrm{Hom}_{\mathscr{C}}(B, A)$, and we get

$$\alpha_C(-) = [- \circ f] : \mathrm{Hom}_{\mathscr{C}}(A, C) \to \mathrm{Hom}_{\mathscr{C}}(B, C)$$

□

Theorem A.2. *Given a locally small category $\mathscr{C}$, i.e. given $X, Y \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ forms a set. Let $A, B, C \in \mathscr{C}$. Then the sequence $A \to B \to C \to 0$ is exact if and only if $0 \to \mathrm{Hom}_{\mathscr{C}}(C, -) \to \mathrm{Hom}_{\mathscr{C}}(B, -) \to \mathrm{Hom}_{\mathscr{C}}(A, -)$ is exact.*

Proof. Given the sequence $0 \to \mathrm{Hom}_{\mathscr{C}}(C, -) \xrightarrow{\beta} \mathrm{Hom}_{\mathscr{C}}(B, -) \xrightarrow{\alpha} \mathrm{Hom}_{\mathscr{C}}(A, -)$. We have from the previous corollary that, for all $M \in \mathscr{C}$, $\alpha_M = [- \circ f]$ and $\beta_M = [- \circ g]$ for unique $f : A \to B$ and $g : B \to C$. For any M we have, given two $\varphi g, \psi g \in \mathrm{Hom}_{\mathscr{C}}(B, M)$, if $\varphi g = \psi g$ then by the injectivity of β there is a unique object ϕ such that $\phi \circ g = \varphi g = \psi g$. Then $\varphi = \phi = \psi$ and g is an epimorphism. So there exist a unique sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

First let $M = C$, we then have

$$\alpha_C \beta_C(1_C) = \alpha_C(1_C \circ g) = 1_C \circ g \circ f = 0$$

so $gf = 0$ and $\mathrm{Im}(f) \subseteq \ker(g)$.
Now let $M = \mathrm{coker}(f)$, and let $\pi : B \twoheadrightarrow \mathrm{coker}(f)$ be the projection. Then

$$\alpha_{\mathrm{coker}(f)}(\pi) = \pi \circ f = 0$$

so by the exactness of the sequence there exist a $\varphi \in \mathrm{Hom}_{\mathscr{C}}(C, \mathrm{coker}(g))$ such that $\beta_{\mathrm{coker}(g)}(\varphi) = \pi$. We see that $\pi = \varphi \circ g$. Since $\mathrm{coker}(f) = B/\mathrm{Im}(f)$ we have by the first isomorphism theorem that $B/\ker(\pi) \cong B/\mathrm{Im}(f)$ and in particular $\ker(\pi) \cong \mathrm{Im}(f)$. We conclude by seeing that $\ker(g) \subseteq \ker(\pi) \cong \mathrm{Im}(f)$.
For the converse, see [Opp16, Theorem 16.2] □

Corollary A.2.1. *If $\mathrm{Hom}_{\mathscr{C}}(B, -) \cong \mathrm{Hom}_{\mathscr{C}}(A, -)$ then $A \cong B$*

Proof. If $\mathrm{Hom}_{\mathscr{C}}(B, -) \cong \mathrm{Hom}_{\mathscr{C}}(A, -)$. Then it fits in an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(B, -) \to \mathrm{Hom}_{\mathscr{C}}(A, -) \to 0$$

and from the previous theorem there exist a unique exact sequence

$$0 \to A \to B \to 0$$

and $A \cong B$ □

Observation A.2.1. Every statement above has a dual. In particular The sequence $0 \to A \to B \to C$ is exact if and only if

$$0 \to \mathrm{Hom}_{\mathscr{C}}(-, A) \to \mathrm{Hom}_{\mathscr{C}}(-, B) \to \mathrm{Hom}_{\mathscr{C}}(-, C)$$

and if $\mathrm{Hom}_{\mathscr{C}}(-, A) \cong \mathrm{Hom}_{\mathscr{C}}(-, B)$ then $A \cong B$

Definition 21 (Cohomological functor)**.** An additive functor $F : \mathscr{C} \to \mathcal{A}$ from a triangulated category $\mathscr{C}$ to an abelian category $\mathcal{A}$ is called a **cohomological functor** if for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{C}$, the sequence

$$F(X) \to F(Y) \to F(Z)$$

is an exact sequence in $\mathcal{A}$.

Observation A.2.2. By the rotation axiom of triangulated categories the definition of a cohomological functor F is equivalent to for any distinguished triangle $X \to Y \to Z \to X[1]$ the sequence

$$\cdots \to F(X[n]) \to F(Y[n]) \to F(Z[n]) \to F(X[n+1]) \to \cdots$$

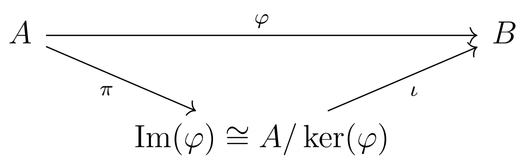
is long exact.

Lemma A.3. *Let $\mathscr{C}$ be a triangulated category, and $T \in \mathscr{C}$. Then* $\mathrm{Hom}_{\mathscr{C}}(T, -)$ *and* $\mathrm{Hom}_{\mathscr{C}}(-, T)$ *are cohomological.*

The proof of the lemma can be found in [Opp16, Theorem 32.4].

Lemma A.4. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between abelian categories. If F reflects 0-objects (i.e. $F(X) = 0$ implies $X = 0$), then F is faithful.*

Proof. Let $\varphi \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ such that $F(\varphi) = 0$. We know φ factors through the image



where π is an epimorphism, and ι is a monomorphism. Since F is assumed to be exact $F(\pi)$ is still an epimorphism, and $F(\iota)$ is a monomorphism. Thus $0 = F(\varphi) = F(\iota)F(\pi)$ which implies that $F(\iota) = 0$ since $F(\pi)$ is an epimorphism. Since $F(\iota)$ is mono we have $F(A/\ker(\varphi)) = 0$ and thus from the assumption that F reflects 0-objects we conclude that $A = \ker(\varphi)$ and $\varphi = 0$. Thus the kernel of the map $\mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(B))$ is zero and F is faithful. $\square$

Lemma A.5. *Let $\mathscr{C} \to \mathscr{D}$ be triangulated categories, and let $F : \mathscr{C} \to \mathscr{D}$ be a triangulated functor. If F is full and* $\ker(F) \cong 0$ *then F is faithfull.*

Proof. Let $X, Y \in \mathscr{C}$, and let $f : X \to Y$ be a morphism such that $F(f) = 0$. We can complete f to a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$$

Since F is triangulated we get a triangle

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to F(X)[1]$$

where $F(g)$ is an isomorphism. Then, as F is full, there exist a map $h : Z \to Y$ such that $F(hg) = \mathrm{id}_{F(Y)}$. Now we complete $Y \xrightarrow{hg} Y$ to the triangle

$$Y \xrightarrow{hg} Y \to \tilde{Z} \to Y[1]$$

Then $F(\tilde{Z}) \cong 0$, and since $\ker(F) = 0$ we have $\tilde{Z} \cong 0$, and hg is an isomorphism. Thus we can conclude that g is a split monomorphism, and in particular $f = 0$. Thus F is faithful. □

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Realization functors and HRS-tilting

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