On cluster-tilting modules for some symmetric algebras

Tor Kringeland
torkrin@protonmail.com

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ロビンとフィンに

Acknowledgments

Tout ce qui est nouveau est, de ce fait, automatiquement traditionnel.

—Odile, *Bande à part* (1964)

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Contents

1 Introduction

Are you ready to rumble?

—Mahdi

The recent years have seen a profusion in the research on cluster-tilting modules — and, more generally, cluster-tilting subcategories. Of interest is both which algebras possess cluster-tilting modules, and what are the ramifications, so to speak, for an algebra having a cluster-tilting module?

In this thesis we look at both questions. We recall and prove in section 4 a result by Erdmann and Holm stating that if a self-injective algebra possess a non-trivial cluster-tilting module, then it has complexity 0 or 1 (which is to say that projective resolutions are either finite or of bounded dimensions). For the other question, we classify the cluster-tilting modules of finite-dimensional symmetric Nakayama algebras in section 3 and those — with the exception of $d = 4$ — for trivial extensions of quiver algebras of Dynkin type $\mathbb{D}$, another class of symmetric algebras, in section 5. Of particular interest is the latter case as we reduce the problem of looking for cluster-tilting modules in the module category to that of looking for cluster-tilting subcategories of a certain factor category $\mathfrak{T}(\mathbb{D}_n)$ of the repetitive quiver of $\mathbb{D}_n$. This allows the search to be performed combinatorially.

In section 2 we recall some basic definitions and results about symmetric algebras and repetitive algebras as well as about cluster-tilting modules. In the first part of section 5, we recall some definitions and results about translation quivers and derived categories, which we need in the sequel.

We assume basic knowledge of homological algebra and representation theory of algebras as prerequisites. Moreover, we do on several occasions refer to some well-known results and definitions within the framework we are working, without giving explicit reference. In these cases, we refer the meticulous reader to Happel's book [14], in which most of these results appear.

By 'algebra' we shall mean 'associative algebra with identity,' and we fix an algebraically closed field k and assume all algebras to be k-algebras.

2 Background theory

Yeah, this one is cheaper

—Athit

2.1 Symmetric algebras

In this section we recall some basic definitions results regarding symmetric algebras.

Definition 1 An algebra A is **symmetric** if it is isomorphic to its dual,

$$A \simeq D(A),$$

as an A-A-bimodule, where $D(A) := \operatorname{Hom}_k(A, k)$. $\square$

Definition 2 An algebra A is **weakly symmetric** if for each indecomposable projective module P of A,

$$\operatorname{top} P \simeq \operatorname{soc} P$$ $\square$

Proposition 1 *If A is representation-finite, then weakly symmetric is equivalent symmetric.* $\square$

PROOF See, e.g., [22, Folgerung 2]. $\blacksquare$

Proposition 2 *If A is a self-injective algebra and M is an A-module, then*

$$\tau M \simeq \Omega^2 \nu M$$ $\square$

PROOF See, e.g., [24, p. 161]. $\blacksquare$

An important construction in section 5 will be that of trivial extensions, which we now define.

Definition 3 The *trivial extension* of the algebra A is the algebra $T(A)$, whose additive structure is given by $T(A) = A \oplus D(A)$, where $D(A)$ is the dual of A, and whose multiplicative structure is given by

$$(a, \phi) \cdot (a', \phi') = (aa', a\phi' + \phi a'),$$

where $(a\phi)(b) := \phi(ba)$ and $(\phi a)(b) := \phi(ab)$. $\square$

In particular, we have the following

Proposition 3 *The trivial extension algebra $T(A)$ is symmetric.* $\square$

PROOF See, e.g., [24, p. 162]. $\blacksquare$

Lastly, we mention a well-known fact about triangle automorphisms of stabilised module categories.

Proposition 4 *If A is a symmetric algebra, the functors ν and $[1]$ commute with all triangle automorphisms of $\underline{\operatorname{mod}}\, A$—in particular, they commute with each other.* $\square$

2.2 Translation quivers and mesh categories

References for this section are [1] and [24], [14] and [13].

In this section we recall some basic definitions and results on quivers and quiver algebras, mostly to fix notation. In section 2.2, we do the same for translation quivers and mesh categories.

By a *quiver* we mean a tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 is the set of vertices and Q_1 the set of directed arrows between vertices in Q_0. For a given arrow $\alpha \in Q_1$, $s(\alpha) \in Q_0$ is the source of α and $t\,(\alpha) \in Q_0$ is the terminus of α.

For a vertex $x \in Q_0$, $x^- \subseteq Q_0$ denotes the set of predecessors of x. That is, vertices y such that there is an error $\alpha \in Q_1$ with $s\,(\alpha) = y$ and $s\,(\alpha) = x$.

Definition 4 A *translation quiver* is a tuple (Q, τ) where Q is a quiver and τ is a bijection between two subsets of Q_0 such that for each $x \in Q_0$ such that τx is defined and each $y \in x^-$, there is an equal amount of arrows from y to x as from τx to y. In case τ is defined on all of Q_0, (Q, τ) is a *full translation quiver*. ◻

Example 1 The prototypical example is the Auslander–Reiten quiver of an algebra along with the Auslander–Reiten translate. ◻

Definition 5 Let Q be a connected, acyclic quiver. We define the *stable translation quiver* (also *infinite translation quiver*) $\mathbb{Z}Q$ of Q as follows. The set of vertices is given as $(\mathbb{Z}Q)_0 = \mathbb{Z} \times Q_0$ and for each arrow $\alpha : x \to y$ in Q_1 and each $n \in \mathbb{Z}$ two arrows $(n, \alpha) : (n, x) \to (n, y)$ and $(n, \widetilde{\alpha}) : (n, y) \to (n + 1, x)$. The translation τ is full and defined as $\tau(n, x) = (n - 1, x)$. This makes $(\mathbb{Z}Q, \tau)$ into a translation quiver.

We define a bijection σ on $(\mathbb{Z}Q)_1$ given by $\sigma(n, \alpha) = (n - 1, \widetilde{\alpha})$ and $\sigma(n, \widetilde{\alpha}) = (n, \alpha)$.

Now consider $\mathbb{Z}Q$ as a category whose objects are the vertices and whose morphisms are paths (as well as the identity morphisms). In this category we consider the *mesh ideal* generated by the *mesh relations*

$$m_x = \sum_{\alpha : y \to x} \alpha \circ \sigma(\alpha),$$

for each $x \in \mathbb{Z}Q$ and for arrows α ending in x. Let us denote the quotient category by $\mathcal{T}\,(Q)$; it will be of importance in section 5. ◻

2.3 Cluster-tilting modules and subcategories

Cluster-tilting modules were first introduced by Osamu Iyama in [18] and [19] as a generalisation of the famous Auslander correspondence given by Maurice Auslander in [2]. We recall, the Auslander correspondence gives a bijection between the set of finite-dimensional algebras of finite representation type A and so-called Auslander algebras B. These are the algebras satisfying

$$\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B,$$

and the bijection is given by $A \mapsto \mathrm{End}_A(M)$, where $M \in \mathrm{mod}\, A$ is an additive generator, $\mathrm{add}(M) = \mathrm{mod}\, A$. Iyama considers in [18] finite-dimensional algebras B such that

$$\mathrm{gl.\,dim}\, B \leq n+1 \leq \mathrm{dom.\,dim}\, B,$$

and proves that the Morita equivalence class of these are in bijection with finite-dimensional algebras A with a so-called n-**cluster-tilting module** M.

A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ in which extensions are defined, is n-cluster-tilting of A precisely if

$$\begin{aligned}\mathcal{C} &= \{X \in \mathrm{mod}\, \mathcal{D} : \mathrm{Ext}^k_D(X, M) = 0, \text{for } k = 1, 2, \ldots, n-1\} \\ &= \{X \in \mathrm{mod}\, \mathcal{D} : \mathrm{Ext}^k_D(M, X) = 0, \text{for } k = 1, 2, \ldots, n-1\}.\end{aligned}$$

The bijection is given by sending a pair (A, M), for M a n-cluster-tilting module, to $\mathrm{End}_A(M)$, which will be a n-Auslander algebra. Note that the above definition readily generalises to other categories where extensions are defined.

Now we come to an important invariant for cluster-tilting subcategories. Define

Definition 6

$$\nu_d := \nu \circ [-d]$$

in a triangulated category, where d is a positive integer and ν is a Serre functor. □

From this we have an important invariant on cluster-tilting subcategories.

Proposition 5 *Suppose X is a d-cluster-tilting subcategory of a derived category $\mathcal{D}^b(\Lambda)$. Then*

$$\nu_d(X) = X.$$

PROOF See, e.g., [21, Proposition 3.4]. ■

3 d-cluster-tilting modules of symmetric Nakayama algebras

…with my Duke!

—Ken

In this section we explicitly classify all d-cluster-tilting modules of symmetric Nakayama algebras. In [6], Darpö and Iyama proves a numerical criterion for a self-injective Nakayama algebra to have a d-cluster-tilting module. Though a full classification of d-cluster-tilting modules of self-injective Nakayama algebras from this criterion is intractable, we may obtain a complete classification if we restrict ourselves to *symmetric* Nakayama algebras. We leverage the numerical criterion along with a result we prove on the Ω-periodicity of a putative cluster-tilting module to obtain a countable one-parameter family of symmetric Nakayama algebras possessing a non-trivial d-cluster-tilting module as well as three possibilities outside this family. Moreover, we prove that this list is exhaustive.

It is known (see, e.g., [1, p. 171]) that a basic, connected finite-dimensional Nakayama algebra Λ that is not isomorphic to the base field k is self-injective if and only if it is isomorphic to an algebra of the form kQ_n/R^h for $h \geq 2$ where Q_n is the quiver of n vertices $1, 2 \ldots, n$ with arrows $i \to (i+1)$ for $i = 1, 2 \ldots, n-1$ and an arrow $n \to 1$. It is well-known that any self-injective finite-dimensional Nakayama algebra is Morita equivalent to one of this one, so we do not lose generality by restricting our attention to algebras of this form.

Now consider $\Lambda = kQ_n/R^h$. Recall from proposition 1 that Λ is symmetric if and only if $\operatorname{soc} P_i = S_i = \operatorname{top} P_i$ for each $i = 1, 2, \ldots, n$. By symmetry, it is sufficient to consider the case $i = 1$. Notice that this is true precisely when $h = an + 1$ for some $a \geq 1$. We thus have the following

Theorem 1 *Let Λ finite-dimensional symmetric Nakayama algebra that is basic, connected and not isomorphic to the base field k. Then*

$$\Lambda \simeq kQ_n/R^{an+1}$$

for some a and n. Moreover, any algebra of the above form is a symmetric Nakayama algebra. □

It is a well-known fact that a finite-dimensional algebra is Morita equivalent to its basic algebra; consequently the existence of a cluster-tilting module of the basic algebra is equivalent to the existence of a cluster-tilting module for said algebra.

Let us now fix $\Lambda = kQ_n/R^{an+1}$ and proceed to find the d-cluster-tilting modules for Λ. We start by proving two numerical lemmas that will be key to proving the classification theorem.

Lemma 1 *Let d, a, n be positive integers, with $d \geq 2$ and $n \geq 3$. If*

$$((an+1)(d-1)+2) \mid 2n, \tag{1}$$

then $(d, a, n) = (2, 1, 3)$ □

PROOF If $a \geq 2$, then

$$(an+1)(d-1)+2 \geq (2n+1)(d-1)+2 > 2n+1$$

which contradicts the assumption, so $a = 1$. If $d \geq 3$, then

$$(n+1)(d-1)+2 > (n+1)(d-1) \geq 2(n+1) > 2n$$

which also contradicts eq. (1), so $d = 2$. Then eq. (1) is reduced to

$$(n+3) \mid 2n$$

Clearly, this is true only if $n+3 = 2n$. In this case, $n = 3$ and we are done. ■

Lemma 2 *Let d, a, n be positive integers, with $d \geq 2$ and $n \geq 3$. Then*

$$((an+1)(d-1)+2) \mid (d+1)n, \tag{2}$$

□

if and only if (d, a, n) is one of

- $(2, 1, 6)$*;*
- $(2, 2, 3)$*; or*
- $(2n-1, 1, n)$.

PROOF Suppose $a \geq 3$. Then

$$\begin{aligned}(an+1)(d-1)+2 > 3n(d-1) &\geq n(d-1)+2n(d-1)\\ &\geq n(d-1)+2n = (d+1)n\end{aligned}$$

which contradicts eq. (2). Now let $a = 2$. Then we have

$$((2n+1)(d-1)+2) \mid (d+1)n$$

Clearly, the LHS grows faster than the RHS as functions of d. Thus if the LHS is larger than the RHS for some fixed d, it will be larger for larger values of d. To this end, fix $d = 4$. Then

$$(2n+1)(d-1)+2 = (2n+1)(3)+2 = 6n+7 > 5n = (d+1)n$$

which contradicts eq. (2). It remains to check $d = 2$ and $d = 3$. In the former case,

$$(2n+5) \mid 3n$$

Then there must exist an integer $r \geq 1$ such that

$$\begin{aligned}r(2n+5) &= 3n\\ 5r &= n(3-2r)\end{aligned}$$

Then clearly $r = 1$ is the only possibility and $n = 5$. We have thus exhausted the options for $a = 2$. Now let $a = 1$. Then eq. (2) becomes

$$((n+1)(d-1)+2) \mid (d+1)n$$

In the first place, set $d = 2$. Then

$$(n+3) \mid 3n$$

Then there must be some integer $r \geq 1$ such that

$$\begin{aligned} (n+3)r &= 3n \\ 3r &= n(3-r) \end{aligned}$$

Clearly $r = 2$ is the only possibility. In that case, $n = 6$. Finally suppose $d \geq 3$. There must exist an integer $r \geq 1$ such that

$$r((n+1)(d-1)+2) = (d+1)n \tag{3}$$

If $r \geq 2$, then

$$\begin{aligned} r((n+1)(d-1)+2) &> 2(n+1)(d-1) \\ &= 2((d-1)n + (d-1)) \\ &> (d-1)n + (d-1)n \\ &\geq (d-1)n + 2n \\ &= (d+1)n, \end{aligned}$$

a contradiction. Rearranging eq. (3), we obtain

$$d = 2n - 1,$$

and the proof is complete. ■

Further, we need the following result about the Ω-periodicity of modules in the stable module category.

Lemma 3 *If for some Λ-module M,*

$$\Omega^a M \simeq \Omega^b M \simeq M$$

in the stable module category, for some positive integers a and b, then

$$\Omega^{\gcd(a,b)} M \simeq M$$

in the stable module category. □

PROOF The functor

$$\Omega : \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$$

is an auto-equivalence of the stable module category of Λ. Thus Ω^{-1} and Ω are inverses. The lemma then follows from a simple algebraic manipulation. ■

We will also need the following useful lemma.

Lemma 4 *Let Λ be a self-injective finite-dimensional algebra. Then for all finitely-generated Λ-modules M and N and positive integers i,*

$$\operatorname{Ext}^i_\Lambda(M,N) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^i M, N) \simeq \underline{\operatorname{Hom}}_\Lambda(M, \Omega^{-i} N) \qquad \square$$

PROOF See, e.g., [26, p. 409]. ■

We can now prove our result, which combined with Darpö and Iyama's result theorem 2 below gives us enough restrictions on d and n to permit a classification of the d-cluster-tilting modules.

Proposition 6 *If Λ has a d-cluster-tilting module X, then $(d+1) \mid 2n$.* □

PROOF Let $r := \gcd(d+1, 2n)$. Combining theorem 5 from section 3.1 and lemma 3 above,

$$\Omega^{d+1}X \simeq \Omega^{2n}X \simeq X;$$

whence

$$\Omega^r X \simeq X.$$

Clearly, $r \geq d$, for if $r < d$, then by lemma 4

$$\operatorname{Ext}^r_\Lambda(X,X) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^r X, X) \simeq \underline{\operatorname{Hom}}_\Lambda(X,X) \neq 0$$

which would contradict X being a d-cluster-tilting module. Now suppose $r = d$. Then

$$\Omega^d X \simeq \Omega^{d+1}X \simeq X$$

and thus by lemma 3,

$$\Omega^{\gcd(d,d+1)}X = \Omega^1 X \simeq X$$

but this contradicts with X being a d-cluster-tilting module, $d \geq 2$. We may then conclude that

$$\gcd(d+1, 2n) = d+1,$$

which implies that $(d+1) \mid 2n$. ■

We now state the result obtained by Darpö and Iyama in [6, Proposition 5.4] and then proceed to stating and proving our classification theorem.

Theorem 2 *Let $\Lambda \simeq kQ_n/R^\ell$ for some integer $\ell \geq 2$. Then there is a d-cluster-tilting module of Λ if and only if at least one of the following two conditions are satisfied.*

- $(\ell(d-1)+2) \mid (2n)$
- $(\ell(d-1)+2) \mid (tn)$

where $t = \gcd(d+1, 2(\ell-1))$. □

Theorem 3 *Let $\Lambda \simeq kQ_n/R^{an+1}$ be a symmetric Nakayama algebra. Then there exists a d-cluster-tilting module of Λ if and only if (d,a,n) is one of the following triples:*

- $(2, 1, 3)$
- $(2, 1, 6)$
- $(2, 2, 3)$
- $(2n - 1, 1, n)$

That is, there are three special cases and one countable family. In particular, every symmetric Nakayama algebra ($n \geq 2$) has at least one non-trivial d-cluster-tilting module, namely a $(2n-1)$-cluster-tilting module. □

PROOF By theorem 2, there are two possibilities on (d, a, n) that are equivalent to the existence of a d-cluster-tilting module. The first is

$$((an + 1)(d - 1) + 2) \mid (2n)$$

By lemma 1, $(d, a, n) = (2, 1, 3)$ is the only possibility in this case. The second case is

$$((an + 1)(d - 1) + 2) \mid (tn)$$

where $t = \gcd(d + 1, 2an)$. By proposition 6, $t = d+1$ and this yields the remaining three cases by lemma 2. ■

Example 2 The two special cases $(a, n) = (1, 3)$ and $(a, n) = (1, 6)$ have two distinct non-trivial cluster-tilting modules for different values of d. Namely, the former has a 2-cluster-tilting module and a 5-cluster-tilting module, while the latter has a 2-cluster-tilting module and a 11-cluster-tilting module. Up to isomorphism these are the only two symmetric Nakayama algebras possessing two distinct cluster-tilting modules. □

3.1 Periodicity of modules of symmetric Nakayama algebras

In this section we will prove the result we used in our proof that any indecomposable module of a symmetric Nakayama algebra has Ω-periodicity (at most) $2n$. As before, fix $\Lambda = kQ_n/R^{an+1}$.

We denote by $\ell\ell(\Lambda)$ is the *Loewy length* of Λ; that is, the length of the radical series of Λ. In our case, $\ell(P) = \ell\ell(\Lambda) = an + 1$, for any indecomposable projective Λ-module P.

Theorem 4 *Let $M \in \operatorname{ind} \Lambda$ with $\ell\ell(M) = t$. Then*

$$M \simeq P_i / \operatorname{rad}^t P_i$$

for the indecomposable projective module P_i corresponding to some vertex i. Consequently, M is uniquely determined by its length and $\operatorname{top} M \simeq S_i$, the simple module corresponding to vertex i. □

PROOF See, e.g., [1, p. 169] and [4, p. 113]. ■

Corollary 1 *For a given $t \in \{1, 2, \ldots, \ell\ell(\Lambda)\}$, there are n indecomposable modules of length t, up to isomorphism.* □

Lemma 5 *Let $t \in \{1, \ldots, \ell\ell(\Lambda) - 1\}$ and let X_t be the set of isomorphism classes of non-projective indecomposable modules of length t. Then Ω^2 induces a permutation on X_t.* □

PROOF Let us remark first that the indecomposable projective modules P of Λ all have the same length; we denote this length by l.

Note that since Λ is symmetric, $\tau \simeq \Omega^2$ and consequently $M, N \in X_t$ are isomorphic if and only if $\Omega^2 M$ and $\Omega^2 N$ are isomorphic. Moreover, $\Omega^2 M$ is indecomposable whenever M is and X_t has cardinality n by corollary 1.

In fact, $\Omega^2 M \in X_t$. To see this, take the canonical projective cover P of M. We then have a short exact sequence

$$0 \to \Omega M \to P \to M \to 0$$

and $\ell(\Omega M) = \ell(P) - \ell(M) = l - t$. Now take a projective cover P' of ΩM. We then get a short exact sequence

$$0 \to \Omega^2 M \to P' \to \Omega M \to 0$$

and consequently

$$\ell\left(\Omega^2 M\right) = \ell(P') - \ell(\Omega M) = l - (l - t) = t$$ ■

Now we can state and prove the main theorem for this subsection.

Theorem 5 *Let $M \in \operatorname{ind} \Lambda$ be non-projective. Then*

$$\Omega^{2n} M \simeq M$$ □

PROOF By lemma 5, Ω^2 induces a permutation on the set X_t of isomorphism classes of the non-projective indecomposable modules of fixed length t. Moreover by corollary 1 the cardinality of X_t is n. Consequently,

$$\Omega^{2n} M \simeq M$$

for any $M \in X_t$. Finally by theorem 4, any $M \in \operatorname{ind} \Lambda$ belong, up to isomorphism, to X_t for some t. ■

4 On a paper by Erdmann and Holm

I love it!

—Jay

In this section we prove a well-known result on the complexity of a self-injective algebra possessing a cluster-tilting module due to Erdmann and Holm [8]. We mostly follow the arguments they gave in their paper.

Throughout this section we fix a finite-dimensional algebra Λ. By ν in this section we refer both to the Nakayama automorphism of Λ to the Nakayama functor of the module category $\operatorname{mod}\Lambda$ — context will make the distinction clear.

To begin, we first need a rather technical result that plays a central role in the proof of the main theorem.

Theorem 6 *Let Λ be a finite-dimensional algebra and let X be an Λ-module with with $\operatorname{Ext}^1_\Lambda(X,X) = 0$. Moreover, let V be an Λ-module such that $\operatorname{Ext}^1_\Lambda(X,V) \neq 0$; let $n = \dim \operatorname{Ext}^1_\Lambda(X,V)$. Then there exists an Λ-module U with $\operatorname{Ext}^1_\Lambda(X,U) = 0$, along with a short exact sequence*

$$0 \to V \to U \to X^n \to 0.$$

□

PROOF See [8, p. 6] or [5, p. 33]. ■

Recall that the *complexity* of a module M is a measure of the size of a minimal projective resolution of M. Concretely, if $\cdots \to P_1 \to P_0 \to 0$ is a minimal projective resolution of M, the complexity of M is defined as

$$\operatorname{cx}(M) \coloneqq \inf\left\{ b \in \mathbb{N} \mid \exists\, C > 0 : \dim P_n \leq C n^{b-1}\, \forall n \in \mathbb{N} \right\},$$

where $\dim P_n$ is the dimension of P_n as a vector space. Note in particular that $\operatorname{cx}(M) = 0$ iff the minimal projective resolution of M is finite (that is, there is some positive integer N such that $P_n = 0$ for all $n \geq N$) and that $\operatorname{cx}(M) = 1$ iff the minimal projective resolution of M is bounded (that is, there is a constant $D > 0$ such that $\dim P_n < D$ for each $n \in \mathbb{N}$).

We will need the following well-known lemma from homological algebra for the proof of the next lemma.

Lemma 6 (Horseshoe lemma) *Let M, M' and M'' be finitely-generated modules over a ring R. Suppose $\ldots \to P'_1 \to P'_0 \to 0$ is a projective resolution of M' and $\ldots \to P''_1 \to P''_0 \to 0$ is a projective resolution of M'' and let $P_i = P'_i \oplus P''_i$. Then there is a projective resolution $\ldots \to P_1 \to P_0 \to 0$ of M.* □

First, give a lemma relating the complexity of a module and short exact sequences.

Lemma 7 *Suppose M, N and L are finite-dimensional modules of a finite-dimensional, self-injective algebra Λ and that we have a short exact sequence*

$$0 \to L \to M \to N \to 0$$

If two of M, N, L have complexity 0 or 1, then so does the third. □

Proof Suppose that L and N have finite (resp. bounded) projective resolutions. Construct minimal projective resolutions. By the horseshoe lemma the direct sum of these two resolutions is a projective resolution for M. It follows that this is finite (resp. bounded).

Alternatively, suppose L and M have finite (resp. bounded) projective resolutions. Construct minimal projective resolutions for L and N. Then the direct sum of these will be a projective resolution of M and thus finite (resp. bounded). This implies that the projective resolution for N is finite (resp. bounded). The same follows for the case with M and N having finite (resp. bounded) projective resolutions. ■

The next lemma concerns the complexity of a module that is $\Omega^{k+2}\nu$-periodic.

Lemma 8 *Suppose a Λ-module M is $\Omega^{k+2}\nu$-periodic, for some nonzero integer $k \geq 1$, then M has complexity 0 or 1.* □

Proof Here, $\nu = \nu_* : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ is the Nakayama functor induced from the Nakayama automorphism (also refered to as ν) of Λ.

Now, it is well-known that the Nakayama functor ν is right exact [1, p. 83] and that $\nu(M) = M$ as vector spaces, when Λ is self-injective. Moreover, ν induces a twisted ation on M; it is well-known that a module M with a twisted action has the same complexity as M with the untwisted action [8, p. 6]. Combining these facts, we get that $\nu(M)$ and M have the same complexity, given $\mathrm{cx}(M) \leq 1$.

Suppose that $M \in \mathrm{mod}\ \Lambda$ is $\Omega^{k+2}\nu$-periodic. If $\mathrm{cx}(M) \neq 0$, construct a minimal projective resolution of M:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0. \tag{4}$$

Let $1 \leq \ell < \infty$ be the $\Omega^{k+2}\nu$-period of M. Then

$$\begin{aligned}(\Omega^{k+2}\nu)^{\ell}(M) &\simeq \Omega^{\ell(k+2)}\nu^{\ell}(M)\\ &= \Omega^{\ell(k+2)}(\widetilde{M})\\ &\simeq M,\end{aligned}$$

where the first isomorphisms follows from the fact that Ω and ν commute, since Λ is self-injective, and where we define $\widetilde{M} := \nu^{\ell}(M)$. By our previous discussion, M and $\widetilde{M}$ have the same complexity, given $\mathrm{cx}(\widetilde{M}) \leq 1$. Thus it is sufficient to show that $\mathrm{cx}(\widetilde{M}) = 1$.

But this is clear. Since $\Omega^{\ell(k+2)}(\widetilde{M}) \simeq M$ and $\widetilde{M}$ and M are isomorphic as vector spaces, only finitely many projective modules will appear in eq. (4). Then the supremum of the dimension of these (which is necessarily finite), will be an upper bound, which shows that $\mathrm{cx}(\widetilde{M}) = \mathrm{cx}(M) = 1$. ■

We now come to the main theorem of [8].

Theorem 7 *Suppose Λ is a self-injective algebra with an n-cluster-tilting module X, $n \geq 2$. Then all Λ modules have complexity 0 or 1.* □

PROOF Let X be a n-cluster-tilting module of Λ. Suppose U_0 is a Λ-module with $\mathrm{cx}(M) \geq 2$. Construct modules $U_1, \ldots, U_n$ inductively as follows.

If $\mathrm{Ext}^1_\Lambda(\Omega^{n-i}X, U_{i-1}) = 0$ for $i \geq 1$, define $U_i = U_{i-1} \oplus \Omega^{n-i}X$. Otherwise we can apply theorem 6 with $V = U_{i-1}$ to construct an universal extension

$$0 \to U_{i-1} \to U_i \to (\Omega^{n-i}X)^{r_i} \to 0 \tag{5}$$

with $\mathrm{Ext}^1_\Lambda(\Omega^{n-i}X, U_i) = 0$ and r_i defined implicitly. We want to show that $\mathrm{Ext}^1_\Lambda(\Omega^j X, U_i) = 0$ for $n - i \leq j \leq n - 1$. Clearly, this is true for $i = 1$. For the inductive step, note first that

$$\mathrm{Ext}^1_\Lambda(\Omega^j X, \Omega^{n-i}X) = \mathrm{Ext}^1_\Lambda(\Omega^{i+j-n}X, X) = \mathrm{Ext}^{i+j+1-n}_\Lambda(X, X) = 0$$

The last equality follows since X is n-cluster-tilting. By induction, $\mathrm{Ext}^1_\Lambda(\Omega^j X, U_{i-1}) = 0$. By applying $\mathrm{Hom}_\Lambda(\Omega^j X, -)$ to eq. (5) and writing down the long exact sequence in homology, we conclude that $\mathrm{Ext}^1_\Lambda(\Omega^j X, U_i) = 0$ for $n - i \leq j \leq n - 1$.

Particularly for U_n, we have

$$\mathrm{Ext}^j_\Lambda(X, U_n) = \mathrm{Ext}^1_\Lambda(\Omega^{j-1}X, U_n) = 0$$

for $1 \leq j \leq n$. As X is n-cluster-tilting, this implies that $U_n \in \mathrm{add}(X)$. Subsequently, U_n is then $\Omega^2\nu$-periodic and, by lemma 8, has complexity 0 or 1. Now suppose $U_n, \ldots, U_{n-i}, 0 \leq i \leq n-1$ all have complexity 0 or 1. There is by assumption the short exact sequence

$$0 \to U_{n-i} \to U_{n-i-1} \to (\Omega^{i+1}X)^{r_{i+1}} \to 0$$

in which the middle two terms have complexity 0 or 1; by lemma 7, U_{n-i-1} has the same complexity. By downward induction, we conclude that U_0 has either complexity 0 or 1. This shows that every Λ-module has complexity at most 1. ∎

5 Cluster-tilting modules of trivial extensions algebras of Dynkin type

That guy doesn't have green!

—Lewis

5.1 Derived categories

References for this section are [11] and [28].

We begin by recalling the definition of the derived category and recall some results on the derived category of an algebra of Dynkin type $\mathbb{D}_n$.

Intuitively, one may think of the (bounded) derived category $\mathcal{D}^b(\Lambda)$ as identifying Λ-modules M their resolutions — that is, with complexes of homology M in the zeroth position and 0 elsewhere. In particular, we consider the category of complexes of Λ-modules with bounded homology. We then identify two complexes if their homologies are isomorphic, in a process reminiscent of localization of rings at, say, prime ideals. For instance, if the Λ-module M has a projective resolution

$$\cdots \to P_1 \to P_0 \to 0 \to 0 \to \cdots,$$

which is exact at every point except at $P_0 \to 0$, then this complex — this resolution of M — has the same homology as the complex consisting only of M in the zeroth position:

$$\cdots \to 0 \to M \to 0 \to \cdots.$$

The exact details of the localization procedure and more arguments for why derived categories are interesting, although interesting, are beyond the scope of this thesis. Thus we will confine ourselves to recalling some results on derived categories relevant to our needs. The following result is well known and true in more generality, although we will confine us to the one specific case relevant to us.

Proposition 7 *Let A and A' be two algebras of Dynkin quivers of the same type (say, $\mathbb{D}_n$), but with different orientations. Then A and A' are derived equivalent.*

That is, if $\mathcal{D}^b(A)$ is the bounded derived category of A and $\mathcal{D}^b(A')$ is the derived category of A', there is an equivalence of categories

$$\mathcal{D}^b(A) \simeq \mathcal{D}^b(A'). \qquad \square$$

PROOF This follows from theorem 9 in section 5.3. ■

As all of our results in the sequel will depend only on derived categories (and not module categories directly) related to $k\mathbb{D}_n$, we may safely pick any orientation on $\mathbb{D}_n$ and be sure that our results will be true for any other orientation.

We recall also that for any algebra A, the stabilised module category $\underline{\mathrm{mod}}\, A$ has a triangulated structure with shift or suspension funtor $[1] := \Omega^{-1}$.

Finally, we recall the definition of a Serre functor.

Definition 7 Let $\mathcal{C}$ be a triangulated category with suspension functor $[1]$. An additive equivalence $\mathbb{S}$ of $\mathcal{C}$ is said to be a **Serre functor** if

$$[1] \circ \mathbb{S} \simeq \mathbb{S} \circ [1]$$

and for each pair $A, B \in \mathcal{C}$, there is an equivalence

$$\operatorname{Hom}_{\mathcal{C}}(A, B) \simeq D \operatorname{Hom}_{\mathcal{C}}(B, \mathbb{S}A). \qquad \square$$

We remark that the Nakayama autofunctor ν on $\mathcal{D}^b(A)$, induced from the Nakayama automorphism on $\underline{\operatorname{mod}}\, A$ for an algebra A, is a Serre functor.

5.2 Notation and Calabi–Yau property

This section serves to introduce and clarify some notation as well as introducing the fractional Calabi–Yau property for derived categories of quiver algebras of type $\mathbb{D}_n$. Further, we prove some results following from the fractional Calabi–Yau property, which we will need in the sequel.

Our reference for this section is [16], to which we refer to reader for more on the fractional Calabi–Yau property.

Throughout this section, we are working in the derived category $\mathcal{D}^b(k\mathbb{D}_n)$; $[1]$ is the shift functor in this category and ν is the Nakayama functor in this category, which is induced by the Nakayama automorphism of $k\mathbb{D}_n$. We mention that ν is a *Serre functor* on $\mathcal{D}^b(k\mathbb{D}_n)$.

Definition 8 An algebra A is **twisted fractionally $\frac{a}{b}$-Calabi–Yau** (abbreviated **$\frac{a}{b}$-CY**) if

$$\nu^b \simeq [a] \circ \phi^*$$

as functors for some integers a and b, where $a \neq 0$ and ϕ^* is the functor induced by an endomorphism ϕ of A.

In the special case $\phi = \mathbb{1}_A$, A is **fractionally $\frac{a}{b}$-Calabi–Yau** (abbreviated **twisted $\frac{a}{b}$-CY**). $\square$

By [16, Proposition 3.1], $k\mathbb{D}_n$ is fractionally $\frac{n-2}{n-1}$-CY if n is even and fractionally $\frac{2n-4}{2n-2}$-CY if n is odd, so we have the following

Proposition 8 *If n is even then,*

$$\nu^{n-1} \simeq [n-2],$$

and if n is odd then,

$$\nu^{2n-2} \simeq [2n-4]. \qquad \square$$

Also, by [16, Proposition 3.2], if n is odd, then $k\mathbb{D}_n$ is twisted $\frac{n-2}{n-1}$-CY where $\sigma := \phi^*$ is induced by the involution of $\mathbb{D}_n$, given by $(n-1) \mapsto n$, $n \mapsto (n-1)$ and otherwise $i \mapsto i$, so we have the following

Proposition 9 *If n is odd, then*

$$\nu^{n-1} \simeq [n-2] \circ \sigma,$$

where σ is the aforementioned involution. □

The following proposition on the characterization of the Auslander–Reiten translation in the derived category is well known; see [14, p. 37].

Proposition 10

$$\tau \simeq \nu \circ [-1].$$ □

From the above two propositions, we have the following corollaries.

Corollary 2

$$[2] \simeq \tau^{2-2n}$$ □

PROOF By the fractional Calabi–Yau property and proposition 10,

$$\begin{aligned} \tau^{2-2n} &\simeq \nu^{2-2n} \circ [2n-2] \\ &\simeq [-(2n-4)] \circ [2n-2] \\ &\simeq [2]. \end{aligned}$$ ■

Corollary 3

$$\tau^{3-2n} \simeq \nu \circ [1]$$ □

PROOF By proposition 10 and corollary 2 above,

$$\begin{aligned} \nu \circ [1] &\simeq \tau \circ [2] \\ &\simeq \tau \circ \tau^{2-2n} \\ &\simeq \tau^{3-2n}. \end{aligned}$$ ■

5.3 Repetitive algebras and Happel's theorem

Before we state Happel's theorem, we need to introduce the notion of the repetitive algebra of an algebra.

Definition 9 Let A be an algebra. We define its repetitive algebra $\widehat{A}$ as follows. The additive structure of $\widehat{A}$ is

$$\widehat{A} \coloneqq \bigoplus_{i \in \mathbb{Z}} (A \oplus D(A)),$$

where $D(A)$ is the dual of A; the multiplicative structure is given by

$$(a_i, \phi_i)_i \cdot (a'_i, \phi'_i) = (a_i a'_i, a_i \phi'_i + \phi_i a'_i).$$

where $D(A)$ is given the same A-A^{op}-bimodule structure as the one defined in section 2.1. □

Note how both the additive and multiplicative structures of $\widehat{A}$ mimic those of the trivial extension algebra $T(A)$ of A (defined in section 2.1), only componentwise. In fact, we note in passing, it is possible to show that $\operatorname{mod} \widehat{A}$ is equivalent to the algebra of $\mathbb{Z}$-graded $T(A)$-modules. (See [14, p. 64].)

Repetitive categories were first introduced in [17]; also see [14].

Endowed with the repetitive algebra, we now state Happel's theorem.

Theorem 8 *Let A be an algebra of finite global dimension. Then there is a triangle equivalence of categories*

$$\mathcal{D}^b(A) \simeq \underline{\operatorname{mod}}\, \widehat{A}. \qquad \square$$

PROOF See [14, p. 88]. ∎

Moreover, we have the following theorem, also due to Happel.

Theorem 9 *Let Q be a quiver of Dynkin type and let $\mathcal{T}(Q)$ be its mesh category, as defined in section 2.2. There is an equivalence*

$$\mathcal{T}(Q) \simeq \operatorname{ind} \mathcal{D}^b(kQ),$$

where the latter is the category of indecomposable objects in $\mathcal{D}^b(kQ)$. Moreover, the τ-functors in the two categories correspond to each other. □

PROOF See [14, p. 55]. ∎

5.4 Last step of the equivalence

Throughout this section, let Q be of Dynkin type type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{D}_n$ or $\mathbb{A}_n, n \geq 3$.

In the previous section we considered two now classical equivalences due to Happel. If we limit our attention to kQ — which we will in the sequel — the two equivalences can be stated as

$$\mathcal{D}^b(kQ) \simeq \underline{\operatorname{mod}}\, \widehat{kQ}$$

and

$$\mathcal{T}(Q) \simeq \operatorname{ind} \mathcal{D}^b(kQ).$$

Our goal in this subsection is to prove that finding a d-cluster-tilting module of $T(kQ)$ is equivalent to finding a d-cluster-tilting subcategory of a certain factor category $\mathfrak{T}$ (defined in the end of this subsection) of the mesh category $\mathcal{T}(Q)$, defined in section 2.2.

To do so, we will need a few more recent results by Darpö and Iyama, given in [6], as well as one result by Gabriel given in [10]. In the former paper, the authors prove their results in greater generality than what is needed here: we will confine us to $\widehat{kQ}$.

Lemma 9 *Let G be a group acting on $\widehat{kQ}$.*

The push-down functor $F_* : \operatorname{mod} \widehat{kQ} \to \operatorname{mod}(\widehat{kQ}/G)$ induces an equivalence

$$(\operatorname{mod} \widehat{kQ})/G \simeq \operatorname{mod}(\widehat{kQ}/G) \qquad \square$$

PROOF Combine [6, Lemma 3.5(c)] to get that the induced functor is full and faithful and [10, Theorem 3.6] to get that it is dense. ■

The following lemma is adopted from [6, Corollary 2.5]. Given an autoequivalence ϕ of $\widehat{kQ}$, we define the autofunctor $\phi_* : \operatorname{mod} \widehat{kQ} \to \widehat{kQ}$ by $\phi_*(M) = M \circ \phi^{-1}$. A subcategory $\mathcal{U}$ is said to be *G-equivariant* for a group G if $g_*(\mathcal{U}) = \mathcal{U}$ for each $g \in G$.

Lemma 10 *Let k be an algebraically closed field and $G = \langle \phi \rangle$ be the group generated by an admissible automorphism ϕ of $\widehat{kQ}$. Then the push-down functor*

$$F_* : \operatorname{mod} \widehat{kQ} \to \operatorname{mod}(\widehat{kQ}/G)$$

induces a bijection from the class of G-equivariant d-cluster-tilting subcategories of $\operatorname{mod} \widehat{kQ}$ to the class of d-cluster-tilting modules of $\operatorname{mod}(\widehat{kQ}/G)$. □

Proposition 11 (Proposition 2.17) *Let ν be the Nakayama functor of $\widehat{kQ}$. $\underline{\operatorname{mod}}\, \widehat{kQ}$ has a Serre functor $\nu_* \circ \Omega$.* □

With the help of the above results, we can prove the following results.

Lemma 11

$$\underline{\operatorname{mod}}\, T(A) \simeq \underline{\operatorname{mod}}\, \widehat{kQ}/\widehat{\nu},$$

where $\widehat{\nu}$ is the Nakayama automorphisms of $\widehat{kQ}$. □

PROOF The repetitive category $\widehat{kQ}$ is self-injective and its Nakayama automorphism $\widehat{\nu} = \nu_{\widehat{kQ}}$ is given by degree-one shift (see [6, p. 11]). Thus the functor

$$F : \underline{\operatorname{mod}}\, T(kQ) \to \underline{\operatorname{mod}}\, \widehat{kQ}/\widehat{\nu},$$

given by inclusion into the first factor,

$$F(a, f) = (0, \ldots, 0, a, f, 0, \ldots, 0)\,,$$

is clearly an equivalence. ■

Lemma 12 *There is an equivalence*

$$\nu_* \simeq \nu \circ [1],$$

where ν is the Nakayama functor induced by the Nakayama automorphism of $\widehat{kQ}$ and ν_* is the functor defined in proposition 11 above. □

PROOF By proposition 11, $\nu_* \circ \Omega$ is a Serre functor for $\underline{\operatorname{mod}}\, \widehat{kQ}$. Moreover, it is well known that if a Serre functor exists, then it is unique up to equivalence. In particular, ν is also a Serre functor on $\underline{\operatorname{mod}}\, \widehat{kQ}$, and so

$$\nu_* \circ \Omega \simeq \nu.$$

Now note that $\Omega^{-1} = [1]$ is the shift functor with the triangulated structure on $\underline{\mathrm{mod}}\, \widehat{A}$, so

$$\nu_* \circ [-1] \simeq \nu,$$

and since $[1]$ an autoequivalence, we have

$$\nu_* \simeq \nu \circ [1]. \qquad \blacksquare$$

Now let $\mathfrak{T} := \mathfrak{T}(Q) := \mathcal{T}(Q)/(\tau^{3-2n})$. To summarise, we have the following

Theorem 10 *There is a triangle equivalence*

$$\underline{\mathrm{mod}}\, T(kQ) \simeq \mathcal{D}^b(kQ)/(\nu \circ [1])$$

and a triangle equivalence

$$\mathfrak{T}(Q) \simeq \mathrm{ind}\, \mathcal{D}^b(kQ)/(\nu \circ [1]). \qquad \square$$

Proof For the first triangle equivalence, combine lemma 11, lemma 9, lemma 12 and finally theorem 8.

For the second triangle equivalence, combine theorem 9 and corollary 3. $\blacksquare$

Since by taking finite direct summands of $\mathrm{ind}\, \mathcal{D}^b(kQ)/(\nu\circ[1])$, we obtain $\mathcal{D}^b(kQ)/(\nu\circ[1])$, the existence of a d-cluster-tilting module in $\underline{\mathrm{mod}}\, T(kQ)$ is equivalent to the existence of a d-cluster-tilting subcategory of $\mathfrak{T}$ by theorem 10. Since any d-cluster-tilting module in $\mathrm{mod}\, T(kQ)$ is also a d-cluster-tilting module considered as an object of $\underline{\mathrm{mod}}\, T(kQ)$ and — by possibly adding the indecomposable projectives as direct summands — vice versa, we have the following

Theorem 11 *There exists a d-cluster-tilting module of $T(kQ)$ if and only if there exists a d-cluster-tilting subcategory of $\mathfrak{T}(Q)$.* $\square$

Our strategy in the sequel will be to leverage theorem 11 to obtain results on the existence or non-existence of cluster-tilting modules of $T(kQ)$, by working in $\mathfrak{T}$. Given that $\mathfrak{T}$ has only a finite number of objects and morphisms of objects, this allows us to work combinatorially.

5.5 Looking for cluster-tilting subcategories of $\mathfrak{T}(\mathbb{D}_n)$

For this subsection, we will be working exclusively in the category $\mathfrak{T} = \mathfrak{T}(\mathbb{D}_n)$. By proposition 7, results proved in $\mathfrak{T}$ with one specific orientation on $\mathbb{D}_n$ will remain true for other orientations on $\mathbb{D}_n$ (since these categories are all equivalent). Thus we fix the following orientation on $\mathbb{D}_n$ throughout this section: there are arrows $i \to (i+1)$ for $i = 1, 2 \ldots, n-2$ as well as an arrow $(n-2) \to n$. The orientation on the repetitive quiver $\mathcal{T}(\mathbb{D}_n)$ follows that of definition 5. In this section, τ denotes the translation in $\mathfrak{T}$.

In $\mathfrak{T}$, the translation functor $[1]$ is induced from the syzygy functor Ω^{-1} in $\underline{\mathrm{mod}}\, T(kQ)$ by the triangle equivalence in theorem 10 and ν in $\mathfrak{T}$ is defined by $\nu = \tau \circ [1]$, as $\tau = \nu \circ [-1]$ in $\mathcal{D}^b(kQ)$.

We note that since $\nu \circ [1] \simeq \mathbb{1}_{\mathfrak{T}}$ and ν and $[1]$ are equivalences, we have $\nu \simeq [-1]$. Since ν is a Serre functor, then so is $[-1]$.

Whenever we mention a **nonzero path** (or simply a **path**) in $\mathfrak{T}(\mathbb{D}_n)$, we refer to a nonzero morphism in $\mathfrak{T}(\mathbb{D}_n)$. That is, one that is out canceled out by the relations m_x in $\mathfrak{T}(\mathbb{D}_n)$ (see definition 5).

5.5.1 Some results in $\mathfrak{T}$

In this subsection, we mention several results that we will need in the sequel. These results may be referred to without reference.

Lemma 13 *If n is even, then*

$$[2n-3] \simeq \mathbb{1}_{\mathfrak{T}}$$

and if n is odd, then

$$[4n-6] \simeq \mathbb{1}_{\mathfrak{T}}. \qquad \square$$

PROOF By proposition 8, if n is even, then

$$\begin{aligned} \mathbb{1}_{\mathfrak{T}} &\simeq (\nu \circ [1])^{n-1} \\ &\simeq \nu^{n-1} \circ [n-1] \\ &\simeq [n-2] \circ [n-1] \\ &\simeq [2n-3]. \end{aligned}$$

Otherwise, if n is odd, we have

$$\begin{aligned} \mathbb{1}_{\mathfrak{T}} &\simeq (\nu \circ [1])^{2n-2} \\ &\simeq \nu^{2n-2} \circ [2n-2] \\ &\simeq [2n-4] \circ [2n-2] \\ &\simeq [4n-6]. \end{aligned} \qquad \blacksquare$$

Lemma 14

$$\tau^{2n-3} \simeq \mathbb{1}_{\mathfrak{T}} \qquad \square$$

PROOF By lemma 13 above and the facts that $\tau = \nu \circ [-1]$ and $\nu \simeq [-1]$, we have

$$\begin{aligned} \tau^{2n-3} &\simeq (\nu \circ [-1])^{2n-3} \\ &\simeq ([-2])^{2n-3} \\ &\simeq [-2(2n-3)] \\ &\simeq ([4n-6])^{-1} \\ &\simeq \mathbb{1}_{\mathfrak{T}}. \end{aligned} \qquad \blacksquare$$

Lemma 15 *If n is even, then*

$$[1] \simeq \tau^{1-n},$$

and if n is odd, then

$$[1] \circ \sigma \simeq \tau^{1-n},$$

where σ is the involution given in definition 8. □

PROOF By the fractional Calabi–Yau property definition 8, we have the following if n is even:

$$\begin{aligned}\tau^{1-n} &\simeq ([-2])^{1-n}\\ &\simeq [2n-2]\\ &\simeq [2n-3] \circ [1]\\ &\simeq [1].\end{aligned}$$

By the twisted fractional Calabi–Yau property definition 8, we have the following if n is odd:

$$\begin{aligned}\tau^{1-n} &\simeq ([-2])^{1-n}\\ &\simeq [2n-2]\\ &\simeq [2n-3] \circ [1] \circ \sigma\\ &\simeq [1] \circ \sigma.\end{aligned}$$ ■

In particular, we have a useful characterisation for suspensions of vertices by even integers.

Lemma 16 *Let $k \in \mathbb{Z}$ and $(i,j) \in \mathfrak{T}(\mathbb{D}_n)$. Then*

$$(i,j)[2k] = (i+k, j).$$ □

PROOF Since σ has order two, we have the following by lemma 15, regardless of the parity of n.

$$\begin{aligned}(i,j)[2] &\simeq \tau^{2(1-n)}(i,j)\\ &\simeq (\tau^{2-2n} \circ \tau^{2n-3})(i,j)\\ &\simeq \tau^{-1}(i,j)\\ &\simeq (i+1,j).\end{aligned}$$

Hence

$$\begin{aligned}(i,j)[2k] &\simeq \tau^{-k}(i,j)\\ &\simeq (i+k,j).\end{aligned}$$ ■

The following proposition is well-known, and will reduce the work on showing if a subcategory $\mathcal{U}$ of $\mathfrak{T}(\mathbb{D}_n)$ is d-cluster-tilting or not.

Proposition 12 *A subcategory $\mathcal{U}$ is a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$ if and only if*

$$\mathcal{U} = \left\{ x \in \mathfrak{T}(\mathbb{D}_n) : \operatorname{Ext}_{\mathfrak{T}}^{1,2,\dots,d-1}(x, \mathcal{U}) = 0 \right\},$$

which is true if and only if

$$\mathcal{U} = \left\{ x \in \mathfrak{T}(\mathbb{D}_n) : \operatorname{Ext}_{\mathfrak{T}}^{1,2,\dots,d-1}(\mathcal{U}, x) = 0 \right\}, \qquad \square$$

5.5.2 Restrictions on d and n

Using the results in the previous subsection, we obtain some restrictions on d and n the existence of a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$ would imply. Let $\mathcal{U}$ be a putative d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. By proposition 5, then

$$\nu_d(\mathcal{U}) = \nu\left(\mathcal{U}[-d]\right) = \mathcal{U}$$

Since $\nu \simeq [-1]$ in $\mathfrak{T}$, this is equivalent to

$$\mathcal{U}[-(d+1)] = \mathcal{U} = \mathcal{U}[d+1].$$

Assuming for the moment that n is even, by lemma 13 we have

$$\mathbb{1}_{\mathfrak{T}} \simeq [2n-3],$$

from which it follows that

$$\mathcal{U}[2n-3] \simeq \mathcal{U}$$

also; whence,

$$\begin{aligned} \mathcal{U} &\simeq \mathcal{U}[2n-3] \\ &\simeq \mathcal{U}[2n-3-(d+1)] \\ &\simeq \cdots \\ &\simeq \mathcal{U}[2n-3-r(d+1)] \\ &\simeq \mathcal{U}[a], \end{aligned}$$

where a is $2n-3$ modulo $(d+1)$. Now if $(d+1)$ does not divide $(2n-3)$, then $1 \leq a \leq d$, and so

$$\begin{aligned} 0 &= \operatorname{Ext}_{\mathfrak{T}}^{a}(\mathcal{U}, \mathcal{U}) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U}, \mathcal{U}[a]) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U}, \mathcal{U}) \\ &\neq 0, \end{aligned}$$

as long as $a \neq d$. If $a = d$, then

$$\begin{aligned} 0 &= \operatorname{Ext}^1_{\mathfrak{T}}(\mathcal{U},\mathcal{U}) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U}[-1],\mathcal{U}) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U}[-1],\mathcal{U}[d]) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U}[-(d+1)],\mathcal{U}) \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}(\mathcal{U},\mathcal{U}) \\ &\neq 0. \end{aligned}$$

Either case yields self-extensions of $\mathcal{U}$, which contradicts it being a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. Thus, we have proved the following

Theorem 12 *If n is even and there is a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$, then*

$$(d+1) \mid (2n-3). \qquad \square$$

Corollary 4 *If d is odd and n is even, there is no d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$.* $\square$

If on the other hand n is odd, we can repeat a similar argument to the one above, noting that

$$\mathcal{U}[4n-6] \simeq \mathcal{U}$$

by lemma 13, to obtain the following

Theorem 13 *If n is odd and there is a d-cluster-tilting subcategory, then*

$$(d+1) \mid 2(2n-3). \qquad \square$$

Lemma 17 *Suppose, for $d \geq 2$ and n odd, that there is a d-cluster-tilting subcategory $\mathcal{U}$ of $\mathfrak{T}(\mathbb{D}_n)$ that is invariant under the functor σ from proposition 9. Then*

$$(d+1) \mid (2n-3). \qquad \square$$

Proof If n is odd, we have

$$\nu^{n-1} \simeq [n-2] \circ \sigma$$

by proposition 9, and since

$$\sigma(\mathcal{U}) = \mathcal{U},$$

where σ is the functor described in proposition 9, we get

$$\mathcal{U}[2n-3] = \mathcal{U}.$$

Now apply the proof of theorem 12. ■

5.5.3 Restrictions on d

Suppose $d \geq 3$. For what values of $1 \leq j \leq n$ can the vertex (i, j) be part of a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$? In general,

$$\begin{aligned} 0 &= \operatorname{Ext}^2_{\mathfrak{T}}[(i, j), (i, j)] \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), (i, j)[2]] \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), \tau^{-1}(i, j)] \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), (i+1, j)], \end{aligned}$$

and if $2 \leq j \leq n-2$, then there is a nonzero path

$$(i, j) \to (i, j+1) \to (i+1, j),$$

which implies that $\operatorname{Hom}_{\mathfrak{T}}[(i, j), (i+1, j)] \neq 0$. Thus if (i, j) was part of a d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$, said subcategory would have a self-extension, a contradiction. Whence we have proved the following

Lemma 18 *If $d \geq 3$ and $(i, j) \in \mathcal{U}$ for a d-cluster-tilting subcategory $\mathcal{U}$ of $\mathfrak{T}(\mathbb{D}_n)$, then $j = 1$, $j = n-1$ or $j = n$.* □

If we moreover assume that $d \geq 5$ and $j = n-1$ or $j = n$, we necessarily need that the following be zero

$$\begin{aligned} \operatorname{Ext}^4_{\mathfrak{T}}[(i, j), (i, j)] &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), (i, j)[4]] \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), \tau^{-2}(i, j)] \\ &\simeq \operatorname{Hom}_{\mathfrak{T}}[(i, j), (i+2, j)]. \end{aligned}$$

But if $j = n-1$, there is a nonzero path

$$(i, n-1) \to (i+1, n-2) \to (i+1, n) \to (i+2, n-1),$$

contradicting $\operatorname{Hom}_{\mathfrak{T}}[(i, n-1), (i+2, n-1)] = 0$. Similarly, if $j = n$, there is a nonzero path

$$(i, n) \to (i+1, n-2) \to (i+1, n) \to (i+2, n),$$

contradicting $\operatorname{Hom}_{\mathfrak{T}}[(i, n), (i+2, n)] = 0$. Whence we have proved the following

Lemma 19 *If $d \geq 5$ and $(i, j) \in \mathcal{U}$ for a d-cluster-tilting subcategory $\mathcal{U}$ of $\mathfrak{T}(\mathbb{D}_n)$, then $j = 1$.* □

Lemmas 18 and 19 hint at dividing the classification problem into four parts: $d \geq 5$, $d = 4$, $d = 3$ and finally $d = 2$. In the next four subsections, we consider each of these cases. Combined, these sections prove the following

Theorem 14 *Let $d \geq 2$. If $d \neq 4$, there is no d-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. If $n = 4$, there is a 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$.* □

Combining the above theorem with theorem 11, we arrive at our main theorem for this section.

Theorem 15 *Let $A = T(k\mathbb{D}_n)$ be the trivial extension of the algebra $k\mathbb{D}_n$, for some orientation on $\mathbb{D}_n$, and let $d \geq 2$. If $d \neq 4$, there is no d-cluster-tilting module of A; if $n = 4$, then there is a 4-cluster-tilting module of A.* □

5.5.4 $d \geq 5$

We begin by considering the case $d \geq 5$. First, we have the following

Lemma 20 *If $d \geq 4$, there is no subcategory $\mathcal{U}$ of $\mathfrak{T}(\mathbb{D}_n)$ consisting only of vertices of the form $(i, 1)$ that is 4-cluster-tilting.* □

Proof We may assume without loss of generality that

$$\mathcal{U} = \left\{\nu_d^\ell(0, 1) : \ell \in \mathbb{Z}\right\}$$

by symmetry and since otherwise, if there were a vertex $(i, 1) \in \mathcal{U}$, it would clearly have extensions with the above. Now suppose that for some vertex $(i, 1) \in \mathfrak{T}(\mathbb{D}_n)$ we have

$$\mathrm{Ext}^k_{\mathfrak{T}}((i, 1), (0, n)) \neq 0,$$

where $1 \leq k \leq d - 1$. Then

$$\mathrm{Ext}^k_{\mathfrak{T}}((i, 1), (0, n)) \simeq \mathrm{Hom}_{\mathfrak{T}}((i, 1)[-k], (0, n)) \neq 0,$$

which implies that $(i, 1)[-k] = (0, 1)$. Consequently,

$$\mathrm{Ext}^k_{\mathfrak{T}}((i, 1), (0, 1)) \simeq \mathrm{Hom}_{\mathfrak{T}}((i, 1)[-k], (0, 1)) \neq 0,$$

so $(i, 1) \notin \mathcal{U}$. But then

$$\mathrm{Ext}^k_{\mathfrak{T}}(\mathcal{U}, (0, n)) = 0,$$

which implies $(0, n) \in \mathcal{U}$, contradicting the hypothesis. ■

The above lemma yields a short proof to the following

Theorem 16 *If $d \geq 5$, there is no d-cluster-tilting subcategories of $\mathfrak{T}(\mathbb{D}_n)$.* □

Proof By lemma 19, a putative d-cluster-tilting subcategory $\mathcal{U}$ would consist only of vertices of the form $(i, 1)$. By lemma 20, this is not possible. ■

5.5.5 $d = 4$

Now we investigate the case $d = 4$. Since then $d + 1 = 5$, the existence of a 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$ implies that

$$5 \mid (2n - 3) \tag{6}$$

by theorems 12 and 13. A simple modulo calculation shows that eq. (6) is equivalent to

$$5 \mid (n + 1). \tag{7}$$

Now let $\mathcal{U}$ be a putative 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. If $(i, j) \in \mathcal{U}$, then $j \in \{1, n - 1, n\}$ by lemma 18. Moreover, if n is odd we have the following

Lemma 21 *If n is odd and $(i, j) \in \mathcal{U}$, then $j = 1$.* □

PROOF Without loss of generality, we may assume that $i = 0$. By definition 8,,

$$[1] \simeq \tau^{1-n} \circ \sigma,$$

where $\sigma(0, n-1) = (0, n)$, $\sigma(0, n) = (0, n-1)$ and $\sigma(0, 1) = (0, 1)$. If $(0, j) \in \mathcal{U}$, then $(0, j)[2n-3] \in \mathcal{U}$. Now

$$(0, j)[2n-3] = \sigma((2n-3)(n-1), j) = \sigma(0, j).$$

If $j = n-1$, then

$$(0, n-1)[2n-3] = (0, n) \in \mathcal{U},$$

since $5 \mid 2n-3$ and $\mathcal{U}$ is invariant under $\nu_d = [5]$ by proposition 5. Now note that there is a path

$$(0, n) \to (1, n-1) = (0, n-1)[2],$$

which gives $\mathcal{U}$ a self-extension:

$$\operatorname{Ext}^2_{\mathfrak{T}}((0, n), (0, n-1)) \simeq \operatorname{Hom}_{\mathfrak{T}}((0, n), (0, n-1)[2]) \neq 0,$$

contradicting $\mathcal{U}$ being a 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. Conversely, suppose that $j = n$. Then

$$(0, n)[2n-3] = (0, n-1) \in \mathcal{U},$$

and there is a path

$$(0, n-1) \to (1, n) = (0, n)[2],$$

which again implies that $\mathcal{U}$ has a self-extension. Hence only $j = 1$ is possible. ■

If n is odd still, then $\mathcal{U}$ must consist only of vertices of the form $(i, 1)$ (by lemma 21, this is the only possibility). But this is not possible by lemma 20. Thus we have proven the following

Lemma 22 *If n is odd, then there is no 4-cluster-tilting subcategories of $\mathfrak{T}(\mathbb{D}_n)$.* □

Combined with eq. (7), lemma 22 gives that

$$5 \mid (n+1)$$

and

$$2 \mid n,$$

whence

$$n = 4 + 10k$$

for some $k \in \mathbb{N}$ are the only possible values for n, given the existence of a 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$. If $k = 0$, we have an explicit example.

Example 3 $\mathcal{U} = \{(0, 1), (0, 4)\}$ defines a 4-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_4)$. □

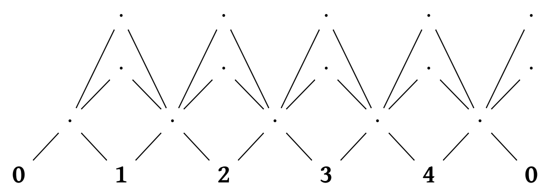

Figure 1: Repetitive quiver $\mathfrak{T}(\mathbb{D}_4)$ for $\mathbb{D}_4$

The verification of example 3 is easy and left to the reader; see fig. 1.

Our conjecture is that $n = 4$ is the only example where $\mathfrak{T}(\mathbb{D}_n)$ possess a 4-cluster-tilting subcategory (hence, *any* d-cluster-tilting subcategory, for $d \geq 2$). However, we have not succeeded in proving this yet. Collecting our results for this subsection, however, we have obtained a partial result.

Proposition 13 *Suppose there is a* 4*-cluster-tilting subcategory of* $\mathfrak{T}(\mathbb{D}_n)$*. Then* $n = 4 + 10k$ *for some* $k \in \mathbb{N}$ *and if* $(i, j) \in \mathcal{U}$*, then* $j = 1$*,* $j = n - 1$ *or* $j = n$*.* □

5.5.6 $d = 3$

In this subsection, we investigate the case $d = 3$. By corollary 4, there can be no 3-cluster-tilting subcategories of $\mathfrak{T}$ if n is even, so we may limit ourselves to the case where n is odd. In this case, theorem 13 gives

$$4 \mid 2(2n - 3),$$

from which we get

$$2 \mid (2n - 3),$$

which is a contradiction since $2n-3$ is always odd. Thus we have proven the following

Theorem 17 *There are no* 3*-cluster-tilting subcategories of* $\mathfrak{T}(\mathbb{D}_n)$*.* □

5.5.7 $d = 2$

In [12], Grimeland classifies all 2-cluster-tilting subcategories of the category $\mathcal{D}^b\left(k\mathbb{D}_n\right)/F$, for $n \geq 4$. Her result is that said category has a 2-cluster-tilting subcategory iff F is one of the following.

- τ^{tn} for $n \geq 5$ odd
- $\tau^{tn-1}[1]$ for $n \geq 5$ odd
- τ^{2t} for $n \geq 4$
- $\tau^{2t-1}[1]$ for $n \geq 5$ odd

where $t \in \mathbb{Z}$. We leverage this result to immediately obtain a full classification of the 2-cluster-tilting subcategories of $\mathfrak{T}(\mathbb{D}_n)$.

Suppose first that $n \geq 4$ is even. Then $F = \tau^{2n-3}$, but $2n-3$ is not even, so by the above result there cannot be a 2-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$.

Now suppose that $n \geq 5$ is odd. Then still $F = \tau^{2n-3}$, but we have more options. Note first that $2n-3 = tn$ is clearly never possible for $n \geq 5$. The other two cases involve $[1]$. From lemma 15, we know that

$$[1] \simeq \tau^{1-n} \circ \sigma,$$

but then $F = \tau^{2n-3}$ cannot involve a term with $[1]$. This takes care of the two remaining cases and shows that no 2-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$ exists for any $n \geq 4$. We write up our discussion as a

Theorem 18 *If $n \geq 4$, then there is no 2-cluster-tilting subcategory of $\mathfrak{T}(\mathbb{D}_n)$.* □

5.6 Looking for cluster-tilting categories of $\mathfrak{T}(\mathbb{A}_n)$ and $\mathfrak{T}(\mathbb{E}_n)$

We mention briefly that the machinery developed in section 5.4 can be applied also to trivial extensions of algebras of Dynkin type $\mathbb{A}$ or $\mathbb{E}$.

Trivial extensions of algebras of Dynkin type $\mathbb{A}$ are known to be symmetric Nakyama algebras; these are already classified in section 3.

However, the author — that is, I — am not sure if trivial extensions of algebras of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ are classified, yet. We do not provide a classification here, but show that we can use the machinery already developed to get only one possible value for d in each. That is, we have the following

Proposition 14 *Let $d \geq 2$ and suppose that there is a d-cluster-tilting subcategory of $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, respectively. Then d is 10, 16 or 28, respectively.* □

Proof It is known (see [16]) that $\mathbb{E}_6$ is $\mathbb{E}_6$ is fractional $\frac{10}{12}$-Calabi–Yau, while $\mathbb{E}_7$ is fractional $\frac{8}{9}$-Calabi–Yau and $\mathbb{E}_8$ is fractional $\frac{14}{15}$-Calabi–Yau.

Carrying out a similar argument to the one we gave in the proof of theorem 12 and applying proposition 5, yields the result. ■

The above proposition is not conclusive, so further research could look at this case or expand the machinery to other algebras that have a similar representation in terms of truncated repetitive quivers.

References

[1] Ibrahim Assem, Andrzej Skowronski, and Daniel Simson. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Cambridge University Press, 2006.

[2] Maurice Auslander. Representation theory of Artin algebras, 1971.

[3] Maurice Auslander. Representation dimension of artin algebras. *Selected works of Maurice Auslander*, 1:505–574, 1999.

[4] Maurice Auslander, Idun Reiten, and Sverre O Smalo. *Representation theory of Artin algebras*, volume 36. Cambridge university press, 1997.

[5] K Bongartz. Tilted algebras, representations of algebras (puebla, 1980), 26-38. *Lecture Notes in Math*, 903.

[6] Erik Darpö and Osamu Iyama. d-representation-finite self-injective algebras. *Advances in Mathematics*, 362:106932, 2020.

[7] Karin Erdmann. *Blocks of Tame Representation Type and Related Algebras*. Springer, 1990.

[8] Karin Erdmann and Thorsten Holm. Maximal n-orthogonal modules for self-injective algebras. *Proceedings of the American Mathematical Society*, 136(9):3069–3078, 2008.

[9] Karin Erdmann, Sira Gratz, and Lisa Lamberti. Cluster tilting modules for mesh algebras, 2019.

[10] Pierre Gabriel. The universal cover of a representation-finite algebra. In *Representations of algebras*, pages 68–105. Springer, 1981.

[11] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Science & Business Media, 2013.

[12] Benedikte Grimeland. Periodicity of cluster tilting objects, 2016.

[13] Benedikte Grimeland and Karin M Jacobsen. Realizing orbit categories as stable module categories: a complete classification. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 58(4):795–818, 2017.

[14] Dieter Happel. *Triangulated categories in the representation of finite dimensional algebras*, volume 119. Cambridge University Press, 1988.

[15] Martin Herschend and Osamu Iyama. n-representation-finite algebras and twisted fractionally calabi–yau algebras. *Bulletin of the London Mathematical Society*, 43(3):449–466, 2011.

[16] Martin Herschend and Osamu Iyama. n-representation-finite algebras and twisted fractionally Calabi–Yau algebras. *Bulletin of the London Mathematical Society*, 43(3):449–466, 2011.

[17] David Hughes and Josef Waschbüsch. Trivial extensions of tilted algebras. *Proceedings of the London Mathematical Society*, 3(2):347–364, 1983.

[18] Osamu Iyama. Auslander correspondence. *Advances in Mathematics*, 210(1):51–82, 2007.

[19] Osamu Iyama. Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories. *Advances in Mathematics*, 210(1):22–50, 2007.

[20] Osamu Iyama. Auslander–Reiten theory revisited. *Trends in representation theory of algebras and related topics*, pages 349–398, 2008.

[21] Osamu Iyama and Yuji Yoshino. Mutation in triangulated categories and rigid Cohen–Macaulay modules. *Inventiones mathematicae*, 172(1):117–168, 2008.

[22] Herbert Kupisch. Basisalgebren symmetrischer Algebren und eine Vermutung von Gabriel. 1978.

[23] Jun-ichi Miyachi and Amnon Yekutieli. Derived picard groups of finite-dimensional hereditary algebras. *Compositio Mathematica*, 129(3):341–368, 2001.

[24] Ralf Schiffler. *Quiver Representations*. Springer, 2014.

[25] Andrzej Skowronski. Selfinjective algebras: finite and tame type. *Contemporary Mathematics*, 406:169–238, 2006.

[26] Andrzej Skowroński and Kunio Yamagata. *Frobenius algebras I*. European Mathematical Society, 2011.

[27] Hiroyuki Tachikawa. Representations of trivial extensions of hereditary algebras. In *Representation theory II*, pages 579–599. Springer, 1980.

[28] Charles A. Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1995.