Viktor Balch Barth

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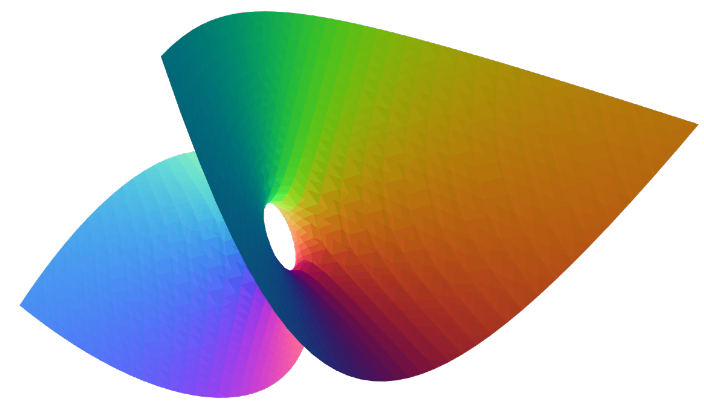
July 2020



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Summary

We study naive motivic homotopy classes of endomorphisms of the projective line over a field. We first give an account of the result in [Caz12] that the canonical map from naive to motivic homotopy classes is a group completion. We proceed to study maps from the Jouanolou device to the projective line, where there is a bijection between the naive and motivic homotopy classes. Looking at which of these maps factor through the Hopf map gives us a partial classification of the naive homotopy classes.

Sammendrag

Vi studerer naive motiviske homotopiklasser av endomorfier av den projektive linja over en kropp. Først redegjør vi for resultatet fra [Caz12] om at den kanoniske funksjonen fra naive til motiviske homotopiklasser er en gruppekomplettering. Så fortsetter med å studere morfier fra Jouanolou-anordningen til den projektive linja, der det er en bijeksjon mellom de naive og motiviske homotopiklassene. Når vi ser på hvilke av disse funksjonene som faktoriserer gjennom Hopf-funksjonen, gir det oss en delvis klassifisering av de naive homotopiklassene.

Preface

This thesis concludes the course MA3911 – Master Thesis in Mathematical Sciences, and marks my completion of the Master's Programme in Mathematical Sciences with specialization in topology at NTNU. The thesis is based on work done in collaboration with William Hornslien, but we have written separate theses.

I am very grateful and indebted to my advisors Glen Wilson and Gereon Quick for their guidance, patience, advice and mathematical discussions. Special thanks to Arvind Asok for offering hospitality and mathematical advice during my stay at the University of Southern California (USC). I must also thank the Norwegian Mathematical Society for granting me the Abel scholarship, which enabled me to travel to USC.

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I'm extremely grateful to Åsil for her unwavering support and patience. Many thanks also to my friends and family for their support, and for all the fun and good times, which has been especially important to me in these times of pandemic and thesis writing.

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Chapter 1

Introduction

To do algebraic topology in an algebro-geometric setting, we need to make some adjustments. For instance, we can not use the unit interval $[0, 1]$ to define homotopies, as it is not a scheme. Replacing it with $\mathbb{A}^1$ allows us to define naive homotopies. Morel and Voevodsky's work [MV99] gives a way to do $\mathbb{A}^1$-homotopy theory in the category $\mathbf{Sm}_S$ of smooth schemes of finite type over a finite dimensional noetherian base scheme S. We are interested in the case of $\mathbf{Sm}_k$, where the base scheme is $\mathrm{Spec}\,(k)$ for some field k. We define the less technical notion of a naive homotopy, which we can then compare to the actual motivic homotopy theory.

Definition 1.0.1 (Definition 1.1 in [Caz12])**.** Let X and Y be two spaces in $\mathbf{Sm}_k$. A naive homotopy is a morphism

$$F : X \times \mathbb{A}^1 \to Y.$$

The restriction $\sigma(F) := F_{|X\times\{0\}}$ is the source of the homotopy and $\tau(F) := F_{|X\times\{1\}}$ is its target. When X and Y have base points, say x_0 and y_0, we say that F is pointed if its restriction to $\{x_0\} \times \mathbb{A}^1$ is constant equal to y_0.

There exists a monoid structure on the set of naive homotopy classes of endomorphisms $\left(\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right)$. There is also a group structure on the set of motivic homotopy classes $\left(\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$. Christophe Cazanave showed the following theorem in his article "Algebraic homotopy classes of rational functions".

Theorem 1.0.2 (Theorem 1.2 in [Caz12])**.** *The canonical map from the monoid of pointed naive homotopy classes of endomorphisms* $\left(\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right)$ *to the group*

$\left(\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1} \right)$ *of* $\mathbb{A}^1$*-homotopy classes of endomorphisms of* $\mathbb{P}^1$ *is a group completion.*

The Jouanolou device of $\mathbb{P}^1$ is an affine scheme which surjects onto $\mathbb{P}^1$ with affine fibers. It is $\mathbb{A}^1$-homotopy equivalent to $\mathbb{P}^1$, and so the homotopy classes of maps $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ are in bijection with the homotopy classes of maps from $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. The canonical map $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \longrightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection, due to a result of Asok, Hoyois and Wendt in [AHW18]. This suggests the existence of a group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$. We arrive at the following theorem.

Theorem 1.0.3. *Over a field k, the bijection* $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} \longrightarrow \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}$ *induces a group structure on* $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$. *The group* $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} = \left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}} \oplus \operatorname{Pic} \mathcal{J}$ *is a direct sum.*

This master's thesis consists of two parts. In the first part (Chapter 2 and 3) we present Cazanave's article and prove the main result. Our exposition follows his, quoting some definitions and theorems verbatim. We also state additional definitions, give illustrating examples, prove some extra lemmata and expand on explanations and proofs. Chapter 2 covers a lot of background material, the purpose of which is to make this thesis accessible for nonexpert readers. Chapter 3 is an exposition of the proof of Theorem 1.0.2. Throughout this first part, some results and definitions are stated in greater generality than needed, in order to refer back to them in the second part of the thesis.

In the second part (Chapter 4 and 5) we approach the problem in a completely different way, using the Jouanolou device to attempt to understand the group completion geometrically. Chapter 4 covers the construction of the Jouanolou device and morphisms from it to $\mathbb{P}^1$. In Chapter 5 we prove Theorem 1.0.3, as well as some results about $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ by looking at which of the morphisms factor through the Hopf map η. We also find a naive homotopy invariant on subfields of $\mathbb{R}$.

Chapter 2

Preliminary theory

2.1 Naive homotopies

Unlike in algebraic topology, a composition of naive homotopies is not necessarily a naive homotopy itself. Hence we need to define naive homotopy classes in the following way to ensure transitivity.

Definition 2.1.1 (Definition 2.5 in [Caz12])**.** Let f and g be two pointed rational functions over k. We say that f and g are in the same pointed naive homotopy class, and we write $f \overset{\mathrm{p}}{\sim} g$, if there exists a finite sequence of pointed homotopies, say (F_i) with $0 \leq i \leq N$, such that

- $\sigma(F_0) = f$ and $\tau(F_N) = g$;
- for every $0 \leq i \leq N-1$, we have $\tau(F_i) = \sigma(F_{i+1})$.

2.2 Pointed scheme endomorphisms of $\mathbb{P}^1$

Recall that the projective line over a commutative ring S is $\mathbb{P}^1_S = \mathrm{Proj}(S[x_0, x_1])$, where $\mathrm{Proj}(-)$ denotes the set of homogeneous prime ideals of a graded commutative ring, and the sheaf structure is as in [Har13, p. 76]. We may cover $\mathbb{P}^1$ by $U_0 = S[s] \simeq \mathbb{A}^1$ and $U_1 = S[t] \simeq \mathbb{A}^1$, by gluing by the map $s \longmapsto t^{-1}$ on the intersection $U_0 \cap U_1 \simeq \mathbb{A}^1 \setminus \{0\}$. We will often write this as $U_0 = S[x_1/x_0]$ and $U_1 = S[x_0/x_1]$, sometimes even shortening x_i/x_j to $x_{i/j}$.

Morphisms $X \longrightarrow \mathbb{A}^n$ are in one-to-one correspondence with elements of $\Gamma(X, \mathcal{O}_X)^n$. Morphisms to $\mathbb{P}^n$ are slightly more subtle. Hartshorne states:

Theorem 2.2.1 (II, Theorem 7.1 (b) in [Har13]). *Let A be a ring, and let X be a scheme over A. If $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, s_1, ..., s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique A-morphism φ: $X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.*

Proof. $\mathbb{P}^n_A$ is covered by the open affine schemes $D_+(x_i) = \operatorname{Spec}\left(A[x_{0/i}, x_{1/i}, ..., x_{n/i}]\right)$, while X is covered by open sets $X_i = \{P \in X | s_i(P) \neq 0\}$. Using the local isomorphisms $\varphi_i : \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$, we can view the fraction s_j/s_i as an element of $\Gamma(X_i, \mathcal{O}_{X_i})$. Using the duality of rings and affine schemes, each ring homomorphism

$$\begin{aligned} A[x_{0/i}, x_{1/i}, ..., x_{n/i}] &\longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \\ \frac{x_j}{x_i} &\longmapsto \frac{s_j}{s_i} \end{aligned}$$

corresponds to a morphism of schemes $X_i \longrightarrow D_+(x_i)$. Observe that the maps agree on the overlaps $X_i \cap X_j$, so they glue to give a morphism φ: $X \longrightarrow \mathbb{P}^n_A$. □

Remark 2.2.2. We will slightly abuse notation in the following way. A morphism $\varphi : X \longrightarrow \mathbb{P}^n$ is given by $(\mathcal{L}, s_0, s_1, ..., s_n)$ and can be written in "homogeneous coordinates" as

$$x \longmapsto [s_0(x) : s_1(x) : ... : s_n(x)].$$

Cazanave considers pointed endomorphisms of $\mathbb{P}^1_k$, where k is a field. We consider $\mathbb{P}^1$ to be pointed at $[1 : 0] = \infty$. By the theorem, an endomorphism corresponds to an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1$, and two generating sections s_0, s_1 of it.

Definition 2.2.3 (Picard group). Recall that an invertible sheaf (line bundle) on X is a locally free $\mathcal{O}_X$-module of rank 1. The Picard group of X, $\operatorname{Pic} X$, is a group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$. Notice that the group operation is well defined, the unit is $\mathcal{O}_X$, and that each $\mathcal{L}$ has as its inverse the dual sheaf $\mathcal{L}^\vee = \operatorname{Hom}(\mathcal{L}, \mathcal{O}_X)$.

Proposition 2.2.4. *The Picard group of $\mathbb{P}^1_k$ is isomorphic $\mathbb{Z}$.*

Proof. For each $n \in \mathbb{Z}$ there is a different invertible sheaf $\mathcal{O}(n)$ with global sections $\Gamma(\mathbb{P}^1, \mathcal{O}(n)) = k[x_0, x_1]_n$, homogeneous polynomials in x_0, x_1 of degree n. Take any line bundle $\mathcal{L}$ on $\mathbb{P}^1$. The restrictions $\mathcal{L}|_{U_0}$ and $\mathcal{L}|_{U_1}$ must be trivial. Restricting further to $U_0 \cap U_1$, there is a gluing map, i.e. an isomorphism of two trivial $k[s^{\pm 1}]$-modules. Such a module isomorphism is given by multiplication by an invertible element cs^n of $k[s^{\pm 1}]^\times$, where $c \in k^\times$, $n \in \mathbb{Z}$. This implies that the global sections are exactly the homogeneous polynomials in x_0, x_1 of degree n, hence $\mathcal{L} \simeq \mathcal{O}(n)$. □

A line bundle $\mathcal{O}(n)$ and two generating sections give rise to a morphism

$$[x_0 : x_1] \longmapsto [a_n x_0^n + a_{n-1} x_0^{n-1} x_1 + \cdots + a_0 x_1^n : b_n x_0^n + b_{n-1} x_0^{n-1} x_1 + \cdots + b_0 x_1^n].$$

A pointed endomorphism f sends $[1 : 0]$ to $[1 : 0] = [a_n : b_n]$, so we need $a_n \neq 0, b_n = 0$. Having taken care of $f([x_0 : 0]$, we may assume $x_1 \neq 0$ and work in coordinates $X = \frac{x_0}{x_1}$. Dividing through by $a_n x_1^n$, we get the following.

Lemma 2.2.5. *Any pointed endomorphism $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is given by*

$$f([x_0 : x_1]) = [X^n + a_{n-1} X^{n-1} + \cdots + a_0 : b_{n-1} X^{n-1} + \cdots + b_0]. \quad (2.1)$$

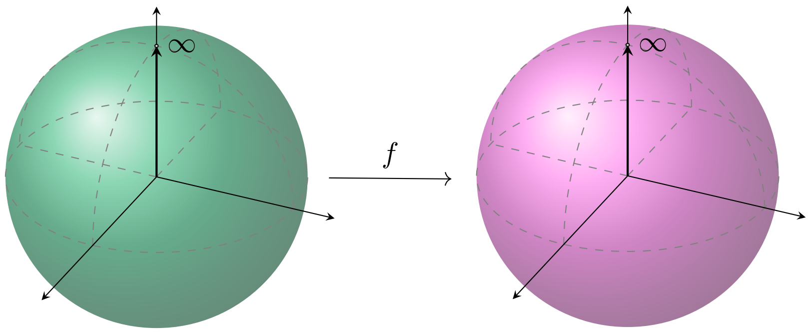


Figure 2.1: $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere. A pointed endomorphism $f : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ sends ∞ to ∞.

2.3 The scheme of pointed rational functions

The preceding discussion motivates the definition of the scheme of pointed degree n rational functions. In order to state it, we first need to define the resultant.

Definition 2.3.1 (Sylvester matrix and resultant)**.** Let $\mathrm{Syl}_{n,m}(A, B)$ denote the Sylvester matrix of the polynomials A and B (considered as polynomials of degree less or equal to n and m). It is a square $(n + m) \times (n + m)$-matrix with entries corresponding to the polynomial coefficients in the following way:

$$\mathrm{Syl}_{n,m}(A,B) = \begin{pmatrix} a_n & 0 & \cdots & 0 & b_m & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{m-1} & b_m & \cdots & 0 \\ a_{n-2} & a_{n-1} & \ddots & 0 & b_{m-2} & b_{m-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & a_n & \vdots & \vdots & \ddots & b_m \\ a_0 & a_1 & \cdots & \vdots & b_0 & b_1 & \cdots & \vdots \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{pmatrix}.$$

The resultant is defined to be the determinant of this matrix.

$$\mathrm{res}_{n,m}(A,B) = \det\left(\mathrm{Syl}_{n,m}(A,B)\right).$$

The resultant is 0 if and only if A and B share a common factor. We prove this (for general k-algebras) in Lemma 2.4.6.

Definition 2.3.2 (Definition 2.1. in [Caz12])**.** For an integer $n \geq 1$, the scheme $\mathcal{F}_n$ of pointed degree n rational functions is the open subscheme of the affine space $\mathbb{A}^{2n} = \mathrm{Spec}\,(k[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}])$ complementary to the hypersurface of equation

$$\mathrm{res}_{n,n}(X^n + a_{n-1}X^{n-1} + \cdots + a_0, b_{n-1}X^{n-1} + \cdots + b_0) = 0.$$

By convention $\mathcal{F}_0 := \mathrm{Spec}\,(k)$.

2.4 Schemes as functors

In this section we define some categorical notions. The functor of points gives a very useful perspective by using the Yoneda lemma to view schemes as functors. While the functor of points construction works for general schemes, we get a stronger result when restricting to the case of schemes over a commutative ring S. Since we are working in the category of smooth schemes over a field k, it is useful to state this stronger version of the definition.

Definition 2.4.1 (Representable functor)**.** Let $\mathscr{C}$ be a category. A functor $F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called representable if it is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, c)$ for some object $c \in \mathscr{C}$. We say that F is represented by c.

Similarly a functor $G : \mathscr{D} \longrightarrow \mathbf{Set}$ is corepresentable if it is naturally isomorphic to $\mathrm{Hom}_{\mathscr{D}}(d, -)$ for some $d \in \mathscr{D}$.

The functor of points is an example of a representable functor.

Definition 2.4.2 (Functor of points)**.** The functor h given by

$$\begin{aligned} h : \mathbf{Sch} &\longrightarrow \mathbf{Fun}\big(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}\big) \\ X &\longmapsto h_X \\ \big(f : X \longrightarrow Y\big) &\longmapsto \big(h(f) : h_X \longrightarrow h_Y\big) \end{aligned}$$

is an equivalence of the category of schemes and a full subcategory of the category of functors. Now, what does the functor h_X do? Let Y, Z be schemes. h_X is defined by

$$\begin{aligned} h_X : Y &\longrightarrow \mathrm{Hom}(Y, X) \\ (f : Y \longrightarrow Z) &\longmapsto (h_X(f) : h_X(Z) \longrightarrow h_X(Y)). \end{aligned}$$

h_X is represented by X.

Since our schemes are all in $\mathbf{Sm}_k$, the following result [EH06, Prop VI-2] will be useful. Recall that the category of S-algebras is dual to the category of affine S-schemes.

Proposition 2.4.3 (Restricted functor of points)**.** *Fix a commutative ring S. The functor of points h_X of the S-scheme X is completely determined by where it sends affine S-schemes.*

$$\begin{aligned} h_X : \mathbf{Aff}_S^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ Y &\longmapsto \mathrm{Hom}(Y, X) \end{aligned}$$

For S-algebras Z^{op}, we call $h_X(Z)$ the set of Q-valued points of Y. An element of this set is called a Q-valued point of Y.

Using this proposition, we may now pass freely back and forth between the perspective of smooth k-schemes, and the perspective of functors from smooth affine k-schemes to sets. We may even pass to the perspective of corepresentable functors from k-algebras to sets, as this the same thing.

Remark 2.4.4. Notice that this restricted functor of points h_X is representable, and represented by the S-scheme X. The above definition simply states that restricting to the full subcategory $\mathbf{Alg}_S = \mathbf{Aff}_S^{\mathrm{op}}$ of $\mathbf{Sch}_S^{\mathrm{op}}$ still gives us sufficient information for the scheme and functor to uniquely determine each other.

Notational remark. We use $\mathcal{F}_n$ to denote this scheme of rational functions, but also to denote its functor of points, meaning that we write $\mathcal{F}_n(Q) = h_{\mathcal{F}_n}(Q)$.

Applying Proposition 2.4.3 of an S-valued point to the scheme $\mathcal{F}_n$ gives us the following.

Definition 2.4.5. Let S be a k-algebra and n a non-negative integer. An S-point of $\mathcal{F}_n$ is a pair (A, B) of polynomials of $S[X]$, where

- A is monic of degree n,
- B is of degree strictly less than n,
- the scalar $\mathrm{res}_{n,n}(A, B)$ is invertible in S.

Such a point is denoted by $\frac{A}{B}$ and is called a pointed degree n rational function with coefficients in S.

Lemma 2.4.6 (Second part of Remark 2.2 in [Caz12])**.** *The above condition* $\mathrm{res}_{n,n}(A, B) \in S^\times$ *is equivalent to the existence of a (necessarily unique) Bézout relation*

$$AU + BV = 1$$

with U *and* V *polynomials in* $S[X]$ *such that* $\deg U \leq n-2$ *and* $\deg V \leq n-1$.

Proof. Notice that $a_n = 1, b_n = 0$ implies $\mathrm{res}_{n,n}(A, B) = \mathrm{res}_{n,n-1}(A, B)$. We look at $\mathrm{res}_{n,n-1}(A, B)$ to make this proof more natural. The Sylvester matrix is a linear operator $\mathrm{Syl}_{n,n-1}(A, B)$, which sends pairs (U, V) of polynomials with $\deg U \leq n-2$ and $\deg V \leq n-1$ to $AU + BV$, a polynomial of degree at most $2n-2$.
The resultant is invertible if and only if the Sylvester matrix has full rank. Then there exists a unique vector $\mathbf{v} = [u_{n-2}, \ldots, u_0, v_{n-1}, \ldots, u_0]$ such that $\mathrm{Syl}_{n,n-1}(A, B) \cdot \mathbf{v} = [0, \ldots, 0, 1]$, which corresponds to $AU + BV = 1$, where $U = u_{n-2}X^{n-2} + \ldots + u_0$ and $V = v_{n-1}X^{n-1} + \ldots + v_0$.
Conversely, if a Bézout relation exists, then $\gcd(A, B) = 1$ and there exists no nonzero (U, V) such that $AU + BV = 0$. This implies that the kernel is trivial, so the Sylvester matrix has full rank and $\mathrm{res}_{n,n}(A, B) = \mathrm{res}_{n,n-1}(A, B) \in S^\times$. $\square$

2.5 Naive connected components functor

Let us define the naive connected components functor on the functors of points. In the next section we will state a more explicit definition of naive homotopy classes on $\mathcal{F}_n$ (as a scheme) and see that these definitions coincide.

Definition 2.5.1 (Coequalizer)**.** Given a diagram with two objects X, Y and two morphisms $f, g : X \rightrightarrows Y$, a coequalizer is a universal pair (Q, q) where Q is an object and $q : Y \longrightarrow Q$ a morphism such that $q \circ f = q \circ g$, making this diagram commute:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "\exists ! u"] \\ & & Q' \end{tikzcd}$$

Definition 2.5.2 (Naive connected components functor)**.** Let G be an object in $\mathbf{Fun}\,(\mathbf{Alg}_k, \mathbf{Set})$. We define a functor π_0^{N} sending G to its naive connected components as follows

$$\begin{aligned} \pi_0^{\mathrm{N}} : \mathbf{Fun}\big(\mathbf{Alg}_k, \mathbf{Set}\big) &\longrightarrow \mathbf{Fun}\big(\mathbf{Alg}_k, \mathbf{Set}\big) \\ G &\longmapsto \pi_0^{\mathrm{N}} G \\ \big(f : G \longrightarrow H\big) &\longmapsto \big(\pi_0^{\mathrm{N}}(f) : \pi_0^{\mathrm{N}} G \longrightarrow \pi_0^{\mathrm{N}} H\big). \end{aligned}$$

Here $\pi_0^{\mathrm{N}} G$ sends a k-algebra S to the coequalizer of the diagram $G(S[T]) \rightrightarrows G(S)$, where the two morphisms are given by evaluation at $T = 0$ and at $T = 1$.

Lemma 2.5.3. *Any k-scheme morphism $\mathcal{F}_n \longrightarrow X$ produces a naive homotopy invariant $(\pi_0^{\mathrm{N}} \mathcal{F}_n)(k) \longrightarrow (\pi_0^{\mathrm{N}} X)(k)$.*

Proof. This is a consequence of the functoriality of π_0^{N}. □

2.6 Pointed morphisms are rational functions

Pointed naive homotopies of rational functions are algebraic paths (parameterized by T) in the scheme of pointed rational functions. Pointed k-scheme morphisms $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ and pointed naive homotopies $F : \mathbb{P}^1 \times \mathbb{A}^1 = \mathbb{P}^1_{k[T]} \longrightarrow \mathbb{P}^1$ are described in terms of rational functions as follows.

Proposition 2.6.1 (Proposition 2.3 in [Caz12])**.** *Let $S = k$ or $S = k[T]$. The datum of a pointed k-scheme morphism $f : \mathbb{P}^1_S \longrightarrow \mathbb{P}^1_k$ is equivalent to the datum of a non-negative integer n and of an element $\frac{A}{B} \in \mathcal{F}_n(S)$. The integer n is called the* degree *of f and is denoted* $\deg(f)$*; the scalar* $\mathrm{res}_{n,n}(A, B) \in S^\times = k^\times$ *is called the* resultant *of f and is denoted* $\mathrm{res}(f)$.

Proof. If $S = k$, just combine Lemma 2.2.5 which states what pointed $\mathbb{P}^1$-endomorphisms look like, with Definition 2.3.2 of the scheme $\mathcal{F}_n$. Observe that s_0, s_1 generate $\mathcal{O}(n)$ if and only if they have no common factor if and only if $\mathrm{res}(f) \neq 0$. If $S = k[T]$, then Definition 2.4.5 of S-points tells us that the argument for $S = k$ still works, since $k[T]^\times = k^\times$. □

Example 2.6.2. Let us calculate some examples. Let n be a positive integer, b_0 be a unit in $k^\times$ and $A = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a monic degree n polynomial of $k[X]$. The element

$$\frac{X^n + Ta_{n-1}X^{n-1} + \cdots + Ta_0}{b_0} \in \mathcal{F}_n(k[T])$$

gives a pointed naive homotopy between $\frac{A}{b_0}$ and $\frac{X^n}{b_0}$. That is, any polynomial is homotopic to its leading term.

Example 2.6.3. Let $B = b_{n-1}X^{n-1} + \cdots + b_0$ be a polynomial of degree $\leq n-1$ such that $B(0) = b_0$. Then $\frac{X^n}{B}$ is a k-point of $\mathcal{F}_n$ and the element

$$\frac{X^n}{Tb_{n-1}X^{n-1} + \cdots + Tb_1X + b_0} \in \mathcal{F}_n(k[T])$$

gives a pointed naive homotopy between $\frac{X^n}{B}$ and $\frac{X^n}{b_0}$.

We call the examples above "trivial homotopies." It is in general difficult to find homotopies between arbitrary rational functions.

Proposition 2.6.1 implies that two pointed rational functions which are in the same pointed naive homotopy class have same degree and same resultant. In particular, the set $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}$ splits as the disjoint union of its components of a given degree

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}} = \coprod_{n \geq 0} \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}_n.$$

Lemma 2.6.4. *For every non-negative integer n, we have a bijection*

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}_n \simeq (\pi_0^{\mathrm{N}}\mathcal{F}_n)(k).$$

Proof. Combine Proposition 2.6.1 with Definition 2.5.2. $\square$

2.7 Monoid structure on rational functions

In this section, we define a graded monoid structure on $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}$ by using the graded monoid structure that already exists on the disjoint union scheme

$$\mathcal{F} := \coprod_{n \geq 0} \mathcal{F}_n.$$

We follow Cazanave in stating this at the generality of any k-algebra S, but note that the k-algebras of interest are just k and $k[T]$.

Two rational functions $\frac{A_i}{B_i} \in \mathcal{F}_{n_i}(S)$ for $i = 1, 2$, uniquely define two pairs (U_i, V_i) of polynomials of $S[X]$ with $\deg U_i \leq n_i - 2$ and $\deg V_i \leq n_i - 1$ and satisfying Bézout identities $A_i U_i + B_i V_i = 1$ (by Lemma 2.4.6). We define polynomials A_3, B_3, U_3 and V_3 by setting

$$\begin{bmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{bmatrix} := \begin{bmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{bmatrix}.$$

Since the matrices $\begin{bmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{bmatrix}$ belong to $\mathrm{SL}_2(S[X])$, so does $\begin{bmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{bmatrix}$. This means that $A_3 U_3 + B_3 V_3 = 1$, so we also have a Bézout relation for A_3 and B_3. Moreover, observe that $A_3 = A_1 A_2 - V_1 B_2$ is monic of degree $n_1 + n_2$ and that $B_3 = B_1 A_2 + U_1 B_2$ is of degree strictly less than $n_1 + n_2$. So $\frac{A_3}{B_3}$ is in $\mathcal{F}_{n_1+n_2}(S)$. Since matrix multiplication is associative, so is this operation.

Proposition 2.7.1. *Let $\mathcal{F} = \prod_{n\geq 0} \mathcal{F}_n$ be the scheme of pointed rational functions. Then the naive sum $\oplus^{\mathrm{N}}$ defines a graded monoid structure on $\mathcal{F}$:*

$$\begin{aligned} \oplus^{\mathrm{N}} : \mathcal{F} \times \mathcal{F} &\longrightarrow \mathcal{F} \\ \left(\frac{A_1}{B_1}, \frac{A_2}{B_2}\right) &\longmapsto \frac{A_3}{B_3}. \end{aligned}$$

The above graded monoid structure on $\mathcal{F}$ induces a graded monoid structure on the connected components $(\pi_0^{\mathrm{N}}\mathcal{F})(k) := \prod_{n\geq 0} (\pi_0^{\mathrm{N}}\mathcal{F}_n)(k)$, and thus on $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}}$ by Lemma 2.6.4. The monoid operation on these sets is again denoted by $\oplus^{\mathrm{N}}$.

Example 2.7.2. Here are some naive sums of rational functions in k and $k[T]$. Let u be in $k^\times$. Then

$$\frac{X}{1} \oplus^{\mathrm{N}} \frac{X}{u} = \frac{X^2 - u}{X} \quad \text{and} \quad \frac{X}{u} \oplus^{\mathrm{N}} \frac{X}{1} = \frac{X^2 - u^{-1}}{uX}.$$

This shows that the naive sum is *not* commutative.

Example 2.7.3. The sum of homotopies

$$\frac{X + T}{1} \oplus^{\mathrm{N}} \frac{X + 2T}{1} = \frac{X^2 + 3TX + 2T^2 - 1}{X + 2T},$$

gives us a homotopy between

$$\frac{X^2 - 1}{X} \quad \text{and} \quad \frac{X^2 + 3X + 1}{X + 2}.$$

This illustrates that the naive sum of trivial homotopies may give a nontrivial homotopy.

Example 2.7.4. For every monic polynomial $P \in k[X]$, and for every unit $b_0 \in k^\times$, we have

$$\frac{P}{b_0} \oplus^{\mathrm{N}} \frac{A}{B} = \frac{AP - \frac{B}{b_0}}{b_0 A} = \frac{P}{b_0} - \frac{1}{b_0^2 \frac{A}{B}}.$$

This last example motivates the next lemma.

Lemma 2.7.5. *Every rational function $f \in \mathcal{F}_n(k)$ admits a unique twisted continued fraction expansion, which allows us to write*

$$f = \frac{P_0}{b_0} \oplus^{\mathrm{N}} \frac{P_1}{b_1} \oplus^{\mathrm{N}} \ldots \oplus^{\mathrm{N}} \frac{P_r}{b_r}.$$

Proof. In degree $n = 1$, every pointed rational function is a polynomial. Let $n \geq 2$ and assume that the lemma holds for all f of degree strictly less than n. Since k is a field, the ring of polynomials $k[x]$ is a Euclidean domain, and so $f = \frac{A}{B} \in \mathcal{F}_n(k)$ admits an expansion of the following form:

$$\frac{A}{B} = \frac{P_0}{b_0} - \frac{Q}{B} \qquad ,$$

where $P_0 \in k[X]$ is a monic polynomial of positive degree and b_0 is the leading non-zero coefficient of B. Crucially, the degree of Q is strictly less than the degree of B, and so $\frac{B}{Q} \in \mathcal{F}_m(k)$ is a pointed rational function of degree $m < n$. By induction, we are done. Using this Euclidean algorithm repeatedly yields the unique twisted continued fraction expansion:

$$\frac{A}{B} = \frac{P_0}{b_0} - \frac{1}{b_0^2 \left(\frac{P_1}{b_1} - \frac{1}{b_1^2(\ldots)} \right)} \qquad ,$$

where for each i, $P_i \in k[X]$ is a monic polynomial of positive degree and b_i is a non-zero scalar in $k^\times$. Such an expansion always stops, as the sum of the degrees of the P_i equals the degree of A. $\square$

Remark 2.7.6. Note that Cazanave uses the assumption that k is a field here. If $S[x]$ is a Euclidean domain, then $S[x]$ is a PID. But $S[x]$ is a PID if and only if S is a field. It immediately follows that if S is not a field, then $S[x]$ is not a Euclidean domain. Hence the technique used above to write $f = \frac{A}{B} \in \mathcal{F}_n(k)$ as a naive sum of polynomials would not work in general.

2.8 Theory of symmetric bilinear forms

Symmetric bilinear forms play an important role in Cazanave's proof. This section goes quickly through the definitions and theorems that we will need, following [Aso16, Lam05, EKM08, Lam10]. Note that we need more than just the theory of bilinear forms over a field k — in order to account for homotopies, we need to understand bilinear forms over $k[T]$. For this reason the following definitions are stated for commutative rings.

Let S be a commutative unital ring and M be an S-module. A bilinear form is a map $B : M \times M \longrightarrow S$ that is linear in both variables. We call the pair (M, B) a bilinear S-module. If (M, B) and (M', B') are bilinear modules, then an S-module map $f : M \longrightarrow M'$ is a morphism of bilinear modules if $B'(f(m_1), f(m_2)) = B(m_1, m_2)$ for all $m_1, m_2 \in M$. These objects and morphisms form a category Bil_S. An isomorphism in this category is called an *isometry*. A *symmetric bilinear form* is a bilinear form satisfying $B(u, v) = B(v, u)$.

Definition 2.8.1. A bilinear module is *nondegenerate* if the map $M \longrightarrow M^\vee$ sending $m \in M$ to $B(-, m)$ is an isomorphism of S-modules.

Definition 2.8.2. If M is a finitely generated projective module, and the bilinear form B is nondegenerate, then (M, B) is called an *inner product space*.

Definition 2.8.3 (Orthogonal sum)**.** Let (M, B) and (M', B') be bilinear S-modules. Their orthogonal sum $(M, B) \perp (M', B')$ consists of the direct sum module $M \oplus M'$ equipped with the bilinear form $B'' : ((x_1, x_2), (y_1, y_2)) \longrightarrow B(x_1, y_1) + B'(x_2, y_2)$. We will also write this as $B'' = B \perp B'$. Note that $B \perp B' \simeq B' \perp B$.

Definition 2.8.4 (Tensor product)**.** Let (M, B) and (M', B') be bilinear S-modules. Their tensor product $(M, B) \otimes (M', B')$ consists of the module $M \otimes M'$ equipped with the bilinear form $B'' : ((x_1 \otimes x_2), (y_1 \otimes y_2)) \longrightarrow B(x_1, y_1) \cdot B'(x_2, y_2)$. We will also write this as $B'' = B \otimes B'$. Note that $B \otimes B' \simeq B' \otimes B$.

Definition 2.8.5 (Witt monoid/semiring)**.** The set of isometry classes of symmetric inner product spaces equipped with the orthogonal sum form a commutative monoid. The unit is the module 0 equipped with the trivial bilinear form. This is called the Witt monoid of S, and is denoted $\mathrm{WM}(S)$. The *stable* Witt monoid $\mathrm{WM}^s(S)$ has stable isometry classes as objects. B and B' are called stably isometric if there exists a B'' such that $B \perp B'' \simeq B' \perp B''$. When $\mathrm{char}(S) \neq 2$, we have $\mathrm{WM}^s(S) = \mathrm{WM}(S)$. Equipping the Witt monoid with the tensor product, we get a commutative semiring.

Given any commutative monoid N, its Grothendieck group $\mathrm{Groth}(N)$ is an abelian group satisfying the universal property that any monoid morphism $N \longrightarrow A$, where A is an abelian group, factors through $\mathrm{Groth}(N)$. There is also an explicit construction, reminiscent of how you might construct $\mathbb{Z}$ from $\mathbb{N}$. On the set $N \times N$, define addition coordinate-wise. Then mod out by the relation that $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + y_2 + c = y_1 + x_2 + c$, for some $c \in N$. The elements can be thought of as formal differences, so (x_1, x_2) "is" $x_1 - x_2$.

Definition 2.8.6 (Grothendieck-Witt group/ring)**.** The Grothendieck-Witt group $\mathrm{GW}(S)$ is the Grothendieck group of the stable Witt monoid $\mathrm{WM}^s(S)$. This group completion is compatible with the tensor product, so $(\mathrm{GW}(S), \oplus, \otimes)$ is a ring.

Notational remark. The multiplicative structure is not important for Cazanave's argument. We will refer to $\mathrm{WM}^s(S)$ and $\mathrm{GW}(S)$ respectively as a monoid and a group when the product structure is irrelevant.

2.8.1 Symmetric bilinear forms over a field

We restrict our attention to the case of symmetric bilinear forms over a field k, following [EKM08] and [Lam05]. Any inner product space (V, f) over k (or over $k[T]$ by the Quillen-Suslin theorem) is free as a module and admits a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. We may express f with respect to $\mathcal{B}$ as a symmetric matrix B_f. Using a matrix $C \in \mathrm{GL}_n(k)$ to change bases, we see that f can be represented by any congruent matrix $C^t B_f C$. Since this transformation may only change the determinant by a factor $(\det C)^2$ in $k^{\times 2}$, we define the *discriminant* $\operatorname{disc} f := \det B_f \cdot k^{\times 2} \in k^\times / k^{\times 2}$ which is independent of choice of basis. This allows us to switch back and forth between the perspective of symmetric bilinear forms and of symmetric matrices.

By [Lam05, §VII.1], $\mathrm{GW}(-)$ is a functor from the category of fields of characteristic not 2, to the category of rings.

Definition 2.8.7 (Isotropicity)**.** Let (V, f) be a symmetric bilinear form over k. We call $v \in V$ isotropic if $f(v, v) = 0$. We call a subspace $W \subset V$ totally isotropic if $f(W, W) = 0$. We call f isotropic if there exists an isotropic vector $v \in V$. We call f anisotropic otherwise.

Definition 2.8.8 (Scheme of nondegenerate symmetric matrices)**.** Let $\mathcal{S}_n(k)$ denote the scheme of nondegenerate symmetric $n \times n$-matrices over k. That is, the open subscheme of $\mathbb{A}^{n^2} = \operatorname{Spec}(k[a_0, \ldots, a_{n^2}])$ complementary to the hypersurface of equation $\det = 0$.

Remark 2.8.9. A matrix in $\mathcal{S}_n$ determines a nondegenerate symmetric bilinear form on the vector space k^n. Similarly, picking a basis associates a matrix to a bilinear form, and the isometry class of the form is invariant under change of basis. The orthogonal sum $\perp$ corresponds to the direct sum $\oplus$, and the tensor product of bilinear forms $\otimes$ corresponds to the Kronecker product $\otimes$ of matrices. For this reason, we may change perspectives back and forth, and we may use matrix arguments to prove statements about bilinear forms.

Definition 2.8.10 (Diagonal forms)**.** Let n be a positive integer. For a sequence of units $u_1, \ldots, u_n \in k^\times$, let $\langle u_1, \ldots, u_n\rangle$ denote the diagonal symmetric bilinear form $\langle u_1\rangle \perp \cdots \perp \langle u_n\rangle$.

Proposition 2.8.11 (§1, Corollary 1.9 in [EKM08])**.** *Any nondegenerate symmetric bilinear form (V, f) over k may be written as $f \simeq \langle u_1, \ldots, u_n\rangle \perp W$, where W is hyperbolic, meaning it may be written as a block diagonal matrix with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ blocks. However, if $\mathrm{char}(k) \neq 2$, then f is diagonalizable, meaning $f \simeq \langle u_1, \ldots, u_n\rangle$.*

Proposition 2.8.12 (§I, Theorem 4.7 in [EKM08])**.** *The Grothendieck-Witt group GW(k) is generated by the isometry classes of 1-dimensional symmetric bilinear forms $\langle a\rangle$ that are subject to the defining relation*

$$\langle a\rangle \perp \langle b\rangle = \langle a+b\rangle \perp \langle ab(a+b)\rangle \quad \textit{for all } a, b \in k^\times \textit{ such that } a+b \neq 0.$$

Remark 2.8.13. Note that the isometry classes of 1-dimensional symmetric bilinear forms are generated by units modulo squares. That is, units $a \in k^\times$ modulo the relation $\langle a\rangle = \langle ab^2\rangle$. Also note that in order to generate the Grothendieck-Witt ring, it suffices to add the relations $\langle 1\rangle = 1$ and $\langle ab\rangle = \langle a\rangle \otimes \langle b\rangle$.

Examples 2.8.14.

1. For any algebraically closed field, $k^\times/k^{\times 2}$ is the trivial group, so the map $\mathrm{rank} : \mathrm{GW}(k) \longrightarrow \mathbb{Z}$ is an isomorphism. In fact, it is sufficient for the field to be quadratically closed for this to hold.

2. Over $\mathbb{R}$, by Proposition 2.8.11 any form is isometric to a diagonal form. Sylvester's law of inertia states that a complete isometry class invariant is the *indices of inertia* (n_+, n_-), where n_+ and n_- denote the number of positive and negative entries on the diagonal. The rank n equals the sum $n_+ + n_-$, and the difference $n_+ - n_-$ is called the *signature*. Hence we get $\mathrm{WM}(\mathbb{R}) = \mathbb{N} \times \mathbb{N}$. Group completing, we get $\mathrm{GW}(\mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$.

   If (V, f) and (W, g) are forms with indices of inertia (n_+, n_-) and (m_+, m_-) respectively, then the class of $((V, f), (W, g))$ in $\mathrm{GW}(\mathbb{R})$ is given by the componentwise difference of indices of inertia $(n_+ - m_+, n_- - m_-)$. We may apply a group automorphism of $(\mathbb{Z} \times \mathbb{Z}, +)$ to get another useful interpretation. We get $(n_+ + n_- - (m_+ + m_-), n_+ - n_- - (m_+ - m_-))$, where the first component is rank $f -$ rank g, and the second component is the difference of the signatures. The canonical inclusion $\mathbb{R} \longrightarrow \mathbb{C}$ then induces (by functoriality of $\mathrm{GW}(-)$) a projection to the first component $\mathrm{GW}(\mathbb{R}) = \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} = \mathrm{GW}(\mathbb{C})$.

3. Over a finite field $\mathbb{F}_q$, the rank and the discriminant determine a group isomorphism $\mathrm{GW}(\mathbb{F}_q) \simeq \mathbb{Z} \times \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$. The multiplicative group $\mathbb{F}_q^\times$ is cyclic. Hence, when q is even, there is only one square class. When q is odd, there are two.

2.8.2 Symmetric bilinear forms over $k[T]$

The following definition and theorem is from Lam's book on Serre's problem [Lam10, p. 236–246].

Definition 2.8.15. Let (P, B) be an inner product space over S. If $f : S \longrightarrow S'$ is a homomorphism of commutative rings, we can define, by "scalar extension", a new pair (P', B') over S', where $P' = S' \otimes_S P$, and B' is given by

$$B'(s_1' \otimes p_1, s_2' \otimes p_2) = s_1' s_2' \cdot f(B(p_1, p_2)).$$

We say that the resulting inner product space (P', B') over S' is extended from (P, B).

Theorem 2.8.16 (Harder). *Let k be a field of characteristic not 2. Then any inner product space (L, B) over $k[T]$ has an orthogonal $k[T]$-basis, and is therefore extended from an inner product space over k.*

If the characteristic is 2, then any inner product space (L, B) over $k[T]$ will decompose into an orthogonal sum $L_0 \perp L_1 \perp \ldots \perp L_m$, where L_0 is extended from k, and all other L_i have rank 2, with matrices of the type $S_i = \begin{pmatrix} s_i & 1 \\ 1 & 0 \end{pmatrix}$, for $s_i \in k[T]$.

2.9 Bézout form

Now that we have reviewed the basics of bilinear forms, it is time to connect this to our scheme of rational functions $\mathcal{F}_n$. Bézout described a way to associate a nondegenerate symmetric matrix to any rational function. That is, for each integer n, a scheme morphism

$$\mathrm{Béz}_n : \mathcal{F}_n \longrightarrow \mathcal{S}_n.$$

Observe that the polynomial $A(X)B(Y)-A(Y)B(X) \in S[X,Y]$ has no constant term. Notice also that for any term that can be written as CX for some $C \in S[X,Y]$, there is a corresponding term $-CY$. Hence, $A(X)B(Y)-A(Y)B(X)$ is divisible by $X-Y$.

Definition 2.9.1. Let S be k or $k[T]$, n be a positive integer and $f = \frac{A}{B}$ be an element of $\mathcal{F}_n(S)$. Let

$$\delta_{A,B}(X,Y) := \frac{A(X)B(Y)-A(Y)B(X)}{X-Y} =: \sum_{1\le p,q\le n} c_{p,q} X^{p-1}Y^{q-1}.$$

Observe that the coefficients of $\delta_{A,B}(X,Y)$ are symmetric in the sense that one has

$$c_{p,q} = c_{q,p} \qquad \forall\, 1 \le p,q \le n.$$

The Bézout form of f is the symmetric bilinear form over S^n whose matrix (corresponding to the canonical basis of S^n) is the $n \times n$-symmetric matrix $(c_{p,q})$. We denote it $\mathrm{Béz}_n(A,B)$ or $\mathrm{Béz}_n(f)$.

The following lemma implies that $\mathrm{Béz}_n(f)$ is a non-degenerate bilinear form.

Lemma 2.9.2. *This equality holds for any pointed rational function:*

$$\det \mathrm{Béz}_n(f) = (-1)^{\frac{n(n-1)}{2}} \mathrm{res}_{n,n}(f). \tag{2.2}$$

Proof. Observe that the $2n\times 2n$ Sylvester matrix can be split up into $n\times n$-matrix blocks.

$$\mathrm{res}_{n,n}(A,B) = \det \mathrm{Syl}_{n,n}(A,B) = \begin{vmatrix} A^- & B^- \\ A^+ & B^+ \end{vmatrix},$$

where

$$A^- = \begin{pmatrix} a_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \quad \text{and} \quad A^+ = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \ddots & a_{n-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix},$$

and B^-, B^+ are defined similarly. We have $a_n = 1$,which implies $\det A^- = 1$.We would like to multiply $\mathrm{Syl}_{n,n}(A,B)$ by some matrix to make its determinant more easily comparable to $\det \mathrm{Béz}_n(A,B)$. Notice that $A^-B^- - B^-A^- = 0$, since lower triangular matrices commute. Hence,

$$\begin{pmatrix} A^- & B^- \\ A^+ & B^+ \end{pmatrix} \cdot \begin{pmatrix} I_n & B^- \\ 0_n & -A^- \end{pmatrix} = \begin{pmatrix} A^- & 0_n \\ A^+ & C \end{pmatrix},$$

where $C := A^+B^- - B^+A^-$. Recall that the determinant of a block triangular matrix is the product of the determinants of its diagonal blocks, which gives

$$\mathrm{res}_{n,n}(f) \cdot 1 \cdot (-1)^n = 1 \cdot \det C$$

We denote by $-J_n$ the $n \times n$-matrix that has -1 along the anti-diagonal and zeros everywhere else. Notice that $-J_n \cdot C = \text{Béz}_n(f)$. Finally, because $\det -J_n = (-1)^n(-1)^{\frac{n(n-1)}{2}}$, we get

$$\mathrm{res}_{n,n}(f) = (-1)^{\frac{n(n-1)}{2}} \det \text{Béz}_n(f).$$

□

The above construction describes for every positive integer a natural transformation of functors of points $\mathcal{F}_n(-) \longrightarrow S_n(-)$ and thus a morphism of schemes

$$\text{Béz}_n : \mathcal{F}_n \longrightarrow S_n.$$

Example 2.9.3. Any rational function of degree 1 corresponds to the 1×1 identity matrix. For instance,

$$\text{Béz}\left(\frac{X}{1}\right) = \begin{pmatrix}1\end{pmatrix} \quad \text{and} \quad \text{Béz}\left(\frac{X+T}{1}\right) = \begin{pmatrix}1\end{pmatrix}.$$

Example 2.9.4. Recalling that

$$\frac{X}{1} \oplus^{\mathrm{N}} \frac{X}{1} = \frac{X^2-1}{X},$$

we might ask: is the direct sum $\oplus$ of matrices compatible with the naive sum $\oplus^{\mathrm{N}}$ of rational functions? We calculate

$$\text{Béz}\left(\frac{X}{1}\right) \oplus \text{Béz}\left(\frac{X}{1}\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \neq \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \text{Béz}\left(\frac{X^2-1}{X}\right),$$

so this is not the case. However, as we shall prove in the next chapter, the monoid structures are indeed compatible after passing to naive homotopy classes.

Chapter 3

Cazanave's proof

Cazanave uses a series of graded monoid isomorphisms in order to arrive at the following theorem.

Theorem 3.0.1 (Corollary 3.10 in [Caz12])**.** *There is a canonical isomorphism of graded monoids:*

$$\left(\left[\mathbb{P}^1,\mathbb{P}^1\right]^{\mathrm{N}},\oplus^{\mathrm{N}}\right)\simeq\Big(\prod_{n\geq 0} WM^s_n(k)\underset{k^\times/k^{\times 2}}{\times} k^\times,\oplus\Big).$$

Combining this with Theorem 6.36 in [Mor12] gives the abstract isomorphism

$$\mathrm{Groth}\left(\left[\mathbb{P}^1,\mathbb{P}^1\right]^{\mathrm{N}},\oplus^{\mathrm{N}}\right)\simeq\big(\left[\mathbb{P}^1,\mathbb{P}^1\right]^{\mathbb{A}^1},\oplus^{\mathbb{A}^1}\big).$$

Cazanave then checks (in the appendix) that the isomorphism is in fact induced by the canonical map, which proves Theorem 1.0.2. To get an overview, here are all the graded monoid isomorphisms used to prove Theorem 3.0.1 assembled in one diagram.

$$\begin{aligned}\left(\left[\mathbb{P}^1,\mathbb{P}^1\right]^{\mathrm{N}},\oplus^{\mathrm{N}}\right)&\simeq\left(\pi_0^{\mathrm{N}}\mathcal{F}_n(k),\oplus^{\mathrm{N}}\right)\\&\simeq\left(\pi_0^{\mathrm{N}}\mathcal{S}_n(k),\perp\right)\simeq\Big(\mathrm{WM}^s_n(k)\underset{k^\times/k^{\times 2}}{\times} k^\times,\oplus\Big).\end{aligned}\tag{3.1}$$

We have already proved the first isomorphism in Proposition 2.6.1. The third isomorphism will be proven in Proposition 3.1.4, and it draws heavily on our discussion of the Witt monoid in Section 2.8. The second isomorphism (Theorem 3.3.9) requires a few lemmata. The idea of the proof is that of generators and relations. That is, we think about each monoid $\pi_0^{\mathrm{N}}\mathcal{F}$ and $\pi_0^{\mathrm{N}}\mathcal{S}$ as a free monoid on a set of generators, modulo some set of relations. We will first prove that there is a bijection on the sets of generators of $\pi_0^{\mathrm{N}}\mathcal{F}$ and $\pi_0^{\mathrm{N}}\mathcal{S}$, and prove that this map of generators induces a monoid morphism. This establishes surjectivity. We then check that any relations that hold in $\pi_0^{\mathrm{N}}\mathcal{S}$ corresponds to relations in $\pi_0^{\mathrm{N}}\mathcal{F}$. This correspondence of relations ensures injectivity.

3.1 The third isomorphism

In this section, we prove the isomorphism $\left(\pi_0^{\mathrm{N}}\mathcal{S}_n(k), \perp\right) \simeq \left(\mathrm{WM}_n^s(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\right)$.

Let us first prove a lemma about transvection matrices. Transvection matrices (also called elementary SL_n-transformations) add a multiple λ of a row i to another row j when you multiply by them. We denote by $A_{ij\lambda}$ the transvection matrix

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & \lambda & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

where entry $a_{ji} = \lambda$. Note that $I = A_{ij0}$ and $A_{ij\lambda}^{-1} = A_{ij(-\lambda)}$, so the transvection matrices form a group TV_n.

Lemma 3.1.1. *Let S be a Euclidean domain. Then $SL_n(S) = TV_n(S)$.*

Proof. Observe that this is true for $n = 1$. If there exists a sequence of transvection matrices transforming $M \in \mathrm{SL}_n(S)$ into I_n, then $M \in \mathrm{TV}_n$. Let $n \geq 2$ and let $M \in \mathrm{SL}_n(S)$. There exists at least one nonzero entry $m_{1j} = u$ in the first column. The determinant is a linear combination of the elements in the first column, and since S is a Euclidean domain there exists a linear combination that equals 1. Multiply by $A_{1j\lambda}$ and $A_{i1\lambda}$ until $m_{11} = 1$, and then by $A_{1j\lambda}$ until $m_{1j} = 0$ for all $j \neq 1$. The problem has now been reduced to the case $n - 1$. The lemma follows by induction. □

Lemma 3.1.2. *Any matrix M in $SL_n(k)$ is naively homotopic to the identity matrix.*

Proof. Decomposing M into a product of transvection matrices as in the proof of Lemma 3.1.1 and replacing each $A_{i,j,\lambda}$ by $A_{i,j,(\lambda - T\lambda)} \in \mathrm{SL}_n(k[T])$ yields a homotopy $M \sim I_n$. □

Lemma 3.1.3. *Let n be a positive integer. The canonical quotient map q_n : $\mathcal{S}_n(k) \longrightarrow WM^s_n(k)$ factors through $(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k)$:*

$$\begin{tikzcd} \mathcal{S}_n(k) \arrow[rr, "q_n"] \arrow[dr, "\pi_0^{\mathrm{N}}"'] & & WM^s_n(k) \\ & (\pi_0^{\mathrm{N}}\mathcal{S}_n)(k) \arrow[ur, "\overline{q}_n"'] & \end{tikzcd}$$

Proof. From Harder's theorem (Theorem 2.8.16), we get that $\mathrm{WM}(k) \longrightarrow \mathrm{WM}(k[T])$ is an isomorphism, and the inverse is given by evaluating T at any $a \in k$. Hence, for any $M(T) \in \mathcal{S}_n(k[T])$, we must have $q(M(0)) = q(M(1))$, so $\overline{q}$ is well defined. $\square$

Proposition 3.1.4. *Let*

$$WM^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$$

be the canonical fibre product induced by the discriminant map $WM^s_n(k) \longrightarrow k^\times/k^{\times 2}$.

$$\begin{tikzcd} WM^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times \arrow[r] \arrow[d] & k^\times \arrow[d, "p"] \\ WM^s_n(k) \arrow[r, "\mathrm{disc}"] & k^\times/k^{\times 2} \end{tikzcd}$$

Then the map

$$\Big(\prod_{n\geq 0}(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k), \oplus\Big) \xrightarrow{\prod_{n\geq 0}\overline{q}_n \times \det} \Big(\prod_{n\geq 0} WM^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\Big).$$

is a monoid isomorphism. (The right term is endowed with the canonical monoid structure induced by the orthogonal sum in $WM^s(k)$ and the product in $k^\times$).

Proof. We know that $\overline{q} \times \det$ is well defined since $\overline{q}$ and $\det$ are, and for $M(T) \in \mathcal{S}_n(k[T])$, $\det M(T) = \det M(0) = \det M(1) \in k^\times$.

To prove injectivity, assume $P, Q \in \mathcal{S}_n(k)$ define isometric forms, and that $\det(P) = \det(Q)$. This implies $P = M^tQM$, for some $M \in \mathrm{SL}^{\pm}_n(k)$. We want to show that we can always have $M \in \mathrm{SL}_n(k)$.

In characteristic 2, $\mathrm{SL}_n^{\pm}(k) = \mathrm{SL}_n(k)$, so we are done. When $\mathrm{char}(k) \neq 2$, we may have $\det M = -1$, in which case we do the following. Every nondegenerate symmetric matrix P is congruent to a diagonal matrix D by $N^t P N = D$, where N is invertible. We define $C^t = C = \mathrm{diag}(-1, 1, \ldots, 1)$, and observe that $C^t D C = D$. Hence we get a congruence $P \sim Q$ given by

$$P = (N^t)^{-1} C^t D C N^{-1} = (MNCN^{-1})^t Q MNCN^{-1}.$$

We calculate $\det(MNCN^{-1}) = 1$, which means that $(MNCN^{-1}) \in \mathrm{SL}_n(k)$. By Lemma 3.1.2 the congruence $P \sim Q$ gives rise to a homotopy, so $\pi_0^{\mathrm{N}} P = \pi_0^{\mathrm{N}} Q$ in $\pi_0^{\mathrm{N}} \mathcal{S}_n(k)$.

To prove surjectivity, assume we are given $(\beta, d) \in \Big(\coprod_{n \geq 0} \mathrm{WM}_n^s(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus \Big)$.

By the definition of the fiber product, $\operatorname{disc} \beta \cong d \pmod{k^{\times 2}}$. We know that q is surjective, so we may pick a preimage $P \in q^{-1}(\beta) \subset \mathcal{S}_n(k)$. We have $\det P = p = u^2 d$, where $u \in k^\times$. Using $U = \mathrm{diag}(u^{-1}, 1, \ldots, 1)$ to change bases, we obtain $Q = U^t P U$. We get that $\overline{q} \times \det : \pi_0^{\mathrm{N}} Q \longmapsto (\beta, d)$, so $\overline{q} \times \det$ is surjective. $\square$

3.2 Surjectivity and monoid compatibility

In this section we want to show that $\pi_0^{\mathrm{N}} \mathcal{F}(k)$ surjects onto $\pi_0^{\mathrm{N}} \mathcal{S}(k)$. We do this by giving a surjective map of generators onto generators, and showing that the monoid structures are compatible.

The following lemma shows that, up to naive homotopy, any symmetric bilinear form is diagonal.

Lemma 3.2.1 (Lemma 3.13 (1) in [Caz12]). *Let n be a positive integer. For any symmetric bilinear form $B \in \mathcal{S}_n(k)$ there exists units $u_1, \ldots, u_n \in k^\times$ such that B is homotopic to the diagonal form $\langle u_1, \ldots, u_n \rangle$.*

Proof. If $\mathrm{char}(k) \neq 2$, then by Proposition 2.8.11 $B \in \mathcal{S}_n(k)$ is conjugate by an element $P \in \mathrm{SL}_n(k)$ to a diagonal matrix. Decomposing P into a product of transvection matrices as in the proof of Lemma 3.1.1 and replacing each $A_{i,j,\lambda}$ by $A_{i,j,(\lambda - T\lambda)}$ yields a homotopy to a diagonal matrix.
If $\mathrm{char}(k) = 2$, by Proposition 2.8.11 B is conjugate by an element $P \in \mathrm{SL}_n(k)$ to a block diagonal matrix, with possible $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ terms. In addition to the preceding argument, we can use the homotopy $\begin{bmatrix} T & 1 \\ 1 & 0 \end{bmatrix}$ to link B to a diagonal matrix. $\square$

Definition 3.2.2. Let n be a positive integer. For a sequence of units $u_1, \dots, u_n \in k^\times$, let $[u_1, \dots, u_n]$ denote the pointed rational function $\frac{X}{u_1} \oplus^{\mathrm{N}} \dots \oplus^{\mathrm{N}} \frac{X}{u_n} \in \mathcal{F}_n(k)$.

Lemma 3.2.3 (Lemma 3.13 (2) in [Caz12])**.** *For any pointed rational function $f \in \mathcal{F}_n(k)$ there exists units $u_1, \dots, u_n \in k^\times$ such that we have*

$$f \overset{\mathrm{p}}{\sim} [u_1, \dots, u_n].$$

Proof. We prove this by induction on the degree n of f. As shown in Lemma 2.7.5, any rational function $f \in \mathcal{F}_n(k)$ is the naive sum of polynomials $P_1 \oplus^{\mathrm{N}} \dots \oplus^{\mathrm{N}} P_k$. Thus we may assume that f is a polynomial. Example 2.6.2 then shows that a polynomial is always homotopic to its leading term. So it's enough to treat the case of a monomial $\frac{X^n}{u}$, with $u \in k^\times$. Now, Example 2.6.3 shows that the element $\frac{X^n}{TX^{n-1}+u} \in \mathcal{F}_n(k[T])$ defnines a homotopy between $\frac{X^n}{u}$ and $\frac{X^n}{X^{n-1}+u}$. But this last rational function decomposes as $X \oplus^{\mathrm{N}} \frac{X^{n-1}+u}{uX}$, where $\frac{X^{n-1}+u}{uX} \in \mathcal{F}_{n-1}(k)$. By the inductive hypothesis on $\frac{X^{n-1}+u}{uX}$ we are done. $\square$

The monoids $((\pi_0^{\mathrm{N}}\mathcal{F})(k), \oplus^{\mathrm{N}})$ and $((\pi_0^{\mathrm{N}}\mathcal{S})(k), \oplus)$ are generated by their degree 1 components, and the map on generators $\pi_0^{\mathrm{N}}\mathrm{Béz}_1 : (\pi_0^{\mathrm{N}}\mathcal{F}_1)(k) \longrightarrow (\pi_0^{\mathrm{N}}\mathcal{S}_1)(k)$ sending $[u]$ to $\langle u \rangle$ is a bijection. The next lemma shows monoid compatibility, and hence that the Bézout form of $[u_1, \dots, u_n] \in \mathcal{F}_n(k)$ is homotopic to the diagonal form $\langle u_1, \dots, u_n \rangle \in \mathcal{S}_n$.

Lemma 3.2.4 (Lemma 3.14 in [Caz12])**.** *Let $\frac{A}{B} \in \mathcal{F}_n(k)$ and $u \in k^\times$. Then the Bézout form of $\frac{X}{u} \oplus^{\mathrm{N}} \frac{A}{B}$ is conjugate (hence homotopic) by an element in $SL_{n+1}(k)$ to the block diagonal form $\langle u \rangle \oplus \mathrm{Béz}_n(A, B)$.*

Proof. By definition, one has $\frac{X}{u} \oplus^{\mathrm{N}} \frac{A}{B} = \frac{XA - \frac{B}{u}}{uA}$. Using the notation from Definition 2.9.1, we have

$$\delta_{XA - \frac{B}{u}, uA}(X, Y) = uA(X)A(Y) + \delta_{A,B}(X, Y) \qquad .$$

In the basis $(1, X, \dots, X^{n-1}, A(X))$, the matrix of the Bézout form is diagonal. $\square$

3.3 Injectivity

Let n be a positive integer. To prove the injectivity of the map $\pi_0^{\mathrm{N}}\mathrm{Béz}_n$, we prove the injectivity of the composite

$$(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k) \xrightarrow{\pi_0^{\mathrm{N}}\mathrm{Béz}_n} (\pi_0^{\mathrm{N}}\mathcal{S}_n)(k) \xrightarrow{\overline{q}_n \times \det} \mathbf{WM}_n^s(k) \underset{k^\times/k^{\times 2}}{\times} k^\times .$$

By Lemma 3.2.3, any rational function is up to homotopy a naive sum of monomials. Proving injectivity amounts to proving that any equivalence of forms on the right side corresponds to an equivalence of rational functions on the left. We know by Proposition 2.8.12 that $\mathrm{GW}(k)$ is defined by the relation $\langle a\rangle \perp \langle b\rangle = \langle a+b\rangle \perp \langle ab(a+b)\rangle$. Since this relation exists in degree 2, we should expect corresponding relations to take place in $\mathcal{F}_2(k)$.

The *Chain Equivalence Theorem* [Lam05, §I, Theorem 5.2] tells us that two quadratic forms are isometric exactly when they can be connected by a chain of simple equivalences. Two diagonal forms $\langle a_1,\ldots,a_n\rangle$, $\langle b_1,\ldots,b_n\rangle$ are *simply equivalent* if $a_k = b_k$, for all k except for two distinct i, j, where $\langle a_i, a_j\rangle \simeq \langle b_i, b_j\rangle$.

When the field has characteristic 2, this gets slightly more complicated to prove. There is however an analogous result [MH73, §III, Lemma 5.6] which reduces the problem to checking degree 2. We will do that in the next section, and then we may conclude that the following proposition holds.

Proposition 3.3.1. *Let $u_1,\ldots,u_n,v_1,\ldots,v_n$ be a sequence of units in $k^\times$. If the classes in $\mathrm{WM}^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ of the diagonal forms $\langle u_1,\ldots,u_n\rangle$ and $\langle v_1,\ldots,v_n\rangle$ are equal, then $[u_1,\ldots,u_n] \overset{\mathrm{p}}{\sim} [v_1,\ldots,v_n]$ holds in $\mathcal{F}_n(k)$.*

Proof. Follows from degree 2 case in the next section. $\square$

3.3.1 Injectivity in degree 2

When proving injectivity in degree 2, we need the scheme-theoretic analogs of a group (group scheme) and of a principal bundle (torsor over a base space). We will define these terms rigorously, although we only need the simplest case of a group scheme, $\mathbb{G}_a$, which is $\mathbb{A}^1$ with additive group structure.

If a functor of points $h_X : \mathbf{Aff}_k^{\mathrm{op}} \longrightarrow \mathbf{Set}$ factors through the forgetful functor $F : \mathbf{Grp} \longrightarrow \mathbf{Set}$, then we may write $h_X = F \circ G$, where $G : \mathbf{Alg}_k \longrightarrow \mathbf{Grp}$. A scheme X with such a factorization $h_X = F \circ G$ is called a group scheme [EH06, §VI.1.4].

More explicitly, we can ensure that such a factorization exists by equipping a functor of points with additional structure. This is how Waterhouse defines an affine group scheme over k in [Wat12, §1]:

Definition 3.3.2 (Affine group scheme over k)**.** Let $H : \mathbf{Alg}_k \longrightarrow \mathbf{Set}$ be a functor corepresented by A. Let A be equipped with natural maps that make it into a Hopf algebra. That is, we have k-algebra morphisms

$$\begin{array}{ll} \text{comultiplication} & \Delta : A \longrightarrow A \otimes A \\ \text{counit} & \varepsilon : A \longrightarrow k \\ \text{coinverse} & S : A \longrightarrow A, \end{array}$$

which correspond to multiplication, unit and inverse. These morphisms satisfy $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$, and if we write $\Delta(a) = \sum b_i \otimes c_i$, then they satisfy $a = \sum \varepsilon(b_i)c_i$ and $\varepsilon(a) = \sum S(b_i)c_i$. This corresponds to the usual group axioms. These data define a factorization $H = F \circ G : \mathbf{Alg}_k \longrightarrow \mathbf{Grp} \longrightarrow \mathbf{Set}$, which we call an affine group scheme over k.

Notational remark. We just write $G : \mathbf{Alg}_k \longrightarrow \mathbf{Grp}$, but $H = F \circ G$ is implied. All group schemes considered are affine and over k, so we write "group scheme" for short.

Lemma 3.3.3. *The k-algebra $k[X]$ equipped with $\Delta : X \longmapsto X \otimes 1 + 1 \otimes X$, $\varepsilon : X \longmapsto 0$ and $S : X \longmapsto -X$ is an group scheme $\mathbb{G}_a : \mathbf{Alg}_k \longrightarrow \mathbf{Grp}$.*

Proof. We only need to check that the relations hold for the generator X of $k[X]$. We get $(\mathrm{id}\otimes\Delta)\circ\Delta = X\otimes1\otimes1+1\otimes X\otimes1+1\otimes1\otimes X = (\Delta\otimes\mathrm{id})\circ\Delta$. The second condition is $X = \varepsilon(X)\cdot 1+\varepsilon(1)\cdot X = X$. The third is $0 = (-X)\cdot 1+1\cdot X$. □

Definition 3.3.4. Let $G : \mathbf{Alg}_k \longrightarrow \mathbf{Grp}$ be a group scheme over k, and let $h_X : \mathbf{Alg}_k \longrightarrow \mathbf{Set}$ be a corepresentable functor. An action of G on h_X is a natural map $G \times h_X \longrightarrow h_X$ such that each map $G(S) \times h_X(S) \longrightarrow h_X(S)$ is a group action. h_X is called a G-torsor over a point[1] if the map $h_G \times h_X \longrightarrow h_X \times h_X$ sending (g, x) to (gx, x) is bijective and if there exists an S such that $k \longrightarrow S$ is faithfully flat and $h_X(S) \neq \emptyset$.

Note that any module over a field is free, and hence $k \longrightarrow S$ is faithfully flat for any k-algebra S, and $\mathcal{F}_n(S)$ is nonempty, so this last part is okay.

Definition 3.3.5. A G-torsor over a base space B is a bundle $\pi : P \longrightarrow B$ together with a group action

$$\rho : G \times_B P \longrightarrow P,$$

such that the induced map

$$\begin{aligned} (\rho, \mathrm{id}) \circ (\mathrm{id}, \Delta) : G \times_B P &\longrightarrow G \times_B P \times_B P \longrightarrow P \times_B P \\ (g, p) &\longmapsto (g, p, p) \longmapsto (g \cdot p, p) \end{aligned}$$

is an isomorphism.

[1] A G-torsor over a point is sometimes called a principal homogeneous G-space

Definition 3.3.6. Let G be a group scheme, and let X and Y be G-torsors. A scheme morphism $f : X \longrightarrow Y$ is G-equivariant if the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "G"'] & Y \arrow[d, "G"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

We would like to show that $\mathcal{F}_n$ is a $\mathbb{G}_a$-torsor when equipped with the action

$$h \cdot \frac{A}{B} := \frac{A + hB}{B}.$$

One way to show that would be to show that $\mathcal{F}_n(k)$ splits into some product $C \times D$, in such a way that $\mathbb{G}_a$ acts simply transitively on C and trivially on D. This would make $\mathcal{F}_n(k)$ into a $\mathbb{G}_a$-torsor over the base space D.

Lemma 3.3.7. *Let S be a k-algebra and $\frac{A}{B}$ be an element of $\mathcal{F}_n(S)$. There exists a unique pair of polynomials (U_1, V_1) of $S[X]$ with $\deg(U_1) = n-1$, $\deg(V_1) \leq n-1$ and such that $AU_1 + BV_1 = X^{2n-1}$. Let $\varphi_n\left(\frac{A}{B}\right)$ be the additive inverse of the coefficient of X^{n-1} in V_1. Then the associated scheme morphism*

$$\varphi_n : \mathcal{F}_n \longrightarrow \mathbb{A}^1$$

is $\mathbb{G}_a$-equivariant. In particular, $\mathcal{F}_n$ splits as the product $\varphi_n^{-1}(0) \times \mathbb{A}^1$.

Proof. Observe that the existence of such (U_1, V_1) follows from the fact that the linear operator $\mathrm{Syl}_{n,n}(A, B)$ has full rank, as explained in the proof of Lemma 2.4.6. Looking at the highest degree terms, we see that $AU_1 + BV_1 = u_{n-1}X^{2n-1} + (a_{n-1}u_{n-1} + b_{n-1}v_{n-1} + u_{n-2})x^{n-1} + \ldots$, which implies that U_1 is monic. Acting on (A, B) by h to give $(A + hB, B)$, turns (U_1, V_1) into $(U_1, V_1 - hU_1)$, since $(A + hB)U_1 + B(V_1 - hU_1) = X^{n-1}$. The following diagram commutes, and the lemma follows.

$$\begin{tikzcd} (A, B) \arrow[r, "\varphi_n"] \arrow[d, "h\cdot"'] & -v_{n-1} \arrow[d, "h\cdot"'] \\ (A + hB, B) \arrow[r, "\varphi_n"] & -v_{n-1} + h \end{tikzcd}$$

$\square$

Notice that $\text{Béz}_n(A+hB, B) = \text{Béz}_n(A, B)$, because $hB(Y)B(X) - hB(X)B(Y) = 0$. This means that Béz_n is $\mathbb{G}_a$-equivariant if we let $\mathbb{G}_a$ act trivially on $\mathcal{S}_n$. Thus Béz_2 induces a scheme isomorphism $\varphi_2^{-1}(0) \longrightarrow \mathcal{S}_2$, and we get the following proposition.

Proposition 3.3.8. *The morphism*

$$\varphi_n^{-1}(0) \times \mathbb{A}^1 = \mathcal{F}_2 \xrightarrow{\text{Béz}_2 \times \varphi_2} \mathcal{S}_2 \times \mathbb{A}^1$$

is a $\mathbb{G}_a$*-equivariant isomorphism of schemes.*

Proof. We find the inverse morphism $\psi : \mathcal{S}_2 \longrightarrow \varphi_2^{-1}(0)$ by solving a system of two equations with two unknowns. The formula is

$$\psi\left(\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\right) = \frac{X^2 + \frac{\alpha\beta}{\beta^2 - \alpha\gamma} X + \frac{\alpha^2}{\beta^2 - \alpha\gamma}}{\gamma X + \beta}.$$

□

Having checked degree 2, we may now conclude that injectivity (Proposition 3.3.1) holds. Since we have also proven surjectivity and graded monoid compatibility, we conclude with the main theorem of Cazanave's article.

Theorem 3.3.9. *The map* $\pi_0^{\mathrm{N}}\text{Béz}$ *is an isomorphism of graded monoids:*

$$\left(\pi_0^{\mathrm{N}} \mathcal{F}_n(k), \oplus^{\mathrm{N}}\right) \simeq \left(\pi_0^{\mathrm{N}} \mathcal{S}_n(k), \perp\right).$$

Having shown all of the necessary graded monoid isomorphisms, we may conclude that Theorem 3.0.1 holds.

Chapter 4

Group completing by way of Jouanolou's device

In this chapter, we define the Jouanolou device of $\mathbb{P}^1$. We then define morphisms and naive homotopies of morphisms from the Jouanolou device to the projective line.

In his paper "Une suite exacte de Mayer-Vietoris en K-théorie algébrique," Jouanolou states the following lemma (my translation, quotation marks in original):

Lemma 4.0.1 (Lemme 1.5 in [Jou73])**.** *Let X be a quasi-projective scheme. Then there exists a vector bundle E on X, and a torsor*

$$p : W \longrightarrow X$$

on E, with W affine. In particular, the fibers of W are vector spaces, and we can say, in a sense that should be clarified, that any quasi-projective scheme has "the same homotopy type" as an affine scheme.

To clarify, W and X are $\mathbb{A}^1$-homotopy equivalent, as stated in [AØ19, Lemma 3.1.4]. We call this W an affine vector bundle torsor on X, meaning that it is a torsor over a base space in the sense of Definition 3.3.5, and that the group action is that of a vector space [Wei89, Definition 4.2]. Such an affine torsor bundle will be called a *Jouanolou device*. In our case, the base space is $\mathbb{P}^1_k$ over some field k.

Definition 4.0.2 (Jouanolou device of $\mathbb{P}^1_k$)**.** Denote by R the ring

$$R := \frac{k[x, y, z, w]}{(x + w - 1, xw - yz)} = \frac{k[x, y, z]}{(x\,(1 - x) - yz)}. \tag{4.1}$$

The Jouanolou device of $\mathbb{P}^1$ is $\mathcal{J} := \mathrm{Spec}\,(R)$. We consider $\mathcal{J}$ to be pointed at $(x-1, y, z, w)$.

Notational remark. Throughout this thesis, we will use R for referring to this specific ring, and never as a placeholder for an arbitrary ring.

Since $\mathcal{J}$ and $\mathbb{P}^1$ are $\mathbb{A}^1$-homotopy equivalent, it is true that

$$[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} = [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}. \tag{4.2}$$

Asok, Hoyois and Wendts paper "Affine representability results in $\mathbb{A}^1$-homotopy theory II: principal bundles and homogeneous spaces" gives us some really useful results. In [AHW18, Definition 2.1.1] they define the notion of being $\mathbb{A}^1$-naive. If a scheme $Y \in \mathbf{Sm}_k$ is $\mathbb{A}^1$-naive, then the canonical map $[X, Y]^{\mathrm{N}} \longrightarrow [X, Y]^{\mathbb{A}^1}$ is a bijection for all smooth, affine k-schemes $X \in \mathbf{Sm}_k^{\mathit{aff}}$. In the same paper, [AHW18, Theorem 4.2.2] states that $\mathcal{J}$ is $\mathbb{A}^1$-naive. [AHW18, Lemma 4.2.4] states that an affine torsor bundle over a base space is $\mathbb{A}^1$-naive if and only if the base space is. Since $\mathcal{J}$ is $\mathbb{A}^1$-naive and an affine torsor bundle over $\mathbb{P}^1$, we get that $\mathbb{P}^1$ is $\mathbb{A}^1$-naive.

Since $\mathcal{J} \in \mathbf{Sm}_k^{\mathit{aff}}$ and $\mathbb{P}^1$ is $\mathbb{A}^1$-naive, the canonical map is a bijection

$$c : [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \longrightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}. \tag{4.3}$$

Combining Eq. (4.2) with Eq. (4.3) gives the bijection

$$d : [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \longrightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}. \tag{4.4}$$

The remainder of this thesis is dedicated to understanding Eq. (4.4). It is a bijection of sets, and the right side is a group. Hence there is a group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ induced by the canonical bijection c in Eq. (4.3). Our goal for the remainder of this thesis is to understand the group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ geometrically.

Remark 4.0.3. A path one could take, but which we will not explore is the one Cazanave points out in his PhD-thesis [Caz09, p. 31]. One could study the canonical bijection $[\mathcal{J}, \mathcal{J}]^N \longrightarrow [\mathcal{J}, \mathcal{J}]^{\mathbb{A}^1} \simeq [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. This would be interesting, but is beyond the scope of this thesis.

4.1 The canonical projection

In order to properly define morphisms $\mathcal{J} \longrightarrow \mathbb{P}^1$, we will use Theorem 2.2.1 and line bundles on $\mathcal{J}$ and two generating sections. However, it is nice to gain some intuition first. As $\mathcal{J}$ is a torsor bundle over $\mathbb{P}^1$, there is a canonical morphism $\pi : \mathcal{J} \longrightarrow \mathbb{P}^1$, which we will properly define in Example 4.3.5.

4.1.1 Intuition for $k = \mathbb{C}$

We informally explore this canonical morphism in the special case of $k = \mathbb{C}$ to gain some visual intuition.

Since $\mathbb{C}$ is algebraically closed, maximal ideals of R are all of the form $(x-a, y-b)$. The ideal $(x-a, y-b, xw-yz, x+w-1)$ is maximal in $k[x,y,z,w]$ and determines a unique point $(x,y,z,w) = (a,b,c,d) \in k^4$. The canonical morphism π acts on points by sending (a,b,c,d) to $[a : b]$ or $[c : d]$, depending on which one is well-defined. Note that they are equal when both are defined. The ideal $(x+w-1)$ ensures that we will never have $[a : b] = [0 : 0] = [c : d]$.

If these were the entries of a matrix $M = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, it would be idempotent. The ideal $(xw-yz)$ ensures that the determinant is 0, and ideal $(x+w-1)$ ensures that the trace is 1. Idempotence combined with trace 1 implies rank 1, and we see that it is idempotent by squaring the matrix:

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix}^2 = \begin{bmatrix} x^2+yz & z(x+w) \\ y(x+w) & w^2+yz \end{bmatrix} = \begin{bmatrix} x(x+w) & z(x+w) \\ y(x+w) & w(x+w) \end{bmatrix} = \begin{bmatrix} x & z \\ y & w \end{bmatrix}.$$

Looking at such a matrix M, there are two canonical ways of assigning a 1-dimensional subspace of k^2 to it: its image and kernel. Note that if $\operatorname{im} M = \ker M$, then $M^2 = M = 0$, which contradicts idempotence, so this can't be the case. Thus M corresponds to $(\ker M, \operatorname{im} M)$, which is a point in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$, where Δ is the diagonal.

The canonical map $\pi : \mathcal{J} \longrightarrow \mathbb{P}^1$ acts on points by sending $\pi : M \longmapsto \operatorname{im} M$. We see that the fiber consists of the different possible kernels, and it is parameterized by $\mathbb{A}^1_{\mathbb{C}}$. This illustrates how $\mathcal{J}$ is a $\mathbb{G}_a$-torsor over $\mathbb{P}^1$.

Let $\pi : \mathcal{J} \longrightarrow \mathbb{P}^1$ be given in homogeneous coordinates by

$$\pi : \begin{bmatrix} x & y \\ z & w \end{bmatrix} \longrightarrow \left[\begin{bmatrix} x \\ z \end{bmatrix} : \begin{bmatrix} y \\ w \end{bmatrix} \right].$$

Note the position of y and z here. Observe that $\operatorname{im} M = \operatorname{Span} \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right\}$, and that the span of $\begin{bmatrix} x \\ y \end{bmatrix}$ corresponds to the point $[x : y]$ in $\mathbb{P}^1$, and similarly $\begin{bmatrix} z \\ w \end{bmatrix}$ corresponds to $[z : w]$.

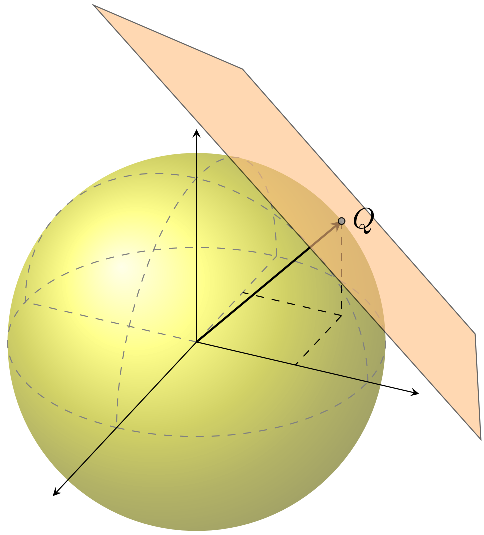


Figure 4.1: A very incomplete illustration of $\mathcal{J}$ over $\mathbb{C}$. $\mathbb{P}^1_{\mathbb{C}}$ is topologically a 2-sphere, and each point in $\mathbb{P}^1_{\mathbb{C}}$ has a complex affine line $\mathbb{A}^1_{\mathbb{C}}$ (i.e. a plane) as a fiber. A "correct" illustration would show an attached plane at every point of the sphere, twisting in the appropriate way. Because of the insufficient available dimensions, this illustration only shows the fiber over the point Q.

4.1.2 Preimages of the cover of $\mathbb{P}^1$

We have explored where π sends points (when $k = \mathbb{C}$). We now do the converse for arbitrary fields k, finding the preimages under π of the open affines that cover $\mathbb{P}^1$.

Proposition 4.1.1. *$\mathcal{J}$ is covered by two copies of the affine plane $\mathbb{A}^2$.*

We prove this by first showing the following lemma.

Lemma 4.1.2. *The preimage $\pi^{-1}(U_0)$ is $\mathbb{A}^2 \simeq D(x) \cup D(z)$.*

Proof. We have a ring isomorphism $\alpha : k[a,b,c]/(c(1-ab)-1) \longrightarrow R[x^{-1}]$, defined by

$$\begin{aligned} \alpha : a &\longmapsto yx^{-1} \\ b &\longmapsto z \\ c &\longmapsto x^{-1}, \end{aligned}$$

which induces a scheme isomorphism $D(x) \longrightarrow D(1-ab) \subset \mathbb{A}^2$. Similarly, the ring isomorphism $\beta : k[d, e, e^{-1}] \longrightarrow R[z^{-1}]$, defined by

$$\begin{aligned} \beta : d &\longmapsto wz^{-1} \\ e &\longmapsto z \\ e^{-1} &\longmapsto z^{-1} \end{aligned}$$

induces a scheme isomorphism $D(z) \longrightarrow D(e) \subset \mathbb{A}^2$. Gluing on the intersection by $a \longmapsto d$, $b \longmapsto e$, we get the following pushout diagram.

$$\begin{array}{ccc} D(1-ab) \cap D(b) \simeq D(1-de) \cap D(e) & \longrightarrow & D(e) \subset \mathbb{A}^2 \\ \downarrow & & \downarrow \\ D(1-ab) \subset \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \end{array}$$

We know that $D(1-ab) \cup D(a) = \mathbb{A}^2$, since this is equivalent to the existence of $c_i \in k[a, b]$ such that $c_1(1-ab) + c_2 a = 1$. We have $1(1-ab) + ba = 1$. Each term in the proceeding diagram is isomorphic to the corresponding term in the following diagram.

$$\begin{array}{ccc} D(x) \cap D(z) & \longrightarrow & D(z) \\ \downarrow & & \downarrow \\ D(x) & \longrightarrow & D(x) \cup D(z) \end{array}$$

Hence, $D(x) \cup D(z) \simeq \mathbb{A}^2$. □

Lemma 4.1.3. *The preimage $\pi^{-1}(U_1)$ is $\mathbb{A}^2 \simeq D(y) \cup D(w)$.*

Proof. By symmetry, this follows from the proof of Lemma 4.1.2. □

Proof of Proposition 4.1.1. We have an isomorphism $\mathrm{Spec}\,(k[a, b]) \simeq D(x) \cup D(z)$ given by $y/x = w/z \longmapsto a$ and $z \longmapsto b$. The other isomorphism $\mathrm{Spec}\,(k[u, v]) \simeq D(y) \cup D(w)$ is given by $x/y = z/w \longmapsto u$ and $y \longmapsto v$.

On the intersection, we glue by the following map, and we prove it is an isomorphism by constructing its inverse.

$$\begin{aligned} \psi : k[a, a^{-1}, b] &\longrightarrow k[u, u^{-1}, v] \\ a &\longmapsto u^{-1} \\ a^{-1} &\longmapsto u \\ b &\longmapsto u - u^2 v \end{aligned} \qquad \begin{aligned} \psi^{-1} : k[u, u^{-1}, v] &\longrightarrow k[a, a^{-1}, b] \\ u^{-1} &\longmapsto a \\ u &\longmapsto a^{-1} \\ v &\longmapsto a - a^2 b. \end{aligned}$$

□

Corollary 4.1.4. *$\mathcal{J}$ is an affine vector bundle torsor on $\mathbb{P}^1$.*

Proof. $\mathcal{J}$ is an affine scheme and $\pi : \mathcal{J} \longrightarrow \mathbb{P}^1$ is an affine map. $\mathcal{J}$ is locally trivializable and hence locally isomorphic to a vector bundle on $\mathbb{P}^1$. The patching maps ψ, ψ^{-1} are affine. □

Corollary 4.1.5. *The Krull dimension of R is 2.*

Proof. For any covering of the scheme X by affine open schemes X_i we have $dim(X) = sup_i(dim(X_i))$. Hence, by Proposition 4.1.1, $\mathcal{J}$ is two-dimensional. □

4.2 Understanding R and its modules

To study morphisms $\mathcal{J} \longrightarrow \mathbb{P}^1$ we should understand the line bundles on $\mathcal{J}$. Serre's theorem [Ser55, Corollaire to Proposition 4, p. 242] states that for an affine scheme $(X, \mathcal{O}_X)$, there is an equivalence of categories between the category of algebraic vector bundles on X and the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules. And an algebraic line bundle $\mathcal{L}$ over a commutative ring S is a finitely-generated S-module of constant rank 1 [Wei13, p.15].

In this section we first show that R is an integral domain, which implies that all finitely generated projective modules have constant rank. We then find enough rank 1 finitely generated projective modules to have a line bundle in each class of Pic $(\mathcal{J})$. The Picard group is an $\mathbb{A}^1$-homotopy invariant, hence the $\mathbb{A}^1$-homotopy equivalence $\pi : \mathcal{J} \longrightarrow \mathbb{P}^1$ induces an isomorphism of Picard groups $\pi^* : \operatorname{Pic} \mathbb{P}^1 \longrightarrow \operatorname{Pic} \mathcal{J} \cong \mathbb{Z}$.

Lemma 4.2.1. *R is an integral domain.*

Proof. This is equivalent to showing that the ideal $(x(1-x) - yz) \subset k[x, y, z]$ is prime. Since $k[x, y, z]$ is a domain, it suffices to show that $x(1-x) - yz$ is an irreducible element. The grading on $k[x, y, z]$ tells us that if $pq = x(1-x) - yz$, then $\deg p = 1 = \deg q$. We write $p = ax + by + cz + d$ and $q = ex + fy + gz + h$. Now, we need these coefficients to satisfy (among others) the following equations:

$$dh = 0 \qquad ah + de = 1 \qquad ch + dg = 0$$
$$bh + df = 0 \qquad bg + cf = -1.$$

Since (0) is prime, $dh = 0$ implies that $d = 0$ or $h = 0$, and $ah + de = 1$ implies that d, h are not both 0. Without loss of generality, $h = 0, d \neq 0$. Then $ch + dg = 0$

implies $g = 0$, and $bh + df = 0$ implies $f = 0$. But then $0 = bg + cf = -1$, which is absurd, hence p, q can not exist. $\square$

Since R is an integral domain it contains no nontrivial idempotents. Since local rings have no nontrivial idempotents, there is a correspondence between idempotents $e \in R$ and clopen sets $D(e) = \{\mathfrak{p} \in \operatorname{Spec}(R) \,|e \notin \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec}(R) \,|e_{\mathfrak{p}} = 1\}$. The only idempotents of R are 1 and 0, which correspond to the only clopen sets $\mathcal{J}$ and $\emptyset$. Hence $\mathcal{J} = \operatorname{Spec}(R)$ is connected. Finitely generated projective modules have locally constant rank, so if $\mathcal{J}$ is connected, then they all have constant rank.

4.2.1 Finitely generated rank 1 projective modules on R

Any finitely generated projective module P is by definition a direct summand of $P \oplus Q = R^n$. Hence, the projection-inclusion $R^n \longrightarrow P \longrightarrow R^n$ is an idempotent matrix $e \in M_n(R)$ [Wei13, p. 8]. We have $\operatorname{im} e = P$ and $\ker e = Q$. We will use this method to find enough modules to describe each isomorphism class of line bundles over $\mathcal{J}$.

Let $\mathcal{P}_1, \mathcal{Q}_1$ be the images of the following idempotent matrices.

$$\mathcal{P}_1 = \operatorname{Im} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
$$\mathcal{Q}_1 = \operatorname{Im} \begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

These images are modules of rank 1, since the columns of each matrix are linearly dependent. Thus they are line bundles over $\mathcal{J}$. We show that they are inverses in $\operatorname{Pic}(\mathcal{J})$ by calculating their tensor product. An element of $\mathcal{P}_1 \otimes \mathcal{Q}_1$ can be written as

$$\sum \left(\alpha_i \begin{bmatrix} x \\ z \end{bmatrix} + \beta_i \begin{bmatrix} y \\ w \end{bmatrix} \right) \otimes_R \left(a_i \begin{bmatrix} x \\ y \end{bmatrix} + b_i \begin{bmatrix} z \\ w \end{bmatrix} \right), \quad \alpha_i, \beta_i, a_i, b_i \in R.$$

We see that $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is generated by the elements

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix} \right\}.$$

We define a module homomorphism $\varphi : R^2 \otimes R^2 \longrightarrow R^2$ by

$$\varphi \left(\begin{bmatrix} a \\ c \end{bmatrix} \otimes \begin{bmatrix} b \\ d \end{bmatrix} \right) = \begin{bmatrix} ab \\ cd \end{bmatrix}. \tag{4.5}$$

Restricting, we get $\varphi|_{\mathcal{P}_1\otimes\mathcal{Q}_1} : \mathcal{P}_1 \otimes \mathcal{Q}_1 \longrightarrow R^2$. We see that the four basis elements correspond to the following elements in R^2

$$\left\{x\begin{bmatrix}x\\w\end{bmatrix}, z\begin{bmatrix}x\\w\end{bmatrix}, y\begin{bmatrix}x\\w\end{bmatrix}, w\begin{bmatrix}x\\w\end{bmatrix}\right\}.$$

Since $(x+w)=1$, these elements generate a rank 1 submodule of R^2. This submodule $R\begin{bmatrix}x\\w\end{bmatrix}$ is isomorphic to R, so $\mathcal{P}_1\otimes\mathcal{Q}_1 \cong \mathcal{O}_{\mathcal{J}}$.

Similarly, for each n, we define the $(n+1)\times(n+1)$-matrix

$$M_{\mathcal{P}_n} := \begin{pmatrix} \binom{n}{0}x^n & \cdots & \binom{n}{i}x^{n-i}y^i & \cdots & \binom{n}{n}y^n \\ \vdots & \ddots & & & \vdots \\ \binom{n}{0}x^{n-j}z^j & & \binom{n}{i}x^\alpha y^\beta z^\gamma w^\delta & & \binom{n}{n}y^{n-j}w^j \\ \vdots & & & \ddots & \vdots \\ \binom{n}{0}z^n & \cdots & \binom{n}{i}z^{n-i}w^i & \cdots & \binom{n}{n}w^n \end{pmatrix},$$

where $\alpha=\min\{n-i,n-j\}$, $\beta=n-j-\alpha$, $\delta=\min\{i,j\}$, and $\gamma=j-\delta$, and $\binom{n}{i}$ denotes n choose i. This matrix is idempotent. Since the rows are all linearly dependent, the projective module $\operatorname{im}(M_{\mathcal{P}_n})$ has rank 1. It is isomorphic to the module generated by the elements $\left\{\begin{bmatrix}x^{n-i}y^i\\z^{n-i}w^i\end{bmatrix}\right\}_{0\le i\le n}$. This module is denoted by $\mathcal{P}_n$.

Similarly, for each n there exists an idempotent matrix $M_{\mathcal{Q}_n}$ which is obtained by by interchanging y and z in the matrix $M_{\mathcal{P}_n}$. The image of $M_{\mathcal{Q}_n}$ is a projective module isomorphic to the one generated by $\left\{\begin{bmatrix}x^{n-i}z^i\\y^{n-i}w^i\end{bmatrix}\right\}_{0\le i\le n}$. This module is denoted by $\mathcal{Q}_n$.

Remark 4.2.2. For the generators of $\mathcal{P}_n$ and $\mathcal{Q}_n$ respectively, observe that $y\begin{bmatrix}x^{n-i}y^i\\z^{n-i}w^i\end{bmatrix} = x\begin{bmatrix}x^{n-i-1}y^{i+1}\\z^{n-i-1}w^{i+1}\end{bmatrix}$ and $w\begin{bmatrix}x^{n-i}y^i\\z^{n-i}w^i\end{bmatrix} = z\begin{bmatrix}x^{n-i-1}y^{i+1}\\z^{n-i-1}w^{i+1}\end{bmatrix}$, as well as

$z\begin{bmatrix}x^{n-i}z^i\\y^{n-i}w^i\end{bmatrix} = x\begin{bmatrix}x^{n-i-1}y^{i+1}\\z^{n-i-1}w^{i+1}\end{bmatrix}$ and $w\begin{bmatrix}x^{n-i}z^i\\y^{n-i}w^i\end{bmatrix} = y\begin{bmatrix}x^{n-i-1}z^{i+1}\\y^{n-i-1}w^{i+1}\end{bmatrix}$.

Lemma 4.2.3. *The involutive automorphism $\tau : R \longrightarrow R$ is defined by*

$$\begin{aligned} \tau : x &\longmapsto x & \qquad y &\longmapsto z \\ z &\longmapsto y & w &\longmapsto w. \end{aligned}$$

Pulling back along τ gives R-module isomorphisms $\tau^*\mathcal{P}_n \simeq \mathcal{Q}_n$ and $\tau^*\mathcal{Q}_n \simeq \mathcal{P}_n$.

Proof. To more easily distinguish between them, we give the domain and codomain different names: $\tau : R \longrightarrow R'$. Pulling back the R'-module $\mathcal{P}_n$, we get the R-module $\tau^*\mathcal{P}_n$, where the multiplication is defined by $r \cdot_R p = \tau(r) \cdot_{R'} p$. The map $f : \tau^*\mathcal{P}_n \longrightarrow \mathcal{Q}_n$ is defined on basis of elements by

$$f : \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \longmapsto \begin{bmatrix} x^{n-i}z^i \\ y^{n-i}w^i \end{bmatrix}.$$

It is easily checked that f is bijective and R-linear and hence an R-module isomorphism.

To see that $\tau^*\mathcal{Q}_n \simeq \mathcal{P}_n$, we pull back along τ on both sides, getting $\tau^*\tau^*\mathcal{Q}_n \simeq \tau^*\mathcal{P}_n$. Since $\tau \circ \tau = \mathrm{id}$, this simplifies to $\mathcal{Q}_n \simeq \tau^*\mathcal{P}_n$, which we just proved. □

Proposition 4.2.4. *$\mathcal{P}_1^{\otimes n} \cong \mathcal{P}_n$ and $\mathcal{Q}_1^{\otimes n} \cong \mathcal{Q}_n$.*

Proof. It suffices to prove this for one of them, since interchanging z and y would be a proof for the other by Lemma 4.2.3. We prove $\mathcal{P}_1^{\otimes n} \cong \mathcal{P}_n$ by induction. Observe that $\mathcal{P}_1^{\otimes 1} \cong \mathcal{P}_1$ is true by definition. Assume that the proposition holds for all $m \leq n$. Calculating $\mathcal{P}_1 \otimes \mathcal{P}_n$ yields the following set of generators

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \right\}_{0\leq i\leq n} \bigcup \left\{ \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \right\}_{0\leq i\leq n}.$$

By applying the same homomorphism φ we used earlier, we get the set

$$\left\{ \begin{bmatrix} x^{n+1-i}y^i \\ z^{n+1-i}w^i \end{bmatrix} \right\}_{0\leq i\leq n+1},$$

which by definition generates $\mathcal{P}_{n+1}$. □

This gives us the following result.

Theorem 4.2.5. *The line bundle $\mathcal{P}_1$ generates Pic $(\mathcal{J}) = \mathbb{Z}$, and $\mathcal{Q}_1$ is its inverse.*

Proposition 4.2.6. *The R-module generated by $\left\{ \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \dots, \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix}, \dots, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\}$ equals the R-module generated by the set $\left\{ \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\}$, and hence these two elements generate $\mathcal{P}_n$. The analogous statement holds for $\mathcal{Q}_n$.*

Proof. We will only prove it for $\mathcal{P}_n$, because by Lemma 4.2.3 the other proof follows. Containment in one direction is obvious. For the other direction, fix n and pick a number $0 \leq i \leq n$. Consider the element

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = (x+w)^n \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = \sum_{d=0}^{n} \binom{n}{d} x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix}.$$

For each d, one of the following hold:

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-i-d}y^i w^d \begin{bmatrix} x^n \\ z^n \end{bmatrix} \qquad \text{if } i+d \leq n,$$

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-d}z^{n-i}w^{d+i-n} \begin{bmatrix} y^n \\ w^n \end{bmatrix} \quad \text{if } i+d > n.$$

The proposition follows. □

Corollary 4.2.7. *The ideal* $(x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$ *equals* $(x^n, y^n) \subset R$*, and similarly for all other pairs* $\{z, w\}$*,* $\{x, z\}$*,* $\{y, w\}$*,* $\{y, z\}$*, and* $\{x, w\}$*.*

Proof. All of the pairs are immediate consequences of Proposition 4.2.6 except for $\{y, z\}$ and $\{x, w\}$, which follow from calculating $(x+w)^n y^{n-i}z^i$ and $(x+w)^n x^{n-i}w^i$. □

Proposition 4.2.8. *The ideal* (x^n, w^n) *equals* R*.*

Proof. By Corollary 4.2.7, the ideal $(x^n, x^{n-1}w, \ldots, w^n) = (x^n, w^n) \subseteq R$. Since $1 = (x+w)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}w^i \in (x^n, x^{n-1}w, \ldots, w^n)$, we get equality $(x^n, w^n) = R$. □

4.2.2 Generating global sections

Using the concept of a unimodular row, we get a simple description of $\mathcal{J}$ and its line bundles.

Definition 4.2.9 (Unimodular row)**.** Let S be a ring. We call $\sigma = (s_1, \ldots, s_n)$, where $s_i \in S$, a unimodular row if any of the following equivalent conditions hold:

- $S^n \cong P \oplus S$, where $P = \ker \sigma$ and the projection $S^n \longrightarrow S$ is σ.
- $S = s_1 S + \ldots + s_n S$.
- $1 = s_1 r_1 + \ldots + s_n r_n$.

Weibel writes about using unimodular rows to build projective modules by patching free modules in "The K-book" [Wei13, p. 11]. A unimodular row with $s_i \in S$ gives rise to a covering of $\mathrm{Spec}\,(S)$. Given transition functions $g_{ij} \in GL_n(S[s_i^{-1}s_j^{-1}])$ that satisfy some compatibility conditions, this data determines a finitely generated projective S-module.

A pair of generating sections of $\mathcal{O}_{\mathcal{J}}$ is by definition a unimodular row. It is also practical to use unimodular rows to check whether a pair of sections of $\mathcal{P}_n$ or $\mathcal{Q}_n$ generate. Observe that (x, w) is a unimodular row since $x + w = 1$. This implies that $\mathcal{J} = \mathrm{Spec}\,(R)$ is covered by the open sets $D(x)$ and $D(w)$, a subcover of the cover in Proposition 4.1.1.

If a module M is finitely presented and satisfies that for all prime ideals $\mathfrak{p} \in R$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module, then (and only then) is M a finitely generated projective R-module [Wei13, p. 10]. Let $s_0 = f_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + f_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$, $s_1 = g_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + g_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$ be a pair of sections of $\mathcal{P}_n$. They generate $\mathcal{P}_n$ if they generate $R[x^{-1}]$ on $D(x)$ and $R[w^{-1}]$ on $D(w)$. We express this in a single equation in Proposition 4.3.1.

4.3 Morphisms from $\mathcal{J}$ to $\mathbb{P}^1$

We have found a line bundle for each element of $\mathrm{Pic}\,(\mathcal{J})$. If we pick a line bundle, and a pair of generating sections, we get a scheme morphism $\mathcal{J} \longrightarrow \mathbb{P}^1$. We give an algebraic criterion before giving some examples. We then look at the homotopy classes $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$.

4.3.1 Criterion for being a morphism

To verify that a pair of sections give a morphism, we need to check that they generate the line bundle. It will be convenient to reformulate that condition in the following way.

Proposition 4.3.1. *Let* $s_0 = f_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + f_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = g_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + g_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$ *be a pair of sections of* $\mathcal{P}_n$*. The following are equivalent:*

1. *The pair defines a morphism* $(f_0, f_1 : g_0, g_1)_{\mathcal{P}_n}$.
2. *The sections generate* $\mathcal{P}_n$.
3. *There exist* $A, B, C, D \in R$ *such that*

$$A(x^n f_0 + y^n f_1) + B(x^n g_0 + y^n g_1) + C(z^n f_0 + w^n f_1) + D(z^n g_0 + w^n g_1) = 1.$$

The analogous statement for $\mathcal{Q}_n$ also holds.

Proof. We already know that the first two statements are equivalent. By Lemma 4.2.3 it suffices to prove the proposition for $\mathcal{P}_n$. We will first prove that (2) implies (3), and then that (3) implies (2).

Assume that the sections generate $\mathcal{P}_n$. By looking at $\mathcal{P}_n$ over the open cover consisting of $D(x)$ and $D(w)$, we get to use the elements x^{-1} and w^{-1} respectively. Applying Remark 4.2.2 and localizing away from x we get

$$\begin{aligned} s_0 &= f_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + f_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} = \left(f_0 + f_1 \frac{y^n}{x^n} \right) \begin{bmatrix} x^n \\ z^n \end{bmatrix} \\ s_1 &= g_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + g_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} = \left(g_0 + g_1 \frac{y^n}{x^n} \right) \begin{bmatrix} x^n \\ z^n \end{bmatrix}. \end{aligned}$$

Localizing away from w gives

$$\begin{aligned} s_0 &= \left(f_0 \frac{z^n}{w^n} + f_1 \right) \begin{bmatrix} y^n \\ w^n \end{bmatrix} \\ s_1 &= \left(g_0 \frac{z^n}{w^n} + g_1 \right) \begin{bmatrix} y^n \\ w^n \end{bmatrix}. \end{aligned}$$

Since we assumed that (s_0, s_1) generates $\mathcal{P}_n$, there exist $U_x, V_x, U_w, V_w \in R$ such that

$$U_x \left(f_0 + f_1 \frac{y^n}{x^n} \right) + V_x \left(g_0 + g_1 \frac{y^n}{x^n} \right) = 1 \tag{$*$}$$

and

$$U_w \left(f_0 \frac{z^n}{w^n} + f_1 \right) + V_w \left(g_0 \frac{z^n}{w^n} + g_1 \right) = 1. \tag{$**$}$$

Multiplying $(*)$ by x^n and $(**)$ by w^n yields the following equations, which no longer contain x^{-1} or w^{-1}, and hence hold over all of $\mathcal{J}$.

$$\begin{aligned} U_x(f_0 x^n + f_1 y^n) + V_x(g_0 x^n + g_1 y^n) &= x^n \\ U_w(f_0 z^n + f_1 w^n) + V_w(g_0 z^n + g_1 w^n) &= w^n. \end{aligned}$$

By Proposition 4.2.8, $(x^n, w^n) = R$. Hence there exist $A, B, C, D \in R$ such that

$$A(x^n f_0 + y^n f_1) + B(x^n g_0 + y^n g_1) + C(z^n f_0 + w^n f_1) + D(z^n g_0 + w^n g_1) = 1. \tag{$***$}$$

To prove the converse, we assume there exist $A, B, C, D \in R$ such that $(***)$ holds. Multiplying $(***)$ by x^n gives

$$(Ax^n + Cz^n)(x^n f_0 + y^n f_1) + (Bx^n + Dz^n)(x^n g_0 + y^n g_1) = x^n,$$

a relation for creating a unit in $R[x^{-1}]$. Similarly, multiplying $(***)$ by w^n gives

$$(Ay^n + Cw^n)(z^n f_0 + w^n f_1) + (By^n + Dw^n)(z^n g_0 + w^n g_1) = w^n,$$

a relation for creating a unit in $R[w^{-1}]$. Which means $(f_0, f_1 : g_0, g_1)_{\mathcal{P}_n}$ is a morphism. $\square$

Let us define some notation.

Definition 4.3.2 (Compact morphism notation)**.** We use morphism notation similar to in Remark 2.2.2. The subscript $\mathcal{P}_n$, $\mathcal{O}_{\mathcal{J}}$ or $\mathcal{Q}_n$ indicates the line bundle. Let $f_i, g_i \in R$. The morphism defined by $\mathcal{P}_n$ and the sections

$$s_0 = f_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + f_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$$
$$s_1 = g_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + g_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$$

will be denoted by

$$(f_0, f_1 : g_0, g_1)_{\mathcal{P}_n},$$

and similarly for $\mathcal{O}_{\mathcal{J}}$ and $\mathcal{Q}_n$.

Definition 4.3.3 (Degree of a morphism)**.** We define a degree map $\deg : \operatorname{Pic} \mathcal{J} \longrightarrow \mathbb{Z}$ which sends $\mathcal{P}_n$ to n, $\mathcal{O}_{\mathcal{J}}$ to 0, and $\mathcal{Q}_n$ to $-n$. We say that the degree of a morphism is the degree of the line bundle that defines it.

Notational remark. The line bundles $\mathcal{P}_n$ and $\mathcal{Q}_n$ admit (by Proposition 4.2.6) a "short" 2-element basis as well as a "long" $n+1$-element generating set. We will use both. They coincide when $2 = n + 1$, and are otherwise clearly distinct since the number of elements differ, so we will not do anything else to distinguish them. For the trivial line bundle $\mathcal{O}_{\mathcal{J}}$, we simply use the canonical 1-element basis, and denote morphisms by $(f_0 : g_0)_{\mathcal{O}_{\mathcal{J}}}$.

We consider $\mathcal{J}$ to be pointed at $(x - 1, y, z, w)$, and a morphism $\mathcal{J} \longrightarrow \mathbb{P}^1$ is pointed if it sends $(x - 1, y, z, w)$ to $[1 : 0]$.

Proposition 4.3.4 (Pointed criterion)**.** *The morphism $(f_0, f_1 : g_0, g_1)_{\mathcal{P}_n}$ is pointed if $f_0 \notin (x - 1, y, z, w)$ and $g_0 \in (x - 1, y, z, w)$. The condition for $(f_0, f_1 : g_0, g_1)_{\mathcal{Q}_n}$ is the same. Similarly, $(f_0 : g_0)_{\mathcal{O}_{\mathcal{J}}}$ is pointed if $f_0 \notin (x - 1, y, z, w)$ and $g_0 \in (x - 1, y, z, w)$.*

Proof. This is immediate, as $\mathcal{J}$ is pointed at $(x - 1, y, z, w)$, and $\mathbb{P}^1$ is pointed at $[1 : 0]$. $\square$

4.3.2 Examples of morphisms

Example 4.3.5. The canonical morphism π is defined by $\mathcal{P}_1$ and the sections

$$s_0 = \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{and} \quad s_1 = \begin{bmatrix} y \\ w \end{bmatrix},$$

which in compact notation is written as $(1, 0 : 0, 1)_{\mathcal{P}_1}$. We may check that this defines a morphism by using the criterion in Proposition 4.3.1. $A = D = 1, B = C = 0$ satisfy the equation $Ax + By + Cz + Dw = 1$, since $x + w = 1$.

Example 4.3.6. Another morphism is defined by $\mathcal{P}_1$ and the sections

$$s_0 = \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{and} \quad s_1 = y^4 \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} y \\ w \end{bmatrix},$$

which in compact notation is written as $(1, 0 : y^4, 1)_{\mathcal{P}_1}$.

Example 4.3.7. The $\mathcal{Q}_1$-analog of the canonical morphism is called $\widetilde{\pi}$ and is defined by $\mathcal{Q}_1$ and the sections

$$s_0 = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad s_1 = \begin{bmatrix} z \\ w \end{bmatrix}.$$

In compact notation it is written as $(1, 0 : 0, 1)_{\mathcal{Q}_1}$.

Example 4.3.8. An example over $\mathcal{Q}_2$ is given by

$$s_0 = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2 \begin{bmatrix} z^2 \\ w^2 \end{bmatrix} \quad \text{and} \quad s_1 = \begin{bmatrix} z^2 \\ w^2 \end{bmatrix},$$

which in short compact notation is written as $(1, 2 : 0, 1)_{\mathcal{Q}_2}$ and in long compact notation as $(1, 0, 2 : 0, 0, 1)_{\mathcal{Q}_2}$.

Example 4.3.9. The terms f_0, f_1, g_0, g_1 do not have to be elements of the ground field k to define a morphism. An example of this is the morphism $(x, 0 : 0, w)_{\mathcal{P}_1}$. We see that it is a morphism by using the condition in Proposition 4.3.1. Picking $A = 1 + 2w, B = C = 0, D = 1 + 2x$ satisfies the equation.

Example 4.3.10. Being expressible in compact notation with only coefficients from the ground field k depends on whether you use short or long notation. An example is the morphism $(1, 0, 1 : 0, 1, 0)_{\mathcal{P}_2} = (1, 0, 1 : y, 0, z)_{\mathcal{P}_2} = (1, 1 : y, z)_{\mathcal{P}_2}$.

Example 4.3.11. The degree 0-morphisms are given by a pair of elements of R that generate the unit ideal, i.e., a unimodular row (Definition 4.2.9). Two examples are $(1 - 2y : y)_{\mathcal{O}_{\mathcal{J}}}$ and $(x^2 + 2xw : w^2)_{\mathcal{O}_{\mathcal{J}}}$.

4.3.3 Factoring through π or $\widetilde{\pi}$

An interesting question is when morphisms $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ factor as $f = g \circ \pi$ or as $f = g \circ \widetilde{\pi}$, where $g : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. In this section we give a sufficient condition for this to hold by defining a resultant for morphisms $\mathcal{J} \longrightarrow \mathbb{P}^1$. It is not nearly as well behaved as the one defined in Definition 2.3.1 for pointed morphisms $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$, but it will still be useful.

Proposition 4.3.12. *Let $s_0 = a_n x_0^n + \ldots + a_0 x_1^n$ and $s_1 = b_n x_0^n + \ldots + b_0 x_1^n$ be two homogeneous polynomials in variables x_0, x_1 with coefficients a_i, b_i from a ring S. Then*

$$\mathrm{res}_{n,n}\left(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}\right) = (-1)^{n+1}\mathrm{res}_{n,n}\left(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}\right).$$

Proof. Consider the Sylvester matrix.

$$\mathrm{Syl}_{n,n}\left(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}\right) = \begin{pmatrix} a_n & & 0 & b_n & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_1 & \ldots & a_n & b_1 & \ldots & b_n \\ a_0 & \ldots & a_{n-1} & b_0 & \ldots & b_{n-1} \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_0 & 0 & & b_0 \end{pmatrix}.$$

We denote the zero matrix by 0_n, the identity matrix by I_n, and the matrix with 1s on the anti-diagonal and 0s everywhere else by J_n. We calculate

$$\begin{aligned} &\begin{pmatrix} J_n & 0_n \\ 0_n & J_n \end{pmatrix} \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} \mathrm{Syl}_{n,n}\left(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}\right) \begin{pmatrix} J_n & 0_n \\ 0_n & J_n \end{pmatrix} \\ &\qquad = \mathrm{Syl}_{n,n}\left(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}\right) = \begin{pmatrix} a_0 & & 0 & b_0 & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_{n-1} & \ldots & a_0 & b_{n-1} & \ldots & b_0 \\ a_n & \ldots & a_1 & b_n & \ldots & b_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_n & 0 & & b_n \end{pmatrix}. \end{aligned}$$

Calculating the determinant of both sides yields

$$\mathrm{res}_{n,n}\left(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}\right) = (-1)^{n+1}\mathrm{res}_{n,n}\left(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}\right).$$

□

On $\mathcal{J}$ we use this in the following way. Consider, without loss of generality, $\mathcal{P}_n$, and a pair of global sections s_0, s_1 of given by

$$s_0 = \begin{bmatrix} a_0x^n + \ldots + a_ny^n \\ a_0z^n + \ldots + a_nw^n \end{bmatrix} \quad \text{and} \quad s_1 = \begin{bmatrix} b_0x^n + \ldots + b_ny^n \\ b_0z^n + \ldots + b_nw^n \end{bmatrix}.$$

They define a map $\mathcal{J} \longrightarrow \mathbb{P}^1$. On $D(x)$ the map can be written as

$$(x, y, z, w) \longmapsto [a_0x^n + \ldots + a_ny^n : b_0x^n + \ldots + b_ny^n],$$

which simplifies to

$$(x, y, z, w) \longmapsto [a_0 + \ldots + a_n(y/x)^n : b_0 + \ldots + b_n(y/x)^n].$$

After simplifying on $D(w)$, we get

$$(x, y, z, w) \longmapsto [a_0(z/w)^n + \ldots + a_n : b_0(z/w)^n + \ldots + b_n].$$

Hence a morphism $(a_0, \ldots, a_n : b_0, \ldots, b_n)_{\mathcal{P}_n}$ can be written on the form as a homogeneous polynomial in x, y and as a homogeneous polynomial in z, w. We can use Proposition 4.3.12 to associate a resultant up to sign to that specific presentation of the morphism.

However, there are two big problems with this resultant. It is possible for the same morphism to have multiple representations that have different resultants. It is also possible for a pair of generating sections to have the same resultant as a pair of non-generating sections.

Example 4.3.13. We have the equality $x \begin{bmatrix} y \\ w \end{bmatrix} = y \begin{bmatrix} x \\ z \end{bmatrix}$, but the resultant is not invariant when using this equality to represent a global section differently.

Example 4.3.14. The two pairs of sections $(x, 0 : 0, w)_{\mathcal{P}_1}$ and $(x, 2x : w, w)_{\mathcal{P}_1}$ both have resultant xw, but only $(x, 0 : 0, w)_{\mathcal{P}_1}$ generates a morphism. Let $k = \mathbb{C}$ and localize at the maximal ideal $\mathfrak{m} = (x - 1, y + \frac{1}{2})$. The value of the first section of $(x, 2x : w, w)_{\mathcal{P}_1}$ is $x^2 - 2yx = 0$, and the value of the second section is $wx + wy = 0$. Since both vanish, they do not define a morphism.

These problems only arise when the coefficients are not in the field k. With those caveats out of the way, the resultant is useful when we only have field coefficients.

Proposition 4.3.15. *$f : \mathcal{J} \longrightarrow \mathbb{P}^1$ factors as $f = g \circ \pi : \mathcal{J} \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ if the sections $(s_0 : s_1)_{\mathcal{P}_n}$ defining f can be written in long compact notation with only field coefficients. $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ factors as $f = g \circ \widetilde{\pi} : \mathcal{J} \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ if $(s_0 : s_1)_{\mathcal{Q}_n}$ defining f can be written in long compact notation with only field coefficients.*

Proof. We only prove this for $\mathcal{P}_n$. As we have seen in Lemma 4.1.2, π acts by

$$\begin{aligned} \pi : y/x &\longmapsto x_1/x_0 \\ z/w &\longmapsto x_0/x_1 \end{aligned}$$

on $D(x)$ and $D(w)$ respectively. If all the coefficients a_i, b_i are in k, then f factors as $g \circ \pi$, where $g : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the map

$$g : [x_0, x_1] \longmapsto [a_0(x_0/x_1)^n + \ldots + a_n : b_0(x_0/x_1)^n + \ldots + b_n].$$

The resultant of f on $D(w)$ and the resultant of g are equal. □

4.4 Naive homotopies of morphisms

Recall the definition of a naive homotopy (Definition 1.0.1). A naive homotopy of maps $\mathcal{J} \longrightarrow \mathbb{P}^1$ is a morphism $H : \mathcal{J} \times \mathbb{A}^1 \longrightarrow \mathbb{P}^1$.

$$\mathcal{J} \underset{s_1}{\overset{s_0}{\rightrightarrows}} \mathcal{J} \times \mathbb{A}^1 \xrightarrow{H} \mathbb{P}^1$$

The inclusions s_0, s_1 of $\mathcal{J}$ are induced by the ring maps $R[T] \longrightarrow R$ given by evaluating at $T = 0$ and $T = 1$. The modules on R can be pulled back under these ring maps to give modules on $R[T]$. The pullback of any $\mathcal{L}$ is $\mathcal{L} \otimes_R R[T]$, which we will denote $\mathcal{L}(R[T])$. It is given by the same generators as $\mathcal{L}$, but allows for coefficients in $R[T]$. Since $\operatorname{Pic} \mathcal{J} = \operatorname{Pic} \mathcal{J} \times \mathbb{A}^1$ (the Picard group is $\mathbb{A}^1$-homotopy invariant), we obtain all line bundles on $\mathcal{J} \times \mathbb{A}^1$ in this way.

Proposition 4.4.1. *The degree (Definition 4.3.3) of a morphism $\mathcal{J} \longrightarrow \mathbb{P}^1$ is a naive homotopy invariant.*

Proof. This follows from the fact that the line bundles on $\mathcal{J} \times \mathbb{A}^1$ are $\mathcal{L}(R[T])$, where $\mathcal{L}$ is a line bundle on $\mathcal{J}$. □

Two global sections of a line bundle $\mathcal{L}(R[T])$ generate the line bundle if they locally generate a module isomorphic to $R[T, x^{-1}]$ on $D(x) \subset \operatorname{Spec}(R[T])$ and similarly a module isomorphic to $R[T, w^{-1}]$ on $D(w) \subset \operatorname{Spec}(R[T])$. The following analog of Proposition 4.3.1 holds.

Proposition 4.4.2. *Let* $s_0 = f_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + f_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = g_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + g_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$ *be a pair of sections of $\mathcal{P}_n(R[T])$. The following are equivalent:*

1. $(f_0, f_1 : g_0, g_1)_{\mathcal{P}_n(R[T])}$ *defines a morphism.*

2. *The sections generate $\mathcal{P}_n(R[T])$.*

3. *There exist $A, B, C, D \in R[T]$ such that*

$$A(x^n f_0+y^n f_1)+B(x^n g_0+y^n g_1)+C(z^n f_0+w^n f_1)+D(z^n g_0+w^n g_1) = 1.$$

The analogous statement for $\mathcal{Q}_n(R[T])$ also holds.

Proof. The proof is the same as for Proposition 4.3.1. □

4.4.1 Examples of naive homotopies

Example 4.4.3. Recall Examples 4.3.5 and 4.3.6. They are homotopic, and a homotopy connecting them is $(1, 0 : Ty^4, 1)_{\mathcal{P}_1(R[T])}$. Picking $A = D = 1, B = C = 0$ satisfies the condition in Proposition 4.4.2.

Proposition 4.4.4. *Generalizing the last example, let $\mathcal{L}$ be $\mathcal{P}_n$ or $\mathcal{Q}_n$ for some n. For any $r \in R$, $(1, 0 : Tr, 1)_{\mathcal{L}(R[T])}$ and $(1, Tr : 0, 1)_{\mathcal{L}(R[T])}$ are morphisms.*

Proof. To see this, consider, without loss of generality, $\mathcal{L} = \mathcal{P}_n$. By Proposition 4.2.8, it is possible to find A and D such that $Ax^n + Dw^n = 1$. To ensure that the entire equation equals 1, set $B = 0$ and $C = -rTD$. This yields

$$Ax^n - Cz^n + D(z^n rT + w^n) = Ax^n + Dw^n = 1.$$

By symmetry $(1, Tr : 0, 1)_{\mathcal{L}(R[T])}$ is a morphism. □

Corollary 4.4.5. *The property of being a pointed morphism is not preserved by naive homotopies.*

Proof. Example: $(1, 0 : 0, 1)_{\mathcal{L}}$ is pointed and $(1, 0 : 1, 1)_{\mathcal{L}}$ isn't, but $(1, 0 : 0, 1)_{\mathcal{L}} \sim (1, 0 : 1, 1)_{\mathcal{L}}$ by Proposition 4.4.4. □

Example 4.4.6. There is a naive homotopy $(1-Tyz, -Tz : y, 1)_{\mathcal{P}_1(R[T])}$ connecting $(1, 0 : y, 1)_{\mathcal{P}_1}$ and $(1 - yz, -z : y, 1)_{\mathcal{P}_1}$. Hence by using Proposition 4.4.4, we obtain $(1 - yz, -z : y, 1)_{\mathcal{P}_1} \sim (1, 0 : 0, 1)_{\mathcal{P}_1}$. However, $(1 - Tyz, -Tz : Ty, 1)_{\mathcal{P}_1(R[T])}$ is not a naive homotopy.

Example 4.4.7. There is a naive homotopy $(x, 0 : 0, w)_{\mathcal{P}_1} \sim (1, 0 : 0, 1)_{\mathcal{P}_1}$, but it is not given by $(x + wT, 0 : 0, w + xT)_{\mathcal{P}_1(R[T])}$. Instead, we have two steps

$$(x, 0 : 0, w)_{\mathcal{P}_1} \sim (x, 0 : 0, 1)_{\mathcal{P}_1} \sim (1, 0 : 0, 1)_{\mathcal{P}_1},$$

given by

$$(x, 0 : 0, w + Tx)_{\mathcal{P}_1(R[T])} \quad \text{and} \quad (x + Tw, 0 : 0, 1)_{\mathcal{P}_1(R[T])}.$$

We show that $(x, 0 : 0, w + Tx)_{\mathcal{P}_1(R[T])}$ is a homotopy by showing that the module generated by the sections is locally free. On $D(x) \subset R[T]$ we see that $(x^2, wy + Txy)$ generates the unit ideal $R[x^{-1}, T]$. On $D(w)$ we need to check that $(xz, w^2 + Txw) = R[w^{-1}, T]$. We calculate

$$\frac{z}{w^2}(w^2 + Txw) = \frac{z}{w}(w + Tx) = z + xz\frac{T}{w},$$

so the module contains z. Since $w^2 + Txw = w^2 + Tyz$, it contains w^2, which is a unit in $R[w^{-1}, T]$.

The case of $(x+Tw, 0 : 0, 1)_{\mathcal{P}_1(R[T])}$ is similar. It contains a unit on $D(w)$. To see that $(x^2 + Twx, y)$ is the unit ideal in $R[x^{-1}, T]$ we calculate $Txw = Tyz \ni (y)$, hence $x^2 \in (x^2 + Twx, y)$. Since x^2 is a unit in $R[x^{-1}, T]$ we are done.

Example 4.4.8. There is a naive homotopy $(1, 1 : 0, 1)_{\mathcal{P}_1} \sim (1, 1-z : z, 1-z^2)_{\mathcal{P}_1}$. However, carelessly multiplying each difference of coefficients by T results in $(1, 1 - Tz : Tz, 1 - Tz^2)_{\mathcal{P}_1(R[T])}$, which is not a valid morphism. A homotopy is given by $(1, 1 - Tz : Tz, 1 - T^2z^2)_{\mathcal{P}_1(R[T])}$, where the exponents of the Ts are crucial.

Examining morphisms and homotopies quickly becomes unwieldy. We automated the process using a Macaulay2-script, but this did not reveal enough families of homotopies to determine the homotopy classes.

There are three heuristic reasons why automating seems intractable. Firstly, the set of morphisms is huge, and the set of naive homotopies is even huger. Secondly, morphisms can be homotopic without being connected by a single homotopy, as remarked in Definition 2.1.1, and illustrated in Examples 4.4.7 and 4.4.6. Thirdly, homotopies may require powers of T greater than 1, as shown in Example 4.4.8.

4.5 Conjectures over a quadratically closed field

In this section, we assume that k is a quadratically closed field, and summarize what we know in this case. After stating it, we also assume Conjecture 4.5.1, and derive some consequences from that. We know the group structure on $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}$ is $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$, which is $\mathbb{Z} \times k^\times$ since k is quadratically closed. From Eq. (4.4) we know we should expect this group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$. We know from Proposition 4.3.15 that the elements of the families of morphisms $\mathcal{P}_n(k)$ and $\mathcal{Q}_n(k)$ factor through π and $\widetilde{\pi}$ respectively. $\mathcal{L}(k)$ denotes morphisms given by the line bundle $\mathcal{L}$ and global sections with only field coefficients. This leads us to the following conjectures.

Conjecture 4.5.1. *Any pointed morphism $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ is naively homotopic to a morphism $(a_n, \dots, a_0 : 0, b_{n-1}, \dots, b_0)_{\mathcal{L}}$ with $\mathcal{L}$ being one of $\mathcal{P}_n, \mathcal{O}_{\mathcal{J}}$ or $\mathcal{Q}_n$, and $a_i, b_i \in k$. Moreover, if two morphisms in $\mathcal{L}(k)$ are homotopic as elements of $\mathcal{L}(R)$, then there exists a homotopy by elements of $\mathcal{L}(k[T])$ connecting them.*

We define some ad hoc notation in order to state the next conjecture.

Definition 4.5.2. Denote the naive homotopy classes of maps from $\mathcal{J} \longrightarrow \mathbb{P}^1$ arising from sections of $\mathcal{L}$ by $[\mathcal{L}, \mathbb{P}]^{\mathrm{N}}$. We define

$$P^{\mathrm{N}} := \coprod_{i \geq 0} [\mathcal{P}_n, \mathbb{P}^1_k]^{\mathrm{N}} \quad \text{and} \quad Q^{\mathrm{N}} := \coprod_{i \geq 0} [\mathcal{Q}_n, \mathbb{P}^1_k]^{\mathrm{N}}.$$

Conjecture 4.5.3. *The following are module isomorphisms.*

$$P^{\mathrm{N}} \cong [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \quad \textit{and} \quad Q^{\mathrm{N}} \cong [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}.$$

Proof that Conjecture 4.5.1 $\implies$ Conjecture 4.5.3. Fix a morphism in P^{N}. By assumption it is naively homotopic to the pair $(a_n, \dots, a_0 : 0, b_{n-1}, \dots, b_0)_n$. On $D(w)$, we can now calculate the resultant of $(a_n \frac{z^n}{w^n} + \dots a_0 \frac{w^n}{w^n}, b_{n-1} \frac{z^{n-1}}{w^{n-1}} + \dots + b_0 \frac{w^n}{w^n})$. Replacing $\frac{z}{w}$ with X yields Cazanave's rational functions.

One can then use Cazanave's results for naive homotopies of pointed rational functions to figure out the naive homotopy classes. □

Since k is quadratically closed, $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} = \mathbb{N} \times k^{\times}$. Group completing gives $\mathbb{Z} \times k^{\times}$, which amounts to adjoining inverses.[1] We can define the group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ by combining the group operations on P^{N} and Q^{N}.

Conjecture 4.5.4. *Let $(p, \alpha)_P \in P^{\mathrm{N}}$ and $(q, \beta)_Q \in Q^{\mathrm{N}}$. The group operation is as follows.*

$$(p, \alpha) \oplus^{\mathrm{N}} (q, \beta) := \begin{cases} (0, \frac{\alpha}{\beta})_P = (0, \frac{\alpha}{\beta})_Q & \textit{if } p = q, \\ (p - q, \frac{\alpha}{\beta})_P & \textit{if } p > q, \\ (q - p, \frac{\beta}{\alpha})_Q & \textit{if } q < p. \end{cases}$$

[1] In general, the group completion of a monoid contains more elements than just the elements of the monoid and their inverses.

When $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ *is equipped with this group operation, the bijection in Eq.* (4.4) *is a group isomorphism.*

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1} \simeq \mathrm{Groth}\left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}\right) \simeq [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \simeq \mathbb{Z} \times \mathbb{C}^{\times}.$$

Chapter 5

Hopf map fibration sequence

In this chapter, we break down $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ into components that are more easily described. We decompose $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ into a degree-part and a part factoring through the algebraic Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$. We then compare this to results from Morel's book "$\mathbb{A}^1$-algebraic topology over a field."

5.1 Morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$

We write $\mathbb{A}^2 = \operatorname{Spec}(k[s,t])$ and consider $\mathbb{A}^2 \setminus \{0\}$ to be pointed at $(s-1, t)$.

Definition 5.1.1 (Hopf map)**.** The Hopf map is a scheme morphism $\eta : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$. It is defined (using Theorem 2.2.1) by the structure sheaf $\mathcal{O}_{\mathbb{A}^2 \setminus \{0\}}$, and the pair (s,t) of generating global sections. That is, $\eta : (s,t) \longmapsto [s : t]$.

To look at the maps $\mathcal{J} \longrightarrow \mathbb{P}^1$ which factor through the Hopf map, we simply look at all maps $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ and compose them with η. We in turn understand the maps to $\mathbb{A}^2 \setminus \{0\}$ in terms of maps to $\mathbb{A}^2$ that factor through the inclusion $i : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{A}^2$.

$$\begin{tikzcd} \mathcal{J} \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^2 \\ & \mathbb{A}^2 \setminus \{0\} \arrow[u, "i"'] \end{tikzcd}$$

To have a scheme morphism $(f, f^{\#}) : (\mathcal{J}, \mathcal{O}_{\mathcal{J}}) \longrightarrow (\mathbb{A}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2 \setminus \{0\}})$, we need a continuous map of points $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ as well as a map of sheaves on

$\mathbb{A}^2 \setminus \{0\}$, $f^\# : \mathcal{O}_{\mathbb{A}^2\setminus\{0\}} \longrightarrow f_*\mathcal{O}_{\mathcal{J}}$.

A continuous map defining a scheme morphism $g : \mathcal{J} \longrightarrow \mathbb{A}^2 = \operatorname{Spec}(k[s,t])$ is given by a ring map $\alpha : k[s,t] \longrightarrow R$. The map g factors through $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ if and only if α is defined by $s \longmapsto p, t \longmapsto q$ such that there exist $u, v \in R$ such that $up + vq = 1$. In other words, $g = i \circ f$ if $(p,q) \in R^2$ is a unimodular row.

Observe that if $U \subseteq \mathbb{A}^2 \setminus \{0\}$ is open, then $U \subseteq \mathbb{A}^2$ is open too. The ring $k[s,t] = \Gamma\left(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}\right)$ is a unique factorization domain, which implies that the maximal set on which a regular function is defined is of the form $D(g) = \operatorname{Spec}(k[s,t]_g)$. Since $\mathbb{A}^2 \setminus \{0\}$ is neither of that form nor contained in anything of that form, there exist no regular functions on $\mathbb{A}^2 \setminus \{0\}$ that do not extend to $\mathbb{A}^2$. Hence $\Gamma\left(U, \mathcal{O}_{\mathbb{A}^2\setminus\{0\}}\right) = \Gamma\left(U, \mathcal{O}_{\mathbb{A}^2}\right)$ for all $U \subseteq \mathbb{A}^2 \setminus \{0\}$, and $i^\#$ is an isomorphism of sheaves $i^\# : \mathcal{O}_{\mathbb{A}^2} \longrightarrow i_*\mathcal{O}_{\mathbb{A}^2\setminus\{0\}}$ on $\mathbb{A}^2$.

We now define the map of sheaves $f^\#$ by using $g^\# : \mathcal{O}_{\mathbb{A}^2} \longrightarrow g_*\mathcal{O}_{\mathcal{J}}$. We require $g = i \circ f$, which implies

$$g^\# = f^\# \circ i^\# : \mathcal{O}_{\mathbb{A}^2} \longrightarrow i_*\mathcal{O}_{\mathbb{A}^2\setminus\{0\}} \longrightarrow i_*f_*\mathcal{O}_{\mathcal{J}}.$$

Since $i^\#$ is an isomorphism of sheaves on $\mathbb{A}^2$, we have $g^\#|_{\mathbb{A}^2\setminus\{0\}} = f^\#$.

Proposition 5.1.2. *The data of a k-scheme morphism $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row $(A, B) \in R^2$.*

Proposition 5.1.3. *The data of a naive homotopy $H : \mathcal{J} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row $(A, B) \in R[T]^2$.*

Proof. Recall Definition 1.0.1 and Section 4.4. A naive homotopy is a morphism H as in the diagram.

$$\mathcal{J} \overset{s_0}{\underset{s_1}{\rightrightarrows}} \mathcal{J} \times \mathbb{A}^1 \xrightarrow{H} \mathbb{A}^2 \setminus \{0\}.$$

The argument that a morphism $X \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is given by a length 2 unimodular row in $\Gamma(X, \mathcal{O}_X)$ holds for any affine scheme. In particular it holds for $\operatorname{Spec}(R[T])$. $\square$

Lemma 5.1.4. *A morphism $(A, B) : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is pointed if $A - 1 \in (x-1, y, z, w)$ and $B \in (x-1, y, z, w)$.*

Proof. $\mathcal{J}$ is a pointed at $(x-1, y, z, w)$, and if (A, B) evaluates to $(1, 0)$ at that ideal, it sends the point $(x-1, y, z, w)$ to $(s-1, t)$. $\square$

We define a binary operation (M-sum) on morphisms $\mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$. This is completely analogous to Cazanave's naive sum of pointed rational functions, defined in Proposition 2.7.1.

Definition 5.1.5 (M-sum and group structure)**.** Let $(A_i, B_i) \in R^2$ (respectively $(A_i, B_i) \in R[T]^2$) for $i = 1, 2$, define a morphism $\mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ (respectively $\mathcal{J} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^2 \setminus \{0\}$). Their M-sum is $(A_1, B_1) \oplus^{\mathrm{M}} (A_2, B_2) = (A_3, B_3)$, given by the matrix product

$$\begin{pmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{pmatrix} := \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}.$$

The operation is associative, but in not commutative. The identity is $(1, 0)$. The inverse is given by

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix}^{-1} = \begin{pmatrix} U & V \\ -B & A \end{pmatrix}.$$

This equips $\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}$ with a group structure.

Remark 5.1.6. The group $\left(\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}, \oplus^{\mathrm{M}}\right)$ might not be a subgroup of $\left(\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right)$, in the sense that the group operation $\oplus^{\mathrm{M}}$ may differ from the operation $\oplus^{\mathrm{N}}$. We conjecture that they are equal, but have not been able to prove it yet.

Example 5.1.7. We calculate

$$\begin{pmatrix} 2x-1 & -2z \\ 2y & 2x-1 \end{pmatrix}^2 = \begin{pmatrix} 4x^2 - 4x + 1 - 4yz & -4z(2x-1) \\ 4y(2x-1) & 4x^2 - 4x + 1 - 4yz \end{pmatrix}.$$

Simplifying, we get $(2x-1, 2y) \oplus^{\mathrm{M}} (2x-1, 2y) = (1 - 8yz, 8xy - 4y)$.

Proposition 5.1.8. *If (A_1, B_1) and (A_2, B_2) are pointed, then so is $(A_3, B_3) = (A_1A_2 - V_2B_1, B_1A_2 + B_2U_1)$.*

Proof. Assume (A_1, B_1) and (A_2, B_2) are two pointed morphisms. We have

$$\begin{aligned} (A_1, B_1) \oplus^{\mathrm{M}} (A_2, B_2) &= \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1A_2 - V_2B_1 & -(A_1V_2 + V_1U_2) \\ B_1A_2 + B_2U_1 & U_1U_2 - B_1V_2 \end{pmatrix} \\ &= (A_1A_2 - V_2B_1, B_1A_2 + B_2U_1). \end{aligned}$$

Since for $i = 1, 2$, we have $A_i - 1 \in (x-1, y, z, w)$ and $B_i \in (x-1, y, z, w)$, it follows that $A_1A_2 - V_2B_1 - 1 \in (x-1, y, z, w)$ and $B_1A_2 + B_2U_1 \in (x-1, y, z, w)$. $\square$

5.1.1 Naive homotopies of morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$

We may associate each unpointed morphism $(A, B) : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ with the matrix with determinant $AU + BV = 1$. We proved in Lemma 3.1.2 that any matrix $\mathrm{SL}_n(k)$ is naively homotopic to the identity matrix. Hence if $(A, B) \in k^2$, then it is homotopic to $(1, 0)$.

Allowing coefficients in R complicates things. There are examples of rings S where there are elements of $\mathrm{SL}_2(S)$ that are naively homotopy trivial that are not in $\mathrm{TV}(S)$. We have not been able to show whether R is such a ring.

5.1.2 The Picard group of the punctured plane

The Picard group of $\mathbb{A}^2 \setminus \{0\}$ can be calculated using Čech cohomology. Intuitively, this is because a line bundle is completely determined by how its locally trivial parts are glued together. They are glued by units of the structure sheaf, and the first Čech cohomology group of the sheaf of units of the structure sheaf measures exactly which units give rise to "interesting" ways to glue.

Proposition 5.1.9. *The Picard group of* $\mathbb{A}^2 \setminus \{0\}$ *is trivial.*

Proof. We have

$$\mathrm{Pic}\left(\mathbb{A}^2 \setminus \{0\}\right) = \mathrm{H}^1\left(\mathbb{A}^2 \setminus \{0\}, \mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}\right) \simeq \check{\mathrm{H}}^1\left(\mathcal{U}, \mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}\right),$$

where $\mathcal{U} = \{D(s), D(t)\}$ is a cover of $\mathbb{A}^2 \setminus \{0\}$. The sections of the sheaf $\mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}(U)$ are the units in the ring $\mathcal{O}_{\mathbb{A}^2\setminus\{0\}}(U)$. The relevant part of the Čech complex is $\check{C}^0 \xrightarrow{\delta^0} \check{C}^1 \xrightarrow{\delta^1} \check{C}^2$, which is

$$\mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}(D(s)) \oplus \mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}(D(t)) \xrightarrow{\delta^0} \mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}(D(s) \cap D(t)) \xrightarrow{\delta^1} 0.$$

Calculating what the units are, we get the following cochain complex.

$$\{as^n\} \oplus \{bt^m\} \xrightarrow{\delta^0} \{cs^pt^q\} \xrightarrow{\delta^1} 0,$$

where $a, b, c \in k^\times, n, m, p, q \in \mathbb{Z}$. The map δ^1 sends everything to 0, so the kernel is $\{cs^pt^q\}$. The image of δ^0 is $\{as^nb^{-1}t^{-m}\}$ which equals $\{cs^pt^q\}$. We calculate the quotient $\check{\mathrm{H}}^1\left(\mathcal{U}, \mathcal{O}^\times_{\mathbb{A}^2\setminus\{0\}}\right) = 0$. $\square$

Proposition 5.1.10. *A map* $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ *factors through* $\eta : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$ *if and only if it induces a trivial map of Picard groups.*

Proof. The map $f : \mathcal{J} \xrightarrow{g} \mathbb{A}^2 \setminus \{0\} \xrightarrow{\eta} \mathbb{P}^1$ induces a map on Picard groups $f^* : \mathbb{Z} \xrightarrow{\eta^*} 0 \xrightarrow{g^*} \mathbb{Z}$. Conversely, a map $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ that induces a trivial map of Picard groups must be given by the trivial bundle $\mathcal{O}_{\mathcal{J}}$ and two generating sections. That is exactly the data of a unimodular row $(A, B) \in R^2$, thus it defines a map $g : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$. Composing with η, we obtain $\eta \circ g = f$. □

5.2 Fibration sequence

5.2.1 Milnor Witt K-theory

$K_*^{MW}(k)$ is the Milnor Witt K-theory of the field k. It is the associative graded ring defined by the following generators and relations. The generators are η of degree -1, and elements $[a]$ of degree 1 for each $a \in k^\times$.

Definition 5.2.1 (Definition 3.1 in [Mor12])**.** Let k be a commutative field. The Milnor-Witt K-theory of k is the graded associative ring $K_*^{MW}(k)$ generated by the symbols $[u]$ of degree $+1$, for each unit $u \in k^\times$, and one symbol η of degree -1 subject to the following relations:

1. (Steinberg relation) For each $a \in k^\times \setminus \{1\} : [a].[1-a] = 0$
2. For each pair $(a, b) \in (k^\times)^2 : [ab] = [a] + [b] + \eta.[a].[b]$
3. For each $u \in k^\times : [u].\eta = \eta.[u]$
4. Set $h := \eta.[-1] + 2$. Then $\eta.h = 0$

The Grothendieck-Witt ring is isomorphic to the degree 0 subring $K_0^{MW}(k)$, and the isomorphism is given by $\langle a \rangle \longmapsto 1 + \eta.[a]$.

Morel defines in [Mor12, §3.2] for each $n \in \mathbb{Z}$ an explicit sheaf $\underline{\mathbf{K}}_n^{MW}$ on $\mathbf{Sm}_k$. (Notice that it is bold and underlined.) The sections of this sheaf on any field k is the group $K_n^{MW}(k)$.

5.2.2 Fibration sequence

In [Mor12, p. 191] we find the following fibration sequence

$$\mathbb{A}^2 \setminus 0 \xrightarrow{\eta} \mathbb{P}^1 \longrightarrow \mathbb{P}^\infty.$$

This sequence gives rise to a long exact sequence of homotopy sheaves, from which one can get a short exact sequence. By applying contraction — which is an exact functor — to the short exact sequence we get the following central extension of sheaves.

$$1 \longrightarrow \underline{\mathbf{K}}_1^{MW} \longrightarrow (F_{\mathbb{A}^1}(1))_{-1} \longrightarrow (\mathbb{G}_m)_{-1} \longrightarrow 1. \tag{5.1}$$

Since $\operatorname{Pic}\mathbb{P}^1 \simeq (\mathbb{G}_m)_{-1} \simeq \mathbb{Z}$, the sequence splits, which leads Morel to the following corollary.

Proposition 5.2.2 (Corollary 7.34 in [Mor12])**.** *The sheaf of groups $(F_{\mathbb{A}^1}(1))_{-1}$ is abelian and is canonically isomorphic to $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW}$.*

Evaluating Seq. (5.1) at a field k gives a short exact sequence of groups.

$$1 \longrightarrow K_1^{MW}(k) \longrightarrow \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1} \longrightarrow \operatorname{Pic}\mathbb{P}^1 \longrightarrow 1. \tag{5.2}$$

Using that $\mathcal{J}$ and $\mathbb{P}^1$ are $\mathbb{A}^1$-homotopy equivalent, we may rewrite this as

$$1 \longrightarrow K_1^{MW}(k) \longrightarrow \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathbb{A}^1} \longrightarrow \operatorname{Pic}\mathcal{J} \longrightarrow 1.$$

We compare this to $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$. The group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ is induced by the canonical bijection $c : \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} \longrightarrow \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}$, which tautologically makes c a group isomorphism.

Lemma 5.2.3. *The following square commutes and the vertical maps are isomorphisms.*

$$\begin{tikzcd}
\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} \arrow[r, "\deg"] \arrow[d, "c"] & \operatorname{Pic}\mathcal{J} \arrow[d, "\mathrm{id}"] \\
\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathbb{A}^1} \arrow[r, "\deg"] & \operatorname{Pic}\mathcal{J}
\end{tikzcd}$$

Proof. The identity map on morphisms induces the map c on homotopy classes. The map $\deg$ is a well-defined map on naive homotopy classes (Proposition 4.4.1) as well as on $\mathbb{A}^1$-homotopy classes (Seq. (5.2.2)) and their definitions coincide. They both take the homotopy class of a morphism f to the element in $\operatorname{Pic}\mathcal{J}$ used to define f. □

Theorem 5.2.4. *The following diagram commutes, and the vertical maps are isomorphisms.*

$$\begin{tikzcd}
1 \arrow[r] & \left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}} \arrow[r] \arrow[d] & \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} \arrow[r, "\deg"] \arrow[d, "c"] & \operatorname{Pic}\mathcal{J} \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & K_1^{MW}(k) \arrow[r] & \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathbb{A}^1} \arrow[r, "\deg"] & \operatorname{Pic}\mathcal{J} \arrow[r] & 1
\end{tikzcd} \tag{5.3}$$

Proof. The first row is a short exact sequence by Proposition 5.1.10. The two rightmost maps are isomorphisms by Lemma 5.2.3, so by the five lemma the leftmost map must be too. $\square$

Remark 5.2.5. In the diagram, $\mathbb{A}^2 \setminus \{0\}$ is equipped with the group structure it has as a subgroup of $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$, which may differ from the group structure defined in Definition 5.1.5.

Since Pic $\mathcal{J} \simeq \mathbb{Z}$, the first row in Diagram (5.3) splits. Since the group $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} \simeq \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1}$ is abelian, we may use the splitting lemma to obtain the following.

$$\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}} = \left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}} \oplus \operatorname{Pic} \mathcal{J}.$$

This proves Theorem 1.0.3.

5.3 Group completion arguments

If k is a quadratically closed field, we have an isomorphism of monoids $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}} \simeq \mathbb{N} \times k^\times$. Group completing gives us $\mathbb{Z} \times k^\times$, which amounts to adjoining inverses to the elements of the monoid, as discussed in Section 4.5.

However, if we look at a non-algebraically closed field, the group completion may adjoin more than just the inverses. As a simple example, consider $\mathbb{R}$. We calculate $\left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathrm{N}} \simeq \mathbb{N} \times \mathbb{N} \times \mathbb{R}^\times$, and the group completion is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^\times$. This creates elements (a, b, u) where $ab < 0$, which are neither elements of the monoid, nor of the monoid of inverses. This shows that not all elements of $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ are expressible as pointed rational functions with field coefficients. The best we can hope for is to be able to express them as formal differences of rational functions.

In the case of $\mathbb{R}$, it is reasonable to expect that any such formal difference may be turned into an element expressible as a rational function, by shifting by the homotopy class of π. In other words, for any homotopy class in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^\times$, and any function $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ representing that class, there exists an $n \in \mathbb{Z}$ such that $n \cdot \pi + f$ is homotopic to a rational function. This leads us to the following conjecture.

Conjecture 5.3.1. *If $GW(k) \times_{k^\times / k^{\times 2}} k^\times$ is a finitely generated group, we may find a "shifting element" ρ, such that for any f there exists an $n \in \mathbb{Z}$ such that $n \cdot \rho + f$ is homotopic to a rational function.*

The Witt ring is a quotient of the Grothendieck-Witt ring by the fundamental ideal I. This ideal is generated by the isometry class of the hyperbolic form,

$I = (\langle 1, -1\rangle)$. The Hasse-Minkowski theorem states that quadratic forms over $\mathbb{Q}$ are equivalent if and only if they are equivalent over all local fields. As a consequence, the Witt ring of $\mathbb{Q}$ is the product $\prod_{p\leq\infty} \mathrm{W}(\mathbb{Q}_p)$, where p ranges over the primes, $\mathbb{Q}_p$ is the field of p-adic numbers, and by convention we write $\mathbb{Q}_\infty = \mathbb{R}$.[1]

$$\mathrm{W}(\mathbb{Q}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \prod_{2<p<\infty} \mathrm{W}(\mathbb{F}_p),$$

where p ranges over the primes greater than 2.

If the Grothendieck-Witt group has an infinite number of $\mathbb{Z}$-factors that stem from corresponding $\mathbb{N}$-factors in the Witt monoid, then the shifting element ρ of Conjecture 5.3.1 can not exist. The condition that $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$ be finitely generated is included for this reason.

Another strategy for understanding the group structure on $\left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$ is looking at the nice submonoids. For instance, the family of naive homotopy classes of morphisms $P^{\mathrm{N}} := \prod_{0\leq n} \mathcal{P}_n(R)$ is a submonoid consisting of homotopy classes of positive degree. A submonoid of P^{N} is $P^{\mathrm{N}}(k) := \prod_{0\leq n} \mathcal{P}_n(k)$, where the generating sections only have field coefficients. Elements of $P^{\mathrm{N}}(k)$ factor as π composed with a pointed rational function $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by Proposition 4.3.15.

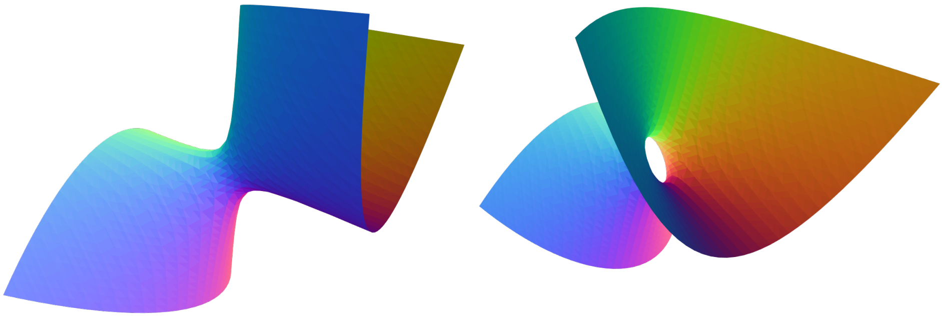
Similarly, by symmetry, there are nice submonoids $Q^{\mathrm{N}}(k) \subseteq Q^{\mathrm{N}} \subset \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathrm{N}}$. The subgroup of morphisms defined by the structure sheaf $\mathcal{O}_{\mathcal{J}}$ is exactly the group $\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}$, as shown in Proposition 5.1.10.

5.4 Real realization

If k is a field and $k \hookrightarrow \mathbb{R}$ is an embedding, then sending a smooth k-scheme X to the topological space $X(\mathbb{R})$ equipped with its usual structure of a real manifold extends to a functor $\Re : \mathscr{H}_\bullet(k) \longrightarrow \mathscr{H}$ [AFW20, p. 14]. $\mathscr{H}_\bullet(k)$ is the homotopy category of smooth k-schemes and $\mathscr{H}$ is the homotopy category of topological spaces. This is shown to be a functor is in [DI04, Section 5.3]. This allows us deduce properties of $\mathcal{J}$ from properties of the topological space it corresponds to.

Proposition 5.4.1. *A naive homotopy $\mathcal{J} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^2 \setminus \{0\}$ over $k \subseteq \mathbb{R}$ corresponds to a homotopy of continuous maps from $\mathcal{J}(\mathbb{R})$ to $\mathbb{R}^2 \setminus \{0\}$ in the real realization.*

[1] See for example [Gou97, p. 46] for this notation.



(a) Side view. **(b)** Inside view.

Figure 5.1: The real realization of the Jouanolou device of $\mathbb{P}^1$ is the surface in $\mathbb{R}^3$ defined by $x(1-x) - yz = 0$.

Proof. Let $f : \mathscr{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ be a map defined in the category $\mathbf{Sm}_k$ where $i : k \hookrightarrow \mathbb{R}$. Then f is determined by a unimodular row $(f_1, f_2) \in R^2$. Algebraic maps are continuous, and any naive homotopy is sent to a homotopy of continuous maps.

If f, g are naively homotopic in $\mathbf{Sm}_k$, and k is a subfield of $\mathbb{R}$, then the realizations $\Re(f), \Re(g) : \mathbb{R}[x, y, z]/(x - x^2 - yz) \longrightarrow \mathbb{R}^2 \setminus (0, 0)$ are homotopic in **Top**. $\square$

Corollary 5.4.2. *If $\Re(f)$ and $\Re(g)$ are not homotopic in* **Top**, *then f and g are not naively $\mathbb{A}^1$-homotopic.*

Proof. This is simply the contrapositive of Proposition 5.4.1. $\square$

5.4.1 A homotopy invariant

As a topological space, $\mathscr{J}(\mathbb{R})$ retracts to S^1. $\mathbb{R}^2 \setminus \{0\}$ retracts to S^1 as well. We know that homotopy classes of continuous maps $S^1 \longrightarrow S^1$ are completely classified by their winding number. Thus we expect the topological homotopy classes of maps from $\mathscr{J}(\mathbb{R})$ to $\mathbb{R}^2 \setminus \{0\}$ to correspond to different winding numbers.

We define the loop $\gamma : S^1 \longrightarrow \mathscr{J}(\mathbb{R})$ by

$$\gamma : \theta \longmapsto \left(\frac{\cos(\theta) + 1}{2}, \frac{-\sin(\theta)}{2}, \frac{-\sin(\theta)}{2} \right).$$

The scheme morphism $F : \mathscr{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is defined by $(2x - 1, 2y)$. Its real realization is the map $\Re(F) : \mathscr{J}(\mathbb{R}) \longrightarrow \mathbb{R}^2 \setminus \{0\}$ defined by

$$\Re(F) : (x, y, z) \longmapsto (2x - 1, 2y).$$

It acts on $\operatorname{im}\gamma \in \mathcal{J}(\mathbb{R})$ by

$$\Re(F) : \left(\frac{\cos(\theta)+1}{2}, \frac{-\sin(\theta)}{2}, \frac{-\sin(\theta)}{2}\right) \longmapsto (\cos(\theta), -\sin(\theta)), \qquad (5.4)$$

which implies that $\Re(F) \circ \gamma : S^1 \longrightarrow \mathcal{J}(\mathbb{R}) \longrightarrow \mathbb{R}^2 \setminus (0,0)$ is homotopically nontrivial. By Corollary 5.4.2, F is a homotopically nontrivial scheme morphism. Using the classical result that loops with different winding numbers are not homotopy equivalent, we get the following.

Proposition 5.4.3. *For k a subfield of $\mathbb{R}$, there exists a surjection from the naive homotopy classes of maps $\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}$ to $\mathbb{Z}$.*

The scheme morphism $F^{-1} : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is defined by $(2x-1, -2y)$. We calculate

$$\begin{pmatrix} 2x-1 & -2z \\ 2y & 2x-1 \end{pmatrix} \begin{pmatrix} 2x-1 & 2z \\ -2y & 2x-1 \end{pmatrix} = \begin{pmatrix} 1+4x^2-4x+4yz & 0 \\ 0 & 1+4x^2-4x+4yz \end{pmatrix}.$$

Since $x^2 - x + yz = 0$ this means that $(2x-1, 2y) \oplus^{\mathrm{M}} (2x-1, -2y) = (1,0)$, so F and F^{-1} are inverses. Denote by F^n the n-fold M-sum $F \oplus^{\mathrm{M}} \ldots \oplus^{\mathrm{M}} F$, by F^{-n} the n-fold M-sum $F^{-1} \oplus^{\mathrm{M}} \ldots \oplus^{\mathrm{M}} F^{-1}$, and by F^0 the map given by $(1,0)$. In general the following proposition holds.

Proposition 5.4.4. *Let k be a subfield of $\mathbb{R}$. For integers $n \neq m$, the morphisms F^n and F^m are not naively homotopic. The subgroup of $\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}$ generated by F and F^{-1} is isomorphic to $\mathbb{Z}$.*

Proof. F^0 is the identity of $\left[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}\right]^{\mathrm{N}}$, and $F \oplus^{\mathrm{M}} F^{-1} = F^{-1} \oplus^{\mathrm{M}} F = (1,0)$. Hence F, F^{-1} generate a cyclic group. To show that this group is isomorphic to $\mathbb{Z}$ amounts to showing that F^n and F^m are never naively homotopic when $n \neq m$. To do this, we show that $\Re(F^n)$ and $\Re(F^m)$ are not homotopic.

$\Re(F)$ sends $\operatorname{im}\gamma \in \mathcal{J}(\mathbb{R})$ to the left column of

$$M = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

That is, to $(\cos(\theta), -\sin(\theta))$. The function $\Re(F^n)$ sends $\operatorname{im}\gamma$ to the left column of M^n. Recall that complex numbers can be represented by 2×2-matrices over

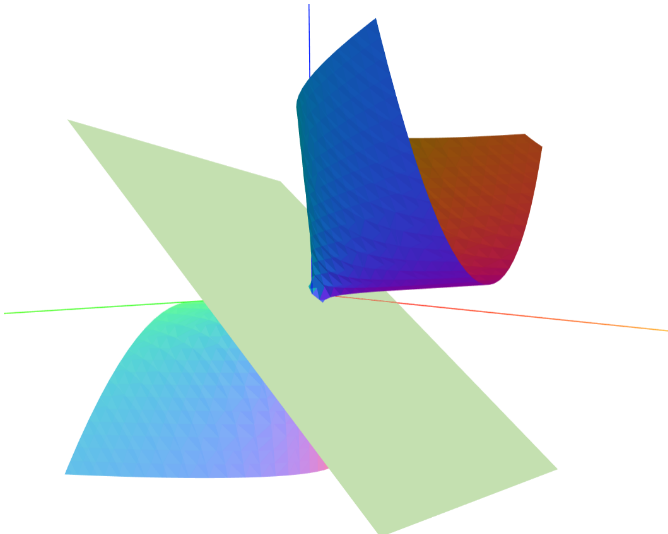


Figure 5.2: The real realization of the Jouanolou device of $\mathbb{P}^1_{\mathbb{R}}$ and the plane $y - z = 0$. Their intersection is a circle.

$\mathbb{R}$ with the correspondence $a + bi \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and observe that our matrix M corresponds to $e^{i\theta}$. Since $(e^{i\theta})^n = e^{ni\theta}$, we get

$$M^n = \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

Hence $\Re(F^n) \circ \gamma$ has winding number n, and is not homotopic to $\Re(F^m) \circ \gamma$ for any $m \neq n$. $\square$

Example 5.4.5. We calculated in Example 5.1.7 that

$$(2x - 1, 2y) \oplus^{\mathrm{M}} (2x - 1, 2y) = (1 - 8yz, 8xy - 4y).$$

By applying the real realization functor to this morphism, we obtain

$$\Re(F \oplus^{\mathrm{M}} F) : (x, y, z) \longmapsto (1 - 8yz, 8xy - 4y).$$

This acts on $\operatorname{im}\gamma \in \mathcal{J}(\mathbb{R})$ by

$$\Re(F) : \left(\frac{\cos(\theta) + 1}{2}, \frac{-\sin(\theta)}{2}, \frac{-\sin(\theta)}{2}\right) \longmapsto (\cos(2\theta), -\sin(2\theta)).$$

5.5 Outlook

We have obtained some partial results about the group structure on $\left[\mathcal{J},\mathbb{P}^1\right]^{\mathrm{N}}$. Theorem 1.0.3 states that it is a direct sum $\left[\mathcal{J},\mathbb{A}^2\setminus\{0\}\right]^{\mathrm{N}}\oplus\mathbb{Z}$. Further research is needed to understand the group structure on $\left[\mathcal{J},\mathbb{A}^2\setminus\{0\}\right]^{\mathrm{N}}$, and in particular to investigate whether the identity map on this set is an isomorphism of groups $\left(\left[\mathcal{J},\mathbb{A}^2\setminus\{0\}\right]^{\mathrm{N}},\oplus^{\mathrm{M}}\right)\simeq\left(\left[\mathcal{J},\mathbb{A}^2\setminus\{0\}\right]^{\mathrm{N}},\oplus^{\mathrm{N}}\right)$.

It might be possible to generalize beyond fields. The Jouanolou device of $\mathbb{P}^1$ is $\mathbb{A}^1$-naive when working in $\mathbf{Sm}_S$, where S is an ind-smooth Dedekind ring with perfect residue fields [AHW18, Theorem 4.2.2]. In particular this holds for $S=\mathbb{Z}$. It is also true that $\operatorname{Pic}\mathbb{P}^1_{\mathbb{Z}}=\mathbb{Z}$, since $\mathbb{Z}$ is a UFD.

Conjecture 5.5.1. *There is a group structure on* $\left[\mathcal{J},\mathbb{P}^1\right]^{\mathrm{N}}$ *in* $\mathbf{Sm}_{\mathbb{Z}}$*. There is a split short exact sequence*

$$1\longrightarrow\left[\mathcal{J},\mathbb{A}^2\setminus\{0\}\right]^{\mathrm{N}}\longrightarrow\left[\mathcal{J},\mathbb{P}^1\right]^{\mathrm{N}}\longrightarrow\operatorname{Pic}\mathcal{J}\longrightarrow 1.$$

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