Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Optimal Control of a Semi-Linear Parabolic System Related to Sustainable Marine Fishery

An attempt at finding better fishing strategies by non-convex optimization

Master's thesis in Industrial Mathematics
Supervisor: Dietmar Hömberg
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Summary

Marine fisheries around the world are facing the increasing problem of overfishing. In this thesis, we will attempt to find optimal and sustainable fishing strategies by formulating a non-convex optimal control problem. We begin by setting up a nonlinear parabolic diffusion-reaction equation describing the behavior of fish biomass and its response to a harvesting effort. Existence and uniqueness of solutions to this equation is proven. We then formulate the optimal control problem and derive optimality conditions, and provide rigorous justification for these. Finally, we will solve the optimal control problem numerically using a gradient based method. The results we obtain are given an interpretation in terms of marine policy.

Preface

Upon completing this master's thesis, I finish my years of study at NTNU. It has been five fantastic years that have taught me a lot and given me experiences and memories that I will cherish for the rest of my life. The people that I have met here have become some of my closest friends, and I cannot imagine having gone through this part of my life without them.

I would like to give a special thanks to all of my parents who have always been extremely supportive, without ever putting any pressure on me. They have always been there, either to help with practical arrangements or for emotional support on the phone. A second thanks goes to all of my siblings, who am I very grateful for having.

The task of writing a master's thesis has been demanding, and it became especially difficult after the coronavirus pandemic hit the world earlier this year. It has been a strange spring/early summer, and the transition from being able to go to school to having to work from home has been hard to tackle. I would not have been able to finish my work without the help and guidance I have received from my supervisor, Dietmar Hömberg. I am very thankful for the assistance he has given me.

Finishing this work fills me with a mixture of anticipation and sadness. It will be exciting to start a new chapter of my life, but I also know that I will look back upon my time as a student with great nostalgia for many years to come. I will never forget the last two weeks that I have spent with Vetle trying to complete this thesis in the midst of moving to Oslo. I am very happy that we did that together.

I would like to end by giving a thanks to my girlfriend, Nora, who has made a tough semester a little easier to get through.

Table of Contents

Summary **i**

Preface **ii**

Table of Contents **iii**

List of Figures **v**

1 Introduction **1**

2 Model Description **3**

2.1 State Equation 3

2.2 Cost Functional 4

2.3 Control Problem Formulation 4

3 The State Equation **7**

3.1 An Auxiliary State Equation 7

3.2 Analysis on the Modified Equation 8

3.2.1 Existence of the Finite-Dimensional Approximations 9

3.2.2 Bounds on the Finite-Dimensional Approximations 10

3.2.3 Existence of a Solution 12

3.2.4 Boundedness of a Solution 14

3.3 Returning to the Auxiliary State Equation 14

3.3.1 Existence of a Solution 15

3.3.2 Nonnegativity of a Solution 15

3.3.3 Uniqueness of a Solution 16

3.4 Returning to the State Equation 17

4 The Control Problem **19**

4.1 Existence of a Minimizer to the Control Problem 19

4.2 Analysis of the Solution Operator 22

4.2.1 Stability of the Solution Operator 22
4.2.2 Differentiability of the Solution Operator 25
4.3 Derivation of First Order Necessary Optimality Conditions 29

5 Numerical Simulations 31
5.1 Discretization of the State Equation 31
5.1.1 Time Discretization of the State Equation 31
5.1.2 Solving the Elliptic Equations 32
5.1.3 An Example with a Stationary Fishing Vessel 33
5.1.4 An Example with a Moving Fishing Vessel 34
5.1.5 An Example with Two Moving Fishing Vessels 35
5.2 Solving the Control Problem Numerically 38
5.2.1 Discretization of the Cost Functional 38
5.2.2 A Projected Gradient Method 38
5.2.3 An Example with High Transportation Costs 40
5.2.4 An Example with Low Transportation Costs 41

6 Conclusions 45
6.1 Discussion 45
6.2 Ideas for Further Work 46

Bibliography 47

List of Figures

5.1 A plot of the logistic growth threshold a used in the first and second examples. 34
5.2 A plot of the stationary control used in the first example. 35
5.3 Plots of the solution to the state equation for the first example with a stationary control. Solutions are shown at times $t \in \{0.0, 0.2, 0.4, 0.6\}$ . . . 36
5.4 Plots of states and control for the second example. The top four panels show the state at times $t \in \{0.0, 0.2, 0.4, 0.6\}$. The bottom four panels show the control at corresponding times. 37
5.5 Plots of states and control for the third example. The top four panels show the state at times $t \in \{0.0, 0.2, 0.4, 0.6\}$. The bottom four panels show the control at corresponding times. 39
5.6 A plot of the values of the objective function at different iterations of the projected gradient algorithm. 42
5.7 Plots of the results from the first example. The left column shows the state, the middle column shows the control and the right column shows the adjoint state. Each variable is plotted at times $t \in \{0.0, 1.0, 2.0, 3.0\}$. 43
5.8 Plots of the results from the first example. The left column shows the state, the middle column shows the control and the right column shows the adjoint state. Each variable is plotted at times $t \in \{0.0, 1.0, 3.0, 4.0\}$. 44

Chapter 1

Introduction

Marine fisheries play an essential role in the world economy. They provide millions of people with both food security and job opportunities. Activity is increasing in this sector, and from 1950 to 1990, the total production volume grew by 8-9% every year. In 2016 the total marine catch was 79.3 million tonnes globally, and today fish and fish products make up 1% of the world's total merchandise trade in terms of value [8]. Marine fish stocks represent a valuable resource, and their preservation in the long term is of high importance. However, overfishing is an ever-growing problem and is threatening to deplete several fish stocks. From 1974 to 2015, the fraction of fish stocks exploited at a sustainable level decreased from 90.0% to 66.9% [17]. With about a third of the fish stocks subject to overfishing today, it is evident that unsustainable harvesting is a big problem. According to Ye et al. [20], rebuilding overfished stocks could increase yearly production by as much as 16.5 million tonnes, suggesting that dealing with this problem would yield huge benefits.

In an attempt to combat the overfishing problem, the harvest in many regions is regulated. The most common policy tools include quotas on the Total Allowable Catch (TAC) and the imposition of Marine Protected Areas (MPAs), sometimes called No-Take Zones (NTZs), where fishing is not allowed at all [8, 9]. To facilitate the design of such policies, the development of good mathematical models for how the fish biomass develops over time would be helpful, and these models could describe fishing strategies by formulation optimization problems.

In research economics, a majority of models are based on ordinary differential equations (ODEs) [3]. These models only capture the temporal evolution of fish biomass and neglect the spatial dimensions. The importance of including spatial dimensions in such analysis is apparent when considering e.g., harvesting in different parts of a region where fish is located.

In recent years, some attempts have been made to include one or two spatial dimensions in such models, see e.g. Xepapadeas [19] or Faugeras and Maury [7]. There has also been work examining the possibility of finding optimal fishing strategies by solving optimal control problems. Ding and Lenhart [5] do this for a control problem where the governing partial differential equation (PDE) is elliptic, i.e., the model is stationary. A control problem where the governing PDE includes both spatial variables and a time variable is studied by Joshi et al. [10]. Here, the aim was to maximize the harvest yield's present-day value for a certain period.

A particularly interesting optimal control problem is presented by Braack et al. [2]. Here, a nonlinear parabolic diffusion-reaction equation is used to model fish biomass and its response to a harvesting effort. This model incorporates two spatial dimensions in addition to the time dimension. The cost functional in the control problem is formulated so that a solution to the control problem describes a fishing strategy that maximizes the harvest while reducing costs and avoiding overfishing. In our work, we will expand on the research done on this optimal control problem.

The main goal of this thesis will be to develop our own theoretical analysis of both the state equation and the control problem. This will also be the most significant contribution of this work. A second crucial goal for this thesis will be to solve numerically both the state equation and the control problem, and illustrate how the results can be used by policymakers who wish to regulate marine fisheries.

In Chapter 2, we present the state equation along with the cost functional, and state the control problem formally. Chapter 3 deals with the state equation. We show here, by introducing an auxiliary equation, the existence of a unique solution to the state equation. We use Galerkin's method to obtain this result. We also prove that such a solution has to be nonnegative. Analysis of the cost functional and the control problem is done in Chapter 4. We show here that the control problem attains a global minimum. We also define a solution operator, which maps a control to the corresponding state, and prove that it is Fréchet differentiable. This allows us to derive a first-order optimality system for the control problem. To do so, we use the Lagrangian technique. In Chapter 5, we describe the numerical methods used to solve the state equation and the control problem. To solve the state equation, we use Rothe's method. The control problem is solved with a projected gradient method. The results obtained using these methods are presented, and we analyze and discuss them. We give our concluding remarks in Chapter 6.

Chapter 2

Model Description

In this chapter we present the semilinear parabolic PDE, proposed by Braack et al. [2], modeling the fish biomass, and derive the weak form of this equation. Next, we introduce the cost functional through which we specify what is meant by an optimal fishing strategy. Finally, we state the optimal control problem formally.

2.1 State Equation

The state equation for the fish biomass in some Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, at time $t \in [0, T]$ can be described as follows

$$\begin{aligned} u_t - \Delta u + u\,(u - a) = -qu \quad & \text{in } (0,T) \times \Omega, \\ \nabla u \cdot n = 0 \quad & \text{on } (0,T) \times \partial\Omega, \\ u(x,0) = u_0(x) \quad & \text{in } \Omega. \end{aligned} \tag{2.1}$$

Here $u = u(x,t)$ represents the biomass of the fish, $q = q(x,t)$ represents the harvesting effort on the fish, and a is a time-independent function describing the growth rate of the biomass. The second term on the left hand side represents diffusion of the biomass due to isotropic movement of the fish. The third term is a classical logistic growth term. It is assumed that the population can only sustain growth until a certain threshold is reached. This threshold can be caused by e.g. food shortage. On the right hand side we have the term representing depletion of the biomass due to harvesting. The assumption that no fish can leave our specified domain is included in the Neumann boundary condition. Throughout this work we will assume that $u_0 \in L^\infty(\Omega)$, and that $a(x) > 0$ for every $x \in \Omega$.

A weak form of the state equation is found by multiplying (2.1) with a test function v and using integration by parts to get rid of higher order derivatives. We say that a weak solution of (2.1) is a function $u \in L^2\left(0,T; H^1(\Omega)\right)$ with $u_t \in L^2\left(0,T; H^{-1}(\Omega)\right)$ for which

$$\int_\Omega u_t v\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega u\,(u + q - a)\,v\,dx = 0, \tag{2.2}$$

for each function $v \in H^1(\Omega)$, for a.e. $t \in [0, T]$, and for which $u(x, 0) = u_0(x)$.

To make it less tedious to discuss solutions to the state equation we define the space

$$W(0,T) := \left\{ v \in L^2(0,T; H^1(\Omega)) : \partial_t v \in L^2(0,T; H^{-1}(\Omega)) \right\}, \tag{2.3}$$

and we will later use the well known result that $W(0, T)$ is compactly embedded into the space $L^2(0, T; L^2(\Omega))$. We also define the space

$$Q := [0, T] \times \Omega, \tag{2.4}$$

to further simplify our discussion. Note also that throughout this work we use Lagrange's prime notation for derivatives to mean a derivative with respect to time.

2.2 Cost Functional

In the optimal control problem we aim to minimize a cost functional with respect to the biomass (state) u and the harvesting effort (control) q. We define the cost functional to be

$$J(u,q) := \int_0^T e^{-\rho t} \int_\Omega q(r-u)\,dx\,dt - \lambda \int_\Omega u(T)\,dx + \frac{\alpha}{2} \int_0^T \int_\Omega q^2\,dx\,dt, \tag{2.5}$$

for $\alpha, \lambda \in \mathbb{R}$, with $\alpha > 0$ and $\lambda \geq 0$. The exponential represents a discounting factor, where $\rho > 0$ is the market interest rate. The purpose of this factor is to specify that it is beneficial to acquire more fish/revenue at an earlier point in time. The function r is intended to describe the cost of moving to different parts of the domain from the harbor. This implies that the term qr gives the cost of fishing in different parts of the domain. Typically, r is proportional to the distance to the harbor, and we require $r \in L^2(\Omega)$. The term qu represents the amount of fish that is harvested. One could choose to multiply this term by the market price of the fish to represent the revenue associated with the harvest, however, here price is normalized to be 1. The integral of the biomass at end time expresses our desire to avoid causing the population to go extinct. The parameter λ determines the emphasis put on this goal. The final term in the cost functional is a classical regularization term on the control, where α determines the level of regularization.

2.3 Control Problem Formulation

Before stating the control problem we need to specify the set of admissible controls. Obviously, the control, or harvesting effort, cannot become negative. Furthermore, it is not allowed to become arbitrarily large. Therefore, we set an upper limit q_{max}. Consequently, the set of admissible controls is defined as

$$\mathcal{Q}_{ad} = \{q \in L^\infty(Q) : 0 \leq q \leq q_{max}\}. \tag{2.6}$$

Now, we are ready to formulate the optimal control problem. It is stated as

$$\begin{aligned} \underset{u,q}{\text{minimize}} \quad & J(u,q) \\ \text{subject to} \quad & u \text{ solves (2.1)}, \\ & q \in \mathcal{Q}_{ad}. \end{aligned} \tag{2.7}$$

Our intention is that, given the definition of the cost functional, a solution to (2.7) will describe an optimal fishing strategy in space and time.

Chapter 3

The State Equation

In this chapter we do extensive analysis on the state equation, with the purpose of showing that there exists a solution to this equation, and that such a solution is unique. Since the state equation has a non-monotone term, it is not straightforward to obtain this result. In order to facilitate our analysis we first begin by studying an auxiliary equation, where the non-monotone term has been replaced by a monotone one. The intention here is that, after a transformation, an existence and uniqueness result for the auxiliary equation will imply that the same holds for the state equation. However, the auxiliary is not simple to analyze either. We therefore modify it slightly by introducing a cut-off function. We then show existence of a solution to the modified equation by using Galerkin's method, as outlined in Evans [6]. Once this result has been established we can, via the auxiliary equation, prove the existence and uniqueness of a solution to the state equation.

3.1 An Auxiliary State Equation

We begin our analysis by studying an auxiliary state equation, where we replace the non-monotone term $u(u-a)$ in (2.1) by the monotone $\gamma|u|u$, where γ is some time dependent function with $0 \leq \gamma(t) < \infty$ for all $t \in [0,T]$, and $\gamma(t) \in C^1([0,T])$. Thus, we have

$$\begin{aligned} y_t - \Delta y + \gamma|y|y &= -qy && \text{in } (0,T)\times\Omega, \\ \nabla y \cdot n &= 0 && \text{on } (0,T)\times\partial\Omega, \\ y(x,0) &= y_0(x) && \text{in } \Omega. \end{aligned} \tag{3.1}$$

We assume $y_0 \in L^\infty(\Omega)$.

Similarly as before, we can obtain a weak form of this equation. We say that a weak solution of (3.1) is a function $y \in L^2\left(0,T;H^1\left(\Omega\right)\right)$ with $y_t \in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$ for which

$$\int_\Omega y_t v\,dx + \int_\Omega \nabla y \cdot \nabla v\,dx + \int_\Omega (\gamma|y|y + qy)\,v\,dx = 0, \tag{3.2}$$

for each function $v \in H^1(\Omega)$, for a.e. $t \in [0,T]$, and for which $y(x,0) = y_0(x)$.

To simplify our analysis of the auxiliary state equation we modify it slightly to obtain a third equation. We begin by introducing the cut-off function $\tau_\mu(x)$ defined by

$$\tau_\mu(x) = \begin{cases} x, & \text{if } |x| \leq y, \\ -\mu, & \text{if } x < -\mu, \\ \mu, & \text{if } x > \mu. \end{cases} \tag{3.3}$$

In the modified equation we replace the non-monotone term $u(u-a)$ in (2.1) by the monotone term $\gamma\tau_\mu(|u|)u$, where γ is the same as it is in the auxiliary state equation. We obtain

$$\begin{aligned} y_t - \Delta y + \gamma\tau_\mu(|y|)y &= -qy && \text{in } (0,T)\times\Omega, \\ \nabla y \cdot n &= 0 && \text{on } (0,T)\times\partial\Omega, \\ y(x,0) &= y_0(x) && \text{in } \Omega. \end{aligned} \tag{3.4}$$

Again we assume that $y_0(x) \in L^\infty(\Omega)$. Again we obtain a weak form of (3.4) by multiplying with a test function v and using integration by parts. We say that a weak solution of (3.4) is a function $y \in L^2\left(0,T;H^1(\Omega)\right)$ with $y_t \in L^2\left(0,T;H^{-1}(\Omega)\right)$ for which

$$\int_\Omega y_t v\,dx + \int_\Omega \nabla y \cdot \nabla v\,dx + \int_\Omega \left(\gamma\tau_\mu(|y|)y + qy\right) v\,dx = 0, \tag{3.5}$$

for each function $v \in H^1(\Omega)$, for a.e. $t \in [0,T]$, and for which $y(x,0) = y_0(x)$.

3.2 Analysis on the Modified Equation

In the following we will show that the modified equation has a weak solution for any $0 < \mu < \infty$. We do so by using Galerkin's method, and begin by constructing finite-dimensional approximations to these solutions, and subsequently show that in the limit we obtain a solution to (3.5).

A finite-dimensional approximation to a solution of (3.5) is expressed as

$$y_m = \sum_{k=1}^{m} d_m^k(t) w_k, \tag{3.6}$$

where the functions $\{w_k\}_{k=1}^\infty$ form an orthogonal basis for $H^1(\Omega)$ and an orthonormal basis for $L^2(\Omega)$. The coefficients $d_m^k(t)$ $(k = 1,\ldots,m)$ are time dependent, and we want to choose them such that

$$d_m^k(0) = \int_\Omega y_0 w_k\,dx, \tag{3.7}$$

and

$$\int_\Omega y_m' w_k\,dx + \int_\Omega \nabla y_m \cdot \nabla w_k\,dx + \int_\Omega \left(\gamma\tau_\mu(|y_m|)y_m + qy_m\right) w_k\,dx = 0, \tag{3.8}$$

holds for $k = 1,\ldots,m$.

3.2.1 Existence of the Finite-Dimensional Approximations

The first step in the Galerkin approach is to show that, for any m, the finite-dimensional approximation y_m exists. This result is presented below.

Lemma 3.2.1. *For each $m = 1, 2, \ldots$ there exists a unique function y_m of the form* (3.6) *for which the conditions* (3.7) *and* (3.8) *hold.*

Proof. We begin by noting that due to orthonormality of the functions $\{w_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ we have that

$$\int_\Omega y_m' w_k \, dx = d_m^{k\prime}(t). \tag{3.9}$$

Inserting this into (3.8) yields

$$d_m^{k\prime}(t) + \sum_{l=1}^{m} d_m^l(t) \left[\int_\Omega \nabla w_l \cdot \nabla w_k + q w_l w_k \, dx \right] + \int_\Omega \gamma(t) \tau_\mu \left(\left| \sum_{r=1}^{m} d_m^r(t) w_r \right| \right) \left(\sum_{s=1}^{m} d_m^s(t) w_s \right) w_k \, dx = 0. \tag{3.10}$$

By taking advantage of the properties of the functions w_k we can rewrite this as

$$d_m^{k\prime}(t) + d_m^k(t) \int_\Omega |\nabla w_k|^2 \, dx + \sum_{l=1}^{m} d_m^l(t) \int_\Omega q w_l w_k \, dx + \int_\Omega \gamma(t) \tau_\mu \left(\left| \sum_{r=1}^{m} d_m^r(t) w_r \right| \right) \left(\sum_{s=1}^{m} d_m^s(t) w_s \right) w_k \, dx = 0. \tag{3.11}$$

This can be expressed as the system of ODEs

$$\frac{d\boldsymbol{d}}{dt} + \boldsymbol{F}(t, \boldsymbol{d}) = 0, \tag{3.12}$$

with $\boldsymbol{d}(t) = [d_m^1(t), \ldots, d_m^m(t)]^T$ and $\boldsymbol{F}(t, \boldsymbol{d}) = [f_1(t, \boldsymbol{d}), \ldots, f_m(t, \boldsymbol{d})]^T$, where

$$f_k(t, \boldsymbol{d}) = d_m^k(t) \int_\Omega |\nabla w_k|^2 \, dx + \sum_{l=1}^{m} d_m^l(t) \int_\Omega q w_l w_k \, dx + \int_\Omega \gamma(t) \tau_\mu \left(\left| \sum_{r=1}^{m} d_m^r(t) w_r \right| \right) \left(\sum_{s=1}^{m} d_m^s(t) w_s \right) w_k \, dx, \tag{3.13}$$

for $k = 1, \ldots, m$. Since the function q is in general not continuous we cannot use the standard Picard-Lindelöf theorem to argue that this ODE system has a solution. However, one can check that $\boldsymbol{F}$ satisfies the Carathéodory condition given on p. 800 of Zeidler [21], and that it can be bounded by a real integrable function. This is sufficient to show that there exists a continuous function $\boldsymbol{d}$ such that (3.12) holds for almost all $t \in [0, T]$ [21].

□

3.2.2 Bounds on the Finite-Dimensional Approximations

We want to show that a subsequence of $\{y_m\}_{m=1}^{\infty}$ converges to a solution of (3.5). In order to do so we need to obtain some bounds on these functions in the relevant spaces. This result is presented in the following lemma.

Lemma 3.2.2. *There exists a constant C, depending only on the harvesting effort q, T and the initial value $y_m(0)$, such that*

$$\max_{0\le t\le T} \|y_m(t)\|^2_{L^2(\Omega)} + \|y_m\|^2_{L^2(0,T;H^1(\Omega))} + \|y_m'\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C, \tag{3.14}$$

for any $m = 1, 2, \ldots$.

Proof. To obtain the first bound on y_m we start by multiplying (3.8) by $d_m^k(t)$ and sum over indices $k = 1, \ldots, m$ to get

$$\int_\Omega y_m' y_m \, dx + \int_\Omega |\nabla y_m|^2 \, dx + \int_\Omega (\gamma(t)\tau_\mu(|y_m|)y_m + q y_m)\, y_m \, dx = 0, \tag{3.15}$$

which implies that

$$\int_\Omega y_m' y_m \, dx + \int_\Omega |\nabla y_m|^2 + y_m^2 \, dx + \gamma(t) \int_\Omega \tau_\mu(|y_m|) y_m^2 \, dx = \int_\Omega (1-q)\, y_m^2 \, dx. \tag{3.16}$$

From this we infer that

$$\frac{1}{2}\frac{\partial}{\partial t} \int_\Omega y_m^2 \, dx + \|y_m\|^2_{H^1(\Omega)} + \gamma(t) \int_\Omega \tau_\mu(|y_m|) y_m^2 \, dx \le \|1-q\|_{L^\infty(\Omega)} \int_\Omega y_m^2 \, dx, \tag{3.17}$$

where we have used that $\partial_t(y_m^2/2) = y_m' y_m$. Integrating from 0 to t yields

$$\int_\Omega y_m(t)^2 \, dx + 2\int_0^t \|y_m\|^2_{H^1(\Omega)} \, dt + 2\int_0^t \gamma(t) \int_\Omega \tau_\mu(|y_m|) y_m^2 \, dx \, dt \le \underbrace{2\|1-q\|_{L^\infty(Q)}}_{=:A} \int_0^t \int_\Omega y_m^2 \, dx \, dt + \underbrace{\int_\Omega y_m(0)^2 \, dx}_{=:B}, \tag{3.18}$$

where the constants A, B depend only on q, T and the initial value $y_m(0)$. By nonnegativity of the terms on the left hand side we get

$$\int_\Omega y_m(t)^2 \, dx \le A \int_0^t \int_\Omega y_m^2 \, dx \, dt + B. \tag{3.19}$$

Now, Gronwall's inequality implies that

$$\int_\Omega y_m(t)^2 \, dx \le B\left(1 + Ate^{At}\right). \tag{3.20}$$

Since this argument holds for any $t \in [0,T]$, we conclude that there exists a constant C_1, depending only on q, T and $y_m(0)$, such that

$$\max_{0\le t\le T} \|y_m(t)\|^2_{L^2(\Omega)} \le C_1. \tag{3.21}$$

To obtain the next bound we return to inequality (3.18). Again, nonnegativity of the terms on the left hand side gives

$$\int_0^t \|y_m\|^2_{H^1(\Omega)}\,dt \le \frac{A}{2}\int_0^t\int_\Omega y_m^2\,dx\,dt + \frac{B}{2}. \tag{3.22}$$

Extending the integration to end time T and employing (3.21) yields

$$\|y_m\|^2_{L^2(0,T;H^1(\Omega))} \le C_2, \tag{3.23}$$

for some constant C_2 depending only on q, T and the initial value $y_m(0)$.

For the final bound we take any $v \in H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \le 1$, and write it as $v = v^1 + v^2$, with $v^1 \in \text{span}\{w_k\}_{k=1}^m$ and v^2 being orthogonal to the functions $\{w_k\}_{k=1}^m$. Due to orthogonality of the functions $\{w_k\}_{k=1}^\infty$, it must hold that $\|v^1\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)}$. From equation (3.8) we infer that

$$\int_\Omega y_m' v^1\,dx + \int_\Omega \nabla y_m\cdot\nabla v^1\,dx + \int_\Omega \left(\gamma\tau_\mu(|y_m|)y_m + qy_m\right)v^1\,dx = 0. \tag{3.24}$$

Consequently, we have that

$$\begin{aligned}
\left|\int_\Omega y_m' v^1\,dx\right| &= \left|\int_\Omega \nabla y_m\cdot\nabla v^1\,dx + \int_\Omega \left(\gamma(t)\tau_\mu(|y_m|)y_m + qy_m\right)v^1\,dx\right| \\
&= \left|\int_\Omega \nabla y_m\cdot\nabla v^1 + y_m v^1\,dx + \int_\Omega (q-1)y_m v^1\,dx\right. \\
&\left. + \int_\Omega \gamma(t)\tau_\mu(|y_m|)y_m v^1\,dx\right| \\
&\le \|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)} + \|q-1\|_{L^\infty(\Omega)}\|y_m\|_{L^2(\Omega)}\|v^1\|_{L^2(\Omega)} \\
&+ \|\gamma\|_{L^\infty([0,T])}\int_\Omega \tau_\mu(|y_m|)|y_m||v^1|\,dx \\
&\le \|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)} + \|q-1\|_{L^\infty(\Omega)}\|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)} \\
&+ \mu\|\gamma\|_{L^\infty([0,T])}\|y_m\|_{L^2(\Omega)}\|v^1\|_{L^2(\Omega)}, \\
&\le \|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)} + \|q-1\|_{L^\infty(\Omega)}\|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)} \\
&+ \mu\|\gamma\|_{L^\infty([0,T])}\|y_m\|_{H^1(\Omega)}\|v^1\|_{H^1(\Omega)},
\end{aligned} \tag{3.25}$$

where we have used the Cauchy-Schwarz inequality, the Hölder inequality and the bound $\tau_\mu(|y_m|) \le \mu$. Next, observe that by (3.6) and the definition of v it must hold that

$$\int_\Omega y_m' v\,dx = \int_\Omega y_m' v^1\,dx. \tag{3.26}$$

Consequently, combining (3.25) and the bound $\|v^1\|_{H^1(\Omega)} \leq 1$ gives

$$\begin{aligned} \|y_m'\|_{H^{-1}(\Omega)} \leq \|y_m\|_{H^1(\Omega)} + \|q-1\|_{L^\infty(\Omega)}\|y_m\|_{H^1(\Omega)} \\ + \mu\|\gamma\|_{L^\infty([0,T])}\|y_m\|_{H^1(\Omega)}. \end{aligned} \tag{3.27}$$

For some constant K we have

$$\begin{aligned} \|y_m'\|^2_{H^{-1}(\Omega)} \leq K\Big[\|y_m\|^2_{H^1(\Omega)} + \|q-1\|^2_{L^\infty(\Omega)}\|y_m\|^2_{H^1(\Omega)} \\ + \mu^2\|\gamma\|^2_{L^\infty([0,T])}\|y_m\|^2_{H^1(\Omega)}\Big]. \end{aligned} \tag{3.28}$$

Integrating from 0 to T and applying (3.23) yields

$$\|y_m'\|^2_{L^2(0,T;H^{-1}(\Omega))} \leq C_3, \tag{3.29}$$

for some constant C_3, depending only on q, T and the initial value $y_m(0)$. □

3.2.3 Existence of a Solution

We are now ready to establish that the weak form of the auxiliary state equation has a solution. The result is stated in the following theorem.

Theorem 3.2.3. *There exists a solution y to equation* (3.5)*, and it holds that $y \in W(0,T)$.*

Proof. In lemma 3.2.2 we established that the sequences $\{y_m\}_{m=1}^\infty$ and $\{y_m'\}_{u=1}^\infty$ are bounded in $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$ respectively. Therefore, there must exist a subsequence $\{y_{m_l}\}_{l=1}^\infty \subset \{y_m\}_{m=1}^\infty$, and a function $y \in L^2(0,T;H^1(\Omega))$ with derivative $y' \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\begin{cases} y_{m_l} \longrightarrow y & \text{weakly in } L^2(0,T;H^1(\Omega)) \\ y_{m_l}' \longrightarrow y' & \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{3.30}$$

We now take a function $v \in L^2(0,T;H^1(\Omega))$ of the form

$$v(t) = \sum_{k=1}^{N} d^k(t) w_k, \tag{3.31}$$

for given smooth functions $\{d^k\}_{k=1}^N$ and a fixed integer N. Taking $m \geq N$, multiplying (3.8) by $d^k(t)$, summing over indices $k = 1, \ldots, N$, and integrating from 0 to T yields

$$\int_0^T \int_\Omega y_m' v \, dx\, dt + \int_0^T \int_\Omega \nabla y_m \cdot \nabla v + (\gamma\tau_\mu(|y_m|)y_m + qy_m)\, v \, dx\, dt = 0. \tag{3.32}$$

We now set $m = m_l$ with the intention of passing to weak limits. Before doing so, we need to establish that the nonlinear term converges to the desired limit. We set $\varphi_m =$

$\gamma(t)\tau_\mu(|y_m|)v$ and wish to show that $\varphi_m \to \varphi$ strongly in $L^2(Q)$. Since the space $W(0,T)$ is compactly embedded into the space $L^2(Q)$, we have $y_m \to y$ in $L^2(Q)$. Consequently, there exists a subsequence $\{y_{m'}\}_{m'=1}^{\infty}$ such that $y_{m'} \to y$ pointwise, for a.e. $(x,t) \in Q$. By continuity of φ_m we have that $\varphi_{m'} \to \varphi$ pointwise, for a.e. $(x,t) \in Q$. Furthermore, we have the bound

$$\begin{aligned}\|\varphi_{m'}\|^2_{L^2(Q)} &= \int_0^T \gamma(t)^2 \int_\Omega \tau_\mu(|y_{m'}|)^2 v^2\,dx\,dt \\ &\le \|\gamma\|^2_{L^\infty([0,T])}\mu^2 \int_0^T \|v\|^2_{L^2(\Omega)}\,dt \\ &\le \|\gamma\|^2_{L^\infty([0,T])}\mu^2 \int_0^T \|v\|^2_{H^1(\Omega)}\,dt,\end{aligned} \tag{3.33}$$

which is constant due to $v \in L^2(0,T;H^1(\Omega))$. Consequently, by the Lebesgue dominated convergence theorem we have $\varphi_m \to \varphi$ strongly in $L^2(Q)$. Therefore, we get

$$\int_0^T \int_\Omega y_m\varphi_m\,dx\,dt \to \int_0^T \int_\Omega y\varphi\,dx\,dt. \tag{3.34}$$

We return now to equation (3.32) where we set $m = m_l$ and pass to weak limits. By (3.34) we obtain

$$\int_0^T \int_\Omega y'v\,dx\,dt + \int_0^T \int_\Omega \nabla y \cdot \nabla v + (\gamma\tau_\mu(|y|)y + qy)\,v\,dx\,dt = 0. \tag{3.35}$$

This equality holds for any $v \in L^2(0,T;H^1(\Omega))$. Thus, for every $v \in H^1(\Omega)$ it holds that

$$\int_\Omega y'v\,dx + \int_\Omega \nabla y \cdot \nabla v + (\gamma\tau_\mu(|y|)y + qy)\,v\,dx = 0, \tag{3.36}$$

for a.e. $0 \le t \le T$.

We now wish to show that y satisfies the initial condition $y(x,0) = y_0(x)$. First, using integration by parts on (3.35) gives

$$\int_0^T \int_\Omega -yv'\,dx\,dt + \int_0^T \int_\Omega \nabla y \cdot \nabla v + (\gamma\tau_\mu(|y|)y + qy)\,v\,dx\,dt = \int_\Omega y(0)v(0)\,dx, \tag{3.37}$$

for any $v \in L^2(0,T;H^1(\Omega))$ for which $v(T) = 0$. In a similar fashion, we obtain from equation (3.32) that

$$\begin{aligned}\int_0^T \int_\Omega -y_m v'\,dx\,dt + \int_0^T \int_\Omega \nabla y_m \cdot \nabla v + (\gamma\tau_\mu(|y_m|)y_m + qy_m)\,v\,dx\,dt \\ = \int_\Omega y_m(0)v(0)\,dx,\end{aligned} \tag{3.38}$$

for any $v \in L^2(0,T;H^1(\Omega))$ for which $v(T) = 0$. We now set $m = m_l$ and pass to weak limits to obtain

$$\int_0^T \int_\Omega -yv'\,dx\,dt + \int_0^T \int_\Omega \nabla y \cdot \nabla v + (\gamma\tau_\mu(|y|)y + qy)\,v\,dx\,dt = \int_\Omega y_0 v(0)\,dx, \tag{3.39}$$

since $y_{m_l} \to y_0$ in $L^2(\Omega)$. Noting that this holds for any $v(0)$, we find that $y(0) = y_0$ by comparing (3.37) and (3.39). □

3.2.4 Boundedness of a Solution

Before establishing the existence of a weak solution to (3.1), we need to assert that a weak solution to (3.4) has to be bounded. The result is given below.

Lemma 3.2.4. *There exists a constant C, independent of μ, and depending on the initial value y_0, such that for a weak solution y of* (3.4) *we have $\|y\|_{L^\infty(Q)} \leq C$.*

Proof. We take $v = y^{2p-1}$ in (3.5), with $p \geq 1$, and obtain

$$\int_\Omega y' y^{2p-1}\, dx + \int_\Omega |\nabla y|^2 y^{2p}\, dx + \gamma \int_\Omega \tau_\mu(|y|) y^{2p}\, dx + \int_\Omega q y^{2p}\, dx = 0. \tag{3.40}$$

Next, we note that for the first term we have

$$\int_\Omega y' y^{2p-1}\, dx = \frac{1}{2p} \frac{\partial}{\partial t} \int_\Omega y^{2p}\, dx. \tag{3.41}$$

Observing that all the other terms are nonnegative we deduce that

$$\frac{1}{2p} \frac{\partial}{\partial t} \int_\Omega y^{2p}\, dx \leq 0. \tag{3.42}$$

Integrating from 0 to t yields

$$\frac{1}{2p} \int_\Omega y^{2p}(t)\, dx \leq \frac{1}{2p} \int_\Omega y_0^{2p}\, dx. \tag{3.43}$$

Since this argument holds for any $t \in (0, T)$, we infer that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|y(t)\|_{L^{2p}(\Omega)} \leq \|y_0\|_{L^{2p}(\Omega)}, \tag{3.44}$$

for any $p \geq 1$. Next, we pass to the limit $p \to \infty$.This yields

$$\|y\|_{L^\infty(Q)} \leq \|y_0\|_{L^\infty(\Omega)}. \tag{3.45}$$

By recalling $y_0 \in L^\infty(\Omega)$ we reach the desired conclusion. □

3.3 Returning to the Auxiliary State Equation

In the following we will show that the auxiliary state equation (3.1) has a weak solution, that such a solution is unique, and that it remains nonnegative for a.e. $t \in [0, T]$ given that $y_0 \geq 0$.

3.3.1 Existence of a Solution

We now use the results we have obtained for the modified equation to show that a solution to the auxiliary state equation exists.

Theorem 3.3.1. *There exists a weak solution to equation* (3.1).

Proof. By lemma 3.2.4 there is a constant C such that for a weak solution y to (3.4) we have $\|y\|_{L^\infty(Q)} \leq C$. Since C is independent of μ, we may take $\mu > C$. Observe now that for any such μ it must hold that $\tau_\mu(|y|) = |y|$, and thus equation (3.4) becomes equivalent to (3.1). By theorem 3.2.3 we must also have existence of a solution to (3.1). □

3.3.2 Nonnegativity of a Solution

In order to prove that a solution to the auxiliary equation is unique, we first need to show that a solution remains nonnegative given that the initial value is also nonnegative. Another motivation for this is that we will later associate a solution of this equation with a solution to the state equation, and we cannot have negative solutions to this problem, as we cannot have a negative fish biomass. The result is given in the theorem below.

Theorem 3.3.2. *For a solution to* (3.2) *with* $y_0 \geq 0$*, it holds that* $y(x,t) \geq 0$ *for a.e.* $(x,t) \in \Omega \times [0,T]$.

Proof. We begin by taking advantage of the fact that a function can be split into its positive and negative part in the following way

$$v = v^+ - v^-,$$

where

$$\begin{aligned} v^+ &= \max\{v, 0\}, \\ v^- &= \max\{-v, 0\}. \end{aligned}$$

Next, note that for any function v it holds that

$$\begin{aligned} \int_0^T \int_\Omega v_t v^- \, dx \, dt &= \int_0^T \int_\Omega \left(v_t^+ - v_t^-\right) v^- \, dx \, dt \\ &= -\int_0^T \int_\Omega v_t^- v^- \, dx \, dt \\ &= -\frac{1}{2} \int_0^T \int_\Omega \frac{\partial}{\partial t} \left(v^-(t)\right)^2 \, dx \, dt \\ &= -\frac{1}{2} \int_\Omega \left(v^-(t)\right)^2 \, dx \Big|_0^T \\ &= -\frac{1}{2} \|v^-(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|v^-(0)\|_{L^2(\Omega)}^2, \end{aligned} \tag{3.46}$$

where we have taken advantage of the fact that $\int_\Omega v_t^+ v^- \, dx = 0$ for a.e. $t \in [0,T]$. We now proceed by taking $v = y^-$ in (3.2). This gives

$$\int_0^T \int_\Omega y_t y^- \, dx \, dt + \int_0^T \int_\Omega \nabla \left(y^+ - y^-\right) \cdot \nabla y^- \, dx \, dt \\ + \int_0^T \int_\Omega \left[\gamma \left(y^+ + y^-\right)\left(y^+ - y^-\right) + q \left(y^+ - y^-\right)\right] y^- \, dx \, dt = 0. \quad (3.47)$$

By (3.46) and

$$\int_0^T \int_\Omega y^+ y^- \, dx \, dt = \int_0^T \int_\Omega \nabla y^+ \cdot \nabla y^- \, dx \, dt = 0, \quad (3.48)$$

we have

$$\begin{aligned} \|y^-(T)\|^2_{L^2(\Omega)} &= \|y^-(0)\|^2_{L^2(\Omega)} \\ &\quad - 2\int_0^T \|\nabla y^-\|^2_{L^2(\Omega)} + \gamma \|y^-\|^3_{L^3(\Omega)} + \|q^{1/2} y^-\|^2_{L^2(\Omega)} \, dt \\ &\leq \|y^-(0)\|^2_{L^2(\Omega)}, \end{aligned} \quad (3.49)$$

where the inequality holds due to nonnegativity of the integrand. If the initial value is nonnegative then $y^-(0) = 0$, meaning that $y^-(T) \equiv 0$ a.e. in Ω. Since the same argument would hold if we integrated to any time $0 < t \leq T$, we conclude that $y \geq 0$ a.e. in Ω for any $t \in [0,T]$. □

3.3.3 Uniqueness of a Solution

We now establish that a weak solution to the auxiliary state equation is unique. This result will be important later.

Theorem 3.3.3. *A weak solution to equation* (3.1) *is unique.*

Proof. Suppose we have two solutions y_1, y_2 satisfying (3.2) for two different controls q_1 and q_2 respectively. We note that $y_1(0) = y_2(0) = y_0$. Subtracting the equations for the respective solutions yields

$$\int_\Omega (y_1' - y_2') \, v \, dx + \int_\Omega \nabla (y_1 - y_2) \cdot \nabla v \, dx \\ + \gamma \int_\Omega (|y_1| y_1 - |y_2| y_2) \, v \, dx + \int_\Omega (q_1 y_1 - q_2 y_2) \, v \, dx = 0 \quad (3.50)$$

Next, we choose $v = y_1 - y_2$ as our test function. This gives

$$\int_\Omega (y_1' - y_2')(y_1 - y_2) \, dx + \int_\Omega |\nabla (y_1 - y_2)|^2 \, dx \\ + \gamma \int_\Omega (|y_1| y_1 - |y_2| y_2)(y_1 - y_2) \, dx = \int_\Omega (q_2 y_2 - q_1 y_1)(y_1 - y_2) \, dx. \quad (3.51)$$

Now, observe that for the integrand in the third integral we have (due to nonnegativity)

$$\left(y_1^2 - y_2^2\right)(y_1 - y_2) = (y_1 - y_2)^2 (y_1 + y_2) \geq 0. \tag{3.52}$$

Consequently, it follows that

$$\int_\Omega (y_1' - y_2')(y_1 - y_2)\, dx \leq \int_\Omega (q_2 y_2 - q_1 y_1)(y_1 - y_2)\, dx. \tag{3.53}$$

We now set $y = y_1 - y_2$. If the controls are equal, i.e. $q_1 = q_2 =: q$, we have

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\left(\|y\|^2_{L^2(\Omega)}\right) &\leq - \int_\Omega q y^2\, dx \\ &\leq \|q\|_{L^\infty(Q)} \|y\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.54}$$

By Gronwall's inequality we have

$$\|y(t)\|^2_{L^2(\Omega)} \leq K \|y(0)\|^2_{L^2(\Omega)}, \tag{3.55}$$

for every $t \in [0, T]$, and for some constant $K > 0$. Since we have $y(0) = y_1(0) - y_2(0) = 0$, we conclude that $y \equiv 0$ for a.e. x in Ω, for every $t \in [0, T]$. □

3.4 Returning to the State Equation

We can now utilize the results we have obtained in the previous sections to show that the state equation has a unique nonnegative solution.

Theorem 3.4.1. *Let the harvesting effort $q \in \mathcal{Q}_{ad}$, and the initial value $u_0 \in L^\infty(\Omega)$. Furthermore, let u_0 be nonnegative for a.e. x in Ω. Then, the sate equation* (2.1) *has a unique weak solution which is nonnegative a.e. in Q. It further holds that $u \in W(0,T) \cap L^\infty(Q)$.*

Proof. This proof is due to Braack et al. [2]. Let $\gamma = e^{at}$ and $y_0 = u_0$ in (3.1), and let y be the unique solution to (3.2). Now, we set $u = \gamma y$ and insert this into (2.1). This yields

$$\begin{aligned} u_t - \Delta u + u^2 - au + qu &= \gamma y_t + a\gamma y - \gamma \Delta y + \gamma^2 y^2 - a\gamma y + \gamma q y \\ &= \gamma\left[y_t - \Delta y + \gamma y^2 + qy\right] \\ &= \gamma\left[y_t - \Delta y + \gamma |y| y + qy\right] \\ &= 0. \end{aligned} \tag{3.56}$$

Note that we also have $\nabla u \cdot n = \gamma \nabla y \cdot n = 0$ on $(0,T) \times \partial\Omega$. Thus, this u clearly solves (2.1). Conversely, for any nonnegative solution u of (2.1) it is easy to check that $y = \gamma^{-1} u$ solves (3.1). By uniqueness of solutions to (3.1) we therefore conclude that (2.1) also has a unique solution. Nonnegativity is given by $u = \gamma y \geq 0$. Finally, it is simple to verify that u belongs to the function spaces stated above by using the same result for y. □

Chapter 4

The Control Problem

In this chapter we begin by introducing a solution operator, mapping a control to the corresponding solution of the state equation. This allows us to state the control problem in terms of a reduced cost functional, depending only on the control variable. We proceed by showing that this control problem admits a unique solution. Next, we do analysis on the solution operator to show that it is Fréchet differentiable. This result is necessary for the final part of the chapter, where we derive a first order necessary optimality condition for the control problem.

4.1 Existence of a Minimizer to the Control Problem

Now that we have shown the existence of a unique nonnegative solution to state equation (2.1), we can define the solution operator

$$S : \mathcal{Q}_{ad} \to W(0,T), \quad q \mapsto u. \tag{4.1}$$

The solution operator maps a control q to the corresponding solution u of (2.2). Now, define the reduced cost functional by

$$j : \mathcal{Q}_{ad} \to \mathbb{R}, \quad q \mapsto J(S(q), q). \tag{4.2}$$

This allows us to formulate the optimal control problem as

$$\min_{q \in \mathcal{Q}_{ad}} j(q). \tag{4.3}$$

It now remains to show that this control problem admits a solution.

Theorem 4.1.1. *There exists a minimizer $\bar{q}$ to control problem* (4.3)*, and a corresponding state $\bar{u} = S(\bar{q})$, for which $\bar{u} \in W(0,T) \cap L^\infty(0,T;L^\infty(\Omega))$ and $\bar{q} \in \mathcal{Q}_{ad}$.*

Proof. This proof is due to Braack et al. [2]. We begin by showing that the reduced cost functional is bounded from below. Trivially we have

$$j(q) \geq \int_0^T e^{-\rho t} \int_\Omega q(r-u)\,dx\,dt - \lambda \int_\Omega u(T)\,dx. \tag{4.4}$$

For the first term we have

$$\begin{aligned}\int_0^T e^{-\rho t} \int_\Omega q(r-u)\,dx\,dt &= \int_0^T e^{-\rho t} \int_\Omega qr\,dx\,dt - \int_0^T e^{-\rho t} \int_\Omega qu\,dx\,dt \\ &\geq -\int_0^T e^{-\rho t} \int_\Omega qu\,dx\,dt \\ &\geq -\int_0^T \int_\Omega qu\,dx\,dt \\ &\geq -\left(\frac{1}{2}\int_0^T \int_\Omega q^2\,dx\,dt + \frac{1}{2}\int_0^T \int_\Omega u^2\,dx\,dt\right),\end{aligned} \tag{4.5}$$

where we have used Young's inequality. This can be bounded by some constant due to $q \in \mathcal{Q}_{ad}$ and $u \in W(0,T)$. For the second term we have

$$\lambda \int_\Omega u(T)\,dx = \lambda \|u(T)\|_{L^1(\Omega)} \leq \lambda |\Omega| \|u(T)\|_{L^\infty(\Omega)}. \tag{4.6}$$

Again, this term is bounded by some constant. Thus, we have that j is bounded from below.

Consequently, since the set of admissible controls is non-empty there must exist a (minimizing) sequence $\{q_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} j(q_n) = \inf_{q \in \mathcal{Q}_{ad}} j(q). \tag{4.7}$$

Since $\mathcal{Q}_{ad}$ is a bounded subset of the space $L^\infty(Q)$, there must exist a subsequence $\{q_k\}_{k=1}^\infty$ which converges weakly to some $\bar{q}$ in $L^\infty(Q)$. Because $\mathcal{Q}_{ad}$ is closed and convex we have $\bar{q} \in \mathcal{Q}_{ad}$.

Next, we look at the sequence $\{u_k\}_{k=1}^\infty$ defined by $u_k = S(q_k)$. This means that we have

$$\int_0^T \int_\Omega u_k' v\,dx\,dt + \int_0^T \int_\Omega \nabla u_k \cdot \nabla v\,dx\,dt \\ + \int_0^T \int_\Omega u_k\,(u_k + q_k - a)\,v\,dx\,dt = 0, \tag{4.8}$$

and by theorem 3.4.1 we have

$$\|u_k\|_{L^2(0,T;H^1(\Omega))} + \|u_k\|_{L^2(0,T;H^{-1}(\Omega))} + \|u_k\|_{L^\infty(Q)} \leq C, \tag{4.9}$$

for some constant C. Therefore, there exists a subsequence $\{u_{k'}\}_{k'=1}^\infty$ which converges weakly to some $\bar{u}$ in the space $W(0,T)$. We now want to show that $\bar{u} = S(\bar{q})$ by setting $k = k'$ in (4.8) and passing to weak limits. This requires special treatment of two terms.

First, we note that the space $W(0,T)$ is compactly embedded into $L^2(Q)$. Thus, $u_{k'} \to \bar{u}$ strongly in $L^2(Q)$. We wish to show that

$$\int_0^T \int_\Omega u_{k'} q_{k'} v \, dx \, dt \to \int_0^T \int_\Omega \bar{u}\bar{q} v \, dx \, dt. \tag{4.10}$$

We have

$$\begin{aligned} \int_0^T \int_\Omega (u_{k'} q_{k'} - \bar{u}\bar{q})\, v \, dx \, dt = &\int_0^T \int_\Omega (q_{k'} - \bar{q})\, \bar{u} v \, dx \, dt \\ &+ \int_0^T \int_\Omega q_{k'} \,(u_{k'} - \bar{u})\, v \, dx \, dt. \end{aligned} \tag{4.11}$$

The first term converges to 0 by weak convergence of $q_{k'}$ in $L^2(Q)$. The second term we estimate by

$$\begin{aligned} \int_0^T \int_\Omega q_{k'} \,(u_{k'} - \bar{u})\, v \, dx \, dt &\leq \|q_{k'}\|_{L^\infty(Q)} \|u_{k'} - \bar{u}\|_{L^2(Q)} \|v\|_{L^2(Q)} \\ &\to 0, \end{aligned} \tag{4.12}$$

since $u_{k'} \to \bar{u}$ in $L^2(Q)$. Next, we want so show that

$$\int_0^T \int_\Omega u_{k'}^2 v \, dx \, dt \to \int_0^T \int_\Omega \bar{u}^2 v \, dx \, dt. \tag{4.13}$$

We use the same strategy as before, and write

$$\begin{aligned} \int_0^T \int_\Omega \left(u_{k'}^2 - \bar{u}^2\right) v \, dx \, dt = &\int_0^T \int_\Omega (u_{k'} - \bar{u})\, \bar{u} v \, dx \, dt \\ &+ \int_0^T \int_\Omega u_{k'} \,(u_{k'} - \bar{u})\, v \, dx \, dt. \end{aligned} \tag{4.14}$$

Again, the first term converges by weak convergence of $u_{k'}$ in $L^2(Q)$, and for the second term we have

$$\begin{aligned} \int_0^T \int_\Omega u_{k'} \,(u_{k'} - \bar{u})\, v \, dx \, dt &\leq \|u_{k'}\|_{L^\infty(Q)} \|u_{k'} - \bar{u}\|_{L^2(Q)} \|v\|_{L^2(Q)} \\ &\to 0, \end{aligned} \tag{4.15}$$

by the same argument as before. Now, passing to weak limits in (4.8) gives

$$\int_0^T \int_\Omega \bar{u}' v \, dx \, dt + \int_0^T \int_\Omega \nabla \bar{u} \cdot \nabla v \, dx \, dt + \int_0^T \int_\Omega \bar{u}\,(\bar{u} + \bar{q} - a)\, v \, dx \, dt = 0. \tag{4.16}$$

Thus, we have $\bar{u} = S(\bar{q})$.

Finally, we want to show that the pair $(\bar{u}, \bar{q})$ minimizes the cost functional. Before passing to the limit in the cost functional we show convergence for the terms

$$\int_0^T e^{-\rho t} \int_\Omega q_{k'}\,(r - u_{k'})\, dx \, dt, \quad \lambda \int_\Omega u_{k'}(T)\, dx. \tag{4.17}$$

First, we take $\varphi = e^{-\rho t}$ and write

$$\int_0^T e^{-\rho t} \int_\Omega q_{k'} \left(r - u_{k'}\right) \, dx \, dt = \int_0^T \int_\Omega q_{k'} r \varphi \, dx \, dt - \int_0^T \int_\Omega q_{k'} u_{k'} \varphi \, dx \, dt. \quad (4.18)$$

The first of these terms converges by weak convergence of $q_{k'}$ in $L^2(Q)$, while the second converges by a similar argument as above. Furthermore, since the trace operator from $W(0,T)$ to $L^2(\Omega)$ is continuous, it follows that $u_{k'}(T) \rightharpoonup \bar{u}(T)$ in $L^2(\Omega)$. Consequently, we have

$$\lambda \int_\Omega u_{k'}(T) \, dx \to \lambda \int_\Omega \bar{u}(T) \, dx. \quad (4.19)$$

Therefore, we get

$$\begin{aligned}
\liminf_{k' \to \infty} j(q_{k'}) &= \liminf_{k' \to \infty} J(u_{k'}, q_{k'}) \\
&= \int_0^T e^{-\rho t} \int_\Omega \bar{q} \left(r - \bar{u}\right) - \lambda \int_\Omega \bar{u}(T) \, dx \\
&+ \frac{\alpha}{2} \liminf_{k' \to \infty} \|q_{k'}\|^2_{L^2(\Omega_T)} \\
&\geq \int_0^T e^{-\rho t} \int_\Omega \bar{q} \left(r - \bar{u}\right) - \lambda \int_\Omega \bar{u}(T) \, dx + \frac{\alpha}{2} \|\bar{q}\|^2_{L^2(\Omega_T)} \\
&= J(\bar{u}, \bar{q}) \\
&= j(\bar{q}),
\end{aligned} \quad (4.20)$$

where we have used the weak lower semi-continuity of the norm $\| \cdot \|_{L^2(Q)}$. Thus, we conclude that

$$j(\bar{q}) = \inf_{q \in \mathcal{Q}_{ad}} j(q). \quad (4.21)$$

□

Since the solution operator S is non-linear, we can not guarantee that the reduced cost functional is convex. Therefore, we will not be able to show uniqueness of a solution to the control problem in this work.

4.2 Analysis of the Solution Operator

In order to obtain first order necessary optimality conditions for the control problem (2.7), which we will use to characterize a solution, we need the solution operator to be differentiable. In this section we will do this explicitly, but first we need a preliminary result.

4.2.1 Stability of the Solution Operator

To show that the solution operator is differentiable we need a stability result for the solution operator, which is presented below.

Lemma 4.2.1. *Let u_1 and u_2 be solutions to equation* (2.2) *for controls q_1 and q_2, i.e. $u_1 = S(q_1)$ and $u_2 = S(q_2)$. Then, for some constant C it holds*

$$\|u_1 - u_2\|_{L^4(Q)} \le C\|q_1 - q_2\|_{L^2(Q)}. \tag{4.22}$$

Proof. We begin by showing that

$$\max_{0\le t\le T} \|u(t)\|^2_{L^2(\Omega)} \le C_1\|q_1 - q_2\|^2_{L^2(Q)}, \tag{4.23}$$

for some constant C_1. To do this we subtract the weak equations for u_1 and u_2. This gives

$$\begin{aligned}\int_\Omega (u_1' - u_2')v\,dx + \int_\Omega \nabla(u_1 - u_2)\cdot\nabla v\,dx + \int_\Omega (u_1^2 - u_2^2)v\,dx \\ + \int_\Omega (u_1q_1 - u_2q_2)v\,dx - \int_\Omega a(u_1 - u_2)v\,dx = 0.\end{aligned} \tag{4.24}$$

Now, we set $u = u_1 - u_2$ and take $v = u$. We obtain

$$\begin{aligned}\int_\Omega u'u\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2(u_1 + u_2)\,dx \\ + \int_\Omega [(q_1 - q_2)u_1 + q_2u]\,u\,dx - \int_\Omega au^2\,dx = 0.\end{aligned} \tag{4.25}$$

We can rewrite this as

$$\begin{aligned}\int_\Omega u'u\,dx + \int_\Omega |\nabla u|^2 + u^2\,dx + \int_\Omega u^2(u_1 + u_2)\,dx + \int_\Omega q_2u^2\,dx \\ = -\int_\Omega (q_1 - q_2)u_1u\,dx + \int_\Omega (a+1)u^2\,dx.\end{aligned} \tag{4.26}$$

Next, we apply Young's inequality on the right hand side and obtain

$$\begin{aligned}\int_\Omega u'u\,dx + \int_\Omega |\nabla u|^2 + u^2\,dx + \int_\Omega u^2(u_1 + u_2)\,dx + \int_\Omega q_2u^2\,dx \\ \le \underbrace{\frac{\|u_1\|_{L^\infty(Q)}}{2}}_{=:A}\int_\Omega (q_1 - q_2)^2\,dx + \underbrace{\left[\frac{\|u_1\|_{L^\infty(Q)}}{2} + \|a+1\|_{L^\infty(\Omega)}\right]}_{=:B}\int_\Omega u^2\,dx.\end{aligned} \tag{4.27}$$

Now, we integrate from 0 to t, and use $\partial_t(u^2/2) = u'u$ in combination with $u(0) = u_1(0) - u_2(0) = 0$. This yields

$$\begin{aligned}\int_\Omega u(t)^2\,dx + \int_0^t\int_\Omega |\nabla u|^2 + u^2\,dx\,dt + \int_0^t\int_\Omega u^2(u_1 + u_2)\,dx\,dt \\ + \int_0^t\int_\Omega q_2u^2\,dx\,dt \le A\int_0^t\int_\Omega (q_1 - q_2)^2\,dx\,dt + B\int_0^t\int_\Omega u^2\,dx\,dt.\end{aligned} \tag{4.28}$$

From this we deduce, by positivity of the terms on the left hand side, that

$$\int_\Omega u(t)^2\,dx \le A\int_0^t\int_\Omega (q_1-q_2)^2\,dx\,dt + B\int_0^t\int_\Omega u^2\,dx\,dt. \tag{4.29}$$

By applying Gronwall's inequality we get

$$\int_\Omega u(t)^2\,dx \le A\|q_1-q_2\|^2_{L^2(Q)}\left(1+Bte^{Bt}\right). \tag{4.30}$$

By extending the integration to end time T we conclude that for some constant C_1 we have

$$\max_{0\le t\le T}\|u(t)\|^2_{L^2(\Omega)} \le C_1\|q_1-q_2\|^2_{L^2(Q)}. \tag{4.31}$$

Now we are ready to prove that the desired stability estimate holds. We begin by rewriting (4.24) as

$$\begin{aligned}\int_\Omega u_t v\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx \\ = \int_\Omega \left[a-q_2-(u_1+u_2)\right]uv\,dx - \int_\Omega u_1(q_1-q_2)v\,dx.\end{aligned} \tag{4.32}$$

Next, we would like choose $v = u_t$ as our test function. Note here that the following derivation is then only formal, since we are using a test function which does not belong to $H^1(\Omega)$. However, the proof can be made rigorous by the use of difference quotients. See Kačur [11] for an illustration of this. In this work we continue with the formal estimate.

The choice $v = u_t$ gives

$$\begin{aligned}\int_\Omega u_t^2\,dx + \int_\Omega \nabla u\cdot\nabla u_t\,dx &= \int_\Omega \left[a-q_2-(u_1+u_2)\right]uu_t\,dx \\ &\quad - \int_\Omega u_1(q_1-q_2)u_t\,dx \\ &\le \frac{1}{2}\int_\Omega \left[a-q_2-(u_1+u_2)\right]^2u^2\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx \\ &\quad + \frac{1}{2}\int_\Omega u_1^2(q_1-q_2)^2\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx \\ &\le \underbrace{\frac{\|a-q_2-(u_1+u_2)\|^2_{L^\infty(Q)}}{2}}_{=:A}\int_\Omega u^2\,dx \\ &\quad + \frac{1}{2}\int_\Omega u_t^2\,dx \\ &\quad + \underbrace{\frac{\|u_1\|^2_{L^\infty(Q)}}{2}}_{=:B}\int_\Omega (q_1-q_2)^2\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx,\end{aligned} \tag{4.33}$$

where we have used Young's inequality. Thus, we have

$$\int_\Omega \nabla u\cdot\nabla u_t\,dx \le A\int_\Omega u^2\,dx + B\int_\Omega (q_1-q_2)^2\,dx. \tag{4.34}$$

Noting that $\partial_t(|\nabla u|^2/2) = \nabla u \cdot \nabla u_t$, we integrate in time to get

$$\int_\Omega |\nabla u(t)|^2\,dx \le 2A \int_0^t \int_\Omega u^2\,dx\,ds + 2B \int_0^t \int_\Omega (q_1 - q_2)^2\,dx\,ds, \tag{4.35}$$

since $\nabla u(0) = 0$ due to $u(0) = 0$. By applying the result (4.31) we obtain that for some constant $\tilde{C}$ we have

$$\max_{0\le t\le T} \|\nabla u\|^2_{L^2(\Omega)^2} \le \tilde{C}\|q_1 - q_2\|^2_{L^2(Q)}. \tag{4.36}$$

Combining this with (4.31) we get

$$\|u\|_{L^\infty(0,T;H^1(\Omega))} \le C\|q_1 - q_2\|_{L^2(Q)}, \tag{4.37}$$

for some constant C.

Finally, we have for some constant $\hat{C}$ that

$$\begin{aligned}
\|u\|^4_{L^4(Q)} &= \int_0^T \|u\|^4_{L^4(\Omega)}\,dt \\
&\le \hat{C}\int_0^T \|u\|^4_{H^1(\Omega)}\,dt \\
&\le \hat{C}T\|u\|^4_{L^\infty(0,T;H^1(\Omega))} \\
&\le \bar{C}\|q_1 - q_2\|^4_{L^2(Q)},
\end{aligned} \tag{4.38}$$

due to the continuous embedding of $H^1(\Omega)$ into $L^4(\Omega)$ [12]. □

4.2.2 Differentiability of the Solution Operator

With the stability result at hand we are ready to establish differentiability of the solution operator.

Theorem 4.2.2. *The solution operator defined in* (4.1) *is Fréchet differentiable. Denote its derivative at a point q in a direction h by $y = S'(q)h$. Then y solves the linearized state equation*

$$\begin{aligned}
y_t - \Delta y + 2uy + hu + qy - ay &= 0 \quad \textit{in } (0,T)\times\Omega, \\
\nabla y \cdot n &= 0 \quad \textit{on } (0,T)\times\partial\Omega, \\
y(x,0) &= 0 \quad \textit{in } \Omega.
\end{aligned} \tag{4.39}$$

Proof. The weak form of (4.39) is given by

$$\int_\Omega y_t v\,dx + \int \nabla y \cdot \nabla v\,dx + \int_\Omega (2uy + hu + qy - ay)\,v\,dx = 0. \tag{4.40}$$

Now, let $u^h = S(q+h)$ and $u = S(q)$. Subtracting the weak equations for u and y from the weak equation for u^h gives

$$\int_\Omega (u^h_t - u_t - y_t) v\,dx + \int_\Omega \nabla(u^h - u - y)\cdot\nabla v\,dx + \int_\Omega \big((u^h)^2 + qu^h + hu^h - au^h\big)\,v\,dx \\ - \int_\Omega \big(u^2 + qu - au\big)\,v\,dx - \int_\Omega (2uy + hu + qy - ay)\,v\,dx = 0. \quad (4.41)$$

Setting $p = u^h - u - y$ we get

$$\int_\Omega p_t v\,dx + \int_\Omega \nabla p\cdot\nabla v\,dx + \int_\Omega \big[(u^h)^2 - u^2 - 2uy\big]\,v\,dx \\ + \int_\Omega \big[(q+h)u^h - qu - qy - hu\big]\,v\,dx - \int_\Omega apv\,dx = 0. \quad (4.42)$$

Now, let $f(u) = u^2$. Then, by Taylor's theorem we have

$$\begin{aligned}(u^h)^2 - u^2 - 2uy &= f(u^h) - f(u) - f'(u)y \\ &= f(u) + f'(u + \xi(u^h - u))(u^h - u) - f(u) - f'(u)y,\end{aligned} \quad (4.43)$$

for some $0 < \xi < 1$. Using that $y = u^h - u - p$ we obtain

$$\begin{aligned}(u^h)^2 - u^2 - 2uy &= \big[f'(u + \xi(u^h - u)) - f'(u)\big]\,(u^h - u) + f'(u)p \\ &= 2\xi(u^h - u)^2 + 2up\end{aligned} \quad (4.44)$$

Inserting this into (4.42) and using $y = u^h - u - p$ again we obtain

$$\int_\Omega p_t v\,dx + \int_\Omega \nabla p\cdot\nabla v\,dx + 2\int_\Omega upv\,dx + 2\xi\int_\Omega (u^h - u)^2 v\,dx \\ + \int_\Omega \big[qp + h(u^h - u)\big]\,v\,dx - \int_\Omega apv\,dx = 0. \quad (4.45)$$

We now take $v = p$ as our test function. This yields

$$\int_\Omega p_t p\,dx + \int_\Omega |\nabla p|^2 + p^2\,dx + 2\int_\Omega up^2\,dx + \int_\Omega qp^2\,dx \\ = -2\xi\int_\Omega (u^h - u)^2 p\,dx - \int_\Omega h(u^h - u)p\,dx + \int_\Omega (a+1)p^2\,dx, \quad (4.46)$$

where we have added an extra quadratic term on both sides. For the right hand side we have

$$
\begin{aligned}
&-2\xi\int_\Omega (u^h-u)^2 p\,dx - \int_\Omega h(u^h-u)p\,dx \int_\Omega (a+1)p^2\,dx \\
&\qquad \le \xi\int_\Omega (u^h-u)^4\,dx + \xi\int_\Omega p^2\,dx + \frac{1}{2}\int_\Omega h^2(u^h-u)^2\,dx \\
&\qquad\qquad + \frac{1}{2}\int_\Omega p^2\,dx + \|a+1\|_{L^\infty(\Omega)}\int_\Omega p^2\,dx \\
&\qquad\qquad = \left[\xi + \frac{1}{2} + \|a+1\|_{L^\infty(\Omega)}\right]\|p\|^2_{L^2(\Omega)} \\
&\qquad\qquad\quad + \xi\|u^h-u\|^4_{L^4(\Omega)}\,dx + \frac{1}{2}\|h\|^2_{L^4(\Omega)}\|u^h-u\|^2_{L^4(\Omega)}
\end{aligned} \tag{4.47}
$$

By nonnegativity of all terms on the left hand side in (4.46) except for the first, we deduce

$$
\int_\Omega p_t p\,dx \le C_1\|p\|^2_{L^2(\Omega)} + C_2\|u^h-u\|^4_{L^4(\Omega)} + C_3\|h\|^2_{L^4(\Omega)}\|u^h-u\|^2_{L^4(\Omega)}, \tag{4.48}
$$

for appropriate constants C_1, C_2, C_3. Note that $\partial_t(p^2/2) = p_t p$ and that $p(0) = u^h(0) - u(0) - y(0) = u_0 - u_0 - 0 = 0$. Consequently, by integrating from 0 to t we obtain

$$
\begin{aligned}
\|p(t)\|^2_{L^2(\Omega)} &\le C_1\int_0^t \|p\|^2_{L^2(\Omega)}\,dt + C_2\int_0^t \|u^h-u\|^4_{L^4(\Omega)}\,dt \\
&\quad + C_3\int_0^t \|h\|^2_{L^4(\Omega)}\|u^h-u\|^2_{L^4(\Omega)}\,dt \\
&\le C_1\int_0^t \|p\|^2_{L^2(\Omega)}\,dt + C_2\|u^h-u\|^4_{L^4(0,T;L^4(\Omega))} \\
&\quad + C_3\|h\|^2_{L^4(Q)}\|u^h-u\|^2_{L^4(Q)}.
\end{aligned} \tag{4.49}
$$

Now, we apply Gronwall's inequality to find that for some constant $\hat{C}$ we have

$$
\max_{0<t<T}\|p(t)\|^2_{L^2(\Omega)} \le \hat{C}\left(\|u^h-u\|^4_{L^4(Q)} + \|h\|^2_{L^4(Q)}\|u^h-u\|^2_{L^4(Q)}\right). \tag{4.50}
$$

We now note that we have

$$
\|h\|_{L^4(Q)} \le A\|h\|_{L^\infty(Q)}, \tag{4.51}
$$

for some constant A. From theorem 4.2.2 we get

$$
\|u^h-u\|_{L^4(Q)} \le B\|h\|_{L^2(Q)} \le \bar{B}\|h\|_{L^\infty(Q)}, \tag{4.52}
$$

for constants B and $\bar{B}$.

Using (4.51) and (4.52), we get by using (4.50) that

$$
\|p\|_{L^\infty(0,T;L^2(\Omega))} = o\left(\|h\|_{L^\infty(Q)}\right). \tag{4.53}
$$

To obtain the next bound we return to (4.46) where we integrate over the time domain. By (4.46) and (4.47), we deduce that

$$\int_0^T \|p\|^2_{H^1(\Omega)} \leq C_1 \int_0^t \|p\|^2_{L^2(\Omega)}\,dt + C_2\|u^h - u\|^4_{L^4(Q)} \\ + C_3\|h\|^2_{L^4(Q)}\|u^h - u\|^2_{L^4(Q)}. \tag{4.54}$$

Using (4.51), (4.52) and (4.53), we infer that

$$\|p\|_{L^2(0,T;H^1(\Omega))} = o\left(\|h\|_{L^\infty(Q)}\right). \tag{4.55}$$

For the final bound let v be any function in $H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \leq 1$. By (4.45) we have

$$\begin{aligned}\left|\int_\Omega p_t v\,dx\right| &\leq \left|\int_\Omega \nabla p\cdot\nabla v\,dx\right| + 2\left|\int_\Omega upv\,dx\right| + 2\xi\left|\int_\Omega (u^h-u)^2 v\,dx\right| \\ &+ \left|\int_\Omega apv\,dx\right| + \left|\int_\Omega qpv\,dx\right| + \left|\int_\Omega h(u^h-u)v\,dx\right| \\ &\leq \|\nabla p\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + 2\|u\|_{L^\infty(Q)}\|p\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &+ 2\xi\|u^h-u\|^2_{L^4(\Omega)}\|v\|_{L^2(\Omega)} + \|a\|_{L^\infty(\Omega)}\|p\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &+ \|q\|_{L^\infty(Q)}\|p\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|h(u^h-u)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \|p\|_{H^1(\Omega)} + 2\|u\|_{L^\infty(Q)}\|p\|_{H^1(\Omega)} \\ &+ 2\xi\|u^h-u\|^2_{L^4(\Omega)} + \|a\|_{L^\infty(\Omega)}\|p\|_{H^1(\Omega)} \\ &+ \|q\|_{L^\infty(Q)}\|p\|_{H^1(\Omega)} + \|h\|_{L^4(\Omega)}\|u^h-u\|_{L^4(\Omega)},\end{aligned} \tag{4.56}$$

where we have used the Hölder inequality. Thus, for some constant $\tilde{C}$ it must hold that

$$\|p_t\|^2_{H^{-1}(\Omega)} \leq \tilde{C}\left(\|p\|^2_{H^1(\Omega)} + \|u^h-u\|^4_{L^4(\Omega)} + \|h\|^2_{L^4(\Omega)}\|u^h-u\|^2_{L^4(\Omega)}\right). \tag{4.57}$$

Integrating in time gives

$$\begin{aligned}\|p_t\|^2_{L^2(0,T;H^{-1}(\Omega))} &\leq \tilde{C}\Bigg(\|p\|^2_{L^2(0,T;H^1(\Omega))} + \|u^h-u\|^4_{L^4(Q)} \\ &+ \int_0^T \|h\|^2_{L^4(\Omega)}\|(u^h-u)\|^2_{L^4(\Omega)}\,dt\Bigg) \\ &\leq \tilde{C}\big(\|p\|^2_{L^2(0,T;H^1(\Omega))} + \|u^h-u\|^4_{L^4(Q)} \\ &+ \|h\|^2_{L^4(Q)}\|u^h-u\|^2_{L^4(Q)}\big).\end{aligned} \tag{4.58}$$

By (4.51), (4.52) and (4.55) we conclude that

$$\|p_t\|_{L^2(0,T;H^{-1}(\Omega))} = o\left(\|h\|_{L^\infty(Q)}\right). \tag{4.59}$$

Thus, it holds that

$$\|p\|_{W(0,T)} = o\left(\|h\|_{L^\infty(Q)}\right), \tag{4.60}$$

and so the definition of Fréchet differentiability is satisfied. □

4.3 Derivation of First Order Necessary Optimality Conditions

We need to be able to characterize a local optimum for the control problem when solving it numerically. In this section we will derive a fist order optimality system using the standard Lagrangian technique. To this end, we define the Lagrangian function as

$$\begin{aligned}
\mathcal{L}(u,p,q) &= J(u,q) - \int_0^T \int_\Omega u_t p \, dx \, dt - \int_0^T \int_\Omega \nabla u \cdot \nabla p \, dx \, dt \\
&- \int_0^T \int_\Omega \left(u^2 - au\right) p \, dx \, dt - \int_0^T \int_\Omega qup \, dx \, dt \\
&= \int_0^T e^{-\rho t} \int_\Omega q(r-u) \, dx \, dt - \lambda \int_\Omega u(T) \, dx + \frac{\alpha}{2} \int_0^T \int_\Omega q^2 \, dx \, dt \\
&- \int_0^T \int_\Omega u_t p \, dx \, dt - \int_0^T \int_\Omega \nabla u \cdot \nabla p \, dx \, dt \\
&- \int_0^T \int_\Omega \left(u^2 - au\right) p \, dx \, dt - \int_0^T \int_\Omega qup \, dx \, dt
\end{aligned} \tag{4.61}$$

Here we have integrated over the time domain in the weak form (2.2), and subtracted the result from the cost functional. We will obtain the optimality system by taking Fréchet derivatives of the Lagrangian. We know that we can do this thanks to theorem 4.2.2.

To obtain the weak form of the adjoint equation, we differentiate the Lagrangian with respect to the state u in a direction h, and set this equal to zero. Doing this yields

$$\begin{aligned}
\mathcal{L}_u(u,p,q)h &= -\int_0^T \int_\Omega e^{-\rho t} qh \, dx \, dt - \lambda \int_\Omega h(T) \, dx - \int_0^T \int_\Omega h_t p \, dx \, dt \\
&- \int_0^T \int_\Omega \nabla h \cdot \nabla p \, dx \, dt - \int_0^T \int_\Omega (2uhp - ahp) \, dx \, dt \\
&- \int_0^T \int_\Omega qhp \, dx \, dt \\
&= \int_0^T \int_\Omega h \left[-qe^{-\rho t} + p_t + \Delta p - 2up + ap - qp\right] \, dx \, dt \\
&+ \int_\Omega h(x,T) \left[-\lambda - p(x,T)\right] \, dx \, dt \\
&- \int_0^T \int_{\partial\Omega} h \frac{\partial p}{\partial \nu} \, dx \, dt \\
&= 0,
\end{aligned} \tag{4.62}$$

where we have used integration by parts and assumed $h(x,0) = 0$. From this we deduce the strong form of the adjoint equation

$$\begin{aligned} -p_t - \Delta p + (2u - a + q)\, p &= -qe^{-\rho t} && \text{in } (0,T) \times \Omega, \\ \nabla p \cdot n &= 0 && \text{on } (0,T) \times \partial\Omega, \\ p(x,T) &= -\lambda && \text{in } \Omega, \end{aligned} \tag{4.63}$$

where p is the adjoint variable. Using a similar approach as the one we used for the state equation it is possible to show that the adjoint equation has a unique solution $p \in W(0,T)$. This proof is omitted here.

Next, we derive the variational inequality by taking the derivative of the Lagrangian with respect to the control q in a direction h We obtain

$$\begin{aligned} \mathcal{L}_q(u,p,q)h &= \int_0^T \int_\Omega e^{-\rho t}\,(r-u)\,h\,dx\,dt + \alpha \int_0^T \int_\Omega qh\,dx\,dt \\ &\quad - \int_0^T \int_\Omega hup\,dx\,dt \\ &= \int_0^T \int_\Omega h\left[e^{-\rho t}\,(r-u) + (\alpha q - up)\right]\,dx\,dt. \end{aligned} \tag{4.64}$$

Let $h = q - \bar{q}$, and $(\bar{u}, \bar{p}, \bar{q})$ be a solution triple to the control problem. We then infer that the variational inequality reads

$$\int_0^T \int_\Omega (q - \bar{q})\left[e^{-\rho t}\,(r - \bar{u}) + (\alpha\bar{q} - \bar{u}\bar{p})\right]\,dx\,dt \geq 0, \quad \forall q \in \mathcal{Q}_{ad}. \tag{4.65}$$

Furthermore, we note that the gradient of the reduced cost functional can be expressed as

$$j'(q) = e^{-\rho t}\,(r-u) + (\alpha q - up)\,. \tag{4.66}$$

In summary, for the control problem (2.7) a first order necessary optimality condition for a solution triple $(\bar{u}, \bar{p}, \bar{q})$ is given by the variational inequality (4.65), where the variables solve the state equation (2.1) as well as the adjoint equation (4.63).

Chapter 5

Numerical Simulations

In this chapter we present a numerical method to solve the weak equation (2.2) using Rothe's method. We implement the method using the FEniCS library for Python. Specifically, we have used FEniCS version 1.5, which is presented in Alnæs et al. [1]. An extensive introduction to the FEniCS library can be found in Logg et al. [14]. For simplicity we have taken the domain Ω to be the unit square in the numerical simulations. Some examples of solutions to the state equation are presented.

Next, we describe how we have discretized the cost functional (2.5). This is also done with the help of functions available in the FEniCS library. We then present the projected gradient method, which is used to solve the control problem (2.7) numerically. We end by showing some of the results we have obtained using these methods, and illustrate how they could be useful in designing marine policy.

5.1 Discretization of the State Equation

In order to solve the state equation numerically we have chosen to follow the horizontal method of lines (also known as Rothe's method) [11], in which we begin by discretizing the time variable. This yields a series of elliptic problems, which we can solve using the finite element method. The reason we have chosen this method over the vertical method of lines, outlined in Knabner and Angermann [13], is due to ease of implementation FEniCS, since this library makes solving the elliptic PDEs arising from the time discretization of (2.1) quite simple. We begin by describing the discretization in time.

5.1.1 Time Discretization of the State Equation

In this work we have chosen to do the time discretization according to the Backward Euler method. This choice of method has been made due to its simplicity and desirable stability properties (see [16, 18]).

We divide the interval $[0, T]$ into N subintervals of equal length, with $N+1$ equidistant nodes $\{t^n\}_{n=0}^{N}$ with internodal distance $\Delta t = T/N$. In the following we let a superscript denote the time at which a quantity is measured, e.g. $u^n = u(x, t^n)$. When using the Backward Euler method on an equation of the form

$$y_t + f(t, y) = 0, \tag{5.1}$$

one obtains the discretized equation

$$\frac{y^{n+1} - y^n}{\Delta t} + f(t^{n+1}, y^{n+1}) = 0. \tag{5.2}$$

Thus, a Backward Euler discretization of the weak equation (2.2) yields

$$\int_\Omega \frac{u^{n+1} - u^n}{\Delta t} v \, dx + \int_\Omega \nabla u^{n+1} \cdot \nabla v \, dx + \int_\Omega u^{n+1} \left(u^{n+1} + q^{n+1} - a \right) v \, dx = 0. \tag{5.3}$$

In order to obtain a linear equation, which is easy to solve with the FEniCS software, we have approximated the quadratic term by $\left(u^{n+1}\right)^2 \approx u^{n+1}u^n$, which is reasonable for sufficiently large values of N. In doing so we have reduced the parabolic equation to N linear elliptic equations of the form

$$\int_\Omega u^{n+1} v \, dx + \Delta t \int_\Omega \nabla u^{n+1} \cdot \nabla v + u^{n+1} \left(u^n + q^{n+1} - a \right) v \, dx = \int_\Omega u^n v \, dx, \tag{5.4}$$

where $u^0 = u_0(x)$. In the next section we describe how to solve these equations numerically.

5.1.2 Solving the Elliptic Equations

The functions u^{n+1} now depend only on the spatial variable. For simplicity we will write $u := u^{n+1}$ and consider functions u^n from previous time steps to be known. We want $u \in H^1(\Omega)$, but in the numerical implementation we will have to look for a finite dimensional approximation. Let u^h be an approximation of u, lying in a finite dimensional space X_h with $X_h \subset H^1(\Omega)$. In order to describe such a solution we begin by specifying X_h. Let $\{\varphi_k\}_{k=1}^\infty$ be a basis for $H^1(\Omega)$. Then we define

$$X_h := \left\{ v \in H^1(\Omega) \middle| v = \sum_{i=1}^{M} v_i(t)\varphi_i(x) \right\}, \tag{5.5}$$

for some integer M. In our implementation we have taken the functions $\{\varphi_k\}_{k=1}^M$ to be nodal polynomial basis functions of degree one, on triangular elements in the unit square. We write

$$u^h = \sum_{i=1}^{M} u_i \varphi_i(x), \tag{5.6}$$

for coefficients u_i. As test function we choose

$$v^h = \sum_{i=1}^{M} \varphi_i(x). \tag{5.7}$$

Inserting this into (5.4) gives

$$\begin{aligned}\sum_{i,j} u_i \int_\Omega \varphi_i\varphi_j\,dx + \Delta t \sum_{i,j} u_i \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j + \left(u^n + q^{n+1} - a\right)\varphi_i\varphi_j\,dx \\ = \sum_j \int_\Omega u^n \varphi_j\,dx.\end{aligned} \tag{5.8}$$

Define now the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{A}$, the matrix $\mathbf{Z}$ as well as the load vector $\boldsymbol{F}$ as

$$M_{i,j} = \int_\Omega \varphi_i\varphi_j\,dx, \qquad A_{i,j} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx, \tag{5.9}$$

$$Z_{i,j} = \int_\Omega \left(u^n + q^{n+1} - a\right)\varphi_i\varphi_j\,dx, \qquad F_i = \int_\Omega u^n\varphi_i\,dx. \tag{5.10}$$

We can now write (5.8) as

$$\left[\mathbf{M} + \Delta t\left(\mathbf{A} + \mathbf{Z}\right)\right]\boldsymbol{u} = \boldsymbol{F}, \tag{5.11}$$

where $\boldsymbol{u} = [u_1, \ldots, u_M]^T$. This is a linear system which can be solved easily using a range of well known methods. The implementation of this, as well as the computation of $\mathbf{M}$, $\mathbf{A}$, $\mathbf{Z}$ and $\boldsymbol{F}$, is done by FEniCS.

In the following subsections we present some results obtained by solving the state equation using this method. Each example presents a different scenario, which can be interpreted to represent some real world situation. In all cases the domain has been taken to be the unit square in two dimensions.

5.1.3 An Example with a Stationary Fishing Vessel

In the first simulation we have taken the initial value to be a Gaussian function with its maximum located in the lower left corner of the domain. Specifically, we have set

$$u_0(x) = A\exp\left(-B((x_1 - 0.25)^2 + (x_2 - 0.25)^2)\right), \tag{5.12}$$

where x_1 and x_2 are the first and second coordinates of x, respectively. The parameters A and B were set to be $A = 1.0$, $B = 10.0$. This indicates that the distribution of the fish is initially centered around the point $(0.25, 0.25)$, and other parts of the domain have very few fish.

The function $a(x)$ specifies the threshold for the logistic growth of the fish population. In this example we have taken it to be the Gaussian function

$$a(x) = C\exp\left(-D((x_1 - 0.75)^2 + (x_2 - 0.75)^2)\right) + 0.5, \tag{5.13}$$

where we set $C = 1.0$ and $D = 20.0$. A plot of this function can be seen in figure 5.1 The intention here was to simulate a case where the upper right corner of the domain has a more favorable environment for the fish than other parts of the domain. This could for example be due to a higher availability of food in some areas.

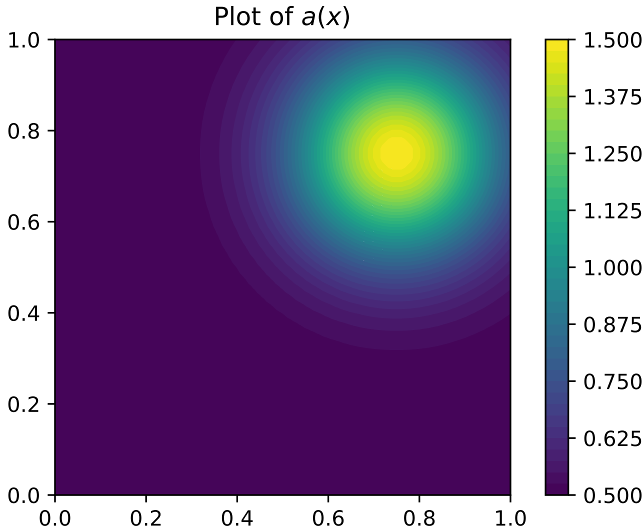

Figure 5.1: A plot of the logistic growth threshold a used in the first and second examples.

For this example we took the control to be the stationary Gaussian function

$$q(x) = E \exp\left(-F((x_1 - 0.5)^2 + (x_2 - 0.5)^2)\right), \tag{5.14}$$

with $E = 1.5$ and $F = 50.0$. A plot of this function can be seen in figure 5.2. This could indicate the presence of a fishing fleet in the middle of the domain. Then end time was set to be $T = 1.0$ in this example.

The results of this simulation can be seen in figure 5.3. We observe that the majority of the fish biomass moves from the lower left corner of the domain to the upper right corner, while there is hardly any fish located in the center. Clearly, there is less fish where the fishing effort is higher. This is a type of behavior for the fish that we would expect given the setup of this experiment.

5.1.4 An Example with a Moving Fishing Vessel

In the second example we took u_0 and a to be as in (5.12) and (5.13), respectively. However, we now study the effect of a time dependent control on the fish biomass. At each time t we still have a Gaussian control, but its center moves around the domain as the time increases. For this simulation we chose

$$\begin{aligned} q(x,t) = E \exp\Big[- F\Big(&(x_1 - (0.5 - 0.25\sin(3.14t)))^2 \\ &+ (x_2 - (0.5 - 0.25\cos(3.14t)))^2 \Big) \Big], \end{aligned} \tag{5.15}$$

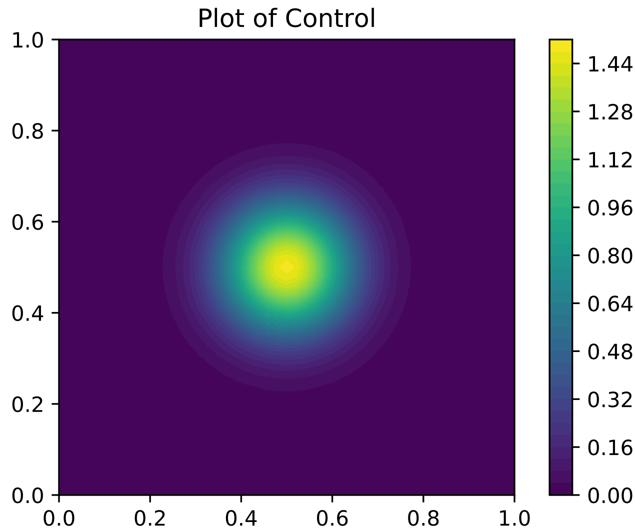

Figure 5.2: A plot of the stationary control used in the first example.

where E and F are as before. This could describe a fishing fleet moving clockwise on a circular path in the domain. Again we took the end time to be $T = 1.0$.

The results of this simulation can be seen in figure 5.4. The bottom four panels show how the fishing fleet moves around the domain as time passes. The top four panels show the corresponding states. We clearly observe the combined effect of the moving fishing fleet and the higher availability of food in the top right corner. This causes the fish to be relocated to the area with more suitable living conditions, as determined by the growth threshold a.

5.1.5 An Example with Two Moving Fishing Vessels

In the final example we add a second vessel to our fleet, to study a setting with more than one vessel doing the harvest. To this end, we take the control to be the sum of two Gaussian functions which are both time dependent. We chose the function

$$\begin{aligned} q(x,t) = E \exp \Big[&- F\Big((x_1 - (0.5 - 0.25\sin(3.14t)))^2 \\ &+ (x_2 - (0.5 - 0.25\cos(3.14t)))^2 \Big) \Big] \\ &+ E \exp \Big[- F\Big((x_1 - (0.5 + 0.25\sin(3.14t)))^2 \\ &+ (x_2 - (0.5 + 0.25\cos(3.14t)))^2 \Big) \Big], \end{aligned} \tag{5.16}$$

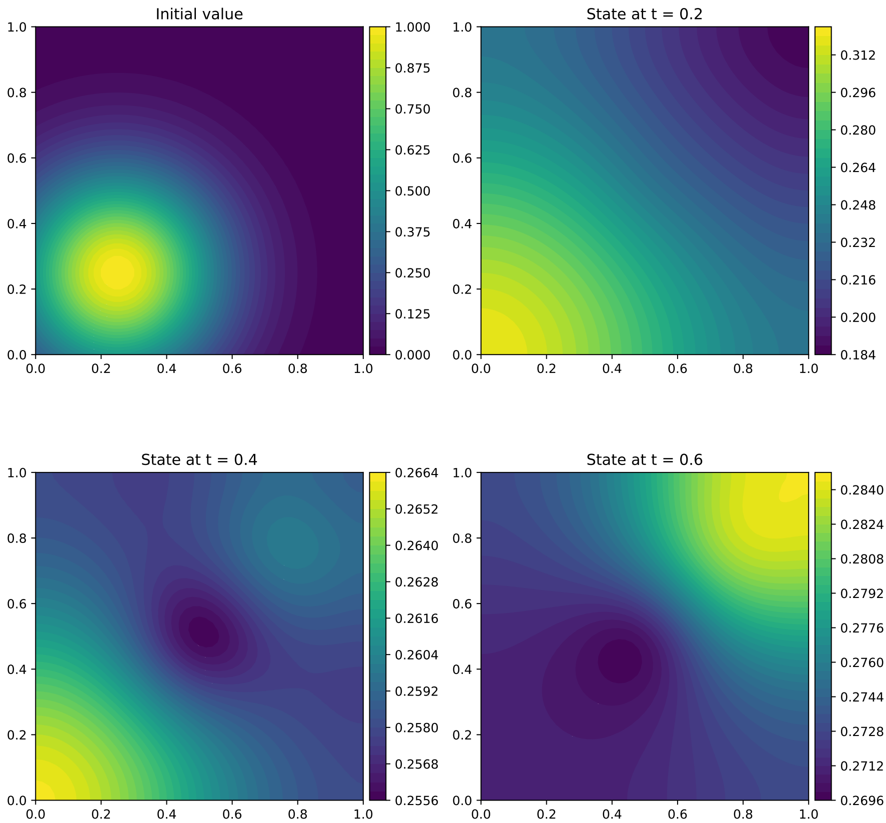

Figure 5.3: Plots of the solution to the state equation for the first example with a stationary control. Solutions are shown at times $t \in \{0.0, 0.2, 0.4, 0.6\}$

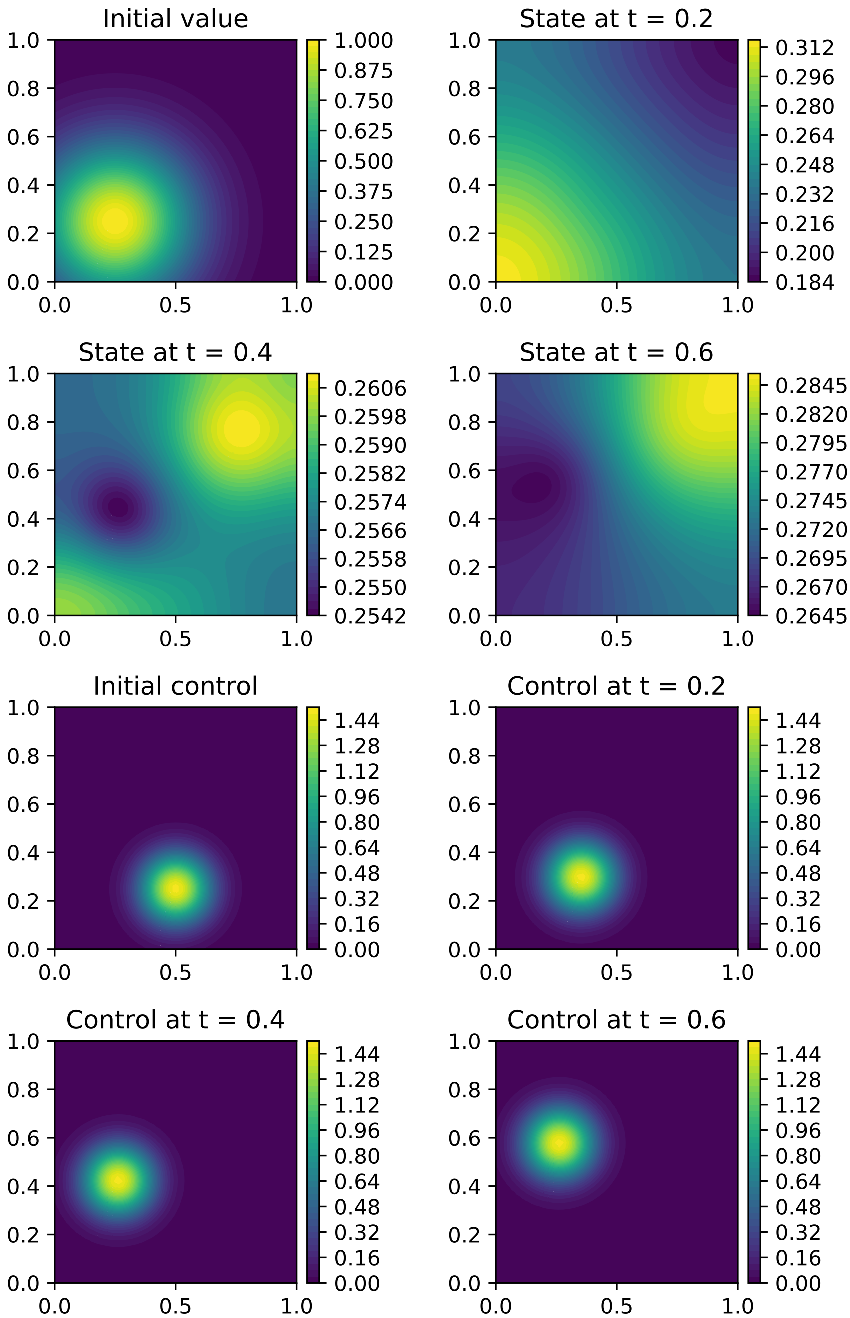

Figure 5.4: Plots of states and control for the second example. The top four panels show the state at times $t \in \{0.0, 0.2, 0.4, 0.6\}$. The bottom four panels show the control at corresponding times.

where, again, E and F are as before. For the initial value and the growth threshold we chose $u_0(x) = 0.7$ and $a(x) = 10.0$, and end time was set to be $T = 1.0$ in this example as well.

The results can be seen in figure 5.5. The bottom four panels show the movement of the fishing vessels as time passes. In the top four panels we clearly see their impact on the distribution of the fish biomass. However, we note that due to a very high value of a, the population grows rapidly in all parts of the domain. The effect of the harvest could be increased by making q and a be closer in magnitude. This could be done by e.g. increasing the value of E in the expression for q. The main point is, however, that as the fishing fleet moves around the domain, the biomass is depleted in the appropriate places.

5.2 Solving the Control Problem Numerically

In this section we describe how the control problem can be solved numerically. We begin by showing how the cost functional can be discretized. Next, we present a projected gradient method for finding a solution to the problem (4.3).

5.2.1 Discretization of the Cost Functional

Although the cost functional is not used directly in the projected gradient method, it is used in order to determine the step lengths for the descent. Therefore, it is necessary to be able to compute its function values. Consequently, we need to discretize it.

We divide the time interval in the same way as we did for the state equation. The state u is also represented in the same way as was done previously. Furthermore, the functions q and r are for each $t^n, n = 0, \dots N$, projected onto the finite dimensional space X_h defined in (5.5), i.e. these functions get represented in a similar manner as u.

We approximate the time integrals in the cost functional by a simple trapezoidal rule. This choice was made due to simplicity of implementation. The resulting approximation to the cost functional becomes

$$\begin{aligned} J(u,q) \approx \frac{\Delta t}{2} \sum_{n=0}^{N-1} \Bigg[\int_\Omega \left(e^{-\rho t^n} q^n \left(r^n - u^n \right) \right) \, dx \\ + \int_\Omega \left(e^{-\rho t^{n+1}} q^{n+1} \left(r^{n+1} - u^{n+1} \right) \right) \, dx \Bigg] \\ - \lambda \int_\Omega u^N \, dx + \frac{\alpha \Delta t}{4} \sum_{n=0}^{N-1} \int_\Omega \left(q^n \right)^2 + \left(q^{n+1} \right)^2 \, dx. \end{aligned} \tag{5.17}$$

The spatial integrals are approximated using Gaussian quadrature. This is done with integration routines available in FEniCS.

5.2.2 A Projected Gradient Method

Since we know the gradient of the reduced cost functional (4.2), we can use a gradient based method to solve (4.3). Because our problem has box constraints, we will use a

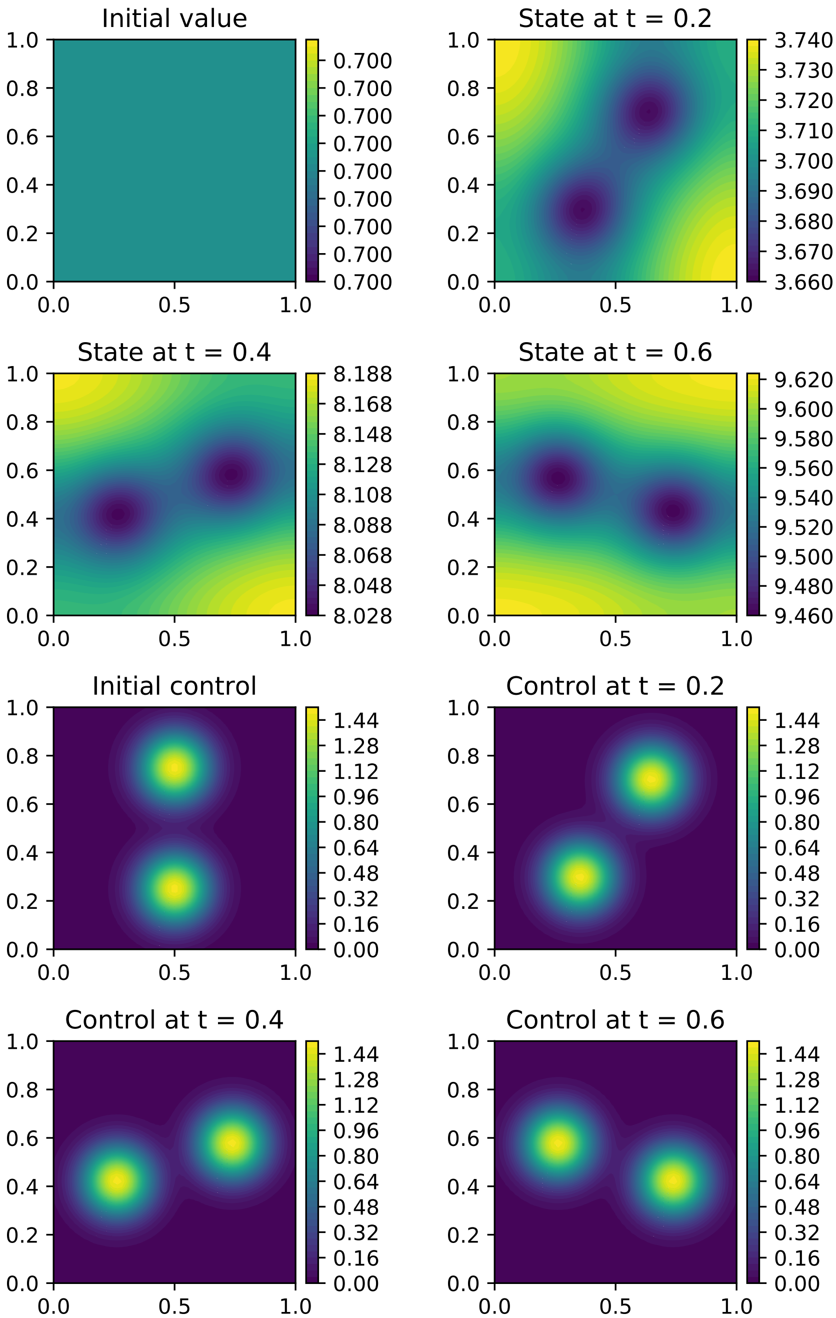

Figure 5.5: Plots of states and control for the third example. The top four panels show the state at times $t \in \{0.0, 0.2, 0.4, 0.6\}$. The bottom four panels show the control at corresponding times.

projected gradient method, as outlined in De los Reyes [4]. A description of the algorithm is given below:

Algorithm 1: Projected Gradient Method for Solving (4.3)

Result: Optimal control $\bar{q}$
Initialize $q_0 \in \mathcal{Q}_{ad}$, set $k = 0$;
while *Stopping criteria not met* **do**
  Compute u_k as solution to the state equation (2.1);
  Compute p_k as solution to the adjoint equation (4.63);
  Set $\nu_k = -j'(q_k)$;
  Compute step length s_k based on an Armijo condition;
  Set q_{k+1} to be the projection of $q_k + s_k\nu_k$ onto $\mathcal{Q}_{ad}$;
  Set $k = k + 1$;
end

In algorithm 1 we need to specify what stopping criteria we would like to use, how to compute the step length and how we project onto the admissible set.

Since we are likely to obtain optimal controls that lie on the boundary of the admissible set, we cannot expect the gradient of the reduced cost functional to be zero for these solutions. Therefore, this is not a useful stopping criterion. Instead, we will terminate our gradient search when there is a small relative change in the value of the reduced cost functional between two consecutive steps.

We compute the step length s_k for the descent based on the Armijo condition, see e.g. Nocedal and Wright [15]. Specifically, we start with a relatively large step size, e.g. set $s_k = 1$. Then we check whether the condition

$$j(q_k + s_k\nu_k) \leq j(q_k) - cs_k|\nu_k|^2, \tag{5.18}$$

holds. Here c is some constant with $c \in (0, 1)$. If it does not, we multiply s_k by some factor $\delta \in (0, 1)$ and check again. This is repeated until (5.18) is satisfied, or we reach a certain number of iterations. In our implementation we set the Armijo search to end after 10 iterations.

Our control problem has box constraints. Consequently, projection onto the admissible set is quite simple. We define the projection operator by

$$\mathcal{P}_{\mathcal{Q}_{ad}} : L^\infty(Q) \to \mathcal{Q}_{ad}, \quad q \mapsto \min\{q_{max}, \max\{0, q\}\}. \tag{5.19}$$

In the implementation the projection is done component-wise. That is, for each element in the vector representation of the function we check whether its value lies in $[0, q_{max}]$ or not. If it does not, we project it onto this interval.

In the following subsections we will present some of the results obtained using this method.

5.2.3 An Example with High Transportation Costs

In the first simulation we set the initial value for the state to be $u_0(x) = 0.7$. The growth parameter was also set to be a constant, with $a(x) = 0.5$. In the cost functional we took the interest rate to be $\rho = 0$, the sustainability constant $\lambda = 1.4$, and the regularization parameter $\alpha = 10^{-5}$. The function r, which describes the cost of moving around the domain, was set to be equal to the distance from the harbor. The location of the harbor was taken to be at the origin, i.e. we have $r(x) = \sqrt{x_1^2 + x_2^2}$. The upper bound on the control was set to be $q_{max} = 1.0$, and we initialized the control to have the value $q_0 = q_{max}/2$ everywhere in Ω. We set the end time for this simulation to be $T = 4.0$.

Figure 5.6 shows the value of the objective function for different iterations of the projected gradient algorithm. It is very clear that the first iteration causes the biggest change in the function value, while subsequent iterations only make minor adjustments. We also note that the second iteration actually increases the function value by a small amount. This is likely due to an unsuccessful Armijo search, which ended because the maximum number of iterations was reached. This increase in the function value is compensated for by subsequent iterations.

The results of the simulation can be seen in figure 5.7. The three columns show, from left to right, the plots of the state, the control and the adjoint variable at four different times. We observe that for all the times shown we get a bang-bang control. Clearly, the transportation cost is dominant in this example, since the control is only active in a quarter circle centered at the harbor. The value of λ is relatively high in this example, which emphasizes the importance of a large fish population at end time. In line with this, we observe that the size of the region with an active control decreases with time. Consequently, the population is still quite large after the harvesting is done.

These results can be translated into policy. One could place a No-Take Zone (NTZ) in the region where the initial control is inactive. Ideally one would make the boundary of this NTZ dynamic such that it at all times coveres the part of the domain where the control is zero. The part of the domain where the control is active at maximal capacity could be open to unrestricted harvesting. In practice, NTZs that change with time might be difficult to enforce. An alternative could be to impose a Total Allowable Catch (TAC) in this region, possibly splitting it into subregions with a decreasing number of catches allowed further away from the harbor.

5.2.4 An Example with Low Transportation Costs

We chose the same initial value for the state in this example as in the previous one. However, we changed the growth parameter a from being a constant to being a Gaussian function. We set

$$a(x) = G \exp\left(-H((x_1 - 0.3)^2 + (x_2 - 0.4)^2)\right), \tag{5.20}$$

with $G = 1.4$ and $H = 20.0$. This was done to study a situation where most of the food was located near the point $(0.3, 0.4)$. The parameters in the cost functional were all taken to be the same as in the first example, except for the function r, which was changed to be $r(x) = 0.01\sqrt{x_1^2 + x_2^2}$. Thus, in this example the harbor is located in the same point, but the cost of moving around the domain is significantly reduced. The maximal control was again set to be $q_{max} = 1.0$, and we initialized the control in the same way as described earlier. The end time for this simulation was also $T = 4.0$.

The results from this example can be seen in figure 5.8. The layout of this figure is the same as before. This time we do not get a bang-bang control. Rather, we get a seemingly continuous output, and we note that the control is active in all parts of the domain. We

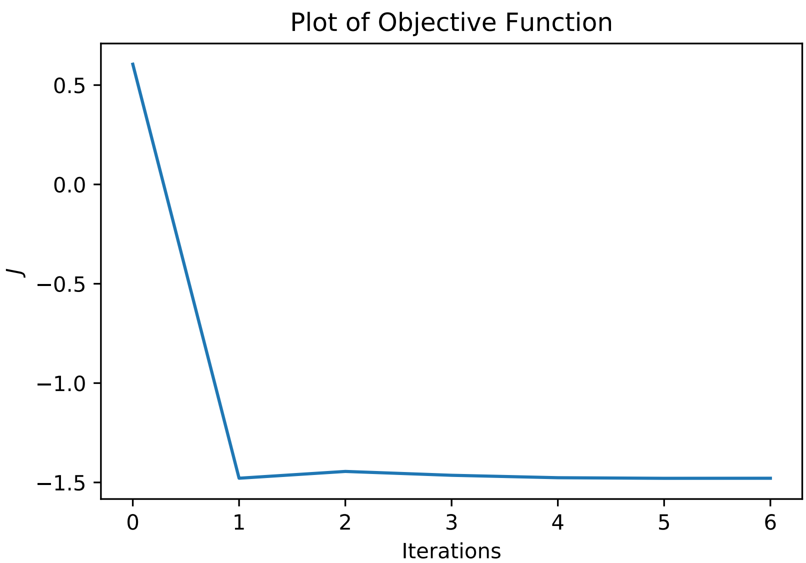

Figure 5.6: A plot of the values of the objective function at different iterations of the projected gradient algorithm.

clearly observe the effect of the reduced transportation costs. This is evident from the fact that the control initially has its highest values in areas that lie far away from the harbor. As time passes and the biomass is decreased in these areas, we see that the combination of less fish and high transportation costs make it inefficient to fish in these parts of the domain. Consequently, the highest harvesting effort is gradually shifted towards the harbor. We also observe here that for all times the control has its lowest values in the part of the domain where living conditions are most favorable for the fish. This suggests that a lower fishing effort here is crucial for the conservation of the species.

Translating a continuous control into policy is not as simple as for the bang-bang control we had in the previous example. Since the control is never zero in this example, it might not make sense to enforce NTZs. One possible solution could be to divide the domain into regions according to the level of the control in different areas. For example, parts of the domain where the control is in the interval $[0.37, 0.40]$ could form one region with a given TAC. Another region could have controls in the interval $[0.40, 0.43]$ and, therefore, a higher TAC. Since the control is time dependent these regions would potentially have to change over time. The feasibility of this would have to be evaluated by policymakers.

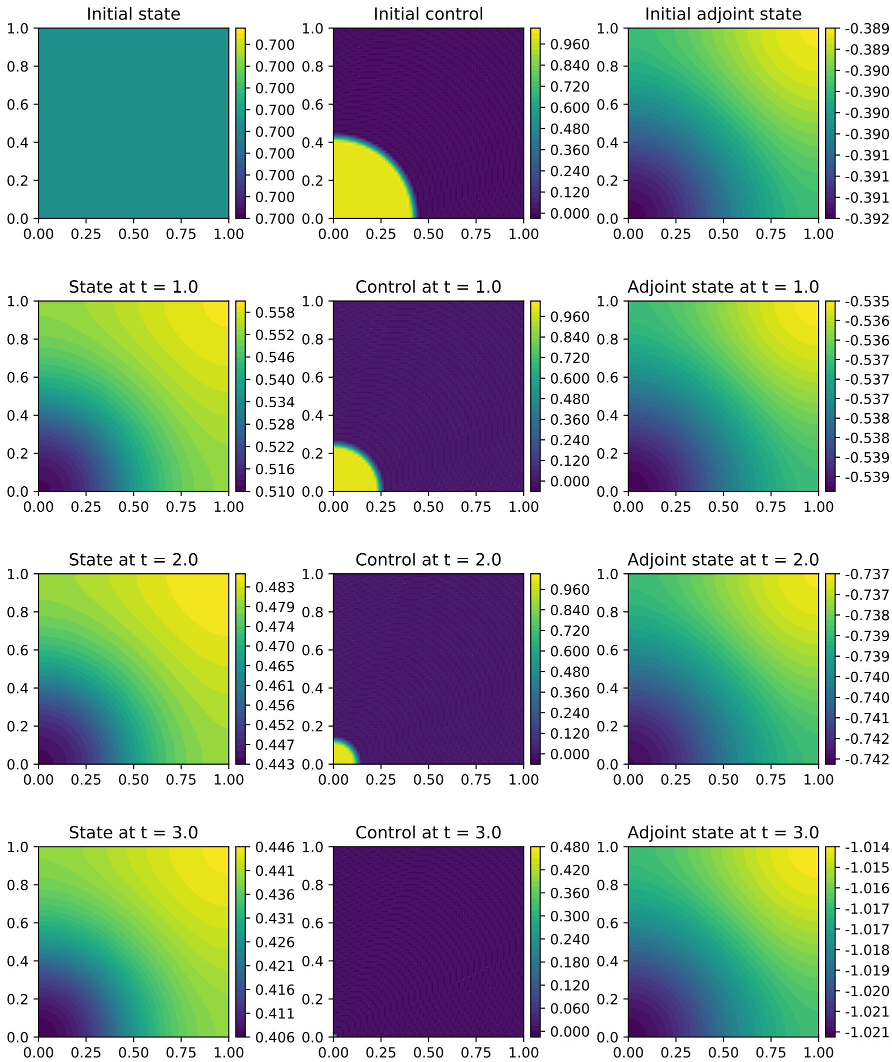

Figure 5.7: Plots of the results from the first example. The left column shows the state, the middle column shows the control and the right column shows the adjoint state. Each variable is plotted at times $t \in \{0.0, 1.0, 2.0, 3.0\}$.

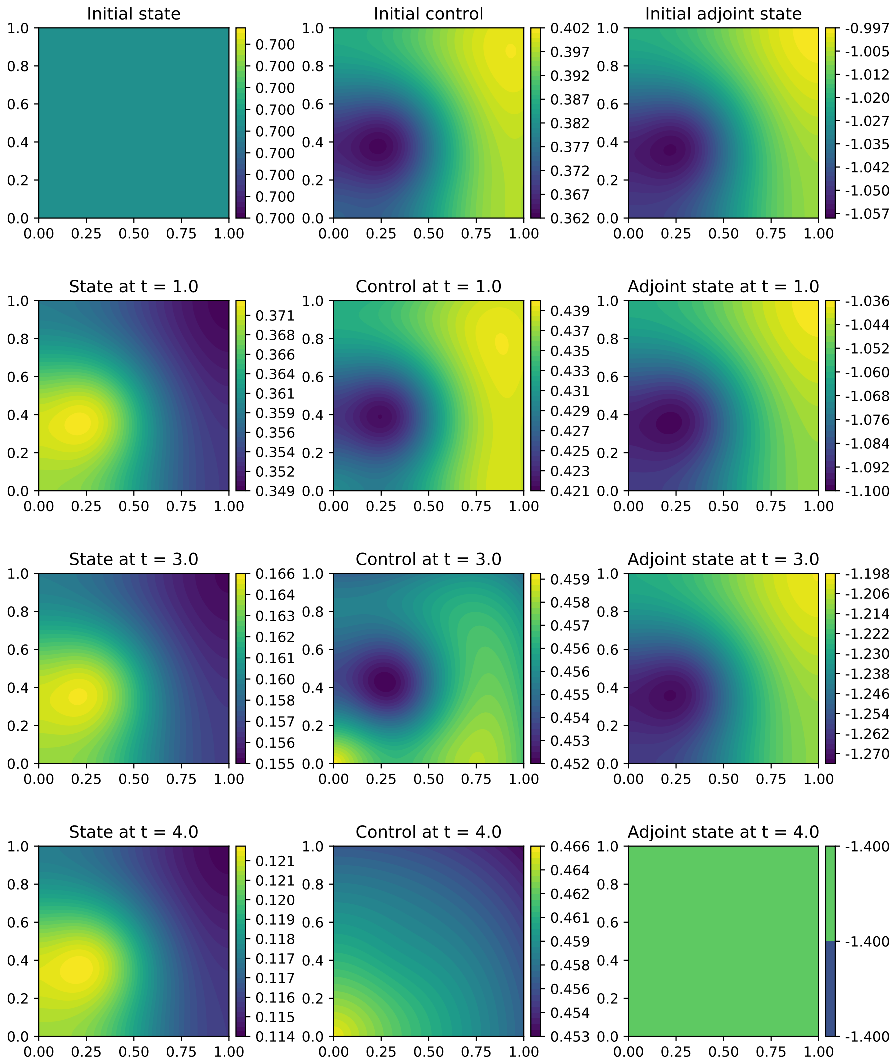

Figure 5.8: Plots of the results from the first example. The left column shows the state, the middle column shows the control and the right column shows the adjoint state. Each variable is plotted at times $t \in \{0.0, 1.0, 3.0, 4.0\}$.

Chapter 6

Conclusions

6.1 Discussion

In this work we have studied a control problem intended to describe fishing strategies in space and time. The cost functional related to this problem was designed such that a solution to the problem would be an optimal fishing strategy. Here, we mean optimal in the sense that one would get a large catch, while reducing costs and ensuring that the resource is not depleted when the harvesting is terminated.

The control problem is governed by a non-linear parabolic partial differential equation. In order to facilitate the analysis of this equation we introduced an auxiliary state equation. After modifying the auxiliary equation with a cut-off function, we were able to show the existence of a unique solution to this problem. We also showed that a solution to the auxiliary equation is nonnegative. After a simple transformation we were able to demonstrate that the same results hold for the state equation. In turn, this allowed us to define a solution operator, which we subsequently proved to be Fréchet differentiable.

After showing that the solution operator was well defined, we were able to restate the optimal control problem in terms of a reduced cost functional. We then established the existence of a solution to the control problem. The proof of this was carried out using a direct approach. We also derived a first order optimality system by using the Lagrangian technique. This system serves as a necessary optimality condition for a solution to the control problem.

Finally, we described how both the state equation and the control problem can be solved numerically. For the state equation we chose to use Rothe's method. Following this approach, we first did a semidiscretization in time, which resulted in a series of elliptic problems. These were solved using standard finite element techniques. A projected gradient method was used to solve the optimal control problem. We provided some examples for solutions to both the state equation and the control problem in a variety of different settings. The results for the control problem were given a possible interpretation in terms of feasible policies for the regulation of marine fisheries. We discovered that giving such an interpretation was rather straightforward in cases where the output was a bang-bang control. However, we learned that a continuous control does not as easily translate directly into policy. Still, it might be used as a guideline even in these cases.

6.2 Ideas for Further Work

An interesting next step for research in this area would be to incorporate real world data into the model. If one could obtain an estimate for a diffusion coefficient, as well as an estimate for the growth rate and carrying capacity of a fish population, one could potentially expand on this model to represent a real world setting.

In the theoretical analysis we allowed our spatial domain to be of either two or three dimensions. Therefore, second possible extension of this work would be to expand the numerical simulations to include three spatial dimensions. This might provide a more realistic picture for how the fish moves around the ocean, and could perhaps make it possible to capture differences in movement of the biomass in both shallow and deep waters.

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