Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Computing motivic homotopy classes on the projective line by algebro-geometric methods

Master's thesis in Applied Physics and Mathematics
Supervisor: Gereon Quick
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Summary

Let k be a field. We investigate an algebraic description of the set $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of $\mathbb{A}^1$-*homotopy classes* of pointed k-scheme endomorphisms of the projective line $\mathbb{P}^1$. Inspired by the methods of Cazanave in [6], we look for a group structure on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ from the Jouanoulou device $\mathcal{J}$ associated to $\mathbb{P}^1$. Since $\mathcal{J}$ is an affine k-scheme, a theorem of Asok–Hoyois–Wendt implies that the $[\mathcal{J}, \mathbb{P}^1]^N$ is isomorphic to $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Our main result is a new description of the set $[\mathcal{J}, \mathbb{P}^1]^N$ by use of concrete algebro-geometric methods avoiding the abstract $\mathbb{A}^1$-homotopy machinery.

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Chapter 1

Introduction

$\mathbb{A}^1$-homotopy theory, introduced by Morel and Voevodsky, gives a convenient framework to use homotopy theory in the setting of algebraic geometry. Morel and Voevodsky defined for a fixed field k the notion of homotopies of morphisms between smooth schemes over k. Thus, given two smooth schemes X and Y, the set $[X,Y]^{\mathbb{A}^1}$ of $\mathbb{A}^1$-homotopy classes of *pointed* morphisms from X to Y is well defined. However, the homotopy relation arises from an abstract construction which makes computation of homotopy classes of morphisms rather mysterious.

A potential starting point for $\mathbb{A}^1$-homotopy is the *naive homotopy*. It mimics the definition of homotopies from algebraic topology. However, since the unit interval, $[0,1]$, is not an algebraic variety, one replaces it by its algebraic analogue, the affine line $\mathbb{A}^1$.

Definition 1.0.1. Let X and Y be two smooth schemes over k. A *naive homotopy* is a morphism schemes

$$F : X \times \mathbb{A}^1 \longrightarrow Y.$$

The restriction $\sigma(F) := F_{|X\times\{0\}}$ is the source of the homotopy and $\tau(F) := F_{|X\times\{1\}}$ is its target. When X and Y have base points, say x_0 and y_0, we say that F is pointed if its restriction to $\{x_0\} \times \mathbb{A}^1$ is constant equal to y_0.

When $X = \operatorname{Spec}(R)$ for some ring R, and the homotopy can be expressed as an element F of some $R[T]$-module. We can think of the source homotopy as F evaluated at $T = 0$, and the target as F evaluated at $T = 1$.

With this definition we define the set $[X,Y]^N$ of pointed naive homotopy classes of morphisms from X to Y as the quotient of the set of pointed morphisms from X to Y with the equivalence relation generated by pointed naive homotopies.

Consider a base field k, and let $\mathbb{P}^n = \mathbf{Proj}(k[x_0,\dots,x_n])$ denote projective n-space as a k-scheme. Cazanave's paper [6] computes the set of naive homotopy classes $[\mathbb{P}^1,\mathbb{P}^1]^N$ of pointed k-scheme endomorphisms and finds that the set $[\mathbb{P}^1,\mathbb{P}^1]^N$ admits a monoid structure. In the end, he also proves that the group completion of the monoid $[\mathbb{P}^1,\mathbb{P}^1]^N$ coincides with the group $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$.

The group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ has been calculated by Morel using the machinery provided by $\mathbb{A}^1$-homotopy theory. The main goal of this thesis is to give an alternative description of this group making use of the following two observations:

1. In [3, Theorem 5.1.3] Asok, Hoyois and Wendt prove the following theorem:

   Theorem 1.0.2. For a smooth affine scheme X and for a smooth scheme Y satisfying some technical conditions, we have an isomorphism
   $$[X, Y]^N \cong [X, Y]^{\mathbb{A}^1}.$$
   The proof of this theorem is beyond the scope of this thesis and will therefore not be discussed. We only point out that the projective line $\mathbb{P}^1$ and the punctured affine plane $\mathbb{A}^2 \setminus \{0\}$ satisfy the technical conditions on the scheme Y in this theorem. Hence we can apply the theorem to $Y = \mathbb{P}^1$. However, $\mathbb{P}^1$ is not affine and therefore Theorem 1.0.2 does not apply to $X = \mathbb{P}^1$. Nevertheless, there is a well known trick that remedies this defect.

2. Associated to $\mathbb{P}^1$ there is a Jouanoulou device $\mathcal{J}$ defined as follows:
   $$\mathcal{J} := \operatorname{Spec}\Big(\frac{k[x, y, z, w]}{(x + w - 1, xw - yz)}\Big).$$
   The key point for us is that $\mathcal{J}$ is an affine scheme and the canonical morphism $\mathcal{J} \to \mathbb{P}^1$ is an $\mathbb{A}^1$-homotopy equivalence [10]. Hence, we obtain a chain of isomorphisms:
   $$[\mathcal{J}, \mathbb{P}^1]^N \cong [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1} \cong [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}.$$

The main achievement of the thesis is calculating $[\mathcal{J}, \mathbb{P}^1]^N$ in an algebro-geometric way, refraining from using as much $\mathbb{A}^1$-homotopy theory as possible. The work is inspired by Cazanave's approach in [6]. We first describe morphisms from $\mathcal{J}$ to $\mathbb{P}^1$ through the use of line bundles. We then try to find a group structure on $[\mathcal{J}, \mathbb{P}^1]^N$. We do get the following description of homotopy classes in $[\mathcal{J}, \mathbb{P}^1]^N$. Throughout this thesis we will assume all rings to be commutative with 1 and all fields to be perfect.

Theorem 1.0.3. The datum of a k-scheme morphism $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ up to naive homotopy is equivalent to an integer n and an element $(A, B) \in R^2$ where there exists $(U, V) \in R^2$ such that $AU + BV = 1$. A group structure on this description of $[\mathcal{J}, \mathbb{P}^1]^N$ can be created. However, describing which morphisms lie in which homotopy class, or find representatives of homotopy classes is difficult.

We would like to point out that the results on $[\mathcal{J}, \mathbb{P}^1]^N$ are original and have not appeared in the literature to the best of our knowledge.

1.0.1 Thesis structure

Chapter 2 covers background material needed to understand Cazanave's article [6]. In section §2.1, the Sylvester matrix and the Bézout form are described, connecting them both to the resultant of two polynomials. Section §2.2 covers some basic properties of bilinear forms and the Witt monoid.

In Chapter 3, we proceed with a literature review of the paper "Algebraic homotopy classes of rational functions" [6] by Cazanave. In section §3.1 naive homotopies are introduced and a description of $[\mathbb{P}^1, \mathbb{P}^1]^N$ as a set of homotopy classes of rational functions over a field k is made. In section §3.2 a monoid law $\oplus^N$ on the scheme of rational function $\mathcal{F}$, is defined. In section §3.3, the main result of Cazanave's paper is treated. The monoid of pointed rational functions is connected to the monoid of symmetric non-degenerate bilinear forms through the Bézout map from section §2.1. The main theorem 3.3.1 shows that this correspondence distinguishes exactly all homotopy classes of rational functions. Lastly, we use the result to compute some examples of $[\mathbb{P}^1, \mathbb{P}^1]^N$ over various fields.

Chapter 4 is the beginning of original material in this thesis. In §4.1 we prove some general properties of the scheme $\mathcal{J}$. Section §4.2 focuses on computing the line bundles of $\mathcal{J}$, and we get a description of the line bundles $\mathcal{P}_n$ and $\mathcal{Q}_n$. In section §4.3 we describe morphisms from $\mathcal{J}$ to $\mathbb{P}^1$. Theorem 4.3.1 gives the initial description, but throughout the section we establish several other equivalent conditions.

Chapter 5 covers the study of homotopy classes of morphisms from $\mathcal{J}$ to $\mathbb{P}^1$. In section §5.1, we examine some candidates for a group operation on $[\mathcal{J}, \mathbb{P}^1]^N$. In section §5.2 we make a conjecture that connects the naive homotopy classes of morphisms from $\mathcal{J}$ to $\mathbb{P}^1$ to the rational functions discussed in chapter 3.

In chapter 6 we study morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$. In §6.1 we explain why these morphisms are of interest. In §6.2 we describe morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$. We do it through scheme theory, but also present a way by using homotopy theory. In §6.3 we explain the connection between morphisms of degree 0 from $\mathcal{J}$ to $\mathbb{P}^1$ and morphims from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$. In §6.4 we turn our problem into a problem in $SL_2(R)$. A problem with morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$ is that it is difficult to determine if a morphism is homotopically trivial or not. In §6.5 we prove that certain morphisms are not homotopically trivial through the use of realization over the real numbers.

1.0.2 Acknowledgements

The results presented in this thesis have been obtained through joint work with my fellow student Viktor Balch Barth. I would like to thank my supervisors Gereon Quick and Glen Wilson for all the advice and discussions during the past six months.

Chapter 2

Resultants and bilinear forms

2.1 Resultants and Bézout relations

Determining when two polynomials are coprime can be done in many ways. A common way of doing it can be by using Euclid's algorithm to figure out what the greatest common divisor is. Another way of doing it is due to Sylvester.

Definition 2.1.1. Let R be an integral domain. Let $A, B \in R[x]$ where $n = deg(A) \geq deg(B)$. Write $A = \sum_{i=0}^{n} a_i x^i$ and $B = \sum_{j=0}^{n} b_i x^i$, where $b_i = 0$ when $i > deg(B)$. The Sylvester matrix $S(A, B)$ is the $2n \times 2n$ matrix given by

$$S_{ij} = \begin{cases} a_{n-i+j} & 0 \leq j \leq n-1, \\ b_{j-i} & n \leq j \leq 2n-1. \end{cases}$$

Define $\operatorname{res}(A, B) := \det S(A, B)$.

Sylvester proved the following theorem about the matrix S.

Theorem 2.1.2. We have $\operatorname{res}(A, B) \in R^{\times}$ if and only if A and B are coprime.

Proof. The matrix $S(A, B)$ corresponds to a linear map

$$\varphi : \mathcal{P}_{n-1} \times \mathcal{P}_{n-1} \longrightarrow \mathcal{P}_{2n-1}$$
$$\varphi(U, V) = AU + BV$$

where $\mathcal{P}_n$ is the $n+1$ dimensional vector space of polynomials of degree less than or equal to n. We have

$$\begin{aligned} \gcd(A, B) \notin R^{\times} &\iff \exists U, V \in \mathcal{P}_{n-1} \text{ such that } AU + BV = 0, \\ &\iff \text{nullspace of } S(A, B) \text{ is nontrivial}, \\ &\iff \operatorname{res}(A, B) \notin R^{\times}. \end{aligned}$$

□

To get a better intuition of how this works, we will calculate some examples. Consider the general case where $A = a_2x^2+a_1x+a_0$ and $B = b_2x^2+b_1x+b_0$ are two polynomials of degree 2 with arbitrary coefficients. The matrix $S(A, B)$ is then given by

$$S(A,B) = \begin{pmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{pmatrix}.$$

Now let us look at a couple of polynomials in $\mathbb{Z}[x]$. Let $A = x^2 + x$ and $B = x$. We can see that B divides A. We expect the determinant of $S(A, B)$ to be 0.

$$S(x^2+x,x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

which due to the whole bottom row being 0's makes it clear that $\operatorname{res}(x^2 + x, x) = \det S(x^2 + x, x) = 0$. Now for an example where A and B are coprime. Let $A = x^2$, $B = 1$. One can see that $S(A, B) = \mathbb{I}_4$, which has determinant 1, hence A and B are coprime.

Proposition 2.1.3. Let $A, B \in R[x]$ where $n = \deg(A) \geq \deg(B)$. If $\operatorname{res}(A, B) \in R^\times$, then there exist polynomials U, V of degree strictly less than n such that $1 = AU + BV$

Proof. $S(A, B)$ is invertible since $\det S(A, B) \in R^\times$. One can write

$$S(A,B)^{-1} = \frac{1}{\operatorname{res}(A,B)}\text{adj}(S(A,B)),$$

where adj$(S(A, B))$ is the adjugate matrix of $S(A, B)$, which is the transopose of the cofactor matrix of $S(A, B)$. Now

$$S(A,B)\text{adj}(S(A,B)) = \operatorname{res}(A,B)\mathbb{I}_{2n}.$$

Let y be the $(2n - 1)$th column vector of adj$(S(A, B))$. The vector y corresponds to two polynomials in $R[x]$ with degree strictly less than n. Multiplying y by the scalar $\operatorname{res}(A, B)$ gives the two desired polynomials U, V such that

$$AU + BV = 1.$$

□

Another way of calculating the resultant for a pair of polynomials is through the Bézout matrix. Before explaining how the matrix is constructed, we need the following result.

Proposition 2.1.4. Let $A, B \in R[X]$ then $X - Y$ divides $A(X)B(Y) - A(Y)B(X)$ in $R[X, Y]$.

Proof. Write $A = \sum_{i=0}^{n} a_i X^i$ and $B = \sum_{j=0}^{n} b_j X^j$, where $b_j = 0$ when $j > deg(B)$. We have

$$A(X)B(Y) - A(Y)B(X) = \sum_{\substack{i,j \leq n \\ i \neq j}} = a_i b_j (X^i Y^j - X^j Y^i).$$

If $(X - Y)$ divides each term in the sum, it is a divisor of the sum. When $i > j$ one can write

$$X^i Y^j - X^j Y^i = (XY)^j (X^{i-j} - Y^{i-j}).$$

Let $d = i - j$. In the case where $d = 1$, and we have

$$(XY)^j (X^d - Y^d) = (XY)^j (X - Y),$$

which is divisible by $(X - Y)$. In the case $d = 2$, $(X^2 - Y^2) = (X - Y)(X + Y)$, which is also divisible by $(X - Y)$. Assume that $(X - Y)$ is a divisor of $(X^k - Y^k)$ for all numbers less than d. Carrying out Euclid's algorithm yields

$$(X^d - Y^d) : (X - Y) = X^{d-1} + Y^{d-1} + XY \frac{X^{d-2} - Y^{d-2}}{X - Y}.$$

By assumption, $(X^{d-2} - Y^{d-2})$ is divisible by $(X - Y)$ and so the the claim holds by induction. $\square$

Definition 2.1.5. Let A and B be two polynomials and $n = \max(\deg A, \deg B)$. The Bézout matrix of A and B denoted $\text{Béz}(A, B)$ is the symmetric matrix given by the coefficients of the polynomial

$$\delta_{A,B}(X.Y) := \frac{A(X)B(Y) - A(Y)B(X)}{X - Y} =: \sum_{0 \leq p,q \leq n-1} c_{p,q} X^p Y^q.$$

$\text{Béz}(A, B)$ is the $(n \times n)$ symmetric matrix $[c_{p,q}]_{0 \leq p,q \leq n-1}$.

It can be shown [4] that the Bézout matrix can be written as

$$\text{Béz}(A, B) = \begin{pmatrix} a_1 & \dots & a_n \\ \vdots & \iddots & \\ a_n & & 0 \end{pmatrix} \begin{pmatrix} b_0 & \dots & b_{n-1} \\ & \ddots & \vdots \\ 0 & & b_0 \end{pmatrix} - \begin{pmatrix} b_1 & \dots & b_n \\ \vdots & \iddots & \\ b_n & & 0 \end{pmatrix} \begin{pmatrix} a_0 & \dots & a_{n-1} \\ & \ddots & \vdots \\ 0 & & a_0 \end{pmatrix}.$$

The coefficients in $\text{Béz}(A, B)$ can then be calculated

$$c_{p,q} = \sum_{k=0}^{\min(p,n-1-q)} a_{q+k+1} b_{p-k} - a_{p-k} b_{q+k+1}.$$

In the case where A is a monic polynomial and $\deg(A) > \deg(B)$, we have the following result.

Proposition 2.1.6. Let A be a monic polynomial of degree n and B be a polynomial with degree strictly lower than A. Then $\det \text{Béz}(A, B) = (-1)^{\frac{n(n-1)}{2}} \text{res}(A, B)$.

Proof. The Sylvester matrix $S(A, B)$ can be broken down into block matrices

$$S(A,B) = \begin{pmatrix} \mathcal{A}^- & \mathcal{B}^- \\ \mathcal{A}^+ & \mathcal{B}^+ \end{pmatrix}.$$

Where the matrices $\mathcal{A}^-$ and $\mathcal{A}^+$ are as follows:

$$\mathcal{A}^- = \begin{pmatrix} a_n & & 0 \\ \vdots & \ddots & \\ a_1 & \dots & a_n \end{pmatrix}, \qquad \mathcal{A}^+ = \begin{pmatrix} a_0 & \dots & a_{n-1} \\ & \ddots & \vdots \\ 0 & & a_0 \end{pmatrix}.$$

$\mathcal{B}^-$ and $\mathcal{B}^+$ are defined similarly. Since A is monic, we have $\det \mathcal{A}^- = 1$. Since $b_n = 0$, the determinant of $\mathcal{B}^-$ is 0. We wish to reduce $S(A, B)$ to a lower diagonal form. Since multiplication of lower triangular matrices commute, we can do it by

$$\begin{pmatrix} \mathcal{A}^- & \mathcal{B}^- \\ \mathcal{A}^+ & \mathcal{B}^+ \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & \mathcal{B}^- \\ 0 & -\mathcal{A}^- \end{pmatrix} = \begin{pmatrix} \mathcal{A}^- & 0 \\ \mathcal{A}^+ & \mathcal{A}^+\mathcal{B}^- - \mathcal{B}^+\mathcal{A}^- \end{pmatrix}.$$

Denote by $\bar{\mathcal{B}} = \mathcal{A}^+\mathcal{B}^- - \mathcal{B}^+\mathcal{A}^-$. Since the determinant of a triangular block matrix is the product of the determinant of the diagonal blocks, we have

$$\begin{aligned} \det S(A,B) \cdot \det \mathbb{I}_n \cdot \det(-\mathcal{A}^-) &= \det \mathcal{A}^- \cdot \det \bar{\mathcal{B}}, \\ (-1)^n \mathrm{res}(A,B) &= \det \bar{\mathcal{B}}. \end{aligned}$$

The coefficients $\bar{b}_{p,q}$ of $\bar{\mathcal{B}}$ can be written as

$$\bar{b}_{p,q} = \sum_{k=\max(i,j)+1}^{n} a_{k-q-1} b_{n+p+1-k} - a_{n+p+1-k} b_{k-q-1}.$$

We want to show that $c_{p,q} = -\bar{b}_{p,n-1-q}$. We have

$$\begin{aligned} \bar{b}_{p,n-1-q} &= \sum_{k=\max(p,n-1-q)+1}^{n} a_{k-(n-1-q)-1} b_{n+p+1-k} - a_{n+p+1-k} b_{k-(n-1-q)-1} \\ &= \sum_{k=\max(p,n-1-q)+1}^{n} a_{k+q-n} b_{n+p+1-k} - a_{n+p+1-k} b_{k+q-n}. \end{aligned}$$

Assume $p \geq n - 1 - q$.

$$\begin{aligned} \bar{b}_{p,n-1-q} &= \sum_{k=p+1}^{n} a_{k-(n-1-q)-1} b_{n+p+1-k} - a_{n+p+1-k} b_{k-(n-1-q)-1} \\ &\underset{d=n-k}{=} \sum_{d=0}^{n-p+1} a_{q-d} b_{d+i+1} - a_{d+i+1} b_{q-d} = -c_{q,p} = -c_{p,q}. \end{aligned}$$

In the other case where $p \leq n-1-q$.

$$\begin{aligned}\bar{b}_{p,n-1-q} &= \sum_{k=n-q}^{n} a_{k-(n-1-q)-1} b_{n+p+1-k} - a_{n+p+1-k} b_{k-(n-1-q)-1} \\ &\underset{d=n-k}{=} \sum_{d=0}^{q} a_{q-d} b_{d+p+1} - a_{d+p+1} b_{q-d} = -c_{q,p} = -c_{p,q}.\end{aligned}$$

So $\bar{\mathcal{B}}$ differs from $\text{Béz}(A,B)$ by a factor of -1 and $\lfloor \frac{n}{2} \rfloor$ column shifts. So we have

$$\begin{aligned}(-1)^n \text{res}(A,B) &= \det \bar{\mathcal{B}} = (-1)^{\lfloor \frac{n}{2} \rfloor + n} \det \text{Béz}(A,B) \\ (-1)^{\lfloor \frac{n}{2} \rfloor} \text{res}(A,B) &= \det \text{Béz}(A,B).\end{aligned}$$

At last, we have used the fact that $(-1)^{\lfloor \frac{n}{2} \rfloor} = (-1)^{\frac{n(n-1)}{2}}$. This concludes the proof. $\square$

The last result is about how we can interpret the resultant of a pair of homogeneous polynomials.

Proposition 2.1.7. Let $s_0 = a_n x_0^n + \ldots + a_0 x_1^n$ and $s_1 = b_n x_0^n + \ldots + b_0 x_1^n$ be two homogeneous polynomials in two variables with a_i, b_i coefficients from some ring A. Then

$$\text{res}(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}) = (-1)^n \text{res}(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}).$$

Proof. Consider the Sylvester matrix $S(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n})$

$$S(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}) = \begin{pmatrix} a_n & & 0 & b_n & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_1 & \ldots & a_n & b_1 & \ldots & b_n \\ a_0 & \ldots & a_{n-1} & b_0 & \ldots & b_{n-1} \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_0 & 0 & & b_0 \end{pmatrix}.$$

Switching the top n rows, with the bottom n rows yields

$$S(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}) \sim \begin{pmatrix} a_0 & \ldots & a_{n-1} & b_0 & \ldots & b_{n-1} \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_0 & 0 & & b_0 \\ a_n & & 0 & b_n & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_1 & \ldots & a_n & b_1 & \ldots & b_n \end{pmatrix}.$$

Switching column 1 and n, 2 and $n-1$ etc. and $n+1$ and $2n$, $n+2$ and $2n-1$ etc. and then switching row 1 and n, 2 and $n-1$ etc. and $n+1$ and $2n$, $n+2$ and $2n-1$ etc. gives

$$S(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}) \sim \begin{pmatrix} a_0 & & 0 & b_0 & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_{n-1} & \dots & a_0 & b_{n-1} & \dots & b_0 \\ a_n & \dots & a_1 & b_n & \dots & b_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_n & 0 & & b_n \end{pmatrix} = S(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}).$$

We then have

$$\text{res}(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}) = \det S(\frac{s_0}{x_1^n}, \frac{s_1}{x_1^n}) = (-1)^n \det S(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}) = (-1)^n \text{res}(\frac{s_0}{x_0^n}, \frac{s_1}{x_0^n}).$$

□

2.2 Bilinear forms and the Witt monoid

Let R be a ring. Let $V = R^n$ be a $n-$dimensional vector space. An (R-)bilinear form is a bilinear map $V \times V \longrightarrow R$ such that $\forall u, v, w \in V$ and $\forall \lambda \in R$

1. $B(u + v, w) = B(u, w) + B(v, w)$ and $B(\lambda u, v) = \lambda B(u, v)$.
2. $B(u, v + w) = B(u, v) + B(u, w)$ and $B(u\lambda, v) = \lambda B(u, v)$.

All bilinear forms can be represented by matrices. Let $\{e_1, \dots, e_n\}$ be a basis for R^n. The matrix $A_{i,j} = B(e_i, e_j)$ is the matrix of the bilinear form on the basis $\{e_1, \dots, e_n\}$. The matrix of a bilinear form differs depending on choice of basis. If $\{f_1, \dots, f_n\}$ is a different basis, then there exists an invertible matrix C such that

$$f_i = \sum_{i=1}^{n} C_{i,j} e_i.$$

Then the matrix of the bilinear form in the new basis is $C^T AC$.

Definition 2.2.1. Let B_1 and B_2 be the matrix representation of two n-ary bilinear forms with respect to some bases. We say that B_1 is *isomorphic* to B_2 if there exists an invertible matrix C such that $B_2 = C^T B_1 C$.

Definition 2.2.2. A bilinear form is called *symmetric* if its matrix representation is a symmetric matrix.

Definition 2.2.3. A bilinear form is called *non-degenerate* if its matrix representation is an invertible matrix.

Definition 2.2.4. The *rank* of a bilinear form the rank of its matrix representation.

Example 2.2.5. Consider the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. It is symmetric and non-degenerate of rank 2. It is also isomorphic to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ because

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let $\mathcal{S}_n(R)$ be the scheme of non-degenerate $(n \times n)$ symmetric matrices with coefficients in the ring R. We will write $\mathcal{S}_n$ when the choice of R is obvious.

Definition 2.2.6. A pointed homotopy of symmetric bilinear forms is an $H(T) \in \mathcal{S}_n(R[T])$. It yields a homotopy between the bilinear forms $H(0)$ and $H(1)$. We say that two forms $B_1, B_2 \in \mathcal{S}_n(R)$ are in the same pointed naive homotopy class if there exists a finite sequence $(H_i) \in \mathcal{S}_n(R[T])$ with $0 \leq i \leq N$, such that

- $H_0(0) = B_1$ and $H_N(1) = B_2$;
- $H_i(1) = H_{i+1}(0)$ for every $0 \leq i \leq N-1$.

If B_1 is in the same pointed naive homotopy class B_2, we write $B_1 \overset{\mathrm{p}}{\sim} B_2$.

We denote the set $\mathcal{S}_n/\overset{\mathrm{p}}{\sim}$ by $\pi_0^N \mathcal{S}_n$.

Example 2.2.7. The homotopy $\begin{pmatrix} 1 & T \\ T & 2+T^2 \end{pmatrix}$ gives us $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \overset{\mathrm{p}}{\sim} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

For bilinear forms over a field, we can create a monoid structure on the isomorphism classes.

Definition 2.2.8.
1. Let k be a field. The Witt monoid of the field k is the monoid with the orthogonal sum $\oplus$ as its operation and isomorphism classes of non-degenerate symmetric k-bilinear forms as its elements. The Witt monoid of k is denoted $\mathrm{MW}(k)$.
2. Let $\mathrm{MW}^s(k)$ be the monoid of stable isomorphism classes of non-degenerate symmetric k-bilinear forms. This is the quotient of $\mathrm{MW}(k)$ where two forms b and b' are identified if there exists a form b'' such that $b \oplus b'' \cong b' \oplus b''$. It comes with a natural grading induced by the rank, and for every positive integer n, we denote by $\mathrm{MW}^s_n(k)$ the degree n component of $\mathrm{MW}^s(k)$.

One can construct the Grothendieck–Witt group $\mathrm{GW}(k)$ as the Grothendieck group of the monoid $\mathrm{MW}^s(k)$. It is the group satisfying the following universal property. There exists a monoid morphism $i : \mathrm{MW}^s(k) \to \mathrm{GW}(k)$. Such that for any abelian group A and any monoid morphism $f : \mathrm{MW}^s(k) \to A$, there exists a unique group homomorphism h such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{MW}^s(k) \arrow[r, "i"] \arrow[dr, "f"'] & \mathrm{GW}(k) \arrow[d, "\exists ! h"] \\ & A. \end{tikzcd}$$

Chapter 3

Literature review

Consider a base field k, and let $\mathbb{P}^n = \mathbf{Proj}(k[x_0, \ldots, x_n])$ denote projective n-space as a k-scheme. In [6], Cazanave computes the set of naive homotopy classes $[\mathbb{P}^1, \mathbb{P}^1]^N$ of pointed k-scheme endomorphisms and finds that the set $[\mathbb{P}^1, \mathbb{P}^1]^N$ admits a monoid structure. In the end, he also proves that the group completion of the monoid $[\mathbb{P}^1, \mathbb{P}^1]^N$ coincides with the group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. In this chapter we will cover results with pointed naive homotopies of endomorphisms.

3.1 Homotopies of rational functions

Definition 3.1.1. For an integer $n \geq 1$, the scheme $\mathcal{F}_n$ of pointed degree n rational functions is the open subscheme of the affine space $\mathbb{A}^{2n} = \text{Spec } k[a_0, ..., a_{n-1}, b_0, ..., b_{n-1}]$ complementary to the hypersurface of equation

$$\text{res}_{n,n}(X^n + a_{n-1}X^{n-1} + \ldots + a_0, b_{n-1}X^{n-1} + \ldots + b_0) = 0.$$

By convention, $\mathcal{F}_0 := \text{Spec } k$.

Proposition 3.1.2 ([6, Theorem 2.3])**.** Let $R = k$ or $R = k[T]$. The datum of a pointed k-scheme morphism $f : \mathbb{P}^1_R \longrightarrow \mathbb{P}^1_R$ is equivalent to the datum of a non-negative integer n and of an element $\frac{A}{B} \in \mathcal{F}_n(R)$. The integer n is called the degree of f and is denoted $\deg(f)$; the scalar $\text{res}_{n,n}(A, B) \in R^\times = k^\times$ is called the resultant of f and is denoted $\text{res}(f)$.

Proposition 3.1.3. The datum of a pointed naive homotopy $F : \mathbb{P}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{P}^1$ is equivalent to the datum of a non-negative integer n and of an element in $\mathcal{F}_n(k[T])$. The source $\sigma(F)$ and the target $\tau(F)$ of F are obtained by evaluating the indeterminate T at 0 and 1 respectively.

Example 3.1.4. Let n be a positive integer.

1. Let $A = X^n + a_{n-1}X^{n-1} + \ldots + a_0$ and $b_0 \in k^\times$. The homotopy

$$\frac{X^n + +a_{n-1}TX^{n-1} + \ldots + a_0T}{b_0} \in \mathcal{F}_n(k[T])$$

gives a pointed naive homotopy between $\frac{A}{b_0}$ and $\frac{X^n}{b_0}$.

2. Let $B = b_{n-1}X^{n-1} + \ldots + b_0$, with $b_0 \in k^\times$. The homotopy

$$\frac{X^n}{b_{n-1}TX^{n-1} + \ldots + b_1XT + b_0} \in \mathcal{F}_n(k[T])$$

gives a pointed naive homotopy between $\frac{X^n}{B}$ and $\frac{X^n}{b_0}$.

3.2 Addition of rational functions

A remarkable property of the pointed rational functions is that they create a monoid. Let $\frac{A_i}{B_i} \in \mathcal{F}_{n_i}(R)$ for $i = 1, 2$. These two functions uniquely define two pairs (U_i, V_i) such that $A_iU_i + B_iV_i = 1$. Observe that $\deg U_i \leq n_i - 2$ and $\deg V_i \leq n_i - 1$. We define the polynomials A_3, B_3, U_3 and V_3 by setting

$$\begin{pmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{pmatrix} := \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}.$$

Since the matrices $\begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix}$ and $\begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}$ both have determinant 1, the same holds for their product. This means we have a Bézout relation $A_3U_3 + B_3V_3 = 1$. Since $A_3 = A_1A_2 - B_2V_1$ it is monic of degree $n_1 + n_2$. We also have $B_3 = B_1A_2 + U_1B_2$ which is of degree strictly less than $n_1 + n_2$. Since the polynomials A_3, B_3 have a Bézout relation, it means their resultant is nontrivial. Thus, the pointed rational function $\frac{A_3}{B_3}$ is an element of $\mathcal{F}_{n_1+n_2}(R)$. We write

$$\frac{A_1}{B_1} \oplus^N \frac{A_2}{B_2} = \frac{A_3}{B_3}.$$

Notice that this operation is associative, because matrix multiplication is associative. We have the following result.

Proposition 3.2.1 ([6, Proposition 3.1])**.** Let $\mathcal{F} := \coprod_{n \geq 0} \mathcal{F}_n$ be the scheme of pointed rational functions. The morphism

$$\oplus^N : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$$

defines a graded monoid structure on $\mathcal{F}$.

Example 3.2.2. 1.

$$X \oplus^N X = \frac{X^2 - 1}{X}$$

2. Let $\frac{A}{B}$ be any pointed rational function, one has

$$X \oplus^N \frac{A}{B} = \frac{AX - B}{A} \quad \text{and} \quad \frac{A}{B} \oplus^N X = \frac{AX - V}{BX + U}.$$

3. Given the trivial homotopies $X \overset{p}{\sim} X$ and $X^2 + TX + T \overset{p}{\sim} X^2$ we can produce a new homotopy

$$X \oplus^N \frac{X^2 + TX + T}{1} = \frac{X^3 + TX^2 + TX - 1}{X^2 + TX + T}.$$

Which means $\frac{X^3-1}{X^2} \overset{p}{\sim} \frac{X^3+X^2+X-1}{X^2+X+1}$.

4. Let $P \in k[X]$ be a monic polynomial and $b_0 \in k^\times$, then

$$\frac{P}{b_0} \oplus^N \frac{A}{B} = \frac{AP - \frac{B}{b_0}}{b_0 A} = \frac{P}{b_0} - \frac{1}{b_0^2 \frac{A}{B}}.$$

The examples give rise to the following remarks

Remark 3.2.3. 1. The binary operation $\oplus^N$ is not commutative.

2. The sum of ”trivial” homotopies can yield ”non trivial” homotopies.

3.3 The monoid of naive homotopy classes

Recall the Bézout form of a pair of polynomials A, B is a symmetric matrix. Denote by Béz_n the function that sends a rational function to its Bézout matrix.

$$\begin{aligned} \text{Béz}_n : &\mathcal{F}_n \longrightarrow \mathcal{S}_n \\ &\frac{A}{B} \longrightarrow \text{Béz}(A, B) \end{aligned}$$

This leads us to the main result of Cazanave’s paper.

Theorem 3.3.1 ([6, Theorem 3.6])**.** The following map is an isomorphism of graded monoids:

$$\Big(\coprod_{n\geq 0} (\pi_0^N \mathcal{F}_n)(k), \oplus^N\Big) \xrightarrow{\coprod_{n\geq 0} \pi_0^N \text{Béz}_n} \Big(\coprod_{n\geq 0} (\pi_0^N \mathcal{S}_n)(k), \oplus\Big)$$

Where $\oplus$ is block matrix concatenation.

Combining the theorem with the following proposition lets us describe $[\mathbb{P}^1, \mathbb{P}^1]^N$.

Proposition 3.3.2 ([6, Proposition 3.9])**.** Let n be a positive integer.

1. The canonical quotient map $q_n : \mathcal{S}_n(k) \longrightarrow \mathrm{MW}^s_n(k)$ factors through $(\pi_0^N \mathcal{S}_n)(k)$:

$$\begin{tikzcd} \mathcal{S}_n(k) \arrow[r, "q_n"] \arrow[d] & \mathrm{MW}^s_n(k) \\ (\pi_0^N \mathcal{S}_n)(k). \arrow[ur, "\overline{q}_n"'] & \end{tikzcd}$$

2. Let $\mathrm{MW}^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ be the canonical fibre product induced by the *discriminant* map $\mathrm{MW}^s_n(k) \longrightarrow k^\times/k^{\times 2}$. Then the map

$$\Big(\coprod_{n\geq 0}(\pi_0^N \mathcal{S}_n)(k), \oplus\Big) \xrightarrow{\coprod_{n\geq 0}\overline{q}_n \times \det} \Big(\coprod_{n\geq 0} \mathrm{MW}^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\Big)$$

is a monoid isomorphism. Above, the right-hand term is endowed with the canonical monoid structure induced by the orthogonal sum in $\mathrm{MW}^s(k)$ and the product in $k^\times$.

Proof. A proof can be found in [13, §VII.3]. □

Even though addition of rational functions is not commutative, we have the following result.

Corollary 3.3.3 ([6, Corollary 3.7])**.** The monoid $\Big(\coprod_{n\geq 0}(\pi_0^N \mathcal{S}_n)(k), \oplus\Big)$ is abelian, and thus, so is $\Big([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N\Big)$.

Theorem 3.3.1 combined with Proposition 3.3.2 gives the following description of $[\mathbb{P}^1, \mathbb{P}^1]^N$.

Corollary 3.3.4 ([6, Corollary 3.10])**.** There is a canonical isomorphism of graded monoids:

$$\Big([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N\Big) \cong \Big(\coprod_{n\geq 0} MW^s_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\Big).$$

Example 3.3.5.

1. When k is algebraically closed, we have an isomorphism of monoids

$$[\mathbb{P}^1, \mathbb{P}^1]^N \xrightarrow[\deg \times \text{ res}]{\cong} \mathbb{N} \times k^\times.$$

2. When k is the field of real numbers $\mathbb{R}$, we have an isomorphism of monoids:

$$[\mathbb{P}^1, \mathbb{P}^1]^N \xrightarrow[(\text{sign}\circ\text{Béz})\times \text{ res}]{\cong} (\mathbb{N} \times \mathbb{N}) \times \mathbb{R}^\times,$$

sign denoting the signature of a real symmetric bilinear form. In this case, the Bézout invariant is sharper than the resultant and the degree invariants.

3. When k is the field of two elements $\mathbb{F}_2$, there is only one homotopy class in each degree.
$$[\mathbb{P}^1, \mathbb{P}^1]^N \xrightarrow[\deg]{\cong} \mathbb{N}.$$

Theorem 3.3.6 ([6, Theorem 3.22])**.** The canonical map $\left([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N\right) \to \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$ is a group completion.

Chapter 4

The Jouanoulou Device

The Jouanolou device $\mathcal{J}$ over $\mathbb{P}^1$ is the affine smooth scheme defined as follows:

$$\mathcal{J} = \text{Spec}(\frac{k[x,y,z,w]}{(x+w-1, xw-yz)}) = \text{Spec}(\frac{k[x,y,z]}{(x(1-x)-yz)}).$$

We can think of $\mathcal{J}$ as the algebraic variety of 2×2 matrices over k with trace 1 and rank 1. There is a canonical map from $\mathcal{J}$ to $\mathbb{P}^1$ given by $\pi : \mathcal{J} \to \mathbb{P}^1$, given intuitively by sending a matrix A in $\mathcal{J}$ to its rows. Consider the matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{J}$. It describes the projection to $\mathbb{P}^1$ by sending a matrix in $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathcal{J}$ to $[a : b]$ or $[c : d]$, whichever of them that is nonzero. In the case where they are both nonzero, they describe the same point in $\mathbb{P}^1$ since $\frac{d}{b}a = c$ and $\frac{d}{b}b = d$.

Example 4.0.1.

1. The matrix $\begin{pmatrix} 2 & 4 \\ -\frac{1}{2} & -1 \end{pmatrix}$ maps to the point $[2:4] = [-\frac{1}{2} : -1]$.
2. The matrix $\begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix}$ maps to the point $[5:1]$.

This map exhibits $\mathcal{J}$ as an affine vector bundle torsor over $\mathbb{P}^1$ [18, Proposition 4.3]. We will now study the basic properties of $\mathcal{J}$ in detail.

We will write R for the ring $\frac{k[x,y,z,w]}{(x+w-1,xw-yz)}$. In this section we present new results regarding the Jouanolou device of $\mathbb{P}^1$.

4.1 Some properties of $\mathcal{J}$

Proposition 4.1.1. $\mathcal{J}$ is a Noetherian scheme.

Proof. Since R is a Noetherian ring, $\mathcal{J}$ is a Noetherian scheme. $\square$

Proposition 4.1.2. $\mathcal{J}$ is an integral scheme.

Proof. We need to prove that R is an integral domain, so we want to show that $(x(1-x)-yz)$ is irreducible in $k[x,y,z]$. It is irreducible if it can not be written as the product of two degree 1 polynomials. Assume there exists

$$p(x,y,z) = ax + by + cz + d,$$
$$q(x,y,z) = \alpha x + \beta y + \gamma z + \delta,$$

such that $pq = -x^2 + x - yz$. First all we get that $d = 0$ or $\delta = 0$. Assume $d = 0$, this implies that $b\delta = 0$ and $c\delta = 0$. If we look at the case where $\delta \neq 0$,we get that $b = 0$ and $c = 0$. This is not possible, because we also wish to have $b\gamma + c\beta = -1$. This means $\delta = 0$. However, this is not possible as well since $a\delta + d\alpha = 1$. This means our first assumption of $d = 0$ is wrong. Carrying out a similar argument with the initial assumption that $\delta = 0$ also leads to a contradiction. This means our assumption of the existence of p and q is incorrect, hence proving that $(x(1-x)-yz)$ is irreducible. □

Proposition 4.1.3. $\mathcal{J}$ is separated.

Proof. We need to prove that the diagonal map $\Delta : \mathcal{J} \longrightarrow \mathcal{J} \times_{\mathbb{Z}} \mathcal{J}$ is a closed immersion.

Since $\mathcal{J}$ and $\mathbb{Z}$ are both affine, we have $\mathcal{J} \times_{\mathbb{Z}} \mathcal{J} = \mathrm{Spec}(\mathcal{J} \otimes_{\mathbb{Z}} \mathcal{J})$ and the map $\Delta^* : \mathcal{J} \otimes_{\mathbb{Z}} \mathcal{J} \longrightarrow \mathcal{J}$ on the level of rings, sending the pair (a,b) to the product $ab \in \mathcal{J}$. Since Δ^* is surjective, Δ is a closed immersion. □

Proposition 4.1.4. $\mathcal{J}$ is a smooth scheme.

Proof. Since R is flat over k, we get that $\mathcal{J}$ is flat over $\mathrm{Spec}(k)$. By [2, §10 Theorem 3'], $\mathcal{J}$ is smooth if and only if it is flat over k and that the fiber over any geometric point is smooth. We can check the smoothness of fibers by looking at the Jacobian of the variety defining R. Let $f = x(1-x) - yz$, we have

$$\frac{\partial f}{\partial x} = 2x - 1,$$
$$\frac{\partial f}{\partial y} = -z,$$
$$\frac{\partial f}{\partial z} = -y.$$

When the characteristic of k is 2, there are no singularities, and when the characteristic is different from 2, we get the singularity $(\frac{1}{2}, 0, 0)$. However, this is not a point on $\mathcal{J}$, so all the fibers are smooth. □

Additionally, when k is algebraically closed, we can say even more about $\mathcal{J}$.

Proposition 4.1.5. If k is algebraically closed $\mathcal{J}$ is locally factorial (All local rings are UFD).

Proof. We will need two more lemmas to prove this statement.

Lemma 4.1.6 ([9, Remark II.6.11.1A])**.** All regular local rings are UFD.

Lemma 4.1.7 ([9, Theorem I.5.1])**.** Let $V \subset \mathbb{A}^n$ be an affine variety. Let $P \in V$ be a point. Then V is nonsingular at P if and only if the local ring $\mathcal{O}_{P,V}$ is a regular local ring.

Since $\mathcal{J}$ is smooth it is nonsingular at all geometric points. When k is algebraically closed, all points in $\mathcal{J}$ are geometric points. This means that all its local rings are UFD and and hence it is locally factorial. $\square$

4.2 Line bundles of $\mathcal{J}$

Line bundles are a common concept in differential geometry, but they do in fact have an algebraic geometric counterpart. In algebraic geometry we first need to introduce the concept of sheaves of modules on a ringed space.

Definition 4.2.1.

1. Let $(X, \mathcal{O}_X)$ be a ringed space. *A sheaf of $\mathcal{O}_X$-modules* (or simply an *$\mathcal{O}_X$-module*) is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$.
2. We define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_x$-modules to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U)$.
3. An $\mathcal{O}_X$-module $\mathcal{F}$ is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open sets U for which $\mathcal{F}\big|_U$ is a free $\mathcal{O}_X\big|_U$-module. In that case the rank of $\mathcal{F}$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite).
4. A locally free sheaf of rank 1 is called an *invertible sheaf.*

Invertible sheaves play an essential role for figuring out morphisms $\mathcal{J}$ to $\mathbb{P}^1$, as can be seen in the following theorem.

Theorem 4.2.2 ([9, Theorem II.7.1])**.** Let A be a ring, and let X be a scheme over A.

1. If $\varphi : X \longrightarrow \mathbb{P}^n_A$ is an A-morphism, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i), i = 0, 1, \ldots, n$, where the x_i are global sections of $\mathcal{O}(1)$ of $\mathbb{P}^n_A$.
2. Conversely, if $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique R-morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.

The theorem makes it clear that if we wish to understand the morphisms from $\mathcal{J}$ to $\mathbb{P}^1$, we need to figure out all the invertible sheaves of $\mathcal{J}$ first. The invertible sheaves up to isomorphism on a scheme X create a group with the tensor product over the structure sheaf as its group action. This group is called the Picard group, denoted by Pic(X). There exists motivic cohomology, where the Picard group of a smooth scheme X is the cohomology group $H^{2,1}(X, \mathbb{Z})$ [14, Corollary 4.2]. Since Pic$(\mathbb{P}^1) = \mathbb{Z}$ [17, Tag 0BXJ], naturally since $\mathbb{P}^1$ is $\mathbb{A}^1$-homotopic to $\mathcal{J}$, we get Pic$(\mathcal{J}) = \mathbb{Z}$.

For commutative rings, there exists the notion of *algebraic line bundles*, and it turns out for affine schemes there is a connection between the invertible sheaves and algebraic line bundles of the base ring. Before we get into the details, we need some more defintions.

Definition 4.2.3. Let A be a commutative ring. The rank of a finitely generated A-module M at a prime ideal $\mathfrak{p}$ of A is $\text{rank}_{\mathfrak{p}}(M) = \dim_{k(\mathfrak{p})} M \otimes_A k(\mathfrak{p})$, where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Since $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong k(\mathfrak{p})^{\mathrm{rank}_{\mathfrak{p}}(M)}$, $\mathrm{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_{\mathfrak{p}}$. We say that M has *constant rank* n, if it is $n = \text{rank}_{\mathfrak{p}}(M)$ for all $\mathfrak{p}$.

Definition 4.2.4. We say that an R-module P is projective, if there exists an R-module Q, such that $P \oplus Q$ is a free module.

Definition 4.2.5. An *algebraic line bundle* over a commutative ring A is a finitely generated projective A-module of constant rank 1.

Proposition 4.2.6 ([8, Corollary 7.41])**.** Let $X = \mathrm{Spec}(A)$ be an affine scheme. Each algebraic line bundle on A gives rise to an invertible sheaf. Similarly, each invertible sheaf on X corresponds to an algebraic line bundle on A.

Since $\mathcal{J}$ is affine, we now need to find all the finitely generated projective modules of constant rank 1 up to isomorphism. Since R is a domain, if e is an idempotent in R, then $e(1-e) = 0$. So $e = 0$ or $e = 1$. This lets us use the following lemma.

Lemma 4.2.7 ([19, Exc. 2.4])**.** The following are equivalent for every commutative ring A

1. $\mathrm{Spec}(A)$ is topologically connected
2. Every finitely generated projective A-module has constant rank
3. A has no idempotent elements except 0 and 1.

The lemma above ensures that we only need to focus on finitely generated projective modules, as they all have constant rank. Let P be a finitely generated projective R-module, then the projection-inclusion composition

$$R^n \longrightarrow P \longrightarrow R^n,$$

corresponds to some matrix e in $M_n(R)$. Notice that this composition is in fact idempotent and that P is the image of e. It is not hard to see that $\ker(e)$ is a projective module as well, since $\ker(e) \oplus \mathrm{im}(e) \cong R^n$. This means we can study projective modules over R by studying idempotent matrices instead.

Definition 4.2.8. We define $M_{n+1} = (m_{ij})$ to be the $(n+1) \times (n+1)$ matrix where

$$m_{ij} = \binom{n}{j} x^{\alpha_x(i,j)} y^{\alpha_y(i,j)} z^{\alpha_z(i,j)} w^{\alpha_w(i,j)},$$

with $\alpha_x(i,j), \alpha_y(i,j), \alpha_z(i,j)$ and $\alpha_w(i,j)$ given by

$$\begin{aligned}
\alpha_x(i,j) &= \min(n-i, n-j), \\
\alpha_y(i,j) &= n - j - \alpha_x(i,j), \\
\alpha_w(i,j) &= \min(i,j), \\
\alpha_z(i,j) &= j - \alpha_w(i,j).
\end{aligned}$$

For example when $n = 2$, we have

$$M_3 = \begin{bmatrix} x^2 & 2xz & z^2 \\ xy & 2xw & zw \\ y^2 & 2yw & w^2 \end{bmatrix}.$$

We now need to prove that our constructed matrix is idempotent.

Proposition 4.2.9. For all $n > 0$. The matrix M_{n+1} is idempotent.

Proof. For $f \in \{x, y, z, w\}$, define

$$\beta_f(i,j,k) = \alpha_f(i,k) + \alpha_f(k,j) - \alpha_f(i,j).$$

Let m_{ij}^2 denote the ij-th entry of the matrix M_{n+1}^2. We can write m_{ij}^2 as

$$m_{ij}^2 = \sum_k m_{ik}m_{kj} = m_{ij} \sum_k \binom{n}{k} x^{\beta_x(i,j,k)} y^{\beta_y(i,j,k)} z^{\beta_z(i,j,k)} w^{\beta_w(i,j,k)}$$

It suffices to treat the case when $i \geq j$, the rest follows by symmetry. We will now inspect the exponents. In the case when $i \geq j \geq k$ we have

$$\begin{aligned} \beta_x(i,j,k) &= n - j, \\ \beta_y(i,j,k) &= j - k, \\ \beta_z(i,j,k) &= j - k, \\ \beta_w(i,j,k) &= 2k - j. \end{aligned}$$

In the case when $i \geq k \geq j$ we have

$$\begin{aligned} \beta_x(i,j,k) &= n - i, \\ \beta_y(i,j,k) &= 0, \\ \beta_z(i,j,k) &= 0, \\ \beta_w(i,j,k) &= k. \end{aligned}$$

and finally, when $k \geq i \geq j$:

$$\begin{aligned} \beta_x(i,j,k) &= n + i - 2k, \\ \beta_y(i,j,k) &= k - i, \\ \beta_z(i,j,k) &= k - i, \\ \beta_w(i,j,k) &= i. \end{aligned}$$

Notice that in all of the cases we have that $\beta_y(i,j,k) = \beta_z(i,j,k)$. We can use the relation $xw = yz$ to convert $(yz)^{\beta_y(i,j,k)}$ to $(xw)^{\beta_y(i,j,k)}$. This gives

$$\begin{aligned} \beta_x(i,j,k) + \beta_y(i,j,k) &= n - k, \\ \beta_w(i,j,k) + \beta_y(i,j,k) &= k, \end{aligned}$$

for all k, as long as $i \geq j$. Inserting this into the expression for M_{ij}^2 gives

$$m_{ij}^2 = m_{ij} \sum_k \binom{n}{k} x^{n-k} w^k = m_{ij}(x+w)^n = m_{ij}.$$

This concludes the proof. $\square$

Proposition 4.2.10. For all $n > 0$. The rank of M_{n+1} is 1.

Proof. The trace of M_{n+1} is $(x+w)^{n+1} = 1$. As the trace of an idempotent matrix is equal to its rank, it has rank 1. $\square$

Define the line bundles $\mathcal{P}_1$ and $\mathcal{Q}_1$ as follows:

$$\mathcal{Q}_1 := \text{Im} \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$
$$\mathcal{P}_1 := \text{Im} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

These images correspond to projective modules of rank 1, hence they are line bundles over $\mathcal{J}$. In the following results, all tensor products are taken over R. We will start out by showing that $\mathcal{P}_1$ is the inverse of $\mathcal{Q}_1$ in Pic$(\mathcal{J})$.

Proposition 4.2.11. $\mathcal{P}_1 \otimes \mathcal{Q}_1 \cong R$.

Proof. An element of $\mathcal{P}_1 \otimes \mathcal{Q}_1$ can be written as

$$\sum (\alpha_i \begin{bmatrix} x \\ z \end{bmatrix} + \beta_i \begin{bmatrix} y \\ w \end{bmatrix}) \otimes (a_i \begin{bmatrix} x \\ y \end{bmatrix} + b_i \begin{bmatrix} z \\ w \end{bmatrix}), \quad \alpha_i, \beta_i, a_i, b_i \in R.$$

One can see that the R-module $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is generated by the four elements

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix} \right\}.$$

Using the module homomorphism $\mathcal{P}_1 \otimes \mathcal{Q}_1 \longrightarrow R^2$, sending $\begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x^2 \\ yz \end{bmatrix}$ etc., we get that each of the four basis elements corresponds to the following elements in R^2

$$\left\{ x \begin{bmatrix} x \\ w \end{bmatrix}, z \begin{bmatrix} x \\ w \end{bmatrix}, y \begin{bmatrix} x \\ w \end{bmatrix}, w \begin{bmatrix} x \\ w \end{bmatrix} \right\}.$$

Since $(x+w) = 1$, we can see that they generate a rank 1 submodule of R^2 which is isomorphic to R. Thus, $\mathcal{P}_1 \otimes \mathcal{Q}_1 \cong R$. $\square$

We would like to understand the tensor powers of $\mathcal{P}_1$ and $\mathcal{Q}_1$. It turns out they can be described by the following R-modules. Denote by $\mathcal{Q}_n$, the R-module generated by the elements $\left\{ \begin{bmatrix} x^{n-i} z^i \\ y^{n-i} w^i \end{bmatrix} \right\}_{0 \leq i \leq n}$ and $\mathcal{P}_n$ the one generated by $\left\{ \begin{bmatrix} x^{n-i} y^i \\ z^{n-i} w^i \end{bmatrix} \right\}_{0 \leq i \leq n}$. Next, we want to show that $\mathcal{P}_1^{\otimes n}$ coincides with our definition of $\mathcal{P}_n$.

Proposition 4.2.12. We have $\mathcal{P}_1^{\otimes n} \cong \mathcal{P}_n$ and $\mathcal{Q}_1^{\otimes n} \cong \mathcal{Q}_n$.

Proof. We will prove this by induction for the $\mathcal{P}_n$-case, the $\mathcal{Q}_n$-case is similar. In the case when $n = 2$, we have that the module $\mathcal{P}_1 \otimes \mathcal{P}_1$ is generated by the elements

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix} \right\}.$$

Notice that

$$\begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix} = (x+w) \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix},$$

so one actually just needs three elements to generate the module. We can map these the an R-module morphism sending $\begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}$ to $\begin{bmatrix} x^2 \\ z^2 \end{bmatrix}$ etc. Thus, the element

$$r_0 \begin{bmatrix} x^2 \\ z^2 \end{bmatrix} + r_1 \begin{bmatrix} xy \\ zw \end{bmatrix} + r_2 \begin{bmatrix} y^2 \\ w^2 \end{bmatrix} \in \mathcal{P}_2,$$

corresponds to the element

$$r_0 \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix} + r_1 \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix} + r_2 \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix} \in \mathcal{P}_1 \otimes \mathcal{P}_1.$$

We will now assume it holds for all n and prove that $\mathcal{P}_n \otimes \mathcal{P}_1 \cong \mathcal{P}_{n+1}$. First notice that when $i \geq 1$

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix} = (x+w) \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x^{n-i+1}y^{i-1} \\ z^{n-i+1}w^{i-1} \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix}.$$

The module $\mathcal{P}_n \otimes \mathcal{P}_1$ is generated by the $n+2$ elements

$$\left\{ \begin{bmatrix} x^n \\ z^n \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \otimes \begin{bmatrix} y \\ w \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} xy^{n-1} \\ zw^{n-1} \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix}, \ldots, \begin{bmatrix} x^{n-1}y \\ z^{n-1}w \end{bmatrix} \otimes \begin{bmatrix} x \\ z \end{bmatrix} \right\}.$$

Using a similar R-module morphism as before, we can map these to basis elements of $\mathcal{P}_{n+1}$ and create an isomorphism. $\square$

We can combine Proposition 4.2.11 and 4.2.12 to prove the following theorem. We need to remark that $\mathcal{P}_n \not\cong \mathcal{P}_m$ for $n \neq m$.

Theorem 4.2.13. The line bundle $\mathcal{P}_1$ generates Pic$(\mathcal{J})$, and $\mathcal{Q}_1$ is its inverse.

Elements of interest are the minimal generating bases for various R-modules.

Proposition 4.2.14. The elements $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$ are contained in the ideal $(x^n, y^n) \subset R$, and similarly for the pairs $\{z, w\}$, $\{x, z\}$, and $\{y, w\}$.

Proof. We will only prove it for the pair $\{x, y\}$, because the other proofs are similar. Fix n and pick an integer $0 \leq i \leq n$. Consider the element

$$x^{n-i}y^i = x^{n-i}y^i(x+w)^n = \sum_{d=0}^{n} \binom{n}{d} x^{n-i}y^i x^{n-d}w^d = \sum_{d=0}^{n} \binom{n}{d} x^{2n-i-d}y^i w^d.$$

For each d, if $2n-i-d \geq n$, it then contains a factor of x^n. In the case where $2n-i-d < n$, we have that $n-i < d$. One can then convert $n-i$ pairs of xw into yz.

$$x^{2n-i-d}y^i w^d = x^{2n-i-d-(n-i)}y^{i+(n-i)}z^{n-i}w^{d-(n-i)} = x^{n-d}y^n z^{n-i}w^{d-(n-i)}.$$

It now contains the factor y^n and we are done. $\square$

A similar proof also yields the following result.

Proposition 4.2.15. $\mathcal{P}_n$ is generated by the vectors $\begin{bmatrix} x^n \\ z^n \end{bmatrix}$ and $\begin{bmatrix} y^n \\ w^n \end{bmatrix}$. Similarly we have that $\mathcal{Q}_n$ is generated by the vectors $\begin{bmatrix} x^n \\ y^n \end{bmatrix}$ and $\begin{bmatrix} z^n \\ w^n \end{bmatrix}$.

Using a similar result as Proposition 4.2.15, one can see that we can express $\text{Im}(M_{n+1})$ similarly,

$$\text{Im}(M_{n+1}) = \text{Span}\Bigg(\begin{bmatrix} x^n \\ x^{n-1}y \\ \vdots \\ xy^{n-1} \\ y^n \end{bmatrix}, \begin{bmatrix} z^n \\ z^{n-1}w \\ \vdots \\ zw^{n-1} \\ w^n \end{bmatrix} \Bigg).$$

Now, we can create an isomorphism between $\mathcal{Q}_n$ and $\text{Im}(M_{n+1})$ by mapping the generators to each other. Similarly we can create an isomorphism between $\mathcal{P}_n$ and $\text{Im}(M_{n+1}^T)$.

Up until now, we have only considered $\mathcal{P}_n$ and $\mathcal{Q}_n$ as R-modules. However, since we care about naive homotopies, we also need to consider them as $R[T]$-modules.

Definition 4.2.16. Let $A = R$ or $A = R[T]$. $\mathcal{P}_n(A)$ is the A-module generated by the elements $\begin{bmatrix} x^n \\ z^n \end{bmatrix}$ and $\begin{bmatrix} y^n \\ w^n \end{bmatrix}$. Similarly, $\mathcal{Q}_n(A)$ is the A-module generated by the elements $\begin{bmatrix} x^n \\ y^n \end{bmatrix}$ and $\begin{bmatrix} z^n \\ w^n \end{bmatrix}$.

Note that $\mathcal{P}_n(R[T])$ and $\mathcal{Q}_n(R[T])$ are the line bundles on the scheme $\mathcal{J} \times \mathbb{A}^1_k$. Since $\mathcal{J}$ and $\mathcal{J} \times \mathbb{A}^1_k$ have the same Picard group [9, Proposition II.6.6] we know that we have all the information we need for calculating naive homotopies of morphism from $\mathcal{J}$ to $\mathbb{P}^1$. Also note that Proposition 4.2.11, 4.2.12, 4.2.15 and Theorem 4.2.13 all hold when considering $\mathcal{P}_n(R[T])$ and $\mathcal{P}_n(R[T])$.

4.3 Morphisms from $\mathcal{J}$ to $\mathbb{P}^1$

Theorem 4.2.2 gives us a way to characterise all morphisms from a scheme X to $\mathbb{P}^1$ by knowing all the invertible sheaves of X and their global sections. If we let $X = \mathcal{J}$ and $A = k$ in the Theorem 4.2.2 above, we have already calculated what the invertible sheaves are. The global sections correspond to elements of the projective module. So we have the following description of morphisms from $\mathcal{J}$ to $\mathbb{P}^1$.

Theorem 4.3.1. Let $A = R$ or $A = R[T]$. The datum of a k-scheme morphism $\varphi : \mathcal{J} \longrightarrow \mathbb{P}^1_k$ is equivalent to the datum of one of the following:

1. A positive integer n, and two elements of $f, g \in \mathcal{P}_n(A)$ (resp. $\mathcal{Q}_n(A)$) such that f and g generate $\mathcal{P}_n(A)$ (resp. $\mathcal{Q}_n(A)$).

2. Two elements $f, g \in A$, where there exists $U, V \in A$, such that $fU + gV = 1$.

Definition 4.3.2. $A = R$ or $A = R[T]$. We will say that a morphism (f, g) has degree n (resp. $-n$) if it generates $\mathcal{P}_n(A)$ (resp. $\mathcal{Q}_n(A)$). We will say that it has degree 0 if (f, g) generate A.

Note that the condition describing morphisms of degree 0 arises from using the structure sheaf $\mathcal{O}_{\mathcal{J}}$ as the invertible sheaf.

In [6], Cazanave studies the naive homotopy classes of pointed endomorphisms of $\mathbb{P}^1$. We need to define pointedness for morphisms from $\mathcal{J}$ to $\mathbb{P}^1$.

Definition 4.3.3. We say that a morphism $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ is pointed if it sends the point corresponding to the maximal ideal $(x-1, y, z, w)$ to the point $[a : 0] \in \mathbb{P}^1$, with $a \in k^\times$.

In the following two propositions, we give other ways of describing a morphism $f : \mathcal{J} \longrightarrow \mathbb{P}^1$.

Proposition 4.3.4. Let $A = R$ or $A = R[T]$. For the the pair of sections

$$(f, g) = \Big(\begin{bmatrix} f_0 x^n + f_1 y^n \\ f_0 z^n + f_1 w^n \end{bmatrix}, \begin{bmatrix} g_0 x^n + g_1 y^n \\ g_0 z^n + g_1 w^n \end{bmatrix} \Big) \in \mathcal{P}_n(A)^2,$$

the following statements are equivalent.

1. They generate $\mathcal{P}_n(A)$.

2. The ideals $(f_0 x^n + f_1 y^n, g_0 x^n + g_1 y^n) \subset A[x^{-1}]$ and $(f_0 z^n + f_1 w^n, g_0 z^n + g_1 w^n) \subset A[w^{-1}]$ are unit ideals.

3. There exist $X, Y, Z, W \in A$ such that

$$X(x^n f_0 + y^n f_1) + Y(x^n g_0 + y^n g_1) + Z(z^n f_0 + w^n f_1) + W(z^n g_0 + w^n g_1) = 1.$$

Proof. We will start by proving that (1) implies (2). Assume that (f, g) are a pair of sections that generate $\mathcal{P}_n(A)$. Then there exists some elements $U, V \in A$ such that

$$Uf + Vg = \begin{bmatrix} x^n \\ z^n \end{bmatrix}.$$

In the localization $A[x^{-1}]$ we have that $w = \frac{yz}{x}$. If we work with $Uf + Vg$ in the ring $A[x^{-1}]$, we get that

$$(U(f_0 + f_1 \frac{y^n}{x^n}) + V(g_0 + g_1 \frac{y^n}{x^n})) \begin{bmatrix} x^n \\ z^n \end{bmatrix} = \begin{bmatrix} x^n \\ z^n \end{bmatrix}.$$

We need the coefficient to be equal to 1. So we have

$$\frac{U}{x^n}(x^n f_0 + y^n f_1) + \frac{V}{x^n}(x^n g_0 + y^n g_1) = 1.$$

This is equivalent to saying that the ideal $(f_0 x^n + f_1 y^n, g_0 x^n + g_1 y^n) \subset A[x^{-1}]$ is the unit ideal. Carrying out a similar argument in the localization $A[w^{-1}]$ gives the condition that $(f_0 z^n + f_1 w^n, g_0 z^n + w_1 y^n) \subset A[w^{-1}]$.

We will now prove that (2) implies (3). We know that there exists $U_x, V_x \in A[x^{-1}]$ and $U_w, V_w \in A[w^{-1}]$ such that

$$U_x(x^n f_0 + y^n f_1) + V_x(x^n g_0 + y^n g_1) = 1, \tag{$*$}$$

and

$$U_w(z^n f_0 + w^n f_1) + V_w(z^n g_0 + w^n g_1) = 1. \tag{$**$}$$

For each set of U_x, V_x, U_w, V_w, there exists an integer $d \geq 0$, such that

$$x^d U_x, x^d V_x, w^d U_w, w^d V_w \in A.$$

Multiplying $(*)$ by x^d and $(**)$ by w^d yields the following equations

$$\begin{aligned} x^d U_x(x^n f_0 + y^n f_1) + x^d V_x(x^n g_0 + y^n g_1) &= x^d, \\ w^d U_w(z^n f_0 + w^n f_1) + w^d V_w(z^n g_0 + w^n g_1) &= w^d. \end{aligned}$$

By proposition 4.2.14, we know that x^d and w^d span the space $\{x^{d-i}w^i\}$. This means that $1 = (x+w)^d$ can be expressed as a linear combination of x^d and w^d. So there exists $X, Y, Z, W \in A$ such that

$$X(x^n f_0 + y^n f_1) + Y(x^n g_0 + y^n g_1) + Z(z^n f_0 + w^n f_1) + W(z^n g_0 + w^n g_1) = 1. \tag{4.1}$$

We will finish it off by proving that (3) implies (1). We will do this by proving that

$$(Xx^n + Zz^n)f + (Yx^n + Wz^n)g = \begin{bmatrix} x^n \\ z^n \end{bmatrix},$$

and that

$$(Xy^n + Zw^n)f + (Yy^n + Ww^n)g = \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

Notice that we have

$$\begin{aligned} (Xx^n + Zz^n)f + (Yx^n + Wz^n)g &= \Big(X(x^n f_0 + y^n f_1) + Y(x^n g_0 + y^n g_1) \\ &\quad + Z(z^n f_0 + w^n f_1) + W(z^n g_0 + w^n g_1)\Big) \begin{bmatrix} x^n \\ z^n \end{bmatrix} \\ &= \begin{bmatrix} x^n \\ z^n \end{bmatrix}. \end{aligned}$$

Similarly we get that

$$(Xy^n + Zw^n)f + (Yy^n + Ww^n)g = \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

We can now generate $\mathcal{P}_n(A)$ with these two elements. $\square$

An analogous result for $\mathcal{Q}_n$ also exists.

Proposition 4.3.5. Let $A = R$ or $A = R[T]$. For the the pair of sections

$$(f, g) = \left(\begin{bmatrix} f_0 x^n + f_1 z^n \\ f_0 y^n + f_1 w^n \end{bmatrix}, \begin{bmatrix} g_0 x^n + g_1 z^n \\ g_0 y^n + g_1 w^n \end{bmatrix} \right) \in \mathcal{Q}_n(A)^2,$$

the following statements are equivalent.

1. They generate $\mathcal{Q}_n(A)$.
2. The ideals $(f_0 x^n + f_1 z^n, g_0 x^n + g_1 z^n) \subset A[x^{-1}]$ and $(f_0 y^n + f_1 w^n, g_0 y^n + g_1 w^n) \subset A[w^{-1}]$ are unit ideals.
3. There exist $X, Y, Z, W \in A$ such that

   $$X(x^n f_0 + z^n f_1) + Y(x^n g_0 + z^n g_1) + Z(y^n f_0 + w^n f_1) + W(y^n g_0 + w^n g_1) = 1.$$

Proof. The proof is similar to the result for $\mathcal{P}_n(A)$. $\square$

Before we give some examples of morphisms, we will define some notation for describing a pair of sections.

Definition 4.3.6 (Compact form)**.** Let $A = R$ or $A = R[T]$. For $f_0, f_1, g_0, g_1 \in A$, we define

$$(f_0, f_1 : g_0, g_1)_n^p := \left(\begin{bmatrix} f_0 x^n + f_1 y^n \\ f_0 z^n + f_1 w^n \end{bmatrix}, \begin{bmatrix} g_0 x^n + g_1 y^n \\ g_0 z^n + g_1 w^n \end{bmatrix} \right) \in \mathcal{P}_n(A)^2,$$

and

$$(f_0, f_1 : g_0, g_1)_n^q := \left(\begin{bmatrix} f_0 x^n + f_1 z^n \\ f_0 y^n + f_1 w^n \end{bmatrix}, \begin{bmatrix} g_0 x^n + g_1 z^n \\ g_0 y^n + g_1 w^n \end{bmatrix} \right) \in \mathcal{Q}_n(A)^2.$$

Definition 4.3.7 (Expanded form)**.** Let $A = R$ or $A = R[T]$. For any integer $n > 0$, let $r_{0i}, r_{1i} \in A$, for $0 < i < n$. We define

$$(r_{00}, \ldots, r_{0n} : r_{10}, \ldots, r_{1n})^p$$
$$:= \left[\begin{bmatrix} r_{00} x^n + r_{01} x^{n-1} y + \ldots + r_{0n} y^n \\ r_{00} z^n + r_{01} z^{n-1} w + \ldots + r_{0n} w^n \end{bmatrix} : \begin{bmatrix} r_{10} x^n + r_{11} x^{n-1} y + \ldots + r_{1n} y^n \\ r_{10} z^n + r_{11} z^{n-1} w + \ldots + r_{1n} w^n \end{bmatrix} \right],$$

and

$$(r_{00}, \ldots, r_{0n} : r_{10}, \ldots, r_{1n})^q$$
$$:= \left[\begin{bmatrix} r_{00} x^n + r_{01} x^{n-1} z + \ldots + r_{0n} z^n \\ r_{00} y^n + r_{01} y^{n-1} w + \ldots + r_{0n} w^n \end{bmatrix} : \begin{bmatrix} r_{10} x^n + r_{11} x^{n-1} z + \ldots + r_{1n} z^n \\ r_{10} y^n + r_{11} y^{n-1} w + \ldots + r_{1n} w^n \end{bmatrix} \right].$$

We can also restate Proposition 4.3.4 in terms of expanded forms. A similar restatement of Proposition 4.3.5 also exists.

Corollary 4.3.8. Let $A = R$ or $A = R[T]$. For the the pair of sections

$$(r_{00}, \ldots, r_{0n} : r_{10}, \ldots, r_{1n})^p \in \mathcal{P}_n(A)^2,$$

the following statements are equivalent.

1. They generate $\mathcal{P}_n(A)$.
2. The ideals $(r_{00}x^n + \ldots + r_{0n}y^n, r_{10}x^n + r_{1n}y^n) \subset A[x^{-1}]$ and $(r_{00}z^n + \ldots + r_{0n}w^n, r_{10}z^n + r_{1n}w^n) \subset A[w^{-1}]$ are unit ideals.
3. There exist $X, Y, Z, W \in A$ such that

$$\begin{gathered} X(r_{00}x^n + \ldots + r_{0n}y^n) + Y(r_{10}x^n + \ldots + r_{1n}y^n) \\ +Z(r_{00}z^n + \ldots + r_{0n}w^n) + W(r_{10}z^n + \ldots + r_{1n}w^n) = 1. \end{gathered}$$

Proof. Since all expanded forms can be written as elements of compact form, it follows directly. □

Example 4.3.9. 1. $(1, 0 : 0, 1)^p_n$ and $(1, 0 : 0, 1)^q_n$ for all positive n. These correspond to the basis of $\mathcal{P}_n(R)$ and $\mathcal{Q}_n(R)$ as in Proposition 4.2.15, so they are morphisms.

2. $(x, 0 : 0, 1)^p_n$ and $(x, 0 : 0, 1)^q_n$. Since x is a unit in $R[x^{-1}]$, the ideal (x^{n+1}, y^n) is the unit ideal. Similarly, since w is a unit in $R[w^{-1}]$, the ideal (xz^n, w^n) is the unit ideal. This proves that $(x, 0 : 0, 1)^p_n$ is a morphism. Likewise for $(x, 0 : 0, 1)^q_n$, the ideals $(x^{n+1}, z^n) \subset R[x^{-1}]$ and $(xy^n, w^n) \subset R[w^{-1}]$ are both unit ideals.

Since $[\mathbb{P}^1, \mathbb{P}^1]^N$ naturally sits inside $[\mathcal{J}, \mathbb{P}^1]^N$, one can expect a map sending endomorphisms of $\mathbb{P}^1$ to a morphism from $\mathcal{J}$ to $\mathbb{P}^1$.

Proposition 4.3.10. Any pointed rational function $f = \frac{X^n + a_{n-1}X^{n-1} + \ldots + a_0}{b_{n-1}X^{n-1} + \ldots + b_0} \in \mathcal{F}_n(k)$ describing a morphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ corresponds to two pointed morphisms $P, Q : \mathcal{J} \longrightarrow \mathbb{P}^1$. These are $(1, a_{n-1}, \ldots, a_0 : 0, b_{n-1}, \ldots, b_0)^p$ and $(1, a_{n-1}, \ldots, a_0 : 0, b_{n-1}, \ldots, b_0)^q$.

Proof. We will prove that $(1, a_{n-1}, \ldots, a_0 : 0, b_{n-1}, \ldots, b_0)^p$ is a morphism, the proof for $(1, a_{n-1}, \ldots, a_0 : 0, b_{n-1}, \ldots, b_0)^q$ is similar. We will use method 2 from Corollary 4.3.8. We want to show that

$$I := (z^n + a_{n-1}wz^{n-1} + \ldots + a_0w^n, b_{n-1}wz^{n-1} + \ldots + b_0w^n) \subset R[w^{-1}],$$

generates the unit ideal. If we divide through by w^n, and substitute $\frac{z}{x}$ with X, we get f. The resultant of f is a unit and this implies that there exists $U, V \in k[\frac{z}{w}]$ such that

$$U((\frac{z}{w})^n + a_{n-1}(\frac{z}{w})^{n-1} + \ldots + a_0) + V(b_{n-1}(\frac{z}{w})^{n-1} + \ldots + b_0) = 1.$$

This means that I is in fact the unit ideal. By Proposition 2.1.7 the homogeneous polynomials

$$(x^n + a_{n-1}yx^{n-1} + \ldots + a_0y^n, b_{n-1}yx^{n-1} + \ldots + b_0y^n),$$

also have a unit resultant, so they generate the unit ideal in $R[x^{-1}]$. So $(1, a_{n-1}, \dots, a_0 : 0, b_{n-1}, \dots, b_0)^p$ is a morphism $\mathcal{J} \longrightarrow \mathbb{P}^1$. One can easily see that the morphism is pointed. $\square$

We can define an automorphism τ on R. Defined as follows:

$$\begin{aligned} \tau(1) &= 1, \\ \tau(x) &= x, \\ \tau(w) &= w, \\ \tau(y) &= z, \\ \tau(z) &= y. \end{aligned}$$

This automorphism gives rise to the following proposition.

Proposition 4.3.11. Let $(f_0, f_1 : g_0, g_1)^p_n$ be a morphism, then $(\tau(f_0), \tau(f_1) : \tau(g_0), \tau(g_1))^q_n$ is a morphism as well.

Proof. For any $\alpha \in R$, denote $\bar{\alpha} := \tau(\alpha)$. Since $(f_0, f_1 : g_0, g_1)^p_n$ is a morphism, we know that there exist $X, Y, Z, W \in R$ such that

$$X(x^n f_0 + y^n f_1) + Y(x^n g_0 + y^n g_1) + Z(z^n f_0 + w^n f_1) + W(z^n g_0 + w^n g_1) = 1.$$

Applying τ to this yields

$$\bar{X}(x^n \bar{f_0} + z^n \bar{f_1}) + \bar{Y}(x^n \bar{g_0} + z^n \bar{g_0}) + \bar{Z}(y^n \bar{f_0} + w^n \bar{f_1}) + \bar{W}(y^n \bar{g_0} + w^n \bar{g_1}) = 1.$$

This proves that $(\bar{f_0}, \bar{f_1} : \bar{g_0}, \bar{g_1})^q_n$ is a morphism. $\square$

At last we will define the sets $\mathfrak{P}_n(A)$ and $\mathfrak{Q}_n(A)$. We will define them as the sets containing all morphisms coming from $\mathcal{P}_n(A)$ and $\mathcal{Q}_n(A)$.

$$\mathfrak{P}_n(A) := \left\{ (f, g) \in (\mathcal{P}_n(A))^2 \middle| f, g \text{ generate } \mathcal{P}_n(A) \right\}$$

$$\mathfrak{Q}_n(A) := \left\{ (f, g) \in (\mathcal{Q}_n(A))^2 \middle| f, g \text{ generate } \mathcal{Q}_n(A) \right\}$$

We will use the sets $\mathfrak{P}^\bullet_n(A)$ and $\mathfrak{Q}^\bullet_n(A)$ for the sets consisting of pointed morphisms. Note that τ induces isomorphisms $\mathfrak{P}_n(A) \cong \mathfrak{Q}_n(A)$ and $\mathfrak{P}^\bullet_n(A) \cong \mathfrak{Q}^\bullet_n(A)$.

Chapter 5

Homotopies of morphisms from $\mathcal{J}$ to $\mathbb{P}^1$

Since we want to create something that resembles the monoid structure on $[\mathbb{P}^1, \mathbb{P}^1]^N$, we will start by looking for a group structure on $[\mathcal{J}, \mathbb{P}^1]^N$.

5.1 The hunt for group structures on $[\mathcal{J}, \mathbb{P}^1]^N$

Since one has that the elements of Pic$(\mathcal{J})$ is achieved from tensoring the bundles $\mathcal{P}_1$ and $\mathcal{Q}_1$, it is natural to think that these isomorphisms could be a clue to the group operation. We will first start by looking for a monoid structure on $\coprod \mathfrak{P}_i(R)$. As the group operation would first and foremost be a monoid operation as well. We begin by ruling out a natural first choice.

Proposition 5.1.1. The binary operation sending $M : \mathcal{P}_i^2 \times \mathcal{P}_j^2 \longrightarrow \mathcal{P}_{i+j}^2$, defined

$$M((a_0, a_1 : b_0, b_1)_i^p, (c_0, c_1 : d_0, d_1)_j^p)$$
$$:= \left[\begin{bmatrix} (a_0x^i + a_1y^i)(c_0x^j + c_1x^j) \\ (a_0z^i + a_1w^i)(c_0z^j + c_1w^j) \end{bmatrix} : \begin{bmatrix} (b_0x^i + b_1y^i)(d_0x^j + d_1x^j) \\ (b_0z^i + b_1w^i)(d_0z^j + d_1w^j) \end{bmatrix} \right],$$

does not define a monoid structure on $\coprod \mathfrak{P}_i(R)$ or $\coprod \mathfrak{P}_i^{\bullet}(R)$.

Proof. We will prove it by counterexample. We pick the two morphisms $(1, 1 : y, z)_2^p = (1, 0, 1 : 0, 1, 0)^p$ and $(1, 0 : 0, 1)_1^p$. We then get

$$M((1, 0, 1 : 0, 1, 0)^p, (1, 0 : 0, 1)_1^p) = (1, 0, 1, 0 : 0, 0, 1, 0)^p.$$

When working over $\mathcal{P}_i(k)$, M simply works as multiplication of numerators and denominators of pointed rational functions.

$$M((1, 0, 1 : 0, 1, 0)^p, (1, 0 : 0, 1)_1^p \longleftrightarrow \frac{X^2+1}{X}\frac{X}{1} = \frac{X^3+X}{X} \longleftrightarrow (1, 0, 1, 0 : 0, 0, 1, 0)^p.$$

One can see that $\frac{X^3+X}{X}$ is not a pointed morphism, due to the common root of the numerator and denominator. $\square$

Intuitively, we need some way to ensure that our operation preserves the morphism property. For pointed rational functions, we ensure this by representing morphisms as elements of $SL_2(k[X])$. Recall that for any morphism $(a_0, a_1 : b_0, b_1)_n^p$ there exist related Bézout relations in $R[x^{-1}]$ and $R[w^{-1}]$. We will write

$$U_x(x^n a_0 + y^n a_1) + V_x(x^n b_0 + y^n b_1) = 1$$

and

$$U_w(z^n a_0 + w^n a_1) + V_w(z^n b_0 + w^n b_1) = 1.$$

From these Bézout relations, we can create the matrices

$$M_x := \begin{pmatrix} x^n a_0 + y^n a_1 & -V_x \\ x^n b_0 + y^n b_1 & U_x \end{pmatrix} \in SL_2(R[\frac{1}{x}]),$$
$$M_w := \begin{pmatrix} z^n a_0 + w^n a_1 & -V_w \\ z^n b_0 + w^n b_1 & U_w \end{pmatrix} \in SL_2(R[\frac{1}{w}]).$$

For example, for the morphism $(1, 0 : 0, 1)_1^p$, we get the matrices:

$$M_x = \begin{pmatrix} x & 0 \\ y & \frac{1}{x} \end{pmatrix},$$
$$M_w = \begin{pmatrix} z & -\frac{1}{w} \\ w & 0 \end{pmatrix}.$$

Squaring the matrices yields

$$M_x^2 = \begin{pmatrix} x^2 & 0 \\ yx + \frac{y}{x} & \frac{1}{x^2} \end{pmatrix},$$
$$M_w^2 = \begin{pmatrix} z^2 - 1 & -\frac{z}{w} \\ zw & -1 \end{pmatrix}.$$

These matrices describe new Bézout relations, but they do not seem to determine a unique element of $\mathfrak{P}_n^\bullet(R)$.

5.2 A conjecture and its implications

Observe that if we study $\mathfrak{P}_n^\bullet(k[T])$ and $\mathfrak{Q}_n^\bullet(k[T])$, Propositions 4.3.10 and 4.3.11 give us a bijection between the sets $\mathfrak{P}_n^\bullet(k[T])$, $\mathfrak{Q}_n^\bullet(k[T])$ and $\mathcal{F}_n(k[T])$.

Definition 5.2.1. Let $A = R$ or $A = k$. Denote by $[\mathfrak{P}_n^\bullet(A), \mathbb{P}]^N$ (resp. $[\mathfrak{Q}_n^\bullet(A), \mathbb{P}]^N$) the pointed naive homotopy classes of maps from $\mathcal{J}$ to $\mathbb{P}^1$ arising from elements in $\mathfrak{P}_n^\bullet(A)$ (resp. $\mathfrak{Q}_n^\bullet(A)$). Also denote

$$P^N(A) := \prod_{i \geq 0} [\mathfrak{P}_i^\bullet(A), \mathbb{P}^1]^N,$$

$$Q^N(A) := \prod_{i\geq 0}[\mathfrak{Q}_i^\bullet(A), \mathbb{P}^1]^N.$$

Since every element of $\mathfrak{P}_n^\bullet(k)$ and $\mathfrak{Q}_n^\bullet(k)$ can be expressed as a pointed rational function. We may use the monoid operation $\oplus^N$ to create a monoid structure on $P^N(k)$ and $Q^N(k)$.

Theorem 5.2.2. The follwing are isomorphisms of monoids.

$$P^N(k) \cong [\mathbb{P}^1, \mathbb{P}^1]^N \quad \text{and} \quad Q^N(k) \cong [\mathbb{P}^1, \mathbb{P}^1]^N.$$

Proof. The induced monoid structure on $P^N(k)$ and $Q^N(k)$ creates an isomorphism

$$P^N(k) \cong Q^N(k) \cong \Big(\coprod_{n\geq 0}(\pi_0^N \mathcal{F}_n)(k), \oplus^N\Big) \cong [\mathbb{P}^1, \mathbb{P}^1]^N.$$

$\square$

Since $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is the group completion of $[\mathbb{P}^1, \mathbb{P}^1]^N$, it is reasonable to assume that $P^N(k)$ and $Q^N(k)$ act as inverses to each other in the group structure. However, we need some way of handling $P^N(R)$ and $Q^N(R)$. The following conjectures are a shot at that.

Conjecture 5.2.3.

1. Any element of $\mathfrak{P}_n^\bullet(R)$ (resp. $\mathfrak{Q}_n^\bullet(R)$) is naively homotopic to an element of $\mathfrak{P}_n^\bullet(k)$ (resp. $\mathfrak{Q}_n^\bullet(k)$)
2. Moreover, If $F, G \in \mathfrak{P}_n^\bullet(R)$ (resp. $\mathfrak{Q}_n^\bullet(R)$) and $F \overset{\mathrm{p}}{\sim} G$ in $P^N(R)$ (resp. $Q^N(R)$). F is homotopic to some $\tilde{F} \in \mathfrak{P}_n^\bullet(k)$ (resp. $\mathfrak{Q}_n^\bullet(k)$) and G is homotopic to some $\tilde{G} \in \mathfrak{P}_n^\bullet(k)$ (resp. $\mathfrak{Q}_n^\bullet(k)$). Then $\tilde{F}$ and $\tilde{G}$ lies in the same homotopy class in $P^N(k)$ (resp. $Q^N(k)$).

The conjecture can also be further specified. We will start by an example. Consider the morphism $(x, 1, 0 : 0, y, w)^p$. We have that

$$(x, 1, 0 : 0, y, w)^p = (1 - w, 1, 0 : 0, y, 1 - x)^p.$$

Since $x\begin{bmatrix} y \\ w \end{bmatrix} = y\begin{bmatrix} x \\ z \end{bmatrix}$, and $w\begin{bmatrix} x \\ z \end{bmatrix} = z\begin{bmatrix} y \\ w \end{bmatrix}$, we can "move" the x's to the right and the w's to the left.

$$(1 - w, 1, 0 : 0, y, 1 - x)^p = (1, 1 - z, 0 : 0, y - y, 1)^p = (1, 1 - z, 0 : 0, 0, 1)^p$$

We can do this trick to remove any x's and w's from the expanded form of any morphism.

Conjecture 5.2.4. Let $F := (f_0, \dots, f_n : g_0, \dots, g_n)^p \in \mathfrak{P}_n(R)$ (resp. $\mathfrak{Q}_n(R)$), with $f_i, g_i \in k[y, z]$. Then

$$F \overset{\mathrm{p}}{\sim} (f_0(0,0), \dots, f_n(0,0) : g_0(0,0), \dots, g_n(0,0))^p.$$

We will now proceed with some results assuming that Conjecture 5.2.3 is true.

Theorem 5.2.5 (Assuming Conjecture 5.2.3 is true)**.**

$$P^N(R) \cong [\mathbb{P}^1, \mathbb{P}^1]^N,$$

$$Q^N(R) \cong [\mathbb{P}^1, \mathbb{P}^1]^N.$$

Proof. If Conjecture 5.2.3 is true, it immediately follows that $P^N(R) = P^N(k)$ and $Q^N(R) = Q^N(k)$. □

This allows us to do explicit calculations when working over any algebraically closed field.

As seen in Example 3.3.5,when k is algebraically closed we have

$$P^N(k) \cong Q^N(k) \cong \mathbb{N} \times k^\times.$$

For elements of $(n,\alpha)^P \in P^N$ and $(m,\beta) \in Q^N$,where $n,m \in \mathbb{N}$ and $\alpha,\beta \in k^\times$. We can also look at $(n,\alpha)^P$ as the $(n \times n)$ identity matrix, but with one entry replaced with α. The binary operation is as follows:

$$(n,\alpha)^P \oplus^N (m,\beta)^Q = \begin{cases} (0, \frac{\alpha}{\beta})^P = (0, \frac{\beta}{\alpha})^Q & p = q, \\ (n-m, \frac{\alpha}{\beta})^P & p > q, \\ (m-n, \frac{\beta}{\alpha})^Q & q < p. \end{cases}$$

5.3 What we know about Conjecture 5.2.3

Conjecture 5.2.3 might only be true for algebraically closed fields, as it can not be true for $\mathbb{R}$. The conjecture reduces $[\mathcal{J}, \mathbb{P}^1]^N$ to two copies of the monoid $[\mathbb{P}^1, \mathbb{P}^1]^N$. Over the real numbers we have

$$[\mathbb{P}^1, \mathbb{P}^1]^N \cong (\mathbb{N} \times \mathbb{N}) \times \mathbb{R}^\times,$$

where $\mathbb{N} \times \mathbb{N}$ can be thought of as the signature of a bilinear form. Since $[\mathcal{J}, \mathbb{P}^1]^N$ is isomorphic to the group completion, we have

$$[\mathcal{J}, \mathbb{P}^1]^N \cong (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^\times.$$

Conjecture 5.2.3 says all elements of $\mathfrak{P}^\bullet_n(R)$ and $\mathfrak{Q}^\bullet_n(R)$ can be expressed as pointed rational functions. This means that in the group completion, an element of $P^N(R)$, can be represented by $(n_p, m_p, r_p) \in (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^\times$, where $n_p, m_p \geq 0$. Similarly, an element $Q^N(R)$ can be represented by (n_q, m_q, r_q) with $n_q, m_q \leq 0$. Morphisms of degree 0, will correspond to elements $(a, -a, r)$, with $a \in \mathbb{Z}$. Note that there are elements in this group which can not be expressed as a pointed rational function. An example is $(2, -1, r) \in (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^\times$. This element is not in $P^N(R)$ or $Q^N(R)$, but it can be expressed as a sum of elements. $(2, 0, r) \oplus (0, -1, 1) = (2, -1, r)$.

Chapter 6

Homotopies of morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$

We shift our focus to $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$, because it turns out understanding $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ can help us understand $[\mathcal{J}, \mathbb{P}^1]^N$.

6.1 Why we study $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$

In [16], Morel computes $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. The proof is rather involved and requires more $\mathbb{A}^1$-homotopy theory than we are going to cover here. However, we will paraphrase the proof to highlight why $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ is an object of interest. The full proof can be read in Chapter 7.3 of [16]. Morel starts off with the fibre sequence

$$\mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^\infty.$$

Here $\mathbb{P}^\infty$ is the direct limit of $\mathbb{P}^n$. The fibre sequence gives rise to the following short exact sequence for homotopy classes,

$$1 \longrightarrow [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \longrightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \longrightarrow [\mathbb{P}^1, \mathbb{P}^\infty]^{\mathbb{A}^1} \longrightarrow 1.$$

This can be further simplified to

$$1 \longrightarrow [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \longrightarrow [\mathcal{J}, \mathbb{P}^1]^N \longrightarrow [\mathcal{J}, \mathbb{P}^\infty]^N \longrightarrow 1.$$

Morel then makes the crucial observation that the sequence splits and $[\mathbb{P}^1, \mathbb{P}^\infty]^{\mathbb{A}^1} \cong \mathbb{Z}$. This means we have

$$[\mathcal{J}, \mathbb{P}^1]^N \cong [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \oplus \mathbb{Z}.$$

So it turns out we can describe homotopy classes of maps from $\mathcal{J}$ to $\mathbb{P}^1$ by understanding the homotopy classes of maps from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$.

6.2 Morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$

6.2.1 A scheme theoretic way

Any morphism $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ can be composed with the inclusion morphism $i : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{A}^2$. This gives a morphism $\bar{f} = i \circ f$, as in the commutative diagram below.

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\bar{f}"] \arrow[d, "f"'] & \mathbb{A}^2 \\ \mathbb{A}^2 \setminus \{0\}. \arrow[ur, "i"'] & \end{tikzcd}$$

Since $\mathcal{J}$ and $\mathbb{A}^2$ are affine schemes, a morphism from $\mathcal{J}$ to $\mathbb{A}^2$ is a ring homomorphism

$$\bar{f}^* : k[s,t] \longrightarrow R.$$

We get that $\bar{f}^*$ is determined by where it sends the elements s and t. These elements are denoted by $p := \bar{f}^*(s)$ and $q := \bar{f}^*(t)$. The ideal $(s,t) \subset k[s,t]$ is the ideal corresponding to the point $(0,0) \in \mathbb{A}^2$. If $\bar{f}$ is a morphism coming from a morphism $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$, $\bar{f}^{-1}((s,t))$ should be empty. If there exists a point $\mathfrak{m} \in \mathcal{J}$ such that $\bar{f}(\mathfrak{m}) = (s,t)$, that implies $p, q \in \mathfrak{m}$. Since we want this to never happen, we need that (p,q) is the unit ideal.

Proposition 6.2.1. The datum of a k-scheme morphism $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to an element $(A,B) \in R^2$ where there exists $(U,V) \in R^2$ such that $AU + BV = 1$.

Similar calculations can be done to determine what the naive homotopies are, yielding the following proposition.

Proposition 6.2.2. The datum of a naive homotopy from $f : \mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to an element $(A,B) \in R[T]^2$ where there exists $(U,V) \in R[T]^2$ such that $AU+BV = 1$.

6.2.2 A shortcut using homotopy theory

Proposition 6.2.3 (Corollary 4.45 [1])**.** The scheme $\mathbb{A}^2 \setminus \{0\}$ is $\mathbb{A}^1$-homotopic to the scheme $SL_2(k) := \mathrm{Spec}(k[a,b,c,d]/(ad - bc - 1))$.

As we are now studying morphisms between affine schemes, we can now work with ring homomorphisms from the base rings. In this case we only need to look at ring homomorphisms from $k[a,b,c,d]/(ad - bc - 1))$ to R. The morphisms are decided by where we send the elements a, b, c and d. We can see that these elements are sent to a matrix in $SL_2(R)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & -V \\ B & U \end{pmatrix}$$

where $A, B, U, V \in R$ satisfy the Bézout relation. We can now pick $(A,B) \in R^2$ as our representative of the morphism $\mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$. This is because the homotopy class of

(A, B) is independent of our choice of U and V. Let $A, B, U_1, U_2, V_1, V_2 \in R$ such that $AU_1 + BV_1 = AU_2 + BV_2 = 1$. Then we have that the matrices

$$\begin{pmatrix} A & -V_1 \\ B & U_1 \end{pmatrix} \overset{\mathrm{p}}{\sim} \begin{pmatrix} A & -V_2 \\ B & U_2 \end{pmatrix}.$$

This is done through the matrix

$$\begin{pmatrix} A & -TV_1 - (1-T)V_2 \\ B & TU_1 + (1-T)U_2 \end{pmatrix} \in SL_2(R[T]).$$

6.3 The connection to maps of degree 0

Recall that a morphism of degree 0 from $\mathcal{J}$ to $\mathbb{P}^1$ can be described by two elements of R which generate R as an R-module. As we have just seen, the data of maps $\mathcal{J} \longrightarrow \mathbb{A}^2 \setminus \{0\}$ is the same as that of degree 0 maps from $\mathcal{J}$ to $\mathbb{P}^1$. As elements of $\mathfrak{P}_n^\bullet(R)$ have degree n and elements of $\mathfrak{Q}_n^\bullet(R)$, have degree $-n$. We can create the following exact sequence

$$0 \longrightarrow [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \longrightarrow [\mathcal{J}, \mathbb{P}^1]^N \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The sequence splits because we can create a map $d : \mathbb{Z} \to [\mathcal{J}, \mathbb{P}^1]^N$ by sending an integer n to some homotopy class of elements of degree n in $[\mathcal{J}, \mathbb{P}^1]^N$. Since $\deg \circ d = \mathrm{id}$ on $\mathbb{Z}$, the sequence splits. Because the sequence splits, any element in $[\mathcal{J}, \mathbb{P}^1]^N$ can be represented by an element in $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ and an integer. Since there is a group structure on $[\mathcal{J}, \mathbb{P}^1]^N$, there exists an element f_1 that acts as the identity on $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$, and has degree 1. This means it can effectively be used to translate any element of $\mathfrak{P}_n^\bullet(R)$ and $\mathfrak{Q}_n^\bullet(R)$ to their representative in $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. We will use $\ominus$ to indicate subtraction in $[\mathcal{J}, \mathbb{P}^1]^N$. For $g \in \mathfrak{P}_n^\bullet(R)$, we have

$$g \underbrace{\ominus f_1 \ominus \ldots \ominus f_1}_{n \text{ times}} \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N.$$

Example 6.3.1. Over the complex numbers, we can find a candidate for f_1. Since $[\mathcal{J}, \mathbb{P}^1]^N = \mathbb{C}^\times \times \mathbb{Z}$ we are looking for a morphism from $\mathcal{J}$ to $\mathbb{P}^1$ of degree one which lies in the homotopy class $(1, 1) \in \mathbb{C}^\times \times \mathbb{Z}$. Assuming that Conjecture 5.2.3 is true, we can pick $f_1 := (1, 0 : 0, 1)_1^p$. Our choice corresponds to the pointed rational function $\frac{X}{1}$. This has resultant 1 and degree 1, so it represents the homotopy class we are looking for.

Due to the splitting of the sequence, we get another way of describing a morphism from $\mathcal{J}$ to $\mathbb{P}^1$ up to naive homotopy.

Theorem 6.3.2. The datum of a k-scheme morphism $f : \mathcal{J} \longrightarrow \mathbb{P}^1$ up to naive homotopy is equivalent to an integer n and an element $(A, B) \in R^2$ where there exists $(U, V) \in R^2$ such that $AU + BV = 1$.

6.4 How the problem becomes a problem in $SL_2(R)$

As mentioned in the previous section there is a connection between elements of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ and homotopy classes of matrices in $SL_2(R)$. This means we can study naive homotopies of morphisms from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$ by studying elements of $SL_2(R[T])$. Any element of $SL_2(R[T])$ can be converted to an element of R^2 by "forgetting" the second column. Now that we have a representation in $SL_2(R[T])$, we can create a binary operation akin to $\oplus^N$ for pointed rational functions. Let $(A_i, B_i) \in R^2$ be unimodular rows for $i = 1, 2$. One can find elements $(U_i, V_i) \in R^2$ such that $A_iU_i + B_iV_i = 1$. We define A_3, B_3, U_3 and V_3 by setting

$$\begin{pmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{pmatrix} := \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}.$$

It does not matter which U_i and V_i we pick, since the naive homotopy classes of (A_i, B_i) are independent of our choice. We write

$$(A_1, B_1) \oplus (A_2, B_2) = (A_3, B_3).$$

Example 6.4.1.

1. The matrix multiplication

$$\begin{pmatrix} x-w & -4z \\ y & x-w \end{pmatrix} \begin{pmatrix} x-w & -4z \\ y & x-w \end{pmatrix} = \begin{pmatrix} (x-w)^2 - 4yz & -8z(x-w) \\ 2y(x-w) & (x-w)^2 - 4yz \end{pmatrix},$$

yields that $(x-w, y) \oplus (x-w, y) = \Big((x-w)^2 - 4yz, 2y(x-w)\Big)$.

2. We can use this to create homotopies. For example:

$$\begin{pmatrix} x-w & -4z \\ y & x-w \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 5T & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 3(x-w) - 20zT & -\frac{4}{3}z \\ 3y + 5T(x-w) & \frac{1}{3}(x-w) \end{pmatrix}.$$

The homotopy yields that $\Big(3(x-w), 3y\Big) \overset{\mathrm{p}}{\sim} \Big(3(x-w) - 20z, 3y + 5(x-w)\Big)$.

For $a \in R$, the matrices

$$\begin{pmatrix} 1 & aT \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ aT & 1 \end{pmatrix}$$

yield homotopies between elementary matrices and the identity matrix. This gives us the following proposition.

Proposition 6.4.2. Any element of $SL_2(R)$ which can be written as a product of elementary matrices is homotopic to the identity.

Naturally, one can now ask the question, do there exist matrices that are not homotopically trivial? The following result makes use of homotopy trivial matrices to give us an interesting homotopy relation.

Proposition 6.4.3. Let $(A, B) \in R^2$ be a morphism from $\mathcal{J}$ to $\mathbb{A}^2 \setminus \{0\}$. For all elements $u \in k^\times$, we have $(A, B) \overset{\mathrm{p}}{\sim} (u^2A, B) \overset{\mathrm{p}}{\sim} (A, u^2B)$.

Proof. Consider $\begin{pmatrix} A & -V \\ B & U \end{pmatrix}$, the representation of (A, B) in $SL_2(R)$. For $u \in k^\times$, the matrix $\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$ can be decomposed into elementary matrices, so it is homotopic to the identity matrix. We will use this matrix to create our homotopies. We have

$$\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} = \begin{pmatrix} u^2 A & -V \\ B & \frac{U}{u^2} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} = \begin{pmatrix} A & -\frac{V}{u^2} \\ u^2 B & D \end{pmatrix}.$$

this shows that $(A, B) \overset{p}{\sim} (u^2 A, B) \overset{p}{\sim} (A, u^2 B)$. $\square$

6.5 Real realization

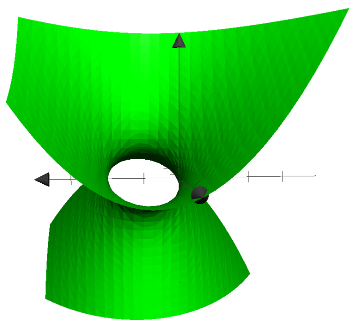

Figure 6.1: Real realization of $\mathcal{J}$

Recall that $\mathcal{J} = \mathrm{Spec}(k[x, y, z]/(x(1-x) - yz))$. We can visualize our scheme $\mathcal{J}$ by realizing it over the real numbers. In Figure 6.1, there is a picture of the real realization of $\mathcal{J}$. Notably, it has a hole in the middle, so it is homotopic to a cylinder, hence having the same fundamental group as $\mathbb{R}^2 \setminus (0, 0)$. With some clever projection from $\mathcal{J}(\mathbb{R})$ to $\mathbb{R}^2 \setminus (0, 0)$, it is possible to gain information about $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. The following proposition gives some insight about relationships between naive homotopies and homotopies in the realization.

Proposition 6.5.1. The real realization of a naive homotopy $H : \mathcal{J} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^2 \setminus \{0\}$ gives a homotopy in $\mathbb{R}^2 \setminus (0, 0)$ with respect to euclidean topology.

Proof. Consider the matrix corresponding to the naive homotopy in $SL_2(R[T])$

$$H = \begin{bmatrix} f(T) & -V(T) \\ g(T) & U(T) \end{bmatrix}.$$

For all T this matrix has determinant 1, so for any ring homomorphism $R[T] \longrightarrow \mathbb{R}[T]$, the matrix is an element of $SL_2(\mathbb{R}[T])$. Any matrix representing $(0,0) \in \mathbb{R}^2 \setminus (0,0)$ is a matrix with determinant 0, so it is not in $SL_2(\mathbb{R}[T])$. This means that the image of $H(T)$ is in $\mathbb{R}^2 \setminus (0,0)$ for all $T \in \mathbb{R}$. So it is a homotopy. $\square$

Proposition 6.5.1 implies that if one can find a nontrivial loop in the realization, its preimage is nontrivial in $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. What is now needed is a suitable projection from the realization to $\mathbb{R}^2 \setminus (0,0)$. We will pick the map $C_\theta : R \to \mathbb{R}$ for $\theta \in [0, 2\pi)$ defined as follows:

$$\begin{aligned}
C_\theta(1) &= 1, \\
C_\theta(x) &= \frac{1}{2}\cos\theta + \frac{1}{2}, \\
C_\theta(y) &= -\frac{1}{2}\sin\theta, \\
C_\theta(z) &= -\frac{1}{2}\sin\theta.
\end{aligned}$$

One can see that C_θ is a ring homomorphism, because

$$C_\theta(x(1-x) - yz) = (\frac{1}{2})^2 - \frac{\cos^2\theta}{4} - \frac{\sin^2\theta}{4} = 0.$$

All that is left is to find a nontrivial morphism. We will first look at the morphism of $F = (x - w, 2y)$ under C_θ,

$$C_\theta(x - w, 2y) = (\cos\theta, -\sin\theta).$$

By letting θ vary from 0 to 2π, we see that this describes a loop in $\mathbb{R}^2 \setminus (0,0)$. Note that if we consider the image of the element in $SL_2(R)$ representing $(x - w, 2y)$, we get

$$C_\theta : \begin{pmatrix} x - w & -2z \\ 2y & x - w \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It follows from the trigonometric identities that one has

$$\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix} = \begin{pmatrix} \cos(n+m)\theta & \sin(n+m)\theta \\ -\sin(n+m)\theta & \cos(n+m)\theta \end{pmatrix}.$$

This implies that $(x - w, 2y)^{\oplus n}$ corresponds to $(\cos n\theta, -\sin n\theta)$. So we do in fact get that the image of $(x - w, 2y)$ generates the fundamental group of $\mathbb{R}^2 \setminus (0,0)$.

Theorem 6.5.2. The morphisms $(x - w, uy)$ and $(x - w, uy)$ are nontrivial morphisms for all $u \in \mathbb{R}^\times$. Their realizations also generate the fundamental group of $\mathbb{R}^2 \setminus (0,0)$.

Proof. We will examine the realization of $(x-w, uy)$ and $(x-w, uz)$.

$$C_\theta(x-w, uy) = C_\theta(x-w, uz) = (\cos\theta, -\frac{u}{2}\sin\theta)$$

The realization parametrizes an ellipse around the origin, so it is nontrivial. The realized matrix of $(x-w, uy)$ is

$$C_\theta : \begin{pmatrix} x-w & -\frac{4z}{u} \\ uy & x-w \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \frac{2}{u}\sin\theta \\ -\frac{u}{2}\sin\theta & \cos\theta \end{pmatrix}.$$

By taking powers of the matrix, we get

$$\begin{pmatrix} \cos\theta & \frac{2}{u}\sin\theta \\ -\frac{u}{2}\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \frac{2}{u}\sin n\theta \\ -\frac{u}{2}\sin n\theta & \cos n\theta \end{pmatrix}.$$

So the realization generates the fundamental group. □

We can use Proposition 6.4.3, to say more about the morphisms $(x-w, uy)$.

Proposition 6.5.3. For all $u > 0$, we have $(x-w, uy) \overset{\mathrm{p}}{\sim} (x-w, y)$. For all $u < 0$, we have $(x-w, uy) \overset{\mathrm{p}}{\sim} (x-w, -y)$.

Proof. Over the $\mathbb{R}$, Proposition 6.4.3 lets us multiply by any positive number. So $(x-w, uy) \overset{\mathrm{p}}{\sim} (x-w, y)$ for all $u > 0$. Similarly for $(x-w, -y)$, we get homotopies to the element $(x-w, -uy)$ for all $u > 0$. □

We will now shift our focus to another realizable morphism, $(x-u, y)$, where $u \neq 0, 1$. Its matrix representation is

$$\begin{pmatrix} x-u & \frac{z}{u^2-u} \\ y & \frac{w-u}{u^2-u} \end{pmatrix} = \begin{pmatrix} x-u & \frac{z}{u^2-u} \\ y & -\frac{x-(1-u)}{u^2-u} \end{pmatrix}.$$

Theorem 6.5.4. The morphisms $(x-u, y)$ and $(x-u, z)$ are nontrivial for when $0 < u < 1$.

Proof. Once again we realize the morphisms.

$$C_\theta(x-u, y) = C_\theta(x-u, z) = (\frac{1}{2}\cos\theta + \frac{1}{2} - u, -\frac{1}{2}\sin\theta)$$

This only describes a loop wrapped around the origin if $0 < u < 1$, so $(x-u, y)$ and $(x-u, z)$ are nontrivial for those values of u. □

We can relate $(x-w, y)$ to these morphisms as well. Since we have $(x-w, y) = (2x-1, y)$ and the $\sqrt{2} \in \mathbb{R}^\times$, we can create the following homotopy

$$(x-w, y) \overset{\mathrm{p}}{\sim} (x - \frac{1}{2}, y).$$

With some more clever conjugation of matrices, we can create more interesting identities. We have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-u & \frac{z}{u^2-u} \\ y & -\frac{x-(1-u)}{u^2-u} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{x-(1-u)}{u^2-u} & -y \\ -\frac{z}{u^2-u} & x-u \end{pmatrix}.$$

This gives us the homotopy $(x-u, y) \overset{\mathrm{p}}{\sim} (-\frac{x-(1-u)}{u^2-u}, -\frac{z}{u^2-u})$. Since $u^2 - u < 0$ for all $0 < u < 1$, we can use Proposition 6.4.3 to get $(x-u, y) \overset{\mathrm{p}}{\sim} (x-(1-u), z)$.

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