Sondre Jensen

Valuation of Single and Multiple Forest Rotations Under Stochastic Prices: A Real Options Approach

Master's thesis in Industrial Economics and Technology Management
Supervisor: Lars Hegnes Sendstad
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Abstract

This paper develops a real options model to value a forest stand. A trinomial tree is used to approximate stochastic timber prices, and subsequently, used in a dynamic programming approach to calculate the stand value and harvest age. The model is specified for both geometric Brownian motion (GBM) and the mean reverting Ornstein–Uhlenbeck process (MR), as well as for a single forest rotation and multiple forest rotations. The multiple rotations problem results in a perpetual compound option, and a heuristic approach is developed to simplify calculations. The approach is conceptually simple and in response to the perceived lack of transparency in real options models, which, perhaps, has hampered widespread use by forestry practitioners. The model is applied in a series of numerical examples. The results suggest that the choice of the stochastic process has a significant impact on the stand value and optimal harvest age.

Sammendrag

Denne artikkelen presenterer en realopsjonsmodell for verdsettelse innen skogbruk. Et trinomisk tre anvendes for å tilnærme stokastiske tømmerpriser, og dynamisk programmering benyttes for å beregne verdien av et skogområde og bestemme optimalt hogsttidspunkt. Modellen er spesifisert for både en enkel skogrotasjon og for multiple skogrotasjoner, samt for to ulike stokastiske prisprosesser: geometric Brownian motion (GBM) og mean reversion (MR). Multiple skogrotasjoner er et eksempel på en compound-opsjon, og en heuristisk tilnærming utvikles for å forenkle verdsettelsen. Realopsjonsmodellen er anvendbar, og er en respons på mangelen av transparens i mange realopsjonsmodeller, hvilket kan ha bidratt til å hemme vidstrakt bruk av realopsjonsanalyse innen skogbruk. Modellen benyttes i en rekke numeriske eksempler, og det demonstreres at valget av stokastisk prosess har stor innvirkning på verdien av et skogområde og hogsttidspunktet.

Preface

I would like to extend my sincere gratitude to my supervisor Lars Hegnes Sendstad for his unparalleled support and valuable guidance. It has truly been a pleasure working with you throughout the process of writing this paper.

Furthermore, I would like to thank Helmer Belbo (Norwegian Institute of Bioeconomy Research) and Roald Sand (Trøndelag Research & Development Institute) for invaluable discussions about the forestry industry.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

A booming home-construction market has contributed to elevated timber prices in the past years (Wall Street Journal, 2020a). However, the coronavirus pandemic is now effectively causing a construction slowdown (Wall Street Journal, 2020b), and the restrictions on international trade, travel, and business have caused significant uncertainty in the forestry industry (Timber Trades Journal, 2020). Although the coronavirus pandemic's implications are currently the most visible, climate-related disturbances remain a significant concern and driver of volatility. On the supply side, vast and multiple wildfires are destroying swathes of trees and leaving behind scorched earth where new trees struggle to grow (National Geographic, 2020). Meanwhile, the rapid expansion of spruce bark beetles, particularly in Central Europe, forces forestry managers to prematurely harvest large areas of attacked forests, causing sudden surges in the supply of timber (Reuters, 2019). The outbreaks of bark beetles are not unique to Europe. Beetles are also ravaging North-American forests (National Geographic, 2020), and the tropical forests face a similar problem with vines, that use other plants as hosts, killing trees (CNBC, 2020). On the demand side, the use of alternative wood-based fuels has proliferated in recent years, providing a novel source of demand (Deutsche Welle, 2020). Concomitantly, forests are increasingly recognized for their role as carbon sinks, and new policies are initiated to reflect this (Gren and Aklilu, 2016). Consequently, within an environment of increasing uncertainty, the challenge of making optimal harvesting decisions and correct valuations become rather formidable.

In the context of irreversible harvesting decisions and uncertainty, the importance of valuing managerial flexibility has long been recognized (Insley and Rollins, 2005). However, traditional valuation methods such as discounted cash flow analysis come short when timber prices are stochastic, and the conventional approaches to determine the optimal time to harvest, which largely build on the pioneering work of Faustmann (1849), may yield significantly wrong conclusions (Gjolberg and Guttormsen, 2002).

A recent strand of the forestry economics literature draws on the real options literature to accurately incorporate the value of flexibility, yet the practical application in forestry management remains rather sparse (Manley and Niquidet, 2010). In other industries, such as the pharmaceutical sector, complexity, and perceived lack of transparency stand out as the main obstacles to more widespread use of real option analysis (Hartmann and Hassan, 2006), and continuous-time analytical models have been characterized by "low practical validity" (Worren et al., 2002). Consequently, the potential of binomial approaches and other tree based models have been emphasized (Hartmann and Hassan, 2006). Therefore, I present a trinomial tree model in this study. The trinomial model is an extension of the binomial model and is able to approximate more complex stochastic processes such as mean reversion (MR), yet it remains conceptually similar. The mathematical simplicity and ease of exposition should help make the real options approach more transparent, and, in turn, the results more palatable to forestry practitioners. I use dynamic programming to calculate the stand value and expected rotation age for a single forest rotation where the forest is assumed to grow deterministically. The model is specified for both geometric Brownian motion (GBM) and MR timber prices. The assumption of GBM is widely used in forestry economics as it makes the problem of determining the optimal harvest age, and stand value more tractable (Insley and Rollins, 2005). However, as noted by Gjolberg and Guttormsen (2002), there are reasons, both theoretical and empirical, to believe that timber prices do not follow GBM, and instead should be modelled as a MR process.

I follow Insley (2002) in investigating the implications of the specification of the price process, and demonstrate that stand value and harvest age is significantly affected by the choice of stochastic process. The stand value is found to be higher under MR compared to under GBM for prices below the equilibrium. Moreover, the critical price is shown to be higher under GBM than MR in the early years of the forest rotation, and conversely, lower under GBM compared to MR when the growth rate has declined. This confirms the findings of Insley (2002); however, it demonstrates a different approach than the finite difference method used in her article. To the best of my knowledge, MR has not been addressed using a trinomial tree in the forestry economics literature. Consequently, I make a contribution to the forestry economics literature by developing an intuitive and transparent model which can handle not only GBM, but also the more complex MR process. Moreover, I estimate the parameters of GBM and MR from historical Norway spruce prices from the Norwegian forestry industry, which presents a novel data set. The results indicate that the choice of stochastic process has a profound impact on the optimal harvesting decisions, and as such, demonstrates the importance of selecting an appropriate stochastic process when making decisions in forestry.

I extend on the single forest rotation by addressing multiple forest rotations. In the multiple rotations problem, the option to replant is made available after harvest. The multiple rotations problem is significantly more complex than the single rotation problem (Insley and Rollins, 2005), and similar in nature to the valuation of a perpetual American compound option. I approach the multiple rotations problem using a trinomial tree to discretize prices, and subsequently employ a dynamic programming approach to calculate the stand value and expected rotation age. I make a methodological contribution to implementing real options approaches in forestry by increasing the practical validity of the compound option model by simplifying the valuation methodology. Specifically, I show that a heuristic approach which uses an option approach for the initial forest rotation, and approximates the subsequent rotations using the Faustmann formula (Faustmann, 1849) yields a similar result as the compound option model. To the best of my knowledge, this presents a novel approach to the valuation of a forest stand with multiple rotations. I demonstrate the applicability of the heuristic approach using historical Norway spruce prices to calculate the value and expected rotation age under both GBM and MR. I show that the stand value and rotation age are higher under GBM than under MR. Moreover, I find that under MR with a sufficiently high speed of reversion, the stand value and rotation age is independent of the current timber price. The results imply the need to revise fixed rotation age policies in forestry to ensure economic efficiency.

The remainder of this study is organized as follows. I discuss related work in Chapter 2, before introducing the data in Chapter 3. Chapter 4.1 reviews the dynamic programming technique, and Chapter 4.2 presents the binomial model as an intuitive introduction to the tree based models. Subsequently, Chapter 4.3 addresses the valuation of a forest stand considering a single rotation using the trinomial tree. I introduce multiple forest rotations in Chapter 4.4. In Chapter 5.1, I present and discuss the results considering a single forest rotation. Chapter 5.2 demonstrates the applicability of the real options approach to multiple forest rotations. Lastly, Chapter 6 concludes the paper, considers potential limitations of the real options approach, and outlines ideas for future work.

Chapter 2

Literature Review

The seminal work of Faustmann (1849) introduces a method to value a piece of land devoted to forestry, yet it remains best known as a benchmark model for determining the optimal rotation age[1]. The optimal rotation problem is one of the oldest and most important in forestry economics (Pearse, 1967), and the pioneering paper of Faustmann (1849) has spawned a substantial body of literature. A strand of this literature illustrates the implications of stochastic prices, which is the focus of this paper and the following literature review.[2]

In the Faustmann setting there is perfect certainty with respect to the model parameters. Specifically, the formulation assumes that interest rates, timber prices and management costs remain constant over time, and that the forest growth can be described by a deterministic function. This may have been justifiable when dealing with a relatively static economy, like that of 19th century Germany. However, in today's economy, technological advances affect the supply of timber, and dynamic market conditions continue to impact demand (Newman, 1988). Consequently, price uncertainty is a necessary extension to the Faustmann model, and the implications for valuation and the optimal rotation age are important research questions in forestry economics.

Stylized models involving price uncertainty emerged in the 1980s (Insley, 2002), with the common assumption of prices following GBM. Clarke and Reed (1989) and Reed and Clarke (1990) extends on these formulations and show that if the cost of harvesting is ignored, a barrier rule is optimal. That is, as soon as the trees reach a certain size, they should be harvested, regardless of prices. As pointed out by Brock and Rothschild (1986), a GBM process has no optimal stopping rule by itself. The barrier rule is a direct consequence of the GBM price process and absence of harvesting costs in the formulation of Clarke and Reed (1989) and Reed and Clarke (1990). As noted by the authors, a barrier

[1] The age at which the forest should be harvested to maximize the value derived from it.

[2] For a broader overview, Newman (1988) and Newman (2002) provides an extensive bibliography and discuss the development of the optimal rotation literature.

rule would not exist for a more complex price process such as MR, or when harvesting costs are included in the model.

As soon as harvesting costs are taken into account, the optimal rotation age cannot be specified in advance as it depends on the current price, however an expected rotation age can be provided. Except for in certain restrictive cases, analytical solutions does not exist. As a result, numerical methods need to be invoked in order to solve the optimal rotation problem. Thomson (1992) includes harvesting costs in his formulation, and use the binomial option pricing model of Cox et al. (1979) to value a forest stand. The study shows that the the optimal rotation age is greater than what the Faustmann rule prescribes and the value of the forest stand is generally higher than in the deterministic Faustmann setting. Moreover, the rotation age decreases and the forest value increases as the price volatility increases. Intuitively, the price uncertainty creates an option value of waiting, which in turn, makes it optimal to wait longer for more information. This application of an option valuation technique to a managerial decision such as deciding between harvesting or letting the forest grow, is known as real options analysis. Real options analysis is commonly used in decision making under uncertainty, and adapts techniques for financial options to real-world decisions (Dixit et al. (1994), Trigeorgis et al. (1996)). The real options approach is inherently more mathematically complex than most of the traditional methods employed in forestry economics and require significantly more computing effort (Kant and Alavalapati, 2014), yet makes up for it by being able to handle uncertainty properly.

The real option models are based on several assumptions, including the assumption of the stochastic process describing the evolution of prices. A common assumption in most of the early work is that the price in any time period is independent of any other period. For instance, Brazee and Mendelsohn (1988) model the prices as independent draws from a normal probability distribution, Haight (1993) assumes the price trend is drawn from a triangular probability distribution, and Lohmander (1987) use a uniform probability distribution for the price. However, empirical evidence suggests that prices exhibit serial correlation (Washburn and Binkley, 1990). As a result, researchers conduct statistical tests in order to gain insight into the the behaviour of timber prices. Several studies find evidence of stationarity in prices (Hultkrantz (1993) and Yin and Newman (1996), while others reject it (Prestemon, 2003). In general, stationary prices, such as MR, have no predictable patterns in the long-run, and the prices are roughly flat (although some cyclic behaviour is possible). This is in contrast to non-stationary process such as GBM where there is a drift over time. However, generally speaking the tests are inconclusive, primarily due to the short span of the time series data available (Kant and Alavalapati, 2014). As a result, the literature on the stochastic optimal rotation problem have investigated the implication of various price processes, and can broadly be separated into two groups according to the price process. The first group assumes a stationary price process, typically the mean reverting Ornstein-Uhlenbeck process, while the second group employ a non-stationary process such as a random walk or GBM.

Insley (2002) and Insley and Rollins (2005) compare and contrast option value and optimal rotation age under the assumption of GBM and MR. The numerical approach taken in

Insley (2002) is an implicit finite difference method in combination with a penalty method (Zvan et al., 1998). The results show that MR yield a higher option value than GBM when the current price is below the equilibrium, reflecting the eventual price increase towards the equilibrium level. Furthermore, in the early years of forest growth, the MR process has lower critical prices than GBM. Intuitively, there is steadily increasing prices under GBM, and thus incentive for the forest manager to delay harvest while the forest is still growing. Additionally, the uncertainty increases with time, leading to a greater value of waiting. In the MR case, the forest manager takes advantage of prices above the mean by harvesting immediately.

This study contributes to the forestry economics literature by developing a trinomial tree model which is able to approximate both GBM and MR. The model is used to investigate the implications of the stochastic process on the stand value and rotation age. Monthly price data for Norway spruce is used to determine the parameters of a GBM- and MR-process, presenting a new source of data. To the best of my knowledge, the trinomial model is a novel approach in forestry economics. The trinomial model provides an extra degree of freedom over the binomial model used by Thomson (1992). As a result, the model can approximate more complex stochastic processes than GBM. Moreover, the trinomial model is also more easily explained and accepted than alternatives such as finite difference and simulation methods (Mun, 2002). Therefore, the trinomial model should be an intuitively appealing tool to forestry practitioners, and potentially contribute to more widespread use of real options analysis in the forestry industry. Note that the methodology is not limited to GBM and the MR process, and can be applied to other processes where a numerical approach is required. For instance, the long-run level to which prices tend to revert is not necessarily static, and a natural extension could follow Pindyck (1999) which incorporates shifts in the slope or level of the long-run level when forecasting energy prices. In addition to addressing the single forest rotation, I contribute to the forestry economics literature by formulating a compound option model to address multiple forest rotations. Furthermore, I develop a heuristic approach, which combines the option approach and the traditional Faustmann value. The heuristic approach should help foster the adoption of compound option valuation techniques in forestry management.

Chapter 3

Data

The data used in this study consist of yield data and historical price data for Norway spruce. The yield data is presented in Section 3.1, and is used to fit a growth function for a specific site index[1] and thinning regime. The growth function is used to model the volume available for harvest in the trinomial model. The historical price data for Norway spruce is described in Section 3.2, and subsequently used in Subsection 5.1.3 to estimate the parameters of a GBM- and MR-process and provide an application of the real options model developed in this study to a real-world scenario.

3.1 Yield Data

Yield tables for Norway spruce (Picea abies (L.) H. Karst) is presented in Braastad (1975) for different site indexes and thinning regimes. The data in Table 3.1 is for an initial stand of 3000 Norway spruce trees per hectare at a site index of H_{40} = 23[2]. The number of trees is decreased to 900 per hectare through 5 thinnings.

The yield table gives the volume at a limited set of ages, and in order to interpolate the volume at the missing age values a function of the form $y(t) = e^{a-b/t}$ is fit to the data using nonlinear regression. The functional form follows Payandeh (1973), and is similar to the growth function used in Thomson (1992). The resulting function is $y(t) = e^{7.52-69.79/t}$. Prior to the initial thinning in year 30, I assume no merchantable volume, and in subsequent years I assume 90% of the grown volume is merchantable. Since, in general, empirical models are only appropriate within the range of data that was used for model development, the constraint that volume does not increase after year 80 is imposed, which

[1] In forestry management, the site index describes the potential for trees to grow at a particular location or "site".

[2] A site index of H_{40} implies that the arithmetic mean height of the top 100 trees is 23 meters when the tree age at breast height is 40 years. The age at breast height is the number of years since the tree reached "breast height", which is 1.3m in most countries.

is close to the range of data used. The growth function is formulated as:

$$Q(t) = \begin{cases} 0, & \text{if } t \leq 30 \\ Q(80), & \text{if } t > 80 \\ 0.9e^{7.52-69.79/t}, & \text{otherwise} \end{cases} \tag{3.1}$$

Table 3.1: Yield table for Norway spruce (H_{40} = 23)

	Volume (m3/ha)		
Age	Before thinning	Thinning	After thinning
23	84	-	84
26	135	-	135
30	216	26	190
34	267	38	229
38	308	44	264
42	343	34	309
46	385	48	337
49	390	-	390
53	459	-	459
57	526	-	526
61	589	-	589
65	649	-	649
69	707	-	707

3.2 Historical Price Data

The historical price data for Norway spruce is from *Skogfondsdatabasen*[3], which is maintained by the Norwegian Agriculture Agency. The prices are deflated using the monthly consumer price index for Norway, and using 2020 as the base year. Further, the historical prices are reported for spruce sawlogs and spruce pulpwood. Sawlogs are generally greater in diameter, and are the most financially valuable part of the trees, and is in contrast to other parts of the tree which are designated as pulpwood. A single spruce price is calculated as the volume weighted average of the sawlogs and pulpwood price, in order to simplify the real options approach by avoiding two correlated price processes. National yearly harvest volumes from *Skogfondsdatabasen* are used to obtain the weights. The historical volumes are shown in Figure 3.1, with the solid line representing share of total volume sold as sawlogs. In general, most of the harvested timber is sold as sawlogs, and the split between sawlogs and pulpwood has remained relatively stable over time.

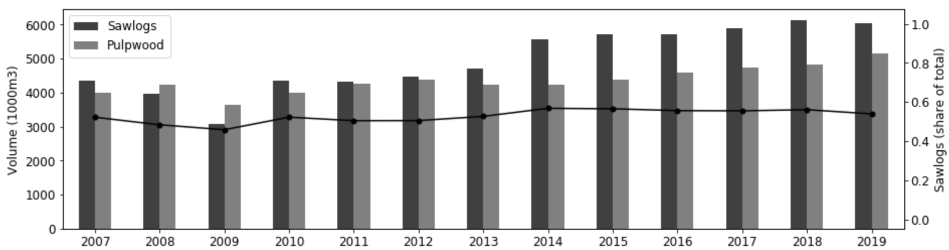


Figure 3.1: Harvested volumes split by sawlogs and pulpwood

The price of both sawlogs and pulpwood have remained relatively stable since 2013 as seen in Figure 3.2. In May 2020, the price of spruce sawlogs was 447 NOK/m3 and spruce pulpwood was 294 NOK/m3. The weighted price was 376 NOK/m3 as implied by a split between sawlogs and pulpwood of 54-46. Descriptive statistics for spruce prices are shown in Table 3.2. Note that the standard deviation of the combined price process is similar to the original time series. Hence, the correlation between pulpwood and sawlogs is high, and our combined price process is unlikely to lead to severe valuation errors.

[3] All data can be found at https://www.landbruksdirektoratet.no/no/statistikk/skogbruk/tommeravvirkning/tommeravvirkning-og-priser-2

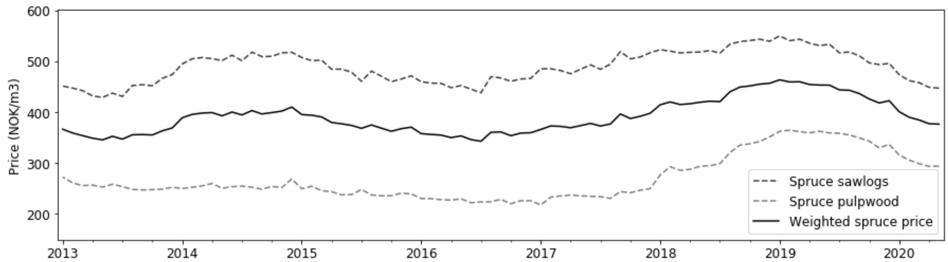


Figure 3.2: Evolution of spruce pulpwood prices, spruce sawlogs prices, and weighted spruce prices from January 2013 to May 2020

Table 3.2: Descriptive statistics for Norway spruce prices

	Mean	Median	St. Dev.	IQR	25th Percentile	75th Percentile	N
Spruce sawlogs	489.0	492.2	31.8	55.7	460.6	516.3	89
Spruce pulpwood	270.2	252.5	44.0	56.5	237.5	294.0	89
Weighted spruce price	391.3	387.5	33.6	51.6	363.3	414.9	89

Chapter 4

Methodology

In the following chapter, I introduce the methodology used to determine the value of a forest stand under stochastic prices. I begin by reviewing the dynamic programming method in Section 4.1, and proceed by presenting the binomial model developed by Cox et al. (1979), which was applied by Thomson (1992) to value a forest stand. As noted by Mun (2002), industry acceptance of the real options approach has mostly been in the use of binomial trees as they are easily explained to and accepted by management. Therefore, the binomial model is included as an intuitive introduction to the approach. Subsequently, the binomial model is extended to the trinomial model in Section 4.3, which can be used to approximate more complex stochastic processes such as MR. Lastly, in Section 4.4, I address multiple forest rotations, an example of a perpetual compound option, and develop a heuristic approach which simplifies calculations.

4.1 Dynamic Programming

Suppose the stochastic timber price at time t is P_t, the deterministic timber volume available for harvest is Q_t, and C is a constant harvesting cost. The forestry manager seeks to maximize the value of the forest stand, denoted by V in the following, by choosing a sequence of actions. The actions available at every time step are *harvest* and *wait*, and as such the problem of maximizing the stand value is a optimal stopping problem due to the binary choice. The dynamic programming method is based on splitting decisions in parts that comprise a sequence in time, and it aims to find the optimal path of decisions. The idea behind this decomposition is stated in Bellman's Principle of Optimality (Bellman, 1966):

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.".

In other words, the principle suggest to consider the decision at time t separately from all later decisions. The result of this decomposition is formally stated in the Bellman equation, which relates the value at time t to the value at time $t+1$. In the forestry setting, harvesting yields an immediate payoff $\Omega(P_t, Q_t) = (P_t - C)Q_t$, which is commonly referred to as as the termination payoff. By waiting, the option to harvest remains alive, which is referred to as the continuation value, and a similar choice is available one time step ahead. By letting E_t denote the expectation calculated using the information available at time t the Bellman equation can be written as:

$$V(P_t, Q_t) = \max\left\{\Omega(P_t, Q_t), \frac{1}{1+r} E_t\big[V(P_{t+1}, Q_{t+1})\big]\right\}, \tag{4.1}$$

where r is the appropriate one period discount rate. Due to the dependence of $V(P_t, Q_t)$ on the values one step ahead, future periods must be evaluated first. Assuming there is a finite time horizon T, we can start at the end and work backward. In this specific application to forestry, it is reasonable to assume a finite horizon due to the eventual decline of tree growth rate, which prohibits postponing harvest indefinitely. At time T there is presumably no value gained from waiting, and therefore the trees are harvested if prices are sufficiently high. Assuming there is a single forest rotation, and ignoring any alternative value for the land, the value at time T is given by:

$$V(P_T, Q_T) = \max\big\{\Omega(P_T, Q_T), 0\big\} \tag{4.2}$$

Subsequently, the value can be calculated at time $T-1$ as:

$$V(P_{T-1}, Q_{T-1}) = \max\left\{\Omega(P_{T-1}, Q_{T-1}), \frac{1}{1+r} E_t\big[V(P_T, Q_T)\big]\right\}, \tag{4.3}$$

which in turn enables calculation of $V(P_{T-2}, Q_{T-2})$ and so on, until $V(P_0, Q_0)$ is reached, which is the present value of the forest stand.

4.2 The Binomial Model

In general, the price of timber is assumed to be governed by a continuous stochastic process. However, in practice, the forestry manager does not observe continuous prices but rather monthly or daily. Therefore, a discrete approximation of the stochastic price process is not only reasonable, but also reflects reality more accurately. The binomial model discretizes the stochastic price process using a binomial tree where the nodes represent potential future prices. Suppose the price starts at P_0. In the next time period it will either increases by a proportionate amount u or decrease by a proportionate amount d. The probability of an up move is π_u, and the probability of a down move is $1 - \pi_u$ as illustrated in Figure 4.1. The choice of parameters, u, d, and π_u, depend on the stochastic process.

A stochastic process commonly used to model economic and financial variables is GBM (Dixit et al., 1994), which is widely used in the forestry economics literature (Thomson (1992) and Insley (2002)). The price of timber P_t is then assumed to satisfy the following stochastic differential equation:

$$dP_t = \alpha P_t dt + \sigma P_t dz_t, \tag{4.4}$$

where $\alpha \geq 0$ is the percentage drift rate, $\sigma \geq 0$ is the percentage volatility, and dz_t is the increment to a Wiener process at time t. By selecting appropriate parameters for the binomial model, (4.4) can be approximated in discrete time. The parameters suggested by Cox et al. (1979) are:

$$u = e^{\sigma\sqrt{\Delta t}}, \tag{4.5}$$

$$d = \frac{1}{u}, \tag{4.6}$$

$$\pi_u = \frac{e^{\alpha \Delta t} - d}{u - d}, \tag{4.7}$$

where Δt is the size of the time steps. By letting $\Delta t \to 0$ the binomial tree converges to (4.4). Note that due to the reciprocal magnitude of the up and down factors ((4.5), (4.6)) the tree is recombining. As a result, the number of nodes increases linearly with the number of time steps as opposed to exponentially. This makes the problem tractable even when the number of time steps is large.

Subsequent to constructing the price tree, a second tree is created where the nodes represent the option values at the various time steps. The value tree is populated using the dynamic programming approach outlined in Section 4.1. In the following, I denote the nodes of the price tree by $P_{t,j}$ and the nodes of the value tree by $V_{t,j}$, where t is a positive integer indicating the time step and j is a positive integer indicating the number of up moves leading to node t, j as illustrated in Figure 4.2. Furthermore, let Q_t be the deterministic volume of timber available for harvest at time t measured in cubic meters (m3), and C a constant harvesting cost per m3.

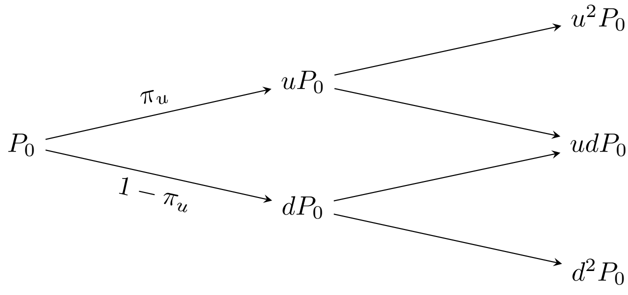


Figure 4.1: Binomial price tree with two time steps

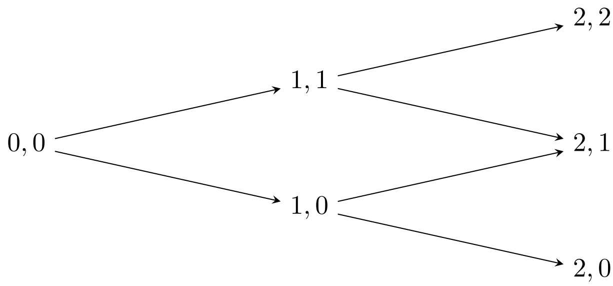


Figure 4.2: Notation used to denote the nodes

The Bellman equation introduced in Section 4.1 can be formulated using the prices and transition probabilities of the binomial tree as:

$$
\begin{aligned}
V(P_{t,j}, Q_t) = \max\Big\{ & (P_{t,j} - C)Q_t, \\
& e^{-r\Delta t}\big[\pi_u V(P_{t+1,j+1}, Q_{t+1}) + (1-\pi_u)V(P_{t+1,j}, Q_{t+1})\big]\Big\}, \\
& t \in \mathbb{Z} : 0 \le t < T,\ \ j \in \mathbb{Z} : 0 \le j \le t,
\end{aligned}
\tag{4.8}
$$

where r is an appropriate discount rate, and u, d and π_u are the parameters given by (4.5), (4.6), and (4.7), respectively. The first argument of the max function is value of immediate harvest, and the second is the expected value of delaying harvest discounted by the factor $e^{-r\Delta t}$. For a given price $P_{t,j}$ and available quantity Q_t harvest is chosen if the termination payoff exceeds the expected continuation value. The present value, $V(P_0, Q_0)$ is obtained using backwards recursion from the finite horizon T. At the finite horizon, the value of the terminal nodes are given by:

$$
V(P_{T,j}, Q_T) = \max\big\{(P_{T,j} - C)Q_T, 0\big\},\ \ j \in \mathbb{Z} : 0 \le j \le T \tag{4.9}
$$

The recursion proceeds by using (4.8) until $t = 0$ is reached, and $V(P_0, Q_0)$ is obtained.

4.3 The Trinomial Model

The trinomial model is an extension of the binomial model, and is conceptually similar. The trinomial model provides an extra degree of freedom by allowing three possible movements of the price at each node. As a result, more complex stochastic processes including MR can be approximated. At each node the price can move up by a proportional amount u, remain the same, or move down by a proportional amount d. Similar to in the binomial model, we require $d = \frac{1}{u}$ to ensure the tree is recombining. The corresponding transition probabilities are denoted by π_u, π_m and π_d, respectively. The general form of the tree for two time steps is illustrated in Figure 4.3.

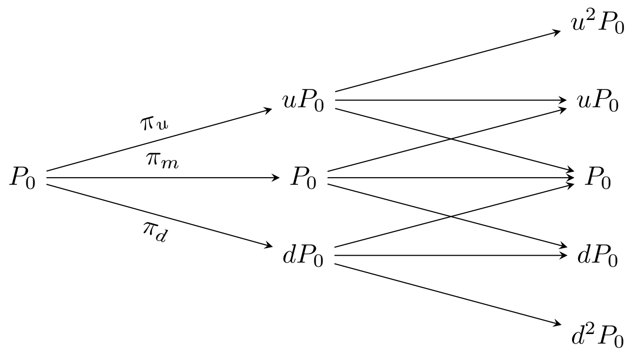


Figure 4.3: Trinomial price tree

The jump sizes and corresponding transition probabilities depend on the specific stochastic process approximated. Subsection 4.3.1 presents the setup for GBM, and Subsection 4.3.2 introduces the more complex setup used to approximate MR.

4.3.1 Geometric Brownian Motion

Suppose the continous time stochastic process is GBM. Again we discretize (4.4), however to accommodate three branches we view a single step on the trinomial tree as a combination of two steps on the binomial tree of Section 4.2. Thus, the parameters for the trinomial tree becomes:

$$u = e^{\sigma\sqrt{2\Delta t}}, \tag{4.10}$$

$$d = \frac{1}{u}, \tag{4.11}$$

$$\pi_u = \left(\frac{e^{\frac{\alpha \Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}} \right)^2, \tag{4.12}$$

$$\pi_d = \left(\frac{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{\frac{\alpha \Delta t}{2}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}} \right)^2, \tag{4.13}$$

$$\pi_m = 1 - \pi_u - \pi_d \tag{4.14}$$

Note that the choice of parameters is not unique, and there exists other choices which also ensure convergence to (4.4) as $\Delta t \to 0$. For instance, Hull (2003) demonstrates that by letting $u = e^{\sigma\sqrt{3\Delta t}}$, $d = 1/u$ and $\pi_m = 2/3$, the trinomial tree model is equivalent to an explicit finite difference approach. In general, any parameterization which ensures that the expected change and variance over a time step in the trinomial tree matches that of (4.4), and results in valid transition probabilities can be used.

Once the stochastic price process is discretized, the dynamic programming approach is used to solve for the stand value. The Bellman equation for the trinomial model is formulated as:

$$\begin{aligned} V(P_{t,j}, Q_t) = \max\Big\{ &(P_{t,j} - C)Q_t, \\ &e^{-r\Delta t}\big[\pi_u V(P_{t+1,j+1}, Q_{t+1}) + \pi_m V(P_{t+1,j}, Q_{t+1}) \\ &+ \pi_d V(P_{t+1,j-1}, Q_{t+1})\big]\Big\}, \\ &t \in \mathbb{Z} : 0 \le t < T, \;\; j \in \mathbb{Z} : -t \le j \le t, \end{aligned} \tag{4.15}$$

where $P_{t,j}$ is the price of timber at node t, j, Q_t is the quantity available, r is the discount rate, and π_u, π_d, π_m are the transition probabilities given by (4.12), (4.13), and (4.14), respectively. The values at the terminal nodes are given by:

$$V(P_{T,j}, Q_T) = \max\big\{(P_{T,j} - C)Q_T, 0\big\}, \;\; j \in \mathbb{Z} : -T \le j \le T \tag{4.16}$$

The recursion starts by calculating the terminal nodes using (4.16), and proceeds backwards using (4.15).

4.3.2 Mean Reversion

Although GBM is commonly used in the forestry economics literature it embodies certain unrealistic implications for timber prices. For instance, the price is allowed to rise indefinitely. As noted by Schwartz (1997), basic microeconomic reasoning suggest that when the price of a commodity is high, new producers will enter the market, thus increasing the supply and lowering the prices. Similarly, when prices fall, producers will exit and as a result prices will increase. In the short-run, timber prices might fluctuate due to adverse weather conditions or one-off events such as wildfires. However, in the long-run the price ought to stabilize at the marginal cost of harvesting. Therefore, timber prices might be more appropriately described by a mean reverting process.

A simple yet reasonable mean reverting process is the Ornstein-Uhlenbeck process:

$$dP_t = \eta(\mu - P_t)dt + \sigma dz_t, \tag{4.17}$$

where η is the speed of reversion, μ is the equilibrium price, that is the level to which P tends to revert, $\sigma \geq 0$ is the volatility, and dz_t is the increment to a Wiener process at time t. To approximate (4.17) in discrete time using a trinomial tree it is necessary to modify the standard branching pattern used everywhere in the tree for GBM. By allowing the branching pattern to vary depending on the price, a more complex tree geometry can be achieved. For most of the nodes the typical branching pattern is appropriate, which I refer to as branching pattern A. Branching pattern A is shown in Figure 4.4. To capture mean reversion we restrict prices from decreasing further when they are low relative to the equilibrium by using branching pattern B, which is shown in Figure 4.5. Similarly, when the price is relatively high compared to the equilibrium, they are restricted from increasing further by employing branching pattern C, which is shown in Figure 4.6.

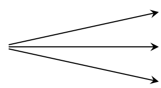


Figure 4.4: Branching pattern A

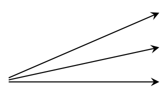


Figure 4.5: Branching pattern B

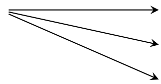


Figure 4.6: Branching pattern C

The branching pattern is determined based on the the number of up and down moves that have occurred on the path from the start node to the current node. Moreover, how quickly the branching pattern switches depend on the speed of reversion, and will become more clear below.

The first step in approximating the MR process is to construct a separate tree for the variable P'_t which is initially 0, and follows the process

$$dP'_t = -\eta P'_t dt + \sigma dz_t, \tag{4.18}$$

where η is the speed of reversion, $\sigma \geq 0$ is the volatility, and dz_t is the increment to a Wiener process at time t. Hull (2003) suggests the relationship between the space and time step to be:

$$\Delta P = \sigma\sqrt{3\Delta t} \tag{4.19}$$

Note that this differs from the trees constructed for GBM where proportionate jump sizes are set as the jumps are now additive. In the following, let node t, j denote the node reached after t time steps and j up moves, where t again is a positive integer, and j is a positive or negative integer. This is similar to the trinomial tree for GBM, but note the subtle difference in how j is now allowed to be negative compared to how it was defined as a positive integer in the binomial setup. This enables us to more easily define the switch between branching patterns in the following.

The tree for P'_t is constructed by initially following branching pattern A. Assuming $\eta > 0$, the branching switches to pattern C when j reaches the integer j_{max}, and to pattern B when j reaches the integer j_{min}. As suggested by Hull (2003), setting $j_{max} = \lceil \frac{0.184}{\eta\Delta t} \rceil$[1] ensures valid transition probabilities. Further, we set $j_{min} = -j_{max}$ in order to have a symmetrical tree which makes the tree construction process more efficient. The transition probabilities π_u, π_m, and π_d are found by matching the expected change and variance of the Ornstein-Uhlenbeck process (4.18) to the expected change and variance over time step Δt in the tree. The expected change of P'_t is $-\eta P'_t \Delta t$ and the variance is $\sigma^2 \Delta t$. Thus, for branching pattern A we require the expected change and variance over time step Δt to match:

$$\pi_u^A \Delta P - \pi_d^A \Delta P = -\eta j \Delta P \Delta t, \tag{4.20}$$

$$\pi_u^A \Delta P^2 + \pi_d^A \Delta P^2 = \sigma^2 \Delta t + \eta^2 j^2 \Delta P^2 \Delta t^2, \tag{4.21}$$

and that the probabilities sum to unity:

$$\pi_u^A + \pi_m^A + \pi_d^A = 1 \tag{4.22}$$

[1] $\lceil x \rceil$ denotes the ceiling function which takes a real valued number, x, and returns the least integer greater than or equal to x.

Using $\Delta P = \sigma\sqrt{3\Delta t}$, Hull (2003) shows that the equations are satisfied by setting

$$\pi_u^A = \frac{1}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 - \eta j \Delta t), \tag{4.23}$$

$$\pi_m^A = \frac{2}{3} - \eta^2 j^2 \Delta t^2 \tag{4.24}$$

$$\pi_d^A = \frac{1}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 + \eta j \Delta t) \tag{4.25}$$

Similarly, matching the expected change and variance over time step Δt for branching pattern B yields:

$$\pi_u^B = \frac{1}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 + \eta j \Delta t), \tag{4.26}$$

$$\pi_m^B = -\frac{1}{3} - \eta^2 j^2 \Delta t^2 - 2\eta j \Delta t, \tag{4.27}$$

$$\pi_d^B = \frac{7}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 + 3\eta j \Delta t) \tag{4.28}$$

Lastly, for branching pattern C:

$$\pi_u^C = \frac{7}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 - 3\eta j \Delta t), \tag{4.29}$$

$$\pi_m^C = -\frac{1}{3} - \eta^2 j^2 \Delta t^2 + 2\eta j \Delta t, \tag{4.30}$$

$$\pi_d^C = \frac{1}{6} + \frac{1}{2}(\eta^2 j^2 \Delta t^2 - \eta j \Delta t) \tag{4.31}$$

Figure 4.7 shows three time steps of a trinomial tree constructed for P'_t with $j_{max} = 2$, and illustrates how the branching switches from pattern A to pattern B at the second time step when the node with value $-2\Delta P$ is reached, and similarly from pattern A to pattern C at the node with value $2\Delta P$.

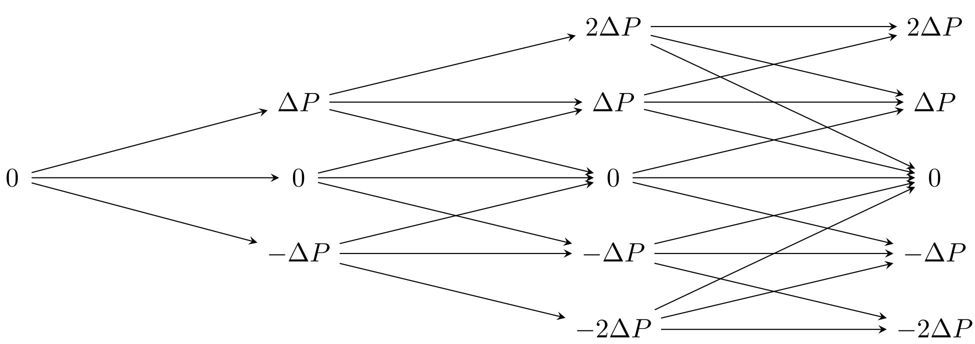


Figure 4.7: Trinomial tree for P' with $j_{max} = 2$

The trinomial tree for P'_t is subsequently transformed into a tree for our variable of interest, P_t. This is done by displacing the nodes $P'_{t,j}$ to adjust for the proper drift. In Hull (2003) the trinomial tree is constructed for an interest rate, and the purpose of the displacement is to fit the initial term structure by using the available forward rates. Kijima and Nagayama (1994) argue that the displacement is simply the expected value of the future interest rate, which makes the tree building process much more efficient. In our case, the underlying variable is the price of timber, and displacing the nodes of the tree using the expected value of the Ornstein-Uhlenbeck process is, therefore, reasonable. The expected value of the Ornstein-Uhlenbeck process in the discrete model form is given by Dixit et al. (1994):

$$E[P_t|P_{t-1}] = \mu + (P_{t-1} - \mu)e^{-\eta\Delta t} \tag{4.32}$$

The tree for P_t is thus obtained by setting the value of each node to $P_{t,j} = P'_{t,j} + E[P_t|P_{t-1}]$. That is, each node $P'_{t,j}$ is displaced by an amount $E[P_t|P_{t-1}]$, which accounts for the proper mean reverting drift. The resulting tree is then used in combination with the dynamic programming approach to calculate the option values. A key difference compared to the dynamic programming on the GBM tree is the dependence of the transition probabilities on the nodes. As such, the Bellman equation can be seen to take a different form depending on the value of j and can be formulated as:

$$V(P_{t,j}, Q_t) = \begin{cases} V^A(P_{t,j}, Q_t), & \text{if } j_{min} < j < j_{max} \\ V^B(P_{t,j}, Q_t), & \text{if } j = j_{min} \\ V^C(P_{t,j}, Q_t), & \text{if } j = j_{max}, \end{cases} \tag{4.33}$$

where

$$\begin{aligned} V^A(P_{t,j}, Q_t) = \max\Big\{ &(P_{t,j} - C)Q_t, e^{-r\Delta t}\big[\pi_u^A V(P_{t+1,j+1}, Q_{t+1}) \\ &+ \pi_m^A V(P_{t+1,j}, Q_{t+1}) + \pi_d^A V(P_{t+1,j-1}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.34}$$

$$\begin{aligned} V^B(P_{t,j}, Q_t) = \max\Big\{ &(P_{t,j} - C)Q_t, e^{-r\Delta t}\big[\pi_u^B V(P_{t+1,j+2}, Q_{t+1}) \\ &+ \pi_m^B V(P_{t+1,j+1}, Q_{t+1}) + \pi_d^B V(P_{t+1,j}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.35}$$

$$\begin{aligned} V^C(P_{t,j}, Q_t) = \max\Big\{ &(P_{t,j} - C)Q_t, e^{-r\Delta t}\big[\pi_u^C V(P_{t+1,j}, Q_{t+1}) \\ &+ \pi_m^C V(P_{t+1,j-1}, Q_{t+1}) + \pi_d^C V(P_{t+1,j-2}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.36}$$

and the transition probabilities are given by (4.23) through (4.31). Note that the expression for the value of immediate harvest is the same in (4.34), (4.35), and (4.36). However, the expression for the expected continuation value differs due to the specific transition probabilities involved, and the prices that can be transitioned to over the next time step. For instance, if the branching pattern emanating from a specific node is pattern A, (4.34) is used and the price one step ahead might increase, decrease or remain the same. In contrast, if the price is already well below the equilibrium level, branching pattern B emanates from the node. As such (4.35) is used, and the price one step ahead can either be higher or remain the same, but it is restricted from going lower.

The dynamic programming uses backwards recursion from the finite horizon T as before. The number of terminal nodes depend on the value of j_{max} and T. If T is sufficiently high, then the price will have reached the ceiling implied by j_{max} and floor implied by j_{min}, and there are $2j_{max}+1$ terminal nodes. Otherwise, there are $2T+1$ terminal nodes. The values at the terminal nodes are given by the maximum of harvesting and abandoning as before:

$$\begin{aligned} V(P_{T,j}, Q_T) = \max\big\{(P_{T,j} - C)Q_T, 0\big\},& \\ j \in \mathbb{Z} : -\max\{j_{max}, T\} \leq j \leq \max\{j_{max}, T\}& \end{aligned} \tag{4.37}$$

The recursion proceeds using (4.33) until the single node at $t = 0$ is reached.

4.4 The Multiple Rotations Problem

The multiple rotations problem extends on the single rotation problem addressed in the preceding sections. The difference arise from the choice made available to the forestry manager after harvesting. Once the trees are harvested, a new choice has to be made between replanting and abandoning. Suppose the cost of replanting is R, and ignoring any abandonment value, the solution is again by approximating the timber prices in discrete time and solving for the option value using dynamic programming. In this case, the immediate payoff from harvesting depends on the value of subsequent rotations. As such, this is a compound real option. The forestry manager has the option at any time step to exercise by paying the harvesting cost, and receive the value of the timber. However, by paying the additional replanting cost, the option to harvest the second tree rotation is made available, and this continues in perpetuity. I address the multiple rotation problem by using a naive approach in Subsection 4.4.1. The naive approach recognizes the similarity between the harvest decision with multiple rotations and the compound American call option, and consequently applies the methodology introduced for the single rotation recursively. Although correct, the approach is computationally expensive. Therefore, a heuristic approach is developed in Subsection 4.4.2, which simplifies calculations significantly.

4.4.1 The Compound Option Approach

Geometric Brownian Motion

Suppose the price of timber P_t follows GBM, and is approximated using the trinomial tree presented in Subsection 4.3.1. Let $P_{t,j}$ denote the price of timber at node t, j, and let Q_k denote the available timber k time steps into the current rotation. As such, initiating a new rotation by replanting causes Q_k to transition to Q_0. Furthermore, as there could be an infinite number of rotations, a finite maximum is set to be $Z \in \mathbb{Z}^+$ to enable backwards recursion. A general rule of thumb in forestry is that the value of any future rotations are roughly $5\% - 15\%$ of the initial rotation due to the effect of discounting. Therefore, it is reasonable to restrict Z in the compound option approach to a low number, say 2 or 3. A restriction is put on the rotation age by imposing the constraint that $k \in \mathbb{Z} : 0 \leq k \leq K$. This is done in order to establish a finite horizon, and solve the problem by dynamic programming. The naive approach to calculate the value involves comparing three different values at each step. The first is the value of immediate harvest and abandonment. The second is the value of immediate harvest and subsequent replanting. The third is the discounted expected continuation value achieved by delaying harvest. Consequently, the Bellman equation can be formulated as:

$$
\begin{aligned}
V(P_{t,j}, Q_k, z) = \max\Big\{ & (P_{t,j} - C)Q_k, \\
& (P_{t,j} - C)Q_k - R + V(P_{t,j}, Q_0, z+1), \\
& e^{-r\Delta t}\big[\pi_u V(P_{t+1,j+1}, Q_{k+1}, z) + \pi_m V(P_{t+1,j}, Q_{k+1}, z) \\
& + \pi_d V(P_{t+1,j-1}, Q_{k+1}, z)\big]\Big\},
\end{aligned}
\tag{4.38}
$$

where z is a positive integer denoting the current rotation number, and π_u, π_d, and π_m are given by (4.12), (4.13), and (4.14), respectively. The first argument to the max function is $(P_{t,j} - C)Q_k$ and represents the value of immediate harvest, and subsequent abandonment. The second argument, $(P_{t,j} - C)Q_k - R + V(P_{t,j}, Q_0, z+1)$, is the value of immediate harvest, and subsequent replanting. By replanting, the option to harvest the following rotation becomes available and is captured by the $V(P_{t,j}, Q_0, z+1)$ term. The last argument is the discounted expected continuation value. Hence, to calculate $V(P_{t,j}, Q_k, z)$, the value of subsequent nodes must be calculated first in addition to the option values of any subsequent rotations. Therefore, the backwards recursion begins at the final rotation Z, where the value at the end of the final rotation is given by the maximum of harvesting or abandoning, and no subsequent replanting is allowed. This value is given by:

$$\begin{aligned} V(P_{t,j}, Q_K, Z) &= \max\big\{(P_{t,j} - C)Q_K, 0\big\}, \\ &\qquad t \in \mathbb{Z} : 0 \le t \le T, \;\; j \in \mathbb{Z} : -T \le j \le T \end{aligned} \tag{4.39}$$

The recursion proceeds backwards to calculate the value at all nodes of the final rotation Z where the choice is either to harvest or wait:

$$\begin{aligned} V(P_{t,j}, Q_k, Z) = \max\Bigg\{ &(P_{t,j} - C)Q_k, \\ &e^{-r\Delta t}\big[\pi_u V(P_{t+1,j+1}, Q_{k+1}, Z) + \pi_m V(P_{t+1,j}, Q_{k+1}, Z) \\ &+ \pi_d V(P_{t+1,j-1}, Q_{k+1}, Z)\big]\Bigg\}, \\ &t \in \mathbb{Z} : 0 \le t \le T, \;\; j \in \mathbb{Z} : -t \le j \le t, \;\; k \in \mathbb{Z} : 0 \le k < K, \end{aligned} \tag{4.40}$$

which in turn enables calculation of all the terminal nodes of prior rotations:

$$\begin{aligned} V(P_{t,j}, Q_K, z) = \max\big\{&(P_{t,j} - C)Q_K, \\ &(P_{t,j} - C)Q_K - R + V(P_{t,j}, Q_0, z+1)\big\}, \\ &t \in \mathbb{Z} : 0 \le t \le T, \;\; j \in \mathbb{Z} : -t \le j \le t, \;\; z \in \mathbb{Z} : 0 \le z < Z, \end{aligned} \tag{4.41}$$

The recursion proceeds via (4.38) until the single node $V(P_{0,0}, Q_0, 0)$ is reached.

An intuitive way to view the above is through the simpler methodology presented for the single rotation case in Section 4.3.1, with the modification that whenever the decision to harvest is considered, a new single rotation model is initiated from the price at the current node to calculate the option value of a potential next rotation. This continues until all Z rotations have been considered.

Mean Reversion

Suppose the price of timber P_t follows MR, and is approximated using the trinomial tree. Now let $P_{t,j}$ be the price of timber at time t after j up jumps, and accounting for the proper drift as described by the MR trinomial tree presented in Subsection 4.3.2. As above, Q_k denotes the available timber k steps into the current rotation, and z the current rotation number. As before, the constraint that Z and K is finite is imposed. As such T is implicitly defined because the largest number of time steps that can possibly occur in the model is Z rotations of age K. The difference in formulating the MR model compared to the previous GBM model is due to the dependence of the transition probabilities on j, and the different branching patterns, similarly to the single rotation case. I formulate the Bellman equation as:

$$V(P_{t,j}, Q_k, z) = \begin{cases} V^A(P_{t,j}, Q_k, z), & \text{if } j_{min} < j < j_{max} \\ V^B(P_{t,j}, Q_k, z), & \text{if } j = j_{min} \\ V^C(P_{t,j}, Q_k, z), & \text{if } j = j_{max}, \end{cases} \tag{4.42}$$

where

$$\begin{aligned} V^A(P_{t,j}, Q_k, z) = \max\Big\{ & (P_{t,j} - C)Q_k, \\ & (P_{t,j} - C)Q_k + V(P_{t,j}, Q_0, z+1), \\ & e^{-r\Delta t}\big[\pi_u^A V(P_{t+1,j+1}, Q_{k+1}, z) \\ & + \pi_m^A V(P_{t+1,j}, Q_{k+1}, z) + \pi_d^A V(P_{t+1,j-1}, Q_{k+1}, z)\big] \Big\}, \end{aligned} \tag{4.43}$$

$$\begin{aligned} V^B(P_{t,j}, Q_k, z) = \max\Big\{ & (P_{t,j} - C)Q_k, \\ & (P_{t,j} - C)Q_k + V(P_{t,j}, Q_0, z+1), \\ & e^{-r\Delta t}\big[\pi_u^B V(P_{t+1,j+2}, Q_{k+1}, z) \\ & + \pi_m^B V(P_{t+1,j+1}, Q_{k+1}, z) + \pi_d^B V(P_{t+1,j}, Q_{k+1}, z)\big] \Big\}, \end{aligned} \tag{4.44}$$

$$\begin{aligned} V^C(P_{t,j}, Q_k, z) = \max\Big\{ & (P_{t,j} - C)Q_k, \\ & (P_{t,j} - C)Q_k + V(P_{t,j}, Q_0, z+1), \\ & e^{-r\Delta t}\big[\pi_u^C V(P_{t+1,j}, Q_{k+1}, z) \\ & + \pi_m^C V(P_{t+1,j-1}, Q_{k+1}, z) + \pi_d^C V(P_{t+1,j-2}, Q_{k+1}, z)\big] \Big\}, \end{aligned} \tag{4.45}$$

and the transition probabilities are given by (4.23) through (4.31). The recursion proceeds backwards from the terminal nodes of the final rotation:

$$\begin{aligned} V(P_{t,j}, Q_K, Z) = \max\big\{ & (P_{t,j} - C)Q_K, 0\big\}, \\ & t \in \mathbb{Z} : 0 \le t \le T, \\ & j \in \mathbb{Z} : -\max\{j_{max}, t\} \le j \le \max\{j_{max}, t\}, \end{aligned} \tag{4.46}$$

via the prior nodes of the final rotation:

$$V(P_{t,j}, Q_k, Z) = \begin{cases} V^A(P_{t,j}, Q_k, Z), & \text{if } j_{min} < j < j_{max} \\ V^B(P_{t,j}, Q_k, Z), & \text{if } j = j_{min} \\ V^C(P_{t,j}, Q_k, Z), & \text{if } j = j_{max}, \end{cases} \tag{4.47}$$

where

$$\begin{aligned} V^A(P_{t,j}, Q_k, Z) = \max\Big\{&(P_{t,j} - C)Q_k, \\ &e^{-r\Delta t}\big[\pi_u^A V(P_{t+1,j+1}, Q_{k+1}, Z) \\ &+ \pi_m^A V(P_{t+1,j}, Q_{k+1}, Z) + \pi_d^A V(P_{t+1,j-1}, Q_{k+1}, Z)\big]\Big\}, \end{aligned} \tag{4.48}$$

$$\begin{aligned} V^B(P_{t,j}, Q_k, Z) = \max\Big\{&(P_{t,j} - C)Q_k, \\ &e^{-r\Delta t}\big[\pi_u^B V(P_{t+1,j+2}, Q_{k+1}, Z) \\ &+ \pi_m^B V(P_{t+1,j+1}, Q_{k+1}, Z) + \pi_d^B V(P_{t+1,j}, Q_{k+1}, Z)\big]\Big\}, \end{aligned} \tag{4.49}$$

$$\begin{aligned} V^C(P_{t,j}, Q_k, Z) = \max\Big\{&(P_{t,j} - C)Q_k, \\ &e^{-r\Delta t}\big[\pi_u^C V(P_{t+1,j}, Q_{k+1}, Z) \\ &+ \pi_m^C V(P_{t+1,j-1}, Q_{k+1}, Z) + \pi_d^C V(P_{t+1,j-2}, Q_{k+1}, Z)\big]\Big\}, \end{aligned} \tag{4.50}$$

and next via the final nodes of prior rotations:

$$\begin{aligned} V(P_{t,j}, Q_K, z) = \max\big\{&(P_{t,j} - C)Q_K, \\ &(P_{t,j} - C)Q_K - R + V(P_{t,j}, Q_0, z+1)\big\}, \\ &t \in \mathbb{Z} : 0 \le t \le T,\;\; j \in \mathbb{Z} : -t \le j \le t,\;\; z \in \mathbb{Z} : 0 \le z < Z, \end{aligned} \tag{4.51}$$

and proceeds using (4.42) until $V(P_{0,0}, Q_0, 0)$ is reached.

4.4.2 The Heuristic Option Approach

A challenge with compound option approach is that the number of nodes that needs to be evaluated grows exponentially with Z. To put it into perspective, consider the single rotation case with 100 time steps. The number of nodes are then $1 + 3 + 5 + ... + 201 = 101(1+201)/2 = 10,201$[2]. In the compound option approach each of these 10,201 nodes requires the calculation of a complete new trinomial tree consisting of 10,201 nodes in order to evaluate the value of the particular node in the first tree, and in turn each of these requires a 10,201 nodes to be evaluated. Even with the number of rotations restricted to, say $Z = 3$, the number of nodes in the problem space is $10,201^3 = 1,061,520,150,601$, compared to the modest $10,201$ in the single rotation case. Consequently, I propose a heuristic approach in the following where the value of the first rotation is calculated using the single rotation option approach, and the value of subsequent rotations are approximated using the deterministic Faustmann value (Faustmann, 1849). This simplification significantly reduces the computational complexity of valuation in the multiple rotations case.

Geometric Brownian Motion

Suppose the stochastic process is GBM. Then the Bellman equation for the heuristic approach is formulated by incorporating the value of any rotations after the initial one by using the Faustmann value:

$$\begin{aligned} V(P_{t,j}, Q_t) = \max\Big\{ &(P_{t,j} - C)Q_t, (P_{t,j} - C)Q_t + F^*(P_{t,j}) - R, \\ &e^{-r\Delta t}\big[\pi_u V(P_{t+1,j+1}, Q_{t+1}) + \pi_m V(P_{t+1,j}, Q_{t+1}) \\ &+ \pi_d V(P_{t+1,j-1}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.52}$$

where $P_{t,j}$ is the price of timber at node t, j, Q_t is the quantity available, r is the discount rate, and π_u, π_d, π_m are the transition probabilities given by (4.12), (4.13), and (4.14), respectively. $F^*(\cdot)$ denotes the Faustmann value. The Faustmann value of a rotation of length t is given by:

$$F(t; P) = \frac{(P - C)Q(t) - R}{e^{rt} - 1}, \tag{4.53}$$

and $F^*(\cdot)$ is given by maximizing (4.53) with respect to t:

$$F^*(P) = \max_T \left\{ \frac{(P - C)Q(T) - R}{e^{rT} - 1} \right\} \tag{4.54}$$

The option at the final age involves the choice between harvesting, and subsequently between replanting or not. The values at the final nodes can thus be expressed as:

$$\begin{aligned} V(P_{T,j}, Q_T) = \max\big\{(P_{T,j} - C)Q_T, 0, F^*(P_{T,j}) - R\big\}, \\ j \in \mathbb{Z} : -T \le j \le T, \end{aligned} \tag{4.55}$$

[2] Initially there is a single node, and in the second time step there are 3 nodes, in the third there are 5 and so on. This forms an arithmetic progression where the difference between any two successive numbers is 2. The sum of the sequence is found by taking the the number of terms and multiplying by the sum of the first and last term and dividing by 2.

and the problem solved by backwards recursion using (4.52). Note, that if the price at T is sufficiently low $F^*(P_{T,j}) < R$, and a new rotation will not be initiated. In the Faustmann formula, replanting is required at the end of each rotation. The heuristic approach thus takes into consideration the value of flexibility in the decision between replanting or abandoning.

Mean Reversion

The formulation when prices follow MR are similar, but as before, the transition probabilities in the MR tree are dependent on the node, and the prices transitioned to. Consequently, the Bellman equation is formulated as:

$$V(P_{t,j}, Q_t) = \begin{cases} V^A(P_{t,j}, Q_t), & \text{if } j_{min} < j < j_{max} \\ V^B(P_{t,j}, Q_t), & \text{if } j = j_{min} \\ V^C(P_{t,j}, Q_t), & \text{if } j = j_{max}, \end{cases} \tag{4.56}$$

where

$$\begin{aligned} V^A(P_{t,j}, Q_t) = \max\Big\{ & (P_{t,j} - C)Q_t, (P_{t,j} - C)Q_t + F^*(P_{t,j}) - R, \\ & e^{-r\Delta t}\big[\pi_u^A V(P_{t+1,j+1}, Q_{t+1}) \\ & + \pi_m^A V(P_{t+1,j}, Q_{t+1}) + \pi_d^A V(P_{t+1,j-1}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.57}$$

$$\begin{aligned} V^B(P_{t,j}, Q_t) = \max\Big\{ & (P_{t,j} - C)Q_t, (P_{t,j} - C)Q_t + F^*(P_{t,j}) - R, \\ & e^{-r\Delta t}\big[\pi_u^B V(P_{t+1,j+2}, Q_{t+1}) \\ & + \pi_m^B V(P_{t+1,j+1}, Q_{t+1}) + \pi_d^B V(P_{t+1,j}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.58}$$

$$\begin{aligned} V^C(P_{t,j}, Q_t) = \max\Big\{ & (P_{t,j} - C)Q_t, (P_{t,j} - C)Q_t + F^*(P_{t,j}) - R, \\ & e^{-r\Delta t}\big[\pi_u^C V(P_{t+1,j}, Q_{t+1}) \\ & + \pi_m^C V(P_{t+1,j-1}, Q_{t+1}) + \pi_d^C V(P_{t+1,j-2}, Q_{t+1})\big]\Big\}, \end{aligned} \tag{4.59}$$

and the transition probabilities are given by (4.23) through (4.31). The expression for the values at the terminal nodes are similar to under GBM:

$$\begin{aligned} V(P_{T,j}, Q_T) &= \max\big\{(P_{T,j} - C)Q_T, 0, F^*(P_{T,j}) - R\big\}, \\ & j \in \mathbb{Z} : -\max\{j_{max}, T\} \leq j \leq \max\{j_{max}, T\}, \end{aligned} \tag{4.60}$$

Chapter 5

Results and Discussion

This chapter is split into two sections; the results from a single forest rotation is introduced in Chapter 5.1, and the results from multiple forest rotations is presented in Chapter 5.2.

5.1 Single Rotation Valuation

As a point of departure I follow Insley (2002), and compare the option value under GBM and MR to the static harvest value for a single forest rotation in Chapter 5.1.1 and Chapter 5.1.2. I do this in order to demonstrate the applicability of the trinomial model, and confirm that the model behaves as expected. Next, in Chapter 5.1.3 the parameters of a GBM- and MR-process is estimated from historical price data, and the trinomial model is applied to a "real-world" scenario assuming a single forest rotation.

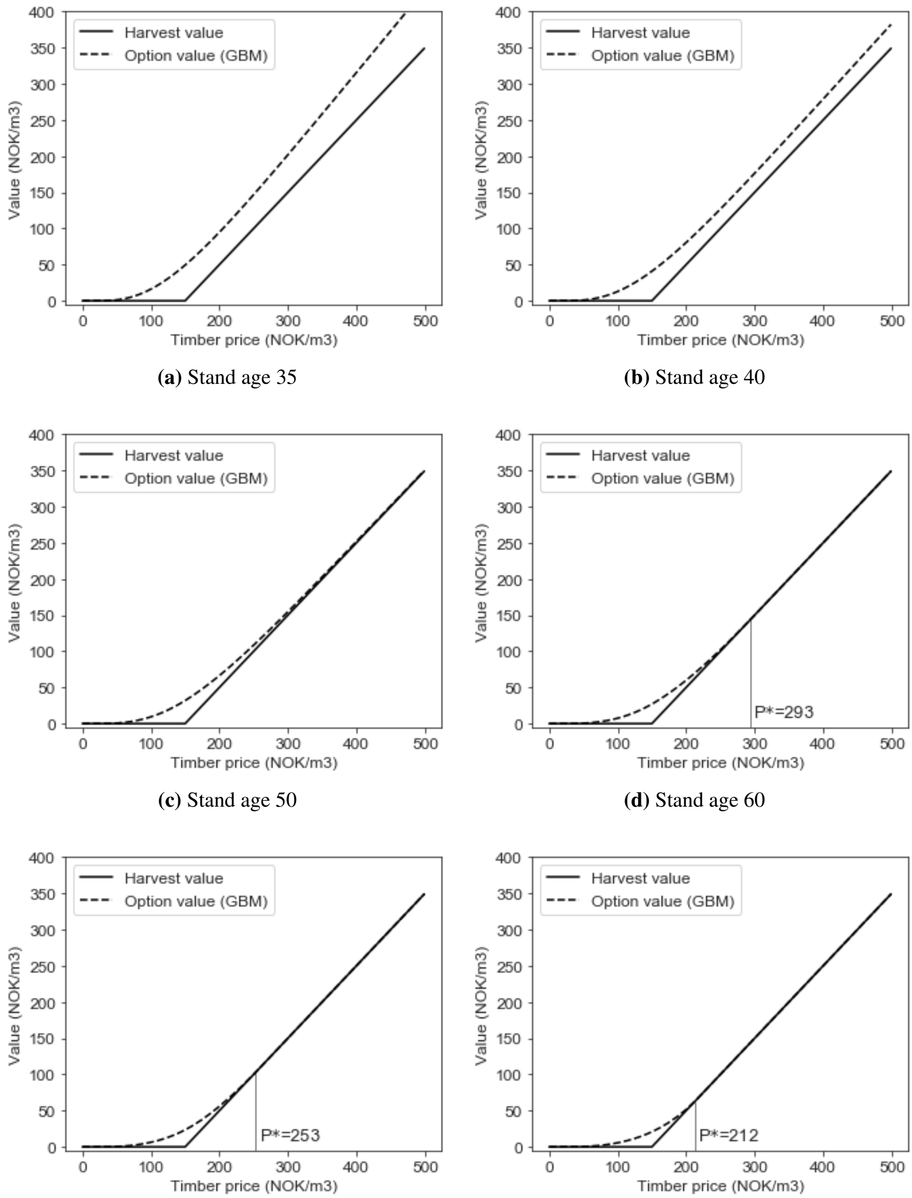
In the following the discount rate is set to $r = 0.04$. This follows the rate used in expropriation cases in Norway (NORSKOG, 2015). Most private forestry practitioners employ a discount rate between 2.5% and 4% in their internal calculations (SkogsNorge, 2014), where the higher rates is used when the investment perspective is long, as is the case with a freshly planted forestry stand. Furthermore, the cost of harvesting is set to 150 NOK/m3, which was provided on a confidential basis and assumed to reflect a typical harvest cost in most parts of Norway. The growth function is given by (3.1). The step size in the tree models is set to 1 year, and the finite horizon to 100 years. Decreasing the step size or increasing the finite horizon was found to have a negligible impact on the results.

5.1.1 Geometric Brownian Motion

Figure 5.1 shows the value of the option to harvest as a function of the timber price for different stand ages. The solid line represents the value of immediate harvest, and the dashed line represents the value under GBM with volatility $\sigma = 0.05$, and drift rate $\alpha = 0.01$. Critical prices are indicated on the plots where they exist. The critical price is the level at which harvest would be optimal for a specific stand age. In other words, should the price in the given year reach the critical price or any price above it, the stand will be harvested.

There are no critical prices when the stand is less than 50 years old. The option value line can be seen to above the harvest value everywhere in Figure 5.1a and 5.1b. In year 50, there is a critical price of 717 NOK/m3, which is not indicated in Figure 5.1c as it falls outside the range of prices that are shown in the plot. The absence of any critical price in the early years can be attributed to two factors. First, the stand is assumed to follow a s-shaped curve as evident from the growth function (3.1). As such, the growth is still rapid when the stand is young, and decreases with time. Therefore, there is incentive for the forestry manager to delay harvest until the growth has slowed down. Second, under GBM the timber price is expected to steadily increase over time due to the positive drift rate and volatility. The uncertainty creates an option value of waiting, which in turn makes it optimal to wait longer for more information. Therefore, it is optimal to delay harvest pending a higher price at a later time. When the stand is 60 years, the critical price has dropped to 293 NOK/m3, meaning that if this price is realized in year 60, the stand will be harvested. The critical price decreases steadily towards 212 NOK/m3 by year 80, where the stand has stopped growing by definition of the growth function. If the prices were assumed to be deterministic, the stand would be harvested in year 80 for any price above the cost of harvesting, as there is no value in waiting. In contrast, under GBM even when the price is above the assumed 150 NOK/m3 cost of harvesting, there is incentive to delay harvest due to the expected price increase, and chance of realizing a sudden price surge due to the volatility.

It is well established in the financial option literature that an increase in volatility leads to an increase in option value. Figure 5.2 shows the option value in year 50 as a function of volatility and timber price. It is clear that an increase in volatility leads to a higher option value, also in the forestry setting. The effect of increasing volatility is most profound for a high timber price. This reflects the proportionate volatility in GBM. Moreover, it is expected that an increase in the drift rate of the GBM process implies a higher option value. This is confirmed in Figure 5.3, which illustrates the sensitivity of the option to the drift rate. Similar to the effect of volatility, the increase in option value due to the drift is most profound for a high timber price, and reflects the nature of the GBM process, where the drift is proportionate to the price.



(a) Stand age 35 **(b)** Stand age 40

(c) Stand age 50 **(d)** Stand age 60

(e) Stand age 70 **(f)** Stand age 80

Figure 5.1: Comparison of immediate harvest value against GBM ($\sigma = 0.05$, $\alpha = 0.01$). The vertical lines indicate critical prices.

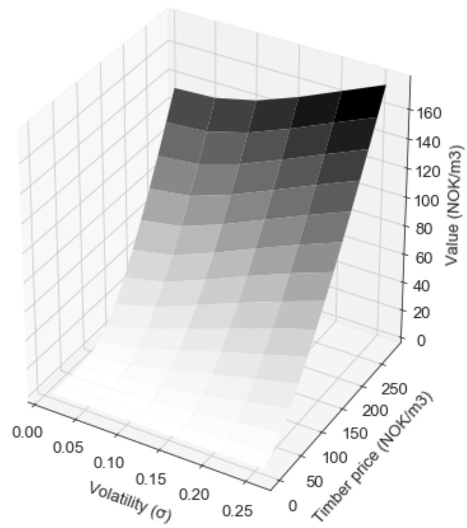


Figure 5.2: Option value in year 50 under GBM as a function of volatility and timber price. The drift rate is $\alpha = 0.01$.

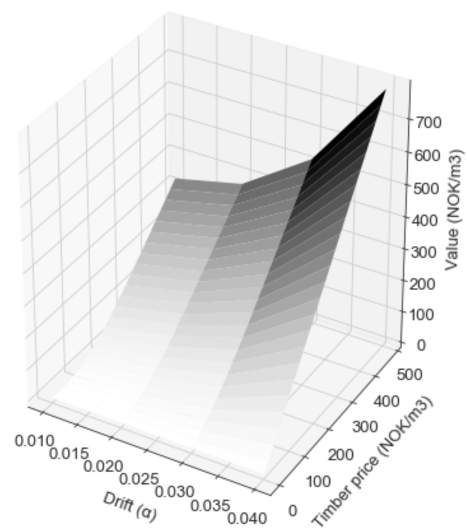


Figure 5.3: Option value in year 50 under GBM as a function of drift rate and timber price. The volatility is $\sigma = 0.05$.

The traditionally prescribed harvest age for a single rotation is the age at which the growth rate falls below the discount rate (Clark, 2010). Intuitively, once the growth is lower than the discount rate, the forestry manager is better off selling the timber and achieving a better return elsewhere. With the growth function used in this article (3.1) the traditional "deterministic" harvest age is 42 years. The option model developed in this article is capable of calculating the expected harvest age simultaneously as calculating the option values in the trinomial tree. This is done by initially setting the expected harvest age at each final node to T. Next, the tree is traversed backwards and at each node the expected harvest age is set to the node's time step t if harvest is optimal, and to the probability weighted expected rotation age of all nodes one time step ahead otherwise. This expected harvest age is analogous to fugit in mathematical finance, which is the expected date to exercise an American option.

Figure 5.4 shows the expected harvest age under GBM with $\alpha = 0.01$ for a freshly planted forest stand. The solid line represents the deterministic harvest age, the dashed line the expected harvest age when $\sigma = 0.05$, and the dotted line represents the expected harvest age when $\sigma = 0.10$. The harvest age under GBM is found to be higher than in the deterministic case, and it increases with volatility. As the initial price is increased the expected harvest age gradually falls towards year 49 in the low volatility case, and towards year 50 in the high volatility case. Intuitively, even when the growth rate has fallen below the discount rate, there is incentive to delay harvest under GBM as prices are expected to be higher in the future. Moreover, when the volatility is high the chance of realizing an exceptionally high price at some point in the future increases, and therefore, the forestry manager has an postpone harvest. This result would seem to suggest that in practice, many forest stands are harvested prematurely, based on the assumption of deterministic prices. Clearly, prices are not deterministic, and hence the option model could be an impactful decision support tool in forestry if used alongside the more "traditional" harvest rules to ensure economic efficiency.

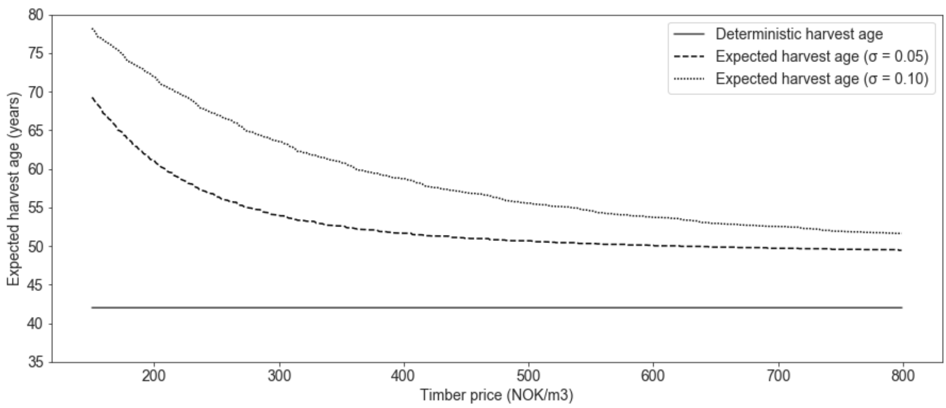


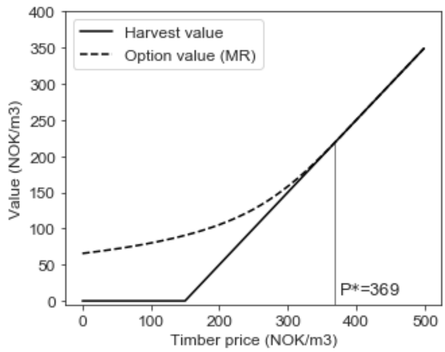
Figure 5.4: Comparison of determinstic harvest age against expected harvest age under GBM with volatility $\sigma = 0.05$, and $\sigma = 0.10$. The drift rate is $\alpha = 0.01$.

5.1.2 Mean Reversion

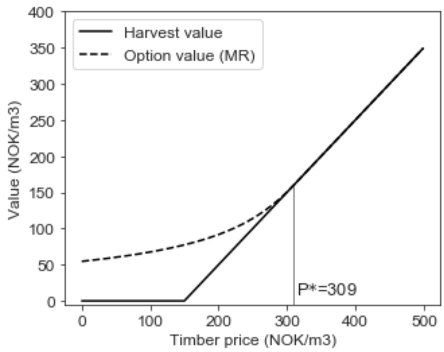
Figure 5.5 shows the option value under MR with speed of reversion $\eta = 0.05$, volatility $\sigma = 0.05$ and the equilibrium $\mu = 300$. The solid line represents the value of immediate harvest, and the vertical line indicates a critical price.

In contrast to GBM, there is a critical price even in the early years when the forest is still growing quickly. This reflects that under MR, any prices well above the equilibrium should be taken advantage of by harvesting immediately before it reverts. Figure 5.5a shows that the option value significantly exceeds the value of immediate harvest when the timber price is low in year 35. As the price is increased towards the equilibrium of 300 NOK/m3 the option value falls towards the harvest value, and coincides at 369 NOK/m3. The critical price eventually declines to 232 NOK/m3 by year 80 as seen in Figure 5.1f. This critical price is below the equilibrium because the forest has stopped growing, and the value of waiting until the price eventually reverts to the equilibrium does not outweigh the opportunity cost of doing so.

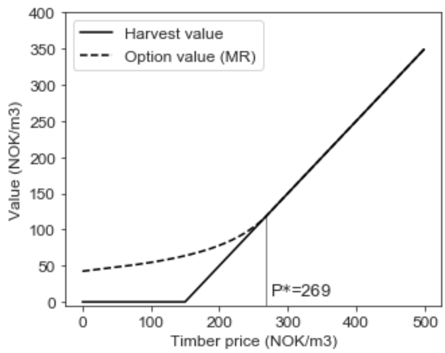
Figure 5.6 shows the sensitivity of the option value to volatility and timber price in year 50. The effect of volatility is most profound for prices below the equilibrium. This reflects that for low prices it is optimal to await a reversion to the equilibrium, and the volatility might lead to a more rapid reversion. In contrast, when the price is above the equilibrium, immediate harvest is optimal to ensure a high price before it reverts. Figure 5.7 illustrates the effect of the speed of reversion on the option value. The option value is most sensitive to an increase in speed of reversion from a low level. For instance, increasing the speed of reversion from 0.05 to 0.10 leads to a notable increase in option value. However, increasing the speed from 0.30 to 0.35 has a negligible impact. This is because once the speed of reversion is sufficiently high, the reversion to the equilibrium is swift, and the opportunity cost of waiting for any potential price spike in the future is too high to justify delaying harvest.



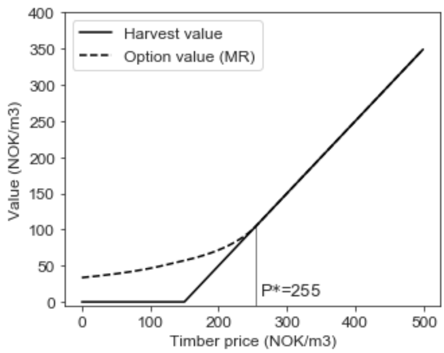
(a) Stand age 35



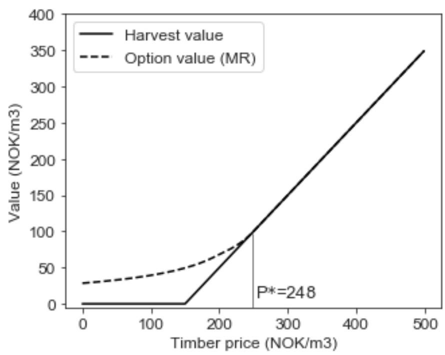
(b) Stand age 40



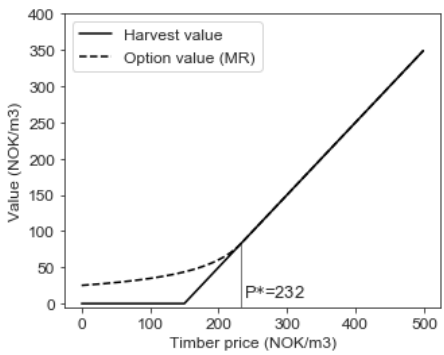
(c) Stand age 50



(d) Stand age 60



(e) Stand age 70



(f) Stand age 80

Figure 5.5: Comparison of immediate harvest value against MR with speed of reversion $\eta = 0.05$, $\sigma = 0.05$, and $\mu = 300$. The vertical lines indicate critical prices.

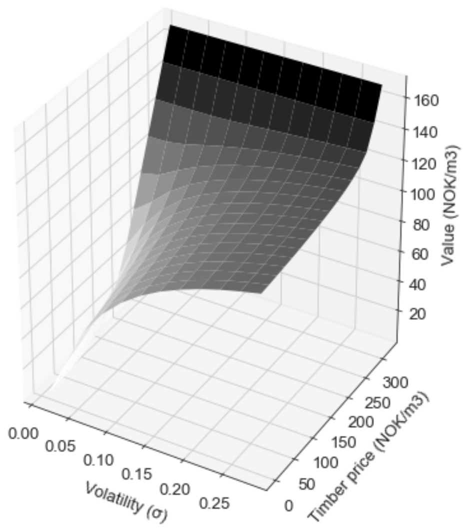


Figure 5.6: Option value in year 50 under under MR as a function of volatility and timber price. The speed of reversion is $\eta = 0.05$ and equilibrium $\mu = 300$.

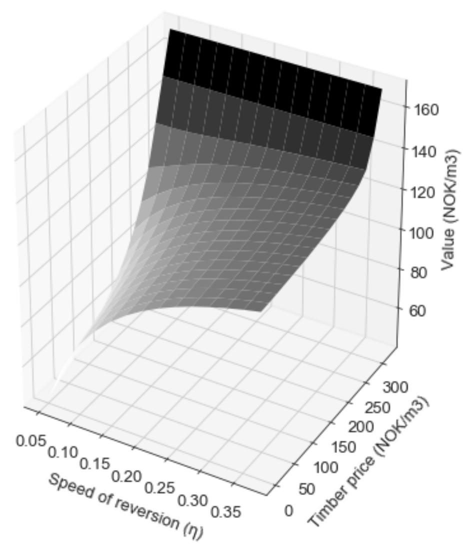


Figure 5.7: Option value in year 50 under under MR as a function of speed of reversion rate and timber price. The volatility is $\sigma = 0.05$ and equilibrium $\mu = 300$.

Figure 5.8 shows the expected harvest age of a freshly planted stand under MR. When the price of timber is above the equilibrium of $\mu = 300$, the expected rotation age is higher than under deterministic prices. This is because the price is expected to revert to the equilibrium in the future, and consequently, the value of delaying harvest is large. Moreover, the expected harvest age is higher when the speed of reversion is low, compared to when it is high. This reflects 1) that it takes longer for the price to eventually revert to the equilibrium, and 2) that any exceptionally high prices realized due to volatility is likely to persist longer than under a high speed of reversion. For an initial price below the equilibrium, the expected rotation age is lower than in the deterministic case. Intuitively, the forestry manager will seek to take advantage of any unusually high prices, and harvest as soon as the growth has somewhat slowed down. This is in contrast to the GBM case, where the forestry manager still expects steadily increasing prices, and therefore, delays harvest. As the initial price is increased, the expected harvest age appears to stabilize at around 39 years when the speed of reversion is low and around 34 years when the speed of reversion is high.

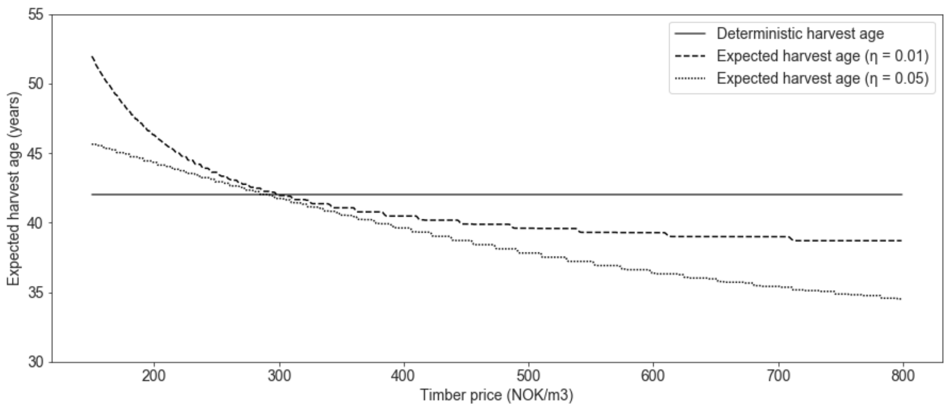


Figure 5.8: Comparison of determinstic harvest age against expected harvest age under MR with speed of reversion $\eta = 0.01$, and $\eta = 0.05$. The volatility is $\sigma = 0.05$.

5.1.3 Comparing Geometric Brownian Motion and Mean Reversion Using Historical Price Data

In order to assess implications of GBM and MR, as well as provide an application of the real options model to a real-world scenario, I compare the option value and critical price of the two using parameters estimated from the historical prices of Norway spruce introduced in Chapter 3.2.

I begin by employing the Augmented Dickey-Fuller (ADF) test to check for the the presence of a unit root in the historical prices as a means to obtain insights into the proper model specification. The presence of a unit root implies that GBM is more appropriate, whereas the absence of a unit root implies a non-stationary process such as MR.

Assuming that prices follow GBM:

$$dP_t = \alpha P_t dt + \sigma P_t dz_t, \tag{5.1}$$

it follows from Itô's lemma that the logarithm of prices follow Brownian motion with drift. Thus, we have that

$$dp_t = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dz_t \tag{5.2}$$

where p_t denotes the logarithm of prices at time t. This can be approximated in discrete time as

$$p_t - p_{t-1} = (\alpha - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon_t\sqrt{\Delta t} \tag{5.3}$$

where $\varepsilon_t \sim N(0,1)$. The ADF test involves running the regression

$$p_t - p_{t-1} = a + bt + \gamma p_{t-1} + \delta_1 \Delta p_{t-1} + ... + \delta_{l-1}\Delta p_{t-l+1} + \epsilon_t, \tag{5.4}$$

where $a = (\alpha - \frac{1}{2}\sigma^2)\Delta t$, $b = \sigma\varepsilon_t\sqrt{\Delta t}$, and l denotes the lag order of an autoregressive process. The inclusion of lagged variables is done to account for serial correlation. The lag order has to be predetermined when applying the test. For the price data for Norway spruce, the lag length is set to 1, which is found to minimize the Akaike information criterion. Consequently, the test runs the regression:

$$p_t - p_{t-1} = a + bt + \gamma p_{t-1} + \delta_1 \Delta p_{t-1} + \epsilon_t, \tag{5.5}$$

where $a = (\alpha - \frac{1}{2}\sigma^2)\Delta t$, and $b = \sigma\varepsilon_t\sqrt{\Delta t}$. The ADF test is then carried out under the null hypothesis $H_0 : \gamma = 0$ (non-stationary) against the alternative hypothesis $H_1 : \gamma < 0$ (stationary).

The ADF test statistic is -2.016 and the critical values are -4.066, -3.462, and -3.157 for the 1%, 5%, and 10% significance level, respectively. Hence, I cannot reject H_0 at any of the significance levels. Although I cannot reject the null hypothesis, it does not prove that the alternative hypothesis is correct. Hence, I proceed to estimate the parameters of both GBM and MR.

To model the price using GBM the drift rate α and volatility σ needs to be estimated. I do this using the ratio between P_t and P_{t-1}. This ratio is lognormally distributed (Hull, 2003). Consequently, the sample mean of the normal distribution is given by:

$$\bar{P} = \frac{1}{n-1}\sum_{t=1}^{n-1}\log\left(\frac{P_t}{P_{t-1}}\right), \tag{5.6}$$

and the sample standard deviation by:

$$\sigma = \sqrt{\frac{1}{n-2}\sum_{t=1}^{n-1}\left(\log\left(\frac{P_t}{P_{t-1}}\right) - \bar{P}\right)^2} \tag{5.7}$$

The standard deviation is used as an estimator for the volatility, and the drift is given by:

$$\alpha = \bar{P} + \frac{1}{2}\sigma^2 \tag{5.8}$$

The obtained estimates are annualized and I find that $\alpha = 0.006$, and $\sigma = 0.067$.

Next, I estimate the parameters of a MR process, specifically the Ornstein-Uhlenbeck process:

$$dP_t = \eta(\mu - P_t)dt + \sigma dz_t \tag{5.9}$$

where η is the speed of reversion, μ is equilibrium price, $\sigma \geq 0$ is the volatility, and dz_t is the increment to a Wiener process at time t.

The continous time (5.9) can be approximated in discrete time as

$$P_t - P_{t-1} = \eta\mu\Delta t - \eta P_{t-1}\Delta t + \sigma\varepsilon_t\sqrt{\Delta t} \tag{5.10}$$

where $\varepsilon_t \sim N(0,1)$. Letting $a = \eta\mu\Delta t$, and $b = -\eta\Delta t$, (5.10) can be rewritten as

$$P_t - P_{t-1} = a + bP_{t-1} + \epsilon_t \tag{5.11}$$

where $\epsilon_t \sim N(0,1)$. To estimate the parameters, the values $P_t - P_{t-1}$ are regressed against the values P_{t-1}, and the parameters are given by: $\mu = -a/b$, $\eta = -b/\Delta t$, and $\sigma = \sigma_\epsilon/\Delta t$, where σ_ϵ is the standard deviation from the regression. As the historical data are monthly, I set $\Delta t = 1/12$, and obtain the annual parameters: $\eta = 0.325$, $\mu = 396$ and $\sigma = 0.067$.

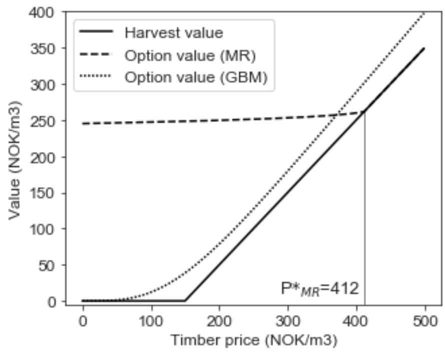
In comparing the option value under GBM and MR, the growth function is again given by (3.1), the cost of harvesting is set to 150 NOK/m3, and the discount rate is 4% as before. Table 5.1 shows the critical prices for various stand ages, and Figure 5.9 presents the option values as a function of initial prices. The value under MR significantly exceeds the GBM value when prices are below the equilibrium for all stand ages, and the difference is largest for the lower prices. This is due to the strong mean reversion at prices low relative to the equilibrium. Notably, even for an initial price of zero, the option value is non-zero under MR. In contrast, the option value is zero under GBM for an initial price of zero. This reflects the different nature of the two stochastic processes. Specifically, under MR, the price is expected to revert to the equilibrium, even if it hits zero. However, under GBM it is evident from the GBM equation (4.4), that if the price ever reaches zero, it will remain there forever, and hence, lead to a worthless option.

In the early years, the option value is higher under MR than GBM for prices below the equilibrium. Moreover, there are no critical prices under GBM in the early years. This is due to the long perspectives in forestry, and the large impact of the drift on the valuation and harvesting decision. In Figure 5.9c, it can be seen that the critical price under GBM has declined below the critical price under MR, and this remains the case for all later years. This indicates that under the assumption of MR the forest is more likely to be harvested in the early years, and confirms the results of (Insley, 2002). Furthermore, it demonstrates that the choice of stochastic process has a large impact on the optimal harvesting decision. For instance, say the price is 376 NOK/m3 in year 50, which was the weighted spruce price in May 2020; if the forestry manager assumes that GBM is the appropriate process to describe timber prices, then the stand is harvested because the current price is above the GBM critical price of 296 NOK/m3. In contrast, if MR was assumed to be the appropriate process, harvest would be suboptimal because the price is below the MR critical price of 387 NOK/m3.

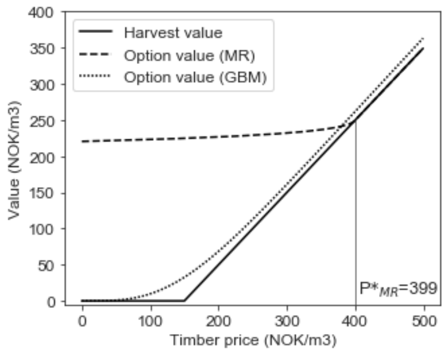
The comparison between GBM and MR demonstrates the applicability of the real options approach to a real-world scenario. Furthermore, it shows that the the choice of a particular stochastic process has a large effect on the option value, and optimal harvesting decision. As noted by Manley and Niquidet (2010), option valuation approaches are difficult to apply in forestry because of the difficulty in determining the appropriate stochastic process, and subsequently, estimate its parameters. In estimating the parameters above, price data from the past 7 years was used. If the prices are all well below some hypothetical equilibrium level, and GBM is assumed to be the appropriate stochastic process, the estimated drift is possibly too high. Moreover, the ADF test would likely suggest a non-stationary process. However, longer time series might reveal that the 7 year of positive drift were partially offset by 7 years of negative drift, and result in a MR process being a better fit. Therefore, it is perhaps most appropriate to use economic intuition to determine the appropriate stochastic process, and subsequently set the parameters based on industry knowledge in the absence of any long time series and conclusive statistical tests.

Table 5.1: Critical prices for a single forest rotation under GBM and MR (NOK/m3)

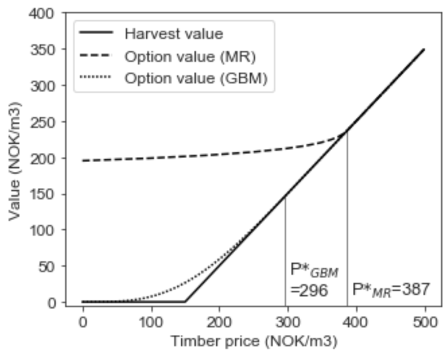
	35	40	50	60	70	80
GBM ($\alpha = 0.006, \sigma = 0.067$)	-	-	296	238	222	202
MR ($\eta = 0.325, \sigma = 0.067, \mu = 396$)	412	399	387	379	375	365



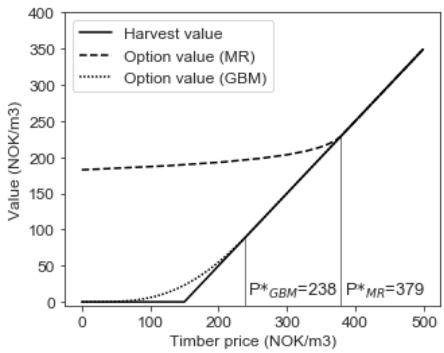
(a) Stand age 35



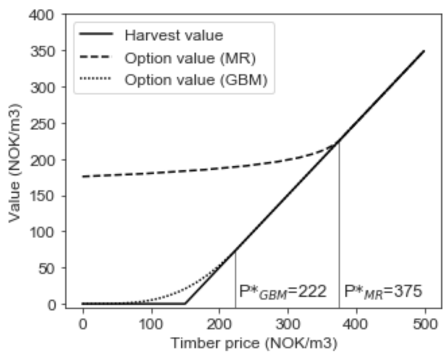
(b) Stand age 40



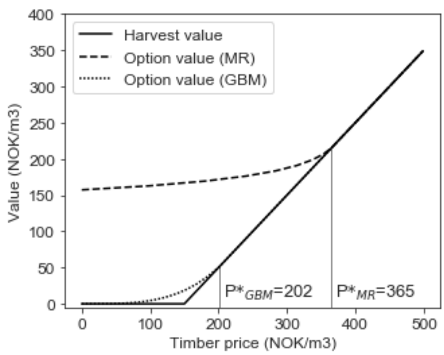
(c) Stand age 50



(d) Stand age 60



(e) Stand age 70



(f) Stand age 80

Figure 5.9: Comparison of immediate harvest value against MR ($\eta = 0.325$, $\sigma = 0.067$, $\mu = 396$) and GBM ($\alpha = 0.006$, $\sigma = 0.067$). Vertical lines indicate critical prices.

5.2 Multiple Rotations Valuation

I compare the heuristic approach and the compound option approach in Subsection 5.2.1 to demonstrate that the heuristic approach is accurate, despite its simplifying assumptions. I proceed by presenting an application of the heuristic approach using the parameters estimated from historical data in 5.2.2.

In the following the cost of harvesting is set to 150 NOK/m3, and the replanting cost to 10,000 NOK/ha based on soil scarification at 2,500 NOK/ha and replanting 3,000 trees at 2.5 NOK/plant. Further, the initial price is set to 376 NOK/m3, which was the weighted average Norway spruce price in May 2020. As before, the discount rate is set to 4%.

5.2.1 Comparing the Heuristic Approach and the Compound Option Approach

Table 5.2 shows the value of of a freshly planted stand on one hectare of land calculated using the compound option approach with Z rotations, the value obtained using the heuristic compound approach, and the Faustmann value calculated using the traditional Faustmann formula[1] (Faustmann, 1849) under five different stochastic processes. The cost of planting the first rotation is ignored. The stand value is seen to be higher when the prices are stochastic as evident by the value from the compound option approach and the heuristic compound approach, which exceeds the Faustmann value. Furthermore, the impact of increasing the volatility is higher under GBM than under MR. Intuitively, if the price deviates from the equilibrium under MR due to the volatility, it is expected to quickly revert. In contrast, the volatility and drift rate can potentially lift the price under GBM indefinitely. By changing the equilibrium level to reflect the initial price of 376 NOK/m3, the option value falls slightly under MR. However, it is still larger than the Faustmann value.

Table 5.2: Stand values (NOK/ha) for multiple forest rotations under GBM and MR

	Compound (Z=2)	Compound (Z=3)	Heuristic option	Faustmann
GBM ($\alpha = 0, \sigma = 0.05$)	14,181	14,364	14,410	14,052
GBM ($\alpha = 0, \sigma = 0.10$)	15,260	15,567	15,601	14,052
MR ($\eta = 0.01, \sigma = 0.05, \mu = 396$)	14,522	14,628	14,730	14,052
MR ($\eta = 0.01, \sigma = 0.10, \mu = 396$)	14,660	14,718	14,919	14,052
MR ($\eta = 0.01, \sigma = 0.10, \mu = 376$)	14,162	14,195	14,430	14,052

The computational complexity of the compound option approach grows exponentially with Z. However, due to the effect of discounting, the present value of future rotations decrease the further into the future they occur. For instance, increasing Z from 2 to 3, leads to a modest increase in value of 1.3% under GBM with 0.05 volatility. Increasing Z any further was found to have an even less impact on the calculated value. Consequently,

[1] As a reminder, the formula is presented in (4.54)

restricting the number of rotations considered in the compound option approach is reasonable. Notably, the difference between the stand value calculated using the compound option approach and the heuristic approach is small. The heuristic approach runs in polynomial time, whereas the runtime of the compound option approach grows exponentially with Z. Moreover, the heuristic approach requires significantly less memory than the compound option approach, as it only constructs a single trinomial tree as opposed to Z trees in the compound approach. Therefore, it has the potential to be a valuable decision support tool for forestry practitioners. The models developed in this study are implemented in Python, and the heuristic approach uses only seconds on cases where the compound approach takes upwards of 10 minutes. Despite the simplifying assumptions the heuristic approach entails, it is relatively accurate in approximating the stand value. Concomitantly, it explicitly models the option like nature of the first rotation, the option to abandon instead of replanting if prices are low, and it incorporates the value of future rotations using the Faustmann value at different future prices as opposed to simply the current price, thereby capturing the value of flexibility in the presence of price uncertainty.

5.2.2 An Application of the Heuristic Approach

The heuristic approach is used to calculate stand value and the expected rotation age under the parameters estimated for historical prices in Chapter 5.1.3. The stand value is the value of a freshly planted stand on one hectare of land, ignoring the cost of planting the first rotation. The results are summarized in Table 5.3, and are for a current price of 376 NOK/m3 which was the weighted average Norway spruce price in May 2020. The rightmost column gives the value of a single forest rotation calculated using the option approach developed for the single rotation for comparison.

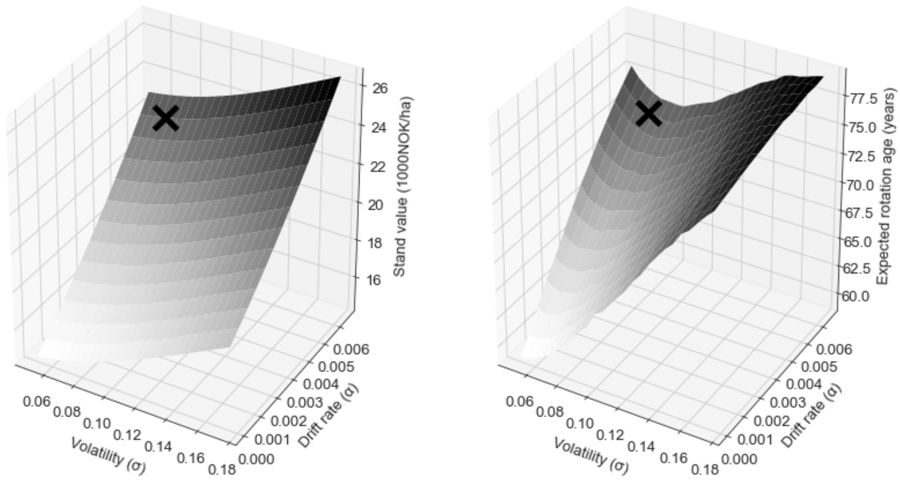
Table 5.3: Expected rotation ages and stand values for multiple forest rotations under GBM and MR using estimated parameters. Single rotation included for comparison.

	Rotation age (years)	Stand value (NOK/ha)	Single rotation value (NOK/ha)
Faustmann	41	14,052	13,273*
GBM ($\alpha = 0.006, \sigma = 0.067$)	73	22,380	20,081
MR ($\eta = 0.325, \sigma = 0.067, \mu = 396$)	42	14,464	14,448

* The Faustmann formula calculates the value of infinite rotations, and is not applicable to a single rotation. The single rotation value listed here is calculated by assuming that harvest occurs when the growth rate declines to the discount rate.

Under GBM the calculated stand value is significantly higher than the Faustmann value, and the rotation age is 73 years. In contrast using the Faustmann rule, the prescribed rotation age is 41 years. This is slightly lower than the 43 years traditional rules prescribe in the single rotation, and reflects the opportunity cost of delaying future rotations. The increase in value and rotation age is unsurprising. Intuitively, even when the forest is no longer growing rapidly, there is benefit to delay harvest and future rotations as the prices are expected to steadily increase. Figure 5.10a shows the value as a function of volatility and drift rate. The X on the surface marks the point which corresponds to the parameters estimated from the historical data. The value increases rapidly as the volatility and drift rate is increased. Note that when the volatility and drift rate is low the stand value coincides with the Faustmann value, which assumes deterministic prices. Similarly, Figure 5.10b shows how the rotation increases with volatility and drift.

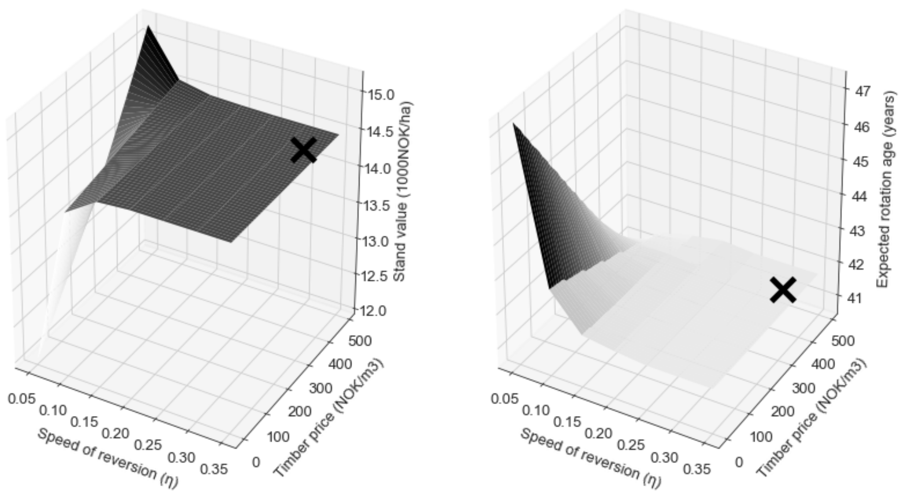
The value and rotation age is found to be significantly lower under MR than GBM. The expected rotation age is 42 years under MR. This reflects that when 42 years have passed the price is expected to have reverted from the initial 376 NOK/m3 to the equilibrium of 396 NOK/m3. The chances of realizing any price significantly above the equilibrium thereafter are slim, consequently, the stand is harvested to initiate a new rotation and benefit from the rapid growth in the early years. Due to the strong speed of reversion estimated from the historical prices the stand value is found to be independent of the current price. This is evident from Figure 5.11a which shows the stand value as a function of speed of reversion and initial price. When the speed of reversion is low the value increases with the price. However, when the speed of reversion is high, the value is seen to be unaffected by the price. Similarly, the rotation age is found to be independent of the current price for a sufficiently high speed of reversion, as seen in Figure 5.11b. Under high speed of reversion, the forestry manager can confidently determine the price far in the future as it has stabilized at the equilibrium. After it has stabilized, the chance of realizing a significantly higher price is low, and therefore the stand is expected to be harvested once the equilibrium is hit, and the growth rate has declined.



(a) Stand value (1000NOK/ha)

(b) Rotation age (years)

Figure 5.10: GBM: Stand value and expected rotation age as a function of volatility and drift rate. X marks the point which represents the parameters estimated from historical data ($\alpha = 0.006, \sigma = 0.067$).



(a) Stand value (1000NOK/ha)

(b) Rotation age (years)

Figure 5.11: MR: Stand value and expected rotation age as a function of speed of reversion and initial price. X marks the point which represents the parameters estimated from historical data ($\eta = 0.325, \sigma = 0.067, \mu = 396$).

Chapter 6

Conclusion

This paper introduced a real options approach to value a forest stand and determine the optimal harvest age for both single- and multiple forest rotations. The approach used a trinomial tree to approximate stochastic timber prices, and subsequently, a dynamic programming approach to calculate the stand value and expected harvest age. The benefit of this approach is its conceptual simplicity and transparency, in addition to its ability to approximate both GBM and MR.

A heuristic approach was developed to simplify calculations in the case of multiple forest rotations using an option approach for the initial rotation, and the Faustmann value for subsequent rotations. The heuristic approach was demonstrated to have a high degree of accuracy. The heuristic approach should enhance the practical validity of the real options approach for multiple rotations in forestry (1) by reducing the mathematical complexity compared to continuous-time analytical option pricing models, finite difference and simulation methods, and (2) by significantly reducing the number of computational steps to value a perpetual compound option using a tree-based model. The approach should, therefore, be of practical applicability, and yet remain palatable to forestry practitioners.

In a series of numerical examples, the value of a single forest rotation was shown to be significantly affected by the choice of stochastic process. Moreover, the expected harvest age was shown to be higher under GBM than the traditionally prescribed deterministic harvest age. In contrast, the expected harvest age under MR was found to be lower than the deterministic case for prices above the equilibrium level, and conversely, lower than the deterministic case for prices above the equilibrium. Historical price data for Norway spruce was used to estimate the parameters of GBM and MR. The value under MR was found to be higher than under GBM when the current timber price is below the equilibrium. For prices above the equilibrium, GBM yields a higher value in the early years of the rotation, but coincides with the MR value as the stand grows older. The critical price under MR is lower than under GBM in the early years of the forest rotation, and conversely higher than under GBM in the later years. This indicates that under the assumption of MR, the stand is more likely to be harvested while it is still growing rapidly, and confirms the results of Insley (2002).

Furthermore, the estimated GBM and MR processes were used to calculate the stand value for multiple forest rotations using the heuristic option approach. The stand value was demonstrated to be higher under GBM than MR. Moreover, it was found that the value and rotation age was higher under both GBM and MR than in the Faustmann setting. The stand value and rotation age were shown to be independent of the current timber price for a sufficiently high speed of reversion MR process. The results suggest that the choice of a particular stochastic process has a significant effect on the stand value and rotation age, also in the case of multiple rotations. Hence, the choice of an appropriate stochastic process is a crucial aspect in determining the stand value and in making optimal decisions in forestry management.

There are certain limitations to the proposed option approach. First, the results are sensitive to the step size set in the trinomial tree. Because the finite horizon was set relatively high in this article, decreasing the step size had little impact on the results. However, if the model is used with a shorter horizon or different growth function, care should be taken when determining the appropriate step size to ensure convergence. Second, approximating the price process in discrete time comes at a cost. Estimating the parameters of the price process is challenging, and as demonstrated, the option value and rotation age is highly sensitive to the choice of parameters. This, however, is not necessarily unique to the real options method proposed in this study and would remain a challenge regardless of the approach when price uncertainty is taken into consideration.

A direction for future work is to incorporate a division between the sawlogs and pulpwood resulting from harvest by modelling two separate price processes and account for the evolution of each product type in the growth function. As a result, this would help alleviate practical concerns regarding the use of a weighted price, as was done to simplify analysis in this article. Moreover, it would be of value to forestry practitioners to consider flexibility in other aspects of forestry management. For instance, an extended model formulation could include the option to use fertilizers resulting in increased growth or incorporate the option to choose a specific thinning regime resulting in a dynamic growth function.

Bibliography

Bellman, R., 1966. Dynamic programming. Science 153, 34–37.

Braastad, H., 1975. Produksjonstabeller og tilvekstmodeller for gran (Yield tables and growth models for Picea abies). Norsk institutt for skogforskning (Norwegian Forest Research Institute).

Brazee, R., Mendelsohn, R., 1988. Timber Harvesting with Fluctuating Prices. Forest Science 34, 359–372.

Brock, W., Rothschild, M., 1986. Comparative statics for multidimensional optimal stopping problems, in: Models of economic dynamics. Springer, pp. 124–138.

Clark, C.W., 2010. Mathematical bioeconomics: the mathematics of conservation. John Wiley & Sons.

Clarke, H.R., Reed, W.J., 1989. The tree-cutting problem in a stochastic environment: The case of age-dependent growth. Journal of Economic Dynamics and Control 13, 569 – 595.

CNBC, 2020. Climate change is driving widespread forest death and creating shorter, younger trees.

Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option pricing: A simplified approach. Journal of Financial Economics 7, 229 – 263.

Deutsche Welle, 2020. Is there enough wood in the world to feed sustainability?

Dixit, A.K., Dixit, R.K., Pindyck, R.S., 1994. Investment under uncertainty. Princeton university press.

Faustmann, M., 1849. On the determination of the value which forest land and immature stands possess for forestry. English edition edited by M. Gane, Oxford Institute Paper 42, 1968, entitled "Martin Faustmann and the Evolution of Discounted Cash Flow," which also contains the prior 1849 paper by E. F. von Gehren.

Gjolberg, O., Guttormsen, A.G., 2002. Real options in the forest: what if prices are mean-reverting? Forest Policy and Economics 4, 13 – 20.

Gren, I.M., Aklilu, A.Z., 2016. Policy design for forest carbon sequestration: A review of the literature. Forest Policy and Economics 70, 128 – 136.

Haight, R.G., 1993. Optimal management of loblolly pine plantations with stochastic price trends. Canadian Journal of Forest Research 23, 41–48.

Hartmann, M., Hassan, A., 2006. Application of real options analysis for pharmaceutical R&D project valuation—Empirical results from a survey. Research Policy 35, 343–354.

Hull, J.C., 2003. Options futures and other derivatives. Pearson Education India.

Hultkrantz, L., 1993. Informational efficiency of markets for stumpage: comment. American Journal of Agricultural Economics 75, 234–238.

Insley, M., 2002. A real options approach to the valuation of a forestry investment. Journal of Environmental Economics and Management 44, 471–492.

Insley, M., Rollins, K., 2005. On Solving the Multirotational Timber Harvesting Problem with Stochastic Prices: A Linear Complementarity Formulation. American Journal of Agricultural Economics 87, 735–755.

Kant, S., Alavalapati, J., 2014. Handbook of forest resource economics. Routledge.

Kijima, M., Nagayama, I., 1994. Efficient numerical procedures for the Hull-White extended Vasicek model. The journal of financial engineering 3.

Lohmander, P., 1987. Pulse Extraction under Risk and a Numerical Forestry Application. IIASA Working Paper.

Manley, B., Niquidet, K., 2010. What is the relevance of option pricing for forest valuation in New Zealand? Forest Policy and Economics 12, 299 – 307.

Mun, J., 2002. Real options analysis: Tools and techniques for valuing strategic investments and decisions. John Wiley & Sons.

National Geographic, 2020. The grand old trees of the world are dying, leaving forests younger and shorter.

Newman, D., 2002. Forestry's golden rule and the development of the optimal forest rotation literature. Journal of Forest Economics 8, 5 – 27.

Newman, D.H., 1988. The optimal forest rotation: a discussion and annotated bibliography. volume 48. US Department of Agriculture, Forest Service, Southeastern Forest Experiment Station.

NORSKOG, 2015. Staten aksepterer 4% kalkulasjonsrente! (The Government accepts a 4% discount rate).

Payandeh, B., 1973. Plonski's yield tables formulated. Department of the Environment, Canadian Forestry Service Publication.

Pearse, P.H., 1967. The optimum forest rotation. The Forestry Chronicle 43, 178–195.

Pindyck, R.S., 1999. The long-run evolutions of energy prices. The energy journal 20.

Prestemon, J.P., 2003. Evaluation of U.S. southern pine stumpage market informational efficiency. Canadian Journal of Forest Research 33, 561–572.

Reed, W.J., Clarke, H.R., 1990. Harvest decisions and asset valuation for biological resources exhibiting size-dependent stochastic growth. International Economic Review 31, 147–169.

Reuters, 2019. Climate change to blame as bark beetles ravage central Europe's forests.

Schwartz, E.S., 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging. The Journal of Finance 52, 923–973.

SkogsNorge, 2014. Skog til salgs - priser (Forests for sale - prices).

Thomson, T.A., 1992. Optimal forest rotation when stumpage prices follow a diffusion process. Land Economics 68, 329–342.

Timber Trades Journal, 2020. Covid-19 and bark beetle feature heavily at EOS general assembly.

Trigeorgis, L., et al., 1996. Real options: Managerial flexibility and strategy in resource allocation. MIT press.

Wall Street Journal, 2020a. The Hottest Home-Construction Market Since 2006 Is Lifting Lumber Prices.

Wall Street Journal, 2020b. Lumber Markets Hint at Housing Slowdown.

Washburn, C.L., Binkley, C.S., 1990. Informational efficiency of markets for stumpage. American Journal of Agricultural Economics 72, 394–405.

Worren, N.A., Moore, K., Elliott, R., 2002. When theories become tools: Toward a framework for pragmatic validity. Human Relations 55, 1227–1250.

Yin, R., Newman, D.H., 1996. Are markets for stumpage informationally efficient? Canadian Journal of Forest Research 26, 1032–1039.

Zvan, R., Forsyth, P., Vetzal, K., 1998. Penalty methods for American options with stochastic volatility. Journal of Computational and Applied Mathematics 91, 199 – 218.

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