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On Excessive Transverse Coordinates for Orbital Stabilization of Periodic Motions ⋆

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Abstract: This paper explores transverse coordinates for the purpose of orbitally stabilizing periodic motions of nonlinear control-affine dynamical systems. It is shown that the dynamics of any (minimal or excessive) set of transverse coordinates, which are defined in terms of a particular parameterization of the motion and a strictly state-dependent projection operator recovering the parameterizing variable, admits a (transverse) linearization along the target motion, with explicit expressions stated. Special focus is then placed on a generic excessive set of orthogonal coordinates, revealing a certain limitation of the "excessive" transverse linearization for the purpose of control design. To overcome this limitation, a linear comparison system is introduced and conditions are stated for when the asymptotic stability of its origin corresponds to the asymptotic stability of the origin of linearized transverse dynamics. This allows for the construction of feedback controllers utilizing this comparison system which, when implemented on the dynamical system, renders the desired motion asymptotically stable in the orbital sense.

Keywords: Orbital stabilization, transverse coordinates, transverse linearization.

1. INTRODUCTION

We consider the task of orbitally stabilizing periodic solutions of nonlinear dynamical systems, defined by

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{1}$$

Here the notion of *asymptotic orbital (Poincaré) stability* simply means the asymptotic convergence to the periodic *orbit* (i.e. the set of all the states along the solution) and not to a specific point-in-time along a trajectory (see e.g. Leonov (2008)). In this regard, we recall the following.

Theorem 1. (Andronov–Vitt). A nontrivial, T-periodic solution $x_*(t) = x_*(t+T)$ of a smooth dynamical system $\dot{x} = F(x)$ on $\mathbb{R}^n$ is asymptotically orbitally stable if the first approximation, $\delta\dot{x} = \frac{\partial F}{\partial x}(x_*(t))\delta x$, has one simple zero characteristic exponent and the remaining $(n-1)$ characteristic exponents have strictly negative real parts.

It thus follows that the stability of a periodic orbit is equivalent to the stability of an $(n-1)$-dimensional subsystem of the first approximation along the nominal solution. At the same time, the Andronov–Vitt theorem also highlights a limitation of the first approximation for the purpose of feedback design for (1) due to its non-vanishing (zero characteristic (Floquet) exponent) solution. It would therefore clearly be beneficial to instead just target the $(n-1)$-dimensional subsystem directly, which it turns out is equivalent to only considering the dynamics *transverse* to the orbit. Indeed, it is known that a periodic solution is asymptotically stable in the orbital sense if (and only if) the dynamics transverse to the flow along the nominal orbit are asymptotically stable (Hauser and Chung (1994)).

The design of orbitally stabilizing feedback controllers can therefore be boiled down to two main steps: 1) Find a (minimal) set of $(n-1)$ independent *transverse coordinates* which vanish on the orbit and are non-zero away from it; and then 2) Design a controller (by some means) which stabilizes the origin of these coordinates. Here the latter step is commonly achieved by linearization of the dynamics of these coordinates along the solution, a so-called *transverse linearization*, allowing for feedback design utilizing well-known linear control techniques.

While there exists constructive procedures for finding such a minimal set of coordinates for certain classes of systems (Shiriaev et al., 2010; Banaszuk and Hauser, 1995)), finding $(n-1)$ independent coordinates can be challenging in the general case. The main contribution of this paper is therefore to show that one instead can utilize an *excessive* set of transverse coordinates. In fact, we show that any such set (minimal or excessive) will do (see Proposition 5). In this regard, we also provide explicit expressions for the linearized transverse dynamics of *any* (minimal or excessive) set of transverse coordinates (see Theorem 6 in Sec. 3).

In order to provide some further insight into- and highlight a limitation of the transverse linearization for an excessive set of coordinates (see Sec. 4.1) with the limited space available, we subsequently focus mainly on a generic set of easy-to-compute orthogonal coordinates introduced in Sec. 4. In this regard, this paper's second major contribution is the introduction of a linear comparison system for these coordinates, which can be used for orbitally stabilizing feedback design for systems of the form (1) (see Proposition 11 in Sec. 4.2). In order to illustrate the proposed scheme, we consider a constructive example in Sec. 5, before, lastly, we state some concluding remarks.

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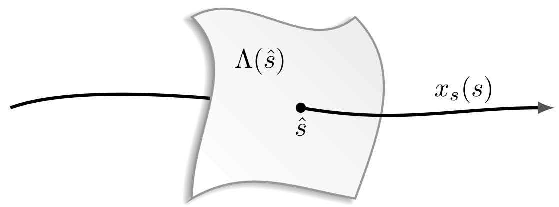

Fig. 1. Illustration of the transverse surface formed by $\Lambda(\cdot)$.

2. PRELIMINARIES AND KEY IDEA

Consider the control-affine system (1) with $f : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable and $g(x) = [g_1(x), \ldots, g_m(x)]$ with (locally) Lipschitz continuous vector fields $g_i : \mathbb{R}^n \to \mathbb{R}^n$. Let $x_*(t) = x_*(t+T)$ denote a bounded, T-periodic solution of the undriven system ($u \equiv 0$) satisfying $\|\dot{x}_*(t)\| > 0$ for all $t \geq 0$, and let

$$\eta_* := \{x \in \mathbb{R}^n : \ x = x_*(t), \ t \in [0, T)\}$$

denote the corresponding closed orbit. Suppose this orbit admits a regular $\mathcal{C}^2$-parameterization, defined by

$$x_s : \mathcal{S} \to \eta_*, \quad s \mapsto x_s(s), \quad x_s(s + s_T) = x_s(s), \qquad (2)$$

such that the parameterizing variable, $s \in \mathcal{S} := [s_0, s_0 + s_T)$, is strictly monotonically increasing along η_* and $\|\frac{d}{ds}x_s(s)\| = \|x_s'(s)\| > 0$ for all $s \in \mathcal{S}$. Further suppose that a *projection operator*, $x \mapsto p(x) \in \mathcal{S}$, in accordance with the following definition is known for this curve.

Definition 2. A mapping $p : \mathbb{R}^n \to \mathcal{S}$ is said to be a *projection operator* onto the orbit η_* if it is twice continuously differentiable within some tubular neighbourhood $\mathcal{X} \subset \mathbb{R}^n$ of η_* and it is a left inverse of the curve (2), that is $s = p(x_s(s))$ for all $s \in \mathcal{S}$. □

The idea behind such a projection operator is simply that, within some tubular neighbourhood, it allows one to project the current states down upon the nominal orbit and consequently define some measure of the distance to it. For instance, consider the set $\Lambda(\hat{s}) := \{x \in \mathcal{X} : p(x) = \hat{s}\}$, that is, the set of states in a neighbourhood of η_* mapped to some particular $\hat{s} \in \mathcal{S}$. As illustrated in Figure 1, it traces out a hypersurface, whose geometry is clearly dependent on the choice of $p(\cdot)$. This surface (manifold) of dimension $(n-1)$ is analogous to a *moving Poincaré section* (Leonov, 2006) which moves along with the trajectory and is locally transverse to its flow. It follows that if one can define a set of coordinates evolving upon- and spanning these sections, and then enforce, by some control action, strict contraction of these coordinates towards their origin (the orbit), then the desired trajectory must be asymptotically stable in the orbital sense.

Note that this concept is in many ways both similar to- and inspired by *Zhukovski stability* (see, e.g., Leonov et al. (1995); Leonov (2008)). Roughly speaking, this notion of stability, which implies orbital stability (Leonov, 2008), utilizes parameterizations to "align" perturbed trajectories in space while not considering their divergence in time. Our approach, however, differs by the fact that, whereas Zhukovski considered reparameterizations of perturbed trajectories in terms of a "rescaling of time", we consider a completely state-dependent projection operator as defined in Def. 2. This has, for the purpose of control design, the benefit that it allows one to define the aforementioned state-dependent distance measure, further allowing for the design of completely state-dependent orbitally stabilizing feedback controllers. Such a feedback, if found, then results in an autonomous closed-loop system which admits the desired solution as an attractive limit cycle.

Notation: $\|\cdot\|$ denotes the Euclidean norm. For a twice-continuously differentiable ($\mathcal{C}^2$-) function $x \mapsto h(x)$, we denote by $Dh(\cdot) = [\frac{\partial h}{\partial x_1}(\cdot), \ldots, \frac{\partial h}{\partial x_n}(\cdot)]$ its Jacobian matrix, while if $h : \mathbb{R}^n \to \mathbb{R}$, we denote by $D^2h(\cdot)$ its symmetric, $n \times n$ Hessian matrix. If $h_s(s) := h(x_s(s))$, then $h_s'(s)$ denotes the derivative $\frac{d}{ds}h_s(s)$.

3. EQUIVALENCE BETWEEN COORDINATES AND THE TRANSVERSE LINEARIZATION

In regards to the aforementioned distance measure, consider

$$z_\perp := x - x_s(p(x)). \qquad (3)$$

In some sense, they are the simplest measure of such a distance, but their definition is also clearly dependent on the choice of the projection operator $p(\cdot)$. In particular, they must evolve upon some hypersurface such as those formed by the set $\Lambda(\cdot)$. But $z_\perp \in \mathbb{R}^n$, and so they are an *excessive* set of coordinates upon this surface. In fact, they are not a valid change of coordinates either, as the map $x \mapsto z_\perp$ is evidently not a diffeomorphism.

To see this more clearly, consider the Jacobian matrix $Dz_\perp(x)$. Taking the time-derivative of (3), we obtain

$$\dot{z}_\perp = Dz_\perp(x)\dot{x} = Dz_\perp(x)f(x) + Dz_\perp(x)g(x)u. \qquad (4)$$

It follows that, sufficiently close the orbit, a variation in the states, δx, relates to a variation in the coordinates (3) through $\Omega(s) := Dz_\perp(x_s(s))$:

$$\delta z_\perp = \Omega(s)\delta x. \qquad (5)$$

Similarly, by defining $\Gamma(s) := Dp(x_s(s))$, we find that

$$\delta s = \Gamma(s)\delta x.$$

Thus for (3) to be a valid (local) change of coordinates, the matrix function $\Omega(s)$ must necessarily be everywhere invertible. However, as is clear by the following statement, which is just a straightforward consequence of the relation

$$\Gamma(s)x_s'(s) \equiv 1 \quad \forall s \in \mathcal{S}, \qquad (6)$$

obtained from $s = p(x_s(s))$ (see Def. 2), this can never be the case for non-constant solutions of the form (2).

Lemma 3. The matrix function

$$\Omega(s) := Dz_\perp(x_s(s)) = I_n - x_s'(s)\Gamma(s) \qquad (7)$$

is a projection matrix (i.e. $\Omega^2(s) = \Omega(s)$), its rank is always $(n-1)$, while $\Gamma(s) := Dp(x_s(s))$ and $x_s'(s) := \frac{d}{ds}x_s(s)$ are its left- and right annihilators, respectively. □ [1]

From Lemma 3 it is clear that we have $\Omega(s)\delta z_\perp = \Omega^2(s)\delta x = \delta z_\perp$, and therefore the relation $\Gamma(s)\delta z_\perp = \Gamma(s)\Omega(s)\delta z_\perp \equiv 0$ must always hold. We can thus infer that, sufficiently close to the nominal orbit, the coordinates (3) are orthogonal to the gradient of the projection operator $p(\cdot)$ and hence locally transverse to the nominal flow of the orbit. Indeed, it is important to note that the relation (6) does not imply that $\Gamma^\mathsf{T}(s)$ is necessarily in the span of

[1] Proofs of all the statements are given in the extended version of this paper which is available on the arXiv: arXiv:1911.06232.

$x_s'(s)$. Rather, if $\theta(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ denotes the angle between $\Gamma^\mathsf{T}(s)$ and $x_s'(s)$ in their common plane, then, as a direct consequence of the inner product $\Gamma x_s' = \|\Gamma\|\|x_s'\|\cos(\theta)$, there exists some continuously differentiable unit vector function $q_\perp^\mathsf{T}(s) : \mathcal{S} \to \mathbb{R}^n$ within $\ker {x_s'}^\mathsf{T}(s)$, such that

$$\Gamma(s) = \frac{{x_s'}^\mathsf{T}(s)}{\|x_s'(s)\|^2} + \tan(\theta(s))\frac{q_\perp(s)}{\|x_s'(s)\|}. \tag{8}$$

Consequently, the coordinates (3) are in general only locally transverse to the flow of the orbit and not necessarily orthogonal to it. Moreover, they must be an excessive set of transverse coordinates as rank $\Omega(s) = n-1$. Nevertheless, we will show shortly that the asymptotic stability of their origin in fact implies the asymptotic stability of any other valid set of transverse coordinates, and, therefore, also the asymptotic stability of the nominal orbit.

Let us start by giving a formal definition of what we mean when we refer to a "valid set of transverse coordinates". In this regard, consider a $\mathcal{C}^2$-function $y_\perp : \mathcal{S} \times \mathbb{R}^n \to \mathbb{R}^N$, together with a projection operator $p(\cdot)$. Note that we will distinguish between the partial- and total derivative of $y_\perp$ with respect to x as follows:

$$Dy_\perp(s,x) = \frac{\partial y_\perp}{\partial x}(s,x) + \frac{\partial y_\perp}{\partial s}(s,x)Dp(x).$$

Definition 4. A $\mathcal{C}^2$-function $y_\perp : \mathcal{S} \times \mathcal{X} \to \mathbb{R}^N$, $N \geq n-1$, is said to be a *valid set of transverse coordinates* for the curve (2) if it vanishes on it, i.e. $y_\perp(s, x_s(s)) \equiv 0$, and for all $s \in \mathcal{S}$ it satisfies rank $\frac{\partial y_\perp}{\partial x}(s, x_s(s)) = \min(N, n)$ and rank $Dy_\perp(s, x_s(s)) = n-1$. □

For the case $N = n-1$, we will refer to $y_\perp$ as a *minimal* set of transverse coordinates by the fact that the mapping $(y_\perp, s) \mapsto x$ is then a diffeomorphism in some non-zero neighbourhood of η_*. One the other hand, whenever $N \geq n$, we will refer to them as *excessive* coordinates.

Suppose $y_\perp$ is a valid set of coordinates by Def. 4. Differentiating, we find that their dynamics are described by

$$\dot{y}_\perp = Dy_\perp(s,x)\left[f(x) + g(x)u\right]. \tag{9}$$

Our task will now be to linearize the dynamics of $y_\perp$ along the orbit η_* in order to obtain a linear (periodic) system, the so-called linearized transverse dynamics, which we then can use to design orbitally stabilizing feedback. Towards this end, we observe that since $y_\perp(s, x_s(s)) \equiv 0$, we must have $\dot{y}_\perp(s, x_s(s)) \equiv 0$. Therefore, by defining

$$\Pi(s) := \frac{\partial y_\perp}{\partial x}(s, x_s(s)),$$

it is implied that the following relation must hold:

$$\frac{\partial y_\perp}{\partial s}(s, x_s(s)) = -\Pi(s)x_s'(s). \tag{10}$$

Thus, sufficiently close to the orbit, it is true that

$$\delta y_\perp = Dy_\perp(s, x_s(s))\delta x = \Pi(s)\Omega(s)\delta x,$$

and hence, by (5), we obtain

$$\delta y_\perp = \Pi(s)\delta z_\perp. \tag{11}$$

This naturally leads us to the following unsurprising statement, which simply shows that there is a certain stability equality between all sets of transverse coordinates.

Proposition 5. The origin of a valid set of transverse coordinates $y_\perp$ is asymptotically stable if and only if the origin of the coordinates $z_\perp$ is asymptotically stable. □

Now, let $\Psi(s) := Dy_\perp(s, x_s(s))$ and consider the differentiable matrix function $\Pi^\dagger : \mathcal{S} \to \mathbb{R}^{n\times N}$, defined by

$$\Pi^\dagger(s) := \begin{cases} \Omega(s)\Psi^\mathsf{T}(s)\left[\Psi(s)\Psi^\mathsf{T}(s)\right]^{-1} & \text{if } N = n-1, \\ \Pi^{-1}(s) & \text{if } N = n, \\ \left[\Pi^\mathsf{T}(s)\Pi(s)\right]^{-1}\Pi^\mathsf{T}(s) & \text{if } N > n. \end{cases} \tag{12}$$

This allows us to state the main result of this section.

Theorem 6. Let $y_\perp \in \mathbb{R}^N$ be a valid set of transverse coordinates together with a projection operator $p(\cdot)$. Then the linearization of their dynamics (9) evaluated along the solution (2) is described by the constrained (differential-algebraic) linear-periodic system

$$\begin{aligned} \frac{d}{dt}\delta y_\perp &= \left[\Pi(s)A_\perp(s) + \Xi(s)\right]\Pi^\dagger(s)\delta y_\perp + \Pi(s)B_\perp(s)u \\ 0 &= \Gamma(s)\Pi^\dagger(s)\delta y_\perp \end{aligned} \tag{13}$$

where

$$A_\perp(s) := \Omega(s)A(s) - x_s'(s){x_s'}^\mathsf{T}(s)D^2p(x_s(s))\rho(s)$$

$$\Xi(s) := \rho(s)\frac{\partial}{\partial x}\left[\frac{\partial y_\perp}{\partial x}(s,x)x_s'(s) + \frac{\partial y_\perp}{\partial s}(s,x)\right]\Bigg|_{x=x_s(s)}$$

$$B_\perp(s) := \Omega(s)B(s)$$

given $A(s) := Df(x_s(s))$, $B(s) := g(x_s(s))$, $\rho(s) := \Gamma(s)f(x_s(s))$ and with $\Pi^\dagger(\cdot)$ as defined in (12).

Remark 7. As $x_s : \mathcal{S} \to \eta_*$ is a regular parameterization, and thus $\rho(s) := \Gamma(s)f(x_s(s)) = \dot{s}_*(s) > 0$, one can utilize the fact that $\frac{d}{ds}\delta y_\perp = \frac{1}{\rho(s)}\frac{d}{dt}\delta y_\perp$ in order to solve (13). □

While there exists several known explicit expressions for transverse linearizations in the literature (see e.g. (Hauser and Chung, 1994, Proposition 1.4), (Mohammadi et al., 2018, Theorem 12), (Shiriaev et al., 2010, Theorem 2), (Leonov et al., 1995, Equation (4.23))), they are all only valid for a specific class of coordinates or for specific choices of the projection operator. Theorem 6, on the other hand, provides explicit expressions valid for any set of transverse coordinates, and just as importantly, for any choice of the projection operator. Also note that, while Theorem 12 in Mohammadi et al. (2018) provides equivalent expressions for the case when $N = n-1$, the proof of their statement is only valid whenever $\theta(s)$, as defined in (8), is exactly zero for all $s \in \mathcal{S}$. This is due to their use of the pseudo-inverse of Ψ as $\Pi^\dagger$, i.e. $\Pi^\dagger(s) = \Psi^\mathsf{T}(s)\left[\Psi(s)\Psi^\mathsf{T}(s)\right]^{-1}$ (cf. $dH^\dagger_{\varphi(\vartheta)}$ therein). While that requires $\Omega(s) = \Omega^\mathsf{T}(s)$ for $\Gamma(s)\Pi^\dagger(s)\delta y_\perp = 0$ to hold, and thus also the relation $\delta x = \Pi^\dagger(s)\delta y_\perp + x_s'(s)\delta s$ between the differentials, it is here satisfied directly by the slight modification of $\Pi^\dagger$ as given by (12).

In order to provide further insight into the transverse linearization of an excessive set of coordinates with the limited space remaining, we will in the sequel focus on a specific set of orthogonal coordinates.

4. A GENERIC SET OF EXCESSIVE ORTHOGONAL COORDINATES

Consider again the excessive coordinates previously defined in (3), namely $z_\perp := x - x_s(s)$. Using the first-order Taylor expansions of $Dp(\cdot)$ and $f(\cdot)$ about η_*, one can show that the transverse dynamics (4) then can be rewritten as

$$\dot{z}_\perp = A_\perp(s)z_\perp + \Omega(x)g(x)u + \Delta(s, z_\perp), \tag{14}$$

where $\|\Delta(\cdot, z_\perp)\| = O(\|z_\perp\|^2)$. The choice of notation in Theorem 6 thus becomes clear by its following corollary.

Corollary 8. The constrained linear-periodic system

$$\frac{d}{dt}\delta z_\perp = A_\perp(s)\delta z_\perp + B_\perp(s)u, \quad \Gamma(s)\delta z_\perp = 0, \tag{15}$$

corresponds to the linearization along (2) of the dynamics of the excessive set of coordinates defined in (3). □

As previously stated, the coordinates $z_\perp$ will depend upon the choice of $p(\cdot)$. While there in general will exist many valid candidates for this projection operator, all with different properties and resulting in different transverse hypersurfaces (moving Poincaré sections) on which the coordinates $z_\perp$ evolve, we will from now on consider those satisfying the orthogonality condition:

$${x'_s}^\mathsf{T}(s) z_\perp \equiv 0. \tag{16}$$

Note that this is locally equivalent to $s = \arg\min_{s\in\mathcal{S}} \|x - x_s(s)\|^2$, and so the Jacobian of this $p(\cdot)$ is given by

$$Dp(x) = \frac{{x'_s}^\mathsf{T}(s)}{\|x'_s(s)\|^2 - {x''_s}^\mathsf{T}(s)\,(x - x_s(s))}, \tag{17}$$

while, moreover, it can be shown that $\Delta(\cdot)$ then satisfies ${x'_s}^\mathsf{T}(s)\Delta(s, z_\perp) \equiv 0$ (Leonov, 2006). In addition, using (16) and that $D^2p(x_s(s))\rho(s)z_\perp = \frac{x'_s(s){x'_s}^\mathsf{T}(s)}{\|x'_s(s)\|^4} A^\mathsf{T}(s)z_\perp$, the matrix function $A_\perp(\cdot)$ can be simplified to

$$A_\perp(s) := \Omega(s)A(s) - \frac{x'_s(s){x'_s}^\mathsf{T}(s)}{\|x'_s(s)\|^2} A^\mathsf{T}(s). \tag{18}$$

Thus the linearized transverse dynamics are given according to Corollary 8 with (18) and $\Gamma(s) = {x'_s}^\mathsf{T}(s)/\|x'_s(s)\|^2$.

Note that the coordinates (3) together with the orthogonality condition (16) have been considered several times times before in relation to the study of the (in-)stability of solutions of autonomous dynamical systems (see e.g. Borg (1960); Hartman and Olech (1962); Zubov (1999); Leonov (2006); Hauser and Chung (1994)). However, they have not, to our best knowledge, been used together for the purpose of designing orbitally stabilizing feedback controllers for nonlinear systems of the form (1). For this purpose, however, the relation ${x'_s}^\mathsf{T}(s)\delta z_\perp \equiv 0$ is of particular interest. This is because, unlike a minimal set of coordinates in which the transversality condition $\Gamma(s)\Pi^\dagger(s)\delta y_\perp = 0$ in (13) is satisfied directly through $\Pi^\dagger$, it must be satisfied through the coordinates themselves for an excessive set.

4.1 Limitations of the excessive transverse linearization

Consider the linear system

$$\dot{y} = A_\perp(s)y + B_\perp(s)u \tag{19}$$

corresponding to (15), with $A_\perp$ as in (18) but without the transversality condition ${x'_s}^\mathsf{T}(s)y \equiv 0$. It can be shown that the undriven system ($u \equiv 0$) then has the solution

$$y_\parallel = \frac{x'_s(s)}{\|x'_s(s)\|^2\rho(s)} = \frac{x'_s(s)}{{x'_s}^\mathsf{T}(s)f_s(s)}, \tag{20}$$

whose characteristic exponent [2] evidently is exactly zero. Moreover, an additional $(n-1)$ linearly independent solutions of the undriven system can be found, which we denote $y_\perp^1(\cdot), \ldots, y_\perp^{n-1}(\cdot)$, and which form a basis of the kernel of $\Gamma(s)$ for a given $s \in \mathcal{S}$ (it can be shown that $\frac{d}{dt}(y_\parallel^\mathsf{T} y_\perp^i) \equiv 0$), and hence satisfy condition (16). Using these solutions, let $\Phi_\perp(s) = [\varphi_\perp^1(s), \ldots, \varphi_\perp^{n-1}(s)]$ denote a smooth normalized basis of the kernel of $\Gamma(s)$, with $\varphi_\perp^i(\cdot)$ defined by $\varphi_\perp^i(s(t)) = y_\perp^i(t)/\|y_\perp^i(t)\|$, and let $\Phi_\perp^\dagger$ denote its pseudo-inverse, that is $\Phi_\perp^\dagger := (\Phi_\perp^\mathsf{T}\Phi_\perp)^{-1}\Phi_\perp^\mathsf{T}$.

Consider now the first approximation (variational) system of (1) along the curve (2):

$$\frac{d}{dt}\delta x = A(s)\delta x + B(s)u. \tag{21}$$

The following statement can be seen as analogous to the Andronov-Vitt theorem for the system (19).

Proposition 9. The system (19) has $(n-1)$ linearly independent solutions of the form $\Phi_\perp(s(t))\xi_\perp(t)$ with $\xi_\perp \in \mathbb{R}^{n-1}$ a solution to the $(n-1)$-dimensional system

$$\dot{\xi}_\perp = \Phi_\perp^\mathsf{T}(s)A(s)\Phi_\perp(s)\xi_\perp + \Phi_\perp^\dagger(s)B(s)(s)u. \tag{22}$$

In addition, it has a solution with a non-vanishing part in the direction of (20) regardless of the control input u. □

An important consequence of Proposition 9 is the fact that the origin of the system (19) can never be asymptotically stabilized. That is to say, even if one can find some feedback asymptotically stabilizing the origin of the system (15), and consequently the periodic orbit, the system (19) will regardless have a non-vanishing solution whose characteristic exponent is zero. Thus the usefulness of this system in terms of control design is limited due to its non-stabilizable subspace. On the other hand, we can infer that if the pair $(\Phi_\perp^\mathsf{T} A\Phi_\perp, \Phi_\perp^\dagger B\Phi_\perp)$ is stabilizable, then we can stabilize the orbit utilizing some controller designed to stabilize the subsystem (22). The obvious alternative is therefore to try to directly stabilize this subsystem. Yet, this requires knowledge of the basis $\Phi_\perp(\cdot)$.

Clearly it would instead be beneficial to find some way of stabilizing the subsystem (22) without the need to form $\Phi_\perp(\cdot)$. In this regard, we will introduce next a linear comparison system of (19), for which, under conditions we state in Proposition 11, the asymptotic stability of its origin implies asymptotic stability of the origin of the subsystem (22) and consequently the asymptotic orbital stability of the nominal solution.

4.2 The existence of a comparison system

Suppose we left-multiply both sides of (19) by the matrix function $\Omega(s)$. Utilizing its properties (see Lemma 3), one can then rewrite the system on several different equivalent forms, with the following among them:

$$\Omega(s)\left[\dot{y} - \Omega(s)\left(A(s)y + B(s)u\right)\right] = 0. \tag{23}$$

Consider, therefore, the linear-periodic system

$$\dot{w} = \Omega(s)A(s)w + \Omega(s)B(s)v, \; w \in \mathbb{R}^n, \; v \in \mathbb{R}^m, \tag{24}$$

corresponding to the terms inside the brackets of the descriptor system (23) being set to zero. Roughly speaking, we will show that if there exists a feedback of the form $v = K(s)w$ which "sufficiently" stabilizes the origin of this *comparison system*, then the controller $u = K(s)\delta z_\perp$ stabilizes the origin of the linearized transverse dynamics (15) as well. Thus this comparison system can allow

[2] The number (or the symbols, $\pm\infty$), given by the formula $\limsup_{t\to+\infty} \frac{1}{t}\ln\|x(t)\|$ is called the characteristic exponent of the continuous function $x : [0,\infty) \to \mathbb{R}^n$ (Leonov, 2006).

one to find a stabilizing feedback for (15) without the need to circumvent the uncontrollable subspace always present in (19) and without having to compute the Hessian $D^2p(\cdot)$. Indeed, there are several connections between these systems, such as the following spectrum condition.

Lemma 10. Consider the system (1) with the feedback $u = K(p(x))[x - x_s(p(x))]$ for some Lipschitz continuous matrix function $K : \mathcal{S} \to \mathbb{R}^{m\times n}$. Then the (minimal) sum of the characteristic exponents of the systems (19), (21) and (24) are the same. □

Suppose, therefore, that a (Lipschitz continuous) matrix function $K : \mathcal{S} \to \mathbb{R}^{m\times n}$ exists such that the largest characteristic exponents, λ_M, of the closed-loop system

$$\dot{w} = \Omega(s)\,(A(s) + B(s)K(s))\,w \tag{25}$$

satisfies $\lambda_M < 0$; i.e. we assume (24) is stabilizable. Let $W(t)$ denote the state transition (Cauchy) matrix for this system. Then, by a small modifications of theorems 2 and 4 in Leonov and Kuznetsov (2007), there exists some number $C > 0$ and a scalar functions $\zeta : [0,\infty) \to \mathbb{R}$ satisfying

$$\lim_{t\to\infty} \frac{1}{t}\int_\tau^t \zeta(\sigma)d\sigma = \lambda_M \quad \forall \tau \geq 0, \tag{26}$$

such that the following inequality

$$\|W(t)W^{-1}(\tau)\| \leq C\exp\left(\int_\tau^t \zeta(\sigma)d\sigma\right) \quad \forall t \geq \tau \geq 0 \tag{27}$$

is satisfied. The main result of this section follows.

Proposition 11. Let $p(\cdot)$ be taken as to satisfy (16). Suppose that $\|A(s)\| \leq \alpha$ for all $s \in \mathcal{S}$ and that the inequality

$$\lambda_M < -C\alpha \leq 0 \tag{28}$$

holds. Then the controller $u = K(s)z_\perp$ with $s = p(x)$ asymptotically stabilizes the origin of the system (14) and consequently renders the periodic solution of the dynamical system (1) asymptotically orbitally stable.

Remark 12. The value of the above statement is not in the condition (28) per se. Rather, its importance is simply due to the fact that it shows the possibility of orbitally stabilizing the solution by designing a stabilizing feedback for the comparison system (24). Indeed, the condition (28) is by no means unique, and similar conditions can be stated using, for example, Lyapunov's second method. □

It is also of practical importance to note that if a controller $v = K(s)w$ stabilizing the origin of the comparison system (24) has been designed, then one does not need to check the conditions of the theorem. That is to say, one can instead utilize the Andronov–Vitt theorem on the first approximation system $\delta\dot{x} = (A(s) + B(s)K(s)\Omega(s))\delta x$ to validate that it will also be a stabilizing controller for (15); or, equivalently, check that the system (19) has $(n-1)$ characteristic multipliers within the unit circle. As yet another alternative, one can utilize the following.

Lemma 13. If the system (24) under the controller $v = K(s)\Omega(s)w$ has one simple zero characteristic exponent and the remaining $(n-1)$ characteristic exponents have strictly negative real parts, then the controller $u = K(s)z_\perp$ asymptotically stabilizes the origin of the system (14). □

This again shows that one does not need to compute the Hessian of $p(\cdot)$ in order to validate the stability of the orbit. Moreover, this has an additional advantage compared to the Andronov–Vitt theorem arising whenever the dynamical system has a periodic solution only in the presence of some non-zero nominal control input $\upsilon(s(t)) \equiv u_*(t)$, i.e. $\frac{d}{dt}x_s(s) = f(x_s(s)) + g(x_s(s))\upsilon(s)$. As then the matrix $A(\cdot)$ of the first approximation is given by

$$A(s) = \left[\frac{\partial f}{\partial x} + g\upsilon'(s)\Gamma(s) + \sum_{i=1}^m \frac{\partial g_i}{\partial x}\upsilon_i(s)\right]\Bigg|_{x=x_s(s)},$$

one needs to compute $\upsilon'(s)$ in order to utilize the Andronov-Vitt Theorem, whereas it can be omitted in the transverse linearization, and consequently for the comparison system (24), due to the condition $\Gamma(s)\delta z \equiv 0$.

We illustrate the above scheme in a simple example next.

5. ILLUSTRATIVE EXAMPLE

Consider the system

$$\dot{x}_1 = x_2 + x_1x_3 + x_1u \tag{29a}$$
$$\dot{x}_2 = -x_1 + x_2x_3 + x_2u \tag{29b}$$
$$\dot{x}_3 = u \tag{29c}$$

which for $u \equiv 0$ has a family of periodic orbits given by

$$\eta_a = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 = a^2, x_3 = 0, a > 0\}. \tag{30}$$

This system has previously been considered in Banaszuk and Hauser (1995), where a (transverse) feedback linearizing approach was utilized in order to find a minimal set of transverse coordinates. More specifically, they showed that by taking $\theta = -\arctan(x_2/x_1)$, there exists a pair of transverse coordinates (σ_1, σ_2), defined as $\sigma_1 := \log\left(\sqrt{x_1^2 + x_2^2}\right) - \log(a) - x_3$ and $\sigma_2 := x_3$, such that $(x_1, x_2, x_3) \mapsto (\theta, \sigma_1, \sigma_2)$ is a diffeomorphism everywhere except $(x_1, x_1) = (0, 0)$. Moreover, the dynamics of θ is trivial ($\dot{\theta} = 1$) while the dynamics of the transverse coordinates (σ_1, σ_2) are linear: $\dot{\sigma}_1 = \sigma_2$, $\dot{\sigma}_2 = u$. While this is clearly a convenient choice of coordinates, and illustrates the possibility of finding a minimal set of coordinate that can greatly simplify control design, it also shows the challenge of finding a (convenient) set of coordinates even for such a simple, low dimensional system.

Let us therefore instead consider $s = p(x) = \operatorname{atan2}(x_1, x_2)$ with $\dot{s}_*(t) = \rho(s(t)) = 1$, which here satisfies the orthogonality condition (16) (atan2($\cdot$) denotes the four-quadrant arctangent function), and which lets us parameterize the orbit η_a by $x_s(s) = [a\sin(s), a\cos(s), 0]^\mathsf{T}$. The linearized transverse dynamics (19) then becomes

$$\frac{dy}{ds} = \begin{bmatrix} 0 & 1 & a\sin(s) \\ -1 & 0 & a\cos(s) \\ 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} a\sin(s) \\ a\cos(s) \\ 1 \end{bmatrix} u, \tag{31}$$

while its comparison system (24) is given by

$$\frac{dw}{ds} = \begin{bmatrix} -\frac{\sin(2s)}{2} & \sin^2(s) & a\sin(s) \\ -\cos^2(s) & \frac{\sin(2s)}{2} & a\cos(s) \\ 0 & 0 & 0 \end{bmatrix} w + \begin{bmatrix} a\sin(s) \\ a\cos(s) \\ 1 \end{bmatrix} v. \tag{32}$$

Taking $a = 1$, we designed a stabilizing controller for the comparison system (24), in which the found controller gains can be seen in Figure 2. These gains correspond to the feedback matrix $K(s) = [k_1(s), k_2(s), k_3] = -B_\perp^\mathsf{T}(s)R(s)$ with $R(s) = R^\mathsf{T}(s)$ the positive definite solution to the periodic Riccati differential equation

$$\frac{dR}{ds} + \Omega A^\mathsf{T} R + R\Omega A + I_3 - RB_\perp B_\perp^\mathsf{T} R = 0.$$

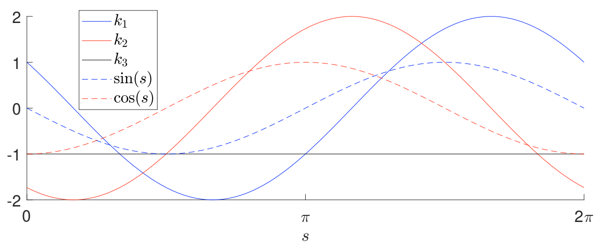

Fig. 2. Controller gains stabilizing (32).

With this controller, the characteristic exponents of (31) were approximately $(0, -1.73, -1)$, implying the asymptotic stability of the orbit by Proposition 9; while for the system (32) they were approximately $(-0.86 \pm 0.5i, -1)$, showing it is indeed an orbitally stabilizing controller as we would expect from Proposition 11.

Let us now also demonstrate a certain limitation of Proposition 11 by instead considering the feedback

$$u(z_\perp) = -\left[\sin(s)\ \cos(s)\ 1\right] z_\perp \tag{33}$$

which stabilizes the system (31), and consequently asymptotically stabilizes the orbit (30) for any $a > 0$. More specifically, it can be shown that the modified periodic Riccati differential equation

$$\Omega^\mathsf{T}\left[\frac{dR_\perp}{ds} + A_\perp^\mathsf{T} R_\perp + R_\perp A_\perp + I_3 - R_\perp B_\perp B_\perp^\mathsf{T} R_\perp\right]\Omega = 0,$$

has a family of solutions given by

$$R_\perp(s) = \Omega^i(s)\begin{bmatrix}\frac{1}{a} & 0 & 0\\ 0 & \frac{1}{a} & 0\\ 0 & 0 & 1\end{bmatrix}\Omega^j(s) + k\begin{bmatrix}\cos^2(s) & -\frac{\sin(2s)}{2} & 0\\ -\frac{\sin(2s)}{2} & \sin^2(s) & 0\\ 0 & 0 & 0\end{bmatrix}$$

for any $k \in \mathbb{R}$ and $i, j \in \{0, 1\}$, such that (33) corresponds to $u(z_\perp) = -B_\perp^\mathsf{T}(s)R_\perp(s)z_\perp$. Therefore, by taking $V = \delta z_\perp^\mathsf{T} R_\perp(s)\delta z_\perp$, we have $\dot{V} \leq -\|\delta z_\perp\|^2$ implying the asymptotic stability of the nominal solution. On other hand, in accordance with Proposition 9, it can be shown that the closed-loop system, i.e. $A_{cl}(s) := A_\perp(s) - B_\perp(s)B_\perp^\mathsf{T}(s)R_\perp(s)$, without the orthogonality condition (16) has the solution $x_s'(s)$ with characteristic exponent equal to zero. Its two other independent solutions are $[0, 0, e^{-t}]^\mathsf{T}$ and $\left[\sin(s(t)), \cos(s(t)), le^{(a-1)t} + \frac{e^{-at}}{a-1}\right]^\mathsf{T}$ with $l \in \mathbb{R}$. Taking $l = 0$, their characteristic exponents equals -1 and $-a$, respectively, again implying the asymptotic stability of the nominal solution.

Consider now the comparison system (32) with the above controller, i.e. $v(w) = -\left[\sin(s)\ \cos(s)\ 1\right] w$. It too has $x_s'(s)$ as a solution, while it can be shown that -1 and $-a$ are the characteristic exponents of the two remaining independent solutions (although note these solutions are different to those of (31) given above). We can therefore utilize Lemma 13 to validate that the controller is asymptotically orbitally stabilizing, but we cannot utilize Proposition 11 for this purpose.

So why is not the origin of the comparison system (32) asymptotically stable under the controller (33)? It turns out that the existence of the solution $x_s'(s)$ is clear simply by noticing that $B_\perp(s)B_\perp^\mathsf{T}(s)R_\perp(s) \equiv \hat{K}(s)\Omega(s)$ given

$$\hat{K}(s) := \begin{bmatrix} a & 0 & a\sin(s)\\ 0 & a & a\cos(s)\\ \sin(s) & \cos(s) & 1\end{bmatrix}.$$

Thus $u(w) = K(s)w \equiv 0$ for any $w \in \operatorname{span}(x_s'(s))$. It follows that a controller asymptotically stabilizing the linearized transverse dynamics (15) will not necessarily asymptotically stabilize the comparison system (24). On the other hand, it is quite interesting to note that all the characteristic exponents of both the systems $\dot{\hat{y}} = \big(A_\perp(s) - \hat{K}(s)\big)\hat{y}$ and $\dot{\hat{w}} = \big(\Omega(s)A(s) - \hat{K}(s)\big)\hat{w}$ have strictly negative real parts and sum to $(-2a - 1)$.

6. CONCLUDING REMARKS

In this paper, we have provided analytical expressions of the linearized transverse dynamics of any valid (minimal or excessive) set of transverse coordinates. In addition, we have defined a generic set of easy-to-compute orthogonal coordinates and shown a certain equivalence between their stability and that of any other valid set. It was further demonstrated that their origin could be stabilized by stabilizing a comparison system of the linearized transverse dynamics. This of course relies on the stabilizability of this comparison system, such that conditions for its stabilizability, as well as the connection to the stabilizability of the linearized transverse dynamics are topics of interest and requiring further study. The presented approach nevertheless lays the foundations for further development and generalizations, such as, for example, its extension to hybrid dynamical systems and to non-periodic motions.

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