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A scalarization scheme for binary relations with applications to set-valued and robust optimization

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Abstract In this paper, a method for scalarizing optimization problems whose final space is endowed with a binary relation is stated without assuming any additional hypothesis on the data of the problem. By this approach, nondominated and minimal solutions are characterized in terms of solutions of scalar optimization problems whose objective functions are the post-composition of the original objective with scalar functions satisfying suitable properties. The obtained results generalize some recent ones stated in quasi ordered sets and real topological linear spaces. Besides, they are applied both to characterize by scalarization approximate solutions of set optimization problems with set ordering and to generalize some recent conditions on robust solutions of optimization problems. For this aim, a new robustness concept in optimization under uncertainty is introduced which is interesting in itself.

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1 Introduction

In vector optimization and other optimization problems formulated in ordered sets, there exist several characterizations of solutions by scalarization that essentially work in the same way. The main aim of this work is to provide a unifying framework for these kinds of results, from which one can derive scalarization techniques in several optimization problems that involve preference relations directly by checking simple properties. In this way, the conditions required on the scalarization mapping to characterize solutions of the problem are clarified.

As far as we are aware, the starting point of this research line arises in Wierzbicki's seminal papers [28–30]. In these works, several so-called order preserving (monotonicity), order representing and order approximating properties were firstly introduced to characterize weak, efficient and proper solutions of a vector optimization problem through solutions of associated scalar optimization problems. Later, in the same setting, weaker order representing properties were formulated by Miglierina and Molho [22]. Recently, Khushboo and Lalitha [15] have redefined the above properties by considering an arbitrary ordering set instead of a cone.

All previous papers focus on real topological linear spaces. In [7], Gutiérrez et al. generalized the above properties to any quasi ordered set. As a result, the main scalarization methods in [22, 28–30] were extended to set optimization with set criteria. In this paper, we try to complete this research line by introducing and studying order preserving and order representing properties that work in any set endowed with a binary relation. Our results therefore generalize known characterizations by scalarization of minimal and nondominated points stated in different settings and problems.

The last part of the paper involves two applications of these results. In the first one, approximate solutions of set optimization problems quasi ordered via the lower set less relation are characterized by approximate solutions of scalar optimization problems. This characterization cannot be derived by the results in [7, 15, 22, 28–30] since the final space of the problem is neither a real topological linear space nor a quasi ordered space.

The second application deals with necessary and sufficient conditions for robust solutions of an optimization problem under uncertainty. First, new concepts of robust solutions are defined that model the uncertainty of the problem suitably, since the different alternatives are compared by functions that are compatible with nondomination criteria. Some examples are given to illustrate this statement and also the derived results.

The remainder of this work is structured as follows: In Section 2, we introduce the main notations and state some preliminary results. Section 3 is dedicated to the characterization of minimal and nondominated points of arbitrary binary relations through scalarization. We pay attention to the particular case of the real linear spaces and, as a consequence, we extend and clarify some recent results of [15]. We finally apply our derived results to set optimization and optimization problems under uncertainty in Sections 4 and 5. Section 6 concludes this paper with some final remarks.

2 Notations and preliminaries

Let $\mathcal{G}$ be a nonempty set and $\trianglelefteq$ be a binary relation on $\mathcal{G}$. By $\trianglelefteq$ one can define the following associated binary relations $\sim$ and $\triangleleft$ on $\mathcal{G}$ (see [1]):

$$y_1, y_2 \in \mathcal{G}, \quad \begin{array}{l} y_1 \sim y_2 :\iff y_1 \trianglelefteq y_2, y_2 \trianglelefteq y_1; \\ y_1 \triangleleft y_2 :\iff y_1 \trianglelefteq y_2, y_1 \not\sim y_2. \end{array}$$

Let $y \in \mathcal{G}$, $r \in \mathbb{R} \cup \{\pm\infty\}$ and $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$. The following sublevel sets are needed:

$$\begin{aligned} S(\mathcal{G}, y, \mathcal{R}) &:= \{z \in \mathcal{G} : z\mathcal{R}y\} \quad (\mathcal{R} \in \{\trianglelefteq, \sim, \triangleleft\}), \\ S_\varphi(\mathcal{G}, r, \mathcal{R}) &:= \{z \in \mathcal{G} : \varphi(z)\mathcal{R}r\} \quad (\mathcal{R} \in \{\leq, <\}). \end{aligned}$$

Observe that $S(\mathcal{G}, y, \triangleleft) \cap S(\mathcal{G}, y, \sim) = \emptyset$ and

$$S(\mathcal{G}, y, \trianglelefteq) = S(\mathcal{G}, y, \triangleleft) \cup S(\mathcal{G}, y, \sim). \tag{1}$$

Recall that $y_0 \in \mathcal{G}$ is a minimal point of $\mathcal{G}$, denoted by $y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq)$, if the following statement is true (see [1]):

$$y \in \mathcal{G}, \quad y \trianglelefteq y_0 \Rightarrow y_0 \trianglelefteq y.$$

By (1) it is clear that

$$\begin{aligned} y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq) &\iff S(\mathcal{G}, y_0, \triangleleft) = \emptyset \\ &\iff S(\mathcal{G}, y_0, \trianglelefteq) = S(\mathcal{G}, y_0, \sim). \end{aligned} \tag{2}$$

In a similar way, we say that $y_0 \in \mathcal{G}$ is a nondominated point of $\mathcal{G}$, denoted by $y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq)$, if the following implication holds:

$$y \in \mathcal{G}, \quad y \trianglelefteq y_0 \Rightarrow y = y_0.$$

These points were named strictly minimal elements by Gutiérrez et al. (see [7, Definition 3.1]) in the setting of quasi ordered sets. By (1) we have that

$$\begin{aligned} y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq) &\iff S(\mathcal{G}, y_0, \trianglelefteq) \subseteq \{y_0\} \\ &\iff S(\mathcal{G}, y_0, \triangleleft) = \emptyset \text{ and } S(\mathcal{G}, y_0, \sim) \subseteq \{y_0\}. \end{aligned} \tag{3}$$

In particular, it follows that $\mathrm{ND}(\mathcal{G}, \trianglelefteq) \subseteq \mathrm{Min}(\mathcal{G}, \trianglelefteq)$ (see (2) and (3)). Furthermore, if the binary relation $\trianglelefteq$ is antisymmetric, then $S(\mathcal{G}, y, \sim) \subseteq \{y\}$, for all $y \in \mathcal{G}$, and so $\mathrm{ND}(\mathcal{G}, \trianglelefteq) = \mathrm{Min}(\mathcal{G}, \trianglelefteq)$.

Throughout, the following abstract optimization problem is considered:

$$\begin{aligned} & f(x) \to \min\nolimits_{\precsim} \\ & \text{s.t. } x \in H, \end{aligned} \tag{P}$$

where $f : X \to Y$, X and Y are nonempty sets, $\emptyset \neq H \subseteq X$ and the image space Y is endowed with a binary relation $\precsim$ that in this optimization setting is called a preference relation. Let us underline that the spaces X, Y and the binary relation $\precsim$ are not required to fulfill any assumption.

In order to solve problem (P), the minimal and nondominated points of the image set $f(H)$ are required. To be precise, a feasible point $x_0 \in H$ is said to be a minimal (resp. nondominated) solution of problem (P), denoted by $x_0 \in \mathrm{Min}(f, H, \precsim)$ (resp. $x_0 \in \mathrm{ND}(f, H, \precsim)$), if $f(x_0) \in \mathrm{Min}(f(H), \precsim)$ (resp. $f(x_0) \in \mathrm{ND}(f(H), \precsim)$). Observe that

$$\begin{aligned} \mathrm{Min}(f, H, \precsim) &= f^{-1}(\mathrm{Min}(f(H), \precsim)) \cap H, \\ \mathrm{ND}(f, H, \precsim) &= f^{-1}(\mathrm{ND}(f(H), \precsim)) \cap H. \end{aligned}$$

Furthermore, the notion of strict solution of problem (P) is also considered. Recall that in scalar optimization (i.e., $Y = \mathbb{R}$), a solution is called strict whenever it is unique. This concept is extended to problem (P) as follows (see [13]): A point $x_0 \in H$ is said to be a strict solution of problem (P), and it is denoted by $x_0 \in \mathrm{Str}(f, H, \precsim)$, if the next condition is fulfilled:

$$x \in H, f(x) \precsim f(x_0) \Rightarrow x = x_0. \tag{4}$$

Lemma 1 *We have that*

$$\{x \in H : S(f(H), f(x), \precsim) = \emptyset\} \subseteq \mathrm{Str}(f, H, \precsim) \subseteq \mathrm{ND}(f, H, \precsim) \subseteq \mathrm{Min}(f, H, \precsim).$$

If $y \notin S(Y, y, \sim)$*, for all* $y \in Y$*, then*

$$\{x \in H : S(f(H), f(x), \precsim) = \emptyset\} = \mathrm{Str}(f, H, \precsim) = \mathrm{ND}(f, H, \precsim). \tag{5}$$

If $S(Y, y, \sim) = \emptyset$*, for all* $y \in Y$*, then*

$$\{x \in H : S(f(H), f(x), \precsim) = \emptyset\} = \mathrm{Str}(f, H, \precsim) = \mathrm{ND}(f, H, \precsim) = \mathrm{Min}(f, H, \precsim). \tag{6}$$

Proof Let us state only the inclusion $\mathrm{Str}(f, H, \precsim) \supseteq \mathrm{ND}(f, H, \precsim)$ in (5), since the other assertions follow easily from the definitions. Let $x_0 \in \mathrm{ND}(f, H, \precsim)$. As $y \notin S(Y, y, \sim)$, for all $y \in Y$, we claim that

$$f(x) \not\preceq f(x_0), \quad \forall x \in H. \tag{7}$$

Indeed, if there exists $x \in H$ such that $f(x) \preceq f(x_0)$, then $f(x) = f(x_0)$ as $x_0 \in \mathrm{ND}(f, H, \precsim)$. Therefore, $f(x_0) \preceq f(x_0)$ and so $f(x_0) \sim f(x_0)$, a contradiction. Clearly, statement (7) implies condition (4) and then $x_0 \in \mathrm{Str}(f, H, \preceq)$.

Assertion (5) motivates to study non-reflexive binary relations. Indeed, notice that $y \not\sim y$ if and only if $y \not\precsim y$. Then, the nondominated solutions of optimization problems whose preference relation is not reflexive coincide with the strict solutions of the problem, and it is well-known that this type of solutions fulfills good properties.

In real-world problems, the feasible set is usually defined by inequality and equality constraints. A mathematical formulation for this kind of feasibility is

$$H = \{x \in M : g(x) \precsim^c z_0\},$$

where $g : X \to Z$, Z is an arbitrary space, $\emptyset \neq M \subseteq X$, $\precsim^c$ is a binary relation on Z and $z_0 \in Z$. In this case, it is possible to study problem (P) through the following unconstrained problem:

$$\begin{aligned} &(f, g)(x) \to \min\nolimits_{\precsim^u} \\ &\text{s.t. } x \in M, \end{aligned} \tag{UP}$$

where $(f, g) : X \to Y \times Z$, $(f, g)(x) := (f(x), g(x))$, for all $x \in X$ and $\precsim^u$ is a binary relation on $Y \times Z$ defined by

$$(y_1, z_1), (y_2, z_2) \in Y \times Z, (y_1, z_1) \precsim^u (y_2, z_2) :\Longleftrightarrow\ y_1 \precsim y_2, z_1 \precsim^c z_2.$$

The next result is an easy consequence of the definitions.

Lemma 2 *Consider problems (P) and (UP). If a nonempty set $G \subseteq X$ satisfies the condition*

$$x_1 \in M, x_2 \in G, (f, g)(x_1) \precsim^u (f, g)(x_2) \Rightarrow x_1 \in G, \tag{8}$$

then we have that

$$\mathrm{Str}(f, G, \precsim) \cap M \subseteq \mathrm{Str}((f, g), M, \precsim^u).$$

Notice that statement (8) holds by considering $G = H$ provided that $\precsim^c$ is transitive. Moreover, observe that assertion $S(Y, y, \sim) = \emptyset$ for all $y \in Y$ implies $S(Y \times Z, (y, z), \sim^u) = \emptyset$, for all $y \in Y$, $z \in Z$. Then, by Lemma 1, by assuming that $S(Y, y, \sim) = \emptyset$, for all $y \in Y$, we have that

$$\begin{aligned} \{x \in M : S((f, g)(M), (f, g)(x), \precsim^u) = \emptyset\} &= \mathrm{Str}((f, g), M, \precsim^u) \\ &= \mathrm{ND}((f, g), M, \precsim^u) \qquad (9) \\ &= \mathrm{Min}((f, g), M, \precsim^u). \qquad (10) \end{aligned}$$

In the literature, scalarizing problem (P) usually means to solve it via an associated scalar optimization problem, whose objective function is the post-composition of f with a suitable scalar function $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$. It is clear that

$$\mathrm{Min}(\varphi \circ f, H, \leq) = f^{-1}(\mathrm{Min}(\varphi, f(H), \leq)) \cap H.$$

Thus, the scalarization of problem (P) can be directly studied on the image space Y, since the elements of $\mathrm{Min}(\varphi \circ f, H, \leq) \subseteq X$ can be obtained via the elements of $\mathrm{Min}(\varphi, f(H), \leq) \subseteq Y$ and the set-valued mapping $f^{-1}(\cdot) \cap H : Y \rightrightarrows X$.

For each mapping $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ and $\varepsilon > 0$, we denote

$$\mathrm{argmin}_{\mathcal{G}}\varphi := \{y_0 \in \mathcal{G} : \forall y \in \mathcal{G}, \varphi(y_0) \leq \varphi(y)\},$$
$$\varepsilon\text{-}\mathrm{argmin}^{<}_{\mathcal{G}}\varphi := \{y_0 \in \mathcal{G} : \forall y \in \mathcal{G}, \varphi(y_0) - \varepsilon < \varphi(y)\}.$$

Given two nonempty subsets A_1, A_2 of a real linear space Y, $y \in Y$, $\emptyset \neq T \subseteq \mathbb{R}$ and $\alpha \in \mathbb{R}$, we denote

$$A_1 + A_2 := \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\},\ \alpha A_1 := \{\alpha a_1 : a_1 \in A_1\},$$
$$y + A_1 := \{y\} + A_1, \qquad Ty := \{ty : t \in T\}.$$

3 Characterization of minimal and nondominated points through scalarization

Next, we define two properties from which one can obtain necessary conditions for minimal and nondominated points via scalarization.

Definition 1 ($\trianglelefteq$-representing property) A mapping $\varphi : \mathcal{G} \to \mathbb{R}\cup\{\pm\infty\}$ is said to be $\trianglelefteq$-representing at $y \in \mathcal{G}$ if one of the following equivalent statements is fulfilled:

(a) $\forall z \in \mathcal{G}\backslash S(\mathcal{G}, y, \lhd)$: $\varphi(z) \geq \varphi(y)$.
(b) $S_\varphi(\mathcal{G}, \varphi(y), <) \subseteq S(\mathcal{G}, y, \lhd)$.
(c) $z \in \mathcal{G}$, $\varphi(z) < \varphi(y) \Rightarrow z \lhd y$.

Remark 1 (i) The notion of $\trianglelefteq$-representing mapping was introduced in [7, Definition 3.2] in the setting of a quasi ordered set. It generalizes the order representation property due to Wierzbicki [30, statement (30)], that was defined in a finite dimensional linear space ordered by components. This property was extended later to ordered linear spaces by Miglierina and Molho [22, assertion (R2)].

(ii) Notice that statements (a)-(c) of Definition 1 do not change if one considers $\mathcal{G}\backslash\{y\}$ instead of $\mathcal{G}$.

Example 1 (Linear spaces equipped with a binary relation) (i) Assume that $\mathcal{G}$ is a nonempty subset of a real linear space Y (observe that Y is not equipped with any topology), and consider the following binary relation $\trianglelefteq_C$ on Y defined by an arbitrary nonempty domination set $C \subseteq Y$:

$$y_1, y_2 \in Y, \quad y_1 \trianglelefteq_C y_2 :\Longleftrightarrow y_1 - y_2 \in -C. \tag{11}$$

Recall that the algebraic interior and the vectorial closure in the direction $q \in Y\backslash\{0\}$ of a set $Q \subseteq Y$ are, respectively, the next sets (see [10, 12, 24]):

$$\mathrm{core}\, Q := \{y \in Y : \forall v \in Y, \exists\lambda > 0 \text{ s.t. } y + [0, \lambda]v \subseteq Q\},$$
$$\mathrm{vcl}_q\, Q := \{y \in Y : \forall\lambda > 0\, \exists\lambda' \in [0, \lambda] \text{ s.t. } y + \lambda' q \in Q\}.$$

Let $e \in Y\backslash\{0\}$, $y_0 \in Y$ and $\varphi^C_{e,y_0} : Y \to \mathbb{R} \cup \{\pm\infty\}$ be the so-called nonconvex separation functional (see [10] and the references therein):

$$\varphi^C_{e,y_0}(y) := \begin{cases} +\infty & \text{if } y \notin y_0 + \mathbb{R}e - C, \\ \inf\{t \in \mathbb{R} : y \in y_0 + te - C\} & \text{otherwise.} \end{cases}$$

Define $\Psi^C_{e,y_0} : Y \to \mathbb{R} \cup \{\pm\infty\}$ as $\Psi^C_{e,y_0}(y) = \varphi^C_{e,y_0}(y)$ for all $y \in Y$, $y \neq y_0$, and $\Psi^C_{e,y_0}(y_0) = 0$. It follows that

$$S_{\Psi^C_{e,y_0}}(\mathcal{G}, \Psi^C_{e,y_0}(y_0), <) = (y_0 + (-\infty, 0)e - \operatorname{vcl}_e C) \cap (\mathcal{G}\backslash\{y_0\}),$$
$$S(\mathcal{G}, y_0, \lhd_C) = (y_0 - C\backslash(C \cap (-C))) \cap (\mathcal{G}\backslash\{y_0\})$$

and so the scalarization mapping Ψ^C_{e,y_0} is $\unlhd_C$-representing at y_0 provided that

$$(0, +\infty)e + \operatorname{vcl}_e C \subseteq C\backslash(C \cap (-C)). \tag{12}$$

For example, property (12) is true if C is an improvement set with respect to a convex cone $D \subset Y$ (i.e., $C + D = C$ and $0 \notin C$, see [3,8,31] and the references therein), $e \in D$ and C is pointed (i.e., $C \cap (-C) = \emptyset$). This particular case follows by applying [9, Lemma 2.3(c)].

Notice that the above condition $C \cap (-C) = \emptyset$ implies $S(\mathcal{G}, y, \sim_C) = \emptyset$, for all $y \in Y$, and then the equalities in (6) hold. Moreover, a simple set satisfying all of them is $C = \operatorname{core} D$ whenever D is an algebraic solid (i.e., $\operatorname{core} D \neq \emptyset$) proper convex cone.

On the contrary, let Y' be the algebraic dual space of Y and consider the strict positive polar cone generated by C, i.e.,

$$C^{\#} := \{\ell \in Y' : \forall\ c \in C\backslash\{0\}, \ell(c) > 0\}.$$

It follows that in general functionals in $C^{\#}$ are not $\unlhd_C$-representing at any $y_0 \in Y$. For instance, if $Y = \mathcal{G} = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ (the nonnegative orthant of $\mathbb{R}^2$) and $\ell = (1/2, 1) \in C^{\#}$, it follows that $\ell(1,0) < \ell(0,1)$, but $(1,0) \not\lhd_C (0,1)$.

(*ii*) Recently, Khushboo and Lalitha [15, Definition 3.1(iii)] defined a kind of order representing property in the setting of a real Hausdorff topological linear space Y equipped with a preference relation given by an arbitrary nonempty domination set $S \subset Y$ as follows:

$$y_1, y_2 \in Y, y_1 \leq_S y_2 \iff y_2 - y_1 \in Y\backslash S. \tag{13}$$

To be precise, the authors say that a mapping $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ is order preserving at a point $y_0 \in Y$ if

$$y \in Y, y_0 \leq_S y \Rightarrow \varphi(y_0) \leq \varphi(y). \tag{14}$$

It is clear that this condition is equivalent to this one:

$$y \in Y, \varphi(y) < \varphi(y_0) \Rightarrow y \unlhd_{-S} y_0. \tag{15}$$

Therefore, property (14) is a kind of order representing property weaker than the $\unlhd_{-S}$-representing property and equivalent to it whenever $S \cap (-S) \subseteq \{0\}$.

Indeed, it is obvious that a point $y \in Y$ such that $\varphi(y) < \varphi(y_0)$ is different from y_0. In addition, for each $y \in Y \backslash \{y_0\}$, condition $S \cap (-S) \subseteq \{0\}$ implies that $y \trianglelefteq_{-S} y_0$ if and only if $y \triangleleft_{-S} y_0$. Thus, property (14), which is equivalent to statement (15), can be rewritten as follows:

$$y \in Y, \varphi(y) < \varphi(y_0) \Rightarrow y \triangleleft_{-S} y_0,$$

that coincides with the order representing property of the binary relation $\trianglelefteq_{-S}$.

Proposition 1 (Necessary condition) *Let $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be $\trianglelefteq$-representing at $y_0 \in \mathcal{G}$. Then:*

$$y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq) \Rightarrow y_0 \in \mathrm{argmin}_{\mathcal{G}} \varphi. \tag{16}$$

Proof Suppose that $y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq)$ and $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ is $\trianglelefteq$-representing at y_0. Then, by (2) we have that $S(\mathcal{G}, y_0, \triangleleft) = \emptyset$ and then the result follows by applying statement (a) of Definition 1.

With respect to the application of the previous result, the following two particular cases must be underlined. First, if $\varphi(y_0) = -\infty$, then φ is trivially $\trianglelefteq$-representing at y_0 for any relation $\trianglelefteq$ and the necessary condition (16) is useless. Secondly, if $\varphi(y_0) = +\infty$ and φ is $\trianglelefteq$-representing at y_0, then $\varphi(y) = +\infty$ for all $y \in \mathcal{G} \backslash S(\mathcal{G}, y_0, \triangleleft)$. Therefore, in this case, the necessary condition (16) actually reduces to know if φ is proper.

Corollary 1 (Necessary condition) *Let $y_0 \in \mathcal{G}$ and $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

$$y \in \mathcal{G}, \varphi(y) < \varphi(y_0) \Rightarrow y \trianglelefteq y_0. \tag{17}$$

Then,

$$y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq) \Rightarrow y_0 \in \mathrm{argmin}_{\mathcal{G}} \varphi.$$

Proof Suppose that $y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq)$. By (3) we see that $S(\mathcal{G}, y_0, \sim) \subseteq \{y_0\}$ and then assertion (17) coincides with the $\trianglelefteq$-representing property of function φ at y_0. Thus, the result is a direct consequence of Proposition 1 as $\mathrm{ND}(\mathcal{G}, \trianglelefteq) \subseteq \mathrm{Min}(\mathcal{G}, \trianglelefteq)$.

Remark 2 Proposition 1 was stated in [7] by considering a quasi ordered set $\mathcal{G}$. In addition, Corollary 1 encompasses [15, Theorem 3.1(i)] via the following data: a real Hausdorff topological linear space Y, a nonempty set $A \subset Y$, a point $\bar{a} \in A$, a set $S \subset Y$ and by defining $\mathcal{G} := A$, $\trianglelefteq := \trianglelefteq_{-S}$ (see (11)) and $\varphi(y) := \phi(y)$, for all $y \in Y$, where $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfies property (14).

Notice by the proof of Corollary 1 that (17) is equivalent to the $\trianglelefteq$-representing property of mapping φ at y_0 whenever y_0 is a nondominated point of $(\mathcal{G}, \trianglelefteq)$.

Analogously, Proposition 1 encompasses [22, Proposition 5.2] by considering a real topological linear space Y, a nonempty set $A \subset Y$, a point $y_0 \in A$, a (topological) solid proper ($K \neq Y$) convex cone $K \subset Y$ and by defining $\mathcal{G} := -A$, $\trianglelefteq := \trianglelefteq_{\mathrm{int}\, K}$ (see (11)) and $\varphi(y) := -s(-y, y_0)$, for all $y \in Y$, where $s : Y \times Y \to \mathbb{R}$ satisfies the following properties: $s(y_0, y_0) = 0$ and

$$\{y \in Y : s(y, y_0) > 0\} \subseteq y_0 + \mathrm{int}\, K.$$

An extension of [22, Proposition 5.2] is possible as a simple consequence of Proposition 1. Next, we state this result, which illustrates the usefulness of Proposition 1.

Corollary 2 *Let Y be a real linear space, $\mathcal{G} \subseteq Y$ be a nonempty set, $y_0 \in \mathcal{G}$, $\bar{y} \in Y$, $\emptyset \neq C \subset Y$ be a domination set and $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that $\phi(\bar{y}) = 0$ and*

$$\{y \in Y : \phi(y) < 0\} \subseteq \bar{y} - (C \backslash (-C)).$$

Then:

$$y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq_C) \Rightarrow y_0 \in \mathrm{argmin}_{\mathcal{G}} \phi(\cdot - y_0 + \bar{y}).$$

Proof The result follows by applying Proposition 1 to the function $\varphi(y) := \phi(y - y_0 + \bar{y})$, for all $y \in Y$.

Next, as an obvious consequence of Proposition 1, a necessary condition for minimal solutions of problem (P) is stated by scalarization.

Corollary 3 *Consider problem (P), a point $x_0 \in H$ and a $\precsim$-representing mapping $\varphi : f(H) \to \mathbb{R} \cup \{\pm\infty\}$ at $f(x_0)$. Then:*

$$x_0 \in \mathrm{Min}(f, H, \precsim) \Rightarrow x_0 \in \mathrm{argmin}_H (\varphi \circ f).$$

Proof Let $x_0 \in \mathrm{Min}(f, H, \precsim)$. By the definition of minimal solution of problem (P) we see that $f(x_0) \in \mathrm{Min}(f(H), \precsim)$. Then, the result follows by applying Proposition 1.

Definition 2 (Strictly $\trianglelefteq$-representing property) A mapping $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be strictly $\trianglelefteq$-representing at $y \in \mathcal{G}$ if one of the following equivalent statements is fulfilled:

(*a*) $\forall z \in (\mathcal{G} \backslash \{y\}) \backslash S(\mathcal{G}, y, \trianglelefteq)$: $\varphi(z) > \varphi(y)$.
(*b*) $S_\varphi(\mathcal{G} \backslash \{y\}, \varphi(y), \leq) \subseteq S(\mathcal{G} \backslash \{y\}, y, \trianglelefteq)$.
(*c*) $z \in \mathcal{G} \backslash \{y\}$, $\varphi(z) \leq \varphi(y) \Rightarrow z \trianglelefteq y$.

Remark 3 The concept of strictly $\trianglelefteq$-representing mapping at a point was firstly defined in [7, Definition 3.3] for a quasi ordered set. In Definition 2, we extend this notion to any binary relation. It is worth underlining that a reformulation of this concept for real Hausdorff topological linear spaces has been introduced by Khushboo and Lalitha [15, Definition 3.1(iv)] (see Example 2(*ii*) below).

Notice that two stronger previous notions were defined by Wierzbicki [29, equation (9)] and Miglierina and Molho [22, equation (R1)] in the setting of a real ordered topological linear space whose domination set C is a convex cone (see (11)).

Example 2 (Linear spaces equipped with a binary relation, see Example 1) (i) By [10, Theorem 4(e)] we have that

$$S_{\Psi^C_{e,y_0}}(\mathcal{G}\backslash\{y_0\}, \Psi^C_{e,y_0}(y_0), \leq) = (y_0 + (-\infty, 0]e - \operatorname{vcl}_e C) \cap (\mathcal{G}\backslash\{y_0\}), \quad (18)$$

$$S(\mathcal{G}\backslash\{y_0\}, y_0, \trianglelefteq_C) = (y_0 - C) \cap (\mathcal{G}\backslash\{y_0\}) \quad (19)$$

and so the scalarization mapping Ψ^C_{e,y_0} is strictly $\trianglelefteq_C$-representing at y_0 provided that

$$[0, +\infty)e + \operatorname{vcl}_e C \subseteq C. \quad (20)$$

This condition is fulfilled, for instance, if C is free disposal with respect to a convex cone $D \subset Y$, $e \in D\backslash\{0\}$ and C is algebraic closed along the direction e (i.e., $\operatorname{vcl}_e C = C$).

Now let $C^+ \subset Y'$ be the (positive) polar cone generated by C, i.e.,

$$C^+ := \{\ell \in Y' : \forall\, c \in C, \ell(c) \geq 0\}.$$

In general, the functionals in C^+ are not strictly $\trianglelefteq_C$-representing at any $y \in \mathcal{G}$. For example, if $Y = \mathcal{G} = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $\ell = (1, 1)$, $y_1 = (1, 0)$ and $y_2 = (0, 1)$, then it is clear that $\ell(y_1) \leq \ell(y_2)$, but $y_1 \not\trianglelefteq_C y_2$.

(ii) The above quoted Khushboo and Lalitha's reformulation (see Remark 3) is as follows: A mapping $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfies the strict order preserving property at a point $y_0 \in Y$ with respect to the ordering $\leq_S$ (see (13)) if:

$$y \in Y\backslash\{y_0\}, y_0 \leq_S y \Rightarrow \varphi(y_0) < \varphi(y). \quad (21)$$

This condition coincides with the following one:

$$y \in Y\backslash\{y_0\}, \varphi(y) \leq \varphi(y_0) \Rightarrow y \trianglelefteq_{-S} y_0.$$

As a result, Khushboo and Lalitha's strict order preserving property is a particular case of the concept of strictly $\trianglelefteq$-representing mapping.

Moreover, in [15, Theorem 3.5(ii)], the authors proved that mapping φ^{-S}_{e,y_0} satisfies this property provided that the following assumptions are fulfilled: S is closed, $\varphi^{-S}_{e,0}(0) = 0$ and $S - (0, +\infty)e \subseteq S$. The first condition implies $\operatorname{vcl}_e(-S) = -S$ and so assertion (20) with $-S$ instead of C is satisfied due to the third assumption. Notice that $\varphi^{-S}_{e,y_0} = \Psi^{-S}_{e,y_0}$ since $\varphi^{-S}_{e,0}(0) = 0$.

However, (20) is a weaker condition to check if Ψ^C_{e,y_0} is strictly $\trianglelefteq_C$-representing. Roughly speaking, one actually needs to find a direction $e \in Y\backslash\{0\}$ such that C is both vectorially closed in that direction and free disposal with respect to the ray $[0, +\infty)e$. In other words, it is possible to fulfill (20) through a non closed set C satisfying $\varphi^C_{e,0}(0) \neq 0$.

For instance, let $Y = \mathbb{R}^2$, $\mathcal{G} = (-1, -1) + \mathbb{R}^2_+$, $C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0, y_2 \geq 0, y_1 + y_2 \geq 1\}$, $S = -C$ and $e = (0, 1)$. It is clear that $C = \operatorname{vcl}_e C$ and $C + [0, +\infty)e = C$. Thus, (20) holds and function $\Psi^C_{e,0}$ is strictly $\trianglelefteq_C$-representing at $(0, 0)$.

Moreover, for each $(y_1, y_2) \in Y$,

$$\varphi^C_{e,0}(y_1, y_2) = \begin{cases} y_2 & \text{if } y_1 < -1, \\ y_1 + y_2 + 1 & \text{if } y_1 \in [-1, 0), \\ +\infty & \text{if } y_1 \geq 0. \end{cases} \tag{22}$$

Then, [15, Theorem 3.5(ii)] cannot be applied to check if function $\varphi^C_{e,0}$ is strictly $\trianglelefteq_C$-representing at $(0,0)$, since C is not closed and $\varphi^C_{e,0}(0,0) = +\infty$.

Next, we denote $\mathrm{NE}(y_0) := (\mathcal{G} \backslash S(\mathcal{G}, y_0, \sim)) \cup \{y_0\}$.

Proposition 2 (Necessary condition) *Let $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be strictly $\trianglelefteq$-representing at $y_0 \in \mathcal{G}$. Then:*

$$y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq) \Rightarrow \mathrm{argmin}_{\mathrm{NE}(y_0)} \varphi = \{y_0\}, \tag{23}$$

$$y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq) \Rightarrow \mathrm{argmin}_{\mathcal{G}} \varphi = \{y_0\}. \tag{24}$$

If, additionally, φ is constant in $S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$, then

$$y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq) \Rightarrow \mathrm{argmin}_{\mathcal{G}} \varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\}. \tag{25}$$

Proof Let us prove statement (23), since (24) is obvious by Definition 2(a) and (25) is a direct consequence of (23) when φ is constant in $S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$.

Consider $y \in \mathrm{NE}(y_0) \backslash \{y_0\}$ and suppose that $\varphi(y) \leq \varphi(y_0)$. Then, $y \in (\mathcal{G} \backslash \{y_0\}) \backslash S(\mathcal{G}, y_0, \sim)$ and $y \trianglelefteq y_0$ by statement (c) of Definition 2, and as $y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq)$ we deduce that $y \sim y_0$, which is a contradiction. Thus, statement (23) is proved, and the proof finishes.

Let us observe that if $y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq)$ then $S(\mathcal{G}, y_0, \sim) \cup \{y_0\} = \{y_0\}$ and so φ is constant in $S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$. Moreover, if $\varphi(y_0) = +\infty$ and φ is strictly $\trianglelefteq$-representing at y_0, then $\mathcal{G} \backslash \{y_0\} \subset S(\mathcal{G}, y_0, \trianglelefteq)$, i.e., y_0 is an upper order bound of $\mathcal{G}$. In particular, we have that $y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq)$ iff $\mathcal{G} = S(\mathcal{G}, y_0 \sim) \cup \{y_0\}$ and $y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq)$ iff $\mathcal{G} = \{y_0\}$. Analogously, if $\varphi(y_0) = -\infty$ and φ is strictly $\trianglelefteq$-representing at y_0, then condition (23) (resp. (24)) reduces to analyze if there exists a point $y \in \mathrm{NE}(y_0)$ (resp. $y \in \mathcal{G}$), different from y_0, such that $\varphi(y) = -\infty$.

Remark 4 Statement (24) of Proposition 2 reduces to [7, Proposition 3.6] when the relation $\trianglelefteq$ is a quasi order. Analogously, it recovers [15, Theorem 3.1(ii)] by considering the following data (see Example 1 and Remark 3): a real topological linear space Y, a nonempty set $A \subset Y$, a point $y_0 \in A$, a set $S \subset Y$, $\mathcal{G} := A$, $\trianglelefteq := \trianglelefteq_{-S}$ (see (11)) and $\varphi(y) := \phi(y)$, for all $y \in Y$, where $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfies property (21).

On the other hand, Proposition 2 encompasses [22, Proposition 5.4] by considering a real topological linear space Y, a nonempty set $A \subset Y$, a point $y_0 \in A$, a pointed convex cone $K \subset Y$ and by defining $\mathcal{G} := -A$, $\trianglelefteq := \trianglelefteq_K$ (see (11)) and $\varphi(y) := -s(-y, y_0)$, for all $y \in Y$, where $s : Y \times Y \to \mathbb{R}$ satisfies the following properties: $s(y_0, y_0) = 0$ and

$$\{y \in Y : s(y, y_0) \geq 0\} \subseteq y_0 + K.$$

Example 3 Consider the problem introduced at the end of Example 2(*ii*). As the set C is pointed, we have that assertions (23) and (24) coincide. On the other hand, notice by Example 2(*ii*) that $\Psi^C_{e,0}$ is strictly $\trianglelefteq_C$-representing at point $(0,0)$ and $\Psi^C_{e,0}(y_1,y_2) = \varphi^C_{e,0}(y_1,y_2)$, for all $(y_1,y_2) \in \mathbb{R}^2\backslash\{(0,0)\}$ (see (22)). In particular, $\Psi^C_{e,0}(-1,y_2) \leq 0$, for all $y_2 \in [-1,0]$. Thus, $\operatorname{argmin}_{\mathcal{G}}\Psi^C_{e,0} \neq \{(0,0)\}$ and so $(0,0) \notin \mathrm{ND}(\mathcal{G}, \trianglelefteq_C)$.

Next, we extend [22, Proposition 5.4] to an arbitrary domination set and a scalarization mapping that vanishes in an arbitrary point $\bar{y}$ different from the nominal point y_0.

Corollary 4 *Let Y be a real linear space, $\mathcal{G} \subseteq Y$ be a nonempty set, $y_0 \in \mathcal{G}$, $\bar{y} \in Y$, $C \subset Y$ be a domination set and $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function satisfying $\phi(\bar{y}) = 0$ and*

$$\{y \in Y : \phi(y) \leq 0\} \subseteq \bar{y} - C.$$

Then:

$$y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq_C) \Rightarrow \operatorname{argmin}_{\mathcal{G}}\phi(\cdot - y_0 + \bar{y}) = \{y_0\}.$$

Proof The result follows by applying Proposition 2 to the function $\varphi(y) := \phi(y - y_0 + \bar{y})$, for all $y \in Y$.

The next result gives by scalarization a necessary condition for minimal and nondominated solutions of problem (P). We denote

$$\mathrm{NE}(H,f,x_0) := \{x \in H : f(x) \not\sim f(x_0), f(x) \neq f(x_0)\} \cup \{x_0\},$$
$$\mathrm{D}(H,f,x_0) := \{x \in H : f(x) \neq f(x_0)\} \cup \{x_0\}.$$

Corollary 5 *Consider problem (P), a point $x_0 \in H$ and a strictly $\precsim$-representing mapping $\varphi : f(H) \to \mathbb{R} \cup \{\pm\infty\}$ at $f(x_0)$. Then:*

$$x_0 \in \mathrm{Min}(f,H,\precsim) \Rightarrow \operatorname{argmin}_{\mathrm{NE}(H,f,x_0)}(\varphi \circ f) = \{x_0\},$$
$$x_0 \in \mathrm{ND}(f,H,\precsim) \Rightarrow \operatorname{argmin}_{\mathrm{D}(H,f,x_0)}(\varphi \circ f) = \{x_0\}.$$

If, additionally, φ is constant in $S(f(H), f(x_0), \sim) \cup \{f(x_0)\}$, then

$$x_0 \in \mathrm{Min}(f,H,\precsim) \Rightarrow \operatorname{argmin}_H(\varphi \circ f) = \{x \in H : f(x) \sim f(x_0)\} \cup \{x_0\}.$$

Proof By definition, $x_0 \in \mathrm{Min}(f,H,\precsim)$ (resp. $x_0 \in \mathrm{ND}(f,H,\precsim)$) if $x_0 \in H$ and $f(x_0) \in \mathrm{Min}(f(H),\precsim)$ (resp. $f(x_0) \in \mathrm{ND}(f(H),\precsim)$). Then, the result is a direct consequence of Proposition 2.

In the following we introduce two properties to derive sufficient conditions for minimal and nondominated points through scalarization.

Definition 3 ($\trianglelefteq$-preserving property) A mapping $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be $\trianglelefteq$-preserving at $y \in \mathcal{G}$ if one of the following equivalent statements is fulfilled:

(a) $\forall z \in \mathcal{G}\backslash S_\varphi(\mathcal{G}, \varphi(y), \leq)$: $z \not\trianglelefteq y$.
(b) $S(\mathcal{G}, y, \trianglelefteq) \subseteq S_\varphi(\mathcal{G}, \varphi(y), \leq)$.
(c) $z \in \mathcal{G}$, $z \trianglelefteq y \Rightarrow \varphi(z) \leq \varphi(y)$.

Remark 5 The $\trianglelefteq$-preserving property is a pointwise concept of monotonicity that was introduced in [7, Definition 3.7] in the setting of a quasi ordered set. However, notice that the same property was defined by Wierzbicki [29, equation (7)] and Miglierina and Molho [22, statement (P1)] in real topological linear spaces ordered by a convex cone. In the same linear framework, Khushboo and Lalitha [18, Definition 3.1(i)] considered the following property for a mapping $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ (see (13)): φ is order representing at $y_0 \in Y$ if

$$y \in Y, y_0 \not\leq_S y \Rightarrow \varphi(y_0) \geq \varphi(y).$$

This condition is equivalent to the following one:

$$y \in Y, y \trianglelefteq_{-S} y_0 \Rightarrow \varphi(y) \leq \varphi(y_0)$$

and so Khushboo and Lalitha's order representing property extends the order preserving properties defined either by Wierzbicki or Miglierina and Molho to an arbitrary domination set.

Example 4 (Linear spaces equipped with a binary relation, see Examples 1 and 2) Returning to Example 1, notice by (18) and (19) that mapping Ψ^C_{e,y_0} is $\trianglelefteq_C$-preserving at y_0. This assertion was stated in [15, Theorem 3.4(ii)] for φ^C_{e,y_0} whenever $\varphi^C_{e,0}(0) = 0$ and $C+[0, +\infty)e = C$. Notice that Ψ^C_{e,y_0} is $\trianglelefteq_C$-preserving at y_0 for any ordering set C. Moreover, if $\varphi^C_{e,0}(0) = 0$ then $\varphi^C_{e,y_0} = \Psi^C_{e,y_0}$ and then condition $C + [0, +\infty)e = C$ in [15, Theorem 3.4(ii)] is superfluous.

On the other hand, it is clear that each mapping $\ell \in C^+$ is $\trianglelefteq_C$-preserving at any $y \in \mathcal{G}$.

Proposition 3 (Sufficient condition) *Let $\varphi : \mathcal{G} \to \mathbb{R}\cup\{\pm\infty\}$ be $\trianglelefteq$-preserving at $y_0 \in \mathcal{G}$. Then:*

$$\{y_0\} = \operatorname{argmin}_{\mathcal{G}}\varphi \Rightarrow y_0 \in \operatorname{ND}(\mathcal{G}, \trianglelefteq), \tag{26}$$

$$\{y_0\} = \operatorname{argmin}_{\operatorname{NE}(y_0)}\varphi \Rightarrow y_0 \in \operatorname{Min}(\mathcal{G}, \trianglelefteq), \tag{27}$$

$$y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi,\ \operatorname{argmin}_{\mathcal{G}}\varphi\backslash\{y_0\} \subseteq S(\mathcal{G}, y_0, \sim) \Rightarrow y_0 \in \operatorname{Min}(\mathcal{G}, \trianglelefteq). \tag{28}$$

Proof Let us prove statement (27), since (26) is clear by Definition 3(c) and (28) follows by (27) since

$$y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi,\ \operatorname{argmin}_{\mathcal{G}}\varphi\backslash\{y_0\} \subseteq S(\mathcal{G}, y_0, \sim) \Rightarrow \{y_0\} = \operatorname{argmin}_{\operatorname{NE}(y_0)}\varphi. \tag{29}$$

Assume that $\{y_0\} = \operatorname{argmin}_{\operatorname{NE}(y_0)}\varphi$ and let $y \in \mathcal{G}$ be such that $y \trianglelefteq y_0$. If $y = y_0$ then it is obvious that $y \sim y_0$. Otherwise, by part (c) of Definition 3 we obtain that $\varphi(y) \leq \varphi(y_0)$ and so $y \in S(\mathcal{G}, y_0, \sim)$, since $y \neq y_0$ and $\{y_0\} = \operatorname{argmin}_{\operatorname{NE}(y_0)}\varphi$. Therefore, implication (27) is true.

Let us clarify a particular case of the previous result. If $\varphi(y_0) = +\infty$ then φ is $\trianglelefteq$-preserving at y_0 for any binary relation $\trianglelefteq$. However, condition $\{y_0\} = \mathrm{argmin}_{\mathcal{G}}\varphi$ (resp. $\{y_0\} = \mathrm{argmin}_{\mathrm{NE}(y_0)}\varphi$) holds only if $\mathcal{G} = \{y_0\}$ (resp. $\mathcal{G} = S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$). Thus, conditions (26)-(28) are useless whenever $\varphi(y_0) = +\infty$.

Remark 6 In statement (29) we have observed that the left hand side of (28) implies the left hand side of (27). The reciprocal implication is not true in general, as we prove in the next example. However, it is not hard to check that the next implication is satisfied:

$$\{y_0\} = \mathrm{argmin}_{\mathrm{NE}(y_0)}\varphi \Rightarrow \mathrm{argmin}_{\mathcal{G}}\varphi \subseteq S(\mathcal{G}, y_0, \sim) \cup \{y_0\}. \tag{30}$$

Furthermore, it is obvious that

$$\mathrm{argmin}_{\mathcal{G}}\varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\} \Rightarrow y_0 \in \mathrm{argmin}_{\mathcal{G}}\varphi,\ \mathrm{argmin}_{\mathcal{G}}\varphi \backslash \{y_0\} \subseteq S(\mathcal{G}, y_0, \sim)$$

and if additionally φ is $\trianglelefteq$-preserving at y_0, then

$$\mathrm{argmin}_{\mathcal{G}}\varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\} \iff y_0 \in \mathrm{argmin}_{\mathcal{G}}\varphi,\ \mathrm{argmin}_{\mathcal{G}}\varphi \backslash \{y_0\} \subseteq S(\mathcal{G}, y_0, \sim).$$

Thus, the assumptions of (28) can be replaced by the left hand side of the previous equivalence.

Example 5 Consider $\mathcal{G} = A_1 \cup A_2$, where $A_1 = \{(0, y) \in \mathbb{R}^2 : y \leq 0\}$ and $A_2 = \{(x, x) \in \mathbb{R}^2 : x > 0\}$, $y_0 = (0, 0)$ and the following binary relation:

$$(y_1, y_2), (z_1, z_2) \in \mathcal{G}, \quad (y_1, y_2) \trianglelefteq (z_1, z_2) \iff y_1 = z_1.$$

Then, the relation $\trianglelefteq$ is reflexive, transitive and

$$S(\mathcal{G}, y_0, \sim) = S(\mathcal{G}, y_0, \triangleleft) = A_1.$$

The function $\varphi : \mathcal{G} \to \mathbb{R}$, $\varphi(y_1, y_2) = y_2$, is $\trianglelefteq$-preserving at y_0. Moreover, it is clear that $\mathrm{NE}(y_0) = A_2 \cup \{(0, 0)\}$. Therefore,

$$\{y_0\} = \mathrm{argmin}_{\mathrm{NE}(y_0)}\varphi$$

and the left hand side of condition (27) is fulfilled. However, $\mathrm{argmin}_{\mathcal{G}}\varphi = \emptyset$ and so the left hand side of condition (28) is not true.

Remark 7 Statement (27) reduces to [7, Proposition 3.11] by assuming that the binary relation $\trianglelefteq$ is a quasi order. Besides, [22, Proposition 5.3] (resp. [15, Theorem 3.1(iv)]) results by (26), Remark 5 and the following data: Y is a real linear space, $K \subset Y$ is a convex cone (resp. $S \subset Y$ is an arbitrary domination set), $\trianglelefteq = \trianglelefteq_{-K}$ (resp. $\trianglelefteq = \trianglelefteq_{-S}$) and $\varphi = -s$ (resp. $\varphi = -\phi$), where the mapping $s : Y \to \mathbb{R}$ (resp. $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$) is $\trianglelefteq_K$-preserving at any point $y \in Y$ (resp. $\trianglelefteq_S$-preserving at the nominal point $y_0 \in Y$).

Next, a sufficient condition via scalarization for strict solutions of problem (P) is obtained by Proposition 3.

Corollary 6 *Consider problem (P), $x_0 \in H$ and let $\varphi : f(H) \to \mathbb{R} \cup \{\pm\infty\}$ be a $\precsim$-preserving mapping at $f(x_0)$. Then:*

$$\{x_0\} = \operatorname{argmin}_H(\varphi \circ f) \Rightarrow x_0 \in \operatorname{Str}(f, H, \precsim).$$

Proof Condition $\{x_0\} = \operatorname{argmin}_H(\varphi \circ f)$ implies that $f(x) \neq f(x_0)$, for all $x \in H\backslash\{x_0\}$ and $\{f(x_0)\} = \operatorname{argmin}_{f(H)}\varphi$. Then, by (26) we see that $f(x_0) \in \operatorname{ND}(f(H), \precsim)$ and this implies that $x_0 \in \operatorname{Str}(f, H, \precsim)$. Indeed, let $x \in H$ such that $f(x) \precsim f(x_0)$. As $f(x_0) \in \operatorname{ND}(f(H), \precsim)$ we deduce that $f(x) = f(x_0)$ and so we have $x = x_0$, since $f(x) \neq f(x_0)$, for all $x \in H\backslash\{x_0\}$.

Definition 4 (Strictly $\trianglelefteq$-preserving property) A mapping $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be strictly $\trianglelefteq$-preserving at $y \in \mathcal{G}$ if one of the following equivalent statements is fulfilled:

(*a*) $\forall z \in \mathcal{G}\backslash S_\varphi(\mathcal{G}, \varphi(y), <)$: $z \ntriangleleft y$.
(*b*) $S(\mathcal{G}, y, \triangleleft) \subseteq S_\varphi(\mathcal{G}, \varphi(y), <)$.
(*c*) $z \in \mathcal{G},\ z \triangleleft y \Rightarrow \varphi(z) < \varphi(y)$.

Remark 8 The strictly $\trianglelefteq$-preserving property was introduced in [7, Definition 3.8] in the setting of a quasi ordered set.

Notice that a non pointwise version of this concept was previously defined by Wierzbicky [29, equation (8)] in a real linear space Y ordered by a convex cone. This notion was extended to a general domination set $S \subset Y$ by Khushboo and Lalitha [18, Definition 3.1(ii)]. To be precise, these authors define the strict order representing property of φ at a point $y_0 \in Y$ as follows (see (13)):

$$y \in Y\backslash\{y_0\}, y_0 \not\leq_S y \Rightarrow \varphi(y_0) > \varphi(y). \tag{31}$$

It is clear that this condition is equivalent to the following one (see (11)):

$$y \in Y\backslash\{y_0\}, y \trianglelefteq_{-S} y_0 \Rightarrow \varphi(y) < \varphi(y_0),$$

that coincides with the strictly $\trianglelefteq_{-S}$-preserving property at y_0 provided that S is pointed.

Example 6 (Linear spaces equipped with a binary relation, see Examples 1, 2 and 4) It is obvious that

$$S(\mathcal{G}, y_0, \triangleleft_C) = (y_0 - (C\backslash(-C))) \cap \mathcal{G}\backslash\{y_0\}.$$

Moreover, by [10, Theorem 4(f)] we see that

$$S_{\Psi^C_{e,y_0}}(\mathcal{G}, \Psi^C_{e,y_0}(y_0), <) = (y_0 + (-\infty, 0)e - \operatorname{vcl}_e C) \cap \mathcal{G}.$$

Therefore, mapping Ψ^C_{e,y_0} is strictly $\trianglelefteq_C$-preserving at y_0 if

$$C\backslash(-C) \subseteq (0, +\infty)e + \operatorname{vcl}_e C.$$

This condition is fulfilled, for instance, if $C\backslash(-C)$ is algebraically open, C is free-disposal with respect to an algebraic solid convex cone D and $e \in \operatorname{core} D$ (see [10, Proposition 18]).

On the other hand, it is clear that each $\ell \in C^\#$ is strictly $\trianglelefteq_C$-preserving at any point $y \in \mathcal{G}$.

Proposition 4 (Sufficient condition) *Let $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be strictly $\trianglelefteq$-preserving at $y_0 \in \mathcal{G}$. Then,*

$$y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi \Rightarrow y_0 \in \operatorname{Min}(\mathcal{G}, \trianglelefteq).$$

Proof If $y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi$, then $S_\varphi(\mathcal{G}, \varphi(y_0), <) = \emptyset$ and the result follows by part (a) of Definition 4.

Remark 9 Proposition 4 was stated in [7, Propostion 3.9] for quasi ordered sets. Notice also that a version of this last result in real topological linear spaces and the preference relation $\trianglelefteq_{\operatorname{int} K}$, where K is a proper convex cone with nonempty topological interior, was firstly obtained by Miglierina and Molho [22, Proposition 5.1]. This particular case can be obtained by applying Proposition 4 to the following data: $\trianglelefteq = \trianglelefteq_{-\operatorname{int} K}$ and $\varphi = -s$, where $s : Y \to \mathbb{R}$ is strictly $\trianglelefteq_{\operatorname{int} K}$-preserving at y, for all $y \in Y$ (observe that $\operatorname{ND}(\mathcal{G}, \trianglelefteq_{-\operatorname{int} K}) = \operatorname{Min}(\mathcal{G}, \trianglelefteq_{-\operatorname{int} K})$ since K is proper). Let us notice that the strictly $\trianglelefteq_{\operatorname{int} K}$-preserving property can be required only to the nominal point y_0.

Recently, Khushboo and Lalitha [15, Theorem 3.1(iii)] have stated the following stronger condition in the setting of a real topological linear space Y: if $y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi$ and φ satisfies (31), then $y_0 \in \operatorname{ND}(\mathcal{G}, \trianglelefteq_{-S})$. Next, we extend this result to a set $\mathcal{G}$ equipped with a binary relation $\trianglelefteq$ via Proposition 4.

Corollary 7 (Sufficient condition) *Let $y_0 \in \mathcal{G}$ and $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

$$y \in \mathcal{G}\backslash\{y_0\}, y \trianglelefteq y_0 \Rightarrow \varphi(y) < \varphi(y_0). \tag{32}$$

Then,

$$y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi \Rightarrow y_0 \in \operatorname{ND}(\mathcal{G}, \trianglelefteq).$$

Proof By assumption (32) and condition $y_0 \in \operatorname{argmin}_{\mathcal{G}}\varphi$ it is obvious that $S(\mathcal{G}, y_0, \sim) \subseteq \{y_0\}$. Thus, y_0 is a nondominated point of $(\mathcal{G}, \trianglelefteq)$ if and only if it is a minimal point and then the result follows by applying Proposition 4.

In the next corollary a sufficient condition for minimal solutions of problem (P) is derived by scalarization as a direct consequence of Proposition 4.

Corollary 8 *Consider problem (P), $x_0 \in H$ and let $\varphi : f(H) \to \mathbb{R} \cup \{\pm\infty\}$ be a strictly $\precsim$-preserving mapping at $f(x_0)$. Then:*

$$x_0 \in \operatorname{argmin}_H(\varphi \circ f) \Rightarrow x_0 \in \operatorname{Min}(f, H, \precsim).$$

Next, we combine the previous necessary and sufficient conditions to characterize minimal and nondominated points via scalarization. The first characterization is based on $\trianglelefteq$-representing and strictly $\trianglelefteq$-preserving mappings, and it is a direct consequence of Propositions 1 and 4. The second one is based on the order preserving and strict order representing properties introduced in statements (14) and (31), respectively, and follows as a consequence of Corollaries 1 and 7 and extends [15, Theorem 3.2(i)] to arbitrary binary relations. The third one considers strictly $\trianglelefteq$-representing and $\trianglelefteq$-preserving mappings and follows by Propositions 2 and 3, statement (30) and Remark 6. Notice that Theorem 3 encompasses [15, Theorem 3.2(ii)] (see Remarks 4 and 7).

Theorem 1 (Characterization) *Let* $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ *and* $y_0 \in \mathcal{G}$ *be such that* $S_\varphi(\mathcal{G}, \varphi(y_0), <) = S(\mathcal{G}, y_0, \lhd)$*. Then,*

$$y_0 \in \mathrm{Min}(\mathcal{G}, \unlhd) \iff y_0 \in \mathrm{argmin}_{\mathcal{G}} \varphi.$$

Theorem 2 (Characterization) *Let* $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ *and* $y_0 \in \mathcal{G}$ *be such that* $S_\varphi(\mathcal{G}\backslash\{y_0\}, \varphi(y_0), <) = S(\mathcal{G}\backslash\{y_0\}, y_0, \unlhd)$*. Then,*

$$y_0 \in \mathrm{ND}(\mathcal{G}, \unlhd) \iff y_0 \in \mathrm{argmin}_{\mathcal{G}} \varphi.$$

Theorem 3 (Characterization) *Let* $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ *and* $y_0 \in \mathcal{G}$ *be such that*

$$S_\varphi(\mathcal{G}\backslash\{y_0\}, \varphi(y_0), \leq) = S(\mathcal{G}\backslash\{y_0\}, y_0, \unlhd). \tag{33}$$

Then,

$$y_0 \in \mathrm{ND}(\mathcal{G}, \unlhd) \iff \mathrm{argmin}_{\mathcal{G}} \varphi = \{y_0\}, \tag{34}$$

$$y_0 \in \mathrm{Min}(\mathcal{G}, \unlhd) \iff \mathrm{argmin}_{\mathrm{NE}(y_0)} \varphi = \{y_0\},$$

$$\mathrm{argmin}_{\mathcal{G}} \varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\} \Rightarrow y_0 \in \mathrm{Min}(\mathcal{G}, \unlhd) \Rightarrow \mathrm{argmin}_{\mathcal{G}} \varphi \subseteq S(\mathcal{G}, y_0, \sim) \cup \{y_0\}.$$

If additionally φ *is constant in* $S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$*, then*

$$y_0 \in \mathrm{Min}(\mathcal{G}, \unlhd) \iff \mathrm{argmin}_{\mathcal{G}} \varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\}.$$

Observe that $S(\mathcal{G}, y_0, \unlhd) \subseteq \mathrm{argmin}_{\mathcal{G}} \varphi$ whenever $y_0 \in \mathrm{argmin}_{\mathcal{G}} \varphi$ and φ is $\unlhd$-preserving at y_0, and then φ is constant in $S(\mathcal{G}, y_0, \sim) \cup \{y_0\}$.

Remark 10 (i) Theorem 1 and assertion (34) of Theorem 3 generalizes, respectively, [7, Corollaries 3.12 and 3.13] from a quasi order to an arbitrary binary relation on a set.

(ii) Theorem 2 (resp., statement (34) of Theorem 3) reduces to [15, Theorem 3.2(i)] (resp. [15, Theorem 3.2(ii)]) by considering a real topological linear space Y and the binary relation $\unlhd = \unlhd_{-S}$ defined by an arbitrary domination set $S \subset Y$ (see (11)).

(iii) Besides, in the same setting, [15, Theorem 3.3(i)] states the equality

$$\mathrm{ND}(\mathcal{G}, \unlhd_{-S}) = \mathrm{argmin}_{\mathcal{G}} \varphi \tag{35}$$

whenever condition $S_\varphi(\mathcal{G}\backslash\{y_0\}, \varphi(y_0), <) = S(\mathcal{G}\backslash\{y_0\}, y_0, \unlhd_{-S})$ is fulfilled for all $y_0 \in \mathcal{G}$. When $\mathcal{G} = Y$ and S is not pointed, this assumption cannot be satisfied. On the other hand, if S is pointed, then $S(\mathcal{G}\backslash\{y_0\}, y_0, \unlhd_{-S}) = S(\mathcal{G}\backslash\{y_0\}, y_0, \lhd)$, $\mathrm{ND}(\mathcal{G}, \unlhd) = \mathrm{Min}(\mathcal{G}, \unlhd)$ and so conclusion (35) could be derived via Theorem 1. Thus, in the setting of real linear spaces and a preference relation defined by an arbitrary domination set, it would be more convenient to consider both the strict $\unlhd$-preserving and the $\unlhd$-representing properties than both properties (14) and (31).

(iv) In the setting of real ordered linear spaces, condition (32) has been frequently considered to state sufficient conditions for the so-called proper minimal points (see [5, 12, 14, 25]).

(v) Analogously, [15, Theorem 3.3(ii)] states that set $\mathrm{ND}(Y, \trianglelefteq_{-S})$ coincides with the set of strict solutions of problem

$$\mathrm{Min}\{\varphi(y) : y \in \mathcal{G}\}$$

provided that condition $S_\varphi(Y, \varphi(y_0), \leq) = S(Y, y_0, \trianglelefteq_{-S})$ is fulfilled for all $y_0 \in \mathcal{G}$. Then, under this condition it is obvious that the set $\mathrm{ND}(Y, \trianglelefteq_{-S})$ is empty or a singleton, which hardly ever happens in real linear spaces and the preference relation $\trianglelefteq_{-S}$. Let us notice that function φ is injective whenever it fulfils the mentioned condition and additionally the set S is pointed. So, the non pointwise version of property (33) seems to be very restrictive and then assertions as [18, Theorem 3.2(ii)] would be useless.

Theorem 4 (Characterization) *Let $\varphi : \mathcal{G} \to \mathbb{R} \cup \{\pm\infty\}$ and $y_0 \in \mathcal{G}$ be such that $S_\varphi(\mathcal{G}\backslash\{y_0\}, \varphi(y_0), <) = S(\mathcal{G}\backslash\{y_0\}, y_0, \lhd)$ and also $S_\varphi(\mathcal{G}\backslash\{y_0\}, \varphi(y_0), \leq) = S(\mathcal{G}\backslash\{y_0\}, y_0, \trianglelefteq)$. Then,*

$$y_0 \in \mathrm{Min}(\mathcal{G}, \trianglelefteq) \iff y_0 \in \mathrm{argmin}_{\mathcal{G}}\varphi \iff \mathrm{argmin}_{\mathcal{G}}\varphi = S(\mathcal{G}, y_0, \sim) \cup \{y_0\},$$
$$y_0 \in \mathrm{ND}(\mathcal{G}, \trianglelefteq) \iff \mathrm{argmin}_{\mathcal{G}}\varphi = \{y_0\}.$$

4 Application to set optimization

The first application concerns with approximate solutions of a set optimization problem. Let $F : X \to 2^{\mathbb{R}^p}$ be a set-valued mapping and $H \subseteq X$ be a nonempty feasible set of an arbitrary decision space X. The set optimization problem

$$F(x) \to \min_{\precsim^l_{\mathbb{R}^p_+}} \quad \text{s.t. } x \in H \tag{SOP}$$

looks for solutions according to the quasi order $\precsim^l_{\mathbb{R}^p_+}$ (the lower set less relation introduced by Kuroiwa in [20], compare [14]):

$$A_1, A_2 \in 2^{\mathbb{R}^p}, \quad A_1 \precsim^l_{\mathbb{R}^p_+} A_2 :\iff A_2 \subseteq A_1 + \mathbb{R}^p_+,$$

where $\mathbb{R}^p_+$ denotes the nonnegative orthant of $\mathbb{R}^p$. In the sequel, $\mathbb{R}^p_{++}$ stands for the topological interior of $\mathbb{R}^p_+$.

In order to approximate nondominated solutions of (SOP), the following concept was recently introduced (see [6, Definition 2.4(c)]).

Definition 5 Given a nonempty set $C \subset \mathbb{R}^p$, a point $x_0 \in H$ is said to be a C-approximate solution of problem (SOP), denoted by $x_0 \in \mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C)$, if $F(x) + C \not\precsim^l_{\mathbb{R}^p_+} F(x_0)$, for all $x \in H$.

Let $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $\varepsilon > 0$ and

$$C(\lambda, \varepsilon) := \left\{ y \in \mathbb{R}^p_+ : \langle \lambda, y \rangle \geq \varepsilon \right\}.$$

For each $x \in H$ we assume that $F(x)$ is nonempty and compact. Thus,

$$\forall x \in H: \ \operatorname{argmin}_{F(x)} \langle \lambda, \cdot \rangle \neq \emptyset.$$

Next, the set $\mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda, \varepsilon))$ of $C(\lambda, \varepsilon)$-approximate solutions of problem (SOP) is characterized by scalarization. First, we show that the compactness assumption implies that a point $x_0 \in H$ is a $C(\lambda, \varepsilon)$-approximate solution of problem (SOP) if and only if it is a nondominated solution of problem

$$\begin{aligned} & F(x) \to \min\nolimits_{\precsim^l_{C(\lambda,\varepsilon)}} \\ & \text{s.t. } x \in H, \end{aligned} \tag{ASOP}$$

where the ordering $\precsim^l_{C(\lambda,\varepsilon)}$ is defined as follows:

$$A_1, A_2 \in 2^{\mathbb{R}^p}, \quad A_1 \precsim^l_{C(\lambda,\varepsilon)} A_2 :\Longleftrightarrow A_2 \subseteq A_1 + C(\lambda, \varepsilon).$$

Recall that notations $\mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ and $\mathrm{Min}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ refer to the sets of nondominated and minimal solutions of problem (ASOP), respectively.

Lemma 3 *For each $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$ and $\varepsilon > 0$ we have that*

$$\mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda, \varepsilon)) = \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)}) = \mathrm{Min}(F, H, \precsim^l_{C(\lambda,\varepsilon)}).$$

Proof Let $x_1, x_2 \in H$. As $C(\lambda, \varepsilon) + \mathbb{R}^p_+ = C(\lambda, \varepsilon)$, condition $F(x_1) + C(\lambda, \varepsilon) \not\precsim^l_{\mathbb{R}^p_+} F(x_2)$ is equivalent to $F(x_1) \not\precsim^l_{C(\lambda,\varepsilon)} F(x_2)$. Then, the inclusion $\mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda, \varepsilon)) \subseteq \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ is clear. For the reciprocal inclusion, let us first prove that

$$F(x) \not\precsim^l_{C(\lambda,\varepsilon)} F(x), \quad x \in H. \tag{36}$$

Indeed, suppose reasoning by contradiction that there exists $x \in H$ such that $F(x) \precsim^l_{C(\lambda,\varepsilon)} F(x)$. In particular, we have $\operatorname{argmin}_{F(x)} \langle \lambda, \cdot \rangle \subseteq F(x) + C(\lambda, \varepsilon)$. Consider an arbitrary point $y \in \operatorname{argmin}_{F(x)} \langle \lambda, \cdot \rangle$. Then, there exists $z \in F(x)$ and $d \in C(\lambda, \varepsilon)$ such that $y = z + d$ and we obtain that

$$\langle \lambda, z \rangle = \langle \lambda, y \rangle - \langle \lambda, d \rangle \leq \min_{F(x)} \langle \lambda, \cdot \rangle - \varepsilon < \min_{F(x)} \langle \lambda, \cdot \rangle,$$

that is contradiction. Therefore, assertion (36) holds true.

Consider $x_0 \in \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ and reasoning again by contradiction suppose that $x_0 \notin \mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda, \varepsilon))$. Then, there exists $x \in H$ such that $F(x) + C(\lambda, \varepsilon) \precsim^l_{\mathbb{R}^p_+} F(x_0)$. Therefore, $F(x) \precsim^l_{C(\lambda,\varepsilon)} F(x_0)$ and it follows that $F(x) = F(x_0)$, since $x_0 \in \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$. Thus, $F(x_0) \precsim^l_{C(\lambda,\varepsilon)} F(x_0)$, that is contrary to (36).

Next, we state that $\mathrm{Min}(F, H, \precsim^l_{C(\lambda,\varepsilon)}) \subseteq \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$. Indeed, let $x_0 \in \mathrm{Min}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ and suppose that there exists $x \in H$ such that $F(x) \precsim^l_{C(\lambda,\varepsilon)} F(x_0)$. As $x_0 \in \mathrm{Min}(F, H, \precsim^l_{C(\lambda,\varepsilon)})$ we deduce that $F(x_0) \subseteq F(x_0) + C(\lambda, \varepsilon) + C(\lambda, \varepsilon) = F(x_0) + C(\lambda, 2\varepsilon)$ and from here we get a contradiction by following the previous reasonings carried out to prove (36). This completes the proof.

To scalarize the nondominated solutions of problem (ASOP) we consider the following two mappings: given $B \in 2^{\mathbb{R}^p}\backslash\{\emptyset\}$ and $q \in \mathbb{R}^p_{++}$, let us define $\Phi^{\lambda,\varepsilon}_{q,B}, \varphi^{\lambda,\varepsilon}_{q,B} : 2^{\mathbb{R}^p}\backslash\{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ as

$$\varphi^{\lambda,\varepsilon}_{q,B}(A) := \sup_{b\in B} \inf_{a\in A} \max\left\{ \max_{1\le j\le p} \left\{ \frac{a_j - b_j}{q_j} \right\}, \frac{\varepsilon + \langle \lambda, a - b\rangle}{\langle \lambda, q\rangle} \right\} \quad (A \in 2^{\mathbb{R}^p}\backslash\{\emptyset\}),$$

where a_j, b_j and q_j denote the jth-component of elements a, b and q, respectively, and $\Phi^{\lambda,\varepsilon}_{q,B}(A) := \varphi^{\lambda,\varepsilon}_{q,B}(A)$, for all $A \in 2^{\mathbb{R}^p}\backslash\{\emptyset, B\}$, $\Phi^{\lambda,\varepsilon}_{q,B}(B) := 0$.

Lemma 4 *Suppose that* $\mathrm{argmin}_B\langle \lambda, \cdot\rangle \neq \emptyset$. *It follows that* $\varphi^{\lambda,\varepsilon}_{q,B}(B) = \varepsilon/\langle \lambda, q\rangle$.

Proof For each $\bar{b} \in B$ it follows that

$$\begin{aligned}
&\inf_{b\in B} \max\left\{ \max_{1\le j\le p} \left\{ \frac{b_j - \bar{b}_j}{q_j} \right\}, \frac{\varepsilon + \langle \lambda, b - \bar{b}\rangle}{\langle \lambda, q\rangle} \right\}\\
&\le \max\left\{ \max_{1\le j\le p} \left\{ \frac{\bar{b}_j - \bar{b}_j}{q_j} \right\}, \frac{\varepsilon + \langle \lambda, \bar{b} - \bar{b}\rangle}{\langle \lambda, q\rangle} \right\}\\
&= \frac{\varepsilon}{\langle \lambda, q\rangle}.
\end{aligned}$$

Thus, it follows that $\varphi^{\lambda,\varepsilon}_{q,B}(B) \le \varepsilon/\langle \lambda, q\rangle$. Reciprocally, consider an arbitrary point $b^0 \in \mathrm{argmin}_B\langle \lambda, \cdot\rangle$. Then,

$$\begin{aligned}
\varphi^{\lambda,\varepsilon}_{q,B}(B) &\ge \inf_{b\in B} \max\left\{ \max_{1\le j\le p} \left\{ \frac{b_j - b^0_j}{q_j} \right\}, \frac{\varepsilon + \langle \lambda, b - b^0\rangle}{\langle \lambda, q\rangle} \right\}\\
&\ge \inf_{b\in B} \frac{\varepsilon + \langle \lambda, b - b^0\rangle}{\langle \lambda, q\rangle}\\
&= \varepsilon/\langle \lambda, q\rangle.
\end{aligned}$$

Therefore, $\varphi^{\lambda,\varepsilon}_{q,B}(B) = \varepsilon/\langle \lambda, q\rangle$ and the proof is complete.

Let us denote $F(H) := \{F(x) : x \in H\} \subseteq 2^{\mathbb{R}^p}\backslash\{\emptyset\}$.

Proposition 5 *For each* $B \in F(H)$, *the mapping* $\Phi^{\lambda,\varepsilon}_{q,B} : F(H) \to \mathbb{R} \cup \{\pm\infty\}$ *is strictly* $\precsim^l_{C(\lambda,\varepsilon)}$*-representing and* $\precsim^l_{C(\lambda,\varepsilon)}$*-preserving at* B.

Proof Let $A \in F(H)\backslash\{B\}$ such that $\Phi^{\lambda,\varepsilon}_{q,B}(A) \le \Phi^{\lambda,\varepsilon}_{q,B}(B)$. Then, we have that $\varphi^{\lambda,\varepsilon}_{q,B}(A) \le 0$ and so

$$\forall b \in B: \inf_{a\in A} \max\left\{\max_{1\le j\le p}\left\{\frac{a_j - b_j}{q_j}\right\}, \frac{\varepsilon + \langle\lambda, a-b\rangle}{\langle\lambda, q\rangle}\right\} \le 0.$$

Fix an arbitrary $b \in B$. For each $n \in \mathbb{N}$ there exists $a^n \in A$ such that

$$\max_{1\le j\le p}\left\{\frac{a^n_j - b_j}{q_j}\right\} \le 1/n, \quad \frac{\varepsilon + \langle\lambda, a^n - b\rangle}{\langle\lambda, q\rangle} \le 1/n. \tag{37}$$

As A is compact we can suppose without loss of generality that $a^n \to a \in A$. Then, by (37) we see that

$$\max_{1\le j\le p}\left\{\frac{a_j - b_j}{q_j}\right\} \le 0, \quad \frac{\varepsilon + \langle\lambda, a-b\rangle}{\langle\lambda, q\rangle} \le 0$$

and we obtain that $b - a \in C(\lambda,\varepsilon)$. As $b \in B$ is arbitrary it follows that $B \subseteq A + C(\lambda,\varepsilon)$, i.e., $A \precsim^l_{C(\lambda,\varepsilon)} B$.

Reciprocally, consider $A \in F(H)\backslash\{B\}$ such that $A \precsim^l_{C(\lambda,\varepsilon)} B$. Let us check that $\Phi^{\lambda,\varepsilon}_{q,B}(A) \le \Phi^{\lambda,\varepsilon}_{q,B}(B)$. Indeed, for each $b \in B$, since $B \subseteq A + C(\lambda,\varepsilon)$, there exist $a^b \in A$ and $d^b \in C(\lambda,\varepsilon)$ such that $b = a^b + d^b$. Thus, $d^b \in \mathbb{R}^p_+$, $\langle\lambda, d^b\rangle \ge \varepsilon$ and then

$$\begin{aligned}&\inf_{a\in A} \max\left\{\max_{1\le j\le p}\left\{\frac{a_j - b_j}{q_j}\right\}, \frac{\varepsilon + \langle\lambda, a-b\rangle}{\langle\lambda, q\rangle}\right\}\\ &\le \max\left\{\max_{1\le j\le p}\left\{\frac{-d^b_j}{q_j}\right\}, \frac{\varepsilon + \langle\lambda, -d^b\rangle}{\langle\lambda, q\rangle}\right\} \le 0.\end{aligned}$$

Therefore,

$$\Phi^{\lambda,\varepsilon}_{q,B}(A) = \varphi^{\lambda,\varepsilon}_{q,B}(A) \le 0 = \Phi^{\lambda,\varepsilon}_{q,B}(B)$$

and the proof is finished.

Theorem 5 *We have that*

$$x_0 \in \mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda,\varepsilon)) \iff x_0 \in \frac{\varepsilon}{\langle\lambda, q\rangle}\text{−}\mathrm{argmin}^{<}_{\mathrm{D}(H,F,x_0)}(\varphi^{\lambda,\varepsilon}_{q,F(x_0)} \circ F),$$

where $\mathrm{D}(H, F, x_0) := \{x \in H : F(x) \ne F(x_0)\} \cup \{x_0\}$.

Proof By Lemma 3 we obtain that

$$\begin{aligned}x_0 \in \mathrm{A}(F, H, \precsim^l_{\mathbb{R}^p_+}, C(\lambda,\varepsilon)) &\iff x_0 \in \mathrm{ND}(F, H, \precsim^l_{C(\lambda,\varepsilon)}) &(38)\\ &\iff x_0 \in H,\ F(x_0) \in \mathrm{ND}(F(H), \precsim^l_{C(\lambda,\varepsilon)}). &(39)\end{aligned}$$

Assume that $x_0 \in H$. By applying Theorem 3 to $\mathcal{G} := F(H)$, $\varphi := \Phi^{\lambda,\varepsilon}_{q,F(x_0)}$ and the binary relation $\precsim^l_{C(\lambda,\varepsilon)}$ it turns that

$$
\begin{aligned}
F(x_0) \in \mathrm{ND}(F(H), \precsim^l_{C(\lambda,\varepsilon)}) &\iff \mathrm{argmin}_{F(H)} \Phi^{\lambda,\varepsilon}_{q,F(x_0)} = \{F(x_0)\} && (40)\\
&\iff \Phi^{\lambda,\varepsilon}_{q,F(x_0)}(F(x_0)) < \Phi^{\lambda,\varepsilon}_{q,F(x_0)}(F(x)), && (41)\\
&\qquad \forall x \in \mathrm{D}(F,H,x_0)\backslash\{x_0\}\\
&\iff 0 < \varphi^{\lambda,\varepsilon}_{q,F(x_0)}(F(x)), && (42)\\
&\qquad \forall x \in \mathrm{D}(F,H,x_0)\backslash\{x_0\}\\
&\iff (\varphi^{\lambda,\varepsilon}_{q,F(x_0)} \circ F)(x_0) - \frac{\varepsilon}{\langle \lambda, q\rangle} && (43)\\
&\qquad < (\varphi^{\lambda,\varepsilon}_{q,F(x_0)} \circ F)(x),\ \forall x \in \mathrm{D}(F,H,x_0)\backslash\{x_0\} && (44)
\end{aligned}
$$

where the last equivalence is a consequence of Lemma 4. Then, the result follows by (38)-(44) and the proof is complete.

Remark 11 Notice that Theorem 5 cannot be deduced by the results of [7, 15, 22, 28–30] since $(2^{\mathbb{R}^p}, +, \cdot, \precsim^l_{C(\lambda,\varepsilon)})$ is neither a real topological linear space nor a quasi ordered space. In this sense, it contributes to the research line suggested by Khushboo and Lalitha in [15, Section 6].

5 Robustness for scalar optimization problems involving uncertainties

In many real world optimization problems, the input data are not completely known. Indeed, in an optimization problem the data are in general inexact due to measurement errors, some parameters have to be estimated since they are unknown or the mathematical formulation of the real problem does not reflect it completely. Such optimization problems, where uncertainties are involved, are very common in practice. One approach for dealing with optimization problems under uncertainty is the concept of robustness. This approach leads us to a deterministic optimization problem where the solution concept is given by certain binary relations introduced in Section 2.

In this section, we will apply our results concerning the characterization of minimal and nondominated points (see Section 3) in order to give a characterization of robust counterpart problems to scalar optimization problems where uncertainties are involved.

Throughout, the following scalar optimization problem under uncertainty is considered:

$$
\begin{aligned}
& f(x,\xi) \to \min && (\mathrm{Q}(\xi))\\
& \text{s.t. } F_i(x,\xi) \le 0, \quad i = 1,\dots,m,\\
& \qquad x \in \mathbb{R}^n,
\end{aligned}
$$

where $f, F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ for all $i = 1, 2, \ldots, m$ and $\xi \in \mathcal{U} := \{\xi_1, \xi_2, \ldots, \xi_q\} \subset \mathbb{R}^N$ is the uncertain parameter, which is assumed to be unknown, but stems from the given uncertainty set $\mathcal{U}$.

In order to solve problem (Q(ξ)), a deterministic counterpart is needed and the so-called robust optimization approach suggests different alternatives (see [16–18] and the references therein). It is well-known that solutions of robust counterparts of problem (Q(ξ)) are solutions in some sense of the following associated multiobjective optimization problems (see [16–18, 23]):

$$\begin{aligned} &\bar{f}(x) \to \min \\ &\text{s.t. } g_{ij}(x) \leq 0, \quad i = 1, \ldots, m,\ j = 1, \ldots, q, \\ &\qquad x \in \mathbb{R}^n, \end{aligned} \tag{MOP}$$

$$\begin{aligned} &\bar{f}^u(x) \to \min \\ &\text{s.t. } x \in \mathbb{R}^n, \end{aligned} \tag{UMOP}$$

where $\bar{f} : \mathbb{R}^n \to \mathbb{R}^q$, $\bar{f}^u : \mathbb{R}^n \to \mathbb{R}^{q(1+m)}$, $g_{ij} : \mathbb{R}^n \to \mathbb{R}$, $\bar{f}_j := f(\cdot, \xi_j)$, $g_{ij} = F_i(\cdot, \xi_j)$, for all $i = 1, \ldots, m$ and $j = 1, 2, \ldots, q$, and $\bar{f}^u = (\bar{f}, \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m)$ with $\bar{g}_i = (g_{i1}, g_{i2}, \ldots, g_{iq}) : \mathbb{R}^n \to \mathbb{R}^q$, for all $i = 1, \ldots, m$.

The image spaces of these problems are ordered by components. To be precise, the relations $\trianglelefteq_{\mathbb{R}^l_+}$ and $\trianglelefteq_{\mathbb{R}^l_{++}}$, $l \in \{q, qm, q(1+m)\}$ are used (recall that notation $\mathbb{R}^l_{++}$ stands for the topological interior of $\mathbb{R}^l_+$).

Thus, the feasible set of problem (MOP) is

$$H := \{x \in \mathbb{R}^n : \bar{g}(x) \trianglelefteq_{\mathbb{R}^{qm}_+} 0\}$$

where $\bar{g} = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m) : \mathbb{R}^n \to \mathbb{R}^{qm}$.

In the sequel, we introduce several robustness notions that generalize the most important robustness concepts of the literature. Then, we relate them with solutions of problems (MOP) and (UMOP) by the results of Section 3. The obtained relationships cover and extend several similar ones stated by well-known robust counterparts.

Definition 6 A function $\rho : \mathbb{R}^l \to \mathbb{R}$ is said to be increasing (resp. strong increasing) if $\rho(y + d) > \rho(y)$ for all $y \in \mathbb{R}^l$ and $d \in \mathbb{R}^l_{++}$ (resp. $d \in \mathbb{R}^l_+ \backslash \{0\}$), and it is said to be nondecreasing if $\rho(y+d) \geq \rho(y)$ for all $y \in \mathbb{R}^l$ and $d \in \mathbb{R}^l_+$).

Remark 12 Let us observe that ρ is increasing (resp. strong increasing) if it is strictly $\trianglelefteq_{\mathbb{R}^l_{++}}$ (resp. $\trianglelefteq_{\mathbb{R}^l_+}$)-preserving at every point $y \in \mathbb{R}^l$, and it is nondecreasing if it is $\trianglelefteq_{\mathbb{R}^l_+}$-preserving at every point $y \in \mathbb{R}^l$.

Definition 7 A point $x_0 \in \mathbb{R}^n$ is said to be a weak (resp. strict) admissible robust solution of (Q(ξ)) if there exists an increasing (resp. nondecreasing) function $\rho : \mathbb{R}^q \to \mathbb{R}$ such that $x_0 \in \operatorname{argmin}_H(\rho \circ \bar{f})$ (resp. $\{x_0\} = \operatorname{argmin}_H(\rho \circ \bar{f})$). The set of all weak (resp. strict) admissible robust solutions of (Q(ξ)) is denoted by WA($Q(\xi)$) (resp. StrA($Q(\xi)$)).

If in the definitions above we consider a surrogate set $G \supseteq H$ instead of H satisfying condition (8) with $f = \bar{f}$, $g = \bar{g}$, $M = \mathbb{R}^n$ and $\precsim^u = \trianglelefteq_{\mathbb{R}^q_{++} \times \mathbb{R}^{qm}_+}$ (resp. $\precsim^u = \trianglelefteq_{\mathbb{R}^{q(1+m)}_+}$) and also the condition $x_0 \in \mathrm{argmin}_G(\rho \circ \bar{f})$ (resp. $\{x_0\} = \mathrm{argmin}_G(\rho \circ \bar{f})$), then we say that x_0 is a surrogate weak (resp. strict) admissible robust solution of (Q(ξ)), denoted by $x_0 \in \mathrm{SWA}(Q(\xi))$ (resp. $x_0 \in \mathrm{SSrtA}(Q(\xi))$).

Analogously, $x_0 \in \mathbb{R}^n$ is said to be an admissible robust solution of (Q(ξ)), denoted by $x_0 \in \mathrm{A}(Q(\xi))$ if there exists a strong increasing function $\rho : \mathbb{R}^q \to \mathbb{R}$ such that $x_0 \in \mathrm{argmin}_H(\rho \circ \bar{f})$.

The following example shows that our new definitions comprise known concepts of robust counterparts.

Example 7 Consider $y^0 \in \mathbb{R}^q$, $w \in \mathbb{R}^q_{++}$ and the problem

$$\begin{aligned} &\max_{j=1,2,\dots,q} w_j(f(x,\xi_j) - y^0_j) \to \min \\ &\text{s.t. } F_i(x,\xi_j) \leq 0,\ i = 1,\dots,m,\ j = 1,2,\dots,q, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{Q(ξ, w, y^0)}$$

Let in Definition 7 the function $\rho : \mathbb{R}^q \to \mathbb{R}$ be given by

$$\forall y \in \mathbb{R}^q : \ \rho(y) := \max_{j=1,2,\dots,q} \{w_j(y_j - y^0_j)\}.$$

It is clear that ρ is nondecreasing as well as increasing, but it is not strong increasing. Therefore, if x is an optimal (resp. strict) solution of problem (Q(ξ, w, y^0)), then x is a weak (resp. strict) admissible robust solution of (Q(ξ)).

The most prominent robust counterpart of (Q(ξ)) is described by the concept of *strict robustness* (also called *minmax robustness*). It has been introduced by Soyster [27] and extensively researched since then, see Ben-Tal et al. [2]. Strict robustness is a very conservative approach, where the worst case objective function is minimized and all constraints have to be fulfilled for every possible uncertain parameter. Formally, the **strictly robust counterpart** of the optimization problem under uncertainty (Q(ξ)) is described by problem Q(ξ, w, y^0) by considering $w = (1, 1, \dots, 1)$ and $y^0 = 0$.

This robustness concept was further generalized by the so-called **weighted robust counterpart** of (Q(ξ)), where weights $w_j > 0$, $j = 1, 2, \dots, q$ are considered. Such a weighted robust approach to an optimization problem under uncertainty was proposed by Kouvelis and Sayin [19, 26] to generate solutions of a vector-valued optimization problem.

If the best possible objective values for each future scenario is taken into account while minimizing the worst possible objective function value at the same time, then the *deviation robustness* is considered; sometimes it is referred to as *minmax regret robustness.* To be precise, if $f^0(\xi_j) \in \mathbb{R}$ denotes the optimal value of problem (Q(ξ_j)) for every $j = 1, 2, \dots, q$, then the **deviation robust counterpart** of (Q(ξ)) is the problem Q($\xi, e, \bar{f}^0$) where $e = (1, 1, \dots, 1) \in \mathbb{R}^q$ and $\bar{f}^0 = (f^0(\xi_1), f^0(\xi_2), \dots, f^0(\xi_q))$.

Theorem 6 *We have the following relationships:*

$$\begin{aligned}
\mathrm{WA}(Q(\xi)) &\subseteq \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}}),\\
\mathrm{SWA}(Q(\xi)) &\subseteq \mathrm{ND}(\bar{f}^u, \mathbb{R}^n, \trianglelefteq_{\mathbb{R}^q_{++}\times\mathbb{R}^{qm}_+}),\\
\mathrm{StrA}(Q(\xi)) &\subseteq \mathrm{Str}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+}),\\
\mathrm{SStrA}(Q(\xi)) &\subseteq \mathrm{Str}(\bar{f}^u, \mathbb{R}^n, \trianglelefteq_{\mathbb{R}^{q(1+m)}_+}),\\
\mathrm{A}(Q(\xi)) &\subseteq \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+}).
\end{aligned}$$

Proof Consider $x_0 \in \mathrm{WA}(Q(\xi))$. Then, there exists an increasing function $\rho : \mathbb{R}^q \to \mathbb{R}$ such that $x_0 \in \mathrm{argmin}_H(\rho \circ \bar{f})$. Then, by Corollary 8 and Lemma 1 we deduce that

$$x_0 \in \mathrm{Min}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}}) = \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}})$$

and the first inclusion follows.

Suppose that $x_0 \in \mathrm{SWA}(Q(\xi))$. Then, reasoning as in the previous paragraph we see that $x_0 \in \mathrm{Min}(\bar{f}, G, \trianglelefteq_{\mathbb{R}^q_{++}})$ for a surrogate set $G \supseteq H$ satisfying condition (8) with $f = \bar{f}$, $g = \bar{g}$, $M = \mathbb{R}^n$ and $\precsim^u = \trianglelefteq_{\mathbb{R}^q_{++}\times\mathbb{R}^{qm}_+}$. Then, the second inclusion is a consequence of (9)-(10) and Lemmas 1 and 2, since $S(\mathbb{R}^q, y, \sim^q) = \emptyset$, for all $y \in \mathbb{R}^q$.

Let $x_0 \in \mathrm{StrA}(Q(\xi))$. Then, there exists a nondecreasing function $\rho : \mathbb{R}^q \to \mathbb{R}$ such that $\{x_0\} = \mathrm{argmin}_H(\rho \circ \bar{f})$. Then, by Corollary 6 we deduce that $x_0 \in \mathrm{Str}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+})$ and the third inclusion follows.

Consider $x_0 \in \mathrm{SStrA}(Q(\xi))$. Then, reasoning as in the previous paragraph, we see that $x_0 \in \mathrm{Str}(\bar{f}, G, \trianglelefteq_{\mathbb{R}^q_+})$ for a surrogate set $G \supset H$ satisfying condition (8) with $f = \bar{f}$, $g = \bar{g}$, $M = \mathbb{R}^n$ and $\precsim^u = \trianglelefteq_{\mathbb{R}^{q(1+m)}_+}$. Then, the fourth inclusion is a consequence of Lemma 2.

Finally, suppose that $x_0 \in \mathrm{A}(Q(\xi))$. Then, there exists a strong increasing function $\rho : \mathbb{R}^q \to \mathbb{R}$ such that $x_0 \in \mathrm{argmin}_H(\rho \circ \bar{f})$. Then, by Corollary 8 we deduce that

$$x_0 \in \mathrm{Min}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+}) = \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+}),$$

since $\trianglelefteq_{\mathbb{R}^q_+}$ is antisymmetric, and the fifth inclusion follows.

Remark 13 The second and fourth inclusions of Theorem 6 recover [18, Theorems 4.1-4.6]. For it, consider $M = \mathbb{R}^n$ and let G be the feasible set of each robust optimization problem. Notice that the assertions corresponding to unique solutions of robust counterparts are improved, since these points are strict solutions of the unconstrained multiobjective optimization problem (UMOP).

Some inclusions of Theorem 6 can be strengthened, as it is stated in the following theorem.

Theorem 7 *We have the following relationship:*

$$\mathrm{WA}(Q(\xi)) = \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}}). \tag{45}$$

Moreover, if $f(\cdot, \xi_j)$ *is a convex function,* $F_i(\cdot, \xi_j)$ *is a semicontinuous quasi-convex function, for all* $j = 1, 2, \ldots, q$ *and* $i = 1, 2, \ldots, m$, $\bar{f}(H) + \mathbb{R}^q_+$ *is closed and* $\mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+}) \neq \emptyset$, *then*

$$\bar{f}(\mathrm{A}(Q(\xi))) \subseteq \mathrm{ND}(\bar{f}(H), \trianglelefteq_{\mathbb{R}^q_+}) \subseteq \mathrm{cl}\, \bar{f}(\mathrm{A}(Q(\xi))).$$

Proof Consider a point $x_0 \in \mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}})$. By Lemma 1, we have $x_0 \in \mathrm{Min}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_{++}})$. Now let us define the increasing function $\rho : \mathbb{R}^q \to \mathbb{R}$,

$$\forall y \in \mathbb{R}^q : \ \rho(y) := \max_{1 \le j \le q} \{y_j - f(x_0, \xi_j)\}.$$

We claim that it is $\trianglelefteq_{\mathbb{R}^q_{++}}$-representing at $\bar{f}(x_0)$. Indeed, let $y \in \mathbb{R}^q$ be such that $\rho(y) < \rho(\bar{f}(x_0)) = 0$. Then, $y_j < f(x_0, \xi_j)$, for all $j = 1, 2, \ldots, q$, i.e., $y \vartriangleleft_{\mathbb{R}^q_{++}} \bar{f}(x_0)$.

Therefore, by applying Corollary 3 we deduce that $x_0 \in \mathrm{argmin}_H(\rho \circ \bar{f})$, and equality (45) is proved.

Notice that each $\ell \in \mathbb{R}^q_{++}$ is a strong increasing function. Then, by [4, Lemma 3.3] and [25, Theorems 3.4.1 and 3.4.2] we have that

$$\begin{aligned} \mathrm{ND}(\bar{f}(H), \trianglelefteq_{\mathbb{R}^q_+}) &= \bar{f}(\mathrm{ND}(\bar{f}, H, \trianglelefteq_{\mathbb{R}^q_+})) \\ &\subseteq \mathrm{cl}\, \bar{f}\left(\bigcup_{\ell \in \mathbb{R}^q_{++}} \mathrm{argmin}_H(\ell \circ \bar{f}) \right) \\ &\subseteq \mathrm{cl}\, \bar{f}(\mathrm{A}(Q(\xi))), \end{aligned}$$

which finishes the proof.

Remark 14 (i) In [4, Lemma 3.3] and [21, Lemma 3.2] one can find conditions that imply the closedness of the set $\bar{f}(H) + \mathbb{R}^q_+$.

(ii) As a result of Theorem 7 and the motivation for dealing with robust counterparts of scalar optimization problems under uncertainty, one can conclude that weak admissible robust solutions should be avoid. On the contrary, for some special problems (for instance, these considered in the second part of Theorem 7), the only kind of robust solutions that should be taken into account are the admissible ones.

6 Conclusions

This paper presents an unifying framework for characterizations of minimal and nondominated points of arbitrary binary relations by scalarization. Our results generalize several corresponding ones from the literature and shed new light to the concept of scalarization. Finally, we have shown that our results have wide applications in the fields of set-valued optimization and scalar optimization under uncertainty and its multi-objective counterpart. Further avenues for future research include the characterization by scalarization of other solution concepts of set-valued optimization problems and the study of robust counterparts of uncertain multiobjective optimization problems (see [11]).

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