Doctoral theses at NTNU, 2021:152

Eirik Skrettingland

Time-Frequency Analysis Meets Quantum Harmonic Analysis

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
Philosophiae Doctor
Faculty of Information Technology and Electrical
Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

Eirik Skrettingland

Time-Frequency Analysis Meets Quantum Harmonic Analysis

Thesis for the Degree of Philosophiae Doctor

Trondheim, May 2021

Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the Degree of Philosophiae Doctor

Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

ISBN 978-82-326-5647-9 (printed ver.)
ISBN 978-82-326-6684-3 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2021:152

Printed by NTNU Grafisk senter

Abstract

The subject of this thesis is the study of quantum harmonic analysis and time-frequency analysis, and in particular the intersection of these two fields. Quantum harmonic analysis is studied abstractly both by obtaining new results and by extending the setting to other abelian and nonabelian groups. Tools and results from quantum harmonic analysis are used to study concepts from time-frequency analysis, for instance localization operators and Cohen's class, obtaining new results and generalizations and reinterpretations of old results in time-frequency analysis. Concepts and results in time-frequency analysis also inspire new directions, results and proofs in quantum harmonic analysis such as the careful study of Fourier series of operators in a general setting.

Sammendrag

I denne avhandlingen studeres de to matematiske teoriene kvante-harmonisk analyse og tid-frekvens-analyse, med et spesielt fokus på skjæringspunktet mellom disse teoriene. Vi studerer kvante-harmonisk analyse abstrakt, både ved å vise nye resultater og gjennom å utvide domenet hvor kvante-harmonisk analyse er gyldig til andre abelske og ikke-abelske grupper. I tillegg bruker vi redskaper og resultater fra kvante-harmonisk analyse til å studere konsepter i tid-frekvens-analyse, som lokaliseringsoperatorer og Cohens klasse, og finner derigjennom både nye resultater samt generaliseringer og nytolkninger av gamle resultater i tid-frekvens-analyse. Konsepter og resultater i tid-frekvens-analyse inspirerer også nye retninger, resultater og bevis i kvante-harmonisk analyse, eksempelvis en grundig studie av Fourierrekker for operatorer.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) in Mathematical Sciences at the Norwegian University of Science and Technology (NTNU). The research presented here was conducted at the Department of Mathematical Sciences at NTNU, under the supervision of Professor Franz Luef and Associate Professor Eduard Ortega.

The main part of the thesis is comprised of seven research papers, five of which have been accepted for publication in research journals. The last two are preprints. All the papers appear in their published or preprint form, except for a small number of clarifications as well as some minor changes in notation for consistency across the thesis. There is also an introduction, serving to provide necessary background for the thesis in a manner that motivates and connects the seven papers. At the end of the introduction, there is a short summary of each of the research papers, where any noteworthy changes from the published version are listed. The reference lists of the papers and introduction have been consolidated to a single list of references at the end of the thesis.

Acknowledgements

First of all, I am very grateful for the support I received from my main supervisor Franz Luef during the last four years. His co-authorship, questions, insights and suggestions have improved this thesis immensely. I would especially like to thank him for keeping the door to his office open whenever possible, as our day-to-day discussions have often improved both my work and my mood. On occasions when an open door was not possible, be it due to travels or pandemics, I have been very grateful for our digital discussions.

I would also like to thank my co-supervisor Eduard Ortega for all the new impulses I got from his hard work organizing the group and seminars in operator algebras; these thanks do of course extend to all the other members of the operator algebras group. My stay at NTNU has been brightened by the value I found in my teaching duties, in large parts due to the mix of support and responsibility I got from the other teaching staff: Franz Luef, Christan Skau, Sigrid Grepstad and Eduard Ortega in particular. In fact, I am very grateful for the support I received from all levels of the Department, including the administrative staff, whose help I have greatly valued in matters both big and small. I would also like to thank Stian Tamlagsrønning and Marius Irgens for giving me the opportunity to contribute within my limited abilities to the guidance of some students together with Johanne Haugland. I truly appreciated the cooperation between the four of us, and would like to extend my heartfelt thanks and best wishes to the students I met during that time.

Sharing an office with Are Austad, Eirik Berge and Stine Marie Berge for most of the last four years has been a pleasure, and I would like to thank them all for always being willing to answer my mathematical questions and for the many coffee breaks. One of the papers in the thesis is coauthored with Eirik, Franz and Stine, and they also deserve to be thanked for making this an interesting and enjoyable collaboration.

I had several shorter stays at the University of Vienna, and would like to thank all those I met at the Faculty of Mathematics for making me feel welcome. The help and invitations I received from Monika Dörfler and Markus Faulhuber were especially appreciated. I would also like to thank Monika for the many enlightening discussions in Vienna, Trondheim and online.

Finally, the ever-present support from my mother and sister helped make this thesis possible. Returning home to them has always allowed me to relax and recharge, which helped me regain my motivation when the task got too overwhelming. Last, but perhaps really first, Ellen deserves all the thanks I can muster for supporting me these years and helping me get my priorities straight when the thesis became even more important for me than it deserved.

Eirik Skrettingland
Trondheim, February 2021

Contents

Part I

Introduction

Chapter 1

Quantum harmonic analysis

The papers of this thesis either directly concern or are motivated by two fields of mathematics: *time-frequency analysis* and *quantum harmonic analysis*. This introduction therefore aims to give a brief overview of the relevant parts of these fields, with the goal of motivating the research papers. Unlike time-frequency analysis, the basics of quantum harmonic analysis are currently not readily available in monographs and surveys. Explaining and motivating this theory will therefore be an important part of the introduction. A brief summary of each of the papers that constitute this thesis is included in the last part of the introduction. After reading the introduction, the reader should have some insight into the motivation for the various papers, how the papers fit together and how the main results fit into the literature. So the introduction aims in no way for completeness, neither in its coverage of background material nor of the papers of this thesis.

1.1 Three fundamental theorems of Wiener

In order to understand the motivation for quantum harmonic analysis, we turn to the well-known theory of harmonic analysis of functions. While harmonic analysis is a vast field of mathematics today, we will mainly be concerned with the circle of ideas and results going back to Wiener's work, as outlined in [253]. This means that we will study properties of convolutions and Fourier transforms of functions. Let us recall that the Fourier transform of a function f on $\mathbb{R}^d$ for some $d \in \mathbb{N}$ is the function $\mathcal{F}(f)$ on $\mathbb{R}^d$ given by

$$\mathcal{F}(f)(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} \, dx,$$

and let T_x for $x \in \mathbb{R}^d$ denote the translation operator which acts on a function f by

$$T_x f(t) = f(t-x) \quad \text{for } t \in \mathbb{R}^d.$$

One of the main results of Wiener is then the following *approximation theorem.*

Theorem (Wiener's approximation theorem)**.** *Let* $f \in L^1(\mathbb{R}^d)$*. The translates* $\{T_x f\}_{x\in\mathbb{R}^d}$ *span a dense subspace (in the norm topology) of* $L^1(\mathbb{R}^d)$ *if and only if* $\mathcal{F}(f)$ *has no zeros.*

One of the aspects of the approximation theorem that makes it rather deep, is the fact that density is in the norm of $L^1(\mathbb{R}^d)$. In fact, there is another version of this result for $L^2(\mathbb{R}^d)$, the proof of which is almost trivial since $\mathcal{F}$ is an isomorphism on $L^2(\mathbb{R}^d)$ by the celebrated Plancherel theorem. Such a trivial proof is not possible when we work with $L^1(\mathbb{R}^d)$, since $\mathcal{F}$ maps $L^1(\mathbb{R}^d)$ into a very different space of functions. This is a phenomenon that will reappear when we start looking at quantum harmonic analysis.

An important consequence of the approximation theorem is Wiener's celebrated Tauberian theorem. To state this result, we need the convolution $f * g$ of $f, g \in L^1(\mathbb{R}^d)$ defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(t)g(x-t)\,dt \qquad \text{for } x \in \mathbb{R}^d.$$

In this context, the intuition to keep in mind when interpreting $f * g$ is, for $d = 1$, that g is a real-valued function supported in a small interval $[-a, a]$. Then

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \\ &= \int_{x-a}^{x+a} f(t)g(x-t)\,dt, \end{aligned}$$

which shows that the value of $f * g(x)$ is obtained as a weighted average of the values of $f(t)$ for $t \in [x-a, x+a]$, where g determines the weights. The relevant interpretation of the term *Tauberian theorems* in this setting is a theorem that allows deductions about a sequence/function to be made based on properties of a weighted average of the sequence/function. If we interpret convolutions as weighted averages, the name of the following seminal theorem is indeed quite fitting.

Theorem (Wiener-Pitt Tauberian theorem)**.** *Suppose* $f \in L^\infty(\mathbb{R}^d)$ *and* $h \in L^1(\mathbb{R}^d)$ *such that* $\mathcal{F}(h)$ *has no zeros. Then the following implication holds for* $A \in \mathbb{C}$*: if*

$$\lim_{|x|\to\infty} (f * h)(x) = A \int_{\mathbb{R}^d} h(y)\,dy, \tag{1.1.1}$$

then for any $g \in L^1(\mathbb{R}^d)$ *we have*

$$\lim_{|x|\to\infty} (f * g)(x) = A \int_{\mathbb{R}^d} g(y)\,dy.$$

Furthermore, if f *is slowly oscillating, then* (1.1.1) *implies that* $\lim_{|x|\to\infty} f(x) = A$.

The Tauberian aspect of theorem is perhaps clearest in the final sentence, which is due to Pitt: it says that for slowly oscillating functions f, we can deduce the behaviour of f as $|x| \to \infty$ from the behaviour of the weighted average $f * h$. Of course, slowly oscillating has a precise meaning, and the eager reader may skip ahead to Paper E for the definition.

Before moving on to quantum harmonic analysis, let us recall another classic result on *Fourier series* by Wiener. We refer to it as Wiener's lemma, as Wiener used it as a lemma to prove the approximation theorem above.

Theorem (Wiener's lemma)**.** *Let $c = \{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, and consider the associated absolutely convergent Fourier series*

$$\hat{c}(x) := \sum_{n\in\mathbb{Z}} c_n e^{2\pi i x n} \quad \text{for } x \in [0,1].$$

If $\hat{c}$ is invertible, meaning that it vanishes nowhere, then the inverse $1/\hat{c}$ is also an absolutely convergent Fourier series $1/\hat{c} = \hat{d}$ for some $d \in \ell^1(\mathbb{Z})$.

As we will soon be studying a version of harmonic analysis for operators, it is worth noting that whereas Wiener's original proof for this result was rather complicated, a slick operator-theory flavoured proof due to Gelfand is today a standard first example in textbooks on Gelfand's theory of Banach algebras. This is an early example of methods with roots in operator theory illuminating harmonic analysis, and many of the results in this thesis fall into a similar category.

The three theorems of Wiener given above have had profound consequences both within harmonic analysis and in other areas — the Tauberian theorem can for instance be used to give a proof of the prime number theorem. We should also mention that formulating the first two theorems for functions and the third for sequences might be somewhat misleading: there are approximation and Tauberian theorems for sequences and a version of Wiener's lemma for functions. In fact, Weil [250] observed that this circle of ideas can be formulated in the abstract context of locally compact abelian groups. Picking the groups $\mathbb{R}^d$ and $\mathbb{Z}^d$ gives results for functions and sequences, respectively. We bring this up because the assumption that the group is abelian is crucial for the theory to work in a straightforward way. For non-abelian groups, the theory of Fourier transforms needs a very abstract formulation, and if the group is not reasonably nice (the technical term is that the group is of type 1), the Fourier theory can become quite horrible. Paper G partly concerns the Fourier analysis on the affine group, which is not abelian yet luckily of type 1.

1.2 Quantum harmonic analysis

We now turn to quantum harmonic analysis. Proofs of many of these results can be found in [203], which is not included in this doctoral thesis as some of the results were already contained in the author's master's thesis. Unlike the classical results of Wiener mentioned above, which only deal with functions, quantum harmonic analysis extends the classical theory to include operators on a Hilbert space. If we look back to the previous section to see what objects we defined there, it is quite clear what we need to extend the theory to include operators:

1. A Banach space of operators that can play the role of $L^1(\mathbb{R}^d)$.
2. A way to translate operators, as an analogue of the translation operator T_x for functions.
3. A convolution operation for operators.
4. A Fourier transform that acts on operators.

The list above leads to more new questions than it answers: Operators on which Hilbert space? What should a "Fourier transform" for operators be? We translate functions by points $x \in \mathbb{R}^d$, but what should we translate operators by? The correct answers to these questions were given by Werner in a seminal paper [251] from 1984. However, before we move to explaining the "how" of quantum harmonic analysis, it is worth dwelling for a moment on the "why": what is there to gain by including operators into the theory?

From the perspective of pure mathematics, this extension is valuable for the simple reason that it works: it is possible to define all the objects listed above, in such a way that results like Wiener's approximation theorem still hold. It is quite remarkable that the theory for operators works in essentially the same way as for functions, even though operator composition is not commutative — unlike function multiplication.

Furthermore, for certain special cases – for instance by picking operators of a specific form — the objects in quantum harmonic analysis become well-studied objects from mathematical physics and time-frequency analysis. The analogues of Wiener's theorems in quantum harmonic analysis then give results for these well-known objects. Some of these results will be new in mathematical physics and time-frequency analysis, others will be familiar. But even when the result is familiar, quantum harmonic analysis might still offer a more natural and convenient expression of the result, and it also comes with a wealth of intuition from harmonic analysis. A familiar, technical result in time-frequency analysis might suddenly express a natural property of convolutions and Fourier transforms, when the convolutions and Fourier transforms are those of *quantum* harmonic analysis.

1.2.1 The basic definitions and results

The basic objects of quantum harmonic analysis are functions f on phase space $\mathbb{R}^{2d}$ and operators $A \in \mathcal{L}(L^2)$, where $\mathcal{L}(L^2)$ always denotes the Banach space of bounded, linear operators on the Hilbert space $L^2(\mathbb{R}^d)$.

Schatten ideals of bounded operators

We start by considering the space of operators that will play the role of $L^1(\mathbb{R}^d)$. As many authors have realized, a natural candidate is the space $\mathcal{S}^1$ of *trace class operators* on $L^2(\mathbb{R}^d)$. For any *positive* bounded, linear operator A on $L^2(\mathbb{R}^d)$, the *trace* of A is the number

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_{L^2}, \tag{1.2.1}$$

where $\{e_n\}_{n=1}^{\infty}$ is any orthonormal basis for $L^2(\mathbb{R}^d)$. We think of the trace as an operator-analogue of the integral of a function, and with this in mind the analogue of $L^1(\mathbb{R}^d)$ is clearly the set $\mathcal{S}^1$ of operators S satisfying $\operatorname{tr}(|S|) < \infty$, with $|S|$ the positive part in the polar decomposition of S. It is well-known that $\mathcal{S}^1$ becomes a Banach space of compact operators with the norm $\|S\|_{\mathcal{S}^1} = \operatorname{tr}(|S|)$, and that the trace extends to a bounded linear functional on all of $\mathcal{S}^1$ by (1.2.1), even for non-positive operators.

As we will see in this thesis, it is possible to define analogues $\mathcal{S}^p$ of all the $L^p(\mathbb{R}^d)$ spaces for $1 \leq p \leq \infty$. For this introduction, we will need one more of these spaces, namely the analogue $\mathcal{S}^2$ of $L^2(\mathbb{R}^d)$: the *Hilbert-Schmidt operators*. The space $\mathcal{S}^2$ consists of those bounded operators T on $L^2(\mathbb{R}^d)$ such that T^*T is a trace class operator. Just like $L^2(\mathbb{R}^d)$, $\mathcal{S}^2$ stands out from the other $\mathcal{S}^p$-spaces by being a Hilbert space when given the inner product

$$\langle S, T \rangle_{\mathcal{S}^2} := \operatorname{tr}(ST^*).$$

Proving results in quantum harmonic analysis for $\mathcal{S}^2$ is often significantly easier than for $\mathcal{S}^1$, because the Fourier transform for operators that we will define turns out to be unitary from $\mathcal{S}^2$ to $L^2(\mathbb{R}^{2d})$, allowing us to work with functions rather than operators. A similar trick is not available for $\mathcal{S}^1$ — for instance, there is no natural bijection from $\mathcal{S}^1$ to $L^1(\mathbb{R}^{2d})$ — so it will often take some effort to deduce results for $\mathcal{S}^1$. This situation is reminiscent of what we saw for functions. As mentioned, proving Wiener's approximation theorem for $L^2(\mathbb{R}^d)$ is almost trivial, while the result for $L^1(\mathbb{R}^d)$ is much deeper.

Translations and convolutions of operators

In order to define a translation for operators, we first need to define the *time-frequency shifts* $\pi(z) \in \mathcal{L}(L^2)$, which for $\psi \in L^2(\mathbb{R}^d)$, $z = (x, \omega) \in \mathbb{R}^{2d}$ are given by

$$\pi(z)\psi(t) = M_\omega T_x \psi(t) = e^{2\pi i \omega \cdot t}\psi(t - x).$$

Here $M_\omega \in \mathcal{L}(L^2)$ denotes the *modulation operator* $M_\omega \psi(t) = e^{2\pi i \omega \cdot t}\psi(t)$, and we have already met the translation operator T_x in the previous section. As we will see later, the time-frequency shifts play a fundamental role in time-frequency analysis, which is where we borrow the terminology from — a physicist would be more inclined to call $\pi(z)$ Weyl operators or the Schrödinger representation. Using the time-frequency shifts, we can define the translation $\alpha_z(S)$ of an operator $S \in \mathcal{L}(L^2)$ by $z \in \mathbb{R}^{2d}$ to be

$$\alpha_z(S) := \pi(z) S \pi(z)^*.$$

The reader wondering why this is a sensible definition of a translation of operators may have a look at Remark 2.1 in Section 2.2.2.

To motivate the definition of the convolutions in quantum harmonic analysis, note that we can rewrite the convolution of $f, g \in L^1(\mathbb{R}^{2d})$ in two ways, namely

$$f * g = \int_{\mathbb{R}^{2d}} f(z) T_z g \, dz,$$
$$f * g(z) = \int_{\mathbb{R}^{2d}} f(z') T_z \check{g}(z') \, dz',$$

where $\check{g}(t) = g(-t)$ and the first integral must be interpreted as a Bochner integral of an integrand taking values in $L^1(\mathbb{R}^{2d})$.

There are two new convolution operations in quantum harmonic analysis. First, we wish to define the convolution $f \star S$ of $f \in L^1(\mathbb{R}^{2d})$ with $S \in \mathcal{S}^1$. To define this, consider the first expression for $f * g$ above. If we replace g by S and T_z by α_z, we end up with

$$f \star S := \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S) \, dz,$$

which is a Bochner integral converging in $\mathcal{S}^1$. In particular, the reader should note that *the convolution of a function with an operator is an operator*. We also define $S \star f := f \star S$. Then, we wish to define the convolution $S \star T$ of two operators $S, T \in \mathcal{S}^1$. For this, we turn to the second expression for $f * g$ above. If we replace f by S, $\check{g}$ by $\check{T}$, T_z by α_z and the integral by a trace, we get

$$S \star T(z) := \operatorname{tr}(S \alpha_z(\check{T})).$$

Of course, $\check{T}$ is more than a formal notation: it is given by $\check{T} = PTP$, where P is the *parity operator* $P\psi(t) = \psi(-t)$. The reader should note that *the convolution of two operators is a function on* $\mathbb{R}^{2d}$.

Remark 1.1. The alert reader will note that we use $\star$ to denote both of these new convolutions. The correct definition can always be deduced from the context.

Calling these new operations convolutions immediately raises a question: do they behave at all like the familiar convolutions of functions? The short answer is that they do, as shown by Werner [251]. One lemma of fundamental importance concerns the integrability of functions $S \star T$.

Lemma 1.2.1. *Let* $S, T \in \mathcal{S}^1$. *Then* $S \star T \in L^1(\mathbb{R}^{2d})$ *with*

$$\int_{\mathbb{R}^{2d}} S \star T(z)\, dz = \mathrm{tr}(S)\mathrm{tr}(T).$$

This is of course an analogue of the fact from harmonic analysis that $f * g \in L^1(\mathbb{R}^d)$ if $f, g \in L^1(\mathbb{R}^d)$, with

$$\int_{\mathbb{R}^d} f * g(x)\, dx = \left(\int_{\mathbb{R}^d} f(x)\, dx\right)\left(\int_{\mathbb{R}^d} g(x)\, dx\right).$$

On the other hand, if we choose S and T to be rank-one operators, one can show that the lemma contains as a special case Moyal's identity (see Section 2.1).

Proposition 1.2.2. *The convolutions introduced above are commutative. Furthermore, they are associative. More precisely, for* $f, g \in L^1(\mathbb{R}^{2d})$ *and* $R, S, T \in \mathcal{S}^1$

$$(f * g) \star S = f * (g \star S)$$
$$f * (S \star T) = (f \star S) \star T$$
$$(R \star S) \star T = R \star (S \star T).$$

This last proposition is deceptively easy to formulate and prove, but it contains many noteworthy aspects.

Remark 1.2.

1. The commutativity of the convolution of two operators, i.e. $S \star T = T \star S$, highlights a very useful property of the trace: it satisfies $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for $A, B \in \mathcal{S}^1$. So even though $S \star T$ is defined in terms of the non-commutative product in $\mathcal{L}(L^2)$, the trace still ensures that the convolutions are commutative.

2. The associativity conditions show that three different convolution operations are compatible, and one of these is the usual convolution $*$ of functions. This suggests that quantum harmonic analysis is not merely an analogue of harmonic analysis, but an *extension*. This perspective will reappear when we consider the quantum harmonic analysis version of Wiener's Tauberian theorem.

3. The last of the associativity conditions has the most involved proof, and it might break if one tweaks the setup in different ways, see Papers C and G.

4. If one writes out some of the associativity conditions for specific kinds of operators or functions, one recovers various known results from the literature, see for instance [109, Prop. 3.10] and [8, Lem. 4.1]. Quantum harmonic analysis allows us to give a simple and illuminating proof of these results as simply associativity of convolutions, whereas the original statements and proofs tend to be rather technical.

Other results for convolutions of functions also generalize to quantum harmonic analysis. An important example is the fact that the domains of convolutions can be extended to other L^p and $\mathcal{S}^p$-spaces, giving an analogue of Young's inequality as stated in Paper A. However, perhaps the most useful property of convolution of functions is its interaction with the Fourier transform. It is therefore time to introduce the Fourier transforms in quantum harmonic analysis.

1.2.2 The Fourier-Wigner transform

As the Fourier transform of an operator $S \in \mathcal{S}^1$, we will use the *Fourier-Wigner transform* $\mathcal{F}_W(S)$, which defines a function on $\mathbb{R}^{2d}$ by

$$\mathcal{F}_W(S)(z) := e^{-\pi i x\cdot\omega}\mathrm{tr}(S\pi(-z)) \quad \text{for } z = (x,\omega).$$

In particular, the Fourier transform of an operator is a function on phase space $\mathbb{R}^{2d}$. To motivate the definition, recall that for $f \in L^1(\mathbb{R}^{2d})$

$$\mathcal{F}(f)(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i z\cdot z'}\,dz'.$$

If we replace $f(z')$ by S, the integral by a trace and $e^{2\pi i z\cdot z'}$ by $e^{-\pi i x\cdot\omega}\pi(z)$, we obtain the definition of $\mathcal{F}_W(S)$. The definition is therefore natural if we use $E(z) = e^{-\pi i x\cdot\omega}\pi(z)$ as an analogue of the characters $\chi_z(z') = e^{2\pi i z\cdot z'}$ — we will solidify this analogue using the so-called Weyl transform soon. Let us also mention that the definition of $\mathcal{F}_W$ is essentially the inverse of the group Fourier transform for the (non-abelian) Heisenberg group.

As one would hope, $\mathcal{F}_W$ shares many properties of the usual Fourier transform. For $S \in \mathcal{S}^1$ there is an analogue of the Riemann-Lebesgue lemma, which says that $\mathcal{F}_W(S)$ belongs to the space $C_0(\mathbb{R}^{2d})$ of continuous functions vanishing at infinity. The Fourier-Wigner transform also extends to a unitary map $\mathcal{F}_W : \mathcal{S}^2 \to L^2(\mathbb{R}^{2d})$, but the its main property in quantum harmonic analysis is the following.

Proposition 1.2.3. *Let $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{S}^1$. Then*

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$$
$$\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T),$$

where $\mathcal{F}_\sigma$ denotes the symplectic Fourier transform given by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i \sigma(z,z')}\, dz'$$

and σ is the symplectic form $\sigma(z, z') = \omega \cdot x' - \omega' \cdot x$ of $z = (x, \omega)$ and $z' = (x', \omega')$.

In words, the Fourier transform of a convolution is the product of Fourier transforms, at least as long as we use the symplectic Fourier transform as the Fourier transform of functions. This last point is of little consequence, as $\mathcal{F}_\sigma$ shares all relevant properties of the usual Fourier transform $\mathcal{F}$. In fact, $\mathcal{F}_\sigma$ appears because phase space $\mathbb{R}^{2d}$ is more correctly written as $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, where $\widehat{\mathbb{R}}$ denotes the dual group. This technicality means that results are typically easier to state using the symplectic Fourier transform, as we will see many times in this thesis.

These facts are all we need to state analogues of Wiener's theorems for quantum harmonic analysis, but before we do so it is worth dwelling for a moment on another property of the Fourier-Wigner transform: the Hausdorff-Young inequality. There is indeed a Hausdorff-Young inequality in this setting, saying that if $S \in \mathcal{S}^p$ for $1 \leq p \leq 2$, then $\mathcal{F}_W(S) \in L^q(\mathbb{R}^{2d})$ for $\frac{1}{p} + \frac{1}{q} = 1$ with

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \|S\|_{\mathcal{S}^p}.$$

Just as the Hausdorff-Young inequality for functions, this inequality is known to be unsharp. However, whereas the constants making the usual Hausdorff-Young inequality sharp have been known for 45 years, the sharp constants in quantum harmonic analysis is still an open question,[1] as recently explored from the perspective of group Fourier transforms on the Heisenberg group by Cowling et al. [66]. We mention this to emphasize that obtaining analogues of known results in quantum harmonic analysis is not necessarily a simple matter.

1.2.3 Wiener's theorems in quantum harmonic analysis

The analogues of Wiener's theorems in quantum harmonic analysis have been proved by different authors over a time span of 35 years. The first of these, the approximation theorem, was proved already by Werner [251] in 1984, in the paper introducing quantum harmonic analysis.

[1]We were able to make some progress on this problem in [203], by showing that a sharp version for trace class operators follows from an inequality due to Lieb [197].

Theorem (Wiener's approximation theorem, QHA)**.** *Let $S \in \mathcal{S}^1$. The translates $\{\alpha_z(S)\}_{z\in\mathbb{R}^{2d}}$ span a dense subspace (in the norm topology) of $\mathcal{S}^1$ if and only if $\mathcal{F}_W(S)$ has no zeros.*

The proof of this result is not independent of Wiener's original approximation theorem, but exploits the fact that quantum harmonic analysis combines functions and operators. This makes it possible to move from the operator setting to the function setting. As an example, $\mathcal{F}_W(S)$ for $S \in \mathcal{S}^1$ has no zeros if and only if the function $\mathcal{F}_\sigma(S \star S) = \mathcal{F}_W(S)^2$ has no zeros.

An important consequence, also noted by Werner, is a result on injectivity and dense ranges of convolutions. The proof is simply a bit of functional analysis.

Corollary 1.2.3.1. *Let $S \in \mathcal{S}^1$. The following are equivalent:*

1. *$\mathcal{F}_W(S)$ has no zeros.*
2. *If $A \in \mathcal{L}(L^2)$ and $A \star S = 0$, then $A = 0$.*
3. *$L^1(\mathbb{R}^{2d}) \star S$ is dense in $\mathcal{S}^1$.*

In its current formulation, the corollary has a simple formulation in terms of convolutions, but is also quite abstract. When we restrict to special classes of operators in Section 3, we will see that the statements get a more concrete form and often reduce to questions previously studied in the literature.

Wiener proved his approximation theorem in order to prove a Tauberian theorem. In quantum harmonic analysis, it would take 35 years before the approximation theorem was used to obtain a Tauberian theorem. This Tauberian theorem and its consequences form the content of Paper E. The long time span from Werner's work to Paper E is not due to any big difficulties in deducing the Tauberian theorem from the approximation theorem — it is not unreasonable to assume that Werner would have included a Tauberian theorem in [251] if he believed that it would be of interest for the community. Rather, it is a consequence of the fact that Werner's theory got little attention besides a few papers in mathematical physics for many years. The aim of Paper E is therefore not just to prove a Tauberian theorem, but also to argue that the developments in time-frequency analysis, Toeplitz operator theory and mathematical physics after Werner's original paper have shown that the questions answered by the Tauberian theorems are of interest. The statement below uses I_{L^2} to denote the identity operator on $L^2(\mathbb{R}^d)$.

Theorem (Wiener Tauberian theorem for a bounded function, QHA)**.** *Let $f \in L^\infty(\mathbb{R}^{2d})$, and assume that one of the following equivalent statements holds for some $A \in \mathbb{C}$:*

(i) There is some $S \in \mathcal{S}^1$ such that $\mathcal{F}_W(S)$ has no zeros and

$$f \star S = A \cdot \mathrm{tr}(S) \cdot I_{L^2} + K$$

for some compact operator K on $L^2(\mathbb{R}^d)$.

(ii) There is some $a \in L^1(\mathbb{R}^{2d})$ such that $\mathcal{F}(a)$ has no zeros and

$$f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h$$

for some $h \in C_0(\mathbb{R}^{2d})$.

Then both of the following statements hold:

1. *For any $T \in \mathcal{S}^1$, $f \star T = A \cdot \mathrm{tr}(T) \cdot I_{L^2} + K_T$ for some compact operator K_T.*
2. *For any $g \in L^1(\mathbb{R}^{2d})$, $f * g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\, dz + h_g$ for some $h_g \in C_0(\mathbb{R}^{2d})$.*

The statement of this Tauberian theorem is long, and still only tells half the story: there is a similar version where $f \in L^\infty(\mathbb{R}^{2d})$ is replaced by $R \in \mathcal{L}(L^2)$, which the reader can find in Paper E. To make the theorem more digestible, we start by noting that the implication $(ii) \implies (2)$ is just Wiener's original Tauberian theorem. Hence the new theorem *extends* the old theorem. Then note that the function statements in (ii) and (2) concern functions that are constants, apart from a perturbation that vanishes at infinity. The operator statements in (i) and (1) concern operators that are constants (times the identity operator), apart from a compact perturbation. Statements (i), (1) are therefore natural analogues of (ii), (2) if we employ the widely used intuition that compact perturbations of an operator do not affect the asymptotics of an operator — the compact operators are, in a sense, the operator analogue of functions vanishing at infinity.

As we have seen, Pitt improved Wiener's Tauberian theorem by showing that if f is slowly oscillating, (ii) implies that $\lim_{|x|\to\infty} f(x) = A$. An important question in Paper E is the analogue of this statement (for $A = 0$) when $f \in L^\infty(\mathbb{R}^{2d})$ is replaced by $R \in \mathcal{L}(L^2)$: under what assumptions on R does $R \star S \in C_0(\mathbb{R}^{2d})$ for $S \in \mathcal{S}^1$ such that $\mathcal{F}_W(S)$ has no zeros imply that R is compact? The answer turns out to be deeply connected to a question about compactness of Toeplitz operators on the so-called Bargmann-Fock space. This question has previously received attention by many researchers in various contexts, see [16, 24, 91, 117].

Periodic operators and Fourier series of operators

Now only one of Wiener's theorems from Section 1.1 remains: Wiener's lemma on Fourier series. Based on our discussion so far, we have a very clear candidate for what a Fourier series of operators should look like: since we claimed that $E(z) = e^{\pi i x\cdot\omega}\pi(z)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$ is the operator analogue of $\chi_z(z') = e^{2\pi i z\cdot z'}$, a Fourier series expansion of an operator should intuitively be

$$\sum_{m,n\in\mathbb{Z}^d} c_{m,n}E(m,n) = \sum_{m,n\in\mathbb{Z}^d} c_{m,n}e^{\pi i m\cdot n}\pi(m,n) \tag{1.2.2}$$

for some sequence $\{c_{m,n}\}_{m,n\in\mathbb{Z}^d}$. This notion turns out to be reasonable, and the corresponding Wiener's lemma was proved in 2004 by Gröchenig and Leinert [140].

Theorem 1.2.4 (Wiener's lemma, QHA)**.** *Let $a, b > 0$ and $S \in \mathcal{L}(L^2)$ of the form*

$$S = \sum_{m,n\in\mathbb{Z}^d} c_{m,n}e^{\pi i(am)\cdot(bn)}\pi(am, bn) \tag{1.2.3}$$

for $\{c_{m,n}\}_{m,n\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^{2d})$. If S is an invertible operator, then there exists $\{d_{m,n}\}_{m,n\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$S^{-1} = \sum_{m,n\in\mathbb{Z}^d} d_{m,n}e^{\pi i(am)\cdot(bn)}\pi(am, bn).$$

Remark 1.3. The reader will note that we have added a couple of parameters a, b. This is not just a trivial extension of considering $a = b = 1$ as in (1.2.2), because the structure of operators given by Fourier series depends heavily on the values of a and b. For instance, if $a = b = 1$, then two operators given by Fourier series will commute, as one easily shows that $\pi(m, n)$ and $\pi(m', n')$ commute for $m, n, m', n' \in \mathbb{Z}^d$. In fact, if we let $\mathcal{A}_{ab}$ be the closure in $\mathcal{L}(L^2)$ of operators of the form (1.2.3) for fixed a, b, then $\mathcal{A}_{ab}$ is the *noncommutative torus with parameter* $\theta = ab$. The first systematic study of noncommutative tori was undertaken by Rieffel [222], who showed that the structure of $\mathcal{A}_{ab}$ as a C^*-algebra is very sensitive to the value of $\theta = ab$. One can also consider more general lattices in $\mathbb{R}^{2d}$, not necessarily of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$, which we will do in Paper D.

Although this result is clearly an analogue of Wiener's lemma, its proof is very far from being a simple translation of Wiener's original result — this should not be too surprising, as invertibility of a Fourier series $\hat{a}$ simply means that $\hat{a}$ vanishes nowhere, while invertibility of an operator is significantly more complicated. In fact, Gröchenig and Leinert did not reference quantum harmonic analysis when proving the result; they proved it to solve an open problem in Gabor analysis on the quality of Gabor frame generators and their dual windows — see Section 2.4 .

The notion of Fourier series of operators above fits well with the theory of periodic operators developed by Feichtinger and Kozek [102]. An operator $S \in \mathcal{L}(L^2)$ is said to be *periodic with respect to* $a\mathbb{Z}^d \times b\mathbb{Z}^d$ if for any $(m, n) \in \mathbb{Z}^{2d}$

we have $\alpha_{(am,bn)}(S) = S$. For instance, one finds that if S is periodic with respect to $b^{-1}\mathbb{Z}^d \times a^{-1}\mathbb{Z}^d$, then S does indeed have an expansion of the form (1.2.3). Part of Paper D concerns revisiting the work of Feichtinger and Kozek from the perspective of quantum harmonic analysis, adding some details to their proofs and extending their results to apply to trace class operators. As usual, many statements are not too difficult to prove for some operators, but extending the results to trace class operators requires some more technical results. In Paper D, these results are then used to deduce properties of a generalization of Gabor frames that we call *Gabor g-frames*.

This is the end of our brief tour of quantum harmonic analysis, which has covered both the background material from our paper [203], not included in the thesis, and hinted at some of the ways in which this thesis extends quantum harmonic analysis. This survey is far from exhaustive, covering neither all background material nor all the directions this thesis will extend quantum harmonic analysis. Notable omissions are the so far unmentioned Paper C, which will combine quantum harmonic analysis with harmonic analysis of lattices, and Paper G which considers the extension of quantum harmonic analysis to a non-unimodular group. We will now turn our attention to the less abstract realm of time-frequency analysis, which also forms an integral part of this thesis. It will allow us to give more concrete examples of the objects we have looked at in quantum harmonic analysis, and also to see how the abstract theorems of quantum harmonic analysis have valuable consequences in this setting.

Chapter 2

Time-frequency analysis

We will now give an introduction to relevant aspects of the field of time-frequency analysis. For more complete treatments we refer to the monographs [57, 113, 131].

2.1 Time-frequency plane, short-time Fourier transform and uncertainty principle

At its core, time-frequency analysis deals with functions ψ on $\mathbb{R}^d$. At least for $d = 1$, the variable in $\mathbb{R}^d$ is thought of as time t, so that the functions ψ are considered to be time-dependent signals. The Fourier transform $\mathcal{F}(\psi)$ is then the frequency distribution of ψ: the size of $|\mathcal{F}(\psi)(\omega)|^2$ shows how much the frequency ω contributes to the energy of the signal ψ. Perhaps the most fundamental objects in time-frequency analysis are the previously introduced time-frequency shifts $\pi(z) = M_\omega T_x$ for $z = (x, \omega)$. The reason for their name is now clear: $\pi(z)$ is the composition of T_x, which shifts the signal ψ in time, and M_ω, which shifts the frequency distribution of the signal since $\mathcal{F}(M_\omega \psi)(\xi) = \mathcal{F}(\psi)(\xi - \omega)$. We will often refer to $\mathbb{R}^{2d}$ as the time-frequency plane, as we think of its elements as $z = (x, \omega)$ for a time x and a frequency ω.

By considering both ψ and its Fourier transform $\mathcal{F}(\psi)$, we can study both the time and the frequency behaviour of a signal ψ. Unfortunately, even when $|\mathcal{F}(\psi)(\omega)|$ is big, this gives no clue about *when* (i.e. for which t) the frequency ω contributes to the energy of the signal. One goal of time-frequency analysis is therefore the construction of *time-frequency distributions*: maps Q sending functions ψ on $\mathbb{R}^d$ to functions $Q(\psi)$ on the time-frequency plane $\mathbb{R}^{2d}$ such that $|Q(\psi)(x, \omega)|$ can be interpreted as the contribution of the frequency ω at the time x to the energy of the signal ψ.

The hope of finding such an ideal time-frequency distribution is severely limited by a fundamental metatheorem in time-frequency analysis: the *uncertainty principle*. Put informally, the uncertainty principle states that

> a function $\varphi \in L^2(\mathbb{R}^d)$ cannot be arbitrarily well concentrated in both time and frequency.

An important consequence of the uncertainty principle is that the concept of instantaneous frequency will never make sense, in other words, we can never say that the signal φ has the frequency ω at the time t. It is therefore unrealistic to hope for a time-frequency distribution Q such that $Q(\psi)(x,\omega)$ gives precisely the contribution of the frequency ω at the time x to the energy of the signal ψ. However, even though this interpretation of $Q(\psi)(x,\omega)$ is (at best) only approximately true, it will still serve as a useful guide for us.

One of the most common time-frequency distributions, which will appear throughout this thesis, is the *short-time Fourier transform* (STFT). Given $\psi, \varphi \in L^2(\mathbb{R}^d)$, the STFT of ψ with window φ is the function $V_\varphi\psi$ on $\mathbb{R}^{2d}$ given by

$$V_\varphi\psi(z) := \langle \psi, \pi(z)\varphi\rangle_{L^2} = \int_{\mathbb{R}^d} \psi(t)e^{-2\pi i\omega\cdot t}\overline{\varphi(t-x)}\,dt.$$

Its motivation is quite simple. The window φ should be picked such that $|\varphi(t)|$ is negligible outside a small neighbourhood of $t = 0$, and $|\mathcal{F}(\varphi)(\omega)|$ is negligible outside a small neighbourhood of $\omega = 0$. We may then think of φ as a *time-frequency atom* concentrated near $(0,0)$ in the time-frequency plane $\mathbb{R}^{2d}$. It follows that $\pi(z)\varphi$ is a time-frequency atom concentrated near z in the time-frequency plane. To measure the contribution to a signal ψ of the frequency ω at time x, we simply project ψ onto our time-frequency atom $\pi(x,\omega)\varphi$, i.e. consider $V_\varphi\psi(z)$. A typical choice for the window φ is the normalized Gaussian

$$\varphi_0(t) = 2^{d/4}e^{-\pi t\cdot t}.$$

If $Q(\psi)(x,\omega)$ is to represent the contribution of frequency ω at time x to the energy of ψ, then $\int_{\mathbb{R}^{2d}} Q(\psi)(x,\omega)\,dx\,d\omega$ should represent the total energy $\|\psi\|_{L^2}^2$ of ψ. We get an important example of such a distribution Q if we square the modulus of the STFT for a normalized window φ, which means that we consider $Q(\psi) = |V_\varphi\psi|^2$, often called the *spectrogram* of ψ with window φ. This fact, known as Moyal's identity, is one of the most fundamental results in time-frequency analysis.

Theorem (Moyal's identity)**.** *Let* $\psi, \varphi, \xi, \eta \in L^2(\mathbb{R}^d)$*. Then*

$$\int_{\mathbb{R}^{2d}} V_\varphi\psi(z)\overline{V_\eta\xi}(z)\,dz = \langle \psi, \xi\rangle_{L^2}\langle \eta, \varphi\rangle_{L^2}.$$

In particular, if $\|\varphi\|_{L^2} = 1$*, then*

$$\int_{\mathbb{R}^{2d}} |V_\varphi \psi(z)|^2 \, dz = \|\psi\|_{L^2}^2.$$

2.2 Operators and time-frequency distributions

As we have now introduced both the basic building blocks of time-frequency analysis and the ever-present influence of the uncertainty principle, we can start to introduce the objects from time-frequency analysis that will be studied in detail in the thesis: localization operators, Weyl transforms and Cohen's class of time-frequency distributions. Each concept will be introduced on its own terms, but we will soon see that connections between these concepts abound.

2.2.1 Localization operators

A straightforward consequence of Moyal's identity is a reconstruction formula: if $\|\varphi\|_{L^2} = 1$ and we let $V_\varphi^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$ denote the operator

$$V_\varphi^*(F) = \int_{\mathbb{R}^{2d}} F(z)\pi(z)\varphi \, dz,$$

where the integral is interpreted weakly, then V_φ^* is the adjoint of the map $V_\varphi : \psi \mapsto V_\varphi \psi$, and $V_\varphi^* \circ V_\varphi = I_{L^2}$. In detail,

$$\psi = \int_{\mathbb{R}^{2d}} V_\varphi \psi(z)\pi(z)\varphi \, dz,$$

where the integral converges weakly. In light of our earlier interpretations, this result is quite reasonable: we considered $\pi(x, \omega)\varphi$ to be a time-frequency atom localized near (x, ω) in the time-frequency plane $\mathbb{R}^{2d}$, and the reconstruction formula says that we can synthesize our signal from all the time-frequency atoms $\pi(x, \omega)\varphi$ if we weight $\pi(x, \omega)\varphi$ with the contribution $V_\varphi \psi(x, \omega)$ of the frequency ω at time x.

Using the reconstruction formula, we can construct a class of operators called *time-frequency localization operators* in a natural way. Given a signal $\psi \in L^2(\mathbb{R}^d)$, we can represent it in the time-frequency plane by forming $V_\varphi \psi$. But instead of using the reconstruction formula to recover ψ, we modify $V_\varphi \psi$ by multiplying it with a function $m \in L^\infty(\mathbb{R}^{2d})$. We then apply V_φ^* to $m \cdot V_\varphi \psi$ — in total we have defined the time-frequency localization operator $\mathcal{A}_m^\varphi$ acting on $\psi \in L^2(\mathbb{R}^d)$ by

$$\mathcal{A}_m^\varphi(\psi) = V_\varphi^*(m V_\varphi \psi) = \int_{\mathbb{R}^{2d}} m(z) V_\varphi \psi(z) \pi(z)\varphi \, dz.$$

Since $\mathcal{A}_m^\varphi(\psi)$ is obtained by integrating up the time-frequency atoms $\pi(z)\varphi$ weighted by $m(z)V_\varphi\psi(z)$, our intuition suggests that

$$V_\varphi\left(\mathcal{A}_m^\varphi(\psi)\right)(z) = m(z)V_\varphi\psi(z).$$

This will not be true. For one thing, the formulas do not work out, but more fundamentally it would contradict the uncertainty principle: if it were true, we could pick m supported in an arbitrarily small ball around 0 in $\mathbb{R}^{2d}$, so interpreting the STFT as a time-frequency distribution we get that $\mathcal{A}_m^\varphi(\psi)$ would be supported in an arbitrarily small subset of the time-frequency plane. Nevertheless, the intuition that $\mathcal{A}_m^\varphi(\psi)$ behaves like $m(z)V_\varphi\psi(z)$ in the time-frequency plane is often useful. Often, m is picked to be a characteristic function χ_Ω for some measurable $\Omega \subset \mathbb{R}^{2d}$, and one then writes $\mathcal{A}_\Omega^\varphi$ for $\mathcal{A}_{\chi_\Omega}^\varphi$. The intuition then suggests that $\mathcal{A}_\Omega^\varphi(\psi)$ picks out the part of ψ "living in the region Ω in the time-frequency plane." For instance, if $d = 1$ and $\Omega = \mathbb{R} \times [-5, 5]$, then $\mathcal{A}_\Omega^\varphi$ is a kind of band-pass filter, resynthesizing a signal using only frequencies from -5 to 5. Even though these intuitions are not strictly correct, they will reappear throughout the thesis.

2.2.2 The Wigner distribution and Weyl transform

Another common time-frequency distribution was first introduced in the context of quantum mechanics by Wigner [254], and it is therefore called the *Wigner distribution*. Given $\psi \in L^2(\mathbb{R}^d)$, the Wigner distribution of ψ is the function $W(\psi)$ defined on $\mathbb{R}^{2d}$ by

$$W(\psi)(x,\omega) = \int_{\mathbb{R}^d} \psi(t + x/2)\overline{\psi(t - x/2)}e^{-2\pi i\omega t}\,dt.$$

We postpone a discussion of the Wigner distribution as a time-frequency distribution to the next subsection, and focus on its role in defining the *Weyl transform.* For this, we first note that the Wigner distribution can be polarized to give the *cross-Wigner distribution* of $\psi, \varphi \in L^2(\mathbb{R}^d)$, given by

$$W(\psi,\varphi)(x,\omega) = \int_{\mathbb{R}^d} \psi(t + x/2)\overline{\varphi(t - x/2)}e^{-2\pi i\omega t}\,dt.$$

The Weyl transform is a map sending functions on $\mathbb{R}^{2d}$ to operators on $L^2(\mathbb{R}^d)$; in physics one might call it a *quantization procedure*. Given $f \in L^2(\mathbb{R}^{2d})$, its Weyl transform L_f can be defined in many equivalent ways, but in this thesis we will follow the convention in time-frequency analysis and use the following weak definition: $L_f \in \mathcal{L}(L^2)$ is the unique operator satisfying

$$\langle L_f\psi, \varphi\rangle_{L^2(\mathbb{R}^d)} = \langle f, W(\varphi,\psi)\rangle_{L^2(\mathbb{R}^{2d})} \quad \text{for all } \psi,\varphi \in L^2(\mathbb{R}^d).$$

Our definition is not very explicit, but has the advantage that it generalizes to much more general settings. By replacing the inner products in the definition by duality brackets, one can even consider the Weyl transform of a tempered distribution on $\mathbb{R}^{2d}$, which will in general not be a bounded operator on $L^2(\mathbb{R}^d)$. In fact, our assumption that $f \in L^2(\mathbb{R}^{2d})$ is very restrictive: Pool [215] has shown that the Weyl transform is unitary from $L^2(\mathbb{R}^{2d})$ onto $\mathcal{S}^2$, so if $f \in L^2(\mathbb{R}^{2d})$ we can only obtain Weyl transforms L_f in $\mathcal{S}^2$. By working with more general functions or distributions f, one gets that any $S \in \mathcal{L}(L^2)$ is given by $S = L_f$ for some unique distribution f on $\mathbb{R}^{2d}$, which will often be exploited in the thesis. We call this f the *Weyl symbol* of $S = L_f$.

Example 2.2.1. Given $z = (x, \omega) \in \mathbb{R}^{2d}$, consider the character χ_z^σ defined on $\mathbb{R}^{2d}$ by $\chi_z^\sigma(z') = e^{2\pi i \sigma(z,z')}$. Its Weyl transform can be shown to be $L_{\chi_z^\sigma} = e^{-\pi i x\cdot\omega}\pi(x,\omega)$. This explains why we considered operators $e^{-\pi i x\cdot\omega}\pi(x,\omega)$ to be characters in quantum harmonic analysis: they are the Weyl transforms of the natural characters on $\mathbb{R}^{2d}$ (recall from the discussion of the symplectic Fourier transform that using χ_z^σ rather than $\chi_z(z') = e^{2\pi i z\cdot z'}$ is merely a matter of convenience).

Remark 2.1 (Weyl transform in quantum harmonic analysis). The Weyl transform motivates the use of α_z as a translation of operators. If $f \in L^2(\mathbb{R}^{2d})$ and $z \in \mathbb{R}^{2d}$, then

$$\alpha_z(L_f) = L_{T_z f}.$$

So the operator translation $\alpha_z(S)$ simply means that we translate the Weyl symbol of S by z. There are similar relations between the Weyl transform and convolutions, for instance one has that

$$L_f \star L_g = f * g.$$

This might suggest that quantum harmonic analysis is simply the image of the usual harmonic analysis of functions under the Weyl transform, but the picture is significantly more complicated as the Weyl transform does not send $L^1(\mathbb{R}^{2d})$ into $\mathcal{S}^1$.

2.2.3 Comparing spectrograms and Wigner distributions

As we mentioned when discussing the uncertainty principle, there is no ideal time-frequency distribution Q. So far, we have met three examples: the STFT, the spectrogram and the Wigner distribution. Both the spectrogram and Wigner distribution[1] are examples of quadratic time-frequency distributions, meaning that $Q(c\psi)(z) = |c|^2 Q(\psi)(z)$ for any $z \in \mathbb{R}^{2d}, c \in \mathbb{C}$. However, they still have

[1] Ville [247] was the first to introduce the Wigner distribution as a time-frequency distribution. In signal analysis it is therefore often called the Wigner-Ville distribution.

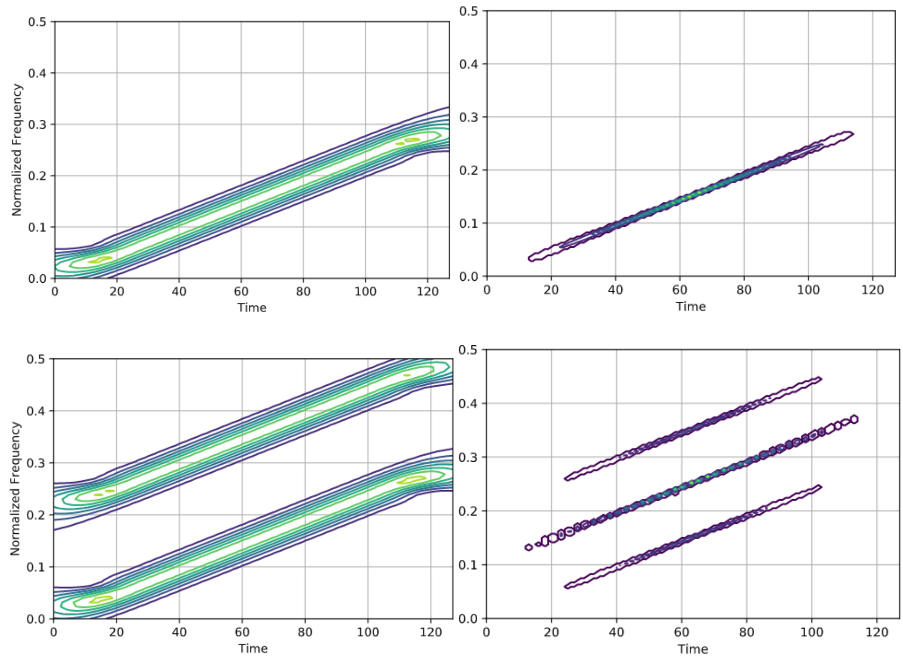

Figure 2.1: Above: Contour plot of the spectrogram with Gaussian window (left) and Wigner distribution (right) of a chirp signal. Below: Contour plot of the spectrogram with Gaussian window (left) and Wigner distribution (right) of superposition of two chirp signals. Plots produced using time-frequency toolbox for Python [82].

somewhat different properties. The most obvious difference is perhaps that the spectrogram $|V_\varphi\psi|^2$ is always a positive function, while Hudson [161] has shown that the Wigner distribution $W(\psi)$ is only positive for certain Gaussians ψ. To get some further intuition about the similarities and differences of spectrograms and Wigner distributions, consider the two upper plots in Figure 2.1. It shows the spectrogram $|V_\varphi\psi|^2$ and Wigner distribution of a *chirp function* ψ, i.e. a function whose frequency increases linearly with time: $\psi(t) = e^{iat^2}$ for $a > 0$.

We see that both the spectrogram and Wigner distribution show the expected behaviour: the frequency increases linearly with time. However, we also see that the Wigner distribution is less spread out along the frequency axis. One might therefore be tempted to say that the Wigner distribution gives a better time-frequency distribution than the spectrogram, but the picture changes drastically if we consider ψ that is a sum of two different chirp signals, see the two lower plots in Figure 2.1.

The spectrogram is still more spread out along the frequency axis, but shows the behaviour we expect from a superposition of chirp signals. The Wigner distribution also shows one straight line for each chirp in the signal, but there is a third line between them which does not correspond to an actual component of the signal. This is the phenomenon of "ghost frequencies".

These pictures show that the spectrograms looks like a somewhat smeared out version of the Wigner distribution. This is not a coincidence, as we have the relation

$$|V_\varphi \psi|^2 = W(\psi) * W(\check{\varphi}).$$

The spectrogram is therefore obtained by convolving the Wigner function $W(\psi)$ with the function $W(\check{\varphi})$. As we have mentioned, taking a convolution with a function — $W(\check{\varphi})$ in this case — corresponds to replacing the value of $W(\psi)(z)$ by a weighted average of the values of $W(\psi)(z')$ for z' in a region Ω around z. Here Ω is the region in $\mathbb{R}^{2d}$ outside of which the function $W(\check{\varphi})$ is negligible, which should not be too large if we pick φ that has little spread in time and frequency.

The fact that the spectrogram is obtained from the Wigner distribution by local averaging, explains the behaviour in Figure 2.1. It is clearly the reason why the spectrogram is more smeared out, but it also the cause of the ghost frequencies vanishing. More information and illustrations from a more applied perspective may be found in Flandrin's recent monograph [113].

2.2.4 Cohen's class of time-frequency distributions

We have seen that the spectrogram of ψ is obtained by convolving the Wigner distribution $W(\psi)$ with another function $\Phi = W(\check{\varphi})$ on $\mathbb{R}^{2d}$. If we allow Φ to vary among all functions (even tempered distributions) on $\mathbb{R}^{2d}$, we get a whole class of time-frequency distributions first defined by Cohen [59]. To be precise, *Cohen's class of time-frequency distributions* consists of time-frequency distributions Q_Φ such that

$$Q_\Phi(\psi) = W(\psi) * \Phi.$$

Here Φ is a function or distribution on $\mathbb{R}^{2d}$, while ψ is a signal on $\mathbb{R}^d$. We need to pick Φ and ψ so that the convolution above is well-defined. For instance, if Φ is a tempered distribution, we must restrict ψ to being a Schwartz function.

Both the Wigner distribution and spectrogram are examples of Cohen's class, corresponding to $\Phi = \delta_0$ (Dirac's delta distribution) and $\Phi = W(\check{\varphi})$, respectively. More examples will be explored in this thesis, for instance the Born-Jordan distribution and Rihaczek distribution. Our main focus will, however, be a more abstract study of the properties of Q_Φ in terms of the properties of Φ. Among the questions we will consider, are the following:

1. For which Φ is $Q_\Phi(\psi)$ a positive function for all signals ψ?

2. Which Q_Φ preserve the energy, i.e. satisfy $\int_{\mathbb{R}^{2d}} Q_\Phi(\psi)(z)\,dz = \|\psi\|_{L^2}^2$?
3. The spectrogram has many interesting properties, and it is known that some of these properties do not hold for a general Cohen's class distribution Q_Φ. What does Φ need to satisfy for these properties of the spectrogram to hold for Q_Φ?

Some answers to these questions were previously known, and Janssen's survey [172] gives a good overview of how properties of Φ influence properties of Q_Φ. However, we will actually study Q_Φ in terms of the Weyl transform L_Φ of Φ. This will allow us to give more precise answers to (1) and (2): (1) becomes a question of whether L_Φ is a positive operator (this was already known [131]) and (2) a question of whether L_Φ is a trace class operator. As there are no simple characterizations of Φ such that L_Φ is positive and/or trace class, this suggests that studying Cohen's class in terms of Weyl operators is a fruitful endeavour. The first steps in this direction, along with answers to (1) and (2), can be found in Paper A. The fact that we work with operators also means that we are in the domain of quantum harmonic analysis. This insight is also first laid out in Paper A, and allows us to attack (3) in Papers B, D and F. We will return to these questions in Section 3.3 in this introduction.

2.3 Modulation spaces

Before we show how quantum harmonic analysis fits with time-frequency analysis, we need to discuss two more aspects of the latter. The first of these are the *modulation spaces* as introduced by Feichtinger [96]. Moyal's identity shows that for any $0 \neq \varphi \in L^2(\mathbb{R}^d)$, the L^2-norm of $V_\varphi\psi$ is an equivalent norm to $\|\psi\|_{L^2}$. The modulation spaces are function spaces on $\mathbb{R}^d$ defined by replacing the L^2-norm of $V_\varphi\psi$ by other L^p-norms. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions.

Definition 2.3.1. Let $0 \neq \varphi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$. The modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $\psi \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|\psi\|_{M^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi\psi(x,\omega)|^p \,dx \right)^{q/p} d\omega \right)^{1/q} < \infty.$$

If $p = \infty$ and/or $q = \infty$, the corresponding integral is replaced by a supremum in the usual way.

With the norm $\|\psi\|_{M^{p,q}}$, $M^{p,q}(\mathbb{R}^d)$ becomes a Banach space of distributions, and it is an important fact that the definition is independent of the window function φ used — picking a different function gives the same space of distributions with an equivalent norm. For this introduction, we have chosen to restrict our attention to unweighted modulation spaces. In Papers D and F we will add a weight function to the setup, which leads to an even larger class of modulation spaces. To achieve this, one needs to use stronger conditions than $\varphi \in \mathcal{S}(\mathbb{R}^d)$ on φ, and to work with an even larger reservoir of distributions than $\mathcal{S}'(\mathbb{R}^d)$. This leads to some technical complications, but even in this very general setting the modulation spaces are Banach spaces, so they can still be studied effectively using the tools of functional analysis.

The modulation spaces contain many interesting examples. An obvious example is that $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, which is a reformulation of Moyal's identity. If we had allowed weight functions, which we do in Papers D and F, we could also obtain the Bessel potential spaces

$$H^s(\mathbb{R}^d) := \left\{ \psi \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\mathcal{F}(\psi)(\omega)|^2 (1 + |\omega|^2)^s \, d\omega < \infty \right\}$$

for $s \in \mathbb{R}$ as examples. Another important example is the space $M^{1,1}(\mathbb{R}^d)$, often called Feichtinger's algebra after its introduction by Feichtinger in [95] prior to the introduction of general modulation spaces. The space $M^{1,1}(\mathbb{R}^d)$ is a convenient test function space in time-frequency analysis. It contains $\mathcal{S}(\mathbb{R}^d)$ as a dense subset, consists of continuous functions and is invariant under both the Fourier transform and the time-frequency shifts. For us, the statement that a function is well-localized in both time and frequency will often be formalized by assuming that the function belongs to $M^{1,1}(\mathbb{R}^d)$. What makes $M^{1,1}(\mathbb{R}^d)$ preferable to $\mathcal{S}(\mathbb{R}^d)$ in many cases is that, in addition to simply containing more functions, $M^{1,1}(\mathbb{R}^d)$ is a Banach space whereas $\mathcal{S}(\mathbb{R}^d)$ is only a Fréchet space. Furthermore, in the definition of modulation spaces we can replace $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by any $\phi \in M^{1,1}(\mathbb{R}^d)$ and still obtain an equivalent norm for the modulation spaces.

Much more can be said about modulation spaces. For instance, $M^{p_1,q_1}(\mathbb{R}^d) \subset M^{p_2,q_2}(\mathbb{R}^d)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$, and they satisfy the natural duality relations $(M^{p,q}(\mathbb{R}^d))' = M^{p',q'}(\mathbb{R}^d)$ for $p, q < \infty$ and $1 = 1/p + 1/p' = 1/q + 1/q'$. In particular, $M^{\infty,\infty}(\mathbb{R}^d)$ is the dual of the test function space $M^{1,1}(\mathbb{R}^d)$, so in this sense $M^{\infty,\infty}(\mathbb{R}^d)$ is a natural reservoir of distributions which will play an important role in Papers D and F. Finally, since we interpret $V_\varphi \psi$ as a time-frequency distribution, the parameters p and q measure respectively the decay and smoothness of ψ — see the introduction to Paper F for more on this perspective.

2.4 Gabor frames

So far we have mainly been working in the continuous setting of $\mathbb{R}^{2d}$, but working over discrete subsets of $\mathbb{R}^{2d}$ is necessary both in real-world applications (for instance to wireless communication [238]) and — more importantly — this thesis. As a step in this direction, we now consider the theory of *Gabor frames*. If $a, b > 0$, a function $\varphi \in L^2(\mathbb{R}^d)$ generates a Gabor frame for $L^2(\mathbb{R}^d)$ with respect to the lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d \subset \mathbb{R}^{2d}$ if there exist constants $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{m,n\in\mathbb{Z}^d} |V_\varphi \psi(am, bn)|^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for all } \psi \in L^2(\mathbb{R}^d).$$

The inequalities above can be viewed as a discretization of Moyal's identity, but whereas all $\varphi \in L^2(\mathbb{R}^d)$ satisfy Moyal's identity, the question of which triples (φ, a, b) give Gabor frames is notoriously difficult. A famous example due to Lyubarskii [207] and Seip-Wallstén [229, 230] says that if $d = 1$, the Gaussian $\varphi(t) = e^{-\pi t^2}$ generates a Gabor frame with respect to a lattice $a\mathbb{Z} \times b\mathbb{Z}$ if and only if $ab < 1$. A more recent contribution by Gröchenig and Stöckler [142] shows that the same holds if φ is any totally positive function, but an example by Janssen [174] shows that for other φ the set of a, b such that φ generates a Gabor frame can be very complicated.

An equivalent way of stating that φ defines a Gabor frame with respect to $a\mathbb{Z}^d \times b\mathbb{Z}^d$ is to say that the *frame operator* S_φ given by

$$S_\varphi(\psi) = \sum_{m,n\in\mathbb{Z}^d} V_\varphi \psi(am, bn)\pi(am, bn)\varphi \quad \text{for } \psi \in L^2(\mathbb{R}^d)$$

defines a bounded, invertible operator on $L^2(\mathbb{R}^d)$. It is easy to see that S_φ is the composition $S_\varphi = D_\varphi C_\varphi$ of the *analysis* operator $C_\varphi : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d})$, given by

$$C_\varphi(\psi)(m, n) = V_\varphi \psi(am, bn) \quad \text{for } (m, n) \in \mathbb{Z}^{2d}, \psi \in L^2(\mathbb{R}^d),$$

and the *synthesis operator* $D_\varphi : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{R}^d)$ given by

$$D_\varphi(c) = \sum_{m,n\in\mathbb{Z}^d} c_{m,n}\pi(am, bn)\varphi \quad \text{for } c = \{c_{m,n}\}_{m,n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^{2d}).$$

In the very special case that S_φ is not only invertible, but $S_\varphi = C \cdot I_{L^2}$ for some non-zero constant C, we immediately obtain that any $\psi \in L^2(\mathbb{R}^d)$ can be expressed as a discrete superposition of time-frequency shifts of φ:

$$\psi = C \sum_{m,n\in\mathbb{Z}^d} V_\varphi \psi(am, bn)\pi(am, bn)\varphi.$$

In this case, we say that φ generates a *tight* Gabor frame. But if S_φ is invertible yet not a multiple of the identity operator, we can still write any signal ψ as a superposition of time-frequency shifts as above. In fact, let $\eta = (S_\varphi)^{-1}(\varphi)$; the so-called canonical dual window of φ. It is not difficult to show that $D_\varphi \circ C_\eta = I_{L^2}$, so that for any $\psi \in L^2(\mathbb{R}^d)$

$$\psi = \sum_{m,n\in\mathbb{Z}^d} V_\eta\psi(am,bn)\pi(am,bn)\varphi.$$

Expansions in terms of frames or orthonormal basis as above are often useful in mathematics, but they become even more useful if the behaviour of the expansion coefficients carry some meaning. What this means in our case is the following question: what do the decay properties of the coefficients $\{V_\eta\psi(am,bn)\}_{m,n\in\mathbb{Z}^d}$ tell us about ψ? To answer this, we need to assume more from φ,ξ than simply that they belong to $L^2(\mathbb{R}^d)$: we need them to belong to $M^{1,1}(\mathbb{R}^d)$. Luckily, the result of Gröchenig and Leinert [140] that gave us Wiener's lemma in quantum harmonic analysis makes achieving this significantly easier: if $\varphi \in M^{1,1}(\mathbb{R}^d)$ generates a Gabor frame, then $\eta = (S_\varphi)^{-1}(\varphi) \in M^{1,1}(\mathbb{R}^d)$. This implies the following result, showing that the decay of the coefficients, as measured by $\ell^{p,q}$-norms, does indeed say something about ψ.

Theorem 2.4.1. *Assume that $\varphi \in M^{1,1}(\mathbb{R}^d)$ generates a Gabor frame with respect to $a\mathbb{Z}^d \times b\mathbb{Z}^d$. Then any $\psi \in M^{p,q}(\mathbb{R}^d)$ for $1 \leq p,q \leq \infty$ has an expansion of the form*

$$\psi = \sum_{m,n\in\mathbb{Z}^d} V_\eta\psi(am,bn)\pi(am,bn)\varphi$$

for some $\eta \in M^{1,1}(\mathbb{R}^d)$, and there exist constants $A,B > 0$ such that

$$A\|\psi\|_{M^{p,q}} \leq \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{m\in\mathbb{Z}^d}|V_\eta\psi(am,bn)|^p\right)^{q/p}\right)^{1/q} \leq B\|\psi\|_{M^{p,q}}.$$

Recall that in Section 2.2.4 we listed three questions about distributions of Cohen's class, and the last asked whether other elements of Cohen's class shared known properties of the spectrogram. One such property of the spectrogram is that it is associated with a class of frames (the spectrogram appears in the definition of Gabor frames), and these frames give equivalent norms for modulation spaces. Showing that this generalizes to a much larger subset of Cohen's class than the spectrogram is the subject of papers D and F.

Chapter 3

Time-frequency analysis meets quantum harmonic analysis

So far, we have only vaguely hinted at the connections between quantum harmonic analysis and time-frequency analysis. To make the connection precise, we start by showing how we can recover familiar objects from time-frequency analysis as special cases of the concepts in quantum harmonic analysis. With this basic connection in place, we can start to look at how results in either of these fields imply and motivate results in the other field — it is in this intersection that a substantial part of the results in this thesis will reside.

3.1 Finding time-frequency analysis within quantum harmonic analysis

The basic objects in quantum harmonic analysis are the convolutions and the Fourier-Wigner transform. We will now consider these objects in the special case of *rank-one operators*. A rank-one operator $\psi \otimes \varphi \in \mathcal{S}^1$ for $\psi, \varphi \in L^2(\mathbb{R}^d)$ is given by

$$\psi \otimes \varphi(\xi) = \langle \xi, \varphi \rangle_{L^2} \psi \quad \text{for } \xi \in L^2(\mathbb{R}^d).$$

Evaluating the operations of quantum harmonic analysis for rank-one operators reproduces concepts of time-frequency analysis, as we first noted in [203]. Recall that $\check{\psi}(t) = \psi(-t)$

Lemma 3.1.1. *Let $f \in L^1(\mathbb{R}^{2d})$ and $\varphi, \psi, \xi, \eta \in L^2(\mathbb{R}^d)$. Then*

1. $f \star (\varphi \otimes \varphi) = \mathcal{A}_f^{\varphi}$.
2. $(\psi \otimes \varphi) \star (\check{\xi} \otimes \check{\eta})(x, \omega) = V_\eta \psi(x, \omega) \overline{V_\xi \varphi(x, \omega)}$.

3. $\mathcal{F}_W(\psi \otimes \varphi)(x, \omega) = e^{i\pi x \cdot \omega} V_\varphi \psi(x, \omega)$.

The first part of the lemma says that localization operators are examples of convolutions of functions with operators. The second part gives that spectrograms are examples of convolutions of two operators, since

$$|V_\varphi \psi|^2 = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi}).$$

Finally, the last part says that the STFT, apart from a phase factor, is an example of a Fourier-Wigner transform.

With these identifications in place, it is clear that we can obtain results on localization operators, spectrograms and STFTs by picking rank-one operators in the abstract results from Section 1.2.3. Examples of this will be prevalent throughout the thesis, but for this introduction we will consider the example that motivated us to investigate this connection in the first place in [203]. In [26], Bayer and Gröchenig proved several injectivity and density results for localization operators and the so-called Berezin transform. Among other results, they proved the following.

Corollary (Bayer and Gröchenig [26])**.** *If $\varphi \in M^{1,1}(\mathbb{R}^d)$ and $V_\varphi \varphi$ has no zeros, then:*

1. *The Berezin transform $\mathfrak{B}_\varphi$ is injective on $\mathcal{L}(L^2)$.*

2. *The set $\{\mathcal{A}_f^\varphi : f \in L^1(\mathbb{R}^{2d})\}$ is dense in $\mathcal{S}^1$.*

Here we have introduced the Berezin transform rather abruptly; a proper introduction can be found in Paper E, but for now it suffices to note that it can be written as

$$\mathfrak{B}_\varphi(A) = A \star (\check{\varphi} \otimes \check{\varphi}) \quad \text{for } A \in \mathcal{L}(L^2).$$

To understand how the result of Bayer and Gröchenig fits into quantum harmonic analysis, let us define $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$. If we formulate Corollary 1.2.3.1 for this particular S, using the lemma above to rewrite convolutions and Fourier-Wigner transforms, we immediately obtain the following.

Theorem ([203])**.** *Let $\varphi \in L^2(\mathbb{R}^d)$. The following are equivalent:*

1. *$V_\varphi \varphi$ has no zeros.*

2. *If $A \in \mathcal{L}(L^2)$ and $\mathfrak{B}_\varphi(A) = 0$, then $A = 0$.*

3. *$L^1(\mathbb{R}^{2d}) \star (\varphi \otimes \varphi) = \{\mathcal{A}_f^\varphi : f \in L^1(\mathbb{R}^{2d})\}$ is dense in $\mathcal{S}^1$.*

We see that Wiener's approximation theorem in quantum harmonic analysis has easily allowed us to improve the result of Bayer and Gröchenig, showing that their assumption on the zeros of $V_\varphi \varphi$ can be added to a list of equivalent statements. Less importantly, we have also replaced the assumption $\varphi \in M^{1,1}(\mathbb{R}^d)$ by the weaker $\varphi \in L^2(\mathbb{R}^d)$. The difficulty in showing the equivalences above lie in the previously mentioned fact that working with $\mathcal{S}^1$ is significantly more complicated than working with the Hilbert space $\mathcal{S}^2$ — the analogous result for $\mathcal{S}^2$ (with equivalences) was already proved in [26].

Parts of Paper E is similarly concerned with exploring the consequences in time-frequency analysis of the *Tauberian* theorem in quantum harmonic analysis. Let us give an example, showing how quantum harmonic analysis can act as a bridge connecting results that seem quite different at first glance. In [109] Fernández and Galbis characterized those functions $f \in L^\infty(\mathbb{R}^{2d})$ such that $\mathcal{A}_f^\varphi$ is compact for all $\varphi \in L^2(\mathbb{R}^d)$: they are precisely the f such that there is a non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi f(x,\omega)| = 0. \tag{3.1.1}$$

Rewriting the localization operator as a convolution, this characterizes those f such that $f \star S$ is compact for all S of the form $\varphi \otimes \varphi$. However, it is easily seen that there is S of this form such that $\mathcal{F}_W(S)$ has no zeros. Hence we are characterizing those f satisfying (i) in the quantum harmonic analysis Tauberian theorem for $A = 0$. Using the equivalence of (i) and (ii) in this Tauberian theorem, we obtain the following.

Theorem. *Let $f \in L^\infty(\mathbb{R}^{2d})$. The following are equivalent:*

1. *There is some $a \in L^1(\mathbb{R}^{2d})$ such that $\mathcal{F}(a)$ has no zeros and $f * a$ vanishes at infinity.*
2. *There is some $S \in \mathcal{S}^1$ such that $\mathcal{F}_W(S)$ has no zeros and $f \star S$ is compact.*
3. *There is a non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$*

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi f(x,\omega)| = 0.$$

The first of these statements is the original condition in Wiener's Tauberian theorem, so the condition of Fernández and Galbis on compactness of localization operators actually gives a new characterization of the functions satisfying Wiener's Tauberian theorem. Implicit in this is also an open problem: does the equivalence of 1. and 3. still hold when $\mathbb{R}^{2d}$ is replaced by $\mathbb{R}^d$? There is no natural analogue of 2. in this case, so it seems unlikely that the methods of [109] — which rely heavily on functional analysis in spaces of operators — can be modified to fit this case.

3.2 Gabor frames in quantum harmonic analysis: Fourier series of operators

Recall that if $\varphi \in L^2(\mathbb{R}^d)$, then the Gabor frame operator S_φ with respect to the lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$ is given by

$$S_\varphi(\psi) = \sum_{m,n\in\mathbb{Z}^d} V_\varphi\psi(am, bn)\pi(am, bn)\varphi.$$

It does not take much work to realize that there is a simple alternative (and well-known) expression, namely

$$S_\varphi = \sum_{m,n\in\mathbb{Z}^d} \alpha_{(am,bn)}(\varphi \otimes \varphi).$$

Since we interpret α as a translation, this shows that the Gabor frame operator is the *periodization* of a rank-one operator φ. For functions, the Fourier series of a periodization is given by *Poisson's summation formula.* Now we consider another way in which quantum harmonic analysis and time-frequency analysis meet: known concepts in time-frequency analysis can inspire generalizations to quantum harmonic analysis. Let us formulate three natural questions of this kind:

1. Is there a Poisson summation formula for operators?
2. What does this formula say for Gabor frame operators?
3. Since Gabor frames correspond to periodizations of rank-one operators, is there a kind of generalized Gabor frame that corresponds to periodizations of general operators?

The answers to these questions are given in Paper D. First, there is indeed a Poisson summation formula for operators. Here $\mathcal{B}'$ is a Banach space of (potentially unbounded) operators on $L^2(\mathbb{R}^d)$, defined using time-frequency analysis by requiring the Weyl symbol to lie in $M^{\infty,\infty}(\mathbb{R}^{2d})$, which includes $\mathcal{L}(L^2)$ as a subspace.

Theorem. *Let $S \in \mathcal{S}^1$. Then*

$$\sum_{m,n\in\mathbb{Z}^d} \alpha_{(am,bn)}(S) = \frac{1}{a^d b^d} \sum_{m,n\in\mathbb{Z}^d} \mathcal{F}_W(S)(m/b, n/a)e^{-\pi i m\cdot n/ab}\pi(m/b, n/a),$$

with weak convergence of both sums in $\mathcal{B}'$.*

The reader should pay particular attention to the Banach spaces $\mathcal{S}^1$ and $\mathcal{B}'$ in the theorem. It is perfectly possible to restrict S to a dense subspace of $\mathcal{S}^1$, and replace $\mathcal{B}'$ by $\mathcal{L}(L^2)$ — for this setting the result was already proved by Feichtinger and Kozek [102]. The extension to $S \in \mathcal{S}^1$ is new in Paper D, and requires a careful study of continuity of certain mappings — again, proving results for $\mathcal{S}^1$ takes extra care. The result also shows that time-frequency analysis can help make sense of natural formulas in quantum harmonic analysis: the space $\mathcal{L}(L^2)$ used in Werner's original formulation of quantum harmonic analysis is simply not big enough for the Poisson summation formula to hold for trace class operators.

To answer the second question, we let $S = \varphi \otimes \varphi$. Inserting this into the Poisson summation formula and using $\mathcal{F}_W(S)(x,\omega) = e^{\pi i x\cdot\omega} V_\varphi\varphi(x,\omega)$, we get that

$$S_\varphi = \frac{1}{(ab)^d} \sum_{m,n\in\mathbb{Z}^d} V_\varphi\varphi(m/b, n/a)\pi(m/b, n/a).$$

This is the famous *Janssen representation* of the Gabor frame operator, first established by Rieffel [223].

The third question is the main question in Paper D. The analogue of Gabor frames are Gabor g-frames, as first defined in the mentioned paper. In short, a Hilbert Schmidt operator S generates a Gabor g-frame with respect to $a\mathbb{Z}^d \times b\mathbb{Z}^d$ if there exist $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{m,n\in\mathbb{Z}^d} \|\alpha_{(am,bn)}(S)\psi\|_{L^2}^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for all } \psi \in L^2(\mathbb{R}^d).$$

The associated g-frame operator on $L^2(\mathbb{R}^d)$ is then

$$\mathfrak{S}_S = \sum_{m,n\in\mathbb{Z}^d} \alpha_{(am,bn)}(S^*S).$$

which also has a Janssen representation by the Poisson summation formula from quantum harmonic analysis. We will have more to say on Gabor g-frames in the next section.

3.3 Cohen's class as operator convolutions

When we introduced Cohen's class of time-frequency distributions, we claimed that we would study it using tools from quantum harmonic analysis. To see how this is possible, consider a Cohen's class distribution

$$Q_\Phi(\psi) = W(\psi) * \Phi.$$

To rewrite this in terms of quantum harmonic analysis, we use a fact from [203]: if f, g are two functions on $\mathbb{R}^{2d}$, then

$$f * g = L_f \star L_g$$

when these convolutions are defined; here L_f is the Weyl transform of f. As one can show that $L_{W(\psi)} = \psi \otimes \psi$, we get in particular that

$$Q_\Phi(\psi) = (\psi \otimes \psi) \star L_\Phi.$$

The Weyl transform is defined and bijective on a large class of distributions (for instance the tempered distributions). Hence nothing is lost by forgetting that L_Φ is the Weyl transform of a function Φ. We will therefore study Cohen's class by studying

$$Q_S(\psi)(z) = (\psi \otimes \psi) \star \check{S}(z)$$

for an operator S — the $\check{}$ is only there to simplify a few equations.

Example 3.3.1.

1. If $S = \varphi \otimes \varphi$ for some $\varphi \in L^2(\mathbb{R}^d)$, then $Q_S(\psi) = |V_\varphi \psi|^2$ — the spectrogram.
2. If $S = P$, the parity operator, then $Q_S(\psi) = W(\psi)$ — the Wigner distribution. The underlying reason is that the Weyl transform L_δ of Dirac's delta distribution is (up to a constant) P.
3. If we pick a localization operator $S = \mathcal{A}_\Omega^\varphi$ for some $\Omega \subset \mathbb{R}^{2d}$, a quick calculation shows that

$$Q_{S^*S}(\psi)(z) = \|\mathcal{A}_{z+\Omega}^\varphi \psi\|_{L^2}^2.$$

By our interpretation of localization operators, the value of $Q_{S^*S}(\psi)(z)$ is the size of the part of ψ living in the domain $\Omega + z$ in the time-frequency plane.

More examples can be found in the papers in the thesis, for instance the τ-Wigner distributions in Paper E.

With this formalism in place, the first two questions in Section 2.2.4 get simple answers: $Q_S(\psi)$ is positive for all ψ if and only if S is a positive operator, and in that case $Q_S(\psi)$ preserves the energy if and only if S is a trace class operator. We also obtain a rather general uncertainty principle, namely:

Theorem. *Let $S \in \mathcal{L}(L^2)$. If $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset such that*

$$\int_\Omega |Q_S(\psi)|\, dz \geq (1-\epsilon)\|S\|_{\mathcal{L}(L^2)}$$

for some $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ and $\epsilon \geq 0$, then

$$\mu(\Omega) \geq 1 - \epsilon.$$

These results can be found in Paper A. Note that the assumption allowing us to prove the uncertainty principle, $S \in \mathcal{L}(L^2)$, has no simple formulation in terms of the Weyl symbol of S, i.e. in terms of Φ for Q_Φ.

3.3.1 Finer properties of the spectrogram in Cohen's class

The more significant question in this thesis, however, is question (3) from Section 2.2.4: which Cohen's class distributions share particular properties of the spectrogram? One property of the spectrogram is that it defines equivalent norms for modulation spaces. After all, we have seen that *any* non-zero $\varphi \in M^{1,1}(\mathbb{R}^d)$ defines an equivalent norm on $M^{p,q}(\mathbb{R}^d)$ by

$$\psi \mapsto \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi \psi(x,\omega)|^p \, dx \right)^{q/p} \, d\omega \right)^{1/q},$$

and if $\varphi \in M^1(\mathbb{R}^d)$ generates a Gabor frame with respect to $a\mathbb{Z}^d \times b\mathbb{Z}^d$ then another equivalent norm on $M^{p,q}(\mathbb{R}^d)$ is

$$\psi \mapsto \left(\sum_{n\in\mathbb{Z}^d} \left(\sum_{m\in\mathbb{Z}^d} |V_\varphi \psi(am,bn)|^p \right)^{q/p} \right)^{1/q}. \tag{3.3.1}$$

If we replace the spectrogram $|V_\varphi \psi|^2$ in these expressions by another Cohen's class distribution Q_T, what must we require from T for the expressions to still define equivalent norms for $M^{p,q}(\mathbb{R}^d)$? The answer is that $T = S^*S$, where S^* is a nuclear operator from $L^2(\mathbb{R}^d)$ to $M^{1,1}(\mathbb{R}^d)$. [1] The continuous case is covered in Paper F, showing that an equivalent norm on $M^{p,q}(\mathbb{R}^d)$ is given by

$$\psi \mapsto \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} Q_{S^*S}(\psi)(x,\omega)^{p/2} \, dx \right)^{q/p} \, d\omega \right)^{1/q}.$$

The discrete case is the culmination of the development of Gabor g-frames in Paper D, where we use the Janssen representation of the Gabor g-frame operator to show that if S generates a Gabor g-frame and S^* is nuclear as above, then we can replace $|V_\varphi \psi(am,bn)|^2$ in (3.3.1) by $Q_{S^*S}(\psi)(am,bn)$. It should be noted that the requirement that the Cohen class is of the form Q_{S^*S} for some nuclear S^* seems very difficult to formalize without the operator formulation of Cohen's class.

By picking different S, one can obtain Cohen's class distributions Q_{S^*S} with different interpretations. Significant parts of Papers D and F therefore deal with interpretations and examples. By picking $S = \xi \otimes \varphi$ for any non-zero $\xi \in L^2(\mathbb{R}^d)$, we recover the usual result that spectrograms with window φ determine equivalent norms for modulation spaces. As another interesting example, picking a localization operator $\mathcal{A}_f^\varphi$ recovers, in the discrete case, a result by Dörfler and Gröchenig [87].

[1] A stronger condition is used in Paper D, only because the author did not realize that this nuclearity condition was sufficient.

In these special cases the nuclearity condition on S^* also gets a much more concrete description.

Finally, Paper B considers another interesting property of the spectrogram from Abreu et al. [8]. If we consider a localization operator $\mathcal{A}_\Omega^\varphi$ for a compact domain $\Omega \subset \mathbb{R}^{2d}$ and $\varphi \in L^2(\mathbb{R}^d)$, then the domain can be approximated by the spectrograms of the first $\lceil |\Omega| \rceil$ eigenfunctions ϕ_n of $\mathcal{A}_\Omega^\varphi$:

$$\chi_\Omega \approx \sum_{n=1}^{\lceil |\Omega| \rceil} |V_\varphi \phi_n|^2.$$

The approximation is shown to hold asymptotically in L^1-norm as the size of Ω increases, and error bounds in L^1-norm and weak L^2-norm are given in the non-asymptotic case. Paper B considers the same question when $\mathcal{A}_\Omega^\varphi = \chi_\Omega \star (\varphi \otimes \varphi)$ is replaced by $\chi_\Omega \star S$ for some other operator S. Do the first $\lceil |\Omega| \rceil$ eigenfunctions ϕ_n^S of $\chi_\Omega \star S$ approximate Ω by

$$\chi_\Omega \approx \sum_{n=1}^{\lceil |\Omega| \rceil} Q_S(\phi_n^S)? \tag{3.3.2}$$

The answer turns out to be yes if one requires that S is a positive trace class operator. It is worth noting that the proofs of Abreu et al. use reproducing kernel Hilbert spaces, which at the time were not available in the more general case. Some of the effort in Paper B is therefore spent circumventing the use of these tools, typically by replacing them by concepts from quantum harmonic analysis.

By analyzing the proofs in Paper B, one sees that the validity of (3.3.2) is closely related to the statement that the eigenvalues λ_n of $\chi_\Omega \star S$ satisfy $\lambda_n \approx 1$ for $1 \leq n \leq \lceil |\Omega| \rceil$. We can therefore not expect (3.3.2) to be a good approximation for small domains Ω: Galbis [120] recently showed that the largest eigenvalue λ_1 of $\mathcal{A}_\Omega^{\varphi_0}$ satisfies

$$\lambda_1 \leq 1 - e^{-|\Omega|}$$

when φ_0 is Gaussian and Ω is radial.

3.4 Extensions of quantum harmonic analysis

One aspect of the thesis that we have mainly ignored until now are the extensions of quantum harmonic analysis developed in Papers C and G. The extension in Paper C is motivated by time-frequency analysis. Some close relatives of localization operators are the *Gabor multipliers*. If $\{c_{m,n}\}_{m,n\in\mathbb{Z}^d}$ is a sequence, $a, b > 0$

and $\varphi \in L^2(\mathbb{R}^d)$, the Gabor multiplier $\mathcal{G}_c^\varphi$ is the operator on $L^2(\mathbb{R}^d)$ given (in a non-standard formulation designed to be suggestive of our purposes) by

$$\mathcal{G}_c^\varphi = \sum_{m,n\in\mathbb{Z}^d} c_{m,n}\pi(am,bn)(\varphi\otimes\varphi)\pi(am,bn)^*.$$

This is very similar to the definition of the localization operator $\mathcal{A}_f^\varphi = f \star (\varphi\otimes\varphi)$, except that f is replaced by a sequence. It therefore suggests a new definition, namely the convolution $c \star_{a,b} S$ of a sequence $\{c_{m,n}\}_{m,n\in\mathbb{Z}^d}$ with $S \in \mathcal{L}(L^2)$ with respect to the lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$ given by

$$c \star_{a,b} S := \sum_{m,n\in\mathbb{Z}^d} c_{m,n}\alpha_{(am,bn)}(S).$$

The question of Paper C is whether this definition can be extended to a sensible theory of quantum harmonic analysis that mixes sequences on lattices with operators on $L^2(\mathbb{R}^d)$. Such a theory is indeed developed, leading to both new results and reinterpretations of old results for Gabor multipliers.

Similarly, as harmonic analysis works in some form on any locally compact group G, one can ask if quantum harmonic analysis could also work on other groups — even nonabelian ones. Due to its connections with wavelet theory, the *affine group* is a natural starting point for such a theory. Quantum harmonic analysis on the affine group is developed in Paper G.

Chapter 4

Summary of papers

4.1 Paper A: Mixed-State Localization Operators: Cohen's Class and Trace Class Operators

The first paper applies the tools of quantum harmonic analysis to problems of signal localization in the time-frequency plane. It is shown that quantum harmonic analysis provides a conceptual framework for many concepts of the mathematics and engineering literature, including localization operators, the spreading function, multi-window STFT multipliers and Cohen's class. This allows a simple yet mathematically rigorous treatment of these concepts.

First, a result is shown on the spreading function of an operator S, i.e. $\mathcal{F}_W(S)$. It is shown that having a well-localized spectrum (as measured by the trace class norm $\|S\|_{\mathcal{S}^1}$) is incompatible with having a well-localized spreading function. This suggests a notion of uncertainty principle for operators that has yet to be explored further.

We then show that the multi-window STFT multipliers from the engineering literature are easily described as convolutions of functions with operators, which allows us to use basic results of quantum harmonic analysis to obtain properties of these multipliers.

By treating Cohen's class using quantum harmonic analysis, we are able to prove precise characterizations of when Cohen's class distributions are positive and energy-preserving, L^p estimates, a phase retrieval result and an uncertainty principle valid for a large subclass of Cohen's class. As we have seen, every Cohen's class distribution is Q_S for some operator S on $L^2(\mathbb{R}^d)$. Associated with S is also a quantization procedure $f \mapsto f \star S$, and we study the relation between the Cohen's class and quantization procedure of a given operator S.

In particular, we are led to introduce the class of *mixed-state localization* *operators*, which arise from quantization procedures associated with positive trace class operators S with $\mathrm{tr}(S) = 1$. They are of the form $\chi_\Omega \star S$ for $\Omega \subset \mathbb{R}^{2d}$, and include the usual localization operators as the special case where S is a rank-one operator. We use an old result by Werner to show that any reasonable notion of assigning $\Omega \subset \mathbb{R}^{2d}$ to a "localization operator" is given by $\Omega \mapsto \chi_\Omega \star S$. The eigenfunctions of a mixed-state localization operator $\chi_\Omega \star S$ are shown to be maximally localized in Ω as measured by the Cohen's class distribution Q_S.

Finally, the quantization procedures of mixed-state localization operators are related to positive operator valued measures, giving a new connection to the associated Cohen's class distribution: Q_S arises as a Radon-Nikodym derivative from the positive operator valued measure of S.

Changes from published version and comments.

- The reader might wish to note that, unlike the rest of the thesis, duality brackets are linear in both coordinates in this paper.
- I have tweaked the definition of singular values in Proposition A.3.2 compared to the published version. Using an infinite index set means that we need to allow $s_n(S) = 0$, whereas the original version said that $s_n(S) > 0$. This minor technicality affects no proofs or results.
- In the published version, Wiener's approximation theorem was called the Tauberian theorem. This terminology sometimes appears in the literature due to the close relationship between the Tauberian theorem and approximation theorem, but I have changed it to "approximation theorem" to better fit the terminology of the introduction and Paper E.
- I have replaced a sentence in the proof of the first part of A.5.1, as the original sentence was too vague.

4.2 Paper B: On Accumulated Cohen's Class Distributions and Mixed-State Localization Operators

Inspired by the definition of mixed-state localization operators in the first paper, the second paper asks whether the theory of accumulated spectrograms can be generalized to this setting. The question is answered in the affirmative: if S is a positive trace class operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain, the first[1] $\lceil |\Omega| \rceil$

[1]When the eigenvalues of $\chi_\Omega \star S$ are ordered non-increasingly.

eigenfunctions h_k^Ω of $\chi_\Omega \star S$ approximate Ω by

$$\chi_\Omega \approx \sum_{k=1}^{\lceil|\Omega|\rceil} Q_S(h_k^\Omega).$$

To be more precise, let

$$\rho_\Omega^S(z) := \sum_{k=1}^{\lceil|\Omega|\rceil} Q_S(h_k^\Omega)$$

be the *accumulated Cohen class distribution* of S and Ω. It is shown that ρ_Ω^S approximates χ_Ω both asymptotically and under stronger assumptions also non-asymptotically. The asymptotic result concerns the dilated domain $R\Omega = \{Rz : z \in \Omega\}$ for large $R > 0$, and states that

$$\|\rho_{R\Omega}^S(R\cdot) - \chi_\Omega\|_{L^1} \to 0 \text{ as } R \to \infty.$$

Non-asymptotic results are shown both Section B.5.2 and Section B.6, and the cleanest statement and proof is likely the sharp result in Section B.6, which states that

$$\|\rho_\Omega^S - \chi_\Omega\|_{L^1} \leq (1/\epsilon + 2\|S\|_{M_{\mathrm{op}}^*}^2)|\partial\Omega|,$$

where $|\partial\Omega|$ is a quantity describing the size of the perimeter of Ω, $|\partial\Omega| \geq \epsilon$ and $\|S\|_{M_{\mathrm{op}}^*}$ is a constant depending on S.

To prove these results it is first necessary to extend results on eigenvalues of localization operators to mixed-state localization operators, and, unlike the case for accumulated spectrograms, we work without the use of reproducing kernel Hilbert spaces. The framework needed to achieve this is quantum harmonic analysis. On our way we will see the central role played by objects such as the function

$$\tilde{S} = S \star \check{S}$$

and the projection functional

$$\mathrm{tr}(T) - \mathrm{tr}(T^2)$$

for trace class operators T. Finally, we consider examples and non-examples of the theory. At the time of writing, interpretations of these results and objects for certain classes of S is work in progress.

Changes from published version and comments.

- The quantity $\|S\|_{M_{\mathrm{op}}^*}$ was somewhat misleadingly called a norm in the published version. I have removed this terminology.

- In the spectral representation of compact self-adjoint operators, I have removed the assumption that the set of eigenvectors is complete. Completeness is not always compatible with non-decreasing order on the eigenvalues. This technicality affects neither results nor proofs, as we never used completeness of the eigenvectors.
- In the published version, Wiener's approximation theorem was called the Tauberian theorem. This has been fixed.

4.3 Paper C: Quantum Harmonic Analysis on Lattices and Gabor Multipliers

Motivated by the theory of Gabor multipliers, we extend quantum harmonic analysis to a hybrid setting of sequences $\{c_\lambda\}_{\lambda\in\Lambda}$ on a full-rank lattice $\Lambda \subset \mathbb{R}^{2d}$ and operators $S \in \mathcal{L}(L^2(\mathbb{R}^d))$, hence we mix the discrete and the continuous setting. Given a sequence c over Λ and operators S, T on $L^2(\mathbb{R}^d)$, we define

- $c \star_\Lambda S = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) S \pi(\lambda)^*$, which gives a new operator on $L^2(\mathbb{R}^d)$,
- $S \star_\Lambda T(\lambda) = \mathrm{tr}(S\pi(\lambda)\check{T}\pi(\lambda)^*)$, which gives a new sequence on Λ.

Properties of these convolutions are studied, including associativity — which no longer holds in general — and interaction with Fourier transforms. The Fourier transform of an operator is the same Fourier-Wigner transform as in Paper A, hence a function on $\mathbb{R}^{2d}$, while the Fourier transform of a sequence is its *symplectic Fourier series*. Great care is taken to specify the sense in which formulas relating Fourier transforms and convolutions are correct. In order for the convolution $S \star_\Lambda T$ to give a summable sequence, we need even stronger assumptions than $S, T \in \mathcal{S}^1$, which leads to the use of a Banach subspace $\mathcal{B}$ of $\mathcal{S}^1$ defined in terms of Feichtinger's algebra of functions.

Analogues of Wiener's approximation theorem, Wiener's lemma and Wiener's division lemma are proved in this setting. It is shown that some of the equivalent statements in Werner's original quantum harmonic analysis version of Wiener's approximation theorem no longer hold. Examples are given in the paper, and we also mention the recent preprint [12] which implies that $T \mapsto T \star_\Lambda \varphi_0 \otimes \varphi_0$ is not injective, where φ_0 is the standard Gaussian. The Hilbert space theory, concerning sequences in $\ell^2(\Lambda)$, is studied by reducing it to the much-studied setting of shift-invariant subspaces of $L^2(\mathbb{R}^{2d})$. Some of these results were recently extended by García [121].

When S is a rank-one operator, $c \star_\Lambda S$ is a Gabor multiplier. Hence we recover as special cases several results on Gabor multipliers and related objects, such as the fundamental theorem of Gabor analysis and recoverability of operators from their channel matrix.

Changes from published version and comments.

- The theorem previously referred to as a Tauberian theorem is now more correctly referred to as an approximation theorem.
- The constant $|\Lambda|$ in the definition of $\mathcal{F}_W(A)$ in Theorems C.7.4 and C.7.5 was missing in the published version.

4.4 Paper D: On Gabor g-frames and Fourier Series of Operators

In this paper, we introduce the notion of Gabor g-frames over full-rank lattices $\Lambda \subset \mathbb{R}^{2d}$, which are generalizations of Gabor frames where the window function φ for the short-time Fourier transform is replaced by a Hilbert-Schmidt operator S. Their theory is developed from scratch, and they are shown to share many attractive properties of Gabor frames.

One such property is the existence of a Janssen representation for Gabor g-frames, which leads us to study a notion of Fourier series for operators. The Janssen representation is then a consequence of a Poisson summation formula for trace class operators. Developing the theory of Fourier series of operators requires the use of Banach subspaces of the trace class operators, and also the dual space which is a Banach superspace of the bounded operators on $L^2(\mathbb{R}^d)$. The use of these Banach spaces is necessary even for establishing results for trace class operators, and a section of the paper is dedicated to defining these Banach spaces and clearing up some subtleties in their definitions and properties.

The paper culminates in showing that if the generator S of a Gabor g-frame belongs to a suitable Banach subspace of the trace class operators, then an equivalent norm of ψ in the weighted modulation space $M^p_m(\mathbb{R}^d)$ is given by weighted ℓ^p norms of the sampled Cohen's class distribution $\{Q_{S^*S}(\psi)(\lambda)\}_{\lambda\in\Lambda}$, equivalently $\{\|S\pi(\lambda)^*\psi\|^2_{L^2}\}_{\lambda\in\Lambda}$. This generalizes both a well-known result for Gabor frames and a result by Dörfler and Gröchenig [87] on localization operators. However, the fact that our results follow from a general theory with a Janssen representation means that we can say more than just a norm equivalence: we obtain a natural reconstruction formula for $\psi \in M^p_m(\mathbb{R}^d)$ from $\{S\pi(\lambda)^*\psi\}_{\lambda\in\Lambda}$, which was previously only available for Gabor frames.

Comments. As we hint at in Remark D.13, we often need only $S^*S \in \mathcal{B}_{v\otimes v}$ (as opposed to $S \in \mathcal{B}_{v\otimes v}$) for results to hold. A weaker condition to ensure this is the condition $S^* \in \mathcal{N}(L^2, M_v^1)$ from Paper F, but this was only realized after Paper D was finished. By working with $S^* \in \mathcal{N}(L^2, M_v^1)$ we would not need the stronger condition on h (see Remark D.16). The proof of Proposition D.7.7 is also unnecessarily complicated — a better proof is given in Proposition F.3.4.

4.5 Paper E: A Wiener Tauberian Theorem for Operators and Functions

The main theorems of this paper are the previously mentioned Tauberian theorems in quantum harmonic analysis. There are two such theorems: one for a fixed function, and another for a fixed operator. The statement for a fixed function extends the original Tauberian theorem of Wiener, and the connection to operators allows us to add an equivalent statement to Wiener's theorem based on the theory of compactness of localization operators.

Applications and interpretations of these Tauberian theorems are then studied in contexts such as Toeplitz operators, compactness of localization operators, Cohen's class and quantization schemes. In particular, a well-known equivalence of localization operators and Toeplitz operators is used to translate results to image spaces of the short-time Fourier transform, Bargmann-Fock space and polyanalytic Bargmann-Fock spaces. The fact that the short-time Fourier transform of a Gaussian never vanishes allows us to prove stronger results for the Bargmann-Fock space. Unlike the classical treatment of Bargmann-Fock space, our arguments contain no complex analysis. The stronger results for Bargmann-Fock space will therefore hold on certain other reproducing kernel Hilbert spaces, and our methods thus illuminate which results in Bargmann-Fock space are not dependent on the connection to complex analysis.

Under a certain continuity assumption on operators, one of the Tauberian theorems reduces to an earlier result by Werner. In our setup, this result by Werner becomes an operator-analogue of Pitt's theorem for functions, which allows us to deduce compactness of operators from asymptotics of related functions on $\mathbb{R}^{2d}$. By translating the problem to the space of Toeplitz operators on the Bargmann-Fock space, this gives an alternative proof (void of complex analysis) of a result of Bauer and Isralowitz, and we obtain an addition to their results by employing Pitt's theorem for functions in addition to Werner's operator-analogue of Pitt's result. The same result by Werner also leads to a sufficient and necessary condition for compactness of a given localization operator.

4.6 Paper F: Equivalent Norms for Modulation Spaces from Positive Cohen's Class Distributions

Using the description of Cohen's class distributions in terms of operators from Paper A, we show that a subclass of Cohen's class can be used to give equivalent norms for modulation spaces. This extends the well-known fact that modulation spaces are invariant under a change of window in the short-time Fourier transform. It also provides the continuous analogue of the equivalent norms for modulation spaces given in terms of Gabor g-frames in Paper D. The main tool of the paper is a generalized short-time Fourier transform where the window function is replaced by a Hilbert-Schmidt operator.

In order for weighted $L^{p,q}$-norms of the Cohen's class distribution $Q_T(\psi)$ to be equivalent to modulation space norms of ψ, the operator T needs to satisfy certain conditions. We show that $T = S^*S$ for some S satisfying a nuclearity condition is sufficient. This implies that Q_T is a positive Cohen's class distribution, and the nuclearity condition on S is interpreted as requiring a certain time-frequency localization of S. We study this nuclearity condition in special cases such as rank-one operators and localization operators, and give sufficient conditions for it to hold.

The main result is formulated somewhat abstractly, but the last sections of the paper are dedicated to examples and interpretations in terms of Cohen's class, Weyl operators and localization operators.

4.7 Paper G: Affine Quantum Harmonic Analysis

In the final paper, we try to extend the theory of quantum harmonic analysis to the affine group. The motivation for this was essentially threefold:

1. A wish to know if and how quantum harmonic analysis could work for some nonabelian group.

2. The coauthors of the paper had recently studied a Weyl transform for the affine group.

3. It seemed likely that quantum harmonic analysis on the affine group could provide a conceptual framework for the theory of *covariant integral quantizations* in physics.

The affine group Aff has underlying set $\mathbb{R} \times \mathbb{R}_+$, and group operation

$$(x, a) \cdot_{\mathrm{Aff}} (y, b) = (x + ay, ab).$$

A unitary representation of Aff on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, r^{-1}dr)$ is given by

$$U(x,a)\psi(t) = e^{2\pi ixt}\psi(at) \quad (x,a) \in \text{Aff}, \psi \in \mathcal{H}.$$

We use this representation to define the convolution $f \star_{\text{Aff}} S$ of a function $f \in L^1(\text{Aff})$ with an operator S on $\mathcal{H}$, and the convolution $S \star_{\text{Aff}} T$ of two operators S, T on $\mathcal{H}$. One would want to obtain that $S \star_{\text{Aff}} T$ is integrable over Aff with respect to Haar measure when S, T are trace class, but this will not be true in general due to the noncommutativity of the group Aff. This leads to the introduction of the class of *admissible operators*, generalizing the notion of admissible vectors in wavelet theory. For the class of admissible operators, a larger part of quantum harmonic analysis still holds on the affine group, hence we study this class in detail.

The natural analogue of the Fourier-Wigner transform, arising from the study of Fourier analysis on nonabelian groups, is also studied, but we are unable to obtain the simple interaction with the convolutions that we desire. We do, however, give an overview of the behaviour of this transform and the ways in which it behaves like a Fourier transform. In particular, we use a result of Führ to obtain a characterization of admissible operators in terms of the Fourier-Wigner transform.

We explore the relation of the affine quantum harmonic analysis to the affine Weyl transform, which turns out to mirror the usual theory quite satisfactory. In particular, the Weyl transform is shown to be given by convolutions of functions with a fixed (unbounded) parity operator.

Finally, we show that quantum harmonic analysis provides a conceptual framework for covariant integral quantizations as studied by Gazeau and his collaborators [13, 41, 42, 123–125] : such quantization procedures are given by mapping f to $f \star_{\text{Aff}} S$ for some operator S. The admissibility of this S means that the quantization procedure gives a desirable resolution of the identity operator. We also use a result from the mathematical physics literature to characterize covariant integral quantizations in terms of abstract properties. A natural Cohen's class is introduced for the affine group, by generalizing the description of Cohen's class using operators in Paper A. In fact, we show that this notion of Cohen's class agrees with the other natural extensions of Cohen's class, namely that of taking convolutions with the affine Wigner distribution. It is also shown that this Cohen's class contains the class of *affine quadratic time-frequency distributions*.

Many of the arguments used in the paper rely only on having an irreducible, square integrable representation of a type 1 locally compact group. This suggests that some form of quantum harmonic analysis can be developed in this setting. In particular, these are all ingredients needed to for the discussion of admissible operators and defining a Cohen's class — without the need for a Wigner distribution.

Part II

Research Papers

Paper A

Mixed-State Localization Operators: Cohen's Class and Trace Class Operators

Franz Luef and Eirik Skrettingland

Published in
Journal of Fourier Analysis and Applications, 2019, volume 25, issue 4.

Paper A

Mixed-State Localization Operators: Cohen's Class and Trace Class Operators

Abstract

We study mixed-state localization operators from the perspective of Werner's operator convolutions which allows us to extend known results from the rank-one case to trace class operators. The idea of localizing a signal to a domain in phase space is approached from various directions such as bounds on the spreading function, probability densities associated to mixed-state localization operators, positive operator valued measures, positive correspondence rules and variants of approximation theorems for operator translates. Our results include a rigorous treatment of multiwindow-STFT filters and a characterization of mixed-state localization operators as positive correspondence rules. Furthermore we provide a description of the Cohen class in terms of Werner's convolution of operators and deduce consequences on positive Cohen class distributions, an uncertainty principle, uniqueness and phase retrieval for general elements of Cohen's class.

A.1 Introduction

We are addressing some key problems of time-frequency analysis: (i) How to measure the time-frequency content of a signal? (ii) What is the effect a (linear) filter has on a signal? Over the years engineers and mathematicians have investigated these questions and have proposed a variety of answers as is demonstrated by the vast literature [46,48,49,63–65,67,112,172,218,219]. We approach these problems from the perspective of quantum harmonic analysis and note that notions and results in [251] provide a unifying umbrella for some of the research in this direction such as localization operators, multiwindow STFT-filters, Cohen's class of quadratic time-frequency representations and the spreading function of a filter.

Harmonic analysis is based on the interplay between the translation of a function, convolution of functions and the Fourier transform. In [251] analogues of these notions are introduced for operators: The *translation* of an operator A on $L^2(\mathbb{R}^d)$ by a point $z = (x, \omega)$ in phase space $\mathbb{R}^{2d}$ is defined by conjugation with the time-frequency shift $\pi(z)$:

$$\alpha_z(A) := \pi(z)A\pi(z)^*,$$

where $\pi(z)\psi(t) = e^{2\pi i\omega t}\psi(t-x)$. In [203] we showed that this yields a natural class of Banach modules. There are two types of convolutions in this noncommutative setting: (i) The convolution between a function $f \in L^1(\mathbb{R}^{2d})$ and a trace class operator S:

$$f \star S := S \star f := \int_{\mathbb{R}^{2d}} f(y)\alpha_y(S)\, dy;$$

(ii) the convolution between two trace class operators S and T is defined by

$$S \star T(z) = \operatorname{tr}(S\alpha_z(\check{T}))$$

for $z \in \mathbb{R}^{2d}$, where $\check{T} = PTP$ is defined by conjugation by the parity operator P. Finally, the analogue of the Fourier transform is given by the *Fourier-Wigner transform* $\mathcal{F}_W(S)$ of a trace class operator S, which is the function given by

$$\mathcal{F}_W(S)(z) = e^{-\pi i x\cdot\omega}\operatorname{tr}(\pi(-z)S)$$

for $z \in \mathbb{R}^{2d}$. Note that the Fourier-Wigner transform and the spreading function differ only by a phase factor. The Fourier-Wigner transform has many properties analogous to those of the Fourier transform of functions [203, 251].
In the case of rank-one operators these concepts of quantum harmonic analysis turn into well-known objects from time-frequency analysis. Suppose $\varphi_2 \otimes \varphi_1$ is the rank-one operator for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then we have

$$f \star (\varphi_2 \otimes \varphi_1) = \int_{\mathbb{R}^{2d}} f(z)V_{\phi_1}\psi(z)\pi(z)\phi_2\, dz,$$

which is a localization operator (or STFT-filter or STFT-multiplier [105, 188]) and is denoted by $\mathcal{A}_f^{\varphi_1,\varphi_2}$, and f is called the mask of the STFT-filter. Similarly, the convolution of two rank-one operators becomes

$$(\phi \otimes \psi) \star (\check{\xi} \otimes \check{\eta})(z) = V_\eta\phi(z)\overline{V_\xi\psi(z)},$$

where $\check{\xi}(x) = \xi(-x)$, which reduces for $\eta = \psi$ and $\psi = \phi$ to the spectrogram [176]. The Fourier-Wigner transform of a rank-one operator is the ambiguity

function. There is also a Hausdorff-Young inequality associated to the Fourier-Wigner transform [203, 251], that in the rank-one case is the non-sharp Lieb's inequality for ambiguity functions [197].
Let us return to the objectives of this paper. Since localization operators are convolutions of a function and a rank-one operator, a natural extension of localization operators are operators of the form $f \star S$ for a trace-class operator S. The results of this paper indicate that these operators describe the time-frequency localization in various ways. For example we are interested in the amount of "spreading" in time and frequency that an operator performs on a function which we describe in form of bounds on the concentration of the spreading function, or equivalently of its Fourier-Wigner transform. The next theorem is an example for the type of statements we have in mind:

Theorem. *Let S be a trace-class operator and let $\Omega \subset \mathbb{R}^{2d}$ have finite Lebesgue measure $|\Omega|$ and assume that*

$$\int_\Omega |\mathcal{F}_W(S)(z)|^2 \, dz \geq 1 - \epsilon$$

for some $\epsilon \geq 0$. For any $p > 2$ we then have

$$|\Omega| \geq \frac{(1-\epsilon)^{p/(p-2)} \left(\frac{p}{2}\right)^{2d/(p-2)}}{\|S\|_{\mathcal{S}^1}^{2p/(p-2)}},$$

where $\|S\|_{\mathcal{S}^1}$ denotes the trace class norm of S.

One interpretation of this uncertainty principle is that a well-concentrated spreading function comes at the cost of a large trace class norm. The proof is a consequence of the Hausdorff-Young inequality for the Fourier-Wigner transform of S.
In the engineering literature [188, 192] one calls an operator

$$H = \sum_n \lambda_n \mathcal{A}_f^{\varphi_{n,1}, \varphi_{n,2}}$$

a *multiwindow STFT-filter*, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers and $\{\varphi_{n,1}\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,2}\}_{n\in\mathbb{N}}$ are sequences of functions in $L^2(\mathbb{R}^d)$, [188]. Multiwindow STFT filters might be thought of as operators that change the signal by some smearing. We give a rigorous treatment of the boundedness of multiwindow STFT filters depending on the sequence $(\lambda_n)_{n\in\mathbb{N}}$ and prove that multiwindow STFT-filters are given by a function convolved with an operator.
In addition we consider the set of multiwindow STFT-filters $f \star S$ for functions f for a fixed trace-class operator S. Using the approximation theorem for convolutions with

operators (Theorem A.3.15), we are able to show (under some assumptions on the Fourier-Wigner spectrum): (i) any Schatten class operator T may be approximated by operators of the form $f \star S$; (2) that the mask f is uniquely determined by the operator $f \star S$. As a sample we have results of the following form: For a trace-class operator S the following are equivalent:

1. The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ is empty.
2. The set of multiwindow STFT-filters $f \star S$ with $f \in L^1(\mathbb{R}^{2d})$ is dense in the set of trace-class operators.
3. Any mask $f \in L^\infty(\mathbb{R}^{2d})$ is uniquely determined by the multiwindow STFT-filter $f \star S$.

In order to gain some understanding of the notion of localization in this context, we focus on operators H_Ω of the form

$$H_\Omega = \chi_\Omega \star S$$

where χ_Ω is the indicator function of a measurable subset Ω of $\mathbb{R}^{2d}$ and S is a positive trace class operator with $\mathrm{tr}(S) = 1$. We refer to these operators as *mixed-state localization operators*.

Given a mixed-state localization operator $\chi_\Omega \star S$, one might ask whether it is possible to recover information about the domain Ω from the operator $\chi_\Omega \star S$. We show that the measure of Ω may be calculated from the eigenvalues of $\chi_\Omega \star S$ and we also consider the problem of reconstructing the domain Ω from H_Ω. Finally we also discuss in which sense an operator H_Ω measures the time-frequency content of a signal in a domain Ω. These questions have received some attention [4, 8] in recent years. Our techniques provide a way to handle unbounded domains, which have not been treated previously in the literature.

The treatment of mixed-state localization operators leads us to the investigation of Cohen class distributions [59]. We show that any Cohen class distribution $Q_S(\psi)$ is of the form

$$Q_S(\psi) = (\psi \otimes \psi) \star \check{S},$$

where S is some possibly unbounded operator on $L^2(\mathbb{R}^d)$. We establish an uncertainty principle for Cohen class distributions and ask whether any square-integrable function is uniquely determined by the associated Cohen class distribution, which in a special case was discussed in [145, Remark A.4] for the spectrogram. In addition we characterize when Cohen class distributions are positive and have the correct total energy properties.

We observe also that mixed-state localization operators define positive operator valued measures (POVMs), a standard tool in quantum mechanics, see [151,152,211] for some relations between POVMs and frame theory. By a theorem of Holevo [158] on positive correspondence rules we have that this is in a sense the only way to produce (covariant) POVMs. We will also argue that the notion of POVM is a natural framework for localization operators and Cohen's class of time-frequency distributions and that a POVM allows one to construct a probability measure on phase space. This measure is absolutely continuous with respect to Lebesgue measure and its Radon-Nikodym derivative is a positive Cohen class distribution.

A.2 Notation and terminology

If X is a Banach space we will denote its dual space by X', and for $x \in X$ and $x^* \in X'$ we write $\langle x^*, x\rangle_{X',X}$ to denote $x^*(x)$. $\langle \cdot, \cdot \rangle_{L^2}$ denotes the inner product on the Hilbert space $L^2(\mathbb{R}^d)$. Note that $\langle \cdot, \cdot \rangle_{X',X}$ is bilinear, whereas $\langle \cdot, \cdot \rangle_{L^2}$ is antilinear in the second argument. Elements of $\mathbb{R}^{2d}$ will often be written in the form $z = (x, \omega)$ for $x, \omega \in \mathbb{R}^d$, and the Lebesgue measure of a subset $\Omega \subset \mathbb{R}^{2d}$ will be denoted by $|\Omega|$. The characteristic function of $\Omega \subset \mathbb{R}^{2d}$ is denoted by χ_Ω. $\sigma(z, z')$ is the standard symplectic form $\sigma(z, z') = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$ of $z = (x_1, \omega_1)$ and $z' = (x_2, \omega_2)$. For two functions ξ, η in the Hilbert space $L^2(\mathbb{R}^d)$, we define the operator $\xi \otimes \eta$ on $L^2(\mathbb{R}^d)$ by $\xi \otimes \eta(\zeta) = \langle \zeta, \eta\rangle_{L^2} \xi$, where $\zeta \in L^2(\mathbb{R}^d)$. The space of Schwartz functions on $\mathbb{R}^{2d}$ is denoted by $\mathscr{S}(\mathbb{R}^{2d})$ and its dual space of tempered distributions by $\mathscr{S}'(\mathbb{R}^{2d})$. We introduce the parity operator P by $\check{\psi}(x) = P\psi(x) = \psi(-x)$ for any $x \in \mathbb{R}^d$ and $\psi : \mathbb{R}^d \to \mathbb{C}$, and define ψ^* by $\psi^*(x) = \overline{\psi(x)}$.

A.3 Preliminaries

A.3.1 Concepts from time-frequency analysis

The symplectic Fourier transform

For functions $f \in L^1(\mathbb{R}^{2d})$ we will use the *symplectic Fourier transform* $\mathcal{F}_\sigma f$, given by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z') e^{-2\pi i \sigma(z,z')} \, dz'$$

for $z \in \mathbb{R}^{2d}$, where σ is the standard symplectic form $\sigma((x_1, \omega_1), (x_2, \omega_2)) = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$. $\mathcal{F}_\sigma$ extends to a unitary operator on $L^2(\mathbb{R}^{2d})$, and this extension satisfies $\mathcal{F}_\sigma^2 = I_{L^2}$, where I_{L^2} is the identity operator [72].

The STFT, Wigner distribution and the Weyl calculus

If $\psi : \mathbb{R}^d \to \mathbb{C}$ and $z = (x, \omega) \in \mathbb{R}^{2d}$, we define the *translation operator* T_x by $T_x\psi(t) = \psi(t - x)$, the *modulation operator* M_ω by $M_\omega\psi(t) = e^{2\pi i \omega \cdot t}\psi(t)$ and the *time-frequency shifts* $\pi(z)$ by $\pi(z) = M_\omega T_x$. For $\psi, \phi \in L^2(\mathbb{R}^d)$ the *short-time Fourier transform* (STFT) $V_\phi\psi$ of ψ with window ϕ is the function on $\mathbb{R}^{2d}$ defined by

$$V_\phi\psi(z) = \langle \psi, \pi(z)\phi\rangle_{L^2}$$

for $z \in \mathbb{R}^{2d}$. By replacing the inner product above with a duality bracket, the STFT may be extended to other spaces, such as $\phi \in \mathscr{S}(\mathbb{R}^d), \psi \in \mathscr{S}'(\mathbb{R}^d)$. We will also refer to the *cross-ambiguity function* $A(\psi, \phi)$ of ψ and ϕ, defined by multiplying the STFT with a phase factor:

$$A(\psi, \phi)(z) = e^{\pi i x\cdot\omega} V_\phi\psi(z).$$

For more background on the ambiguity function and its utility in the theory of radar see [114, 131]. A close relative of the STFT is the *cross-Wigner distribution* of two functions ψ and ϕ on $\mathbb{R}^d$. By definition, the cross-Wigner distribution $W(\psi, \phi)$ is given by

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi\left(x + \frac{t}{2}\right) \overline{\phi\left(x - \frac{t}{2}\right)} e^{-2\pi i\omega\cdot t}\, dt.$$

This expression is similar to the definition of the STFT, and in fact $W(\psi, \phi) = \mathcal{F}_\sigma(A(\psi, \phi))$ [72].

Using the cross-Wigner distribution, we may introduce the *Weyl calculus*. For $f \in \mathscr{S}'(\mathbb{R}^{2d})$ and $\psi, \phi \in \mathscr{S}(\mathbb{R}^d)$, we define the *Weyl transform* L_f of f to be the operator given by

$$\left\langle L_f\psi, \phi\right\rangle_{\mathscr{S}',\mathscr{S}} = \langle f, W(\psi, \phi)\rangle_{\mathscr{S}',\mathscr{S}}\,.$$

f is called the *Weyl symbol* of the operator L_f.

Cohen's class of quadratic time-frequency distributions

A quadratic time-frequency distribution Q is said to be of *Cohen's class* if Q is given by

$$Q(\psi) = Q_\phi(\psi) := W(\psi, \psi) * \phi$$

for some $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$ [59, 131]. The class of functions ψ to which we may apply Q_ϕ clearly depends on the distribution ϕ. The Wigner distribution is obtained by picking $\phi = \delta_0$, where δ_0 is Dirac's delta distribution centered at 0. Cohen's class contains all shift-invariant, weakly continuous quadratic time-frequency distributions, as is made precise by the following Lemma from [131, Thm. 4.5.1].

Lemma A.3.1. *Let Q be a quadratic time-frequency distribution satisfying*

1. $Q(\pi(z)\psi) = T_z(Q(\psi))$,
2. $|Q(\psi_1, \psi_2)(0)| \leq \|\psi_1\|_2\|\psi_2\|_2$,

for all $z \in \mathbb{R}^{2d}$ *and* $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$. *Then* $Q(\psi) = W(\psi, \psi) * \phi$ *for some* $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$.

A.3.2 Concepts from operator theory

The Schatten classes of operators

In classical harmonic analysis one often studies the L^p-spaces of functions, and we will similarly need to introduce classes of operators in $\mathcal{L}(L^2)$ — the space of bounded linear operators on $L^2(\mathbb{R}^d)$ — with different properties. To introduce these classes, we need the *singular value decomposition* of compact operators on $L^2(\mathbb{R}^d)$ [220].

Proposition A.3.2. *Let S be a compact operator on* $L^2(\mathbb{R}^d)$. *There exist two orthonormal sets* $\{\psi_n\}_{n\in\mathbb{N}}$ *and* $\{\phi_n\}_{n\in\mathbb{N}}$ *in* $L^2(\mathbb{R}^d)$ *and a sequence* $\{s_n(S)\}_{n\in\mathbb{N}}$ *of non-negative numbers with* $s_n(S) \to 0$*, such that S may be expressed as*

$$S = \sum_{n\in\mathbb{N}} s_n(S)\psi_n \otimes \phi_n,$$

with convergence in the operator norm. The non-zero numbers among $\{s_n(S)\}_{n\in\mathbb{N}}$ *are called the singular values of S, and are the non-zero eigenvalues of the operator* $|S|$.

For $1 \leq p < \infty$ we define the *Schatten class* $\mathcal{S}^p$ of operators by

$$\mathcal{S}^p = \{T \text{ compact} : (s_n(T))_{n\in\mathbb{N}} \in \ell^p\}.$$

We will also write $\mathcal{S}^\infty = \mathcal{L}(L^2)$ with $\|\cdot\|_{\mathcal{S}^\infty}$ given by the operator norm to simplify the statement of some results. The Schatten class $\mathcal{S}^p$ becomes a Banach space under pointwise addition and scalar multiplication in the norm $\|S\|_{\mathcal{S}^p} = \left(\sum_{n\in\mathbb{N}} s_n(S)^p\right)^{1/p}$. Since these norms are defined in terms of ℓ^p-norms of sequences, we get that $\|\cdot\|_{\mathcal{L}(L^2)} \leq \|\cdot\|_p \leq \|\cdot\|_1$ for $1 \leq p \leq \infty$. Furthermore, the spaces $\mathcal{S}^p$ are ideals in $\mathcal{L}(L^2)$, meaning that $A \in \mathcal{L}(L^2)$ and $T \in \mathcal{S}^p$ implies that $AT, TA \in \mathcal{S}^p$ [232, Thm. 2.7].

The trace and trace class operators

Recall that an operator $S \in \mathcal{L}(L^2)$ is *positive* if $\langle S\psi, \psi\rangle_{L^2} \geq 0$ for any $\psi \in L^2(\mathbb{R}^d)$. For a positive operator $S \in \mathcal{L}(L^2)$, the *trace* of S is defined to be

$$\operatorname{tr}(S) = \sum_{n\in\mathbb{N}} \langle Se_n, e_n\rangle_{L^2}, \tag{A.3.1}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. This definition is independent of the orthonormal basis used, and the trace is linear and satisfies $\operatorname{tr}(ST) = \operatorname{tr}(TS)$ [220]. However, the expression in (A.3.1) may well be infinite, and is not well-defined for a general non-positive operator S. If $S \in \mathcal{S}^1$, then $\operatorname{tr}(S)$ is well-defined and a simple calculation shows that

$$\operatorname{tr}(S) = \sum_{n\in\mathbb{N}} s_n(S),$$

where the sum of singular values converges by the definition of $\mathcal{S}^1$. For this reason the class $\mathcal{S}^1$ is often referred to as *trace class operators*. By a celebrated theorem due to Lidskii, the trace $\operatorname{tr}(S)$ of $S \in \mathcal{S}^1$ equals the sum $\sum_{i=1}^{\infty} \lambda_i$ of the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ of S, where the eigenvalues are counted with algebraic multiplicity [232].

Using the trace we may state the duality relations of the Schatten p-classes [232, Thm 2.8 and 3.2].

Lemma A.3.3. *Let $1 \leq p < \infty$, and let q be the number determined by $\frac{1}{p} + \frac{1}{q} = 1$. The dual space of $\mathcal{S}^p$ is $\mathcal{S}^q$, and the duality may be given by*

$$\langle T, S\rangle_{\mathcal{S}^q,\mathcal{S}^p} = \operatorname{tr}(TS)$$

for $S \in \mathcal{S}^p$ and $T \in \mathcal{S}^q$.

Another well-known Schatten class is $\mathcal{S}^2$, known as the *Hilbert-Schmidt operators*. $\mathcal{S}^2$ is a Hilbert space under the inner product $\langle S, T\rangle_{\mathcal{S}^2} := \operatorname{tr}(ST^*)$ for $S, T \in \mathcal{S}^2$.

Remark A.1. The Schatten classes behave analogously to the L^p-spaces of functions — the duality relations are the same, and both $L^1(\mathbb{R}^{2d})$ and $\mathcal{S}^1$ have a natural linear functional given by the integral and trace, respectively. The intuition that L^p corresponds to $\mathcal{S}^p$ will often be useful, and is strengthened by the convolutions defined in Section A.3.4.

Vector-valued integration

We will need to integrate operator-valued functions $G : \mathbb{R}^{2d} \to \mathcal{L}(L^2)$ of the form $G(z) = g(z)F(z)$, where $g \in L^1(\mathbb{R}^{2d})$ and $F : \mathbb{R}^{2d} \to \mathcal{L}(L^2)$ is measurable, bounded and strongly continuous. The operator-valued integral $\int_{\mathbb{R}^{2d}} g(z)F(z)\, dz \in \mathcal{L}(L^2)$ is defined in a weak and pointwise sense: for any $\psi \in L^2(\mathbb{R}^d)$ we define $\left(\int_{\mathbb{R}^{2d}} g(z)F(z)\, dz\right)\psi$ by

$$\left\langle \left(\int_{\mathbb{R}^{2d}} g(z)F(z)\, dz\right)\psi, \phi\right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} g(z)\, \langle F(z)\psi, \phi\rangle_{L^2}\, dz$$

for any $\phi \in L^2(\mathbb{R}^d)$. This defines an operator $\int_{\mathbb{R}^{2d}} g(z)F(z)\, dz$, and we get the norm estimate $\| \int_{\mathbb{R}^{2d}} g(z)F(z)\, dz\|_{\mathcal{L}(L^2)} \leq \|g\|_{L^1} \sup_{z\in\mathbb{R}^{2d}} \|F(z)\|_{\mathcal{L}(L^2)}$ [203]. For fixed $\psi \in L^2(\mathbb{R}^d)$ the $L^2(\mathbb{R}^d)$-valued function $z \mapsto g(z)F(z)\psi$ is even Bochner integrable, so the reader familiar with this theory may interpret the integral $\int_{\mathbb{R}^{2d}} g(z)F(z)\psi\, dz \in L^2(\mathbb{R}^d)$ as a Bochner integral, see [83] for a reference.

A.3.3 Localization operators and multiwindow STFT-filters

Given a function f on $\mathbb{R}^{2d}$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, we define the *localization operator* (or STFT-filter [189]) $\mathcal{A}_f^{\varphi_1,\varphi_2}$ *with mask* f *and windows* φ_1, φ_2 by

$$\mathcal{A}_f^{\varphi_1,\varphi_2}\psi = \int_{\mathbb{R}^{2d}} f(z)V_{\varphi_1}\psi(z)\pi(z)\varphi_2\, dz$$

for $\psi \in L^2(\mathbb{R}^d)$, where the integral is interpreted in the weak sense discussed above. We will in particular be interested in the case where $\varphi_1 = \varphi_2$ and $f = \chi_\Omega$ is the characteristic function of some measurable subset $\Omega \subset \mathbb{R}^{2d}$, and we will write $\mathcal{A}_\Omega^\varphi := \mathcal{A}_{\chi_\Omega}^{\varphi,\varphi}$ in this case.

We will follow Kozek [189] and call any operator H of the form

$$H = \sum_n \lambda_n \mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}}$$

a *multiwindow STFT-filter*, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers and $\{\varphi_{n,1}\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,2}\}_{n\in\mathbb{N}}$ are sequences of functions in $L^2(\mathbb{R}^d)$. Hence a multiwindow STFT-filter is a possibly infinite linear combination of localization operators with common mask f. We will return to the question of convergence of the sum in equation (A.3.3) in Section A.5. For further information on filters and their use in the engineering literature the reader may consult, for instance, [157,189,192,209].

A.3.4 Convolutions of operators and functions

This section introduces the theory of convolutions of operators and functions due to Werner [251]. In order to introduce these convolution operations, we will first need to define a shift for operators. For $z \in \mathbb{R}^{2d}$ and $A \in \mathcal{L}(L^2)$, we define the operator $\alpha_z(A)$ by

$$\alpha_z(A) := \pi(z)A\pi(z)^*.$$

It is easily confirmed that $\alpha_z\alpha_{z'} = \alpha_{z+z'}$, and we will informally think of α as a shift or translation of operators. The interpretation of α as a shift of operators has also been remarked in the signal processing literature by Kozek [188,189].

Similarly we define the analogue of the involution $f \mapsto \check{f}$ of a function, for an operator $A \in \mathcal{L}(L^2)$ by

$$\check{A} = PAP,$$

where P is the parity operator $P\psi(x) = \psi(-x)$ for $\psi \in L^2(\mathbb{R}^d)$. The intuition that α is a shift of operators is supported by considering the Weyl symbol [188, 203].

Lemma A.3.4. *Let $f \in L^1(\mathbb{R}^{2d})$, and let L_f be the Weyl transform of f.*

- $\alpha_z(L_f) = L_{T_z f}$ *for* $z \in \mathbb{R}^{2d}$.
- $\check{L_f} = L_{\check{f}}$.

Using α, Werner defined a convolution operation between functions and operators [251]. If $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$ we define the *operator* $f \star S$ by

$$f \star S := S \star f := \int_{\mathbb{R}^{2d}} f(y)\alpha_y(S)\, dy$$

where the integral is interpreted as in Section A.3.2. Then $f \star S \in \mathcal{S}^1$ and $\|f \star S\|_{\mathcal{S}^1} \leq \|f\|_{L^1} \|S\|_{\mathcal{S}^1}$ [203, Prop. 2.5].

For two operators $S, T \in \mathcal{S}^1$, Werner defined the *function* $S \star T$ by

$$S \star T(z) = \operatorname{tr}(S\alpha_z(\check{T}))$$

for $z \in \mathbb{R}^{2d}$.

Remark A.2. The notation $\star$ may therefore denote either the convolution of two functions or the convolution of an operator with a function. The correct interpretation will be clear from the context.

The following result shows that $S \star T \in L^1(\mathbb{R}^{2d})$ for $S, T \in \mathcal{S}^1$ and provides an important formula for its integral [251, Lem. 3.1]. In the simplest case where S and T are rank-one operators, this formula is the so-called Moyal identity for the STFT [114, p. 57].

Lemma A.3.5. *Let $S, T \in \mathcal{S}^1$. The function $z \mapsto \operatorname{tr}(S\alpha_z T)$ for $z \in \mathbb{R}^{2d}$ is integrable and $\|\operatorname{tr}(S\alpha_z T)\|_{L^1} \leq \|S\|_{\mathcal{S}^1}\|T\|_{\mathcal{S}^1}$.*

Furthermore,

$$\int_{\mathbb{R}^{2d}} \operatorname{tr}(S\alpha_z T)\, dz = \operatorname{tr}(S)\operatorname{tr}(T).$$

The convolutions can be defined on other L^p-spaces and Schatten p-classes by duality [203, 251]. As an important example, the convolution $f \star S \in \mathcal{L}(L^2)$ for $f \in L^\infty(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$ is defined by the relation

$$\langle f \star S, T \rangle_{\mathcal{L}(L^2), \mathcal{S}^1} = \left\langle f, \check{S} \star T \right\rangle_{L^\infty, L^1} \qquad \text{for any } T \in \mathcal{S}^1, \tag{A.3.2}$$

By writing these dualities explicitly, the definition becomes

$$\mathrm{tr}((f \star S)T) = \int_{\mathbb{R}^{2d}} f(z)(\check{S} \star T)(z)\, dz \qquad \text{for any } T \in \mathcal{S}^1. \tag{A.3.3}$$

When extended to other functions and operator spaces, the convolutions satisfy a version of Young's inequality.

Proposition A.3.6. *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^{2d})$, $S \in \mathcal{S}^p$ and $T \in \mathcal{S}^q$, then the following convolutions may be defined and satisfy the norm estimates*

$$\|f \star T\|_{\mathcal{S}^r} \leq \|f\|_{L^p} \|T\|_{\mathcal{S}^q},$$
$$\|S \star T\|_{L^r} \leq \|S\|_{\mathcal{S}^p} \|T\|_{\mathcal{S}^q}.$$

The convolutions of operators and functions are associative, a fact that is non-trivial since the convolutions between operators and functions can produce both operators and functions as output [203, 251]. Commutativity and bilinearity, however, follows straight from the definitions.

Furthermore, the convolutions preserve positivity and identity elements [234].

Proposition A.3.7.
1. *If $S, T \in \mathcal{L}(L^2)$ are positive operators and f is a positive function, then $f \star S$ is a positive operator and $S \star T$ is a positive function.*
2. *If 1 is the constant function $1(z) = 1$ for $z \in \mathbb{R}^{2d}$ and I_{L^2} is the identity operator on $L^2(\mathbb{R}^d)$, then $1 \star S = I_{L^2}$ and $I_{L^2} \star S = 1$ for every $S \in \mathcal{S}^1$ with* $\mathrm{tr}(S) = 1$.

The convolutions make the Schatten classes $\mathcal{S}^p$ into *Banach modules* over $L^1(\mathbb{R}^{2d})$ if the module multiplication is defined by $(f, S) \mapsto f \star S$ for $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^p$, [203]. By using the Cohen-Hewitt theorem for Banach modules [128], one obtains that any operator in $\mathcal{S}^p$ for $p < \infty$ can be written as a convolution [203, Prop. 7.4].

Proposition A.3.8. *Given $T \in \mathcal{S}^p$ for $p < \infty$, there exists $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^p$ such that $T = f \star S$.*

A.3.5 Localization operators and spectrograms as convolutions

In [203] we established that Werner's convolutions provide a conceptual framework for localization operators, as shown by the following result.

Lemma A.3.9. *Let f be a function on $\mathbb{R}^{2d}$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then the localization operator $\mathcal{A}_f^{\varphi_1,\varphi_2}$ can be expressed as the convolution of the function f and the rank-one operator $\varphi_2 \otimes \varphi_1$,*

$$\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1).$$

Similarly, the convolution of two rank-one operators reduces to a familiar object in the simplest case — namely the spectrogram.

Lemma A.3.10. *Let $\phi, \psi, \xi, \eta \in L^2(\mathbb{R}^d)$. Then the function $V_\eta\phi(z)\overline{V_\xi\psi(z)}$ may be expressed as the convolution of two rank-one operators,*

$$(\phi \otimes \psi) \star (\check{\xi} \otimes \check{\eta})(z) = V_\eta\phi(z)\overline{V_\xi\psi(z)}.$$

for $z \in \mathbb{R}^{2d}$. In particular, if $\eta = \psi$ and $\psi = \phi$, then $(\phi \otimes \phi) \star (\check{\eta} \otimes \check{\eta})$ is the spectrogram $|V_\eta\phi|^2$.

Note that in the physics literature the spectrogram $|V_\eta\phi|^2$ is called the *Husimi function* of ϕ when η is a Gaussian [176].

A.3.6 The Fourier-Wigner transform of operators

For operators $S \in \mathcal{S}^1$, the *Fourier-Wigner transform* $\mathcal{F}_W(S)$ of S is the function given by

$$\mathcal{F}_W(S)(z) = e^{-\pi i x\cdot\omega}\mathrm{tr}(\pi(-z)S)$$

for $z \in \mathbb{R}^{2d}$. In the special case of an operator of rank one, the Fourier-Wigner transform is the ambiguity function [203, Lemma 6.1].

Lemma A.3.11. *If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, then $\mathcal{F}_W(\varphi_2 \otimes \varphi_1)(z) = A(\varphi_2, \varphi_1)(z)$.*

The Fourier-Wigner transform has many properties analogous to those of the Fourier transform of functions [203, 251]. It extends to a unitary operator $\mathcal{F}_W : \mathcal{S}^2 \to L^2(\mathbb{R}^{2d})$, and by the following proposition it interacts with the convolutions defined by Werner in the expected way.

Proposition A.3.12. *Let $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{S}^1$.*

1. $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$.
2. $\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$.

In time-frequency analysis and signal processing, operators are sometimes studied by considering the so-called *spreading function* [102], which expresses the operator as an infinite linear combination of time-frequency shifts. In fact, the Fourier-Wigner transform and the spreading function differ only by a phase factor [203].

Proposition A.3.13. *1. If $S \in \mathcal{S}^1$ has spreading function $f \in L^1(\mathbb{R}^{2d})$, i.e.*

$$S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)\, dz,$$

where the integral is interpreted as in Section A.3.2, then

$$\mathcal{F}_W(S)(z) = e^{i\pi x\cdot\omega} f(z).$$

2. The Weyl symbol a_S of $S \in \mathcal{S}^1$ is given by

$$a_S = \mathcal{F}_\sigma \mathcal{F}_W(S).$$

As for the Fourier transform of functions, there is also a Hausdorff-Young inequality associated to the Fourier-Wigner transform [203, 251].

Proposition A.3.14. *Let $1 \leq p \leq 2$ and let q be the conjugate exponent determined by $\frac{1}{p} + \frac{1}{q} = 1$. If $S \in \mathcal{S}^p$, then $\mathcal{F}_W(S) \in L^q(\mathbb{R}^{2d})$ with norm estimate*

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \|S\|_{\mathcal{S}^p}.$$

Using Lieb's uncertainty principle [131, 197] we can improve this result in a special case [203].

Corollary A.3.14.1. *Let $2 \leq p < \infty$. If $S \in \mathcal{S}^1$, then*

$$\|\mathcal{F}_W(S)\|_{L^p} \leq \left(\frac{2}{p}\right)^{d/p} \|S\|_{\mathcal{S}^1}.$$

Approximation theorems for operators

Werner [251] has proved a version of Wiener's approximation theorem for operators. The theorem was later generalized in [182], and more equivalent statements and a proof may be found in [182, 203]. We state the relevant parts of the theorem for our purposes.

Theorem A.3.15. *Let $S \in \mathcal{S}^1$.*

(a) The following are equivalent.

(a1) The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ is empty.

(a2) If $f \in L^\infty(\mathbb{R}^{2d})$ and $f \star S = 0$, then $f = 0$.

(a3) $L^1(\mathbb{R}^{2d}) \star S$ is dense in $\mathcal{S}^1$.

(a4) If $T \in \mathcal{L}(L^2)$ and $S \star T = 0$, then $T = 0$.

(b) The following are equivalent.

(b1) The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ *has Lebesgue measure* 0.

(b2) If $f \in L^2(\mathbb{R}^{2d})$ *and* $f \star S = 0$*, then* $f = 0$.

(b3) $L^2(\mathbb{R}^{2d}) \star S$ *is dense in* $\mathcal{S}^2$.

(b4) If $T \in \mathcal{S}^2$ *and* $S \star T = 0$*, then* $T = 0$.

(c) The following are equivalent.

(c1) The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ *has dense complement.*

(c2) If $f \in L^1(\mathbb{R}^{2d})$ *and* $f \star S = 0$*, then* $f = 0$.

(c3) $L^\infty(\mathbb{R}^{2d}) \star S$ *is weak* dense in* $\mathcal{L}(L^2)$.

(c4) If $T \in \mathcal{S}^1$ *and* $S \star T = 0$*, then* $T = 0$.

A.3.7 Schwartz operators and tempered distributions

The theory of convolutions and Fourier transforms of operators can be extended to more general objects than bounded operators, just as the convolution and Fourier transform of functions is extended from the L^p-spaces to tempered distributions. To define this extension, we start by defining two classes of operators. We let $\mathfrak{S}$ be the set of pseudodifferential operators with Weyl symbol in the Schwartz class $\mathscr{S}(\mathbb{R}^{2d})$, and we let $\mathfrak{S}'$ be the set of pseudodifferential operators with Weyl symbol in the tempered distributions $\mathscr{S}'(\mathbb{R}^{2d})$. These sets of operators were studied in detail by Keyl et al. in [179]. They show that $\mathfrak{S}$ may be equipped with a topology making it a Frechet space, and that $\mathfrak{S}'$ is the topological dual space of $\mathfrak{S}$ in this topology. Hence one may define convolutions and Fourier transforms on $\mathfrak{S}'$ using duality. We summarize the main results in the following proposition, and refer to Section 5 of [179] for proofs.

Proposition A.3.16. *1. Let* $S, T \in \mathfrak{S}$, $A \in \mathfrak{S}'$, $f \in \mathscr{S}(\mathbb{R}^{2d})$ *and* $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$. *The following convolutions may be defined:*

$$
\begin{aligned}
&S \star T \in \mathscr{S}(\mathbb{R}^{2d}) && f \star S \in \mathfrak{S} \\
&S \star A \in \mathscr{S}'(\mathbb{R}^{2d}) && \phi \star S \in \mathfrak{S}' \\
& && f \star A \in \mathfrak{S}'.
\end{aligned}
$$

2. *The definitions in part (1) are compatible with those in Section A.3.4 whenever both are applicable.*

3. *The Fourier-Wigner transform may be extended to a topological isomorphism* $\mathcal{F}_W : \mathfrak{S}' \to \mathscr{S}'(\mathbb{R}^{2d})$.

4. *The relations $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$ and $\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$ still hold for operators S,T and a function f whenever the convolutions are defined by part (1).*

5. *The Weyl symbol of $A \in \mathfrak{S}'$ is given by $\mathcal{F}_\sigma \mathcal{F}_W(A)$.*

Remark A.3. By the Schwartz kernel theorem (see [159]), we know that we may identify $\mathfrak{S}'$ with the continuous operators from $\mathcal{S}(\mathbb{R}^{2d})$ to $\mathcal{S}'(\mathbb{R}^{2d})$.

A.3.8 Positive operator valued measures

In Section A.9 of this paper we will argue that the notion of a positive operator valued measure is a natural framework for localization operators and Cohen's class of time-frequency distributions. This notion is more commonly used in operator theory and quantum mechanics [33]. We recall the basic concepts.

Definition A.3.1. Let $\mathcal{B}(\mathbb{R}^{2d})$ denote the σ-algebra of Borel subsets of $\mathbb{R}^{2d}$. A positive operator valued measure (POVM) on $\mathbb{R}^{2d}$ is a mapping $F : \mathcal{B}(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ such that

1. $F(M)$ is a positive operator for any $M \in \mathcal{B}(\mathbb{R}^{2d})$,
2. $F(\mathbb{R}^{2d})$ is the identity operator on $L^2(\mathbb{R}^d)$,
3. $F(\cup_{i\in\mathbb{N}} M_i) = \sum_{i\in\mathbb{N}} F(M_i)$ for any countable collection of disjoint, measurable subsets $\{M_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^{2d}$, where the sum converges in the weak operator topology.

Hence a POVM on $\mathbb{R}^{2d}$ assigns a positive operator on $L^2(\mathbb{R}^d)$ to each Borel subset of $\mathbb{R}^{2d}$. Convergence in the weak operator topology of the sum $\sum_{i\in\mathbb{N}} F(M_i)$ to the operator $T := F(\cup_{i\in\mathbb{N}} M_i)$ means that for any $\psi, \phi \in L^2(\mathbb{R}^d)$ we have $\sum_{i\in\mathbb{N}} \langle F(M_i)\psi, \phi\rangle_{L^2} = \langle T\psi, \phi\rangle_{L^2}$. Any POVM F appearing in this text will be *covariant*, meaning that $\alpha_z(F(M)) = F(M+z)$ for any $z \in \mathbb{R}^{2d}$ and $M \in \mathcal{B}(\mathbb{R}^{2d})$, where $M + z = \{m + z : m \in M\}$ and α is the shift of operators defined in Section A.3.4.

Integration and the probability measures associated to a POVM

Let F be a fixed POVM. For each $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$, F allows us to construct a probability measure μ^F_ψ on $\mathbb{R}^{2d}$ by defining

$$\mu^F_\psi(\Omega) = \langle F(\Omega)\psi, \psi\rangle_{L^2}$$

for $\Omega \subset \mathbb{R}^{2d}$.

Using the measures μ^F_ψ, we may define a notion of integration w.r.t. the POVM F [37, Sec. 5 Thm. 9]

Lemma A.3.17. *If $f : \mathbb{R}^{2d} \to \mathbb{C}$ is a measurable, bounded function and F a POVM on $\mathbb{R}^{2d}$, then there exists a unique operator $A_f \in \mathcal{L}(L^2)$ such that $\langle A_f\psi, \psi\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z)d\mu^F_\psi$ for any $\psi \in L^2(\mathbb{R}^d)$.*

We denote the operator A_f by $\int_{\mathbb{R}^{2d}} f(z)dF$. For $f \in L^\infty(\mathbb{R}^{2d})$ and $\Omega \subset \mathbb{R}^{2d}$, we define $\int_\Omega f\, dF := \int_{\mathbb{R}^{2d}} \chi_\Omega f\, dF$. It is easily seen that $\int_\Omega dF = F(\Omega)$.

A.4 The time-frequency concentration of the spreading function

When considering a filter H, it is often of interest to determine the amount of "spreading" in time and frequency that H performs on a signal. By Proposition A.3.13, the Fourier-Wigner function $\mathcal{F}_W(H)$ is, up to a phase factor, the spreading function of H. Hence the Fourier-Wigner transform $\mathcal{F}_W(H)(z)$ is the weight of the time-frequency shift $\pi(z)$ when H is decomposed as a linear combination of time-frequency shifts:

$$H = \int_{\mathbb{R}^{2d}} e^{i\pi x\cdot\omega}\mathcal{F}_W(H)(z)\pi(z)\, dz,$$

where the integral is interpreted in the sense of Section A.3.2 for $H \in \mathcal{S}^1$. For instance, an operator which only shifts signals slightly in time and frequency will have a spreading function concentrated around 0 in $\mathbb{R}^{2d}$.

To measure the effect of H on a signal, we would therefore like to obtain bounds on the concentration of the spreading function, or equivalently of $\mathcal{F}_W(H)$. In fact, the Hausdorff Young inequality in Proposition A.3.14 does exactly this. By this inequality, if $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then if $H \in \mathcal{S}^p$ we get

$$\left(\int_{\mathbb{R}^{2d}} |\mathcal{F}_W(H)|^q\, dz\right)^{1/q} \le \|H\|_{\mathcal{S}^p}. \tag{A.4.1}$$

Hence we can interpret the Hausdorff Young inequality as saying that the Schatten class norm of H provides information on the concentration of the spreading function of H. If H is trace class, then the above inequality holds for all $2 \le q < \infty$, and we may replace $\|H\|_{\mathcal{S}^p}$ by $\|H\|_{\mathcal{S}^1}$, since $\|H\|_{\mathcal{S}^p} \le \|H\|_{\mathcal{S}^1}$ for any $p \ge 1$.

Remark A.4. Since the Fourier-Wigner transform is unitary from $\mathcal{S}^2$ to $L^2(\mathbb{R}^{2d})$ [203], we actually have an equality in equation (A.4.1) for $p = q = 2$.

Following the reasoning used by Gröchenig to prove an uncertainty principle for functions in [131, Thm. 3.3.3.], we can use Corollary A.3.14.1 to obtain an uncertainty principle for spreading functions of filters.

Theorem A.4.1. *Let $S \in \mathcal{S}^1$ and let $\Omega \subset \mathbb{R}^{2d}$ with $|\Omega| < \infty$ and assume that*

$$\int_\Omega |\mathcal{F}_W(S)(z)|^2 \, dz \geq 1 - \epsilon$$

for some $\epsilon \geq 0$. For any $p > 2$ we then have

$$|\Omega| \geq \frac{(1-\epsilon)^{p/(p-2)} \left(\frac{p}{2}\right)^{2d/(p-2)}}{\|S\|_{\mathcal{S}^1}^{2p/(p-2)}}.$$

In particular, for $p = 4$ we obtain

$$|\Omega| \geq \frac{(1-\epsilon)^2 2^d}{\|S\|_{\mathcal{S}^1}^4}.$$

Proof. By Hölder's inequality with $p' = \frac{p}{2}$ and $q' = \frac{p}{p-2}$, we find that

$$\begin{aligned} 1 - \epsilon &\leq \int_\Omega |\mathcal{F}_W(S)(z)|^2 \, dz \\ &\leq \left(\int_{\mathbb{R}^{2d}} |\mathcal{F}_W(S)|^{2\frac{p}{2}} \, dz \right)^{2/p} \left(\int_{\mathbb{R}^{2d}} \chi_\Omega(z)^{\frac{p}{p-2}} \, dz \right)^{(p-2)/p} \\ &\leq \left(\frac{2}{p} \right)^{2d/p} \|S\|_{\mathcal{S}^1}^2 |\Omega|^{(p-2)/p}, \end{aligned}$$

where the last inequality follows from Corollary A.3.14.1. Rearranging this inequality, we obtain

$$|\Omega| \geq \frac{(1-\epsilon)^{p/(p-2)} \left(\frac{p}{2}\right)^{2d/(p-2)}}{\|S\|_{\mathcal{S}^1}^{2p/(p-2)}}.$$

□

One interpretation of this uncertainty principle is that a well-concentrated spreading function comes at the cost of a large trace class norm. As an example, consider the special case of an *underspread* trace class operator S, meaning that the support of S is contained in some bounded subset $\Omega \subset \mathbb{R}^{2d}$ with $|\Omega| << 1$ [190]. Assume that S is normalized in the sense that $\|S\|_{\mathcal{S}^2}^2 = \int_{\mathbb{R}^{2d}} |\mathcal{F}_W(S)|^2 \, dz = 1$. By assumption we then have

$$\int_\Omega |\mathcal{F}_W(S)|^2 \, dz = 1,$$

and by the previous result with $\epsilon = 0$ we conclude that

$$1 >> |\Omega| \geq \frac{2^d}{\|S\|_{\mathcal{S}^1}^4},$$

hence $\|S\|_{\mathcal{S}^1} >> 1$.

A.5 Multiwindow STFT-filters are convolutions

One aim of the recent paper [203] was to apply Werner's theory of convolutions to localization operators (or STFT-filters [104, 189]) using the identity

$$\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1).$$

There are several advantages to this approach. Proposition A.3.6 provides a simple relationship between the properties of the mask f and the operator $\mathcal{A}_f^{\varphi_1,\varphi_2}$, the Fourier-Wigner transform is a useful tool for considering the Weyl symbol of $\mathcal{A}_f^{\varphi_1,\varphi_2}$ and the approximation theorem (Theorem A.3.15) is a powerful tool to deduce new insights into localization operators. We will now show that multiwindow STFT-filters also allow a description in terms of convolutions.

In Section A.3.3, a multiwindow STFT-filter H was defined as a linear combination of localization operators with a fixed mask f:

$$H = \sum_{n=1}^{N} \lambda_n \mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}}$$

for some sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ and sequences $\{\varphi_{n,1}\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,2}\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$. Since any $\mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}}$ may be written as the convolution $f \star (\varphi_{n,2} \otimes \varphi_{n,1})$, we get by the linearity of convolutions that

$$H = f \star \sum_{n=1}^{N} \lambda_n \varphi_{n,2} \otimes \varphi_{n,1}.$$

Hence H is the convolution of f with the operator $\sum_{n=1}^{N} \lambda_n \varphi_{n,2} \otimes \varphi_{n,1}$. When N is finite, the sum $\sum_{n=1}^{N} \lambda_n \varphi_{n,2} \otimes \varphi_{n,1}$ is always a trace class operator, so by Proposition A.3.6 we may pick the mask $f \in L^p(\mathbb{R}^{2d})$ for any $1 \leq p \leq \infty$ and obtain a bounded operator. However, if we follow Hlawatsch and Kozek [192] and introduce infinite linear combinations of localization operators, both the properties of the mask f and convergence must be considered more carefully.

Proposition A.5.1. *Fix $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Let $\{\varphi_{n,1}\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,2}\}_{n\in\mathbb{N}}$ be two orthonormal sequences in $L^2(\mathbb{R}^d)$.*

1. *If $\{\lambda_n\}_{n\in\mathbb{N}} \in \ell^p$ and $f \in L^q(\mathbb{R}^{2d})$, then the sum defining the multiwindow STFT-filter $\sum_{n=1}^{\infty} \lambda_n \mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}}$ converges in $\mathcal{S}^r$. Furthermore,*

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}} = f \star \sum_{n=1}^{\infty} \lambda_n \varphi_{2,n} \otimes \varphi_{1,n}.$$

2. *Conversely, any operator of the form $f \star S \in \mathcal{S}^r$ for $f \in L^q(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^p$ can be written as a multiwindow STFT-filter with mask f. That is, there exists some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \ell^p$ of non-negative numbers and $\{\varphi'_{n,1}\}_{n\in\mathbb{N}}$, $\{\varphi'_{n,2}\}_{n\in\mathbb{N}}$ two orthonormal sequences in $L^2(\mathbb{R}^d)$ such that*

$$f \star S = \sum_{n=1}^{\infty} \lambda_n \mathcal{A}_f^{\varphi'_{n,1},\varphi'_{n,2}}$$

where the sum converges in $\mathcal{S}^r$.

Proof. 1. The sum $\sum_{n=1}^{\infty} \lambda_n \varphi_{2,n} \otimes \varphi_{1,n}$ converges in the norm of $\mathcal{S}^p$ to an operator in $\mathcal{S}^p$ — this follows by showing that the partial sums form a Cauchy sequence in $\mathcal{S}^p$. By Proposition A.3.6 the convolution $(h, S) \mapsto h \star S$ is continuous from $L^q(\mathbb{R}^{2d}) \times \mathcal{S}^p$ into $\mathcal{S}^r$, and we may write

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n \mathcal{A}_f^{\varphi_{n,1},\varphi_{n,2}} &= \sum_{n=1}^{\infty} \lambda_n f \star (\varphi_{2,n} \otimes \varphi_{1,n}) \\ &= f \star \sum_{n=1}^{\infty} \lambda_n (\varphi_{2,n} \otimes \varphi_{1,n}),\end{aligned}$$

where continuity considerations were used in the last step.

2. S has a singular value decomposition

$$S = \sum_{n=1}^{\infty} \lambda_n \varphi'_{2,n} \otimes \varphi'_{1,n}$$

converging in the norm of $\mathcal{S}^p$, with $\{\lambda_n\}_{n\in\mathbb{N}} \in \ell^p$ and $\{\varphi'_{n,1}\}_{n\in\mathbb{N}}$, $\{\varphi'_{n,2}\}_{n\in\mathbb{N}}$ two orthonormal sequences in $L^2(\mathbb{R}^d)$. By the continuity properties of the convolutions, we can write

$$\begin{aligned}f \star S &= f \star \sum_{n=1}^{\infty} \lambda_n \varphi'_{2,n} \otimes \varphi'_{1,n} \\ &= \sum_{n=1}^{\infty} \lambda_n f \star (\varphi'_{2,n} \otimes \varphi'_{1,n}) \\ &= \sum_{n=1}^{\infty} \lambda_n \mathcal{A}_f^{\varphi'_{n,1},\varphi'_{n,2}}.\end{aligned}$$

□

Remark A.5. The setting in [192] is that of square-summable sequences $\{\lambda_n\}_{n\in\mathbb{N}} \in \ell^2$ and masks f with unspecified properties. The above proposition makes the relationship between properties of $\{\lambda_n\}_{n\in\mathbb{N}}$, f and the multi-window STFT-filter more transparent, showing how properties of $\{\lambda_n\}_{n\in\mathbb{N}}$ and f are reflected in Schatten class properties of the multi-window STFT-filter. In particular the proposition gives conditions on $\{\lambda_n\}_{n\in\mathbb{N}}$ and f to guarantee that the filter is a well-defined bounded operator, analogous to the conditions for the convolutions of two functions to be well-defined by Young's inequality.

Remark A.6.

1. By Proposition A.3.8 *any* operator $H \in \mathcal{S}^p$ for $1 \leq p < \infty$ can be written in the form $H = f \star S$ for $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^p$. With this in mind, the study of multiwindow STFT-filters is the study of the Schatten classes $\mathcal{S}^p$ from a certain perspective.
2. By Proposition A.3.16, one might also define multiwindow STFT-filters $f \star S$ when $f \in \mathscr{S}(\mathbb{R}^{2d})$ and $S \in \mathfrak{S}'$, or when $f \in \mathscr{S}'(\mathbb{R}^{2d})$ and $S \in \mathfrak{S}$.

A.5.1 The Fourier-Wigner transform and multiwindow STFT-filters

In [188], Kozek studied multiwindow STFT-filters by considering their Weyl symbols. One advantage from writing multiwindow STFT-filters using convolutions is that the relationship between such filters and their Weyl symbol becomes the relationship between convolutions and Fourier transforms.

Proposition A.5.2. *Let $S \in \mathcal{S}^1$ and $f \in L^1(\mathbb{R}^{2d})$. The Weyl symbol $a_{f\star S}$ of the multiwindow STFT $f \star S$ is given by $f * a_S$, where a_S is the Weyl symbol of S.*

Proof. By Proposition A.3.13, $a_{f\star S} = \mathcal{F}_\sigma \mathcal{F}_W(f \star S)$. From Proposition A.3.12 we know that $\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) * \mathcal{F}_W(S)$. Furthermore, we have the relation $\mathcal{F}_\sigma(gh) = \mathcal{F}_\sigma(g) * \mathcal{F}_\sigma(h)$ for $g, h \in L^1(\mathbb{R}^{2d})$; a fact that follows easily from the corresponding fact for the regular Fourier transform. Hence

$$\begin{aligned} a_{f\star S} = \mathcal{F}_\sigma \mathcal{F}_W(f \star S) &= \mathcal{F}_\sigma\left(\mathcal{F}_\sigma(f)\mathcal{F}_W(S)\right) \\ &= f * \mathcal{F}_\sigma \mathcal{F}_W(S) = f * a_S, \end{aligned}$$

where we have used that $\mathcal{F}_\sigma$ is its own inverse. □

Remark A.7. Proposition 4.2 holds for more general f and S, as long as the convolutions, $\mathcal{F}_\sigma$ and $\mathcal{F}_W$ are interpreted as their extensions to $\mathscr{S}'(\mathbb{R}^{2d})$ and $\mathfrak{S}'$, respectively [179].

Since the Weyl symbol of the operator $\varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ is the Wigner function $W(\varphi,\varphi)$ [203], we get in particular that the Weyl symbol a_Ω of a localization operator $\mathcal{A}_\Omega^\varphi = \chi_\Omega \star (\varphi \otimes \varphi)$ is

$$a_\Omega = \chi_\Omega * W(\varphi,\varphi),$$

as is well-known [63].

Remark A.8. Consider $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. By the same arguments as above we get that the Weyl symbol of the localization operator $f \star (\varphi_2 \otimes \varphi_1)$ is $f * W(\varphi_2, \varphi_1)$. When Kozek and Hlawatsch generalized from localization operators (or STFT-filters) $f \star (\varphi_2 \otimes \varphi_1)$ to multiwindow STFT-filters $f \star S$ for $S \in \mathcal{S}^2(\mathbb{R}^d)$ in [192], they did so by considering the Weyl symbol $f * W(\varphi_2, \varphi_1)$ of a localization operator, and replaced $W(\varphi_2, \varphi_1)$ with an arbitrary function k in $L^2(\mathbb{R}^{2d})$. Hence they considered the operator with Weyl symbol $f * k$, which by Proposition A.5.2 is the operator $f \star L_k$, where L_k is the Weyl transform of k. Since $\mathcal{S}^2(\mathbb{R}^d)$ is exactly the set of bounded operators with Weyl symbol in $L^2(\mathbb{R}^{2d})$ [215], the set of operators $f \star L_k$ for $k \in L^2(\mathbb{R}^{2d})$ equals the set of operators $f \star S$ for $S \in \mathcal{S}^2(\mathbb{R}^d)$.

A.5.2 Density of multiwindow STFT-filters and uniqueness of masks

We will now fix an operator $S \in \mathcal{S}^1$, and consider the corresponding set of multiwindow STFT-filters $f \star S$ for functions f. Using the approximation theorem for convolutions with operators (Theorem A.3.15), we will be able to answer two questions about this set of filters. First, we ask whether any operator T may be approximated by operators of the form $f \star S$, where T belongs some specified Schatten p-class of operators. We then ask whether the mask f is uniquely determined by the operator $f \star S$.

Proposition A.5.3. *Let $S \in \mathcal{S}^1$. The following are equivalent.*

1. *The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ is empty.*
2. *The set of multiwindow STFT-filters $f \star S$ with $f \in L^1(\mathbb{R}^{2d})$ is dense in $\mathcal{S}^1$.*
3. *Any mask $f \in L^\infty(\mathbb{R}^{2d})$ is uniquely determined by the multiwindow STFT-filter $f \star S$.*

Proof. The result is simply a restatement of parts (a1), (a2) and (a3) of Theorem A.3.15 in the terminology of multiwindow STFT-filters. □

Remark A.9. Since the Weyl symbol of S is $a_S = \mathcal{F}_\sigma \mathcal{F}_W(S)$, we see that $\mathcal{F}_W(S) = \mathcal{F}_\sigma a_S$. Hence part (1) of the result could equivalently have been formulated using the set of zeros of $\mathcal{F}_\sigma a_S$ — the symplectic Fourier transform of the Weyl symbol of S.

By relaxing the conditions on the set of zeros of the Fourier-Wigner transform of S, we obtain a result for Hilbert-Schmidt operators from Theorem A.3.15.

Proposition A.5.4. *Let $S \in \mathcal{S}^1$. The following are equivalent.*

1. *The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ has Lebesgue measure zero.*

2. *The set of multiwindow STFT-filters $f \star S$ with $f \in L^2(\mathbb{R}^{2d})$ is dense in $\mathcal{S}^2$.*

3. *Any mask $f \in L^2(\mathbb{R}^{2d})$ is uniquely determined by the multiwindow STFT-filter $f \star S$.*

With an even weaker assumption on the zeros of $\mathcal{F}_W(S)$, Theorem A.3.15 gives yet another result.

Proposition A.5.5. *Let $S \in \mathcal{S}^1$. The following are equivalent.*

1. *The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ has dense complement in $\mathbb{R}^{2d}$.*

2. *The set of multiwindow STFT-filters $f \star S$ with $f \in L^\infty(\mathbb{R}^{2d})$ is weak*-dense in $\mathcal{L}(L^2)$.*

3. *Any mask $f \in L^1(\mathbb{R}^{2d})$ is uniquely determined by the multiwindow STFT-filter $f \star S$.*

If we pick $S = \varphi_2 \otimes \varphi_1$ for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ in the three previous propositions, the conditions on the set of zeros of $\mathcal{F}_W(S)$ becomes a condition on the zeros of the ambiguity function $A(\varphi_2, \varphi_1)$. We noted this in [203], where we generalized previous results from [25,26]. For such rank-one operators, Proposition A.5.3 raises a natural question: Does there exist a pair of windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ such that $A(\varphi_2, \varphi_1)$ has no zeros, except when $\varphi_1 = \varphi_2$ is a Gaussian? In the case where $\varphi_1 = \varphi_2$ Hudson's Theorem [131] requires φ to be a Gaussian. Similarly, Toft [244] has shown that $V_{\varphi_1}\varphi_2$ can only be a positive function if $\varphi_1 = \varphi_2$ is a Gaussian. However, the question of whether one may find $\varphi_1 \neq \varphi_2$ such that $A(\varphi_2, \varphi_1)$ has no zeros has been adressed in [139].

Example A.5.1. Condition (1) of Proposition A.5.4 is much weaker than the corresponding condition in Proposition A.5.3. It will for instance be satisfied by $S = h_n \otimes h_n$, where h_n is the n'th Hermite function. In fact, $A(h_n, h_n)$ has a finite number of zeros, namely the zeros of some n'th Laguerre polynomials [114, p. 64].

A.6 Mixed-state localization operators

Among the localization operators $\mathcal{A}_f^{\varphi_1,\varphi_2}$, those of the form $\mathcal{A}_\Omega^\varphi$ for some measurable $\Omega \subset \mathbb{R}^{2d}$ have a special interpretation: if $\psi \in L^2(\mathbb{R}^d)$, the signal $\mathcal{A}_\Omega^\varphi \psi$ is interpreted as the part of ψ "living on" Ω [63], which explains the "localization" terminology. In Section A.5 we considered multiwindow STFT-filters as a generalization of localization operators — a natural question is then whether we can find some subset of the multiwindow STFT-filters where the "localization" interpretation above is still reasonable. We define a *mixed-state localization operator* to be an operator H_Ω of the form

$$H_\Omega = \chi_\Omega \star S$$

where $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset and S is a positive trace class operator with $\mathrm{tr}(S) = 1$.

Remark A.10.

1. The relation between general localization operators $\mathcal{A}_f^{\varphi_1,\varphi_2}$ and those of the form $\mathcal{A}_\Omega^\varphi$ is the same as the relationship between multiwindow STFT-filters and mixed-state localization operators: A general localization operator may be written as $\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1)$, and the localization operators $\mathcal{A}_\Omega^\varphi$ are exactly those localization operators $f \star (\varphi_2 \otimes \varphi_1)$ such that $f = \chi_\Omega$ for some $\Omega \subset \mathbb{R}^{2d}$ and $\varphi_2 \otimes \varphi_1$ is a positive operator with $\mathrm{tr}(\varphi_2 \otimes \varphi_1) = 1$. This follows from the fact that $\varphi_2 \otimes \varphi_1$ is positive if and only if $\varphi_1 = \varphi_2$, and $\mathrm{tr}(\varphi_2 \otimes \varphi_1) = \langle \varphi_2, \varphi_1 \rangle_{L^2}$.
2. In quantum mechanics, operators $\varphi \otimes \varphi$ with $\|\varphi\|_{L^2} = 1$ describe *pure states* of a system [72]. More general states, the *mixed states* are described by a positive operator $S \in \mathcal{S}^1$ with $\mathrm{tr}(S) = 1$. So a mixed-state localization operator is by definition given by the convolution of χ_Ω with an operator describing a mixed state — hence the name.

Given a mixed-state localization operator $\chi_\Omega \star S$, one might ask whether it is possible to recover information about the domain Ω from the operator $\chi_\Omega \star S$. The next proposition shows that the measure of Ω may be calculated from the eigenvalues $\chi_\Omega \star S$. In Section A.6.2 we will consider the problem of reconstructing the domain Ω in more detail.

Proposition A.6.1. *Let $\Omega \subset \mathbb{R}^{2d}$ be a subset of finite Lebesgue measure, and let $S \in \mathcal{S}^1$ be a positive operator with* $\mathrm{tr}(S) = 1$. *Then*

1. $\mathrm{tr}(\chi_\Omega \star S) = |\Omega|$.

2. If $\{\lambda_i\}_{i=1}^N$ are the eigenvalues of $\chi_\Omega \star S$ counted with algebraic multiplicity, then

$$\sum_{i=1}^N \lambda_i = |\Omega|.$$

Proof.

1. By Proposition A.3.12, we have that

$$\mathcal{F}_W(\chi_\Omega \star S)(0) = \mathcal{F}_\sigma(\chi_\Omega)(0)\mathcal{F}_W(S)(0),$$

and by the definitions of $\mathcal{F}_W$ and $\mathcal{F}_\sigma$ we we have that

$$\mathcal{F}_\sigma(\chi_\Omega)(0)\mathcal{F}_W(S)(0) = |\Omega|\mathrm{tr}(S) = |\Omega|.$$

2. This follows from the first part along with Lidskii's equality from Section A.3.2. □

Remark A.11. 1. The proof of this proposition would work equally well if χ_Ω is replaced by any $f \in L^1(\mathbb{R}^{2d})$, as long as $|\Omega|$ is replaced by $\int_{\mathbb{R}^{2d}} f(z)\, dz$.

2. This result holds in particular for the localization operators $\mathcal{A}_f^\varphi$ by picking $S = \varphi \otimes \varphi$. In this context it is well-known, see for instance [104]. The proposition therefore supports the intuition that $\chi_\Omega \star S$ is a generalized localization operator.

A.6.1 A characterization of mixed-state localization operators

By our definition of mixed-state localization operators, a positive trace class operator S with $\mathrm{tr}(S) = 1$ assigns to each domain $\Omega \subset \mathbb{R}^{2d}$ a mixed-state localization operator $\chi_\Omega \star S$. In fact, $f \star S$ belongs to $\mathcal{L}(L^2)$ for any $f \in L^\infty(\mathbb{R}^{2d})$ by Proposition A.3.6, and in this way S defines a bounded, linear mapping from $L^\infty(\mathbb{R}^{2d})$ to $\mathcal{L}(L^2)$. The next theorem, originally due to Holevo [158], characterizes all bounded linear mappings $L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ of this form in terms of four properties. We provide an outline of the proof in [251] in our notation for completeness. The details may also be found in Proposition 1 and Lemma 3 in [183] where the result is proved in a more general setting.

Theorem A.6.2. *Let $\Gamma : L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ be a linear operator satisfying*

1. *$\Gamma(\chi_{\mathbb{R}^{2d}}) = I_{L^2}$, where I_{L^2} the identity operator,*
2. *$\Gamma(T_z f) = \alpha_z(\Gamma(f))$ for any $z \in \mathbb{R}^{2d}$ and $f \in L^\infty(\mathbb{R}^{2d})$,*
3. *$\Gamma(f)$ is a positive operator whenever f is a positive function,*
4. *Γ is weak* to weak*-continuous.*

Then there exists a positive operator $S \in \mathcal{S}^1(\mathbb{R}^d)$ with $\mathrm{tr}(S) = 1$ such that

$$\Gamma(f) = f \star S$$

for any $f \in L^\infty(\mathbb{R}^{2d})$.

Proof. Before we begin, we remark that assumption (4) is exactly what we need to conclude that Γ is the Banach space adjoint of some bounded linear operator $\Gamma_* : \mathcal{S}^1 \to L^1(\mathbb{R}^{2d})$. The existence of Γ_* is how assumption (4) will be used, although it will not be explicitly mentioned in this brief outline.
The first step of the proof is to show that Γ induces a bounded mapping Γ :

$L^1(\mathbb{R}^{2d}) \to \mathcal{S}^1$. A calculation using all the assumptions of the proposition shows that for a positive $f \in L^\infty(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^d)$ and a positive operator $T \in \mathcal{S}^1$ we have

$$\int_{\mathbb{R}^{2d}} \operatorname{tr}(T\alpha_z(\Gamma(f)))\, dz = \int_{\mathbb{R}^{2d}} f(z)\, dz\, \operatorname{tr}(T).$$

Comparing this with Lemma A.3.5 we see that $\Gamma(f) \in \mathcal{S}^1$ with $\|\Gamma(f)\|_{\mathcal{S}^1} = \operatorname{tr}(\Gamma(f)) = \int_{\mathbb{R}^{2d}} f(z)\, dz$. This result holds for positive $f \in L^\infty(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^{2d})$, and using this we may extend Γ to a well-defined bounded operator from $L^1(\mathbb{R}^{2d})$ to $\mathcal{S}^1$.

Using $\Gamma : L^1(\mathbb{R}^{2d}) \to \mathcal{S}^1$ we can construct a measure on $\mathbb{R}^{2d}$ with values in $\mathcal{S}^1$. For a bounded, Lebesgue measurable subset $\Omega \subset \mathbb{R}^{2d}$, we define a measure by $\Omega \mapsto \Gamma(\chi_\Omega)$. By our previous calculation we have $\|\Gamma(\chi_\Omega)\|_{\mathcal{S}^1} = \int_{\mathbb{R}^{2d}} \chi_\Omega(z)\, dz = |\Omega|$. This shows that our $\mathcal{S}^1$-valued measure is absolutely continuous with respect to Lebesgue measure, and since $\mathcal{S}^1$ has the Radon-Nikodym property [1] it follows that there is some measurable $\bar{S} : \mathbb{R}^{2d} \to \mathcal{S}^1$ such that [2]

$$\Gamma(f) = \int_{\mathbb{R}^{2d}} f(z)\bar{S}(z)\, dz.$$

The proof is now concluded by showing that assumption (2) and uniqueness of Radon-Nikodym derivatives imply that the function $\bar{S}(z)$ is given by $\bar{S}(z) = \alpha_z(S)$ for some fixed $S \in \mathcal{S}^1$ — see [251] or [183] for the details. □

Remark A.12.

1. Mappings $\Gamma : L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ having these four properties are called *positive correspondence rules* by Werner [251].

2. Recently, a similar result for $\Gamma : \mathcal{S}'(\mathbb{R}^{2d}) \to \mathfrak{S}'$ has been proved by Cordero et al. [60, Thm. 4.7] at the level of Weyl symbols.

3. The proof that mappings of the form $\Gamma(f) = f \star S$ are positive correspondence rules, for positive $S \in \mathcal{S}^1$ with $\operatorname{tr}(S) = 1$, is deferred to Section A.9 — see the remark following Proposition A.9.4.

We claim that Theorem A.6.2 shows that our definition of mixed-state localization operators is natural. Consider the case of a localization operator $\mathcal{A}_\Omega^\varphi$ for $\Omega \subset \mathbb{R}^{2d}$ and $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. These localization operators define a mapping $\Gamma_\varphi : L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ by $f \mapsto \mathcal{A}_f^\varphi = f \star (\varphi \otimes \varphi)$. For Γ_φ, the four properties in Theorem A.6.2 are true and have natural interpretations:

[1] This follows from Theorem 1 on page 79 of [83], as $\mathcal{S}^1$ is the dual space of the compact operators.

[2] We have ignored one issue: we need to restrict Ω to bounded subsets to ensure that $\chi_\Omega \in L^1(\mathbb{R}^{2d})$. The technical details needed to circumvent this issue are given in the proof of Lemma 3.1 in [183].

1. We have that $\Gamma_\varphi(\chi_{\mathbb{R}^{2d}})\psi = \mathcal{A}^\varphi_{\mathbb{R}^{2d}}\psi = \psi$ for $\psi \in L^2(\mathbb{R}^d)$, which formalizes the fact that localizing $\psi \in L^2(\mathbb{R}^d)$ to the whole time-frequency plane $\mathbb{R}^{2d}$ should return ψ.

2. For a characteristic function χ_Ω, the property
$$\Gamma_\varphi(T_z\chi_\Omega) = \alpha_z(\Gamma_\varphi\chi_\Omega)$$
says that
$$\mathcal{A}^\varphi_{\Omega-z} = \mathcal{A}^{\pi(z)\varphi}_\Omega$$
where $\Omega - z = \{z' - z : z' \in \Omega\}$. We may interpret this as saying that shifting the domain Ω of a localization operator by $z \in \mathbb{R}^{2d}$ is equivalent to replacing the window φ with the time-frequency shift $\pi(z)\varphi$.

3. Since $\Gamma_\varphi(\chi_\Omega) = \mathcal{A}^\varphi_\Omega$ is interpreted as an operator that picks out the part of a signal living in Ω in the time-frequency plane, it makes sense that $\left\langle \mathcal{A}^\varphi_\Omega\psi, \psi\right\rangle_{L^2} \geq 0$ — i.e. $\Gamma_\varphi(\chi_\Omega)$ is a positive operator.

4. $\Gamma_\varphi : L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ is weak* to weak*-continuous, in particular assigning a localization operator $\mathcal{A}^\varphi_\Omega$ to a domain Ω is continuous in this sense.

It seems natural to require that a generalization of localization operators also satisfies (1), (2), (3) and (4), and Theorem A.6.2 shows that we are then lead to our definition of mixed-state localization operators.

A.6.2 Uniqueness of the domain

In recent years the question of obtaining the domain Ω from the localization operator $\mathcal{A}^\varphi_\Omega$ has received some attention [4,8]. In this section we will consider the theoretical possibility of such reconstruction for the mixed-state localization operators: if $S \in \mathcal{S}^1$, when is the domain $\Omega \subset \mathbb{R}^{2d}$ uniquely determined by the mixed-state localization operator $\chi_\Omega \star S$, up to sets of Lebesgue measure zero?[3] Since the localization operators $\mathcal{A}^\varphi_\Omega$ form a subset of the mixed-state localization operators, our results will also be applicable to such operators. Our results follow from Theorem A.3.15 — the approximation theorem for convolutions with operators. The first result concerns domains Ω with $|\Omega| < \infty$.

Theorem A.6.3. *1. If $S \in \mathcal{S}^1$ is such that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S)(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$, then any $\Omega \subset \mathbb{R}^{2d}$ with finite Lebesgue measure is uniquely determined by the operator $\chi_\Omega \star S$, up to Lebesgue measure zero.*

[3] By "up to sets of Lebesgue measure zero" we mean that we regard two sets Ω, Ω' to be equal if $|\Omega \triangle \Omega'| = 0$, where $\triangle$ is the symmetric difference of sets.

2. *If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ are windows such that the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$, then any $\Omega \subset \mathbb{R}^{2d}$ with finite Lebesgue measure is uniquely determined by the operator $\mathcal{A}_\Omega^{\varphi_1,\varphi_2}$, up to Lebesgue measure zero.*

Proof. Follows from the implication (1) $\implies$ (3) in Proposition A.5.3. □

Remark A.13. In [8] it is shown that the theory of accumulated spectrograms gives a method for reconstructing a compact domain Ω, using the spectrograms of a finite subset of the eigenfunctions of $\mathcal{A}_{R\cdot\Omega}^\varphi$ as $R \to \infty$. Note, however, that this requires knowledge of $\mathcal{A}_{R\cdot\Omega}^\varphi$ as $R \to \infty$, and hence not merely of $\mathcal{A}_\Omega^\varphi$. On the other hand it is also shown in [8] that χ_Ω can be *estimated* using only the spectrograms of a finite number of eigenfunctions of $\mathcal{A}_\Omega^\varphi$. In a coming work we use quantum harmonic analysis to show that this is possible for any mixed-state localization operator with compact domain.

To the knowledge of the authors, the problem of reconstructing *unbounded* domains Ω from localization operators $\mathcal{A}_\Omega^\varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ has not previously been considered in the literature. We will cover the more general question of reconstructing an unbounded domain Ω from a mixed-state localization operator $\chi_\Omega \star S$. Since an unbounded set Ω may have infinite Lebesgue measure, we will not be able to use that $\chi_\Omega \in L^1(\mathbb{R}^{2d})$ as we did in the proof of the previous corollary. We need to consider χ_Ω as an element of $L^\infty(\mathbb{R}^{2d})$. This leads to a stronger condition on the set of zeros of the Fourier-Wigner transform.

Theorem A.6.4. 1. *If $S \in \mathcal{S}^1$ is such that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S)(z) = 0\}$ is empty, then any measurable $\Omega \subset \mathbb{R}^{2d}$ is uniquely determined by the operator $\chi_\Omega \star S$, up to Lebesgue measure zero.*

2. *If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ are windows such that the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty, then any measurable $\Omega \subset \mathbb{R}^{2d}$ is uniquely determined by the operator $\mathcal{A}_\Omega^{\varphi_1,\varphi_2}$, up to Lebesgue measure zero.*

Proof. The proof is the same as in Theorem A.6.3, except that we use Proposition A.5.5. □

A.7 Cohen's class and convolutions of operators

In Section A.3.1 we defined Cohen's class to be those quadratic time-frequency representations Q_ϕ of the form

$$Q_\phi(\psi) = W(\psi, \psi) * \phi \tag{A.7.1}$$

for some $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$ and any $\psi \in \mathscr{S}(\mathbb{R}^d)$. In this section we give a characterization of Cohen's class as convolutions with a fixed operator. We will show that many properties of the Cohen's class distribution may be precisely described as properties of the corresponding operator.

Proposition A.7.1. *For $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$, the associated Cohen's class distribution Q_ϕ is given by*

$$Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi \text{ for } \psi \in \mathscr{S}(\mathbb{R}^d), \tag{A.7.2}$$

where L_ϕ is the Weyl transform of ϕ.
Conversely, any operator $A \in \mathfrak{S}'$ determines a Cohen's class distribution by

$$Q_A(\psi) := (\psi \otimes \psi) \star \check{A} \text{ for } \psi \in \mathscr{S}(\mathbb{R}^d).$$

Proof. We will apply the symplectic Fourier transform twice to equation (A.7.2) and use parts (4) and (5) of Proposition A.3.16. First note that

$$\mathcal{F}_\sigma((\psi \otimes \psi) \star L_\phi) = \mathcal{F}_W(\psi \otimes \psi)\mathcal{F}_W(L_\phi) = A(\psi, \psi)\mathcal{F}_W(L_\phi)$$

using Lemma A.3.11. We now apply $\mathcal{F}_\sigma$ again, and since $\mathcal{F}_\sigma\mathcal{F}_\sigma$ is the identity operator we find

$$\begin{aligned}(\psi \otimes \psi) \star L_\phi &= \mathcal{F}_\sigma(A(\psi, \psi)\mathcal{F}_W(L_\phi)) \\ &= \mathcal{F}_\sigma(A(\psi, \psi)) * \mathcal{F}_\sigma\mathcal{F}_W(L_\phi) \\ &= W(\psi, \psi) * \phi = Q_\phi(\psi),\end{aligned}$$

where we have used that $\mathcal{F}_\sigma(A(\psi, \psi)) = W(\psi, \psi)$ and that $\mathcal{F}_\sigma\mathcal{F}_W L_\phi$ is the Weyl symbol of L_ϕ by part (5) of Proposition A.3.16. The second statement follows easily from the first: let ϕ be the Weyl symbol of $\check{A}$. The first part states that $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi = (\psi \otimes \psi) \star \check{A} = Q_A(\psi)$, showing that Q_A is of Cohen's class. □

Remark A.14. In light of Lemma A.3.1, this proposition shows that any shift-invariant, [4] weakly continuous quadratic time-frequency distribution is given by a convolution with a fixed operator on $L^2(\mathbb{R}^d)$.

By Proposition A.7.1, any Cohen's class distribution may be described by either a distribution $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$ or by an operator $A \in \mathfrak{S}'$, where

$$Q_\phi = Q_A \text{ if } L_\phi = \check{A}.$$

We have defined Q_A in terms of $\check{A}$ to simplify formulas in Section A.8, and the reader should note that A and $\check{A}$ share all relevant properties, such as positivity,

[4] In the sense that $Q(\pi(z)\psi) = T_z(Q(\psi))$ for $z \in \mathbb{R}^{2d}$ and $\psi \in L^2(\mathbb{R}^d)$.

trace and membership of Schatten classes. Using that $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$, we may apply the theory of convolutions of operators to deduce some simple results on Cohen's class distributions.

Proposition A.7.2. *Fix $1 \leq p \leq \infty$. Consider a Cohen's class distribution Q_ϕ for $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$. Let L_ϕ be the Weyl transform of ϕ. If $L_\phi \in \mathcal{S}^p$, then $Q_\phi(\psi)$ is well-defined for any $\psi \in L^2(\mathbb{R}^d)$ and $Q_\phi(\psi) \in L^p(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^p} \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{S}^p}$. In particular, if $L_\phi \in \mathcal{L}(L^2)$, then $Q_\phi(\psi) \in L^\infty(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^\infty} \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)}$.*

Proof. For any $\psi \in L^2(\mathbb{R}^d)$ we have that $\psi \otimes \psi \in \mathcal{S}^1$ with $\|\psi \otimes \psi\|_{\mathcal{S}^1} = \|\psi\|_{L^2}^2$. Since $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$ by Proposition A.7.1, the results follow from Proposition A.3.6. □

Remark A.15. By Pool's Theorem [215], the condition that $L_\phi \in \mathcal{S}^2$ is equivalent to $\phi \in L^2(\mathbb{R}^{2d})$. Unfortunately there is no equally simple characterization of those ϕ such that $L_\phi \in \mathcal{S}^1$ or $L_\phi \in \mathcal{L}(L^2)$.

Example A.7.1.

1. The Wigner distribution $Q_\phi(\psi) = W(\psi, \psi)$ is given by $\phi = \delta_0$ in equation (A.7.1). By Proposition A.7.1, $W(\psi, \psi)$ is also given by

$$W(\psi, \psi) = (\psi \otimes \psi) \star L_{\delta_0}$$

for $\psi \in \mathcal{S}(\mathbb{R}^d)$. By a result of Grossmann, $L_{\delta_0} = 2^d P$, where P is the parity operator [146].

2. Fix a window $\varphi \in L^2(\mathbb{R}^d)$ and consider the operator $S = \varphi \otimes \varphi$. Then $\check{S} = \check{\varphi} \otimes \check{\varphi}$, and by Proposition A.7.1, S defines a Cohen's class distribution Q_S by

$$Q_S(\psi) = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi}) = |V_\varphi \psi|^2,$$

where the last expression follows from Lemma A.3.10. This Cohen's class distribution is therefore the spectrogram. The corresponding function ϕ, i.e. the Weyl symbol of $\check{\varphi} \otimes \check{\varphi}$, is the Wigner distribution $W(\check{\varphi}, \check{\varphi})$.

The idea of using operators to define time-frequency distributions appeared in the work of Wigner [255]. Wigner assumed the existence of a self-adjoint operator $A(z) \in \mathcal{L}(L^2)$ for each $z \in \mathbb{R}^{2d}$, and then defined a distribution $Q_A(\psi)(z) = \langle A(z)\psi, \psi \rangle_{L^2}$ for $\psi \in L^2(\mathbb{R}^d)$. As Janssen notes [172], it follows from Lemma A.3.1 that the desirable property $Q_A(\pi(z')\psi) = T_{z'} Q_A(\psi)$ is only satisfied if $A(z) = \pi(z) A \pi(z)^*$ for some fixed operator $A \in \mathcal{L}(L^2)$. In this case we get by the definition of the convolution of two operators that $Q_A(\psi) = (\psi \otimes \psi) \star \check{A}$. However, this approach is not pursued any further than this remark in [172].

A.7.1 Positive Cohen's class distributions

We say that a Cohen's class distribution Q_ϕ is *positive* if $Q_\phi(\psi)(z) \geq 0$ for all $z \in \mathbb{R}^{2d}$ and ψ in the domain of Q_ϕ. As has been pointed out by Gröchenig [131, Ch. 14.6], positivity of Q_ϕ may be expressed in terms of the corresponding operator L_ϕ.

Proposition A.7.3. *Let Q_ϕ be a Cohen's class distribution such that the Weyl transform L_ϕ is bounded on $L^2(\mathbb{R}^d)$. Then Q_ϕ is positive if and only if L_ϕ is a positive operator.*

Proof. If $Q_\phi(\psi)$ is positive, then we have in particular for any $\psi \in L^2(\mathbb{R}^d)$ that

$$0 \leq Q_\phi(\psi)(0) = \left\langle \check{L_\phi}\pi(0)^*\psi, \pi(0)^*\psi \right\rangle_{L^2} = \left\langle \check{L_\phi}\psi, \psi \right\rangle_{L^2},$$

where we have used that $(\psi \otimes \psi) \star L_\phi(z) = \left\langle \check{L_\phi}\pi(z)^*\psi, \pi(z)^*\psi \right\rangle_{L^2}$, as follows from the definition of the convolution of operators. This shows that $\check{L_\phi}$ is positive, hence L_ϕ is positive. We have used that the function $(\psi \otimes \psi) \star L_\phi(z)$ is in fact a continuous function [203, Prop 3.3 (3)] to ensure that it has a well-defined value at 0. If we assume that L_ϕ is positive, then we get that Q_ϕ is positive since $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$, $\psi \otimes \psi$ is a positive operator and the convolution of positive operators is a positive function by Proposition A.3.7. □

The condition that a time-frequency distribution Q_ϕ should be positive is a natural requirement. One might therefore ask what conditions $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$ should satisfy to ensure that Q_ϕ is positive. By the previous proposition, we may equivalently ask which conditions ϕ must satisfy to ensure that the Weyl transform L_ϕ is a positive operator. This question is of interest in quantum mechanics, and providing a general answer has turned out to be difficult. The so-called KLM conditions due to Kastler [175], and Loupias and Miracle-Sole [199, 200] are the most famous result of this kind, and we now formulate these conditions in our context, using the notation from [61, 72].

Theorem A.7.4. *Let ϕ be a real-valued function on $\mathbb{R}^{2d}$ such that the Weyl transform $L_\phi \in \mathcal{S}^1$. The Cohen's class distribution Q_ϕ is positive if and only if*

1. *ϕ is continuous.*
2. *For every $N \geq 1$ and every N-tuple $(z_1, ..., z_N) \in \left(\mathbb{R}^{2d}\right)^N$ the $N \times N$ matrix with entries*
$$M_{jk} = e^{-2\pi i \sigma(z_j, z_k)} \mathcal{F}_\sigma(\phi)(z_j - z_k)$$
is positive semidefinite.

Proof. The KLM-conditions state that if ϕ is a real-valued function on $\mathbb{R}^{2d}$ such that the Weyl transform $L_\phi \in \mathcal{S}^1$, then the operator L_ϕ is positive if and only if ϕ satisfies the two properties above [72, Prop 306 and Thm 309]. By Proposition A.7.3, Q_ϕ is positive if and only if L_ϕ is a positive operator. □

There are other, more recent results of this nature. Cordero, de Gosson and Nicola [61] have recently proved a version of the KLM-conditions that seems more tractable for numerical verification. In fact, their conditions characterize those $\phi \in L^2(\mathbb{R}^{2d})$ such that the Weyl transform $L_\phi \in \mathcal{S}^2$ is a positive operator. In terms of Cohen's class, their condition characterizes those $\phi \in L^2(\mathbb{R}^{2d})$ such that Q_ϕ is positive. The reader is referred to [61] for precise statements and proofs.

A.7.2 Cohen's class distributions with the correct total energy property

Following Janssen [172], we say that a Cohen's class distribution Q_ϕ has the *correct total energy property* if

$$\int_{\mathbb{R}^{2d}} Q_\phi(\psi)(z)\, dz = \|\psi\|_{L^2}^2$$

for all $\psi \in L^2(\mathbb{R}^d)$. One might think of $Q_\phi(\psi)$ as an energy distribution for the signal ψ, and so one would hope that the total energy $\|\psi\|_{L^2}^2$ equals the integral of the energy distribution $Q_\phi(\psi)$. We now show how this property of the distribution Q_ϕ is related to properties of the Weyl transform L_ϕ.

Proposition A.7.5. *Let Q_ϕ be a Cohen's class distribution, and let L_ϕ be the Weyl transform of ϕ. If $L_\phi \in \mathcal{S}^1$, then*

$$\int_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \|\psi\|_{L^2}^2 \operatorname{tr}(L_\phi) \tag{A.7.3}$$

for any $\psi \in L^2(\mathbb{R}^d)$. If in addition $\phi \in L^1(\mathbb{R}^{2d})$, then

$$\int_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \|\psi\|_{L^2}^2 \int_{\mathbb{R}^{2d}} \phi(z)\, dz \tag{A.7.4}$$

Proof. By Proposition A.7.1

$$Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi.$$

By the definition of the convolution of two operators, $(\psi \otimes \psi) \star L_\phi = \operatorname{tr}((\psi \otimes \psi)\alpha_z(PL_\phi P))$. Since L_ϕ is assumed to be trace class, we may apply Lemma A.3.5 to find that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \operatorname{tr}((\psi \otimes \psi)\alpha_z(PL_\phi P))\, dz &= \operatorname{tr}(\psi \otimes \psi)\operatorname{tr}(PL_\phi P) \\ &= \|\psi\|_{L^2}^2 \operatorname{tr}(P^2 L_\phi) = \|\psi\|_{L^2}^2 \operatorname{tr}(L_\phi).\end{aligned}$$

We have used that $\operatorname{tr}(\psi \otimes \psi) = \|\psi\|_{L^2}^2$, which is a simple consequence of the definition of the trace. If $\phi \in L^1(\mathbb{R}^{2d})$, we may use the well-known relation

$$\operatorname{tr}(L_\phi) = \int_{\mathbb{R}^{2d}} \phi(z)\, dz$$

between a distribution ϕ and its Weyl transform in this case to complete the proof [72, Prop. 286]. □

Remark A.16. There are many examples of Cohen's class distributions Q_ϕ where $L_\phi \in \mathcal{S}^1$ yet $\phi \notin L^1(\mathbb{R}^{2d})$, so that equation (A.7.4) does not apply. For instance, if $\phi = W(\check{\varphi}, \check{\varphi})$ for $\varphi \in L^2(\mathbb{R}^d)$, we saw in Example A.7.1 that Q_ϕ is a spectrogram and $L_\phi = \check{\varphi} \otimes \check{\varphi} \in \mathcal{S}^1$. For $W(\check{\varphi}, \check{\varphi}) \in L^1(\mathbb{R}^{2d})$ to hold, φ must be an element of the so-called *Feichtinger algebra* [72, 95], in particular φ must be continuous. Hence equation (A.7.3) holds for a set of Cohen's class distributions where equation (A.7.4) does not even make sense, and equation (A.7.3) and the connection to the trace of an operator is new to the best of our knowledge.

In the special case of $\phi \in \mathscr{S}(\mathbb{R}^{2d})$, equation (A.7.4) is contained in Section 2.4.2 of Janssen's survey [172]. In this case both $L_\phi \in \mathcal{S}^1$ and $\phi \in L^1(\mathbb{R}^{2d})$ are satisfied, so Janssen needed neither equation (A.7.3) nor the connection to trace class operators.

A recurring theme in Janssen's thorough survey [172] is that positivity for a Cohen's class distribution Q_ϕ is incompatible with many other desirable properties of Q_ϕ. Proposition A.7.5 shows that Q_ϕ may be both positive and have the correct total energy property if $L_\phi \in \mathcal{S}^1$, but the next corollary shows that this fails spectacularly whenever $L_\phi \notin \mathcal{S}^1$.

Corollary A.7.5.1. *Let Q_ϕ a be a positive Cohen's class distribution, and let L_ϕ be the Weyl transform of ϕ. If $L_\phi \in \mathcal{L}(L^2) \setminus \mathcal{S}^1$, then*

$$\int_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \infty$$

for all $\psi \in L^2(\mathbb{R}^d)$.

Proof. A simple approximation argument shows that the relation $\int_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \|\psi\|_{L^2}^2 \operatorname{tr}(L_\phi)$ actually holds when L_ϕ is any bounded *positive* operator on $L^2(\mathbb{R}^d)$, where $\operatorname{tr}(L_\phi) = \infty$ if $L_\phi \notin \mathcal{S}^1$. □

A.7.3 Characterization of positive Cohen's class distributions with correct total energy property

The previous two subsections have introduced two desirable properties in a Cohen's class distribution Q_ϕ, namely positivity and $\int_{\mathbb{R}^{2d}} Q_\phi(\psi)(z)\ dz = \|\psi\|_{L^2}^2$ for any $\psi \in L^2(\mathbb{R}^d)$. Using the results from these subsections, we may now characterize those Cohen's class distributions with both these properties. In short, they are all given as linear combinations of the spectrogram.

Theorem A.7.6. *Let Q_ϕ be a Cohen's class distribution. Q_ϕ is positive and has the correct total energy property if and only if the Weyl transform L_ϕ is a positive trace class operator with* $\operatorname{tr}(L_\phi) = 1$. *If this is the case, there exists an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of non-negative numbers with $\sum_{n=1}^\infty \lambda_n = 1$ such that*

$$Q_\phi(\psi)(z) = \sum_{n=1}^\infty \lambda_n |V_{\varphi_n}\psi|^2(z),$$

and this sum converges uniformly for each $\psi \in L^2(\mathbb{R}^d)$.

Proof. The main idea is to study Q_ϕ using the Weyl transform L_ϕ, since $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$. Since Q_ϕ is positive, Proposition A.7.3 gives that L_ϕ is a positive operator. As we remarked in the proof of Corollary A.7.5.1, we then have that

$$\int_{\mathbb{R}^{2d}} Q_\phi(\psi)\ dz = \|\psi\|_{L^2}^2 \operatorname{tr}(L_\phi),$$

for $\psi \in L^2(\mathbb{R}^d)$, where $\operatorname{tr}(L_\phi) = \infty$ if L_ϕ is not trace class. We easily see from this expression that we need $L_\phi \in \mathcal{S}^1$ with $\operatorname{tr}(L_\phi) = 1$ in order to have that $\int_{\mathbb{R}^{2d}} Q_\phi(\psi)(z)\ dz = \|\psi\|_{L^2}^2$. Hence L_ϕ is a positive trace class operator, so we may use the singular value decomposition of L_ϕ to write

$$L_\phi = \sum_{n=1}^\infty \lambda_n \varphi_n' \otimes \varphi_n'$$

where $\{\varphi_n'\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}^d)$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative numbers with $\sum_{n=1}^\infty \lambda_n = \operatorname{tr}(L_\phi) = 1$. This sum of operators converges to L_ϕ in the operator norm on $\mathcal{L}(L^2)$. In order to make the end results more aesthetic we define $\varphi_n = P\varphi_n'$ for each $n \in \mathbb{N}$, so that $\varphi_n' = \check{\varphi}_n$. The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is clearly also orthonormal, and we have that

$$L_\phi = \sum_{n=1}^\infty \lambda_n \check{\varphi}_n \otimes \check{\varphi}_n.$$

Now recall that $(\psi \otimes \psi) \star (\check{\varphi}_n \otimes \check{\varphi}_n) = |V_{\varphi_n}\psi|^2$ by Lemma A.3.10 . For each $\psi \in L^2(\mathbb{R}^d)$ the operator $\psi \otimes \psi$ is trace class, and since the convolution of operators is continuous $\mathcal{S}^1 \times \mathcal{L}(L^2) \to L^\infty(\mathbb{R}^{2d})$ by Proposition A.3.6 we get that

$$\begin{aligned} Q_\phi(\psi) &= L_\phi \star (\psi \otimes \psi) \\ &= \left(\lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \check{\varphi}_n \otimes \check{\varphi}_n \right) \star (\psi \otimes \psi) \\ &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \lambda_n \check{\varphi}_n \otimes \check{\varphi}_n \star (\psi \otimes \psi) \right) \\ &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \lambda_n |V_{\varphi_n}\psi|^2 \right) \end{aligned}$$

with convergence in the norm of $L^\infty(\mathbb{R}^{2d})$, i.e. uniform convergence. □

A restatement of the previous theorem is that the Cohen's class distributions Q that are positive with the correct total energy property are exactly given by

$$Q(\psi) = (\psi \otimes \psi) \star S_Q$$

for some positive operator $S_Q \in \mathcal{S}^1$ with $\mathrm{tr}(S_Q) = 1$. Operators of the form S_Q are also known as density operators and play a central role in quantum mechanics, see [74] for a modern and rigorous treatment. Examples of Cohen class distributions satisfying this property and characterization have been given in [25].

Example A.7.2. 1. The spectrogram $|V_\varphi\psi(z)|^2 = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi})$ for $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$ is both positive and has the correct total energy property. This agrees with Theorem A.7.6 since the operator $\check{\varphi} \otimes \check{\varphi}$ is positive and $\mathrm{tr}(\check{\varphi} \otimes \check{\varphi}) = \langle \check{\varphi}, \check{\varphi} \rangle_{L^2} = 1$.

2. The Wigner distribution $W(\psi) = (\psi \otimes \psi) \star 2^d P$ is not positive by Proposition A.7.3, as P is not a positive operator. The correct total energy property holds for some, but not all $\psi \in L^2(\mathbb{R}^d)$ [131].

3. Using a result due to Gracia-Bondía and Várilly [127], we may now give a characterization of the Gaussians that give positive Cohen's class distributions with the correct total energy property. To make this precise, let Φ_M be the normalized Gaussian

$$\Phi_M(z) = 2^d \frac{1}{\det(M)^{1/4}} e^{-z^T \cdot M \cdot z} \text{ for } z \in \mathbb{R}^{2d},$$

where M is a $2d \times 2d$-matrix. The result of [127] states that the Weyl transform L_{Φ_M} is a positive trace class operator if and only if

$$M = S^T \Lambda S,$$

where S is a symplectic matrix and Λ is diagonal matrix of the form

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d, \lambda_1, \lambda_2, \ldots, \lambda_d)$$

with $0 < \lambda_i \leq 1$. Hence these Gaussians Φ_M are exactly the Gaussians such that the Cohen's class distribution Q_{Φ_M} is positive with the correct total energy property. Note that this provides examples of positive Cohen's class distributions with the correct total energy property that are *not* spectrograms, since some of the Gaussians above do not correspond to operators of the form $\check{\varphi} \otimes \check{\varphi}$ under the Weyl transform [74, 127]. These results are also linked with the symplectic structure of the phase space [75].

A.7.4 Uncertainty principles for Cohen's class

By using the connection between Cohen's class and convolutions of operators we obtain a weak uncertainty principle for Cohen's class distributions. The result is modeled on uncertainty principles for the spectrogram and Wigner distribution [131].

Proposition A.7.7. *Let $S \in \mathcal{L}(L^2)$ and let Q_S be the Cohen's class distribution determined by $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$ for $\psi \in L^2(\mathbb{R}^d)$. If $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset such that*

$$\int_\Omega |Q_S(\psi)| \, dz \geq (1 - \epsilon) \|S\|_{\mathcal{L}(L^2)}$$

for some $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ and $\epsilon \geq 0$, then

$$|\Omega| \geq 1 - \epsilon.$$

Proof. By Proposition A.7.2, we know that $Q_S(\psi) \in L^\infty(\mathbb{R}^{2d})$ with $\|Q_S(\psi)\|_{L^\infty} \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)}$. It follows that

$$\int_\Omega |Q_S(\psi)| \, dz \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)} \int_\Omega dz = \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)} |\Omega|.$$

Hence if

$$\int_\Omega |Q_S(\psi)| \, dz \geq (1 - \epsilon) \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)},$$

we must have that

$$\|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)} |\Omega| \geq (1 - \epsilon) \|\psi\|_{L^2}^2 \|S\|_{\mathcal{L}(L^2)},$$

and therefore $|\Omega| \geq 1 - \epsilon$. □

A.7.5 Phase retrieval for Cohen's class distribution

Given a Cohen's class distribution Q_ϕ, one might ask whether any $\psi \in L^2(\mathbb{R}^d)$ is uniquely determined by $Q_\phi(\psi)$. Since Proposition A.7.1 shows that ψ enters the expression for $Q_\phi(\psi)$ via $\psi \otimes \psi$, we can at most hope that $\psi \otimes \psi$ is uniquely determined by $Q_\phi(\psi)$. It is simple to show that $\psi_1 \otimes \psi_1 = \psi_2 \otimes \psi_2$ if and only if $\psi_1 = e^{ia}\psi_2$ for some $a \in \mathbb{R}$, so we will ask whether ψ is determined by $Q_\phi(\psi)$ up to some constant phase e^{ia} with $a \in \mathbb{R}$. In the special case where $L_\phi \in \mathcal{S}^1$, a rather weak condition on ϕ is enough to ensure this.

Proposition A.7.8. *Let $\phi \in L^2(\mathbb{R}^{2d})$ be a function such that the Weyl transform L_ϕ is trace class. Assume that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma\phi(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$. If $Q_\phi(\psi_1) = Q_\phi(\psi_2)$ for $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$, then $\psi_1 = e^{ia}\psi_2$ for some constant $a \in \mathbb{R}$.*

Proof. By Proposition A.7.1, we know that $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$. If the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W L_\phi(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$, we get from Theorem A.3.15 that the mapping $\psi \otimes \psi \mapsto (\psi \otimes \psi) \star L_\phi = Q_\phi(\psi)$ is injective. Hence, if $Q_\phi(\psi_1) = Q_\phi(\psi_2)$, then $\psi_1 \otimes \psi_1 = \psi_2 \otimes \psi_2$. By the discussion preceding the Proposition this implies that $\psi_1 = e^{ia}\psi_2$ for some constant $a \in \mathbb{R}$. Furthermore, we know by Proposition A.3.13 that $\phi = \mathcal{F}_\sigma\mathcal{F}_W(L_\phi)$, and applying $\mathcal{F}_\sigma$ to this equation we obtain $\mathcal{F}_\sigma\phi = \mathcal{F}_W(L_\phi)$, so the sets $\{z :\in \mathbb{R}^{2d} : \mathcal{F}_W L_\phi(z) = 0\}$ and $\{z :\in \mathbb{R}^{2d} : \mathcal{F}_\sigma\phi(z) = 0\}$ are equal. □

When $\phi = W(\varphi, \varphi)$ for some $\varphi \in L^2(\mathbb{R}^d)$, the previous result gives a condition on the window $\varphi \in L^2(\mathbb{R}^d)$ to ensure that any $\psi \in L^2(\mathbb{R}^d)$ is determined by its spectrogram $|V_\varphi\psi|^2$ up to a phase e^{ia} for $a \in \mathbb{R}$. This is the problem of *phase retrieval* for the spectrogram [145].

Corollary A.7.8.1. *If $\varphi \in L^2(\mathbb{R}^d)$ and the set $\{z :\in \mathbb{R}^{2d} : A(\varphi, \varphi)(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$ and $|V_\varphi\psi_1| = |V_\varphi\psi_2|$, then $\psi_1 = e^{ia}\psi_2$ for some constant $a \in \mathbb{R}$.*

Proof. Let $\phi = W(\check{\varphi}, \check{\varphi})$. As we saw in Example A.7.1, we then have

$$Q_\phi(\psi) = |V_\varphi\psi|^2.$$

The result now follows from Proposition A.7.8 by noting that $\mathcal{F}_\sigma W(\check{\varphi}, \check{\varphi}) = A(\check{\varphi}, \check{\varphi}) = \overline{A(\varphi, \varphi)}$, where the last equality follows from the definition of $A(\varphi, \varphi)$. □

Remark A.17. This corollary appears in [145, Remark A.4] under the stronger assumption that the set of zeros of the ambiguity function has Lebesgue measure

0. The same paper also proves that when $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and its ambiguity function has *no* zeros, then the same result holds for $\psi_1, \psi_2 \in \mathcal{S}'(\mathbb{R}^{2d})$ [145, Thm. 2.3] — a result that is referred to as "folklore" by [145].

A.8 Multiwindow STFTs and Cohen's class

In Sections A.5 and A.6 we saw that an operator S can be used to assign to a function f on $\mathbb{R}^{2d}$ a multiwindow STFT-filter $f \star S$. On the other hand we saw in Section A.7 that S defines a Cohen's class distribution Q_S by $Q_S(\psi)(z) = (\psi \otimes \psi) \star \check{S}(z)$. In fact, there is a close connection between operators $f \star S$ and Cohen class distribution Q_S.

Proposition A.8.1. *Let $S \in \mathcal{S}^1$, $f \in L^\infty(\mathbb{R}^{2d})$ and $\psi \in L^2(\mathbb{R}^d)$. Let Q_S be the Cohen's class distribution $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$. Then*

$$\langle f \star S, \psi \otimes \psi \rangle_{\mathcal{L}(L^2)), \mathcal{S}^1} = \langle f, Q_S(\psi) \rangle_{L^\infty, L^1}. \tag{A.8.1}$$

More explicitly

$$\langle (f \star S)\psi, \psi \rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z) Q_S(\psi)(z)\, dz. \tag{A.8.2}$$

Proof. When $f \in L^\infty(\mathbb{R}^{2d})$, the operator $f \star S$ is defined by the relation

$$\langle f \star S, T \rangle_{\mathcal{L}(L^2), \mathcal{S}^1} = \left\langle f, \check{S} \star T \right\rangle_{L^\infty, L^1}$$

for any $T \in \mathcal{S}^1$, as we noted in equation (A.3.2). In particular this must hold for $T = \psi \otimes \psi$. Since $Q_S(\psi) = \check{S} \star (\psi \otimes \psi)$, we get that

$$\langle f \star S, \psi \otimes \psi \rangle_{\mathcal{L}(L^2), \mathcal{S}^1} = \left\langle f, \check{S} \star (\psi \otimes \psi) \right\rangle_{L^\infty, L^1} = \langle f, Q_S(\psi) \rangle_{L^\infty, L^1}.$$

To prove equation (A.8.2) we recall that the duality action of $\mathcal{L}(L^2)$ on $\mathcal{S}^1$ is given by

$$\langle f \star S, \psi \otimes \psi \rangle_{\mathcal{L}(L^2), \mathcal{S}^1} = \operatorname{tr}((f \star S)(\psi \otimes \psi)).$$

By picking an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ for $L^2(\mathbb{R}^d)$ we calculate that

$$\begin{aligned} \operatorname{tr}((f \star S)(\psi \otimes \psi)) &= \sum_{n \in \mathbb{N}} \langle (f \star S)(\psi \otimes \psi) e_n, e_n \rangle_{L^2} \\ &= \sum_{n \in \mathbb{N}} \langle e_n, \psi \rangle_{L^2} \langle (f \star S)\psi, e_n \rangle_{L^2} \\ &= \langle (f \star S)\psi, \psi \rangle_{L^2}, \end{aligned}$$

where we have used Parseval's equality to remove the sum. □

Remark A.18. The same result holds whenever $f \star S$ is defined in Proposition A.3.6 and the brackets in equation (A.8.1) may be interpreted as duality. In the most general case we have $S \in \mathfrak{S}'$, and we then have for $f \in \mathcal{S}(\mathbb{R}^{2d})$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\langle (f \star S)\psi, \psi \rangle_{\mathcal{S}',\mathcal{S}} = \langle f, Q_S(\psi) \rangle_{\mathcal{S}',\mathcal{S}}$$

where Q_S is given by $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$.

The proposition shows that the Cohen's class distribution Q_S has a naturally associated operator $f \star S$ for any $f \in L^\infty(\mathbb{R}^{2d})$. The idea of associating operators to Cohen's class distributions has also been considered previously, but in this context it seems to be a novel insight that *both the Cohen's class distribution and the associated operators are given by convolutions with a fixed operator* S (and $\check{S}$). Previous discussions of such results appear in [49,219], and more recently in [46,47] where the operators $f \star S$ are called Cohen operators. In these references a Cohen's class distribution Q was taken as a starting point, and it was shown that one could associate operators to Q using a version of equation (A.8.2).

Proposition A.8.1 generalizes several known relations between pseudodifferential operators and Cohen's class distributions.

Example A.8.1.

1. If we pick $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$, then $S \in \mathcal{S}^1$ and $\check{S} = \check{\varphi} \otimes \check{\varphi}$. For $f \in L^\infty(\mathbb{R}^{2d})$ the operator $f \star S$ is the localization operator $\mathcal{A}_f^\varphi$ by Proposition A.3.9. The Cohen's class distribution determined by S is the spectrogram by Example A.7.1

$$Q_S(\psi)(z) = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi})(z) = |V_\varphi \psi(z)|^2.$$

Equation (A.8.2) states the familiar relation

$$\left\langle \mathcal{A}_f^\varphi \psi, \psi \right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z) |V_\varphi \psi(z)|^2 \, dz.$$

2. For $S = 2^d P$ the proposition describes the Weyl calculus. As we observed in Example A.7.1 the Cohen's class distribution associated to $2^d P = \left(2^d P\right)\check{}$ is the Wigner distribution

$$Q_{2^d P}(\psi) = (\psi \otimes \psi) \star 2^d P(z) = W(\psi)(z).$$

For a function $f \in L^1(\mathbb{R}^{2d})$ the operator $f \star 2^d P$ is the Weyl transform L_f of f: the Weyl symbol of $2^d P$ is δ_0 [146] and hence the Weyl symbol of $f \star 2^d P$ is f by Proposition A.5.2. Equation (A.8.2) becomes

$$\left\langle L_f \psi, \psi \right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z) W(\psi)(z) \, dz$$

which is the equation we used to define the Weyl transform L_f.

3. When $\phi = \mathcal{F}_\sigma \Theta$, where $\Theta(z) = \frac{\sin(\pi x \omega)}{\pi x \omega}$, the Cohen's class distribution Q_ϕ is closely related to Born-Jordan quantization [59, 76]. By Proposition A.7.1 we may write $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$, where L_ϕ is the Weyl transform of ϕ.

   For $f \in \mathcal{S}(\mathbb{R}^{2d})$ we get the associated operators $f \star L_\phi$. In fact, $f \star L_\phi$ is the Born-Jordan quantization of the function f. To prove this, we note that in [61] the Born-Jordan quantization of f is defined to be the operator with Weyl symbol $f * \phi$. By Proposition A.5.2 the Weyl symbol of $f \star L_\phi$ is $f * \phi$, so $f \star L_\phi$ really is the Born-Jordan quantization of f.
   Equation (A.8.1) is a well-known relation between Born-Jordan quantization and the Cohen's class distribution determined by ϕ, in fact this is used to define Born-Jordan quantization in [73].

A.8.1 The localization problem for Cohen's class

The previous section showed that an operator S allows the construction of operators $f \star S$ and a Cohen's class distribution $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$, and that the operators $f \star S$ are related to $Q_S(\psi)$ in a natural way. We will now consider this relationship when f is the characteristic function χ_Ω of some domain $\Omega \subset \mathbb{R}^{2d}$. In this case equation (A.8.2) from the previous section becomes

$$\langle \chi_\Omega \star S\psi, \psi \rangle_{L^2} = \int_\Omega Q_S(\psi)(z)\, dz. \tag{A.8.3}$$

The right hand side of this equation may be interpreted as a measure of the concentration of the energy of ψ in the region Ω of the time-frequency plane, and leads to a natural **localization problem for Cohen's class** [112, 198, 217, 219] : *for a Cohen's class distribution Q and a measurable $\Omega \subset \mathbb{R}^{2d}$, find the signal $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ that maximizes*

$$\int_\Omega Q(\psi)(z)\, dz.$$

Equation (A.8.3) implies that the problem is solved by considering the eigenfunctions of the operator $\chi_\Omega \star S$ by Courant's min-max principle [194, Thm. 28.4], as the next proposition makes formal.

Proposition A.8.2. *Let $\Omega \subset \mathbb{R}^{2d}$ be a measurable subset, let $S \in \mathcal{L}(L^2)$ be a self-adjoint operator and let Q_S be the associated Cohen's class distribution $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$. Assume that $\chi_\Omega \star S$ is a compact operator. Let $\lambda_1 \geq \lambda_2, \ldots$ be the positive eigenvalues of $\chi_\Omega \star S$ (counted with multiplicities) and let ϕ_i be the eigenvector corresponding to λ_i for $i \in \mathbb{N}$. Then*

$$\int_\Omega Q_S(\phi_n)(z)\, dz = \max\left\{ \int_\Omega Q_S(\psi)(z)\, dz : \|\psi\|_{L^2} = 1, \psi \perp \phi_1, \phi_2, \ldots, \phi_{n-1} \right\}.$$

Proof. By the min-max principle [194, Thm. 28.4] we know that

$$\lambda_n = \min_{\psi_1,\dots,\psi_{n-1}} \max_{\substack{\psi\perp\psi_1,\dots,\psi_{n-1}\\ \|\psi\|_{L^2}=1}} \langle (\chi_\Omega \star S)\psi, \psi\rangle_{L^2}\,, \tag{A.8.4}$$

where $\psi_1, \psi_2, \dots, \psi_{n-1}$ is any set of linearly independent vectors in $L^2(\mathbb{R}^d)$. Since $\lambda_n = \langle (\chi_\Omega \star S)\phi_n, \phi_n\rangle_{L^2}$ and $\phi_n \perp \phi_1, \dots\phi_{n-1}$, the minimum in equation (A.8.4) is achieved when $\psi_1 = \phi_1, \psi_2 = \phi_2, \dots, \psi_{n-1} = \phi_{n-1}$, hence

$$\lambda_n = \max_{\substack{\psi\perp\phi_1,\dots,\phi_{n-1}\\ \|\psi\|_{L^2}=1}} \langle (\chi_\Omega \star S)\psi, \psi\rangle_{L^2}\,. \tag{A.8.5}$$

By equation (A.8.3) we know that $\langle (\chi_\Omega \star S)\psi, \psi\rangle_{L^2} = \int_\Omega Q_S(\psi)(z)\, dz$, and since $\lambda_n = \langle (\chi_\Omega \star S)\phi_n, \phi_n\rangle_{L^2}$ equation (A.8.5) states that

$$\int_\Omega Q_S(\phi_n)(z)\, dz = \max\left\{ \int_\Omega Q_S(\psi)(z)\, dz : \|\psi\|_{L^2} = 1, \psi \perp \phi_1, \phi_2, \dots, \phi_{n-1} \right\}.$$

□

Remark A.19. We have formulated the result by requiring that $\chi_\Omega \star S$ is compact. It is easy to find conditions making this true; by Proposition A.3.6 it will be true if $|\Omega| < \infty$ and $S \in \mathcal{S}^p$ for some $p < \infty$. However, $\chi_\Omega \star S$ may well be compact in other cases too.

This idea of solving the localization problem by considering eigenfunctions of operators goes back to the work of Flandrin [112] for the Wigner distribution. Ramanathan and Topiwala [219] later showed that similar techniques were possible for other Cohen's class distributions, by defining the operators $\chi_\Omega \star S$ in equation (A.8.3) using the Weyl calculus. Recently Boggiatto et. al have considered the same problem in [46] using methods very similar to those we consider, but without the convolutions of operators and functions.

Example A.8.2.

1. If $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$, then we know from Examples A.7.1 and A.8.1 that $Q_S(\psi) = |V_\varphi \psi|^2$, the spectrogram, and $\chi_\Omega \star S$ is the localization operator $\mathcal{A}_\Omega^\varphi$. Proposition A.8.2 says that the functions ψ that minimize
$$\int_\Omega |V_\varphi \psi|^2(z)\, dz$$
are the eigenfunctions of the operator $\mathcal{A}_\Omega^\varphi$. This relation is well known [218], and exploited in the recent work of Abreu et al. on accumulated spectrograms [4, 8].

2. When $S = 2^d P$, we have seen in Examples A.7.1 and A.8.1 that $\chi_\Omega \star 2^d P$ is the Weyl transform L_{χ_Ω} and that $Q_S(\psi) = W(\psi)$ — the Wigner distribution of ψ. If we wish to find the functions $\psi \in L^2(\mathbb{R}^d)$ whose Wigner distributions is maximally concentrated in a domain $\Omega \subset \mathbb{R}^{2d}$, Proposition A.8.2 [5] reduces the problem to finding the eigenfunctions of the Weyl transform L_{χ_Ω}. This insight was first formulated in Flandrin's paper [112], and extensions of his results include [217] and [198].

Although Proposition A.8.2 only assumes that $\chi_\Omega \star S$ is compact and self-adjoint, the interpretation of

$$\int_\Omega Q_S(\psi)(z)\, dz$$

as the energy concentration of ψ in Ω is more natural when Q_S is positive and normalized in the sense that

$$\int_{\mathbb{R}^{2d}} Q_S(\psi)(z)\, dz = \|\psi\|_{L^2}^2.$$

As we observed in Section A.7.3, this is satisfied exactly when $S \in \mathcal{S}^1$ is a positive operator with $\mathrm{tr}(S) = 1$. In this case the operators $\chi_\Omega \star S$ are the mixed-state localization operators introduced in Section A.6 using different arguments, and the next section considers this special case in detail.

A.9 Localization operators and positive operator valued measures

In this section we will approach the mixed-state localization operators $\chi_\Omega \star S$ from another perspective, namely that of covariant positive operator valued measures (POVMs). This perspective has been ever-present when the convolutions of operators have been introduced and discussed in quantum physics [158, 182, 183, 251], and we wish to show that it may be of interest also in time-frequency analysis. A POVM F gives two possible measures of the time-frequency content of a signal ψ in a domain Ω in the the time-frequency plane. On the one hand, the *signal* $F(\Omega)\psi$ may be interpreted as the component of ψ with time-frequency components in Ω. On the other hand, we know from Section A.3.8 that ψ induces a probability measure μ_ψ^F, and the *number* $\mu_\psi^F(\Omega)$ measures the time-frequency content of ψ in Ω.
Given a signal $\psi \in L^2(\mathbb{R}^d)$, we wish to show that these two ways of measuring

[5] The proposition requires that $\chi_\Omega \star 2^d P = L_{\chi_\Omega}$ is compact. Even though P is not a compact operator, the operator L_{χ_Ω} is compact whenever $|\Omega| < \infty$, since $\chi_\Omega \in L^2(\mathbb{R}^{2d})$ in this case and $L_f \in \mathcal{S}^2$ whenever $f \in L^2(\mathbb{R}^{2d})$ by Pool's Theorem [215].

the time-frequency content of ψ in a domain Ω lead to the study of mixed-state localization operators and Cohen's class distributions, respectively. The first step in this direction is to note that mixed-state localization operators define POVMs.

Proposition A.9.1. *Let $S \in \mathcal{S}^1$ be a positive operator with* $\operatorname{tr}(S) = 1$. *Then S defines a covariant POVM F by*

$$F(\Omega) = \chi_\Omega \star S$$

for any measurable $\Omega \subset \mathbb{R}^{2d}$.

Proof. We get from Proposition A.3.7 that $F(\Omega) \geq 0$ and $F(\mathbb{R}^{2d}) = I_{L^2}$. The covariance of F follows from the relation $\alpha_z(f \star S) = (T_z f) \star S$ for $f \in L^\infty(\mathbb{R}^{2d})$ [234], since $\alpha_z(F(\Omega)) = \alpha_z(\chi_\Omega \star S) = (T_z\chi_\Omega) \star S = \chi_{\Omega+z} \star S = F(\Omega + z)$.
If $\{\Omega_i\}_{i\in\mathbb{N}}$ is a collection of disjoint, measurable subsets of $\mathbb{R}^{2d}$ and $\Omega := \cup_{i\in\mathbb{N}}\Omega_i$, then $\chi_\Omega = \sum_{i=1}^\infty \chi_{\Omega_i}$, where the sum converges pointwise. We need to show that $\chi_\Omega \star S = \sum_{i=1}^\infty \chi_{\Omega_i} \star S$ with convergence in the weak operator topology, i.e. $\sum_{i=1}^\infty \left\langle \chi_{\Omega_i} \star S\phi, \psi \right\rangle_{L^2} = \langle \chi_\Omega \star S\phi, \psi\rangle_{L^2}$ for all $\phi, \psi \in L^2(\mathbb{R}^d)$. Since $\chi_\Omega \in L^\infty(\mathbb{R}^{2d})$, we know from equation (A.3.3) in Section A.3.4 that the operator $\chi_\Omega \star S \in \mathcal{L}(L^2)$ is defined by the duality relation

$$\operatorname{tr}(T(\chi_\Omega \star S)) = \int_{\mathbb{R}^{2d}} \chi_\Omega(z)\check{S} \star T(z)\, dz$$

for any $T \in \mathcal{S}^1$. In particular, with $T = \phi \otimes \psi$ with $\phi, \psi \in L^2(\mathbb{R}^d)$ we get that

$$\langle \chi_\Omega \star S\phi, \psi\rangle_{L^2} = \int_{\mathbb{R}^{2d}} \chi_\Omega(z)(\check{S} \star (\phi \otimes \psi))(z)\, dz. \tag{A.9.1}$$

This implies that

$$\sum_{i=1}^\infty \left\langle \chi_{\Omega_i} \star S\phi, \psi \right\rangle_{L^2} = \sum_{i=1}^\infty \int_{\mathbb{R}^{2d}} \chi_{\Omega_i}(z)(\check{S} \star (\phi \otimes \psi))(z)\, dz.$$

Since $\sum_{i=1}^\infty \chi_{\Omega_i} = \chi_\Omega$ and $\check{S} \star (\phi \otimes \psi) \in L^1(\mathbb{R}^{2d})$ by Proposition A.3.6, we may use Fubini's theorem to change the order of integration, and we obtain that

$$\begin{aligned}\sum_{i=1}^\infty \left\langle \chi_{\Omega_i} \star S\phi, \psi \right\rangle_{L^2} &= \int_{\mathbb{R}^{2d}} \sum_{i=1}^\infty \chi_{\Omega_i}(z)(\check{S} \star (\phi \otimes \psi))(z)\, dz \\ &= \int_{\mathbb{R}^{2d}} \chi_\Omega(z)(\check{S} \star (\phi \otimes \psi))(z)\, dz \\ &= \langle \chi_\Omega \star S\phi, \psi\rangle_{L^2},\end{aligned}$$

where the final line follows from equation (A.9.1). □

In particular, this result implies that the localization operators $\mathcal{A}_\Omega^\varphi$ may be interpreted as POVMs.

Corollary A.9.1.1. *Let $\varphi \in L^2(\mathbb{R}^d)$ be a window with $\|\varphi\|_2 = 1$. Then φ defines a POVM F by*

$$F(\Omega) = \mathcal{A}_\Omega^\varphi.$$

Proof. Follows from Lemma A.3.9 and the previous Proposition with $S = \varphi \otimes \varphi$. □

Remark A.20. The fact that a localization operator determines a POVM has been remarked by other authors, such as [13].

A.9.1 Cohen's class and POVMs

By Proposition A.9.1, a positive operator $S \in \mathcal{S}^1$ with $\mathrm{tr}(S) = 1$ defines a POVM F via the mixed-state localization operators $F(\Omega) = \chi_\Omega \star S$. Once we have a POVM F, we know from Section A.3.8 that we obtain a probability measure μ_ψ^F for each $\psi \in L^2(\mathbb{R}^d)$. As we mentioned at the start of the section, the time-frequency content of ψ in Ω may be measured either by $F(\Omega)\psi$, or by the induced probability measure μ_ψ^F. When the POVM F is of the form in Proposition A.9.1, the measures μ_ψ^F are given by the positive Cohen's class distribution induced by S as in Proposition A.8.1.

Lemma A.9.2. *Let $S \in \mathcal{S}^1$ be a positive operator with* $\mathrm{tr}(S) = 1$*, and consider the POVM $F(\Omega) = \chi_\Omega \star S$. For $\psi \in L^2(\mathbb{R}^d)$, the induced probability measure μ_ψ^F on $\mathbb{R}^{2d}$ is given by*

$$\mu_\psi^F(\Omega) = \int_\Omega \left((\psi \otimes \psi) \star \check{S} \right)(z)\, dz.$$

In other words, the Radon-Nikodym derivative of μ_ψ^F w.r.t. Lebesgue measure dz is the Cohen class distribution

$$Q_S(\psi) = (\psi \otimes \psi) \star \check{S}.$$

Proof. This is merely a restatement of Proposition A.8.1 in the terminology of POVMs, since μ_ψ^F is defined by $\mu_\psi^F(\Omega) = \langle F(\Omega)\psi, \psi\rangle_{L^2} = \langle (\chi_\Omega \star S)\psi, \psi\rangle_{L^2}$. □

Since S is assumed to be a positive operator with $\mathrm{tr}(S) = 1$, we know that the Cohen's class distribution Q_S in Lemma A.9.2 is positive and has the correct total energy property by Theorem A.7.6. This is exactly what we need to get that μ_ψ^F is a probability measure.

Example A.9.1. Let $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. As we have seen in Example A.8.1, the POVM $F(\Omega) = \chi_\Omega \star S$ is given by the localization operators $F(\Omega) = \mathcal{A}_\Omega^\varphi$, and the associated Cohen's class distribution is the spectrogram: $Q_S(\psi) = |V_\varphi \psi|^2$. By Lemma A.9.2 the induced probability measures μ_ψ^F are given by

$$\mu_\psi^F(\Omega) = \int_\Omega |V_\varphi \psi(z)|^2 \, dz,$$

hence the Radon-Nikodym derivatives of the probability measures induced by localization operators are spectrograms.

Another way of stating the relation between the mixed-state localization operators and the POVM F that they induce, is to express the localization operators as an integral over the POVM.

Proposition A.9.3. *Let F be a POVM given by $F(\Omega) = \chi_\Omega \star S$ for some positive $S \in \mathcal{S}^1$ with* $\mathrm{tr}(S) = 1$. *Then*

$$f \star S = \int_{\mathbb{R}^{2d}} f dF.$$

In particular, the mixed-state localization operators $\chi_\Omega \star S$ may be expressed as

$$\chi_\Omega \star S = \int_\Omega dF.$$

Proof. The operator $\int_{\mathbb{R}^{2d}} f dF$ is by definition the unique operator satisfying

$$\left\langle \int_{\mathbb{R}^{2d}} f dF \psi, \psi \right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f d\mu_\psi^F$$

for each $\psi \in L^2(\mathbb{R}^d)$. We need to show that $f \star S$ satisfies this condition.

$$\begin{aligned}
\langle f \star S\psi, \psi \rangle_{L^2} &= \left\langle \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S) \, dz \, \psi, \psi \right\rangle_{L^2} \\
&= \int_{\mathbb{R}^{2d}} f(z) \left\langle \alpha_z(S)\psi, \psi \right\rangle_{L^2} \, dz \\
&= \int_{\mathbb{R}^{2d}} f(z) \left((\psi \otimes \psi) \star \check{S} \right) (z) \, dz \\
&= \int_{\mathbb{R}^{2d}} f d\mu_\psi^F.
\end{aligned}$$

In the calculation we have moved the inner product inside the integral. This is an instance of the definition of $f \star S$ in equation (A.3.3), when $T = \psi \otimes \psi$. We have also used the equality $\langle \alpha_z(S)\psi, \psi \rangle_{L^2} = \left((\psi \otimes \psi) \star \check{S} \right) (z)$, which follows from the definition of the convolution of two operators. In the last line we used Lemma A.9.2. □

Corollary A.9.3.1. *Let F be a POVM given by $F(\Omega) = \mathcal{A}_\Omega^\varphi$ for some window $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$. Then*

$$\mathcal{A}_f^\varphi = \int_{\mathbb{R}^{2d}} f\,dF.$$

In particular, the localization operators $\mathcal{A}_\Omega^\varphi$ may be expressed as

$$\mathcal{A}_\Omega^\varphi = \int_\Omega dF.$$

A much deeper result than Proposition A.9.1 is that the converse is also true: any covariant POVM F is of the form in Proposition A.9.1 [158, 183, 251]. We provide the proof in our terminology for completeness.

Proposition A.9.4. *Let F be a covariant POVM. There exists some positive $S \in \mathcal{S}^1$ with $\operatorname{tr}(S) = 1$ such that*

$$F(\Omega) = \chi_\Omega \star S$$

for all $\Omega \subset \mathbb{R}^{2d}$.

Proof. We will show that the map $\Gamma : L^\infty(\mathbb{R}^{2d}) \to \mathcal{L}(L^2)$ defined by

$$\Gamma(f) = \int_{\mathbb{R}^{2d}} f\,dF$$

satisfies the conditions of Theorem A.6.2. By that theorem we could then conclude that there is some positive $S \in \mathcal{S}^1$ with $\operatorname{tr}(S) = 1$ such that

$$\int_{\mathbb{R}^{2d}} f\,dF = f \star S$$

for any $f \in L^\infty(\mathbb{R}^{2d})$, and in particular $F(\Omega) = \int_{\mathbb{R}^{2d}} \chi_\Omega dF = \chi_\Omega \star S$. We check that the conditions in Theorem A.6.2 are satisfied.

1. $\Gamma(\chi_{\mathbb{R}^{2d}}) = \int_{\mathbb{R}^{2d}} \chi_{\mathbb{R}^{2d}} dF = F(\mathbb{R}^{2d}) = I_{L^2}$, by the definition of a POVM.

2. Fix $z' \in \mathbb{R}^{2d}$ and $f \in L^\infty(\mathbb{R}^{2d})$. We need to show that $\Gamma(T_{z'} f) = \alpha_{z'}(\Gamma(f))$, and by the uniqueness part of Lemma A.3.17 it suffices to show that

$$\int_{\mathbb{R}^{2d}} T_{z'} f \; d\mu_\psi^F = \langle \alpha_{z'}(\Gamma(f))\psi, \psi\rangle_{L^2}\,. \tag{A.9.2}$$

   Note that

$$\begin{aligned}\langle \alpha_{z'}(\Gamma(f))\psi, \psi\rangle_{L^2} &= \langle \pi(z')\Gamma(f)\pi(z')^*\psi, \psi\rangle_{L^2} \\ &= \langle \Gamma(f)\pi(z')^*\psi, \pi(z')^*\psi\rangle_{L^2} \\ &= \int_{\mathbb{R}^{2d}} f \; d\mu^F_{\pi(z')^*\psi}\end{aligned}$$

by Lemma A.3.17. From the definition of the probability measure $\mu^F_{\pi(z')^*\psi}$ and the covariance of F we find that

$$\begin{aligned}\mu^F_{\pi(z')^*\psi}(\Omega) &= \langle F(\Omega)\pi(z')^*\psi, \pi(z')^*\psi\rangle_{L^2} \\ &= \langle \alpha_{z'}(F(\Omega))\psi, \psi\rangle_{L^2} \\ &= \langle (F(\Omega+z'))\psi, \psi\rangle_{L^2} \\ &= \mu^F_\psi(\Omega+z').\end{aligned}$$

Hence

$$\int_{\mathbb{R}^{2d}} f\, d\mu^F_{\pi(z')^*\psi} = \int_{\mathbb{R}^{2d}} T_{z'}f\, d\mu^F_\psi$$

by a change of variable, which proves equation (A.9.2).

3. By Lemma A.3.17, the operator $\Gamma(f) = \int_{\mathbb{R}^{2d}} f\, dF$ satisfies

$$\langle \Gamma(f)\psi, \psi\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z)\, d\mu^F_\psi.$$

If f is positive, the integral on the right hand side is clearly positive. Hence $\langle \Gamma(f)\psi, \psi\rangle_{L^2} \geq 0$ for all $\psi \in L^2(\mathbb{R}^d)$, so $\Gamma(f)$ is a positive operator.

4. To show that Γ is weak*-weak*-continuous, we assume that a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^{2d})$ converges in the weak*-topology to some $f \in L^\infty$. We need to show that $\mathrm{tr}(S\Gamma(f_n))$ converges to $\mathrm{tr}(S\Gamma(f))$ for each $S \in \mathcal{S}^1$, since $\mathcal{L}(L^2)$ is the dual space of $\mathcal{S}^1$. For each $S \in \mathcal{S}^1$, the expression $\mu^F_S(\Omega) = \mathrm{tr}(SF(\Omega))$ defines a complex, finite measure on $\mathbb{R}^{2d}$, and Lemma A.3.17 may be extended to obtain that

$$\mathrm{tr}\left(S\int_{\mathbb{R}^{2d}} f\, dF\right) = \int_{\mathbb{R}^{2d}} f\, d\mu^F_S$$

for each $f \in L^\infty(\mathbb{R}^{2d})$, see the proof of Lemma 6 in [183]. Furthermore, one can show [183, Lemma 6(b)] that the covariance of F implies that the measures μ^F_S are all absolutely continuous with respect to Lebesgue measure dz. Hence μ^F_S has a Radon-Nikodym derivative $g_S \in L^1(\mathbb{R}^{2d})$ such that $d\mu^F_S = g_S(z)dz$. Using these facts we find that

$$\begin{aligned}\mathrm{tr}(S\Gamma(f)) &= \mathrm{tr}\left(S\int_{\mathbb{R}^{2d}} f_n\, dF\right) \\ &= \int_{\mathbb{R}^{2d}} f_n\, d\mu^F_S \\ &= \int_{\mathbb{R}^{2d}} f_n(z)g_S(z)dz \to \int_{\mathbb{R}^{2d}} f(z)g_S(z)dz = \mathrm{tr}(S\Gamma(f))\end{aligned}$$

by the weak*-convergence of $\{f_n\}_{n\in\mathbb{N}}$.

□

Remark A.21. If $S \in \mathcal{S}^1$ is positive with $\mathrm{tr}(S) = 1$, then $F(\Omega) = \chi_\Omega \star S$ for $\Omega \subset \mathbb{R}^{2d}$ defines a covariant POVM F by Proposition A.9.1, and $\int_{\mathbb{R}^{2d}} f \, dF = f \star S$ for $f \in L^\infty(\mathbb{R}^{2d})$ by Proposition A.9.3. The proof of Proposition A.9.4 shows that $f \mapsto \int_{\mathbb{R}^{2d}} f \, dF = f \star S$ satisfies the four axioms of Theorem A.6.2, as we claimed in Section A.6.

In our terminology this means that any covariant POVM F is given by mixed-state localization operators: $F(\Omega) = \chi_\Omega \star S$ for some positive $S \in \mathcal{S}^1$ with $\mathrm{tr}(S) = 1$. In particular, the induced probability measures μ_ψ^F for $\psi \in L^2(\mathbb{R}^d)$ must be given by positive Cohen's class distributions with the correct total energy property, by Lemma A.9.2.

Paper B

On Accumulated Cohen's Class Distributions and Mixed-State Localization Operators

Franz Luef and Eirik Skrettingland

Published in
Constructive Approximation, 2020, volume 52, issue 1.

Paper B

On Accumulated Cohen's Class Distributions and Mixed-State Localization Operators

Abstract

Recently we introduced mixed-state localization operators associated to a density operator and a (compact) domain in phase space. We continue the investigations of their eigenvalues and eigenfunctions. Our main focus is the definition of a time-frequency distribution which is based on the Cohen class distribution associated to the density operator and the eigenfunctions of the mixed-state localization operator. This time-frequency distribution is called the accumulated Cohen class distribution. If the trace class operator is a rank-one operator, then the mixed-state localization operators and the accumulated Cohen class distribution reduce to Daubechies' localization operators and the accumulated spectrogram. We extend all the results about the accumulated spectrogram to the accumulated Cohen class distribution. The techniques used in the case of spectrograms cannot be adapted to other distributions in Cohen's class since they rely on the reproducing kernel property of the short-time Fourier transform. Our approach is based on quantum harmonic analysis on phase space which also provides the tools and notions to introduce the analogues of the accumulated spectrogram for mixed-state localization operators; the accumulated Cohen's class distributions.

B.1 Introduction

In their study of the spectral behavior of localization operators Abreu et al. introduced the accumulated spectrogram and established interesting results in [8, 9], which revealed some intriguing features of localization operators. We show how the theorems in [8, 9] may be extended to a setting involving (infinite) sums of localization operators, also known as mixed-state localization operators.

The main object of this paper is an in-depth treatment of the mixed-state localization operators from [204] and their associated time-frequency distributions from the perspective developed in [8, 9], and to describe them we first recall some facts about quantum harmonic analysis [203, 251]. Concretely, the convolution between two trace class operators on $L^2(\mathbb{R}^d)$ and the convolution between a function and a trace class operator. Both convolutions are defined in terms of the *translation* of an operator S by a point $z = (x, \omega)$ in phase space $\mathbb{R}^{2d}$:

$$\alpha_z(S) = \pi(z)S\pi(z)^*,$$

where $\pi(z)$ denotes the time-frequency shift of $\psi \in L^2(\mathbb{R}^d)$ by $z = (x, \omega) \in \mathbb{R}^{2d}$, $\pi(z)\psi(t) = e^{2\pi i t\omega}\psi(t - x)$. The convolution between two trace class operators S and T is the function on $\mathbb{R}^{2d}$ given by

$$S \star T(z) = \mathrm{tr}(S\alpha_z(\check{T})) \quad \text{for } z \in \mathbb{R}^{2d},$$

where $\check{T} = PTP$ for $P\psi(x) = \psi(-x)$. An interesting example is the convolution of rank-one operators:

$$(\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi})(z) = |V_\varphi \psi(z)|^2,$$

where $\check{\varphi} = P\varphi$ and $\varphi \otimes \varphi$ is given by $\varphi \otimes \varphi(\xi) = \langle \xi, \varphi \rangle_{L^2} \varphi$.

The convolution between a function $f \in L^1(\mathbb{R}^{2d})$ and a trace class operator S is given by

$$f \star S := \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\, dz.$$

For a rank-one operator $S = \varphi_2 \otimes \varphi_1$ with $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, we have that

$$f \star (\varphi_2 \otimes \varphi_1)(\psi) = \int_{\mathbb{R}^{2d}} f(z)V_{\varphi_1}\psi(z)\pi(z)\varphi_2\, dz,$$

which is a STFT-multiplier [192], also known as a localization operator. In the case of $f = \chi_\Omega$, the characteristic function of a measurable subset Ω of $\mathbb{R}^{2d}$, and $\varphi_1 = \varphi_2$ we obtain Daubechies' localization operator $\mathcal{A}_\Omega^\varphi$ [67]. Interesting results on the relation between the eigenfunctions of a localization operator and its domain Ω have been given in [4], and there is also an extensive literature on the study of localization operators $f \star (\varphi_2 \otimes \varphi_1)$ for more general symbols f [63, 64, 241–243] .

A *mixed-state localization operator* is an operator of the form $\chi_\Omega \star S$, where S is a positive trace class operator with $\mathrm{tr}(S) = 1$ – a *density operator*. The main theme of our paper is the step from rank-one operators $\varphi \otimes \varphi$ to arbitrary density operators, i.e. the step from Daubechies' localization operators to mixed-state localization operators.

The quadratic time-frequency representation associated to localization operators is the spectrogram $|V_\varphi\psi(z)|^2$. In order to extend the results in [8, 9] to mixed-state localization operators we have to find a quadratic time-frequency representation defined by the density operator S. It turns out that elements of Cohen's class provide the desired object.

We have shown in [204] that Q belongs to Cohen's class if it is of the form $Q_S(\psi) = \check{S}\star(\psi\otimes\psi)$, where S is a linear operator mapping the Schwartz class $\mathscr{S}(\mathbb{R}^{2d})$ to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^{2d})$. In particular, density operators S provide distributions in Cohen's class, and Cohen's class distributions defined by density operators have been considered in the physics literature [176, 184, 185]. The relevance of Cohen's class distributions has also been noted by [46, 47, 49, 219].

Furthermore we have given the following characterization in [204]: S is a density operator if and only if $Q_S(\psi)$ is a positive function and $\int_{\mathbb{R}^{2d}} Q_S(\psi)(z)dz = \|\psi\|_2^2$ for any $\psi \in L^2(\mathbb{R}^{2d})$. Note that $Q_{\varphi\otimes\varphi}(\psi)$ is the spectrogram $|V_\varphi\psi(z)|^2$ and thus Q_S is the correct generalization of the spectrogram.

Since the mixed-state localization operator $\chi_\Omega \star S$ is a positive trace class operator, the spectral theorem yields the existence of a sequence of eigenvalues and of eigenfunctions. We will denote the non-zero eigenvalues of $\chi_\Omega \star S$ by $\{\lambda_k^\Omega\}_{k\in\mathbb{N}}$ and the orthonormal system formed by its eigenfunctions by $\{h_k^\Omega\}_{k\in\mathbb{N}}$, thus the spectral representation is

$$\chi_\Omega \star S = \sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega. \tag{B.1.1}$$

We always assume that the eigenvalues are arranged in non-increasing order, i.e. $\lambda_1^\Omega \geq \lambda_2^\Omega \geq \dots$.

Quantum harmonic analysis seems to provide the natural setting for the investigations of eigenvalues and eigenfunctions of (mixed-state) localization operators as in this setup many of the proofs in [8, 9, 80] become natural statements about convolutions between operators. An important aspect of this paper is that one can reformulate the results of [8] in terms of quantum harmonic analysis which then allows us to formulate their results for mixed-state localization operators. Note that our approach provides an alternative proof of results for the accumulated spectrogram as well.

Let us briefly present our results: The first result is that the eigenvalues of a mixed-state localization operator have the same asymptotic behaviour as the one for localization operators [80, 219], see Theorem B.4.4. This is a prerequisite for generalizing the results [8, 9]. A key fact is that the approach in [8, 9] is only feasible in the case of rank-one operators. For a general density operator one has to develop a different strategy. Ours is based on noting that the reproducing kernel techniques can be bypassed if one notes that the replacement of the spectrogram in this case is the function $\tilde{S} = S \star \check{S}$ on phase space $\mathbb{R}^{2d}$, which reduces to the spectrogram for $S = \varphi \otimes \varphi$. A crucial observation is an intrinsic link between mixed-state localization operators and Cohen class distributions:

$$\chi_\Omega * \tilde{S}(z) = \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega)(z), \quad \text{for } z \in \mathbb{R}^{2d}.$$

We are now in the position to introduce the accumulated spectrogram associated to a mixed-state localization operator $\chi_\Omega \star S$ for a compact set $\Omega \subset \mathbb{R}^{2d}$. The *accumulated Cohen class distribution* is defined by

$$\rho_\Omega^S(z) := \sum_{k=1}^{A_\Omega} Q_S(h_k^\Omega)(z) \quad \text{for } z \in \mathbb{R}^{2d},$$

where $A_\Omega = \lceil |\Omega| \rceil$, i.e. the least integer greater than or equal to $|\Omega|$.

Note that $\rho_\Omega^{\psi \otimes \psi}$ is the accumulated spectrogram which is an intriguing object both from a mathematical and application point of view. Our main results are the extension of the theorems in [8, 9] on the accumulated spectrogram to accumulated Cohen's class distributions. Our proofs are non-trivial adaptations of the ones in [8,9] and we have tried to emphasize the modifications required by the mixed-state setting.

In Theorem B.1.1 we demonstrate the asymptotic convergence of accumulated Cohen class distributions to the characteristic function of the domain:

Theorem B.1.1 (Asymptotic convergence)**.** *Let S be a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain. Then*

$$\|\rho_{R\Omega}^S(R\,\cdot) - \chi_\Omega\|_{L^1} \to 0 \text{ as } R \to \infty.$$

We then move on to study the non-asymptotic convergence of accumulated Cohen class distributions, where the bounds depend on the size of the perimeter of the domain $\Omega \subset \mathbb{R}^{2d}$. To quantify the size of the perimeter, we will use the variation of its characteristic function χ_Ω and a subset M_{op}^* of density operators:

$$M_{\text{op}}^* = \{S \text{ trace class operator} : S \geq 0, \operatorname{tr}(S) = 1 \text{ and } \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z|\, dz < \infty\},$$

where $|z|$ is the Euclidean norm of z, with

$$\|S\|_{M_{\text{op}}^*}^2 = \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z|\, dz.$$

This lets us bound the approximation of χ_Ω by $\chi_\Omega * \tilde{S}$. Consequently, we are able to prove the next statement:

Theorem B.1.2 (Non-asymptotic convergence)**.** *If $S \in M^*_{\text{op}}$ and $\Omega \subset$ is a compact domain with finite perimeter such that $\|S\|^2_{M^*_{\text{op}}} |\partial\Omega| \geq 1$, then for any $\delta > 0$*

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho^S_\Omega(z) - \chi_\Omega(z)\right| > \delta\right\}\right| \lesssim \frac{1}{\delta^2}\|S\|^2_{M^*_{\text{op}}} |\partial\Omega|.$$

In [9] the sharpness of this bound for the spectrogram was shown by considering Euclidean balls $B(z, R) = \{z' \in \mathbb{R}^{2d} : |z| < R\}$ as the domain Ω. In Theorem B.1.3 we demonstrate this sharpness for accumulated Cohen class distributions Q_S for $S \in M^*_{\text{op}}$. Our approach is inspired by the spectrogram results in [80, 106] where the projection functional enters in a crucial manner. We give an expression for this projection functional applied to $\chi_\Omega \star S$:

$$\text{tr}(\chi_\Omega \star S) - \text{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_{\mathbb{R}^{2d}\setminus\Omega} \tilde{S}(z - z')\, dz' dz.$$

The results above also shed some light on results in [204], where we considered the question of recovering Ω from $\chi_\Omega \star S$. The approach in [204] was only concerned with establishing conditions on S for this to be possible, and offered no clue as to how Ω could be recovered. Theorem B.1.2 shows that ρ^S_Ω, defined using a finite number of eigenfunctions of $\chi_\Omega \star S$, estimates χ_Ω.The sharpness of the bounds is contained in Theorem B.1.3:

Theorem B.1.3 (Sharpness)**.** *Let $S \in M^*_{\text{op}}$. There exist constants C^1_S and C^2_S such that for $R > 1$*

$$C^1_S R^{2d-1} \leq \|\rho^S_{B(0,R)} - \chi_{B(0,R)}\|_{L^1} \leq C^2_S R^{2d-1}.$$

We close this paper by discussing some examples of Cohen's class distributions suitable for the accumulated Cohen's class construction, namely those given by a density operator S. In particular we show that any such distribution can be used to obtain *new* examples by convolving an operator with a positive function, previously noted in a different setting by Gracia-Bondía and Várilly [127].

B.2 Preliminaries

B.2.1 The short-time Fourier transform

If $\psi : \mathbb{R}^d \to \mathbb{C}$ and $z = (x, \omega) \in \mathbb{R}^{2d}$, we define the *translation operator* T_x by $T_x\psi(t) = \psi(t - x)$, the *modulation operator* M_ω by $M_\omega\psi(t) = e^{2\pi i \omega \cdot t}\psi(t)$ and the *time-frequency shifts* $\pi(z)$ by $\pi(z) = M_\omega T_x$. For $\psi, \phi \in L^2(\mathbb{R}^d)$ the *short-time*

Fourier transform (STFT) $V_\phi\psi$ of ψ with window ϕ is the function on $\mathbb{R}^{2d}$ defined by

$$V_\phi\psi(z) = \langle \psi, \pi(z)\phi\rangle_{L^2} \quad \text{for } z \in \mathbb{R}^{2d},$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the usual inner product on $L^2(\mathbb{R}^d)$. By replacing the inner product above with a duality bracket,[1] the STFT may be extended to other spaces, such as $\phi \in \mathscr{S}(\mathbb{R}^d), \psi \in \mathscr{S}'(\mathbb{R}^d)$ where $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space and $\mathscr{S}'(\mathbb{R}^d)$ its dual space of tempered distributions. We will also meet a close relative of the STFT: the cross-Wigner distribution, defined for $\psi, \xi \in L^2(\mathbb{R}^d)$ by

$$W(\psi,\xi)(x,\omega) = \int_{\mathbb{R}^d} \psi\left(x+\frac{t}{2}\right)\overline{\xi\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega\cdot t}\, dt \quad \text{for } (x,\omega) \in \mathbb{R}^{2d}.$$

B.2.2 Operator theory

Our approach relies heavily on properties of the bounded operators $\mathcal{L}(L^2)$ on $L^2(\mathbb{R}^d)$, and a basic result is the *spectral representation* of self-adjoint compact operators [54, Thm. 3.5].

Proposition B.2.1. *Let S be a self-adjoint, compact operator on $L^2(\mathbb{R}^d)$ with non-zero eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. There exists an orthonormal system $\{\varphi_k\}_{k\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ such that S may be expressed as*

$$S = \sum_{k\in\mathbb{N}} \lambda_k \varphi_k \otimes \varphi_k,$$

with convergence in the operator norm. Here $\varphi_k \otimes \varphi_k$ is the rank-one operator defined by $\varphi_k \otimes \varphi_k(\xi) = \langle \xi, \varphi_k\rangle_{L^2}\varphi_k$ for $\xi \in L^2(\mathbb{R}^d)$.

The notation above suggests that the set of non-zero eigenvalues is infinite, as will almost always be true in this paper, hence we use this notation. If S has finite rank it is a trivial matter to replace the sum by a finite sum.

The trace and trace class operators

For a *positive* operator $S \in \mathcal{L}(L^2)$, one can define the *trace* of S by

$$\mathrm{tr}(S) = \sum_{k\in\mathbb{N}} \langle Se_k, e_k\rangle_{L^2}, \tag{B.2.1}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. The Banach space $\mathcal{S}^1$ of *trace class operators* consists of those compact operators S where $\mathrm{tr}(|S|) < \infty$, with norm

[1] Which we always assume is antilinear in the second coordinate, to be consistent with the inner product on $L^2(\mathbb{R}^d)$.

$\|S\|_{\mathcal{S}^1} = \operatorname{tr}(|S|)$. The trace in (B.2.1) defines a linear functional on $\mathcal{S}^1$ that satisfies $\operatorname{tr}(ST) = \operatorname{tr}(TS)$, and the definition in (B.2.1) is independent of the orthonormal basis used [54]. By a celebrated theorem due to Lidskii, for $S \in \mathcal{S}^1$,

$$\operatorname{tr}(S) = \sum_{k=1}^{\infty} \lambda_k \tag{B.2.2}$$

where the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of S are counted with multiplicity [232].

The Weyl transform

An important concept for associating operators on $L^2(\mathbb{R}^d)$ with functions on $\mathbb{R}^{2d}$ is the *Weyl transform*. If $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$, then we define the Weyl transform L_ϕ as an operator $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by

$$\left\langle L_\phi \xi, \psi \right\rangle_{L^2} = \langle \phi, W(\psi, \xi) \rangle_{L^2} \quad \text{for } \xi, \psi \in \mathscr{S}(\mathbb{R}^d),$$

where the bracket denotes the action of $\mathscr{S}'(\mathbb{R}^d)$ as functionals on $\mathscr{S}(\mathbb{R}^d)$. We call ϕ the *Weyl symbol* of the operator L_ϕ. For more information on the Weyl transform in the same spirit as this short introduction, such as conditions to ensure $L_\phi \in \mathcal{L}(L^2)$, we refer to [131].

B.2.3 Quantum harmonic analysis

This section introduces the theory of convolutions of operators and functions due to Werner [251]. For $z \in \mathbb{R}^{2d}$ and $A \in \mathcal{L}(L^2)$, we define the operator $\alpha_z(A)$ by

$$\alpha_z(A) := \pi(z) A \pi(z)^*.$$

It is easily confirmed that $\alpha_z \alpha_{z'} = \alpha_{z+z'}$, and we will informally think of α as a shift or translation of operators.

Similarly we define the analogue of the involution $\check{f}(z) := f(-z)$ of a function, for an operator $A \in \mathcal{L}(L^2)$ by

$$\check{A} := PAP,$$

where P is the parity operator $P\psi(x) = \psi(-x)$ for $\psi \in L^2(\mathbb{R}^d)$.

Using α, Werner defined a convolution operation between functions and operators [251]. If $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$ we define the *operator* $f \star S$ by

$$f \star S := S \star f := \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S)\, dz$$

where the integral is interpreted in the weak sense by requiring that

$$\langle (f \star S)\psi, \xi \rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z) \langle \alpha_z(S)\psi, \xi \rangle_{L^2}\, dz, \quad \text{for } \psi, \xi \in L^2(\mathbb{R}^d).$$

Then $f \star S \in \mathcal{S}^1$ and $\|f \star S\|_{\mathcal{S}^1} \leq \|f\|_{L^1}\|S\|_{\mathcal{S}^1}$ [203, Prop. 2.5].

For two operators $S,T \in \mathcal{S}^1$, Werner defined the *function* $S \star T$ by

$$S \star T(z) := \operatorname{tr}(S\alpha_z(\check{T})) \quad \text{for } z \in \mathbb{R}^{2d}. \tag{B.2.3}$$

Remark B.1. The notation $\star$ may therefore denote either the convolution of two functions or the convolution of an operator with a function. The correct interpretation will be clear from the context.

The following result relates the convolutions of operators to the standard convolutions of Weyl symbols. The statements follow by combining Propositions 3.12 and 3.16(5) in [204].

Proposition B.2.2. *Let $f \in L^1(\mathbb{R}^{2d})$ and $S,T \in \mathcal{S}^1$. Let a_S and a_T be the Weyl symbols of S and T. Then*

1. *$S \star T(z) = a_S * a_T(z)$ for $z \in \mathbb{R}^{2d}$.*
2. *The Weyl symbol of $f \star S$ is $f * a_S$.*

Here $*$ denotes the usual convolution of functions.

The following result shows that $S \star T \in L^1(\mathbb{R}^{2d})$ for $S,T \in \mathcal{S}^1$ and provides an important formula for its integral [251, Lem. 3.1]. In the simplest case where S and T are rank-one operators, this formula is the so-called Moyal identity for the STFT [114, p. 57].

Lemma B.2.3. *Let $S,T \in \mathcal{S}^1$. The function $z \mapsto S \star T(z)$ for $z \in \mathbb{R}^{2d}$ is integrable and $\|S \star T\|_{L^1} \leq \|S\|_{\mathcal{S}^1}\|T\|_{\mathcal{S}^1}$. Furthermore,*

$$\int_{\mathbb{R}^{2d}} S \star T(z)\, dz = \operatorname{tr}(S)\operatorname{tr}(T).$$

The convolutions can be defined on other L^p-spaces and Schatten p-classes by duality [203, 251]. As a special case we mention that (B.2.3) defines a continuous function even when $T \in \mathcal{L}(L^2)$ [203]; in particular it is clear from (B.2.3) that

$$S \star I_{L^2}(z) = \operatorname{tr}(S) \tag{B.2.4}$$

for any $z \in \mathbb{R}^{2d}$ when I_{L^2} is the identity operator and $S \in \mathcal{S}^1$. The convolutions of operators and functions are associative, a fact that is non-trivial since the convolutions between operators and functions can produce both operators and functions as output [203, 251]. The fact that the convolution of an operator with a function is commutative, i.e. $f \star S = S \star f$, is true by definition. The convolution of two operators is also commutative, see [203, Prop. 4.4] for the simple proof. One may also easily check that the convolutions are bilinear. We will need the following simple property.

Lemma B.2.4. *Let $S \in \mathcal{S}^1$ be a positive operator. If $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, then*

$$\sum_{n=1}^{\infty} S \star (\xi_n \otimes \xi_n)(z) = \operatorname{tr}(S), \quad \textit{for any } z \in \mathbb{R}^{2d}.$$

Proof. A simple calculation (see the proof of [203, Thm. 5.1]) shows that

$$S \star (\xi_n \otimes \xi_n)(z) = \left\langle \check{S}\pi(-z)\xi_n, \pi(-z)\xi_n \right\rangle_{L^2}.$$

By Proposition B.2.1, $\check{S}$ has a spectral representation

$$\check{S} = \sum_{k=1}^{\infty} \lambda_k \varphi_k \otimes \varphi_k,$$

where $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal system in $L^2(\mathbb{R}^d)$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ are the non-zero eigenvalues of $\check{S}$. We insert this into the previous formula and apply Parseval's theorem to get

$$\begin{aligned}
\sum_{n=1}^{\infty} S \star (\xi_n \otimes \xi_n)(z) &= \sum_{n=1}^{\infty} \left\langle \sum_{k=1}^{\infty} \lambda_k \varphi_k \otimes \varphi_k \pi(-z)\xi_n, \pi(-z)\xi_n \right\rangle_{L^2} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k \left\langle \pi(-z)\xi_n, \varphi_k \right\rangle_{L^2} \left\langle \varphi_k, \pi(-z)\xi_n \right\rangle_{L^2} \\
&= \sum_{k=1}^{\infty} \lambda_k \sum_{n=1}^{\infty} \left\langle \varphi_k, \pi(-z)\xi_n \right\rangle_{L^2} \left\langle \pi(-z)\xi_n, \varphi_k \right\rangle_{L^2} \\
&= \sum_{k=1}^{\infty} \lambda_k \left\langle \varphi_k, \varphi_k \right\rangle_{L^2} = \sum_{k=1}^{\infty} \lambda_k = \operatorname{tr}(\check{S}) = \operatorname{tr}(S).
\end{aligned}$$

The final line uses (B.2.2), and that $\operatorname{tr}(\check{S}) = \operatorname{tr}(PSP) = \operatorname{tr}(P^2 S) = \operatorname{tr}(S)$. Note that we used that $\pi(-z)$ is unitary to get that $\{\pi(-z)\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis. □

The convolutions preserve positivity [202, Lem. 4.1] .

Lemma B.2.5. *If $S, T \in \mathcal{L}(L^2)$ are positive operators and f is a positive function, then $f \star S$ is a positive operator and $S \star T$ is a positive function.*

B.3 Cohen's class and mixed-state localization operators

A quadratic time-frequency distribution Q is said to be of *Cohen's class* if Q is given by

$$Q(\psi) = Q_\phi(\psi) := W(\psi, \psi) * \phi$$

for some $\phi \in \mathscr{S}'(\mathbb{R}^{2d})$ [59, 131]. In [204] we emphasized another way of defining Cohen's class, namely that Q belongs to Cohen's class if Q is given by

$$Q(\psi) = Q_S(\psi) = \check{S} \star (\psi \otimes \psi), \tag{B.3.1}$$

where $S : \mathscr{S}(\mathbb{R}^{2d}) \to \mathscr{S}'(\mathbb{R}^{2d})$ is a continuous linear operator. It can be shown using Proposition B.2.2 that these two definitions are equivalent [204], since

$$Q_\phi = Q_S \quad \text{when } L_\phi = \check{S}. \tag{B.3.2}$$

We will be particularly interested in Q_S when S is a *positive trace class operator with* $\mathrm{tr}(S) = 1$. In quantum mechanics, such operators are often called *density operators*, and we will adopt this terminology in this paper. There is a simple characterization of those Cohen's class distributions Q_S where S is a density operator [204].

Proposition B.3.1. *Let $S \in \mathcal{L}(L^2)$. S is a density operator if and only if, for any $\psi \in L^2(\mathbb{R}^d)$, $Q_S(\psi)$ is a positive function and $\int_{\mathbb{R}^{2d}} Q_S(\psi)(z)\, dz = \|\psi\|_{L^2}^2$.*

In light of (B.3.2), the set of Cohen's class distributions Q_S with S a density operator equals $\{Q_\phi : \phi \in \mathcal{W}\}$, where

$$\mathcal{W} := \{\phi \in \mathscr{S}'(\mathbb{R}^{2d}) : L_\phi \text{ is a density operator }\}.$$

Remark B.2. Due to (B.3.2), the operator $\check{S}$ will appear many times. The reader should therefore note that $\check{S}$ and S are unitarily equivalent by definition, and share relevant properties such as positivity and trace.

In [204] we also introduced the notion of a *mixed-state localization operator*, which is an operator of the form $\chi_\Omega \star S$ where S is a density operator and χ_Ω the characteristic function of a domain $\Omega \subset \mathbb{R}^{2d}$. By definition $\chi_\Omega \star S$ acts on $\psi \in L^2(\mathbb{R}^d)$ by

$$(\chi_\Omega \star S)\psi = \int_\Omega \pi(z) S \pi(z)^* \psi\, dz.$$

The simplest examples of density operators are given by the rank-one operators $\varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. In this case, the Cohen class distribution $Q_{\varphi\otimes\varphi}$ is the *spectrogram*:

$$Q_{\varphi\otimes\varphi}(\psi) = (\check{\varphi} \otimes \check{\varphi}) \star (\psi \otimes \psi) = |V_\varphi \psi|^2, \tag{B.3.3}$$

and the mixed-state localization operators $\chi_\Omega \star (\varphi \otimes \varphi)$ are the usual localization operators introduced by Daubechies [67], which act on $\psi \in L^2(\mathbb{R}^d)$ by

$$(\chi_\Omega \star (\varphi \otimes \varphi))(\psi) = \int_\Omega V_\varphi \psi(z) \pi(z) \varphi \, dz.$$

Remark B.3. In quantum mechanics, a rank-one operator $\varphi \otimes \varphi$ describes a so-called pure state of a system [72]. More general states are called mixed states, and are described by density operators – hence the terminology of mixed-state localization operators.

B.3.1 Notation for mixed-state localization operators

In order to fix notation, we briefly consider the spectral representation of mixed-state localization operators. If $\Omega \subset \mathbb{R}^{2d}$ is compact and S is a density operator, we know from Lemma B.2.5 and Section B.2.3 that $\chi_\Omega \star S$ is a positive trace class operator. For the rest of the paper we will denote the non-zero eigenvalues of $\chi_\Omega \star S$ by $\{\lambda_k^\Omega\}_{k\in\mathbb{N}}$ and the orthonormal system formed by its eigenfunctions by $\{h_k^\Omega\}_{k\in\mathbb{N}}$, thus the spectral representation is

$$\chi_\Omega \star S = \sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega. \tag{B.3.4}$$

We always assume that the eigenvalues are in non-increasing order, i.e. $\lambda_1^\Omega \geq \lambda_2^\Omega \geq \dots$. The function $S \star \check{S}$, for some operator $S \in \mathcal{S}^1$, will play an important role in our results. To emphasize this, we introduce the notation

$$\tilde{S}(z) := S \star \check{S}(z).$$

If S is a density operator, it follows from Section B.2.3 that $\tilde{S}$ is a positive, continuous function such that $\int_{\mathbb{R}^{2d}} \tilde{S}(z)\, dz = \mathrm{tr}(S)\mathrm{tr}(S) = 1$. In the special case where $S = \varphi \otimes \varphi$ for some $\varphi \in L^2(\mathbb{R}^d)$, we get by (B.3.3) that $\tilde{S}(z) = |V_\varphi \varphi(z)|^2$.

B.3.2 A consequence of associativity

As we have mentioned, the associativity of the convolutions introduced in Section B.2.3 is non-trivial. It leads to the following relation between Cohen's class distributions and mixed-state localization operators, see [8, Lem. 4.1] for an alternative proof for spectrograms.

Proposition B.3.2. *Let S be a density operator and let $\Omega \subset \mathbb{R}^{2d}$ be a compact set. Then*

$$\chi_\Omega * \tilde{S}(z) = \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega)(z), \quad \textit{for } z \in \mathbb{R}^{2d}.$$

Proof. By the associativity of convolutions, we have that $\chi_\Omega * \tilde{S} = \chi_\Omega * (S \star \check{S}) = (\chi_\Omega \star S) \star \check{S}$ in $L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$. Now insert the spectral representation from (B.1.1):

$$\begin{aligned}(\chi_\Omega \star S) \star \check{S} &= \left(\sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega\right) \star \check{S} \\ &= \sum_{k=1}^{\infty} \lambda_k^\Omega (h_k^\Omega \otimes h_k^\Omega) \star \check{S} \\ &= \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega).\end{aligned}$$

When moving to the second line, we have used that the spectral representation converges in the operator norm and that convolutions with a fixed operator is norm-continuous from $\mathcal{L}(L^2)$ to $L^\infty(\mathbb{R}^{2d})$ [203, Prop. 4.2]. Furthermore, $\chi_\Omega * (S \star \check{S}) = (\chi_\Omega \star S) \star \check{S}$ holds pointwise since both sides are continuous functions – the left side is the convolution of a bounded function χ_Ω with $S \star \check{S} \in L^1(\mathbb{R}^{2d})$, and the right side is the convolution of two trace class operators which is continuous by [203, Prop. 3.3]. □

B.3.3 Approximate identities for $L^1(\mathbb{R}^{2d})$

In the section we will obtain an approximate identity for $L^1(\mathbb{R}^{2d})$ for each normalized trace class operator S. Recall that a family $\{\phi_\mu\}_{\mu>0}$ in $L^1(\mathbb{R}^d)$ is an approximate identity for $L^1(\mathbb{R}^d)$ if it satisfies the following three conditions:

1. $\int_{\mathbb{R}^d} \phi_\mu(x)\, dx = 1$ for any $\mu > 0$,
2. $\sup_{\mu>0} \|\phi_\mu\|_{L^1} < \infty$,
3. For every $\delta > 0$, $\lim_{\mu\to\infty} \int_{|x|>0} |\phi_\mu(x)|\, dx$,

The three conditions above imply that $\lim_{\mu\to\infty} \phi_\mu * \phi = \phi$ in the norm of $L^1(\mathbb{R}^d)$ for any $\phi \in L^1(\mathbb{R}^d)$. The following standard result is easily proved by straightforward calculations.

Proposition B.3.3. *Let $\phi \in L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$. The family $\{\phi_R\}_{R>0}$ of normalized dilations of ϕ defined by $\phi_R(x) = R^d \phi(Rx)$ is an approximate identity for $L^1(\mathbb{R}^d)$.*

As a consequence we obtain the following result, which is Lemma 3 in [218] when ϕ is positive.

Lemma B.3.4. *Let $\phi \in L^1(\mathbb{R}^d)$ be a function with $\int_{\mathbb{R}^{2d}} \phi(z)\,dz = 1$, and let $\Omega \subset \mathbb{R}^d$ be a compact domain. Then*

$$\frac{1}{R^d}\int_{R\Omega}\int_{R\Omega}\phi(x-x')\,dx\,dx' \to |\Omega|$$

as $R \to \infty$.

Proof. By the previous proposition we know that the family $\{\phi_R\}_{R>0}$ is an approximate identity for $L^1(\mathbb{R}^d)$. In particular we have that $\chi_\Omega * \phi_R \to \chi_\Omega$ in $L^1(\mathbb{R}^d)$ as $R \to \infty$. Since $\psi \mapsto \int_{\mathbb{R}^d} \psi(x)\chi_\Omega(x)\,dx$ is a linear functional on $L^1(\mathbb{R}^d)$, we get as a consequence that $\int_\Omega \chi_\Omega * \phi_R(x)\,dx \to |\Omega|$ as $R \to \infty$. It only remains to show that $\int_\Omega \chi_\Omega * \phi_R(x)\,dx$ equals the left hand side in the statement of the theorem:

$$\begin{aligned}\int_\Omega \chi_\Omega * \phi_R\,dx &= \int_\Omega\int_{\mathbb{R}^d}\chi_\Omega(x')R^d\phi(R(x-x'))\,dx'\,dx\\ &= R^d\int_\Omega\int_\Omega \phi(R(x-x'))\,dx'\,dx\\ &= \frac{1}{R^d}\int_{R\Omega}\int_{R\Omega}\phi(u-v)\,du\,dv,\end{aligned}$$

where we have introduced the new variables $u = Rx$ and $v = Rx'$. □

This allows us to introduce an important class of approximate identities based on trace class operators.

Corollary B.3.4.1. *Let $S \in \mathcal{S}^1$ be an operator with $\mathrm{tr}(S) = 1$. The functions $\{\tilde{S}_R\}_{R>0}$ form an approximate identity for $L^1(\mathbb{R}^{2d})$ and*

$$\frac{1}{R^{2d}}\int_{R\Omega}\int_{R\Omega}\tilde{S}(z-z')\,dz\,dz' \to |\Omega|$$

as $R \to \infty$ for any compact domain $\Omega \subset \mathbb{R}^{2d}$.

Proof. By Lemma B.2.3, $\tilde{S} = S \star \check{S} \in L^1(\mathbb{R}^{2d})$ and $\int_{\mathbb{R}^{2d}} S\star\check{S}(z)\,dz = \mathrm{tr}(S)\mathrm{tr}(\check{S}) = 1$, hence the result follows from the previous lemma and proposition. □

B.4 The eigenvalues of mixed-state localization operators

In this section we will be interested in the eigenvalues of mixed-state localization operators $\chi_{R\Omega} \star S$ as $R \to \infty$, where $R\Omega = \{Rz : z \in \Omega\}$. In the case of localization operators, corresponding to $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$, the following behaviour of the eigenvalues $\{\lambda_k^{R\Omega}\}_{k\in\mathbb{N}}$ of $\chi_{R\Omega} \star (\varphi \otimes \varphi)$ has been established in [104, 218] for any fixed $\delta \in (0,1)$:

$$\frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|} \to 1 \text{ as } R \to \infty. \tag{B.4.1}$$

To show that this holds for the eigenvalues $\{\lambda_k^{R\Omega}\}_{k\in\mathbb{N}}$ of any mixed-state localization operator $\chi_{R\Omega} \star S$, we need a few lemmas.

Lemma B.4.1. *If S is a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain, the eigenvalues of $\chi_\Omega \star S$ satisfy* $0 \leq \lambda_k^\Omega \leq 1$.

Proof. As we saw in Section B.3.1, $\chi_\Omega \star S$ is a positive operator, so its eigenvalues are non-negative. By equation (9) in [204], $\langle \chi_\Omega \star S\psi, \psi\rangle_{L^2} = \int_\Omega Q_S(\psi)(z)\, dz$ for $\psi \in L^2(\mathbb{R}^d)$. If we let ψ be the eigenfunction h_k^Ω, Proposition B.3.1 now gives

$$\lambda_k^\Omega = \int_\Omega Q_S(h_k^\Omega)(z)\, dz \leq \int_{\mathbb{R}^{2d}} Q_S(h_k^\Omega)(z)\, dz = \|h_k^\Omega\|_{L^2}^2 = 1. \qquad \square$$

Lemma B.4.2. *Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain, and let $S \in \mathcal{S}^1$.*

$$\operatorname{tr}(\chi_\Omega \star S) = \sum_{k=1}^\infty \lambda_k^\Omega = |\Omega| \operatorname{tr}(S),$$
$$\operatorname{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_\Omega \tilde{S}(z-z')\, dz\, dz'.$$

Proof. The formula $\operatorname{tr}(\chi_\Omega \star S) = \sum_{k=1}^\infty \lambda_k^\Omega$ is Lidskii's theorem from (B.2.2). To prove $\operatorname{tr}(\chi_\Omega \star S) = |\Omega|\operatorname{tr}(S)$, we note that (B.2.4) says that $(\chi_\Omega \star S) \star I_{L^2}(z) = \operatorname{tr}(\chi_\Omega \star S)$ for any $z \in \mathbb{R}^{2d}$. However, by associativity of convolutions and $S \star I_{L^2}(z) = \operatorname{tr}(S)$ we also have that

$$\begin{aligned}(\chi_\Omega \star S) \star I_{L^2}(z) &= \chi_\Omega * (S \star I_{L^2})(z) \\ &= \int_{\mathbb{R}^{2d}} \chi_\Omega(z')(S \star I_{L^2})(z-z')\, dz' = \operatorname{tr}(S)|\Omega|.\end{aligned}$$

For the second part, note that $T \star \check{T}(0) = \operatorname{tr}(T^2)$ for any $T \in \mathcal{S}^1$ by the definition of convolution of operators. In particular[2] $(\chi_\Omega \star S) \star (\check{\chi}_\Omega \star \check{S})(0) = \operatorname{tr}((\chi_\Omega \star S)^2)$.

[2]The alert reader will note that we use $(\chi_\Omega \star S)^\vee = \check{\chi}_\Omega \star \check{S}$. See [234] for the simple proof.

Hence, using associativity and commutativity of convolutions,

$$\begin{aligned}\operatorname{tr}((\chi_\Omega \star S)^2) &= \chi_\Omega * (\check{\chi_\Omega} * (S \star \check{S}))(0) \\ &= \int_\Omega \check{\chi_\Omega} * (S \star \check{S})(-z')\, dz' \\ &= \int_\Omega \int_{\mathbb{R}^{2d}} \chi_\Omega(-z)(S \star \check{S})(-z'-z)\, dz\, dz' \\ &= \int_\Omega \int_\Omega (S \star \check{S})(z-z')\, dz\, dz',\end{aligned}$$

where we substituted $z \mapsto -z$ in the last line. □

Remark B.4. For rank-one operators $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ these formulas are well known and used to obtain the profile of the eigenvalues of localization operators, see for instance [8, 104, 218]. The approach used to obtain the second formula in these papers uses the reproducing kernel Hilbert space associated with the short-time Fourier transform. Our approach does not rely on this property of the STFT, which allows us to prove the result for general trace class operators.

The following is a generalization of [8, Lem 3.3] to mixed-state localization operators. Our proof follows the proof from that paper, which is based on the approach in [104].

Lemma B.4.3. *Let S be a density operator, let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain and fix $\delta \in (0,1)$. Then*

$$\left|\#\{k \geq 1 : \lambda_k^\Omega > 1-\delta\} - |\Omega|\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \left|\int_\Omega \int_\Omega \tilde{S}(z-z')dzdz' - |\Omega|\right|$$

Proof. Following [8] we define the function

$$G(t) := \begin{cases} -t & \text{if } 0 \leq t \leq 1-\delta \\ 1-t & \text{if } 1-\delta < t \leq 1. \end{cases}$$

We may apply G to the eigenvalues in the spectral representation (B.1.1) to obtain a new operator $G(\chi_\Omega \star S)$:

$$G(\chi_\Omega \star S) = \sum_{k=1}^\infty G(\lambda_k^\Omega) h_k^\Omega \otimes h_k^\Omega.$$

Since $\chi_\Omega \star S$ is trace class, $\{\lambda_k^\Omega\}_{k=1}^\infty \in \ell^1$. As $\sum_{k=1}^\infty \lambda_k^\Omega = |\Omega|$, only finitely many λ_k^Ω can satisfy $\lambda_k^\Omega > 1-\delta$, and it follows that $\{G(\lambda_k^\Omega)\}_{k=1}^\infty \in \ell^1$ because $|G(t)| = |t|$

for $t \in [0, 1-\delta]$. Hence $G(\chi_\Omega \star S)$ is a trace class operator with trace

$$\begin{aligned}\operatorname{tr}(G(\chi_\Omega \star S)) &= \sum_{k=1}^{\infty} G(\lambda_k^\Omega) \\ &= \#\{k : \lambda_k^\Omega > 1-\delta\} - \sum_{k=1}^{\infty} \lambda_k^\Omega \\ &= \#\{k \geq 1 : \lambda_k^\Omega > 1-\delta\} - |\Omega|.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\#\{k \geq 1 : \lambda_k^\Omega > 1-\delta\} - |\Omega|\right| &= |\operatorname{tr}(G(\chi_\Omega \star S))| \\ &\leq \operatorname{tr}(|G|(\chi_\Omega \star S)) \\ &\leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \operatorname{tr}\left(\chi_\Omega \star S - (\chi_\Omega \star S)^2\right),\end{aligned}$$

where we have used $|G(t)| \leq \max\{\frac{1}{\delta}, \frac{1}{1-\delta}\}(t-t^2)$ for $t \in [0,1]$. The final result follows from inserting the expressions for $\operatorname{tr}(\chi_\Omega \star S)$ and $\operatorname{tr}((\chi_\Omega \star S)^2)$ from Lemma B.4.2. □

The following is the main result of this section, which shows that (B.4.1) is valid for mixed-state localization operators.

Theorem B.4.4. *Let S be a density operator, let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain and fix $\delta \in (0,1)$. If $\{\lambda_k^{R\Omega}\}_{k \in \mathbb{N}}$ are the non-zero eigenvalues of $\chi_{R\Omega} \star S$, then*

$$\frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|} \to 1 \text{ as } R \to \infty.$$

Proof. By the previous lemma,

$$\left|\#\{k \geq 1 : \lambda_k^{R\Omega} > 1-\delta\} - R^{2d}|\Omega|\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \left|\int_{R\Omega}\int_{R\Omega} \tilde{S}(z-z')\,dz\,dz' - R^{2d}|\Omega|\right|.$$

Hence if we divide by $R^{2d}|\Omega|$

$$\left|\frac{\#\{k \geq 1 : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|} - 1\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \frac{1}{|\Omega|} \left|\frac{1}{R^{2d}} \int_{R\Omega}\int_{R\Omega} \tilde{S}(z-z')\,dz\,dz' - |\Omega|\right|.$$

The result now follows from Corollary B.3.4.1. □

B.5 Accumulated Cohen class distributions

For any density operator S and compact domain $\Omega \subset \mathbb{R}^{2d}$, we define an associated *accumulated Cohen class distribution* by

$$\rho_\Omega^S(z) := \sum_{k=1}^{A_\Omega} Q_S(h_k^\Omega) \quad \text{for } z \in \mathbb{R}^{2d},$$

where $A_\Omega = \lceil |\Omega| \rceil$ and h_k^Ω are the eigenfunctions of $\chi_\Omega \star S$. Note that ρ_Ω^S may also be written as a convolution of operators, since

$$\rho_\Omega^S = \sum_{k=1}^{A_\Omega} \check{S} \star (h_k^\Omega \otimes h_k^\Omega) = \check{S} \star \sum_{k=1}^{A_\Omega} (h_k^\Omega \otimes h_k^\Omega).$$

As a consequence, Lemma B.2.4 gives that $\rho_\Omega^S(z) \leq 1$ for any $z \in \mathbb{R}^{2d}$, since $\{h_k^\Omega\}_{k\in\mathbb{N}}$ can be extended to an orthonormal basis and

$$\rho_\Omega^S(z) = \sum_{n=1}^{A_\Omega} \check{S} \star (h_k^\Omega \otimes h_k^\Omega)(z) \leq \sum_{n=1}^{\infty} \check{S} \star (h_k^\Omega \otimes h_k^\Omega)(z) \leq \operatorname{tr}(S) = 1.$$

In [8], Abreu et al. prove results showing that when Q_S is a spectrogram, ρ_Ω^S is an approximation of the characteristic function χ_Ω. We will show that their results hold when S is *any* density operator. Our presentation and proofs follow those in [8]. The proofs will typically consist of two parts: the easy part is to show that the function $\chi_\Omega * \tilde{S}$ approximates χ_Ω. The more intricate part is to show that $\chi_\Omega * \tilde{S}$ also approximates ρ_Ω^S. We start by generalizing [8, Lem. 4.2, 4.3].

Lemma B.5.1. *Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain and define*

$$E(\Omega) = 1 - \frac{\sum_{k=1}^{A_\Omega} \lambda_k^\Omega}{|\Omega|}.$$

Then

$$\frac{1}{|\Omega|} \|\rho_\Omega^S - \chi_\Omega * \tilde{S}\|_{L^1} \leq \left(\frac{1}{|\Omega|} + 2E(\Omega) \right),$$

and

$$E(R\Omega) \to 0 \text{ as } R \to \infty.$$

Proof. Using Lemma B.2.3 and the associativity of convolutions, we find that

$$\begin{aligned}
\|\rho_\Omega^S - \chi_\Omega * (S \star \check{S})\|_{L^1} &= \left\| \left(\sum_{k=1}^{A_\Omega} h_k^\Omega \otimes h_k^\Omega \right) \star \check{S} - (\chi_\Omega \star S) \star \check{S} \right\|_{L^1} \\
&\leq \left\| \sum_{k=1}^{A_\Omega} h_k^\Omega \otimes h_k^\Omega - \chi_\Omega \star S \right\|_{\mathcal{S}^1} \|\check{S}\|_{\mathcal{S}^1} \\
&= \left\| \sum_{k=1}^{A_\Omega} h_k^\Omega \otimes h_k^\Omega - \sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega \right\|_{\mathcal{S}^1} \\
&= \sum_{k=1}^{A_\Omega} (1 - \lambda_k^\Omega) + \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega.
\end{aligned}$$

We have expanded $\chi_\Omega \star S$ using the spectral representation (B.1.1), and the last equality uses that $\|T\|_{\mathcal{S}^1}$ is the sum of the eigenvalues for positive operators $T \in \mathcal{S}^1$. Since $\sum_{k=1}^\infty \lambda_k^\Omega = |\Omega|$, we further get that

$$\begin{aligned}
\sum_{k=1}^{A_\Omega} (1 - \lambda_k^\Omega) + \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega &= |\Omega| + A_\Omega - 2 \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \\
&= (A_\Omega - |\Omega|) + 2 \left(|\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \right) \\
&\leq 1 + 2E(\Omega)|\Omega|.
\end{aligned}$$

To prove that $E(R\Omega) \to 0$ as $R \to \infty$, we will pick $\delta \in (0,1)$ and find an upper bound of $E(R\Omega)$ in terms of $\frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{|\Omega|}$ – an application of Theorem B.4.4 will then give the desired result. For a fixed $\delta \in (0,1)$ and domain Ω, we define

$$l_\delta(\Omega) = \min\{A_\Omega, \#\{k : \lambda_k^\Omega > 1 - \delta\}\}.$$

By definition $l_\delta(\Omega) \leq A_\Omega$, and since the eigenvalues λ_k^Ω are arranged in non-increasing order we see that $\lambda_k^\Omega > 1 - \delta$ for $k \leq l_\delta(\Omega)$. Using this we estimate that

$$\begin{aligned}
E(\Omega) &= 1 - \frac{\sum_{k=1}^{A_\Omega} \lambda_k^\Omega}{|\Omega|} \\
&\leq 1 - \frac{\sum_{k=1}^{l_\delta(\Omega)} \lambda_k^\Omega}{|\Omega|} \\
&\leq 1 - (1-\delta) \frac{l_\delta(\Omega)}{|\Omega|},
\end{aligned}$$

where we have also used that the eigenvalues λ_k^Ω are non-negative. Note that we always have $E(\Omega) \geq 0$, since $\sum_{k=1}^\infty \lambda_k^\Omega = |\Omega|$. If we replace the domain Ω by the new domain $R\Omega$ in the previous estimate and insert the definition of $l_\delta(R\Omega)$, we obtain

$$0 \leq E(R\Omega) \leq 1 - (1-\delta)\min\left\{\frac{A_{R\Omega}}{R^{2d}|\Omega|}, \frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|}\right\}.$$

By definition of A_Ω we know that $\frac{A_{R\Omega}}{|\Omega|R^{2d}} \geq 1$, hence we get the estimate

$$0 \leq E(R\Omega) \leq 1 - (1-\delta)\min\left\{1, \frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|}\right\}. \tag{B.5.1}$$

The behaviour of the term $\frac{\#\{k:\lambda_k^{R\Omega}>1-\delta\}}{R^{2d}|\Omega|}$ is described by Theorem B.4.4, which says that this fraction approaches 1 as $R \to \infty$. Therefore

$$0 \leq \limsup_{R\to\infty} E(R\Omega) \leq 1 - (1-\delta) = \delta,$$

and by picking δ arbitrarily close to 0 we see that in fact $E(R\Omega) \to 0$ as $R \to \infty$.

□

B.5.1 Asymptotic convergence of accumulated Cohen class distributions

We are now ready to prove the generalization of [8, Thm. 1.3] – the asymptotic convergence of accumulated Cohen's class distributions to the characteristic function of the domain.

Proof of Theorem B.1.1. We will use the estimate

$$\|\rho_{R\Omega}^S(R\cdot) - \chi_\Omega\|_{L^1} \leq \|\rho_{R\Omega}^S(R\cdot) - \chi_\Omega * \tilde{S}_R\|_{L^1} + \|\chi_\Omega * \tilde{S}_R - \chi_\Omega\|_{L^1},$$

where $\tilde{S}_R(z) = R^{2d}\tilde{S}(Rz)$. The second term converges to 0 as $R \to \infty$ by Corollary B.3.4.1. To bound the first term, we note that a straightforward calculation using a change of variable gives that $\chi_\Omega * \tilde{S}_R(z) = \chi_\Omega * (S \star \check{S})_R(z) = \chi_{R\Omega} * (S \star \check{S})(Rz)$. Hence we find, with $z' = Rz$, that

$$\begin{aligned}\|\rho_{R\Omega}^S(R\cdot) - \chi_\Omega * \tilde{S}_R\|_{L^1} &= \int_{\mathbb{R}^{2d}} |\rho_{R\Omega}^S(Rz) - \chi_{R\Omega} * (S \star \check{S})(Rz)|\, dz \\ &= \frac{1}{R^{2d}} \int_{\mathbb{R}^{2d}} |\rho_{R\Omega}^S(z') - \chi_{R\Omega} * (S \star \check{S})(z')|\, dz' \\ &\leq \frac{1}{R^{2d}} + 2E(R\Omega)|\Omega|,\end{aligned}$$

where the last inequality is Lemma B.5.1. By the same lemma, this expression converges to 0 as $R \to \infty$. □

The above result shows that the domain Ω is uniquely determined by $\rho^S_{R\Omega}$ as $R \to \infty$, i.e. from knowledge of S and the first $A_{R\Omega} = \lceil |R\Omega| \rceil$ eigenfunctions of $\chi_{R\Omega} \star S$ for infinitely many R. In [204] we used an approximation theorem for operators due to Werner [251] to establish certain conditions on S, formulated in terms of a Fourier transform for operators, that guarantee that Ω can be recovered from only $\chi_\Omega \star S$. The next two sections will show that we may *estimate* Ω from $\chi_\Omega \star S$, but make no claim that Ω is determined by $\chi_\Omega \star S$ for any density operator S.

B.5.2 Non-asymptotic approximation of χ_Ω by accumulated Cohen class distributions

The bounds for the non-asymptotic convergence of accumulated Cohen class distributions will depend on the size of the perimeter of the domain $\Omega \subset \mathbb{R}^{2d}$. To quantify the size of the perimeter, we will use the variation of its characteristic function χ_Ω. Recall that the variation of function $f \in L^1(\mathbb{R}^d)$ is

$$\mathrm{Var}(f) = \sup \left\{ \int_{\mathbb{R}^d} f(x) \mathrm{div}\phi(x)\, dx : \phi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d), |\phi(x)| \leq 1\ \forall x \in \mathbb{R}^d \right\},$$

where $\mathrm{div}\phi$ is the divergence of ϕ, $C^1_c(\mathbb{R}^d, \mathbb{R}^d)$ is the set of compactly supported differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ and $|\phi(x)|$ denotes the Euclidean norm on $\mathbb{R}^d$. We say that f has bounded variation if $\mathrm{Var}(f) < \infty$. We define

$$|\partial\Omega| = \mathrm{Var}(\chi_\Omega)$$

for a domain $\Omega \subset \mathbb{R}^{2d}$, and say that Ω has finite perimeter if χ_Ω has bounded variation. The reader may find more relevant discussion regarding functions of bounded variation in chapter 5 of [93]. The only way this concept will enter our considerations is via the following lemma, which is proved in [8, Lem. 3.2].

Lemma B.5.2. *Let $f \in L^1(\mathbb{R}^d)$ have bounded variation, and let $\varphi \in L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} \varphi(z)\, dz = 1$. Then*

$$\|f * \varphi - f\|_1 \leq Var(f) \int_{\mathbb{R}^d} |x||\varphi(x)|\, dx,$$

where $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$.

We also define a subset M^*_{op} of density operators by

$$M^*_{\text{op}} = \{S \in \mathcal{S}^1 : S \geq 0, \text{tr}(S) = 1 \text{ and } \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z| \, dz < \infty\},$$

where $|z|$ is the Euclidean norm of z, with

$$\|S\|^2_{M^*_{\text{op}}} = \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z| \, dz.$$

This lets us bound the approximation of χ_Ω by $\chi_\Omega * \tilde{S}$, since Lemma B.5.2 gives

$$\|\chi_\Omega - \chi_\Omega * \tilde{S}\|_{L^1} \leq |\partial\Omega| \|S\|^2_{M^*_{\text{op}}}. \tag{B.5.2}$$

When Q_S is a spectrogram, i.e. $S = \varphi \otimes \varphi$ for some $\varphi \in L^2(\mathbb{R}^{2d})$ by (B.3.3), the quantity $\|S\|^2_{M^*_{\text{op}}}$ becomes $\int_{\mathbb{R}^{2d}} |V_\varphi \varphi(z)|^2 |z| \, dz$, which is the quantity $\|\varphi\|_{M^*}$ introduced in [8] for accumulated spectrograms. We now prove the generalization of [8, Prop. 3.4].

Lemma B.5.3. *Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain with finite perimeter and $S \in M^*_{\text{op}}(\mathbb{R}^d)$. If $\delta \in (0, 1)$, then*

$$\left|\#\{k : \lambda_k^\Omega > 1 - \delta\} - |\Omega|\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \|S\|^2_{M^*_{\text{op}}} |\partial\Omega|$$

Proof. By Lemma B.4.3, it suffices to bound the expression

$$\left|\int_\Omega \int_\Omega \tilde{S}(z - z') dz dz' - |\Omega|\right|.$$

We may rewrite this expression as

$$\begin{aligned}
\left|\int_\Omega \int_{\mathbb{R}^{2d}} \chi_\Omega(z) \tilde{S}(z - z') dz dz' - |\Omega|\right| &= \left|\int_\Omega \chi_\Omega * \tilde{S}(z') dz' - \int_\Omega \chi_\Omega(z') \, dz'\right| \\
&= \left|\int_\Omega \left(\chi_\Omega * \tilde{S}(z') - \chi_\Omega(z')\right) dz'\right| \\
&\leq \int_{\mathbb{R}^{2d}} \left|\chi_\Omega * \tilde{S}(z') - \chi_\Omega(z')\right| \, dz' \\
&= \|\chi_\Omega * \tilde{S} - \chi_\Omega\|_{L^1},
\end{aligned}$$

where we have used $\tilde{S}(z - z') = \tilde{S}(z' - z)$ to write the left summand as a convolution with χ_Ω. This relation holds since $S \star \check{S}(-z) = \check{S} \star \check{\check{S}}(z) = \check{S} \star S(z) = S \star \check{S}(z)$, see [234, Lem. 4.7]. The result now follows from Lemma B.4.3 and (B.5.2). □

The following L^1-bound generalizes [8, Thm. 1.4] to general $S \in M^*_{\text{op}}$.

Theorem B.5.4. *If $S \in M^*_{\text{op}}$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter, then*

$$\frac{1}{|\Omega|}\|\rho^S_\Omega - \chi_\Omega * \tilde{S}\|_{L^1} \leq \left(\frac{1}{|\Omega|} + 4\|S\|_{M^*_{\text{op}}}\sqrt{\frac{|\partial\Omega|}{|\Omega|}}\right).$$

Proof. From Lemma B.5.1,

$$\frac{1}{|\Omega|}\|\rho^S_\Omega - \chi_\Omega * \tilde{S}\|_{L^1} \leq \left(\frac{1}{|\Omega|} + 2E(\Omega)\right).$$

We will prove the theorem by proving the estimate $E(\Omega) \leq 2\|S\|_{M^*_{\text{op}}}\sqrt{\frac{|\partial\Omega|}{|\Omega|}}$, which generalizes [8, Lem. 4.3]. We therefore jump back to our estimate in (B.5.1), which was the estimate for $E(R\Omega)$ we obtained when we did not assume $S \in M^*_{\text{op}}$. For $R = 1$ this equation gives

$$0 \leq E(\Omega) \leq 1 - (1-\delta)\frac{\#\{k : \lambda^\Omega_k > 1-\delta\}}{|\Omega|}. \tag{B.5.3}$$

To bound this expression, we note that Lemma B.5.3 gives

$$\frac{\#\{k : \lambda^\Omega_k > 1-\delta\}}{|\Omega|} \geq 1 - \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}\|S\|^2_{M^*}\frac{|\partial\Omega|}{|\Omega|}.$$

Inserting this estimate into (B.5.3) and setting $\delta = \|S\|_{M^*_{\text{op}}}\sqrt{\frac{|\partial\Omega|}{|\Omega|}}$ now gives the desired estimate – we refer to the proof of [8, Lem 4.3] for the details. □

As a corollary, one can derive an estimate for $\|\rho^S_\Omega - \chi_\Omega\|_{L^1}$. We return to this question in Section B.6.

B.5.3 Weak L^2-convergence of accumulated Cohen class distributions

Finally, we show that the weak-L^2 bounds for $\rho^S_\Omega - \chi_\Omega$ in [8, Thm 1.5] hold in the more general case where S is a density operator. Following the proof in [8] we start by proving a technical lemma.

Lemma B.5.5. *If $S \in M^*_{\text{op}}$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter such that $\|S\|^2_{M^*_{\text{op}}}|\partial\Omega| \geq 1$, then for any $\delta > 0$*

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho^S_\Omega(z) - \chi_\Omega * \tilde{S}(z)\right| > \delta\right\}\right| \lesssim \frac{1}{\delta^2}\|S\|^2_{M^*_{\text{op}}}|\partial\Omega|.$$

Proof. By Proposition B.3.2 we find that

$$\begin{aligned}|\rho_\Omega^S(z) - \chi_\Omega * \tilde{S}(z)| &= \left|\sum_{k=1}^{A_\Omega} Q_S(h_k^\Omega)(z) - \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega)(z)\right| \\ &\leq \sum_{k=1}^{\infty} \mu_k Q_S(h_k^\Omega)(z),\end{aligned}$$

where we have introduced $\mu_k = \lambda_k^\Omega$ for $k > A_\Omega$ and $\mu_k = 1 - \lambda_k^\Omega$ for $k \leq A_\Omega$. To obtain our desired bound, we will split this sum into three parts. Following the lead of the proof in [8, Prop. 4.4], we assume that $0 < \delta \leq \frac{1}{2}$ and define

$$\begin{aligned}a_\delta &:= \#\{k : \lambda_k^\Omega > 1 - \delta\}, \\ b_\delta &:= \#\{k : \lambda_k^\Omega > \delta\}.\end{aligned}$$

Then let

$$\begin{aligned}a'_\delta &:= \min\{a_\delta, A_\Omega\}, \\ b'_\delta &:= \max\{b_\delta, A_\Omega\}.\end{aligned}$$

Now note that $\sum_{k=1}^{\infty} Q_S(h_k^\Omega)(z) \leq 1$ for all $z \in \mathbb{R}^{2d}$ by Lemma B.2.4 and that $\mu_k \leq \delta$ for $k \notin [a'_\delta + 1, b'_\delta]$,

$$\begin{aligned}\sum_{k=1}^{\infty} \mu_k Q_S(h_k^\Omega)(z) &= \sum_{k=1}^{a'_\delta} \mu_k Q_S(h_k^\Omega)(z) + \sum_{a'_\delta+1}^{b'_\delta} \mu_k Q_S(h_k^\Omega)(z) + \sum_{k=b'_\delta+1}^{\infty} \mu_k Q_S(h_k^\Omega)(z) \\ &\leq 2\delta + \sum_{a'_\delta+1}^{b'_\delta} \mu_k Q_S(h_k^\Omega)(z).\end{aligned}$$

As a consequence we clearly get

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho_\Omega^S(z) - \chi_\Omega * \tilde{S}(z)\right| > 3\delta\right\}\right| \leq \left|\left\{z \in \mathbb{R}^{2d} : \sum_{a'_\delta+1}^{b'_\delta} \mu_k Q_S(h_k^\Omega)(z) > \delta\right\}\right|.$$

To control this expression, one may use Lemma B.5.3 and the assumptions $0 < \delta \leq \frac{1}{2}$, $\|S\|_{M_{\mathrm{op}}^*}^2 |\partial\Omega| \geq 1$ to get (see [8, Prop. 4.4] for details)

$$0 \leq b'_\delta - a'_\delta \lesssim \frac{1}{\delta} \|S\|_{M_{\mathrm{op}}^*} |\partial\Omega|.$$

By using $0 \leq \mu_k \leq 1$ and $\|Q_S(h_k^\Omega)\|_{L^1} = \|h_k^\Omega\|_{L^2}^2 = 1$ (see (B.3.1) and Lemma B.2.3), we then get

$$\left|\left\{z \in \mathbb{R}^{2d} : \sum_{a'_\delta+1}^{b'_\delta} \mu_k Q_S(h_k^\Omega)(z) > \delta\right\}\right| \leq \frac{1}{\delta}\left\|\sum_{a'_\delta+1}^{b'_\delta} \mu_k Q_S(h_k^\Omega)\right\|_{L^1}$$
$$= \frac{1}{\delta}\sum_{a'_\delta+1}^{b'_\delta} \mu_k \left\|Q_S(h_k^\Omega)\right\|_{L^1}$$
$$\leq \frac{1}{\delta}\sum_{a'_\delta+1}^{b'_\delta} 1 \lesssim \frac{1}{\delta^2}\|S\|_{\mathcal{M}^*_{\mathrm{op}}}|\partial\Omega|.$$

The substitution $\delta \mapsto \frac{\delta}{3}$ proves the result for $0 < \delta \leq \frac{3}{2}$, and the result is trivial for $\delta > 1$ since we always have the bound $|\rho_\Omega^S(z) - \chi_\Omega * \tilde{S}(z)| \leq \sum_{k=1}^\infty \mu_k Q_S(h_k^\Omega)(z) \leq \sum_{k=1}^\infty Q_S(h_k^\Omega)(z) \leq 1$ by Lemma B.2.4. □

Proof of Theorem B.1.2. By the previous lemma we have the weak-L^2 bound

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho_\Omega^S(z) - \chi_\Omega * \tilde{S}(z)\right| > \delta/2\right\}\right| \lesssim \frac{1}{\delta^2}\|S\|_{\mathcal{M}^*_{\mathrm{op}}}^2|\partial\Omega|,$$

and we obviously have the weak-L^1 bound

$$\left|\left\{z \in \mathbb{R}^{2d} : |\chi_\Omega * \tilde{S}(z) - \chi_\Omega(z)| > \delta/2\right\}\right| \leq \frac{2}{\delta}\|\chi_\Omega * \tilde{S}(z) - \chi_\Omega(z)\|_{L^1}$$
$$\leq \frac{1}{\delta}\|S\|_{\mathcal{M}^*_{\mathrm{op}}}^2|\partial\Omega|,$$

where the last bound is Lemma B.5.2. Combining these bounds, we get

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho_\Omega^S(z) - \chi_\Omega(z)\right| > \delta\right\}\right| \lesssim \frac{1}{\delta^2}\|S\|_{\mathcal{M}^*_{\mathrm{op}}}^2|\partial\Omega| + \frac{1}{\delta}\|S\|_{\mathcal{M}^*_{\mathrm{op}}}^2|\partial\Omega|.$$

When $\delta \leq 2$ we have that $1/\delta \leq 2/\delta^2$, so the result is proved in this case. In fact, this is the only case we need to consider, as $\rho_\Omega^S(z), \chi_\Omega(z) \leq 1$ implies $\{z \in \mathbb{R}^{2d} : |\rho_\Omega^S(z) - \chi_\Omega(z)| > \delta\} = \emptyset$ for $\delta > 2$. □

B.6 Sharp bounds for accumulated Cohen's class distributions

As a simple consequence of Theorem B.5.4 one can derive the bound

$$\|\chi_\Omega - \rho_\Omega^S\|_{L^1} \lesssim \sqrt{|\partial\Omega||\Omega|}$$

for S, Ω satisfying the assumptions of that theorem and $|\Omega| \geq 1$, see [8, Cor. 5.1] for a proof when Q_S is a spectrogram. In [9], Abreu et al. were able to improve this bound in the case of spectrograms to

$$\|\chi_\Omega - \rho_\Omega^S\|_{L^1} \lesssim |\partial\Omega|. \tag{B.6.1}$$

The very elegant proof of (B.6.1) in [9] exploits the spectral theory of localization operators. Since Section B.4 indicates that the spectral theory is largely the same for *mixed-state* localization operators, we will be able to prove (B.6.1) for general density operators S based on the same arguments.

Theorem B.6.1. *Fix $\epsilon > 0$. If $S \in M_{\mathrm{op}}^*$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter satisfying $|\partial\Omega| \geq \epsilon$, then*

$$\|\rho_\Omega^S - \chi_\Omega\|_{L^1} \leq (1/\epsilon + 2\|S\|_{M_{\mathrm{op}}^*}^2)|\partial\Omega|.$$

Proof. To estimate the left hand side, we will split the integral into two parts. First note that since $0 \leq \rho_\Omega^S(z) \leq 1$ for any $z \in \mathbb{R}^{2d}$,

$$\begin{aligned}
\int_\Omega |\rho_\Omega^S(z) - \chi_\Omega(z)|\, dz &= \int_\Omega (1 - \rho_\Omega^S(z))\, dz \\
&= |\Omega| - \int_\Omega \sum_{k=1}^{A_\Omega} \check{S} \star (h_k^\Omega \otimes h_k^\Omega)(z)\, dz \\
&= |\Omega| - \sum_{k=1}^{A_\Omega} \int_{\mathbb{R}^{2d}} \chi_\Omega(z) \left(\check{S} \star (h_k^\Omega \otimes h_k^\Omega)\right)(z)\, dz \\
&= |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega.
\end{aligned}$$

The final equality uses a relation between convolutions and duality, namely the fact that $\left\langle \chi_\Omega, \check{S} \star (h_k^\Omega \otimes h_k^\Omega)\right\rangle_{L^\infty, L^1} = \left\langle \chi_\Omega \star S, h_k^\Omega \otimes h_k^\Omega \right\rangle_{\mathcal{L}(L^2), \mathcal{S}^1}$, where the bracket denotes duality. See [234] for a verification. A simple calculation using the spectral representation of $\chi_\Omega \star S$ gives that

$$\left\langle \chi_\Omega \star S, h_k^\Omega \otimes h_k^\Omega \right\rangle_{\mathcal{L}(L^2), \mathcal{S}^1} = \mathrm{tr}((\chi_\Omega \star S) h_k^\Omega \otimes h_k^\Omega) = \lambda_k^\Omega.$$

The other part of the integral satisfies

$$\begin{aligned}\int_{\mathbb{R}^{2d}\setminus\Omega} |\rho_\Omega^S(z) - \chi_\Omega(z)|\ dz &= \int_{\mathbb{R}^{2d}\setminus\Omega} \rho_\Omega^S(z)\ dz \\ &= \int_{\mathbb{R}^{2d}} \rho_\Omega^S(z)\ dz - \int_\Omega \rho_\Omega^S(z)\ dz \\ &= \sum_{k=1}^{A_\Omega} \int_{\mathbb{R}^{2d}} \check{S} \star (h_k^\Omega \otimes h_k^\Omega)(z)\ dz - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \\ &= A_\Omega - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \leq 1 + |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega,\end{aligned}$$

where we have used Lemma B.2.3 to calculate $\int_{\mathbb{R}^{2d}} \check{S} \star (h_k^\Omega \otimes h_k^\Omega)(z)\ dz = 1$, and used our expression for $\int_\Omega \rho_\Omega^S(z)\ dz$ from the previous calculation. In total

$$\int_{\mathbb{R}^{2d}} |\rho_\Omega^S(z) - \chi_\Omega(z)|\ dz \leq 1 + 2\left(|\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega\right). \tag{B.6.2}$$

To bound $|\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega$ we will look at $\operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2)$. On the one hand it follows easily from Lemma B.4.2 that

$$\begin{aligned}\operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) &= \int_\Omega \left(1 - \chi_\Omega * \tilde{S}(z)\right)\ dz \\ &\leq \|\chi_\Omega * \tilde{S} - \chi_\Omega\|_{L^1} \\ &\leq |\partial\Omega| \|S\|_{M^*_{\mathrm{op}}}^2,\end{aligned}$$

where the last inequality is Lemma B.5.2. On the other hand we know that $\operatorname{tr}(\chi_\Omega \star S) = \sum_{k=1}^\infty \lambda_k^\Omega = |\Omega|$ and $\operatorname{tr}((\chi_\Omega \star S)^2) = \sum_{k=1}^\infty \left(\lambda_k^\Omega\right)^2$, which leads to the following estimate:

$$\begin{aligned}
\operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) &= \sum_{k=1}^{A_\Omega} \lambda_k^\Omega (1-\lambda_k^\Omega) + \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega (1-\lambda_k^\Omega) \\
&\geq \lambda_{A_\Omega}^\Omega \sum_{k=1}^{A_\Omega} (1-\lambda_k^\Omega) + (1-\lambda_{A_\Omega}^\Omega) \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega \\
&= \lambda_{A_\Omega}^\Omega A_\Omega - \lambda_{A_\Omega}^\Omega |\Omega| + \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega \\
&= \lambda_{A_\Omega}^\Omega (A_\Omega - |\Omega|) + |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \\
&\geq |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega .
\end{aligned}$$

We therefore have $|\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \leq \operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) \leq |\partial\Omega| \|S\|_{M^*_{\mathrm{op}}}^2$, and inserting this into (B.6.2) gives us

$$\int_{\mathbb{R}^{2d}} |\rho_\Omega^S(z) - \chi_\Omega(z)| \, dz \leq \left(1/\epsilon + 2\|S\|_{M^*_{\mathrm{op}}}^2 \right) |\partial\Omega|$$

when we also use the assumption $|\partial\Omega|/\epsilon \geq 1$. □

B.6.1 Sharpness of the bound

By considering Euclidean balls $B(z,R) = \{z' \in \mathbb{R}^{2d} : |z| < R\}$, it was shown in [9] that (B.6.1) gives a sharp bound for the convergence of accumulated spectrograms. As we will now show (see Theorem B.1.3), the same is true when the spectrogram is replaced with the Cohen class distribution Q_S for $S \in M^*_{\mathrm{op}}$. Our approach is inspired by [80], which deals with the case of spectrograms using the associated reproducing kernel Hilbert spaces. These Hilbert spaces are not available for general density operators S, so our proofs must instead rely on techniques from quantum harmonic analysis. In the terminology of [106, 107], the following result gives an expression for the *projection functional* applied to $\chi_\Omega \star S$.

Lemma B.6.2. *Let S be a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain. Then*

$$\operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_{\mathbb{R}^{2d} \setminus \Omega} \tilde{S}(z - z') \, dz' dz.$$

Proof. From Lemma B.4.2 we have that

$$\operatorname{tr}(\chi_\Omega \star S) = \int_{\mathbb{R}^{2d}} \chi_\Omega(z)\, dz$$
$$\operatorname{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_\Omega \tilde{S}(z - z')\, dz'\, dz.$$

In order to combine these two formulas, we note that

$$\int_{\mathbb{R}^{2d}} \tilde{S}(z - z')\, dz = \int_{\mathbb{R}^{2d}} S \star \check{S}(z - z')\, dz' = \operatorname{tr}(S)\operatorname{tr}(S) = 1$$

for each $z \in \mathbb{R}^{2d}$ by Lemma B.2.3. Hence we can in fact write

$$\begin{aligned} \operatorname{tr}(\chi_\Omega \star S) &= \int_{\mathbb{R}^{2d}} \chi_\Omega(z) \int_{\mathbb{R}^{2d}} \tilde{S}(z - z')\, dz'\, dz \\ &= \int_\Omega \int_{\mathbb{R}^{2d}} \tilde{S}(z - z')\, dz'\, dz. \end{aligned}$$

We may now combine our formulas to get that

$$\begin{aligned} \operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) &= \int_\Omega \int_{\mathbb{R}^{2d}} \tilde{S}(z - z')\, dz'\, dz - \int_\Omega \int_\Omega \tilde{S}(z - z')\, dz'\, dz \\ &= \int_\Omega \int_{\mathbb{R}^{2d} \setminus \Omega} \tilde{S}(z - z')\, dz'\, dz. \end{aligned}$$

□

We will also need the following technical consequence of the continuity of $\tilde{S}$.

Lemma B.6.3. *Let S be a density operator. There exist constants $r_S > 0$ and $m > 0$ such that whenever $r \leq r_S$*

$$\tilde{S}(z - z') \geq \frac{m}{|B(0, r)|} \int_{\mathbb{R}^{2d}} \chi_{B(z'', r)}(z) \chi_{B(z'', r)}(z')\, dz'' \tag{B.6.3}$$

for all $z, z' \in \mathbb{R}^{2d}$.

Proof. The function $\tilde{S} = S \star \check{S}$ is continuous, positive and satisfies $S \star \check{S}(0) = \operatorname{tr}(S^2) > 0$. Let $m = \operatorname{tr}(S^2)/2 > 0$. By continuity of $\tilde{S}$ at the origin, there must exist a constant $\delta > 0$ such that $S \star \check{S}(z) > m$ whenever $z \in B(0, \delta)$. Now let $r_S = \delta/2$, and consider the integral

$$\int_{\mathbb{R}^{2d}} \chi_{B(z'', r)}(z) \chi_{B(z'', r)}(z')\, dz''$$

for $r \leq r_S$. We note that the integrand is zero whenever $z - z' \notin B(0, 2r)$. When $z - z' \in B(0, 2r) \subset B(0, \delta)$ we know by construction of δ that $S \star \check{S}(z - z') \geq m$. We may also estimate that for any $z, z' \in \mathbb{R}^{2d}$

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \chi_{B(z'',r)}(z)\chi_{B(z'',r)}(z') \, dz'' &\leq \int_{\mathbb{R}^{2d}} \chi_{B(z'',r)}(z) \, dz'' \\ &= |B(0,r)|.\end{aligned}$$

Hence (B.6.3) holds: if $z - z' \notin B(0, 2r)$ it holds trivially as the integrand is zero, and if $z - z' \in B(0, 2r)$, we know that $S \star \check{S}(z - z') \geq m$ and the integral is bounded from above by $B(0, r)$. □

The previous two results lead to a lower bound for the projection functional for mixed-state localization operators with $\Omega = B(0, R)$.

Proposition B.6.4. *Let S be a density operator. Then there exists a constant C_S such that*

$$\operatorname{tr}(\chi_{B(0,R)} \star S) - \operatorname{tr}((\chi_{B(0,R)} \star S)^2) \geq C_S R^{2d-1}, \quad \textit{for } R > 1.$$

Proof. Let r_S be as in Lemma B.6.3, and let $r = \min\{r_S, 1\}$. By Lemma B.6.2 we know that

$$\operatorname{tr}(\chi_{B(0,R)} \star S) - \operatorname{tr}((\chi_{B(0,R)} \star S)^2) = \int_{B(0,R)} \int_{\mathbb{R}^{2d} \setminus B(0,R)} \tilde{S}(z - z') \, dz' dz,$$

and by inserting the estimate from Lemma B.6.3 we get that $\operatorname{tr}(\chi_{B(0,R)} \star S) - \operatorname{tr}((\chi_{B(0,R)} \star S)^2)$ is bounded from below by

$$\frac{m}{|B(0,r)|} \int_{B(0,R)} \int_{\mathbb{R}^{2d} \setminus B(0,R)} \int_{\mathbb{R}^{2d}} \chi_{B(z'',r)}(z)\chi_{B(z'',r)}(z') \, dz'' dz' dz.$$

By changing the order of integration this lower bound becomes

$$\frac{m}{|B(0,r)|} \int_{\mathbb{R}^{2d}} |B(0,R) \cap B(z'',r)| \cdot \left| \left(\mathbb{R}^{2d} \setminus B(0,R) \right) \cap B(z'',r) \right| \, dz''. \quad \text{(B.6.4)}$$

Now assume that z'' lies in the strip in $\mathbb{R}^{2d}$ defined by $R - r/2 \leq |z''| \leq R + r/2$. A simple estimate shows that both $B(0, R) \cap B(z'', r)$ and $\left(\mathbb{R}^{2d} \setminus B(0, R)\right) \cap B(z'', r)$ must contain a ball of radius $r/4$ in this case, so that

$$|B(0,R) \cap B(z'',r)| \cdot \left| \left(\mathbb{R}^{2d} \setminus B(0,R) \right) \cap B(z'',r) \right| \geq |B(0,r/4)|^2.$$

The expression in (B.6.4) is therefore bounded from below by

$$\begin{aligned}
&|B(0,r/4)|^2 \frac{m}{|B(0,r)|} \int_{R-r/2\leq |z''|\leq R+r/2} dz'' \\
&= |B(0,r/4)|^2 \frac{m}{|B(0,r)|} C_d \left((R+r/2)^{2d} - (R-r/2)^{2d} \right) \\
&\geq |B(0,r/4)|^2 \frac{m}{|B(0,r)|} C_d 2drR^{2d-1},
\end{aligned}$$

which finishes the proof by setting $C_S := |B(0,r/4)|^2 \frac{m}{|B(0,r)|} C_d 2dr$. Here C_d is the measure of the unit sphere in $\mathbb{R}^{2d}$, and the fact that $(R+r/2)^{2d} - (R-r/2)^{2d} \geq 2drR^{2d-1}$ is a simple consequence of the binomial theorem. □

Using these results we may now prove the desired sharpness of (B.6.1) with exactly the same arguments that were used to prove it for accumulated spectrograms in [9].

Proof of Theorem B.1.3. Since $|\partial B(0,R)| = C_d R^{2d-1}$, where C_d is the measure of the unit sphere in $\mathbb{R}^{2d}$, the upper bound follows from Theorem B.6.1 with $\epsilon = 1/C_d$. For the lower bound we will bound $\|\rho^S_{B(0,R)} - \chi_{B(0,R)}\|_{L^1}$ by $\mathrm{tr}(\chi_\Omega \star S) - \mathrm{tr}((\chi_\Omega \star S)^2$ from below, which will imply the result by Proposition B.6.4. In the proof of Theorem B.6.1 we derived the equalities

$$\int_\Omega |\rho^S_\Omega(z) - \chi_\Omega(z)|\, dz = |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega$$

$$\int_{\mathbb{R}^{2d}\setminus\Omega} |\rho^S_\Omega(z) - \chi_\Omega(z)|\, dz = A_\Omega - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega,$$

which together give us — when using $\sum_{k=1}^\infty \lambda_k^\Omega = |\Omega|$ by Lemma B.4.2 — that

$$\begin{aligned}
\|\rho^S_{B(0,R)} - \chi_{B(0,R)}\|_{L^1} &= |\Omega| - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega + A_\Omega - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega \\
&= \sum_{k=1}^{\infty} \lambda_k^\Omega - \sum_{k=1}^{A_\Omega} \lambda_k^\Omega + \sum_{k=1}^{A_\Omega} (1-\lambda_k^\Omega) \\
&= \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega + \sum_{k=1}^{A_\Omega} (1-\lambda_k^\Omega) \\
&\geq \sum_{k=A_\Omega+1}^{\infty} \lambda_k^\Omega (1-\lambda_k^\Omega) + \sum_{k=1}^{A_\Omega} \lambda_k^\Omega (1-\lambda_k^\Omega) \\
&= \sum_{k=1}^{\infty} \lambda_k^\Omega - \sum_{k=1}^{\infty} \left(\lambda_k^\Omega\right)^2 = \mathrm{tr}(\chi_\Omega \star S) - \mathrm{tr}((\chi_\Omega \star S)^2.
\end{aligned}$$

As mentioned, the result now follows from Proposition B.6.4. □

Remark B.5. In [9], the previous result is stated for spectrograms when $R > 0$. We have only obtained the result for $R > 1$, but this is not because we consider a more general setting. In fact, the proof for the upper bound in [9] is simply Theorem B.6.1 with $\epsilon = 1$, which needs the assumption $|\partial\Omega| \geq 1$. This is clearly not satisfied for arbitrarily small R.

B.7 Examples and other perspectives

We now turn to examples of Cohen's class distributions such that the theory of accumulated Cohen's class distributions works, namely those given by

$$Q_S(\psi) = \check{S} \star (\psi \otimes \psi)$$

for some density operator S. As we have mentioned, the definition above is equivalent to the more standard definition of Cohen's class, where $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$ defines a Cohen's class distribution by

$$Q_\phi(\psi) = \phi * W(\psi, \psi).$$

In fact, we introduced the set $\mathcal{W}$ such that $\phi \in \mathcal{W}$ if and only if $Q_\phi = Q_S$ for some density operator S. We will therefore look for functions ϕ that belong to $\mathcal{W}$.

A Weyl symbol characterization of M^*_{op}

Before we look at the examples, we reformulate the definition of M^*_{op}. By Proposition B.2.2, $S \in M^*_{\text{op}}$ if and only if the Weyl symbol ϕ of $\check{S}$ satisfies $\phi \in \mathcal{W}$ and

$$\int_{\mathbb{R}^{2d}} \phi * \check{\phi}(z)|z| \, dz < \infty. \tag{B.7.1}$$

B.7.1 Examples

Spectrograms

If $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$, such that $\check{S}$ has Weyl symbol $\phi = W(\check{\varphi}, \check{\varphi})$, then S is a density operator and by (B.3.3) Q_S is the spectrogram $Q_S(\psi) = |V_\varphi \psi|^2$. A calculation using the definition of convolutions of operators reveals that $\tilde{S} = |V_\varphi \varphi|^2$, so that $S \in M^*_{\text{op}}$ if and only if $\int_{\mathbb{R}^{2d}} |V_\varphi \varphi|^2(z)|z| \, dz < \infty$. This is the setting considered in the theory of accumulated spectrograms [8, 9].

Schwartz functions

If ϕ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$, then it is well-known [72, Prop. 286] that L_ϕ is a trace class operator with

$$\mathrm{tr}(L_\phi) = \int_{\mathbb{R}^{2d}} \phi(z)\, dz.$$

Hence any suitably normalized $\phi \in \mathcal{S}(\mathbb{R}^{2d})$ gives us an operator L_ϕ that is trace class with $\mathrm{tr}(L_\phi) = 1$, and it is also clear from (B.7.1) that $L_\phi \in M^*_{\mathrm{op}}$ in this case. The problem of determining whether L_ϕ is positive is much more difficult. The classical conditions on ϕ for L_ϕ to be a positive operator are the KLM-conditions [175, 199, 200], see also the more recent results in [61]. In the case where ϕ is a normalized, generalized Gaussian

$$\phi(z) = 2^d \frac{1}{\det(M)^{1/4}} e^{-z^T \cdot M \cdot z} \text{ for } z \in \mathbb{R}^{2d},$$

for some $2d \times 2d$-matrix M, it is known [127, 233] that the Weyl transform L_ϕ is a positive operator if and only if

$$M = S^T \Lambda S,$$

where S is a symplectic matrix and Λ is diagonal matrix of the form

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_d, \lambda_1, \lambda_2, \dots, \lambda_d)$$

with $0 < \lambda_i \leq 1$. Hence L_ϕ is a density operator in this case, and the theory of accumulated Cohen's class distributions will work for all such Gaussians. One should note that Q_ϕ is *not* a spectrogram for many of these Gaussians [74, 127]. Versions of these results on positivity of mixed Gaussian states have been obtained several times, see Section 4.2 in [172] and references therein, and they are also linked with the symplectic structure of the phase space [75].

A nonexample: the Wigner distribution

The prototype of a Cohen's class distribution is the Wigner distribution $W(\psi, \psi)$. By a result due to Grossmann [146], the Wigner distribution corresponds to $S = 2^d P$ in (B.3.1), i.e.

$$W(\psi, \psi) = Q_P(\psi) = 2^d P \star (\psi \otimes \psi),$$

see [204] for a proof. The parity operator P is not a density operator, and so our approach does not apply to the Wigner distribution. In fact, it has been shown that the operators $\chi_\Omega \star P$ are never trace class for a non-trivial domain Ω [217, Prop. 11]. As a consequence, the methods exploited in this paper, which often consider the sum of eigenvalues of such operators, will fail for the Wigner distribution.

B.7.2 Generating new examples from old

Checking whether a given function ϕ belongs to $\mathcal{W}$ is in general a non-trivial task. However, using quantum harmonic analysis we can use our examples of $\phi \in \mathcal{W}$ to obtain new elements of $\mathcal{W}$.

Lemma B.7.1. *Let S be a density operator, and let $f \in L^1(\mathbb{R}^{2d})$ be a positive function such that $\int_{\mathbb{R}^{2d}} f(z)\, dz = 1$. Then $f \star S$ is a density operator.*

Proof. By Lemma B.2.5, $f \star S$ is a positive operator, and the same proof as for the first part of Lemma B.4.2 gives that

$$\operatorname{tr}(f \star S) = \int_{\mathbb{R}^{2d}} f(z)\, dz \operatorname{tr}(S) = 1. \qquad \square$$

Using associativity of convolutions we see that the Cohen's class distribution associated to $\check{f} \star S$ is given by

$$Q_{\check{f} \star S}(\psi) = (\check{f} \star S)^{\check{}} \star (\psi \otimes \psi) = f * Q_S(\psi).$$

Corollary B.7.1.1. *Let $\phi \in \mathcal{W}$, and let $f \in L^1(\mathbb{R}^{2d})$ be a positive function such that $\int_{\mathbb{R}^{2d}} f(z)\, dz = 1$. Then $f * \phi \in \mathcal{W}$, and the Cohen's class distributions associated to ϕ and $f * \phi$ are related by*

$$Q_{f*\phi}(\psi) = f * Q_\phi(\psi) \quad \text{for } \psi \in L^2(\mathbb{R}^d). \tag{B.7.2}$$

Proof. We know from Proposition B.2.2 that $L_{f*\phi} = f \star L_\phi$, and as L_ϕ is a density operator by assumption the previous lemma gives that $f * \phi \in \mathcal{W}$. By definition

$$Q_{f*\phi}(\psi) = (f * \phi) * W(\psi, \psi) = f * (\phi * W(\psi, \psi)) = f * Q_\phi(\psi). \qquad \square$$

In particular, this works when Q_ϕ is a spectrogram, i.e. $Q_\phi(\psi) = |V_\varphi \psi|^2$ for some $\varphi \in L^2(\mathbb{R}^{2d})$. We then obtain the new Cohen's class distribution

$$Q_{\phi * f}(\psi) = f * |V_\varphi \psi|^2.$$

The non-asymptotic bounds on the convergence of accumulated Cohen's class distributions ρ_Ω^S in Theorems B.5.4, B.1.2 and B.6.1 depend on the quantity $\|S\|_{M^*_{\text{op}}}$. We are therefore interested in how this quantity changes when S is replaced by the new density operator $\check{f} \star S$ discussed above, or equivalently when Q_ϕ is replaced by $f * Q_\phi$.

Proposition B.7.2. *Let $S \in M^*_{\text{op}}$, and let f be a positive function such that $\int_{\mathbb{R}^{2d}} f(z)\, dz = 1$ and $\int_{\mathbb{R}^{2d}} f(z)|z|\, dz < \infty$. Then $\check{f} \star S \in M^*_{\text{op}}$, and*

$$\|\check{f} \star S\|^2_{M^*_{\text{op}}} \leq \|S\|^2_{M^*_{\text{op}}} + 2 \int_{\mathbb{R}^{2d}} f(z)|z|\, dz.$$

Proof. We begin by proving a general result. Assume that g, h are positive functions on $\mathbb{R}^{2d}$ such that $\int_{\mathbb{R}^{2d}} g(z)\ dz = \int_{\mathbb{R}^{2d}} h(z)\ dz = 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^{2d}} g * h(z)|z|\ dz &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} g(z')h(z-z')|z|\ dz'\ dz \\ &= \int_{\mathbb{R}^{2d}} g(z') \int_{\mathbb{R}^{2d}} h(z'')|z''+z'|\ dz''\ dz' \qquad (z'' := z - z') \\ &\leq \int_{\mathbb{R}^{2d}} g(z') \int_{\mathbb{R}^{2d}} h(z'')\left(|z''|+|z'|\right)\ dz''\ dz' \\ &= \int_{\mathbb{R}^{2d}} h(z)|z|\ dz + \int_{\mathbb{R}^{2d}} g(z)|z|\ dz.\end{aligned}$$

Now note that

$$\|\check{f} \star S\|_{M^*_{\text{op}}}^2 = \int_{\mathbb{R}^{2d}} (\check{f} \star S) \star (\check{f} \star S)^{\vee}(z)|z|\ dz = \int_{\mathbb{R}^{2d}} (\check{f} * f) * (\check{S} \star S)(z)|z|\ dz,$$

where we have used $(\check{f} \star S)^{\vee} = f \star \check{S}$ and the commutativity and associativity of convolutions. The functions $g = \check{f} * f$ and $h = S \star \check{S}$ satisfy the assumptions of the calculation above, since they are positive functions by Lemma B.2.5, $\int_{\mathbb{R}^{2d}} S \star \check{S}(z)\ dz = \mathrm{tr}(S)\mathrm{tr}(\check{S}) = 1$ by Lemma B.2.3 and $\int_{\mathbb{R}^{2d}} \check{f} * f(z)\ dz = \left(\int_{\mathbb{R}^{2d}} f(z)\ dz\right)^2 = 1$ by a simple calculation using Tonelli's theorem. So we apply our calculation with these functions, and find that

$$\begin{aligned}\|\check{f} \star S\|_{M^*_{\text{op}}}^2 &\leq \int_{\mathbb{R}^{2d}} \check{f} * f(z)|z|\ dz + \int_{\mathbb{R}^{2d}} S \star \check{S}(z)|z|\ dz \\ &= \|S\|_{M^*_{\text{op}}}^2 + \int_{\mathbb{R}^{2d}} f * \check{f}(z)|z|\ dz.\end{aligned}$$

Furthermore, if we pick $g = \check{f}$ and $h = f$, we get $\int_{\mathbb{R}^{2d}} \check{f} * f(z)|z|\ dz \leq 2\int_{\mathbb{R}^{2d}} f(z)|z|\ dz$. If we insert this into the estimate above, our result follows. □

Remark B.6. The idea of smoothing a time-frequency distribution Q by taking convolutions with a function f on $\mathbb{R}^{2d}$, as in (B.7.2), is useful in practice [172]. In a sense, this is the idea behind Cohen's class, which by definition consists of smoothened versions of the Wigner distribution. Janssen mentions the case where Q has "rapidly alternating positive and negative values" [172, p. 3], where smoothing can remove this behaviour. In fact, we saw in Example B.7.1 that convolving the Wigner distribution with a Gaussian ϕ produces a positive distribution Q_ϕ.

Another simple way of obtaining new examples is to consider convex combinations. If $\phi_n \in \mathcal{W}$ for each $1 \leq n \leq N$ and $\{\lambda_n\}_{n=1}^N$ is a sequence of nonnegative numbers with $\sum_{n=1}^{N} \lambda_n = 1$, then

$$\phi := \sum_{n=1}^{N} \lambda_n \phi_n$$

also belongs to $\mathcal{W}$ since $L_\phi = \sum_{n=1}^{N} \lambda_n L_{\phi_n}$. Using the definition of positivity for operators it is trivial to check that L_ϕ is positive, and

$$\mathrm{tr}(L_\phi) = \sum_{n=1}^{N} \lambda_n \mathrm{tr}(L_{\phi_n}) = \sum_{n=1}^{N} \lambda_n = 1.$$

Paper C

Quantum Harmonic Analysis on Lattices and Gabor Multipliers

Eirik Skrettingland

Published in
Journal of Fourier Analysis and Applications, 2019, volume 26, issue 3.

Paper C

Quantum Harmonic Analysis on Lattices and Gabor Multipliers

Abstract

We develop a theory of quantum harmonic analysis on lattices in $\mathbb{R}^{2d}$. Convolutions of a sequence with an operator and of two operators are defined over a lattice, and using corresponding Fourier transforms of sequences and operators we develop a version of harmonic analysis for these objects. We prove analogues of results from classical harmonic analysis and the quantum harmonic analysis of Werner, including approximation theorems and a Wiener division lemma. Gabor multipliers from time-frequency analysis are described as convolutions in this setting. The quantum harmonic analysis is thus a conceptual framework for the study of Gabor multipliers, and several of the results include results on Gabor multipliers as special cases.

C.1 Introduction

In time-frequency analysis, one studies a signal $\psi \in L^2(\mathbb{R}^d)$ by considering various time-frequency representations of ψ. An important class of time-frequency representations is obtained by fixing $\varphi \in L^2(\mathbb{R}^d)$ and considering the *short-time Fourier transform* $V_\varphi\psi$ of ψ with window φ, which is the function on the time-frequency plane $\mathbb{R}^{2d}$ given by

$$V_\varphi\psi(z) = \langle \psi, \pi(z)\varphi \rangle_{L^2} \qquad \text{for } z \in \mathbb{R}^{2d},$$

where $\pi(z) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the *time-frequency shift* given by $\pi(z)\varphi(t) = e^{2\pi i \omega \cdot t}\varphi(t-x)$ for $z = (x, \omega)$. The intuition is that $V_\varphi\psi(z)$ carries information about the components of the signal ψ with frequency ω at time x.

A question going back to von Neumann [248] and Gabor [119] is the validity of reconstruction formulas of the form

$$\psi = \sum_{\lambda \in \Lambda} V_\varphi \psi(\lambda) \pi(\lambda) \xi \quad \text{for any } \psi \in L^2(\mathbb{R}^d), \tag{C.1.1}$$

where $\Lambda = A\mathbb{Z}^{2d}$ for $A \in GL(2d, \mathbb{R})$ is a lattice in $\mathbb{R}^{2d}$ and $\varphi, \xi \in L^2(\mathbb{R}^d)$. It is known that (C.1.1) is indeed true for certain windows φ, ξ and lattices Λ, and such formulas naturally lead to the concept of *Gabor multipliers*. If $\varphi, \xi \in L^2(\mathbb{R}^d)$ and $m = \{m(\lambda)\}_{\lambda \in \Lambda}$ is a sequence of complex numbers, we define the Gabor multiplier $\mathcal{G}_m^{\varphi,\xi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$\mathcal{G}_m^{\varphi,\xi}(\psi) := \sum_{\lambda \in \Lambda} m(\lambda) V_\varphi \psi(\lambda) \pi(\lambda) \xi.$$

Compared to (C.1.1) we see that $\mathcal{G}_m^{\varphi,\xi}$ modifies the time-frequency content of ψ in a simple way, namely by multiplying the samples of its time-frequency representation with a mask m. Gabor multipliers have been studied in the mathematics literature by [32, 63, 89, 97, 105, 108, 137, 144] among others, and also in more application-oriented contributions [20, 216, 240].

Gabor multipliers are the discrete analogues of the much-studied localization operators [26, 63, 68, 143]. In [203] we showed that the quantum harmonic analysis developed by Werner and coauthors [182, 251] provides a conceptual framework for localization operators, leading to new results and interesting reinterpretations of older results on localization operators. The goal of this paper is therefore to develop a version of quantum harmonic analysis for lattices to provide a similar conceptual framework for Gabor multipliers. Hence we continue the line of research into applications of quantum harmonic analysis from [202–204].

With this aim we introduce two convolutions of operators and sequences in Section C.4. Following [102, 188, 251] we first define the translation of an operator S on $L^2(\mathbb{R}^d)$ by $\lambda \in \Lambda$ to be the operator

$$\alpha_\lambda(S) := \pi(\lambda) S \pi(\lambda)^*.$$

If $c \in \ell^1(\Lambda)$ and S is a trace class operator on $L^2(\mathbb{R}^d)$, the convolution $c \star_\Lambda S$ is defined to be the *operator*

$$c \star_\Lambda S := \sum_{\lambda \in \Lambda} c(\lambda) \alpha_\lambda(S).$$

Gabor multipliers are then given by convolutions

$$\mathcal{G}_m^{\varphi,\xi} = m \star_\Lambda (\xi \otimes \varphi),$$

where $\xi \otimes \varphi$ is the rank-one operator $\xi \otimes \varphi(\psi) = \langle \psi, \varphi \rangle_{L^2} \xi$. Furthermore, we define the convolution $S \star_\Lambda T$ of two trace class operators S and T to be the *sequence* over Λ given by

$$S \star_\Lambda T(\lambda) := \operatorname{tr}(S\alpha_\lambda(\check{T})),$$

where $\check{T} = PTP$ with P the parity operator $P\psi(t) = \psi(-t)$ for $\psi \in L^2(\mathbb{R}^d)$. In Section C.4 we investigate the commutativity and associativity of these convolutions, extend their domains and in Proposition C.4.3 we establish a version of Young's inequality for convolutions of operators and sequences.

An important tool throughout the paper is a Banach space $\mathcal{B}$ of trace class operators, consisting of operators with Weyl symbol in the so-called Feichtinger algebra [95]. The use of $\mathcal{B}$ allows us to obtain continuity results for the convolutions with respect to $\ell^p(\Lambda)$ and Schatten-p classes – an important example is Proposition C.4.1 which states that

$$\|S \star_\Lambda T\|_{\ell^1(\Lambda)} \lesssim \|S\|_{\mathcal{B}} \|T\|_{\mathcal{S}^1}$$

for $S \in \mathcal{B}$ and trace class T, where $\|\cdot\|_{\mathcal{S}^1}$ is the trace class norm. While there are other classes of operators that would ensure that $S \star_\Lambda T \in \ell^1(\Lambda)$, see for instance the Schwartz operators [179], $\mathcal{B}$ has the advantage of being a Banach space, hence allowing the use of tools such as Banach space adjoints. The space $\mathcal{B}$ has previously been studied by [89, 102, 168] among others.

To complement the convolutions, we introduce Fourier transforms of sequences and operators in Section C.5. For a sequence $c \in \ell^1(\Lambda)$ we use its symplectic Fourier series

$$\mathcal{F}_\sigma^\Lambda(c)(z) := \sum_{\lambda \in \Lambda} c(\lambda) e^{2\pi i \sigma(\lambda, z)} \quad \text{for } z \in \mathbb{R}^{2d},$$

where $\sigma(z, z') = \omega \cdot x' - x \cdot \omega'$ for $z = (x, \omega), z' = (x', \omega')$. As a Fourier transform for trace class operators S we use the Fourier-Wigner transform

$$\mathcal{F}_W(S)(z) := e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S) \quad \text{for } z = (x, \omega) \in \mathbb{R}^{2d}.$$

Equipped with both convolutions and Fourier transforms, we naturally ask whether the Fourier transforms turn convolutions into products. We show in Theorem C.5.3 for $z \in \mathbb{R}^{2d}$ that

$$\mathcal{F}_\sigma^\Lambda(S \star_\Lambda T)(z) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} F_W(S)(z + \lambda^\circ) \mathcal{F}_W(T)(z + \lambda^\circ), \tag{C.1.2}$$

where Λ° is the adjoint lattice of Λ defined in Section C.5, and in Propositions C.5.4 and C.5.5 we show that

$$\mathcal{F}_W(c \star_\Lambda S)(z) = \mathcal{F}_\sigma^\Lambda(c)(z) \mathcal{F}_W(S)(z). \tag{C.1.3}$$

These results include as special cases the so-called fundamental identity of Gabor analysis [103,170,223,245] and results on the spreading function of Gabor multipliers due to [89]. Equations (C.1.2) and (C.1.3) hold for general classes of operators and sequences, and we take care to give a precise interpretation of the objects and equalities in all cases.

A fruitful approach to Gabor multipliers due to Feichtinger [97] is to consider the so-called Kohn-Nirenberg symbol of operators. The Kohn-Nirenberg symbol of an operator S on $L^2(\mathbb{R}^d)$ is a function on $\mathbb{R}^{2d}$, and Feichtinger used this to reduce questions about Gabor multipliers in the Hilbert Schmidt operators to questions about functions in $L^2(\mathbb{R}^{2d})$. This approach has later been used in other papers on Gabor multipliers [32, 89, 105]. As Gabor multipliers are examples of convolutions, we show in Section C.6 that this approach can be generalized and phrased in terms of our quantum harmonic analysis, and that one of the main results of [97] finds a natural interpretation as a Wiener's lemma in our setting – see Theorem C.6.3, Corollary C.6.3.1 and the remarks following the corollary.

In Section C.7 we show the extension of some deeper results of harmonic analysis on $\mathbb{R}^d$ to our setting. We obtain an analogue of Wiener's classical approximation theorem in Theorem C.7.3, similar to the results of Werner and coauthors [182, 251] in the continuous setting. As an example we have the following equivalent statements for $S \in \mathcal{B}$:

(i) The set of zeros of $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ contains no open subsets in $\mathbb{R}^{2d}/\Lambda^\circ$.

(ii) If $c \star_\Lambda S = 0$ for $c \in \ell^1(\Lambda)$, then $c = 0$.

(iii) $\mathcal{B}' \star_\Lambda S$ is weak*-dense in $\ell^\infty(\Lambda)$.

These results are related to earlier investigations of Gabor multipliers by Feichtinger [97]. In particular, he showed that if $S = \xi \otimes \varphi$ is a rank-one operator and $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ has *no* zeros, then any $m \in \ell^\infty(\Lambda)$ can be recovered from the Gabor multiplier $\mathcal{G}_m^{\varphi,\xi}$. Since Gabor multipliers are given by convolutions, the equivalence (i) $\iff$ (ii) shows that we can recover $m \in \ell^1(\Lambda)$ from $\mathcal{G}_m^{\varphi,\xi}$ under the weaker condition (i) – this holds in particular for finite sequences m.

Finally, we apply our techniques to prove a version of Wiener's division lemma in Theorem C.7.4. At the level of Weyl symbols this turns out to reproduce a result by Gröchenig and Pauwels [141], but in our context it has the following interpretation:

> If $\mathcal{F}_W(S)$ has compact support for some operator S, and the support is sufficiently small compared to the density of Λ, then there exists a sequence $m \in \ell^\infty(\Lambda)$ such that $S = m \star_\Lambda A$ for some $A \in \mathcal{B}$. If S belongs to the Schatten-p class of compact operators, then $m \in \ell^p(\Lambda)$.

The above result fits well into the common intuition that operators S with compactly supported $\mathcal{F}_W(S)$ (so-called underspread operators) can be approximated by Gabor multipliers [89] – i.e. by operators $c \star_\Lambda T$ where T is a rank-one operator. The result shows that if we allow T to be *any* operator in $\mathcal{B}$, then any underspread operator S is precisely of the form $S = c \star_\Lambda T$ for a sufficiently dense lattice Λ.

We end this introduction by emphasizing the hybrid nature of our setting. In [251], Werner introduced quantum harmonic analysis of functions on $\mathbb{R}^{2d}$ and operators on the Hilbert space $L^2(\mathbb{R}^d)$. We are considering the discrete setting of sequences on a lattice instead of functions on $\mathbb{R}^{2d}$. If we had modified the Hilbert space $L^2(\mathbb{R}^d)$ accordingly, many of our results would follow by the arguments of [251], as already outlined in [182]. However, we keep the same Hilbert space $L^2(\mathbb{R}^d)$ as in the continuous setting. We are therefore mixing the discrete (lattices) and the continuous ($L^2(\mathbb{R}^d)$), which leads to some extra intricacies.

C.2 Conventions

By a lattice Λ we mean a full-rank lattice in $\mathbb{R}^{2d}$, i.e. $\Lambda = A\mathbb{Z}^{2d}$ for $A \in GL(2d, \mathbb{R})$. The volume of $\Lambda = A\mathbb{Z}^{2d}$ is $|\Lambda| := \det(A)$. For a lattice Λ, the Haar measure on $\mathbb{R}^{2d}/\Lambda$ will always be normalized so that $\mathbb{R}^{2d}/\Lambda$ has total measure 1.

If X is a Banach (or Fréchet) space and X' its dual space, the action of $y \in X'$ on $x \in X$ is denoted by the bracket $\langle y, x\rangle_{X',X}$, where the bracket is antilinear in the second coordinate to be compatible with the notation for inner products in Hilbert spaces. This means that we are identifying the dual space X' with *anti*linear functionals on X. For two Banach spaces X, Y we use $\mathcal{L}(X, Y)$ to denote the Banach space of continuous linear operators from X to Y, and if $X = Y$ we simply write $\mathcal{L}(X)$. The notation $P \lesssim Q$ means that there is some $C > 0$ such that $P \leq C \cdot Q$.

C.3 Spaces of operators and functions

C.3.1 Time-frequency shifts and the short-time Fourier transform

For $z = (x, \omega) \in \mathbb{R}^{2d}$ we define the *time-frequency shift* operator $\pi(z)$ by

$$(\pi(z)\psi)(t) = e^{2\pi i \omega \cdot t}\psi(t - x) \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

Hence $\pi(z)$ can be written as the composition $M_\omega T_x$ of a translation operator $(T_x\psi)(t) = \psi(t - x)$ and a modulation operator $(M_\omega\psi)(t) = e^{2\pi i \omega \cdot t}\psi(t)$. The time-frequency shifts $\pi(z)$ are unitary operators on $L^2(\mathbb{R}^d)$. For $\psi, \varphi \in L^2(\mathbb{R}^d)$ we can use the time-frequency shifts to define the *short-time Fourier transform* $V_\varphi\psi$ of ψ with window φ by

$$V_\varphi\psi(z) = \langle \psi, \pi(z)\varphi\rangle_{L^2} \quad \text{for } z \in \mathbb{R}^{2d}.$$

The short-time Fourier transform satisfies an orthogonality condition, sometimes called Moyal's identity [114, 131].

Lemma C.3.1 (Moyal's identity)**.** *If* $\psi_1, \psi_2, \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$*, then* $V_{\varphi_i}\psi_j \in L^2(\mathbb{R}^{2d})$ *for* $i, j \in \{1, 2\}$*, and the relation*

$$\left\langle V_{\varphi_1}\psi_1, V_{\varphi_2}\psi_2 \right\rangle_{L^2} = \langle \psi_1, \psi_2 \rangle_{L^2} \overline{\langle \varphi_1, \varphi_2 \rangle}_{L^2}$$

holds, where the leftmost inner product is in $L^2(\mathbb{R}^{2d})$ *and those on the right are in* $L^2(\mathbb{R}^d)$.

By replacing the inner product in the definition of $V_\varphi\psi$ by a duality bracket, one can define the short-time Fourier transform for other classes of ψ, φ. The most general case we need is that of a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$; we define

$$V_\psi\varphi(z) = \langle \psi, \pi(z)\varphi \rangle_{\mathcal{S}', \mathcal{S}} \qquad \text{for } z \in \mathbb{R}^{2d}.$$

C.3.2 Feichtinger's algebra

An appropriate space of functions for our purposes will be Feichtinger's algebra $S_0(\mathbb{R}^d)$, first introduced by Feichtinger in [95]. To define $S_0(\mathbb{R}^d)$, let φ_0 denote the L^2-normalized Gaussian $\varphi_0(x) = 2^{d/4}e^{-\pi|x|^2}$ for $x \in \mathbb{R}^d$. Then $S_0(\mathbb{R}^d)$ is the space of all $\psi \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|\psi\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)| \, dz < \infty.$$

With the norm above, $S_0(\mathbb{R}^d)$ is a Banach space of continuous functions and an algebra under multiplication and convolution [95]. By [131, Thm. 11.3.6], the dual space of $S_0(\mathbb{R}^d)$ is the space $S_0'(\mathbb{R}^d)$ consisting of all $\psi \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|\psi\|_{S_0'} := \sup_{z \in \mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)| \, dz < \infty,$$

where an element $\psi \in S_0'(\mathbb{R}^d)$ acts on $\phi \in S_0(\mathbb{R}^d)$ by

$$\langle \phi, \psi \rangle_{S_0', S_0} = \int_{\mathbb{R}^{2d}} V_{\varphi_0}\phi(z)\overline{V_{\varphi_0}\psi(z)} \, dz.$$

We get the following chain of continuous inclusions:

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow S_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow S_0'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

One important reason for using Feichtinger's algebra is that it consists of continuous functions, and that sampling them over a lattice produces a summable sequence [95, Thm. 7C)].

Lemma C.3.2 (Sampling Feichtinger's algebra)**.** *Let* Λ *be a lattice in* $\mathbb{R}^{2d}$ *and* $f \in S_0(\mathbb{R}^{2d})$*. Then* $f|_\Lambda = \{f(\lambda)\}_{\lambda\in\Lambda} \in \ell^1(\Lambda)$ *with*

$$\|f|_\Lambda\|_{\ell^1} \lesssim \|f\|_{S_0},$$

where the implicit constant depends only on the lattice Λ*.*

C.3.3 The symplectic Fourier transform

We will use the *symplectic Fourier transform* $\mathcal{F}_\sigma f$ of functions $f \in L^1(\mathbb{R}^{2d})$, defined by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i\sigma(z,z')}\, dz',$$

where σ is the standard symplectic form $\sigma(z,z') = \omega\cdot x' - x\cdot\omega'$ for $z = (x,\omega), z' = (x',\omega')$. $\mathcal{F}_\sigma$ is a Banach space isomorphism $S_0(\mathbb{R}^{2d}) \to S_0(\mathbb{R}^{2d})$, extends to a unitary operator $L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ and a Banach space isomorphism $S_0'(\mathbb{R}^{2d}) \to S_0'(\mathbb{R}^{2d})$ [102, Lem. 7.6.2]. In fact, $\mathcal{F}_\sigma$ is its own inverse, so that $\mathcal{F}_\sigma(\mathcal{F}_\sigma(f)) = f$ for $f \in S_0'(\mathbb{R}^{2d})$ [72, Prop. 144].

C.3.4 Banach spaces of operators on $L^2(\mathbb{R}^d)$

The results of this paper concern operators on various function spaces, and we will pick operators from two kinds of spaces: the Schatten-p classes $\mathcal{S}^p$ for $1 \leq p \leq \infty$ and a space $\mathcal{B}$ of operators defined using the Feichtinger algebra.

The Schatten classes

Starting with the Schatten classes, we recall that any compact operator S on $L^2(\mathbb{R}^d)$ has a singular value decomposition [54, Remark 3.1], i.e. there exist two orthonormal sets $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a bounded sequence of non-negative numbers $\{s_n(S)\}_{n\in\mathbb{N}}$ such that S may be expressed as

$$S = \sum_{n\in\mathbb{N}} s_n(S)\psi_n \otimes \phi_n,$$

with convergence of the sum in the operator norm. Here $\psi\otimes\phi$ for $\psi,\phi \in L^2(\mathbb{R}^d)$ denotes the rank-one operator $\psi\otimes\phi(\xi) = \langle \xi,\phi\rangle_{L^2}\,\psi$.

For $1 \leq p < \infty$ we define the *Schatten-p class* $\mathcal{S}^p$ of operators on $L^2(\mathbb{R}^d)$ by

$$\mathcal{S}^p = \{T \text{ compact} : \{s_n(T)\}_{n\in\mathbb{N}} \in \ell^p\}.$$

To simplify the statement of some results, we also define $\mathcal{S}^\infty = \mathcal{L}(L^2)$ with $\|\cdot\|_{\mathcal{S}^\infty}$ given by the operator norm. The Schatten-p class $\mathcal{S}^p$ is a Banach space with the norm $\|S\|_{\mathcal{S}^p} = \left(\sum_{n\in\mathbb{N}} s_n(S)^p\right)^{1/p}$. Of particular interest is the space $\mathcal{S}^1$; the so-called trace class operators. Given an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}^d)$, the trace defined by

$$\mathrm{tr}(S) = \sum_{n\in\mathbb{N}} \langle Se_n, e_n\rangle_{L^2}$$

is a well-defined and bounded linear functional on $\mathcal{S}^1$, and independent of the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ used. The dual space of $\mathcal{S}^1$ is $\mathcal{L}(L^2)$ [54, Thm. 3.13], and $T \in \mathcal{L}(L^2)$ defines a bounded antilinear functional on $\mathcal{S}^1$ by

$$\langle T, S\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} = \mathrm{tr}(TS^*) \quad \text{for } S \in \mathcal{S}^1.$$

Another special case is the space of Hilbert-Schmidt operators $\mathcal{HS} := \mathcal{S}^2$, which is a Hilbert space with inner product

$$\langle S, T\rangle_{\mathcal{HS}} = \mathrm{tr}(ST^*).$$

The Weyl transform and operators with symbol in $S_0(\mathbb{R}^{2d})$

The other class of operators we will use will be defined in terms of the *Weyl transform.* We first need the *cross-Wigner distribution* $W(\xi, \eta)$ of two functions $\xi, \eta \in L^2(\mathbb{R}^d)$, defined by

$$W(\xi,\eta)(x,\omega) = \int_{\mathbb{R}^d} \xi\left(x+\frac{t}{2}\right)\overline{\eta\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega\cdot t}\,dt \quad \text{for } (x,\omega)\in\mathbb{R}^{2d}.$$

For $f \in S_0'(\mathbb{R}^{2d})$, we define the *Weyl transform* L_f of f to be the operator $L_f : S_0(\mathbb{R}^d) \to S_0'(\mathbb{R}^d)$ given by

$$\left\langle L_f\eta, \xi\right\rangle_{S_0',S_0} := \langle f, W(\xi,\eta)\rangle_{S_0',S_0} \quad \text{for any } \xi,\eta \in S_0(\mathbb{R}^d).$$

f is called the *Weyl symbol* of the operator L_f. By the kernel theorem for modulation spaces [131, Thm. 14.4.1], the Weyl transform is a bijection from $S_0'(\mathbb{R}^{2d})$ to $\mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))$.

Notation. In particular, any $S \in \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))$ has a Weyl symbol, and we will denote the Weyl symbol of S by a_S. By definition, this means that $L_{a_S} = S$.

It is also well-known that the Weyl transform is a unitary mapping from $L^2(\mathbb{R}^{2d})$ to $\mathcal{HS}$ [215]. This means in particular that

$$\langle S, T\rangle_{\mathcal{HS}} = \langle a_S, a_T\rangle_{L^2} \quad \text{for } S, T \in \mathcal{HS},$$

which often allows us to reduce statements about Hilbert Schmidt operators to statements about $L^2(\mathbb{R}^{2d})$.

We then define $\mathcal{B}$ to be the Banach space of continuous operators $S : S_0(\mathbb{R}^d) \to S_0'(\mathbb{R}^d)$ such that $a_S \in S_0(\mathbb{R}^{2d})$, with norm

$$\|S\|_{\mathcal{B}} := \|a_S\|_{S_0}.$$

$\mathcal{B}$ consists of trace class operators $L^2(\mathbb{R}^d)$ and we have a norm-continuous inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{S}^1$ [129, 138].

Example C.3.1. If $\phi, \psi \in L^2(\mathbb{R}^d)$, consider the rank-one operator $\phi \otimes \psi$. Its Weyl symbol is the cross-Wigner distribution $W(\phi, \psi)$ [72, Cor. 207], and $W(\phi, \psi) \in S_0(\mathbb{R}^{2d})$ if and only if $\phi, \psi \in S_0(\mathbb{R}^d)$ [72, Prop. 365]. The simplest examples of operators in $\mathcal{B}$ are therefore $\phi \otimes \psi$ for $\phi, \psi \in S_0(\mathbb{R}^d)$.

The dual space $\mathcal{B}'$ can also be identified with a Banach space of operators. By definition, $\tau : \mathcal{B} \to S_0(\mathbb{R}^{2d})$ given by $\tau(S) = a_S$ is an isometric isomorphism. Hence the Banach space adjoint $\tau^* : S_0'(\mathbb{R}^{2d}) \to \mathcal{B}'$ is also an isomorphism. Since the Weyl transform is a bijection from $S_0'(\mathbb{R}^{2d})$ to $\mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))$, we can identify $\mathcal{B}'$ with operators $S_0(\mathbb{R}^d) \to S_0'(\mathbb{R}^d)$:

$$\mathcal{B}' \xleftrightarrow{\tau^*} S_0'(\mathbb{R}^{2d}) \xleftrightarrow{\text{Weyl calculus}} \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d)).$$

In this paper we will always consider elements of $\mathcal{B}'$ as operators $S_0(\mathbb{R}^d) \to S_0'(\mathbb{R}^d)$ using these identifications. Since $\mathcal{L}(L^2)$ is the dual space of $\mathcal{S}^1$, the Banach space adjoint $\iota^* : \mathcal{L}(L^2) \to \mathcal{B}'$ is a weak*-to-weak*-continuous inclusion of $\mathcal{L}(L^2)$ into $\mathcal{B}'$.

Remark C.1. For more results on $\mathcal{B}$ and $\mathcal{B}'$ we refer to [102, 168]. In particular we mention that we could have defined $\mathcal{B}$ using other pseudodifferential calculi, such as the Kohn Nirenberg calculus, and still get the same space $\mathcal{B}$ with an equivalent norm. We would also like to point out that the statements of this section may naturally be rephrased using the notion of Gelfand triples, see [102].

C.3.5 Translation of operators

The idea of translating an operator $S \in \mathcal{L}(L^2)$ by $z \in \mathbb{R}^{2d}$ using conjugation with $\pi(z)$ has been utilized both in physics [251] and time-frequency analysis [102, 188]. More precisely, we define for $z \in \mathbb{R}^{2d}$ and $S \in \mathcal{B}'$ the translation of S by z to be the operator

$$\alpha_z(S) = \pi(z) S \pi(z)^*.$$

We will also need the operation $S \mapsto \check{S} = PSP$, where P is the parity operator $(P\psi)(t) = \psi(-t)$ for $\psi \in L^2(\mathbb{R}^d)$. The main properties of these operations are listed below, note in particular that part (i) supports the intuition that α_z is a translation of operators. See Lemmas 3.1 and 3.2 in [203] for the proofs.

Lemma C.3.3. *Let $S \in \mathcal{B}'$.*

(i) If a_S is the Weyl symbol of S, then the Weyl symbol of $\alpha_z(S)$ is $T_z(a_S)$.

(ii) $\alpha_z(\alpha_{z'}(S)) = \alpha_{z+z'}(S)$.

(iii) The operations α_z, * and $\check{}$ are isometries on $\mathcal{B}, \mathcal{B}'$ and $\mathcal{S}^p$ for $1 \leq p \leq \infty$.

(iv) $(S^*)\check{} = (\check{S})^*$.

By the last part we can unambiguously write $\check{S}^*$.

C.4 Convolutions of sequences and operators

In [251], the convolution of a function $f \in L^1(\mathbb{R}^{2d})$ and an operator $S \in \mathcal{S}^1$ was defined by the operator-valued integral

$$f \star S := \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\, dz$$

and the convolution of two operators $S, T \in \mathcal{S}^1$ was defined to be the *function*

$$S \star T(z) := \operatorname{tr}(S\alpha_z(\check{T})) \quad \text{for } z \in \mathbb{R}^{2d}.$$

These definitions, along with a Fourier transform defined for operators, have been shown to produce a theory of quantum harmonic analysis with non-trivial consequences for topics such as quantum measurement theory [182] and time-frequency analysis [203]. The setting where $\mathbb{R}^{2d}$ is replaced by some lattice $\Lambda \subset \mathbb{R}^{2d}$ is frequently studied in time-frequency analysis, and our goal is therefore to develop a theory of convolutions and Fourier transforms of operators in that setting.

For a sequence $c \in \ell^1(\Lambda)$ and $S \in \mathcal{S}^1$, we define the operator

$$c \star_\Lambda S := S \star_\Lambda c := \sum_{\lambda \in \Lambda} c(\lambda)\alpha_\lambda(S), \tag{C.4.1}$$

and for operators $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$ we define the sequence

$$S \star_\Lambda T(\lambda) := S \star T(\lambda) \quad \text{for } \lambda \in \Lambda. \tag{C.4.2}$$

Hence $S \star_\Lambda T$ is the *sequence* obtained by restricting the *function* $S \star T$ to Λ.

Remark C.2. We use the same notation $\star_\Lambda$ for the convolution of an operator and a sequence and for the convolution of two operators. The correct interpretation of $\star_\Lambda$ will always be clear from the context.

Since α_λ is an isometry on $\mathcal{S}^1$ and $\mathcal{B}$, $c \star_\Lambda S$ is well-defined with $\|c \star_\Lambda S\|_{\mathcal{S}^1} \leq \|c\|_{\ell^1} \|S\|_{\mathcal{S}^1}$ for $S \in \mathcal{S}^1$ and similarly $\|c \star_\Lambda S\|_{\mathcal{B}} \leq \|c\|_{\ell^1} \|S\|_{\mathcal{B}}$ for $S \in \mathcal{B}$. The fact that $S \star_\Lambda T$ is a well-defined and summable sequence on Λ is less straightforward.

Proposition C.4.1. *If $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$, then $S \star_\Lambda T \in \ell^1(\Lambda)$ with $\|S \star_\Lambda T\|_{\ell^1} \lesssim \|S\|_{\mathcal{B}} \|T\|_{\mathcal{S}^1}$.*

Proof. By [203, Thm. 8.1] we know that $S \star T \in S_0(\mathbb{R}^{2d})$ with $\|S \star T\|_{S_0} \lesssim \|S\|_{\mathcal{B}} \|T\|_{\mathcal{S}^1}$. Hence the result follows from Lemma C.3.2 and $S \star_\Lambda T(\lambda) = S \star T(\lambda)$. □

C.4.1 Gabor multipliers and sampled spectrograms

If we consider rank-one operators, these convolutions reproduce well-known objects from time-frequency analysis. First consider the rank-one operator $\xi_1 \otimes \xi_2$ for $\xi_1, \xi_2 \in L^2(\mathbb{R}^d)$. The operators $c \star_\Lambda (\xi_1 \otimes \xi_2)$ are well-known in time-frequency analysis as *Gabor multipliers* [32, 89, 97, 105]: it is simple to show that

$$\alpha_\lambda(\xi_1 \otimes \xi_2) = (\pi(\lambda)\xi_1) \otimes (\pi(\lambda)\xi_2),$$

so if $c \in \ell^1(\Lambda)$ it follows from the definition (C.4.1) that $c \star_\Lambda (\xi_1 \otimes \xi_2)$ acts on $\psi \in L^2(\mathbb{R}^d)$ by

$$c \star_\Lambda (\xi_1 \otimes \xi_2)\psi = \sum_{\lambda \in \Lambda} c(\lambda) V_{\xi_2}\psi(\lambda) \pi(\lambda) \xi_1, \tag{C.4.3}$$

which is the definition of the Gabor multiplier $\mathcal{G}_c^{\xi_2,\xi_1}$ used in time-frequency analysis [105], i.e. $\mathcal{G}_c^{\xi_2,\xi_1} = c \star_\Lambda (\xi_1 \otimes \xi_2)$.

Remark C.3. In this sense, operators of the form $c \star_\Lambda S$ are a generalization of Gabor multipliers. We mention that this is a different generalization from the *multiple Gabor multipliers* introduced in [89].

If we pick another rank-one operator $\check{\varphi}_1 \otimes \check{\varphi}_2$ for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ (here $\check{\varphi}(t) = \varphi(-t)$), one can calculate using the definition (C.4.2) that

$$(\xi_1 \otimes \xi_2) \star_\Lambda (\check{\varphi}_1 \otimes \check{\varphi}_2)(\lambda) = V_{\varphi_2}\xi_1(\lambda) \overline{V_{\varphi_1}\xi_2(\lambda)}. \tag{C.4.4}$$

In particular, if $\varphi_1 = \varphi_2 = \varphi$ and $\xi_1 = \xi_2 = \xi$, then

$$(\xi \otimes \xi) \star_\Lambda (\check{\varphi} \otimes \check{\varphi})(\lambda) = |V_\varphi \xi(\lambda)|^2. \tag{C.4.5}$$

The function $|V_\varphi \xi(z)|^2$ is the so-called spectrogram of ξ with window φ, hence $(\xi \otimes \xi) \star_\Lambda (\check{\varphi} \otimes \check{\varphi})$ consists of samples of the spectrogram over Λ.

Finally, if $S \in \mathcal{S}^1$ is any operator, then one may calculate that

$$S \star_\Lambda \check{\varphi}_1 \otimes \check{\varphi}_2(\lambda) = \langle S\pi(\lambda)\varphi_1, \pi(\lambda)\varphi_2 \rangle_{L^2}, \tag{C.4.6}$$

often called the lower symbol of S with respect to φ_1, φ_2 and Λ [97].

Remark C.4. In particular, Proposition C.4.1 does not hold for all $S \in \mathcal{S}^1$. By Remark 4.6 in [32], there exists a function $\psi \in L^2(\mathbb{R})$ such that

$$\sum_{(m,n)\in\mathbb{Z}^2} (\psi \otimes \psi) \star_{\mathbb{Z}^2} (\check{\psi} \otimes \check{\psi})(m,n) = \sum_{(m,n)\in\mathbb{Z}^2} |V_\psi \psi(m,n)|^2 = \infty.$$

Since $\psi \otimes \psi, \check{\psi} \otimes \check{\psi} \in \mathcal{S}^1$, this shows that the assumption $S \in \mathcal{B}$ in Proposition C.4.1 is necessary.

C.4.2 Associativity and commutativity of convolutions

Since the convolution $S \star T$ of two operators $S, T \in \mathcal{S}^1$ is commutative in the continuous setting [251, Prop. 3.2], it follows from the definitions that the convolutions (C.4.1) and (C.4.2) are commutative. It is also a straightforward consequence of the definitions that the convolutions are bilinear.

In the original theory of Werner [251], the associativity of the convolution operations is of fundamental importance. Associativity still holds in some cases when moving from $\mathbb{R}^{2d}$ to Λ, but we will later see in Corollary C.7.2.2 that the convolution of three operators over a lattice is not associative in general. In what follows, $c *_\Lambda d$ denotes the usual convolution of sequences

$$c *_\Lambda d(\lambda) = \sum_{\lambda' \in \Lambda} c(\lambda')d(\lambda - \lambda').$$

Proposition C.4.2 (Associativity)**.** *Let $c, d \in \ell^1(\Lambda)$, $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$. Then*

(i) $c *_\Lambda (S \star_\Lambda T) = (c \star_\Lambda S) \star_\Lambda T$,

(ii) $(c *_\Lambda d) \star_\Lambda T = c \star_\Lambda (d \star_\Lambda T)$.

Proof. For the proof of (i), we write out the definitions of the convolutions and use the commutativity $S \star_\Lambda T = T \star_\Lambda S$ to get

$$\begin{aligned}
c *_\Lambda (S \star_\Lambda T)(\lambda) &= c *_\Lambda (T \star_\Lambda S)(\lambda) \\
&= \sum_{\lambda' \in \Lambda} c(\lambda') \mathrm{tr}(T \alpha_{\lambda - \lambda'}(\check{S})) \\
&= \mathrm{tr}\left(T \sum_{\lambda' \in \Lambda} c(\lambda') \alpha_{\lambda - \lambda'}(\check{S}) \right) \\
&= \mathrm{tr}\left(T \alpha_\lambda \left(\sum_{\lambda' \in \Lambda} c(\lambda') \alpha_{-\lambda'}(\check{S}) \right) \right) \quad \text{by Lemma C.3.3} \\
&= \mathrm{tr}\left(T \alpha_\lambda \left(P \sum_{\lambda' \in \Lambda} c(\lambda') \alpha_{\lambda'}(S) P \right) \right) \\
&= T \star_\Lambda (c \star_\Lambda S) \quad \text{by (C.4.1) and (C.4.2)} \\
&= (c \star_\Lambda S) \star_\Lambda T \quad \text{by commutativity.}
\end{aligned}$$

We have used the easily checked relation $\alpha_{-\lambda'}(\check{S}) = P \alpha_{\lambda'}(S) P$. For the second part, we find that

$$\begin{aligned}
(c *_\Lambda d) \star_\Lambda T &= \sum_{\lambda \in \Lambda} (c *_\Lambda d)(\lambda) \alpha_\lambda(T) \\
&= \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} c(\lambda') d(\lambda - \lambda') \alpha_\lambda(T) \\
&= \sum_{\lambda' \in \Lambda} c(\lambda') \sum_{\lambda \in \Lambda} d(\lambda - \lambda') \alpha_\lambda(T) \\
&= \sum_{\lambda' \in \Lambda} c(\lambda') \alpha_{\lambda'}(d \star_\Lambda T) = c \star_\Lambda (d \star_\Lambda T).
\end{aligned}$$

To pass to the last line we have used the relation $\alpha_{\lambda'}(d \star_\Lambda T) = \sum_\lambda d(\lambda - \lambda') \alpha_\lambda(T)$, which is easily verified. □

Remark C.5. Part (*ii*) of this result along with the trivial estimate $\|c \star_\Lambda T\|_{\mathcal{S}^1} \leq \|c\|_{\ell^1} \|T\|_{\mathcal{S}^1}$ shows that $\mathcal{S}^1$ is a *Banach module* (see [128]) over $\ell^1(\Lambda)$ if we define the action of $c \in \ell^1(\Lambda)$ on $T \in \mathcal{S}^1$ by $c \star_\Lambda T$. The same proofs also show that this is true when $\mathcal{S}^1$ is replaced by $\mathcal{B}$ or any Schatten class $\mathcal{S}^p$ for $1 \leq p \leq \infty$.

Example C.4.1. Let $\varphi, \xi \in L^2(\mathbb{R}^d)$ and $c \in \ell^1(\Lambda)$, and define $S = \xi \otimes \xi$ and $T = \check{\varphi} \otimes \check{\varphi}$. If we use (C.4.5) to simplify $S \star_\Lambda T$ and (C.4.6) to simplify $(c \star_\Lambda S) \star_\Lambda T$, the first part of the result above becomes

$$c *_\Lambda |V_\varphi \xi|^2(\lambda) = \langle (c \star_\Lambda \xi \otimes \xi) \pi(\lambda) \varphi, \pi(\lambda) \varphi \rangle_{L^2}. \tag{C.4.7}$$

In words, the convolution of a sequence c with samples of a spectrogram $|V_\varphi \xi|^2$ can be described using the action of a Gabor multiplier $c \star (\xi \otimes \xi)$. In applications of convolutional neural networks to audio processing, one often considers the spectrogram of an audio signal as the input to the network. Convolutions of sequences with samples of spectrograms therefore appear naturally in such networks, and the connection (C.4.7) has been exploited in this context – see the proof of [86, Thm. 1].

C.4.3 Young's inequality

The convolutions in (C.4.1) and (C.4.2) can be defined for more general sequences and operators by establishing a version of Young's inequality [131, Thm. 1.2.1]. In the continuous case such an inequality was established by Werner [251] using the L^p-norms of functions and Schatten-p-norms of operators. In the discrete case, it is not always possible to use the Schatten-p-norms, since Proposition C.4.1 requires $S \in \mathcal{B}$. We will therefore always require that one of the operators belongs to $\mathcal{B}$.

A Young's inequality for Schatten classes can then be established by first extending the domains of the convolutions by duality. If $S \in \mathcal{B}$ and $c \in \ell^\infty(\Lambda)$, we define $c \star_\Lambda S \in \mathcal{L}(L^2)$ by

$$\langle c \star_\Lambda S, R \rangle_{\mathcal{L}(L^2),\mathcal{S}^1} := \left\langle c, R \star_\Lambda \check{S}^* \right\rangle_{\ell^\infty,\ell^1} \text{ for any } R \in \mathcal{S}^1. \tag{C.4.8}$$

and if $S \in \mathcal{B}$ and $T \in \mathcal{L}(L^2) = \mathcal{S}^\infty$ we define $T \star_\Lambda S \in \ell^\infty(\Lambda)$ by

$$\langle T \star_\Lambda S, c \rangle_{\ell^\infty(\Lambda),\ell^1(\Lambda)} := \left\langle T, c \star_\Lambda \check{S}^* \right\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} \text{ for any } c \in \ell^1(\Lambda). \tag{C.4.9}$$

It is a simple exercise to show that these definitions define elements of $\mathcal{L}(L^2)$ and $\ell^\infty(\Lambda)$ satisfying $\|c \star_\Lambda S\|_{\mathcal{L}(L^2)} \lesssim \|c\|_{\ell^\infty} \|S\|_{\mathcal{B}}$ and $\|T \star_\Lambda S\|_{\ell^\infty} \leq \|T\|_{\mathcal{L}(L^2)} \|S\|_{\mathcal{B}}$, and that they agree with (C.4.1) and (C.4.2) when $c \in \ell^1(\Lambda)$ or $T \in \mathcal{S}^1$. A standard (complex) interpolation argument then gives the following result, since $(\ell^1(\Lambda), \ell^\infty(\Lambda))_\theta = \ell^p(\Lambda)$ and $(\mathcal{S}^1, \mathcal{S}^\infty)_\theta = \mathcal{S}^p$ with $\frac{1}{p} = 1 - \theta$ [43]. For Gabor multipliers the second part of this result is well-known [105, Thm. 5.4.1], and a weaker version of the first part is known for $p = 1, 2, \infty$ [105, Thm. 5.8.3].

Proposition C.4.3 (Young's inequality)**.** *Let $S \in \mathcal{B}$ and $1 \leq p \leq \infty$.*

(i) If $T \in \mathcal{S}^p$, then $\|T \star_\Lambda S\|_{\ell^p} \lesssim \|T\|_{\mathcal{S}^p} \|S\|_{\mathcal{B}}$.

(ii) If $c \in \ell^p(\Lambda)$, then $\|c \star_\Lambda S\|_{\mathcal{S}^p} \lesssim \|c\|_{\ell^p} \|S\|_{\mathcal{B}}$.

Remark C.6. If $1 \in \ell^\infty(\Lambda)$ is given by $1(\lambda) = 1$ for any λ, then Feichtinger observed in [97, Thm. 5.15] that $\phi \in S_0(\mathbb{R}^d)$ generates a so-called tight Gabor frame if and only if the Gabor multiplier $1 \star_\Lambda (\phi \otimes \phi)$ is the identity operator I_{L^2} in $\mathcal{L}(L^2)$. A similar result holds in the more general case: if $S \in \mathcal{B}$, then $1 \star_\Lambda S^*S = I_{L^2}$ if and only if S generates a tight *Gabor g-frame*, recently introduced in [236].

We may also use duality to define the convolution $T \star_\Lambda S \in \ell^\infty(\Lambda)$ of $S \in \mathcal{B}$ with $T \in \mathcal{B}'$ by

$$\langle T \star_\Lambda S, c\rangle_{\ell^\infty,\ell^1} := \left\langle T, c \star_\Lambda \check{S}^*\right\rangle_{\mathcal{B}',\mathcal{B}} \text{ for any } c \in \ell^1(\Lambda), \tag{C.4.10}$$

which agrees with (C.4.9) when $T \in \mathcal{L}(L^2) \subset \mathcal{B}'$ and satisfies $\|S \star_\Lambda T\|_{\ell^\infty} \leq \|S\|_{\mathcal{B}}\|T\|_{\mathcal{B}'}$. We end this section by showing that the space $c_0(\Lambda)$ of sequences vanishing at infinity corresponds to compact operators under convolutions with $S \in \mathcal{B}$. The second part of this statement is due to Feichtinger [97, Thm. 5.15] for the special case of Gabor multipliers.

Proposition C.4.4. *Let $S \in \mathcal{B}$. If T is a compact operator, then $T \star_\Lambda S \in c_0(\Lambda)$. If $c \in c_0(\Lambda)$, then $c \star_\Lambda S$ is a compact operator on $L^2(\mathbb{R}^d)$.*

Proof. By [203, Prop. 4.6], the *function* $T \star S$ belongs to the space $C_0(\mathbb{R}^{2d})$ of continuous functions vanishing at infinity. Since $T \star_\Lambda S$ is simply the restriction of $T \star S$ to Λ, it follows that $T \star_\Lambda S \in c_0(\Lambda)$. For the second part, let c_N be the sequence

$$c_N(\lambda) = \begin{cases} c(\lambda) \text{ if } |\lambda| < N \\ 0 \text{ otherwise.} \end{cases}$$

Then $c_N \star_\Lambda S = \sum_{|\lambda|<N} c(\lambda)\alpha_\lambda(S)$ is a compact operator for each $N \in \mathbb{N}$, and by Proposition C.4.3 and the bilinearity of convolutions

$$\|c \star_\Lambda S - c_N \star_\Lambda S\|_{\mathcal{L}(L^2)} \leq \|c - c_N\|_{\ell^\infty}\|S\|_{\mathcal{B}} \to 0 \text{ as } N \to \infty.$$

Hence $c \star_\Lambda S$ is the limit in the operator topology of compact operators, and is therefore itself compact. □

C.5 Fourier transforms

In [251], Werner observed that if one defines a Fourier transform of an operator $S \in \mathcal{S}^1$ to be the function

$$\mathcal{F}_W(S)(z) := e^{-\pi i x\cdot\omega}\mathrm{tr}(\pi(-z)S) \quad \text{for } z = (x,\omega) \in \mathbb{R}^{2d},$$

then the formulas

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S), \qquad \mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T) \tag{C.5.1}$$

hold for $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{S}^1$. The transform $\mathcal{F}_W$, called the *Fourier-Wigner transform* (or the Fourier-Weyl transform [251]) is an isomorphism $\mathcal{F}_W : \mathcal{B} \to S_0(\mathbb{R}^{2d})$, can be extended to a unitary map $\mathcal{F}_W : \mathcal{HS} \to L^2(\mathbb{R}^{2d})$, and to an isomorphism $\mathcal{F}_W : \mathcal{B}' \to S_0'(\mathbb{R}^{2d})$ by defining $\mathcal{F}_W(S)$ for $S \in \mathcal{B}'$ by duality [102, Cor. 7.6.3]:

$$\langle F_W(S), f\rangle_{S_0',S_0} := \langle S, \rho(f)\rangle_{\mathcal{B}',\mathcal{B}} \quad \text{for any } f \in S_0(\mathbb{R}^{2d}). \tag{C.5.2}$$

Here $\rho : S_0(\mathbb{R}^{2d}) \to \mathcal{B}$ is the inverse of $\mathcal{F}_W$. In fact, $\mathcal{F}_W$ and the Weyl transform are related by a symplectic Fourier transform: for any $S \in \mathcal{B}'$ we have

$$\mathcal{F}_W(S) = \mathcal{F}_\sigma(a_S),$$

where a_S is the Weyl symbol of S. As an important special case, the Fourier-Wigner transform of a rank-one operator $\phi \otimes \psi$ is

$$\mathcal{F}_W(\phi \otimes \psi)(x,\omega) = e^{\pi i x\cdot\omega} V_\psi \phi(x,\omega). \tag{C.5.3}$$

Since we have defined convolutions of operators and sequences, it is natural to ask whether a version of (C.5.1) holds in our setting. We start by defining a suitable Fourier transform of sequences.

Symplectic Fourier series

For the purposes of this paper, we identify the dual group $\widehat{\mathbb{R}^{2d}}$ with $\mathbb{R}^{2d}$ by the bijection $\mathbb{R}^{2d} \ni z \mapsto \chi_z \in \widehat{\mathbb{R}^{2d}}$, where χ_z is the *symplectic* character[1] $\chi_z(z') = e^{2\pi i \sigma(z,z')}$. Given a lattice $\Lambda \subset \mathbb{R}^{2d}$, it follows that the dual group of Λ is identified with $\mathbb{R}^{2d}/\Lambda^\circ$ (see [81, Prop. 3.6.1]), where Λ° is the annihilator group

$$\begin{aligned}\Lambda^\circ &= \{\lambda^\circ \in \mathbb{R}^{2d} : \chi_{\lambda^\circ}(\lambda) = 1 \text{ for any } \lambda \in \Lambda\} \\ &= \{\lambda^\circ \in \mathbb{R}^{2d} : e^{2\pi i \sigma(\lambda^\circ,\lambda)} = 1 \text{ for any } \lambda \in \Lambda\}.\end{aligned}$$

The group Λ° is itself a lattice, namely the so-called *adjoint lattice* of Λ from [102, 223]. Given this identification of the dual group of Λ, the Fourier transform of $c \in \ell^1(\Lambda)$ is the symplectic Fourier series

$$\mathcal{F}_\sigma^\Lambda(c)(\dot{z}) := \sum_{\lambda\in\Lambda} c(\lambda) e^{2\pi i \sigma(\lambda,z)}.$$

Here $\dot{z}$ denotes the image of $z \in \mathbb{R}^{2d}$ under the natural quotient map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda^\circ$, so $\mathcal{F}_\sigma^\Lambda(c)$ is a function on $\mathbb{R}^{2d}/\Lambda^\circ$. If we denote by $A(\mathbb{R}^{2d}/\Lambda^\circ)$ the Banach space of functions on $\mathbb{R}^{2d}/\Lambda^\circ$ with symplectic Fourier coefficients in $\ell^1(\Lambda)$, the Feichtinger algebra has the following property [95, Thm. 7 B)].

[1]Phase space, which in this paper is $\mathbb{R}^{2d}$, is more properly described by (the isomorphic) space $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. The symplectic characters appear because they are the natural way of identifying the group $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ with its dual group.

Lemma C.5.1. *If Λ is a lattice, the periodization operator $P_\Lambda : S_0(\mathbb{R}^{2d}) \to A(\mathbb{R}^{2d}/\Lambda)$ defined by*

$$P_\Lambda(f)(\dot{z}) = |\Lambda| \sum_{\lambda \in \Lambda} f(z+\lambda) \quad \text{for } z \in \mathbb{R}^{2d}$$

is continuous and surjective.

Remark C.7. (i) Since $|\Lambda^\circ| = \frac{1}{|\Lambda|}$ [102, Lem. 7.7.4], we have

$$P_{\Lambda^\circ}(f)(\dot{z}) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda} f(z+\lambda^\circ).$$

(ii) One may define Feichtinger's algebra $S_0(G)$ for any locally compact abelian group G [95]. In fact, all our function spaces besides $L^2(\mathbb{R}^d)$ are examples of Feichtinger's algebra, since $S_0(\Lambda) = \ell^1(\Lambda)$ and $S_0(\mathbb{R}^{2d}/\Lambda^\circ) = A(\mathbb{R}^{2d}/\Lambda^\circ)$.

When we identify the dual group of Λ with $\mathbb{R}^{2d}/\Lambda^\circ$, the Poisson summation formula for functions in $S_0(\mathbb{R}^{2d})$ takes the following form.

Theorem C.5.2 (Poisson summation)**.** *Let Λ be a lattice in $\mathbb{R}^{2d}$ and assume that $f \in S_0(\mathbb{R}^{2d})$. Then*

$$\frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} f(z+\lambda^\circ) = \sum_{\lambda \in \Lambda} \mathcal{F}_\sigma(f)(\lambda) e^{2\pi i \sigma(\lambda, z)} \text{ for } z \in \mathbb{R}^{2d}.$$

Proof. This is [81, Thm. 3.6.3] with $A = \mathbb{R}^{2d}$, $B = \Lambda^\circ$ and using $(\Lambda^\circ)^\circ = \Lambda$. To get equality for any $z \in \mathbb{R}^{2d}$, we use that $\sum_{\lambda^\circ \in \Lambda^\circ} f(z+\lambda^\circ)$ defines a continuous function on $\mathbb{R}^{2d}/\Lambda^\circ$ by Lemma C.5.1. □

Since $\mathcal{F}_\sigma^\Lambda$ is a Fourier transform it extends to a unitary mapping $\mathcal{F}_\sigma^\Lambda : \ell^2(\Lambda) \to L^2(\mathbb{R}^{2d}/\Lambda^\circ)$ satisfying

$$\mathcal{F}_\sigma^\Lambda(c *_\Lambda d) = \mathcal{F}_\sigma^\Lambda(c)\mathcal{F}_\sigma^\Lambda(d) \tag{C.5.4}$$

for $c \in \ell^1(\Lambda)$ and $d \in \ell^2(\Lambda)$.

C.5.1 The Fourier transform of $S \star_\Lambda T$

We now consider a version of (C.5.1) for sequences. The formula for $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda T)$ is a simple consequence of the Poisson summation formula.

Theorem C.5.3. *Let $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$. Then*

$$\begin{aligned}\mathcal{F}_\sigma^\Lambda(S \star_\Lambda T)(\dot{z}) &= \frac{1}{|\Lambda|}\sum_{\lambda^\circ \in \Lambda^\circ} F_W(S)(z+\lambda^\circ)\mathcal{F}_W(T)(z+\lambda^\circ) \\ &= P_{\Lambda^\circ}(\mathcal{F}_W(S)\mathcal{F}_W(T))(\dot{z})\end{aligned}$$

for any $z \in \mathbb{R}^{2d}$.

Proof. From [203, Thm. 8.2], we know that $S \star T \in S_0(\mathbb{R}^{2d})$. Hence $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T) \in S_0(\mathbb{R}^{2d})$ since $\mathcal{F}_\sigma : S_0(\mathbb{R}^{2d}) \to S_0(\mathbb{R}^{2d})$ is an isomorphism. By applying Poisson's summation formula from Theorem C.5.2 to $f = \mathcal{F}_W(S)\mathcal{F}_W(T)$, we find that

$$\begin{aligned}\frac{1}{|\Lambda|}\sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_W(S)(z+\lambda^\circ)\mathcal{F}_W(T)(z+\lambda^\circ) &= \sum_{\lambda \in \Lambda} \mathcal{F}_\sigma(\mathcal{F}_W(S)\mathcal{F}_W(T))(\lambda)e^{2\pi i \sigma(\lambda,z)} \\ &= \sum_{\lambda \in \Lambda} S \star_\Lambda T(\lambda)e^{2\pi i \sigma(\lambda,z)},\end{aligned}$$

where we used that $\mathcal{F}_\sigma$ is its own inverse to conclude that

$$\mathcal{F}_\sigma(\mathcal{F}_W(S)\mathcal{F}_W(T))(\lambda) = \mathcal{F}_\sigma(\mathcal{F}_\sigma(S \star T))(\lambda) = S \star T(\lambda) = S \star_\Lambda T(\lambda).$$

Since $\mathcal{F}_W(S)\mathcal{F}_W(T) \in S_0(\mathbb{R}^{2d})$, Theorem C.5.2 says that the equation holds for any $z \in \mathbb{R}^{2d}$. □

Remark C.8. Theorem C.5.3 has also been proved and used in [196, Cor. A.3] in noncommutative geometry, with stronger assumptions on S, T.

Theorem C.5.3 has many interesting special cases. We will frequently refer to the following version, which follows since a short calculation using the definition of the Fourier-Wigner transform shows that

$$\mathcal{F}_W(\check{S}^*)(z) = \overline{\mathcal{F}_W(S)(z)}. \tag{C.5.5}$$

Corollary C.5.3.1. *Let $S \in \mathcal{B}$. Then*

$$\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) = \frac{1}{|\Lambda|}\sum_{\lambda^\circ \in \Lambda^\circ} |\mathcal{F}_W(S)(z+\lambda^\circ)|^2 \quad \text{for any } z \in \mathbb{R}^{2d}.$$

Corollary C.5.3.2. *Let $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$. Then*

$$\sum_{\lambda \in \Lambda} S \star_\Lambda T(\lambda) = \frac{1}{|\Lambda|}\sum_{\lambda^\circ \in \Lambda^\circ} F_W(S)(\lambda^\circ)F_W(T)(\lambda^\circ).$$

Proof. This follows from Theorem C.5.3 with $z = 0$. □

Now assume that S and T are rank-one operators: $S = \xi_1 \otimes \xi_2$ for $\xi_1, \xi_2 \in S_0(\mathbb{R}^d)$ and $T = \check{\varphi}_1 \otimes \check{\varphi}_2$ for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. By (C.4.4)

$$S \star_\Lambda T(\lambda) = V_{\varphi_2}\xi_1(\lambda)\overline{V_{\varphi_1}\xi_2(\lambda)},$$

and noting that $T = \check{T_0}^*$ for $T_0 = \varphi_2 \otimes \varphi_1$, we can use (C.5.3) and (C.5.5) to find

$$\mathcal{F}_W(S)(z) = e^{\pi i x\cdot\omega} V_{\xi_2}\xi_1(z)$$
$$\mathcal{F}_W(T)(z) = e^{-\pi i x\cdot\omega} \overline{V_{\varphi_1}\varphi_2(z)}$$

Hence Theorem C.5.3 says that

$$\mathcal{F}_\sigma^\Lambda(V_{\varphi_2}\xi_1\overline{V_{\varphi_1}\xi_2}|_\Lambda)(\dot{z}) = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} V_{\xi_2}\xi_1(z+\lambda^\circ)\overline{V_{\varphi_1}\varphi_2(z+\lambda^\circ)}.$$

Furthermore, Corollary C.5.3.2 gives

$$\sum_{\lambda\in\Lambda} V_{\varphi_2}\xi_1(\lambda)\overline{V_{\varphi_1}\xi_2(\lambda)} = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} V_{\xi_2}\xi_1(\lambda^\circ)\overline{V_{\varphi_1}\varphi_2(\lambda^\circ)},$$

which is the *fundamental identity of Gabor analysis* [103, 170, 223, 245].

C.5.2 The Fourier transform of $c \star_\Lambda S$

When $c \in \ell^1(\Lambda)$, we obtain the expected formula for $\mathcal{F}_W(c \star_\Lambda S)$.

Proposition C.5.4. *If $c \in \ell^1(\Lambda)$ and $S \in \mathcal{S}^1$, then*

$$\mathcal{F}_W(c \star_\Lambda S)(z) = \mathcal{F}_\sigma^\Lambda(c)(\dot{z})\mathcal{F}_W(S)(z) \quad \textit{for } z \in \mathbb{R}^{2d}.$$

Proof. One easily verifies the formula

$$\mathcal{F}_W(\alpha_\lambda(S))(z) = e^{2\pi i\sigma(\lambda,z)}\mathcal{F}_W(S)(z),$$

showing that the Fourier transform of a translation is a modulation. Hence

$$\begin{aligned}\mathcal{F}_W(c \star_\Lambda S)(z) &= \sum_{\lambda\in\Lambda} c(\lambda)\mathcal{F}_W(\alpha_\lambda(S))\\ &= \sum_{\lambda\in\Lambda} c(\lambda)e^{2\pi i\sigma(\lambda,z)}\mathcal{F}_W(S)(z)\\ &= \mathcal{F}_W(S)(z)\sum_{\lambda\in\Lambda} c(\lambda)e^{2\pi i\sigma(\lambda,z)}.\end{aligned}$$

To move $\mathcal{F}_W$ inside the sum, we use that the sum $\sum_{\lambda\in\Lambda} c(\lambda)\alpha_\lambda(S)$ converges absolutely in $\mathcal{S}^1$, and $\mathcal{F}_W$ is continuous from $\mathcal{S}^1$ to $L^\infty(\mathbb{R}^{2d})$ by the Riemann-Lebesgue lemma for $\mathcal{F}_W$ [203, Prop. 6.6]. □

Technical intermezzo

Let $A'(\mathbb{R}^{2d}/\Lambda^\circ)$ denote the dual space of $A(\mathbb{R}^{2d}/\Lambda^\circ)$, consisting of distributions on $\mathbb{R}^{2d}/\Lambda^\circ$ with symplectic Fourier coefficients in $\ell^\infty(\Lambda)$. To understand the statement in Proposition C.5.4 when $c \in \ell^\infty(\Lambda)$, we need to 'extend' distributions in $A'(\mathbb{R}^{2d}/\Lambda^\circ)$ to distributions in $S_0'(\mathbb{R}^{2d})$. When $f \in A(\mathbb{R}^{2d}/\Lambda^\circ)$ this is achieved by

$$A(\mathbb{R}^{2d}/\Lambda^\circ) \ni f \mapsto f \circ q \in S_0'(\mathbb{R}^{2d}),$$

where $q : \mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda^\circ$ is the natural quotient map. To extend this map to distributions $f \in A'(\mathbb{R}^{2d}/\Lambda^\circ)$, one can use Weil's formula [130, (6.2.11)] to show that for $f \in A(\mathbb{R}^{2d}/\Lambda^\circ)$ and $g \in S_0(\mathbb{R}^{2d})$ one has

$$\langle f \circ q, g\rangle_{S_0',S_0} = \langle f, P_{\Lambda^\circ} g\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d},\Lambda^\circ)} .$$

This shows that the map $f \mapsto f \circ q$ agrees with the Banach space adjoint $P_{\Lambda^\circ}^* : A'(\mathbb{R}^{2d}/\Lambda^\circ) \to S_0'(\mathbb{R}^{2d})$ for $f \in A(\mathbb{R}^{2d}/\Lambda^\circ)$. The natural way to extend $f \in A'(\mathbb{R}^{2d}/\Lambda^\circ)$ is therefore to consider $P_{\Lambda^\circ}^* f \in S_0'(\mathbb{R}^{2d})$, and by an abuse of notation we will use f to also denote the extension $P_{\Lambda^\circ}^* f$ – by definition this means that when $f \in A'(\mathbb{R}^{2d}/\Lambda^\circ)$ is considered an element of $S_0'(\mathbb{R}^{2d})$, it satisfies for $g \in S_0(\mathbb{R}^{2d})$

$$\langle f, g\rangle_{S_0',S_0} = \langle f, P_{\Lambda^\circ} g\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d},\Lambda^\circ)} . \tag{C.5.6}$$

We also remind the reader that for $c \in \ell^\infty(\Lambda)$ one defines $\mathcal{F}_\sigma^\Lambda(c)$ as an element of $A'(\mathbb{R}^{2d}/\Lambda^\circ)$ by

$$\left\langle \mathcal{F}_\sigma^\Lambda(c), g\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d}/\Lambda^\circ)} := \left\langle c, (\mathcal{F}_\sigma^\Lambda)^{-1}(g)\right\rangle_{\ell^\infty(\Lambda),\ell^1(\Lambda)}, \tag{C.5.7}$$

where $(\mathcal{F}_\sigma^\Lambda)^{-1}(g)$ are the symplectic Fourier coefficients of g. This is [167, Example 6.8] for the group $G = \mathbb{R}^{2d}/\Lambda^\circ$. Finally, recall that we can multiply $f \in S_0'(\mathbb{R}^{2d})$ with $g \in S_0(\mathbb{R}^{2d})$ to obtain an element $fg \in S_0'(\mathbb{R}^{2d})$ given by

$$\langle fg, h\rangle_{S_0',S_0} := \langle f, \overline{g}h\rangle_{S_0',S_0} \quad \text{for } h \in S_0(\mathbb{R}^{2d}). \tag{C.5.8}$$

The case $c \in \ell^\infty(\Lambda)$

The technical intermezzo allows us to make sense of the following generalization of Proposition C.5.4. Recall in particular that $\mathcal{F}_\sigma^\Lambda(c)$ is shorthand for the distribution $P_{\Lambda^\circ}^*(\mathcal{F}_\sigma^\Lambda(c)) \in S_0'(\mathbb{R}^{2d})$.

Proposition C.5.5. *If $c \in \ell^\infty(\Lambda)$ and $S \in \mathcal{B}$, then*

$$\mathcal{F}_W(c \star_\Lambda S) = \mathcal{F}_\sigma^\Lambda(c)\mathcal{F}_W(S) \quad \text{in } S_0'(\mathbb{R}^{2d}).$$

Proof. For $h \in S_0(\mathbb{R}^{2d})$, we get from (C.5.2), (C.4.8) and (C.5.7) (in that order)

$$\begin{aligned}\langle \mathcal{F}_W(c \star_\Lambda S), h\rangle_{S_0',S_0} &= \langle c \star_\Lambda S, \rho(h)\rangle_{\mathcal{B}',\mathcal{B}} \\ &= \left\langle c, \rho(h) \star_\Lambda \check{S}^*\right\rangle_{\ell^\infty(\Lambda),\ell^1(\Lambda)} \\ &= \left\langle \mathcal{F}_\sigma^\Lambda(c), \mathcal{F}_\sigma^\Lambda(\rho(h) \star_\Lambda \check{S}^*)\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d}/\Lambda^\circ)}.\end{aligned}$$

By Theorem C.5.3 we find using (C.5.5) that

$$\mathcal{F}_\sigma^\Lambda(\rho(h) \star_\Lambda \check{S}^*) = P_{\Lambda^\circ}(\overline{\mathcal{F}_W(S)}h),$$

where we also used that ρ is the inverse of $\mathcal{F}_W$. On the other hand we find using (C.5.8) and (C.5.6) that

$$\begin{aligned}\left\langle \mathcal{F}_\sigma^\Lambda(c)\mathcal{F}_W(S), h\right\rangle_{S_0',S_0} &= \left\langle \mathcal{F}_\sigma^\Lambda(c), \overline{\mathcal{F}_W(S)}h\right\rangle_{S_0',S_0} \\ &= \left\langle \mathcal{F}_\sigma^\Lambda(c), P_{\Lambda^\circ}(\overline{\mathcal{F}_W(S)}h)\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d}/\Lambda^\circ)}\end{aligned}$$

Hence $\left\langle \mathcal{F}_\sigma^\Lambda(c)\mathcal{F}_W(S), h\right\rangle_{S_0',S_0} = \langle \mathcal{F}_W(c \star_\Lambda S), h\rangle_{S_0',S_0}$, which implies the statement. □

Remark C.9. For Gabor multipliers $c \star_\Lambda (\psi \otimes \psi)$, Propositions C.5.4 and C.5.5 were proved in [89, Lem. 14], and have been used in the theory of convolutional neural networks [86].

C.6 Riesz sequences of translated operators in $\mathcal{HS}$

Two of the useful properties of the Weyl transform $f \mapsto L_f$ are that it is a unitary transformation from $L^2(\mathbb{R}^{2d})$ to the Hilbert-Schmidt operators $\mathcal{HS}$, and that it respects translations in the sense that

$$L_{T_z f} = \alpha_z(L_f) \quad \text{for } f \in L^2(\mathbb{R}^{2d}), z \in \mathbb{R}^{2d}.$$

As a consequence, statements concerning translates of functions in $L^2(\mathbb{R}^{2d})$ can be lifted to statements about translates of operators and convolutions $\star_\Lambda$ in $\mathcal{HS}$. This approach was first used for Gabor multipliers in [97, 105], and has later been explored in other works [32, 89] – we include these results for completeness, and because the proofs and results find natural formulations and generalizations in the framework of this paper.

For fixed $S \in \mathcal{HS}$ and lattice Λ, we will be interested in whether $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a *Riesz sequence in* $\mathcal{HS}$, i.e. whether there exist $A, B > 0$ such that for all finite sequences $c \in \ell^2(\Lambda)$

$$A\|c\|_{\ell^2(\Lambda)}^2 \leq \left\| \sum_{\lambda\in\Lambda} c(\lambda)\alpha_\lambda(S) \right\|_{\mathcal{HS}}^2 \leq B\|c\|_{\ell^2(\Lambda)}^2. \tag{C.6.1}$$

Since the Weyl transform is unitary and preserves translations, if we let a_S be the Weyl symbol of S, then (C.6.1) is clearly equivalent to the fact that $\{T_\lambda(a_S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$, meaning that

$$A\|c\|_{\ell^2(\Lambda)}^2 \leq \left\| \sum_{\lambda\in\Lambda} c(\lambda)T_\lambda(a_S) \right\|_{L^2(\mathbb{R}^{2d})}^2 \leq B\|c\|_{\ell^2(\Lambda)}^2,$$

for finite $c \in \ell^2(\Lambda)$. Following [32, 89, 97, 105] we can use a result from [31] to give a characterization of when (C.6.1) holds in terms of an expression familiar from Corollary C.5.3.1.

Theorem C.6.1. *Let Λ be a lattice and $S \in \mathcal{B}$. Then the following are equivalent.*

(i) The function
$$\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*) = P_{\Lambda^\circ}(|\mathcal{F}_W(S)|^2)$$
has no zeros in $\mathbb{R}^{2d}/\Lambda^\circ$.

(ii) $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$.

Proof. The equality in (i) is Corollary C.5.3.1. By the preceding discussion, $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$ if and only if $\{T_\lambda(a_S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$. The result from [31] (see [32] for a statement for general lattices and symplectic Fourier transform) says that $\{T_\lambda(a_S)\}_{\lambda\in\Lambda}$ is a Riesz sequence if and only if there exist $A, B > 0$ such that

$$A \leq \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{F}_\sigma(a_S)(z+\lambda^\circ)|^2 \leq B \text{ for any } z \in \mathbb{R}^{2d}.$$

Since the Weyl transform and Fourier-Wigner transform are related by $\mathcal{F}_\sigma(a_S) = \mathcal{F}_W(S)$, we we may restate this condition as

$$A \leq \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{F}_W(S)(z+\lambda^\circ)|^2 \leq B \text{ for any } z \in \mathbb{R}^{2d}. \tag{C.6.2}$$

Note that the middle term is $P_{\Lambda^\circ}(|\mathcal{F}_W(S)|^2)(\dot{z})$, and since $S \in \mathcal{B}$ we know that $|\mathcal{F}_W(S)|^2 \in S_0(\mathbb{R}^{2d})$. Therefore $P_{\Lambda^\circ}(|\mathcal{F}_W(S)|^2) \in A(\mathbb{R}^{2d}/\Lambda^\circ)$ by Lemma C.5.1, which in particular means that $P_{\Lambda^\circ}(|\mathcal{F}_W(S)|^2)$ is a continuous function on the compact space $\mathbb{R}^{2d}/\Lambda^\circ$. For a continuous function on a compact space, condition (C.6.2) is equivalent to having no zeros. This completes the proof. □

Remark C.10. (i) Since we assume $S \in \mathcal{B}$, the first condition above is in fact equivalent to $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ generating a *frame sequence* in $\mathcal{HS}$, which is a weaker statement than (2) above. The proof of this in [32] for Gabor multipliers works in our more general setting.

(ii) As mentioned in the introduction, Feichtinger [97] used the Kohn-Nirenberg symbol rather than the Weyl symbol. This makes no difference for our purposes – we have opted for the Weyl symbol as it is related to $\mathcal{F}_W$ by a symplectic Fourier transform.

If $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$, the *synthesis operator* is the map $D_S : \ell^2(\Lambda) \to \mathcal{HS}$ given by

$$D_S(c) = c \star_\Lambda S = \sum_{\lambda\in\Lambda} c(\lambda)\alpha_\lambda(S),$$

and the sum $\sum_{\lambda\in\Lambda} c(\lambda)\alpha_\lambda(S)$ converges unconditionally in $\mathcal{HS}$ for each $c \in \ell^2(\Lambda)$ [57, Cor. 3.2.5]. We also get by [57, Thm. 5.5.1] that

$$\overline{\text{span}\{\alpha_\lambda(S) : \lambda \in \Lambda\}} = \ell^2(\Lambda) \star S, \tag{C.6.3}$$

where the closure is taken with respect to the norm in $\mathcal{HS}$.

C.6.1 The biorthogonal system and best approximation

Any Riesz sequence has a so-called biorthogonal sequence and, by the theory of frames of translates [57, Prop. 9.4.2], if the Riesz sequence is of the form $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ for some $S \in \mathcal{B}$, then the biorthogonal system has the same form. This means that there exists $S' \in \mathcal{HS}$ such that the biorthogonal system is

$$\{\alpha_\lambda(S')\}_{\lambda\in\Lambda},$$

and biorthogonality means that

$$\langle \alpha_\lambda(S), \alpha_{\lambda'}(S')\rangle_{\mathcal{HS}} = \delta_{\lambda,\lambda'},$$

where $\delta_{\lambda,\lambda'}$ is the Kronecker delta. Now note that for $T \in \mathcal{HS}$ the definition (C.4.2) of $T \star_\Lambda S'$ implies that

$$\langle T, \alpha_\lambda(S')\rangle_{\mathcal{HS}} = T \star_\Lambda \check{S'}^*(\lambda),$$

so if we define $R := \check{S'}^*$ we have

$$\langle T, \alpha_\lambda(S') \rangle_{\mathcal{HS}} = T \star_\Lambda R(\lambda). \tag{C.6.4}$$

With this observation we can formulate the standard properties of the biorthogonal sequence using convolutions with R.

Lemma C.6.2. *Assume that $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ with $S \in \mathcal{B}$ is a Riesz sequence in $\mathcal{HS}$. Let*

$$V_S^2 := \overline{span\{\alpha_\lambda(S) : \lambda \in \Lambda\}} = \ell^2(\Lambda) \star S.$$

With R defined as above, we have that

(i) $S \star_\Lambda R(\lambda) = \delta_{\lambda,0}$.

(ii) For any $T \in V_S^2$, $T \star_\Lambda R \in \ell^2(\Lambda)$.

(iii) For any $T \in V_S^2$,
$$T = (T \star_\Lambda R) \star_\Lambda S.$$

Proof. This is simply a restatement of the properties of the biorthogonal sequence of a Riesz sequence using the relation $\langle T, \alpha_\lambda(S') \rangle_{\mathcal{HS}} = T \star_\Lambda R(\lambda)$ – with this observation, parts (i), (ii) and (iii) follow from [57, Thm. 3.6.2]. □

Remark C.11. (i) If the convolution of three operators were associative, we could find for any $T \in \mathcal{HS}$ (not just $T \in V_S^2$ as above) that $T = (T \star_\Lambda R) \star_\Lambda S$, since $T \star_\Lambda (R \star_\Lambda S) = T \star_\Lambda \delta_{\lambda,0} = T$. However, we will soon see that the convolution of three operators is *not* associative.

(ii) For $T, R \in \mathcal{HS}$, we have strictly speaking not defined $T \star_\Lambda R$ (since (C.4.2) has stronger assumptions than simply $\mathcal{HS}$). However, it is clear by the Cauchy Schwarz inequality for $\mathcal{HS}$ that

$$|T \star_\Lambda R(\lambda)| = |\langle T, \alpha_\lambda(S') \rangle_{\mathcal{HS}}| \leq \|T\|_{\mathcal{HS}} \|S'\|_{\mathcal{HS}},$$

so we can define $T \star_\Lambda R \in \ell^\infty(\Lambda)$ by (C.4.2) also in this case.

We will now answer two natural questions. First, to what extent does R inherit the nice properties of S – is it true that $R \in \mathcal{B}$? Then, how is R related to S? The answer is provided by the following theorem, first proved by Feichtinger [97, Thm. 5.17] for Gabor multipliers, and the proof finds a natural formulation using our tools.

Theorem C.6.3. *Assume that $S \in \mathcal{B}$ and that $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$. If R is defined as above, then $R \in \mathcal{B}$ and $R = b \star_\Lambda \check{S}^*$ where $b \in \ell^1(\Lambda)$ are the symplectic Fourier coefficients of*

$$\frac{1}{\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)} = \frac{1}{P_{\Lambda^\circ}\left(|\mathcal{F}_W(S)|^2\right)}.$$

Proof. By [57, Thm. 3.6.2], the generator S' of the biorthogonal system belongs to V_S^2, hence there exists some $b' \in \ell^2(\Lambda)$ such that $S' = b' \star_\Lambda S$. Since $R = \check{S'}^*$, one easily checks by the definitions of $\check{}$ and * that

$$R = (b' \star S)^{\check{}*} = b \star_\Lambda \check{S}^*$$

if we define $b(\lambda) = \overline{b'(-\lambda)}$. By part (i) of Lemma C.6.2 and the associativity of convolutions, we have

$$b *_\Lambda (\check{S}^* \star_\Lambda S) = (b \star_\Lambda \check{S}^*) \star_\Lambda S = R \star_\Lambda S = \delta_{\lambda,0}.$$

Taking the symplectic Fourier series of this equation using (C.5.4) and Corollary C.5.3.1, we find for a.e. $\dot{z} \in \mathbb{R}^{2d}/\Lambda^\circ$

$$\mathcal{F}_\sigma^\Lambda(b)(\dot{z})\mathcal{F}_\sigma^\Lambda(\check{S}^* \star_\Lambda S)(\dot{z}) = \mathcal{F}_\sigma^\Lambda(b)(\dot{z})P_{\Lambda^\circ}\left(|\mathcal{F}_W(S)|^2\right)(\dot{z}) = 1,$$

hence

$$\mathcal{F}_\sigma^\Lambda(b)(\dot{z}) = \frac{1}{P_{\Lambda^\circ}\left(|\mathcal{F}_W(S)|^2\right)},$$

and by assumption on S (see Theorem C.6.1 and its proof) the denominator is bounded from below by a positive constant. Since $S \in \mathcal{B}$, we know that $|\mathcal{F}_W(S)|^2 \in S_0(\mathbb{R}^{2d})$, and therefore Lemma C.5.1 implies that $P_{\Lambda^\circ}\left(|\mathcal{F}_W(S)|^2\right) \in A(\mathbb{R}^{2d}/\Lambda^\circ)$. By Wiener's lemma [221, Thm. 6.1.1], we get $\frac{1}{P_{\Lambda^\circ}\left(|\mathcal{F}_W(S)|^2\right)} \in A(\mathbb{R}^{2d}/\Lambda^\circ)$. In other words, $b \in \ell^1(\Lambda)$. Since $b \in \ell^1(\Lambda)$ and $\check{S}^* \in \mathcal{B}$, it follows that $R = b \star_\Lambda \check{S}^* \in \mathcal{B}$. □

To prepare for the next result, fix $S \in \mathcal{B}$ and let

$$V_S^\infty = \ell^\infty(\Lambda) \star_\Lambda S,$$

hence V_S^∞ is the set of operators given as a convolution $c \star_\Lambda S$ for $c \in \ell^\infty(\Lambda)$. The first part of the next result says that when $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence, then the Schatten-p class properties of $c \star_\Lambda S$ are precisely captured by the ℓ^p properties of c. This result appears to be a new result even for Gabor multipliers. We also determine for any $T \in \mathcal{HS}$ the best approximation (in the norm $\|\cdot\|_{\mathcal{HS}}$) of T by an operator of the form $c \star_\Lambda S$. See [97, Thm. 5.17] and [89, Thm. 19] for the statement for Gabor multipliers.

Corollary C.6.3.1. *Assume that $S \in \mathcal{B}$ and that $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$, and let R be as above.*

(i) *For any $1 \leq p \leq \infty$ the map $D_S : \ell^p(\Lambda) \to \mathcal{S}^p \cap V_S^\infty$ given by*

$$D_S(c) = c \star_\Lambda S$$

is a Banach space isomorphism, with inverse $C_R : \mathcal{S}^p \cap V_S^\infty \to \ell^p(\Lambda)$ given by

$$C_R(T) = T \star_\Lambda R.$$

Hence $V_S^\infty \cap \mathcal{S}^p = \ell^p(\Lambda) \star_\Lambda S$ and $\|c\|_{\ell^p} \lesssim \|c \star_\Lambda S\|_{\mathcal{S}^p} \lesssim \|c\|_{\ell^p}$.

(ii) *For any $T \in \mathcal{HS}$, the best approximation in $\|\cdot\|_{\mathcal{HS}}$ of T by an operator $c \star_\Lambda S$ with $c \in \ell^2(\Lambda)$ is given by*

$$c = T \star_\Lambda R.$$

Equivalently, the symplectic Fourier series of c is given by

$$\mathcal{F}_\sigma^\Lambda(c) = \frac{P_{\Lambda^\circ}\left[\overline{\mathcal{F}_W(S)}\mathcal{F}_W(T)\right]}{P_{\Lambda^\circ}|\mathcal{F}_W(S)|^2}.$$

Proof. (i) By Proposition C.4.3 part (i) we get $\|C_R(T)\|_{\ell^p} \leq \|T\|_{\mathcal{S}^p}\|R\|_{\mathcal{B}}$, and by part (ii) of the same proposition we get $\|D_S(c)\|_{\mathcal{S}^p} \lesssim \|c\|_{\ell^p}\|S\|_{\mathcal{B}}$. Hence both maps in the statement are continuous. It remains to show that the two maps are inverses of each other, which will follow from the associativity of convolutions. First assume that $c \in \ell^p(\Lambda)$. Then

$$C_R D_S(c) = (c \star_\Lambda S) \star_\Lambda R = c *_\Lambda (S \star_\Lambda R) = c,$$

where we have used associativity and part (i) of Lemma C.6.2. Then assume $T \in V_S^\infty \cap \mathcal{S}^p$, so that $T = c \star_\Lambda S$ for $c \in \ell^\infty(\Lambda)$. We find

$$D_S C_R(c \star_\Lambda S) = ((c \star_\Lambda S) \star_\Lambda R) \star_\Lambda S = (c *_\Lambda (S \star_\Lambda R)) \star_\Lambda S = c \star_\Lambda S.$$

Hence D_S and C_R are inverses. In particular $V_S^\infty \cap \mathcal{S}^p = \ell^p(\Lambda) \star_\Lambda S$ as D_S is onto $V_S^\infty \cap \mathcal{S}^p$, and $V_S^\infty \cap \mathcal{S}^p$ is closed in $\mathcal{S}^p$ (hence a Banach space) since $D_S : \ell^p(\Lambda) \to \mathcal{S}^p$ has a left inverse C_R and therefore has a closed range in $\mathcal{S}^p$.

(ii) We claim that the map $T \mapsto (T \star_\Lambda R) \star_\Lambda S$ is the orthogonal projection from $\mathcal{HS}$ onto $\ell^2(\Lambda) \star_\Lambda S$, which is a closed subset of $\mathcal{HS} = \mathcal{S}^2$ by part (i) (or (C.6.3)). If $T = c \star_\Lambda S$ for some $c \in \ell^2(\Lambda)$, then $c = T \star_\Lambda R$ by part (i) –

therefore $T = (T \star_\Lambda R) \star_\Lambda S$. Then assume that $T \in (\ell^2(\Lambda) \star_\Lambda S)^\perp$. As we saw in (C.6.4), we can write

$$T \star_\Lambda R(\lambda) = \langle T, \alpha_\lambda(S') \rangle_{\mathcal{HS}}. \tag{C.6.5}$$

From the proof of Theorem C.6.3, $S' = b' \star_\Lambda S$ for some $b' \in \ell^2(\Lambda)$. One easily checks that

$$\alpha_\lambda(S') = \alpha_\lambda(b' \star_\Lambda S) = T_\lambda b' \star_\Lambda S,$$

where $T_\lambda b'(\lambda') = b'(\lambda' - \lambda)$. It follows that $\alpha_\lambda(S') \in \ell^2(\Lambda) \star_\Lambda S$ for any $\lambda \in \Lambda$. Hence if $T \in (\ell^2(\Lambda) \star_\Lambda S)^\perp$, (C.6.5) shows that $(T \star_\Lambda R) \star_\Lambda S = 0$. Finally, to obtain the equivalent expression recall from Theorem C.6.3 that $R = b \star_\Lambda \check{S}^*$ for $b \in \ell^1(\Lambda)$. Hence by associativity and commutativity of convolutions,

$$c = T \star_\Lambda R = b \star_\Lambda (T \star_\Lambda \check{S}^*).$$

It follows from (C.5.4) that we get

$$\mathcal{F}_\sigma^\Lambda(c) = \mathcal{F}_\sigma^\Lambda(b) \mathcal{F}_\sigma^\Lambda(T \star_\Lambda \check{S}^*).$$

We have a known expression for $\mathcal{F}_\sigma^\Lambda(b)$ from Theorem C.6.3, and a known expression for $\mathcal{F}_\sigma^\Lambda(T \star_\Lambda \check{S}^*)$ from Theorem C.5.3 – inserting these expressions into the equation above yields the desired result.

□

The key to the results of this section is Wiener's lemma, used in the proof of Theorem C.6.3. In fact, we may interpret these results as a variation of Wiener's lemma. To see this, recall that $V_S^2 = \overline{\operatorname{span}\{\alpha_\lambda(S) : \lambda \in \Lambda\}} = \ell^2(\Lambda) \star_\Lambda S \subset \mathcal{HS}$. Then $\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$ is a Riesz sequence if and only if the convolution map $D_S : \ell^2(\Lambda) \to V_S^2$ given by

$$D_S(c) = c \star_\Lambda S$$

has a bounded inverse [57, Thm. 3.6.6]. Corollary C.6.3.1 therefore says the following: if $S \in \mathcal{B}$ and the convolution map $D_S : \ell^2(\Lambda) \to V_S^2$ has a bounded inverse, then the inverse is given by the convolution

$$C_R(T) = R \star_\Lambda T$$

for some $R \in \mathcal{B}$. The similarities with Wiener's lemma are evident when we compare this to the following formulation of Wiener's lemma [136, Thm. 5.18]:

If $b \in \ell^1(\mathbb{Z})$ and the convolution map $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$c \mapsto c *_{\mathbb{Z}} b$$

has a bounded inverse on $\ell^2(\mathbb{Z})$, then the inverse is given by the convolution map

$$c \mapsto c *_{\mathbb{Z}} b'$$

for some $b' \in \ell^1(\mathbb{Z})$.

C.7 Approximation theorems

In the continuous setting, where one considers functions on $\mathbb{R}^{2d}$ and the convolutions briefly introduced at the beginning of Section C.4, a version of Wiener's approximation theorem for operators was obtained by Kiukas et al. [182], building on earlier work by Werner [251]. This theorem consists of a long list of equivalent statements for $\mathcal{S}^p$ and $L^p(\mathbb{R}^{2d})$ for $p = 1, 2, \infty$, and as a starting point for our discussion we state a shortened version for $p = 2$ below.

Theorem C.7.1. *Let $S \in \mathcal{S}^1$. The following are equivalent.*

1. *The span of $\{\alpha_z(S)\}_{z\in\mathbb{R}^{2d}}$ is dense in $\mathcal{HS}$.*
2. *The set of zeros of $\mathcal{F}_W(S)$ has Lebesgue measure 0 in $\mathbb{R}^{2d}$.*
3. *The set of zeros of $\mathcal{F}_\sigma(S \star \check{S}^*)$ has Lebesgue measure 0 in $\mathbb{R}^{2d}$.*
4. *If $f \star S = 0$ for $f \in L^2(\mathbb{R}^{2d})$, then $f = 0$.*
5. *If $T \star S = 0$ for $T \in \mathcal{HS}$, then $T = 0$.*

We wish to obtain versions of this theorem when $\mathbb{R}^{2d}$ is replaced by a lattice Λ, functions on $\mathbb{R}^{2d}$ are replaced by sequences on Λ and we still consider operators on $L^2(\mathbb{R}^d)$. In this discrete setting, statements (3) and (4) in Theorem C.7.1 are still equivalent, mutatis mutandis, while the analogues of (1) and (5) can never be true. First we show that the discrete version of statement (1) can never hold.

Proposition C.7.2. *Let Λ be any lattice in $\mathbb{R}^{2d}$ and let $S \in \mathcal{HS}$. Then the linear span of $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is not dense in $\mathcal{HS}$.*

Proof. As we have exploited on several occasions, the Weyl transform is unitary from from $L^2(\mathbb{R}^{2d})$ to $\mathcal{HS}$ and sends translations of operators using α to translations of functions. It is therefore sufficient to show that $\{T_\lambda(a_S)\}_{\lambda\in\Lambda}$ is not dense in $L^2(\mathbb{R}^{2d})$, where a_S is the Weyl symbol of S. Let $c := \frac{2}{|\Lambda|}$, and define $\Lambda' = c\mathbb{Z}^{2d}$.

Consider the lattice $\Lambda \times \Lambda'$ in $\mathbb{R}^{4d}$. Then we have that $|\Lambda \times \Lambda'| = |\Lambda| \cdot c = 2 > 1$. By the density theorem for Gabor systems [27, 131, 155], this implies that the system $\{\pi(\lambda, \lambda')a_S\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$ cannot be span a dense subset in $L^2(\mathbb{R}^{2d})$, so in particular the subsystem $\{\pi(\lambda, 0)a_S\}_{(\lambda,0)\in\Lambda\times\Lambda'} = \{T_\lambda a_S\}_{\lambda\in\Lambda}$ cannot be complete. □

This implies that we cannot hope to generalize part (5) of Theorem C.7.1 to the discrete setting.

Corollary C.7.2.1. *Let $S \in \mathcal{B}$. There exists $0 \neq T \in \mathcal{HS}$ such that $T \star_\Lambda S = 0$.*

Proof. To obtain a contradiction, we assume that $T \star_\Lambda S = 0 \implies T = 0$ for $T \in \mathcal{HS}$. As we have seen in (C.6.4),

$$T \star_\Lambda S(\lambda) = \left\langle T, \alpha_\lambda(\check{S}^*)\right\rangle_{\mathcal{HS}}.$$

Our assumption is therefore equivalent to

$$\left\langle T, \alpha_\lambda(\check{S}^*)\right\rangle_{\mathcal{HS}} = 0 \text{ for all } \lambda \in \Lambda \implies T = 0,$$

which implies that the linear span of $\{\alpha_\lambda(\check{S}^*)\}_{\lambda\in\Lambda}$ is dense in $\mathcal{HS}$ – a contradiction to Proposition C.7.2 applied to $\check{S}^* \in \mathcal{B}$. □

Proposition C.7.2 also allows us to construct counterexamples to the associativity of convolutions of three operators.

Corollary C.7.2.2. *Assume that $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}$ for $S \in \mathcal{B}$. Then there exist $R \in \mathcal{B}$ and $T \in \mathcal{HS}$ such that*

$$(T \star_\Lambda R) \star_\Lambda S \neq T \star_\Lambda (R \star_\Lambda S).$$

Proof. Choose $R \in \mathcal{B}$ as in Section C.6.1, i.e. such that $S \star_\Lambda R = \delta_{\lambda,0}$. Then use Proposition C.7.2 to pick $T \in \mathcal{HS}$ that does not belong to the closed linear span of $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ in $\mathcal{HS}$. We get that

$$T \star_\Lambda (R \star_\Lambda S) = T \star_\Lambda \delta_{\lambda,0} = T.$$

If we assumed associativity, we would get

$$T = (T \star_\Lambda R) \star_\Lambda S,$$

where $T \star_\Lambda R \in \ell^2(\Lambda)$ by Proposition C.4.3. Hence we could express $T = c \star_\Lambda S$ for $c \in \ell^2(\Lambda)$, which would imply that T belongs to the closed linear span of $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ by (C.6.3) – a contradiction. □

On the positive side, we can use the techniques developed in Section C.5 to prove the following theorem, which shows that parts (3) and (4) of Theorem C.7.1 have natural analogues for sequences. For Gabor multipliers, Feichtinger was interested in the question of recovering c from $c \star_\Lambda (\varphi \otimes \varphi)$, and the continuity of the mapping $c \star_\Lambda (\varphi \otimes \varphi) \mapsto c$. In this case he proved the equivalence $(1)(i) \iff (1)(iv)$ below [97, Thm. 5.17], and that this implies the final statement in part (1) [97, Prop. 5.22 and Prop. 5.23]. In part (3) we show that any $c \in \ell^1(\Lambda)$ (in particular any finite sequence) can be recovered from $c \star_\Lambda S$ under significantly weaker assumptions on S for a fixed lattice Λ, but obtain no continuity statement.

Theorem C.7.3. *Let $S \in \mathcal{B}$.*

1. *The following are equivalent:*
   - *(i)* $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ *has no zeros in* $\mathbb{R}^{2d}/\Lambda^\circ$.
   - *(ii) If* $c \star_\Lambda S = 0$ *for* $c \in \ell^\infty(\Lambda)$, *then* $c = 0$.
   - *(iii)* $\mathcal{B} \star_\Lambda S$ *is dense in* $\ell^1(\Lambda)$.
   - *(iv)* $\{\alpha_\lambda S\}_{\lambda \in \Lambda}$ *is a Riesz sequence in* $\mathcal{HS}$.

   If any of the statements above holds, $c \in \ell^\infty(\Lambda)$ *is recovered from* $c \star_\Lambda S$ *by* $c = (c \star_\Lambda S) \star_\Lambda R$ *for some* $R \in \mathcal{B}$. *In particular, the map* $c \star_\Lambda S \mapsto c$ *is continuous* $\mathcal{L}(L^2) \to \ell^\infty(\Lambda)$.

2. *The following are equivalent:*
   - *(i)* $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ *is non-zero a.e. in* $\mathbb{R}^{2d}/\Lambda^\circ$.
   - *(ii) If* $c \star_\Lambda S = 0$ *for* $c \in \ell^2(\Lambda)$, *then* $c = 0$.
   - *(iii)* $\mathcal{HS} \star_\Lambda S$ *is dense in* $\ell^2(\Lambda)$.

3. *The following are equivalent:*
   - *(i) The set of zeros of* $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ *contains no open subsets in* $\mathbb{R}^{2d}/\Lambda^\circ$.
   - *(ii) If* $c \star_\Lambda S = 0$ *for* $c \in \ell^1(\Lambda)$, *then* $c = 0$.
   - *(iii)* $\mathcal{B}' \star_\Lambda S$ *is weak*-dense in* $\ell^\infty(\Lambda)$.

Proof. 1. The equivalence of (i) and (iv) was the content of Theorem C.6.1. By Corollary C.6.3.1, (iv) implies that $c \mapsto c \star_\Lambda S$ is injective, hence $(i) \iff (iv) \implies (ii)$ holds. Then assume that (ii) holds, and let $\dot{z} \in \mathbb{R}^{2d}/\Lambda^\circ$ – to show (i), we need to show that $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) \neq 0$, which by Corollary C.5.3.1 is equivalent to showing that there exists some $\lambda^\circ \in \Lambda^\circ$ such that $\mathcal{F}_W(S)(z + \lambda^\circ) \neq 0$.

Consider the distribution $\delta_{\dot{z}} \in A'(\mathbb{R}^{2d}/\Lambda^\circ)$ defined by

$$\langle \delta_{\dot{z}}, f\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d}/\Lambda^\circ)} = \overline{f(\dot{z})}$$

(recall that our duality brackets are antilinear in the second coordinate), and let $c^{\dot{z}} = \{c^{\dot{z}}(\lambda)\}_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$ be its symplectic Fourier coefficients, i.e. $\mathcal{F}_\sigma^\Lambda(c^{\dot{z}}) = \delta_{\dot{z}}$. We know that $c^{\dot{z}} \star_\Lambda S \in \mathcal{B}'$ is non-zero by (ii), and Proposition C.5.5 gives for any $f \in S_0(\mathbb{R}^{2d})$ that

$$\begin{aligned}\left\langle \mathcal{F}_W(c^{\dot{z}} \star_\Lambda S), f\right\rangle_{S_0',S_0} &= \langle \delta_{\dot{z}}\mathcal{F}_W(S), f\rangle_{S_0',S_0} \\ &= \left\langle \delta_{\dot{z}}, \overline{\mathcal{F}_W(S)}f\right\rangle_{S_0',S_0} \\ &= \left\langle \delta_{\dot{z}}, P_{\Lambda^\circ}\left[\overline{\mathcal{F}_W(S)}f\right]\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ),A(\mathbb{R}^{2d}/\Lambda^\circ)} \\ &= P_{\Lambda^\circ}\left[\mathcal{F}_W(S)\overline{f}\right](\dot{z}) \\ &= \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S)(z+\lambda^\circ)\overline{f(z+\lambda^\circ)}.\end{aligned} \tag{C.5.6}$$

From this it is clear that if $\mathcal{F}_W(S)(z+\lambda^\circ) = 0$ for all $\lambda^\circ \in \Lambda^\circ$, then $\mathcal{F}_W(c^{\dot{z}} \star_\Lambda S) = 0$ and hence $c^{\dot{z}} \star_\Lambda S = 0$ since $\mathcal{F}_W : \mathcal{B}' \to S_0(\mathbb{R}^{2d})$ is an isomorphism, which cannot hold by (ii).

Before we prove $(ii) \iff (iii)$, note that (i) is unchanged when $S \mapsto \check{S}^*$ by commutativity of the convolutions. Since $(i) \iff (ii)$, this means that (ii) is equivalent to

(ii') If $c \star_\Lambda \check{S}^* = 0$ for $c \in \ell^\infty(\Lambda)$, then $c = 0$.

To prove the equivalence of (ii') and (iii), we will prove that the map $D_{\check{S}^*} : \ell^\infty(\Lambda) \to \mathcal{B}'$ given by $D_{\check{S}^*}(c) = c \star_\Lambda \check{S}^*$ is the Banach space adjoint of $C_S : \mathcal{B} \to \ell^1(\Lambda)$ given by $C_S(T) = T \star_\Lambda S$. This amounts to proving that

$$\left\langle D_{\check{S}^*}(c), T\right\rangle_{\mathcal{B}',\mathcal{B}} = \langle c, C_S(T)\rangle_{\ell^\infty(\Lambda),\ell^1(\Lambda)} \qquad \text{for } T \in \mathcal{B}, c \in \ell^\infty(\Lambda).$$

By writing out the definitions of $D_{\check{S}^*}$ and C_S, we see that we need to show that

$$\left\langle c \star_\Lambda \check{S}^*, T\right\rangle_{\mathcal{B}',\mathcal{B}} = \langle c, T \star_\Lambda S\rangle_{\ell^\infty,\ell^1} \qquad \text{for } T \in \mathcal{B}, c \in \ell^\infty(\Lambda),$$

which is simply the definition of $c \star_\Lambda \check{S}^*$ when $c \in \ell^\infty(\Lambda)$ from (C.4.8), hence true. Since a bounded linear operator between Banach spaces has dense range if and only if its Banach space adjoint is injective (see [227, Corollary to Thm. 4.12], part (b)), this implies that (ii') is equivalent to (iii). Finally, Corollary C.6.3.1 implies the final statement that $c = (c \star_\Lambda S) \star_\Lambda R$.

2. The equivalence $(ii) \iff (iii)$ is proved as above . Assume that (i) holds, and that $c \star_\Lambda S = 0$ for some $c \in \ell^2(\Lambda)$. By associativity of convolutions,

$$c *_\Lambda (S \star_\Lambda \check{S}^*) = 0.$$

Applying $\mathcal{F}_\sigma^\Lambda$ to this, we find using (C.5.4) that

$$\mathcal{F}_\sigma^\Lambda(c)\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*) = 0.$$

By (i) this implies that $\mathcal{F}_\sigma^\Lambda(c) = 0$ in $L^2(\mathbb{R}^{2d}/\Lambda^\circ)$, hence $c = 0$.

Then assume that (i) does not hold, i.e. there is a subset $U \subset \mathbb{R}^{2d}/\Lambda^\circ$ of positive measure where $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)$ vanishes. Pick $c \in \ell^2(\Lambda)$ such that $\mathcal{F}_\sigma^\Lambda(c) = \chi_U$, where χ_U is the characteristic function of U, which is possible since $\mathcal{F}_\sigma^\Lambda : \ell^2(\Lambda) \to L^2(\mathbb{R}^{2d}/\Lambda^\circ)$ is unitary and so in particular onto. Then by Proposition C.5.5, for $f \in S_0(\mathbb{R}^{2d})$,

$$\begin{aligned}
\langle \mathcal{F}_W(c \star_\Lambda S), f\rangle_{S_0', S_0} &= \langle \chi_U \mathcal{F}_W(S), f\rangle_{S_0', S_0} \\
&= \left\langle \chi_U, \overline{\mathcal{F}_W(S)} f \right\rangle_{S_0', S_0} \\
&= \left\langle \chi_U, P_{\Lambda^\circ}\left[\overline{\mathcal{F}_W(S)} f\right]\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda^\circ), A(\mathbb{R}^{2d}/\Lambda^\circ)} \qquad \text{(C.5.6)} \\
&= \int_{\mathbb{R}^{2d}/\Lambda^\circ} \chi_U(\dot{z}) \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_W(S)(z + \lambda^\circ)\overline{f(z + \lambda^\circ)} d\dot{z} \\
&= 0.
\end{aligned}$$

To see why the last integral is zero, note first that if $\dot{z} \notin U$, then $\chi_U(\dot{z}) = 0$. If $\dot{z} \in U$, then we use that by Corollary C.5.3.1,

$$\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} |\mathcal{F}_W(S)(z + \lambda^\circ)|^2 \text{ for any } z \in \mathbb{R}^{2d}.$$

Hence the assumption $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) = 0$ for $\dot{z} \in U$ implies that $\mathcal{F}_W(S)(z + \lambda^\circ) = 0$ for any $\lambda^\circ \in \Lambda^\circ$ when $\dot{z} \in U$. In conclusion we have shown that the integrand above is zero, hence the integral is zero. This means that $\mathcal{F}_W(c \star_\Lambda S) = 0$, so $c \star_\Lambda S = 0$, contradicting (ii) since $c \neq 0$.

3. Assume that (i) holds, and that $c \star_\Lambda S = 0$ for some $c \in \ell^1(\Lambda)$. By associativity, we also have that $c \star_\Lambda (S \star_\Lambda \check{S}^*) = 0$, and by applying $\mathcal{F}_\sigma^\Lambda$ we get from (C.5.4)

$$\mathcal{F}_\sigma^\Lambda(c)(\dot{z})\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) = 0 \quad \text{ for any } \dot{z} \in \mathbb{R}^{2d}/\Lambda^\circ.$$

Since $c \in \ell^1(\Lambda)$, $\mathcal{F}_\sigma^\Lambda(c)$ is a continuous function. So if $c \neq 0$, there must exist an open subset $U \subset \mathbb{R}^{2d}/\Lambda^\circ$ such that $\mathcal{F}_\sigma^\Lambda(c)(\dot{z}) \neq 0$ for $\dot{z} \in U$. But the equation above then gives that $\mathcal{F}_\sigma(S \star \check{S}^*)(\dot{z}) = 0$ for $\dot{z} \in U$; a contradiction to (i). Hence $c = 0$, and (ii) holds. Then assume that (ii) holds, and assume that there is an open set $U \subset \mathbb{R}^{2d}/\Lambda^\circ$ such that $\mathcal{F}_\sigma^\Lambda(S \star_\Lambda \check{S}^*)(\dot{z}) = 0$ for any $\dot{z} \in U$. By Theorem C.5.3, this means that

$$\sum_{\lambda^\circ \in \Lambda^\circ} |\mathcal{F}_W(S)(z + \lambda^\circ)|^2 = 0 \quad \text{when } \dot{z} \in U,$$

which is clearly equivalent to

$$\mathcal{F}_W(S)(z) = 0 \quad \text{whenever } \dot{z} \in U.$$

Then find some non-zero $c \in \ell^1(\Lambda)$ such that $\mathcal{F}_\sigma^\Lambda(c)$ vanishes outside U, which is possible by [221, Remark 5.1.4]. Using Proposition C.5.4, we have

$$\mathcal{F}_W(c \star_\Lambda S)(z) = \mathcal{F}_\sigma^\Lambda(c)(\dot{z})\mathcal{F}_W(S)(z) \quad \text{for } z \in \mathbb{R}^{2d}.$$

If $\dot{z} \notin U$, then $\mathcal{F}_\sigma^\Lambda(c)(\dot{z}) = 0$ by construction of c. Similarly, if $\dot{z} \in U$, then we saw that $\mathcal{F}_W(S)(z) = 0$. Hence $\mathcal{F}_W(c \star_\Lambda S)(z) = 0$ for any $z \in \mathbb{R}^{2d}$, which implies that $c \star_\Lambda S = 0$. But $c \neq 0$, so this is impossible when we assume (ii), so there cannot exist an open subset $U \subset \mathbb{R}^{2d}/\Lambda^\circ$ such that $\mathcal{F}_\sigma^\Lambda(c)(\dot{z}) \neq 0$ for $\dot{z} \in U$.

The equivalence $(ii) \iff (iii)$ is proved as in part (1), with some minor modifications. We note that (i) is unchanged when $S \mapsto \check{S}^*$, so as $(i) \iff (ii)$ we have that (ii) is equivalent to

(ii') If $c \star_\Lambda \check{S}^* = 0$ for $c \in \ell^1(\Lambda)$, then $c = 0$.

By simply writing out the definitions, one sees using (C.4.10) that the map $C_S : \mathcal{B}' \to \ell^\infty(\Lambda)$ given by $C_S(T) = T \star_\Lambda S$ is the Banach space adjoint of $D_{\check{S}^*} : \ell^1(\Lambda) \to \mathcal{B}$ given by $D_{\check{S}^*}(c) = c \star_\Lambda \check{S}^*$. The equivalence $(ii') \iff (iii)$ therefore follows from part (c) of [227, Corollary of Thm. 4.12]: a bounded linear operator between Banach spaces is injective if and only if the range of its adjoint is weak*-dense.

□

Let us rewrite the statements of the theorem in the case that S is a rank-one operator $S = \varphi \otimes \varphi$ for $\varphi \in S_0(\mathbb{R}^d)$. By (C.4.5) we find that

$$S \star_\Lambda \check{S}^*(\lambda) = |V_\varphi \varphi(\lambda)|^2,$$

and by (C.4.3) $c \star_\Lambda S$ is the Gabor multiplier

$$c \star_\Lambda (\varphi \otimes \varphi)\psi = \sum_{\lambda \in \Lambda} c(\lambda) V_\varphi \psi(\lambda) \pi(\lambda)\varphi.$$

Hence the equivalences $(i) \iff (ii)$ provides a characterization using the symplectic Fourier series of $V_\varphi \varphi|_\Lambda$ of when the symbol c of a Gabor multiplier is uniquely determined.

C.7.1 Underspread operators and a Wiener division lemma

For motivation, recall Wiener's division lemma [221, Lem. 1.4.2]: if $f, g \in L^1(\mathbb{R}^{2d})$ satisfy that $\hat{f}$ has compact support ($\hat{f}$ is the usual Fourier transform on $\mathbb{R}^{2d}$) and $\hat{g}$ does not vanish on $\operatorname{supp}(\hat{f})$, then

$$f = f * h * g$$

for some $h \in L^1(\mathbb{R}^{2d})$ satisfying $\hat{h}(z) = \frac{1}{\hat{g}(z)}$ for $z \in \operatorname{supp}(\hat{f})$. The next result is a version of this statement for the convolutions and Fourier transforms of operators and sequences. At the level of Weyl symbols, this result is due to Gröchenig and Pauwels [141] (see also the thesis of Pauwels [213]) using different techniques. We choose to include a proof using the techniques of this paper to show how the the statement fits our formalism. Note that apart from the function g – introduced to ensure $A \in \mathcal{B}$ – Theorem C.7.4 is obtained by replacing the convolutions and Fourier transforms in Wiener's division lemma by the convolutions and Fourier transforms of sequences and operators.

Remark C.12. If $\Lambda^\circ = A\mathbb{Z}^{2d}$, we will pick the fundamental domain $\square_{\Lambda^\circ} = A[-\frac{1}{2}, \frac{1}{2})^{2d}$ which means that any $z \in \mathbb{R}^{2d}$ can be written as $z = z_0 + \lambda^\circ$ for $z_0 \in \square_{\Lambda^\circ}, \lambda^\circ \in \Lambda^\circ$ in a unique way. This choice of fundamental domain implies that $(1-\epsilon)\square_{\Lambda^\circ} = A[-\frac{1}{2} + \frac{\epsilon}{2}, \frac{1}{2} - \frac{\epsilon}{2})^{2d}$, so we may find g in the statement below by [195, Prop. 2.26].

Theorem C.7.4. *Assume that $S \in \mathcal{B}$ satisfies supp$(F_W(S)) \subset (1-\epsilon)\square_{\Lambda^\circ}$ for some $0 < \epsilon < 1/2$. Pick $g \in C_c^\infty(\mathbb{R}^{2d})$ such that $g|_{(1-\epsilon)\square_{\Lambda^\circ}} \equiv 1$ and supp$(g) \subset \square_{\Lambda^\circ}$. If $T \in \mathcal{B}$ satisfies $\mathcal{F}_W(T)(z) \neq 0$ for $z \in$ supp(g), then*

$$S = (S \star_\Lambda T) \star_\Lambda A,$$

where $A \in \mathcal{B}$ is given by $\mathcal{F}_W(A) = |\Lambda| \frac{g}{\mathcal{F}_W(T)}$.

Proof. We first show that $A \in \mathcal{B}$ by showing $\mathcal{F}_W(A) \in S_0(\mathbb{R}^{2d})$. The Wiener-Lévy theorem [221, Thm. 1.3.1] gives $h \in L^1(\mathbb{R}^{2d})$ such that $\hat{h}(z) = 1/\mathcal{F}_W(T)(z)$ for

$z \in \operatorname{supp}(g)$, where $\hat{h}$ denotes the usual Fourier transform. Therefore $\mathcal{F}_W(A) = |\Lambda| g \cdot \hat{h}$, which belongs to $S_0(\mathbb{R}^{2d})$ by [131, Prop. 12.1.7].

To show that $S = (S \star_\Lambda T) \star_\Lambda A$, we will show that their Fourier-Wigner transforms are equal. Using Proposition C.5.4 and Theorem C.5.3 we find that

$$\begin{aligned}\mathcal{F}_W((S \star_\Lambda T) \star_\Lambda A)(z) &= \mathcal{F}_\sigma^\Lambda(S \star_\Lambda T)(\dot{z})\mathcal{F}_W(A)(z)\\ &= \frac{\mathcal{F}_W(A)(z)}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_W(S)(z+\lambda^\circ)\mathcal{F}_W(T)(z+\lambda^\circ).\end{aligned}$$

To show that this equals $\mathcal{F}_W(S)$, we consider three cases.

- If $z \in (1-\epsilon)\square_{\Lambda^\circ}$, then $g(z) = 1$ by construction and

$$\begin{aligned}\frac{\mathcal{F}_W(A)(z)}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} &\mathcal{F}_W(S)(z+\lambda^\circ)\mathcal{F}_W(T)(z+\lambda^\circ)\\ &= \frac{\mathcal{F}_W(A)(z)}{|\Lambda|}\mathcal{F}_W(S)(z)\mathcal{F}_W(T)(z)\\ &= \frac{g(z)}{\mathcal{F}_W(T)(z)}\mathcal{F}_W(S)(z)\mathcal{F}_W(T)(z)\\ &= \mathcal{F}_W(S)(z)\end{aligned}$$

  where we used that the only summand contributing to the sum is $\lambda^\circ = 0$ since $\operatorname{supp}(\mathcal{F}_W(S)) \subset \square_{\Lambda^\circ}$ and $z \in \square_{\Lambda^\circ}$ and $\square_{\Lambda^\circ}$ is a fundamental domain.

- If $z \in \square_{\Lambda^\circ} \setminus (1-\epsilon)\square_{\Lambda^\circ}$, then $\mathcal{F}_W(S)(z) = 0$ and the same argument as above gives

$$\begin{aligned}\frac{\mathcal{F}_W(A)(z)}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} &\mathcal{F}_W(S)(z+\lambda^\circ)\mathcal{F}_W(T)(z+\lambda^\circ)\\ &= \frac{\mathcal{F}_W(A)(z)}{|\Lambda|}\overbrace{\mathcal{F}_W(S)(z)}^{0}\mathcal{F}_W(T)(z) = 0\end{aligned}$$

- If $z \notin (1-\epsilon)\square_{\Lambda^\circ}$, then $\mathcal{F}_W(S)(z) = 0$ since $\operatorname{supp}(\mathcal{F}_W(S)) \subset \square_{\Lambda^\circ}$ and $\mathcal{F}_W((S \star_\Lambda T) \star_\Lambda A)(z) = 0$ since $\frac{\mathcal{F}_W(A)(z)}{|\Lambda|} = \frac{g(z)}{\mathcal{F}_W(T)(z)} = 0$ as $\operatorname{supp}(g) \subset \square_{\Lambda^\circ}$.

□

A similar argument using duality brackets shows that essentially the same result even holds for $S \in \mathcal{B}'$.

Theorem C.7.5. *Assume that $S \in \mathcal{B}'$ satisfies supp$(F_W(S)) \subset (1-2\epsilon)\square_{\Lambda^\circ}$ for some $0 < \epsilon < 1/2$. Pick $g \in C_c^\infty(\mathbb{R}^{2d})$ such that $g|_{(1-\epsilon)\square_{\Lambda^\circ}} \equiv 1$ and supp$(g) \subset \square_{\Lambda^\circ}$. If $T \in \mathcal{B}$ satisfies $\mathcal{F}_W(T)(z) \neq 0$ for $z \in$ supp(g), then*

$$S = (S \star_\Lambda T) \star_\Lambda A,$$

where $A \in \mathcal{B}$ is given by $\mathcal{F}_W(A) = |\Lambda| \frac{g}{\mathcal{F}_W(T)}$.

Proof. We have already seen that $A \in \mathcal{B}$. Let $f \in S_0(\mathbb{R}^{2d})$. Then

$$\begin{aligned}
&\langle \mathcal{F}_W\left[(S \star_\Lambda T) \star_\Lambda A)\right], f\rangle_{S_0',S_0} \\
&\qquad = \langle (S \star_\Lambda T) \star_\Lambda A), \rho(f)\rangle_{\mathcal{B}',\mathcal{B}} \quad \text{by (C.5.2)} \\
&\qquad = \left\langle S \star_\Lambda T, \rho(f) \star_\Lambda \check{A}^*\right\rangle_{\ell^\infty,\ell^1} \quad \text{by (C.4.8)} \\
&\qquad = \left\langle S, (\rho(f) \star_\Lambda \check{A}^*) \star_\Lambda \check{T}^*\right\rangle_{\mathcal{B}',\mathcal{B}} \quad \text{by (C.4.10)} \\
&\qquad = \left\langle \mathcal{F}_W(S), \mathcal{F}_W\left[(\rho(f) \star_\Lambda \check{A}^*) \star_\Lambda \check{T}^*\right]\right\rangle_{S_0',S_0} \quad \text{by (C.5.2)} \\
&\qquad = \left\langle \mathcal{F}_W(S), b \cdot \mathcal{F}_W\left[(\rho(f) \star_\Lambda \check{A}^*) \star_\Lambda \check{T}^*\right]\right\rangle_{S_0',S_0}.
\end{aligned}$$

In the last line we multiplied the right hand side by a bump function $b \in C_c^\infty(\mathbb{R}^{2d})$ such that $b|_{(1-2\epsilon)\square_{\Lambda^\circ}} \equiv 1$ and supp$(b) \subset (1-\epsilon)\square_{\Lambda^\circ}$ – this does not change anything by the assumptions on the supports of $\mathcal{F}_W(S)$ and b. We find using Theorem C.5.3 and Proposition C.5.4 that

$$\begin{aligned}
b \cdot \mathcal{F}_W\left[(\rho(f) \star_\Lambda \check{A}^*) \star_\Lambda \check{T}^*\right] &= b \cdot \mathcal{F}_\sigma^\Lambda(\rho(f) \star_\Lambda \check{A}^*) \cdot \overline{\mathcal{F}_W(T)} \\
&= b \cdot \overline{\mathcal{F}_W(T)} P_{\Lambda^\circ}(f \overline{\mathcal{F}_W(A)}) \quad \text{by (C.5.5).}
\end{aligned}$$

We claim that this last function equals $b \cdot f$: if $z \notin (1-\epsilon)\square_{\Lambda^\circ}$, then $b(z) = 0$, so $b(z)f(z) = 0$ and

$$b(z) \cdot \overline{\mathcal{F}_W(T)(z)} P_{\Lambda^\circ}(f \overline{\mathcal{F}_W(A)})(\dot{z}) = 0.$$

If $z \in (1-\epsilon)\square_{\Lambda^\circ}$, then $g(z) = 1$ and

$$\begin{aligned}
&b(z)\overline{\mathcal{F}_W(T)(z)} P_{\Lambda^\circ}(f \overline{\mathcal{F}_W(A)})(\dot{z}) \\
&\qquad = b(z)\overline{\mathcal{F}_W(T)(z)} \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} f(z+\lambda^\circ)\overline{\mathcal{F}_W(A)}(z+\lambda^\circ) \\
&\qquad = b(z)\overline{\mathcal{F}_W(T)(z)} f(z) \frac{1}{|\Lambda|} \overline{\mathcal{F}_W(A)(z)} \\
&\qquad = b(z)\overline{\mathcal{F}_W(T)(z)} f(z) \frac{\overline{g(z)}}{\overline{\mathcal{F}_W(T)(z)}} \\
&\qquad = b(z)f(z)
\end{aligned}$$

since $\mathcal{F}_W(A)$ vanishes outside of $\square_{\Lambda^\circ}$ by construction. Hence we have shown that

$$\begin{aligned}\langle \mathcal{F}_W\left[(S \star_\Lambda T) \star_\Lambda A)\right], f\rangle_{S_0',S_0} &= \langle \mathcal{F}_W(S), b\cdot f\rangle_{S_0',S_0} \\ &= \langle \mathcal{F}_W(S), f\rangle_{S_0',S_0}\end{aligned}$$

for any $f \in S_0(\mathbb{R}^{2d})$, which implies the result. □

Operators S such that $\operatorname{supp}(\mathcal{F}_W(S)) \subset [-\frac{a}{2}, \frac{a}{2}]^d \times [-\frac{b}{2}, \frac{b}{2}]^d$ where $ab \leq 1$ are called *underspread*, and provide realistic models of communication channels [89, 141, 190, 193, 238]. We immediately obtain the following consequence.

Corollary C.7.5.1. *Any underspread operator $S \in \mathcal{B}'$ can be expressed as a convolution $T = c \star_\Lambda A$ with $c \in \ell^\infty(\Lambda)$ and $A \in \mathcal{B}$ for a sufficiently dense lattice Λ. In particular, S is bounded on $L^2(\mathbb{R}^d)$.*

It is known (see [89]) that for an operator S to be well-approximated by Gabor multipliers – i.e. operators $c \star_\Lambda (\psi \otimes \psi)$ for $\psi \in L^2(\mathbb{R}^d)$ – S should be underspread. The result above shows that any underspread operator S is given precisely by a convolution $S = c \star_\Lambda A$ if we allow A to be any operator in $\mathcal{B}$, not just a rank-one operator. In fact, A as constructed in the theorem will never be a rank-one operator, since $\mathcal{F}_W(A)$ has compact support – this is not possible for rank-one operators [173]. If S satisfies $S \in \mathcal{S}^p$ in addition to the assumptions of Theorem C.7.5, then $c = S \star T \in \ell^p(\Lambda)$ by Proposition C.4.3. Hence the p-summability of c in $S = c \star_\Lambda A$ reflects the fact that $S \in \mathcal{S}^p$.

Theorem C.7.5 also implies that underspread operators S are determined by the sequence $S \star_\Lambda T$ when $T \in \mathcal{B}$ is chosen appropriately. This was a major motivation for [141], since when T is a rank-one operator $T = \varphi \otimes \varphi$, the sequence $S \star_\Lambda \check{T}$ is the diagonal of the so-called channel matrix of S with respect to φ – see [141, 213] for a thorough discussion and motivation of these concepts. Finally, note that Theorem C.7.5 gives a (partial) discrete analogue of part (5) of Theorem C.7.1.

Acknowledgements

We thank Franz Luef for insightful feedback on various drafts of this paper. We also thank Markus Faulhuber for helpful discussions and suggestions, particularly concerning Proposition C.7.2.

Paper D

On Gabor g-frames and Fourier Series of Operators

Eirik Skrettingland

Published in
Studia Mathematica, 2021, volume 259, issue 1.

Paper D

On Gabor g-frames and Fourier Series of Operators

Abstract

We show that Hilbert-Schmidt operators can be used to define frame-like structures for $L^2(\mathbb{R}^d)$ over lattices in $\mathbb{R}^{2d}$ that include multi-window Gabor frames as a special case. These frame-like structures are called Gabor g-frames, as they are examples of g-frames as introduced by Sun. We show that Gabor g-frames share many properties of Gabor frames, including a Janssen representation and Wexler-Raz biorthogonality conditions. A central part of our analysis is a notion of Fourier series of periodic operators based on earlier work by Feichtinger and Kozek, where we show in particular a Poisson summation formula for trace class operators. By choosing operators from certain Banach subspaces of the Hilbert Schmidt operators, Gabor g-frames give equivalent norms for modulation spaces in terms of weighted ℓ^p-norms of an associated sequence, as previously shown for localization operators by Dörfler, Feichtinger and Gröchenig.

D.1 Introduction

The study of Gabor frames is today an essential part of time-frequency analysis. By fixing a window function $\varphi \in L^2(\mathbb{R}^d)$, a signal $\psi \in L^2(\mathbb{R}^d)$ is analyzed by considering its projections onto copies of φ shifted in time and frequency. In other words, one considers the *short-time Fourier transform*

$$V_\varphi\psi(z) = \langle \psi, \pi(z)\varphi \rangle_{L^2} \qquad \text{for } z \in \mathbb{R}^{2d},$$

where $\pi(z)$ is the time-frequency shift operator defined by $\pi(z)\varphi(t) = e^{2\pi i \omega \cdot t}\varphi(t - x)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$. If φ is well-behaved, one interprets $|V_\varphi\psi(x, \omega)|^2$ as a measure of the contribution of the frequency ω at the time x in the signal ψ. Given a lattice $\Lambda = A\mathbb{Z}^{2d}$ for $A \in \mathrm{GL}(2d, \mathbb{R})$, φ generates a *Gabor frame* over Λ if the ℓ^2-norm of the sequence $\{V_\varphi \psi(\lambda)\}_{\lambda\in\Lambda}$ is equivalent to the L^2-norm of ψ, i.e. there should exist constants $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} |V_\varphi \psi(\lambda)|^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d). \tag{D.1.1}$$

In the usual terminology of frames, see for instance the monographs [57, 131, 156], this simply means that $\{\pi(\lambda)\varphi\}_{\lambda\in\Lambda}$ is a frame for $L^2(\mathbb{R}^d)$, and (D.1.1) is equivalent to the fact that the *frame operator*

$$\psi \mapsto \sum_{\lambda\in\Lambda} V_\varphi \psi(\lambda)\pi(\lambda)\varphi$$

is bounded and invertible on $L^2(\mathbb{R}^d)$. Research over the last thirty years has revealed several intriguing features of Gabor frames, among them the Janssen representation of the frame operator [102, 170, 223], the Wexler-Raz biorthogonality conditions [70, 102, 170, 252] and that for well-behaved windows φ summability conditions on the coefficients $\{V_\varphi \psi(\lambda)\}_{\lambda\in\Lambda}$ characterize smoothness and decay properties of ψ [100, 101, 140].

The aim of this paper is to show that Gabor frames over a lattice $\Lambda \subset \mathbb{R}^{2d}$ are a special case of a more general situation, namely that Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$ can be used to define a frame-like structure for $L^2(\mathbb{R}^d)$. These structures are obtained by shifting a "window" operator S over Λ by the operation

$$\alpha_z(S) = \pi(z)S\pi(z)^* \quad \text{for } z \in \mathbb{R}^{2d}.$$

Following Werner [251] and Kozek [188] we consider $\alpha_\lambda(S)$ to be a translation of S by λ. Our main definition is that S generates a *Gabor g-frame* for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\psi\|_{L^2}^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for } \psi \in L^2(\mathbb{R}^d). \tag{D.1.2}$$

When S is a rank-one operator we recover the definition of Gabor frames – more generally we obtain multi-window Gabor frames [257] if S is of finite rank. If (D.1.2) holds, the associated *g-frame operator* $\mathfrak{S}_S$ given by

$$\mathfrak{S}_S(\psi) = \sum_{\lambda\in\Lambda} \alpha_\lambda(S^*S)\psi, \tag{D.1.3}$$

is bounded and invertible on $L^2(\mathbb{R}^d)$, and we show that this operator is the composition of two other natural operators: the analysis and synthesis operators. A major goal of this paper is to show that although Gabor g-frames are not frames, they nevertheless share much of the structure of Gabor frames. Our terminology stems from the fact that Gabor g-frames are examples of g-frames as introduced by Sun [239], but apart from terminology the abstract theory of g-frames does not feature much in this paper.

Fourier series of operators and the Janssen representation

Our investigations into the structure of Gabor g-frames naturally lead to the study of a notion of Fourier series of operators, inspired by the analysis of periodic operators by Feichtinger and Kozek [102] and the quantum harmonic analysis of Werner [251]. By Fourier series for operators we mean that a Λ-periodic operator T – meaning that $\alpha_\lambda(T) = T$ for all $\lambda \in \Lambda$ – has an expansion of the form

$$T = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x \cdot \lambda^\circ_\omega} \pi(\lambda^\circ). \tag{D.1.4}$$

Here Λ° is the adjoint lattice of Λ defined in Section D.6, and we write $\lambda^\circ = (\lambda^\circ_x, \lambda^\circ_\omega)$. Such expansions have also been studied in [102], and the interpretation that this is a Fourier series of operators follows from considering the operator $e^{-i\pi x\cdot\omega}\pi(z)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$ as the operator-analogue of the character $t \mapsto e^{2\pi i z\cdot t}$ on $\mathbb{R}^{2d}$. This interpretation is strengthened by the fact that an analogue of Wiener's classical lemma for absolutely summable Fourier series also holds for operators, by a result of Gröchenig and Leinert [140]. We show that any Λ-periodic bounded operator on $L^2(\mathbb{R}^d)$ has a Fourier series expansion (D.1.4). This is not the only possible approach to Fourier series of operators, see for instance [44, 77–79], and we also remark that periodic operators have been studied in [21, Prop. 5.5].

Due to the form of the Gabor g-frame operator (D.1.3) it is particularly interesting to study the Fourier series expansion of periodic operators T given by a periodization over Λ:

$$T = \sum_{\lambda \in \Lambda} \alpha_\lambda(R)$$

for some operator R. This leads to the following *Poisson summation formula for trace class operators*: if R is a trace class operator, then

$$\sum_{\lambda \in \Lambda} \alpha_\lambda(R) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_W(R)(\lambda^\circ) e^{-\pi i \lambda^\circ_x \cdot \lambda^\circ_\omega} \pi(\lambda^\circ), \tag{D.1.5}$$

where $\mathcal{F}_W$ is the Fourier-Wigner transform of R defined by

$$\mathcal{F}_W(R)(z) = e^{-\pi i x\cdot\omega} \operatorname{tr}(\pi(-z)R) \quad \text{for } z = (x, \omega) \in \mathbb{R}^{2d},$$

which Werner [251] argued is a Fourier transform of operators. Showing that (D.1.5) holds for all trace class operators requires a careful study of the continuity of several mappings. Equation (D.1.5) is an analogue of the usual Poisson summation formula for functions: the Fourier coefficients of a periodization $\sum_{\lambda\in\Lambda} \alpha_\lambda(R)$ is given by the samples of the Fourier transform of R. Comparing (D.1.5) with (D.1.3), we obtain an alternative expression for the g-frame operator of a Gabor g-frame which generalizes the Janssen representation for Gabor frames. This generalized Janssen representation allows us to deduce an extension of the Wexler-Raz biorthogonality conditions to Gabor g-frames, and to establish painless procedures for making Gabor g-frames using *underspread* operators.

Time-frequency localization and Gabor g-frames

The definition (D.1.2) has a particularly interesting interpretation if $\alpha_\lambda(S)\psi$ can, in some sense, be interpreted as the part of the signal ψ localized around the point λ in the time-frequency plane $\mathbb{R}^{2d}$. In this case, one may interpret $\|\alpha_\lambda(S)\psi\|_{L^2}$ as a measure of the part of ψ localized around λ in the time-frequency plane. For instance, picking a rank-one operator $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$, one finds that $\|\alpha_\lambda(S)\psi\|_{L^2} = |V_\varphi\psi(\lambda)|$, which is the measure of localization of ψ around λ used in Gabor frames. Another prime example of operators S where $\alpha_\lambda(S)\psi$ has this interpretation are the localization operators $A^\varphi_{\chi_\Omega}$ with domain $\Omega \subset \mathbb{R}^{2d}$ and window $\varphi \in L^2(\mathbb{R}^d)$ introduced by Daubechies [63, 67, 88], and the inequalities (D.1.2) have been studied for such operators by Dörfler, Feichtinger and Gröchenig [85, 87]. The results of [85, 87] are therefore a second important class examples of Gabor g-frames in addition to (multi-window) Gabor frames.

In our terminology, [85, 87] showed that if $A^\varphi_{\chi_\Omega}$ generates a Gabor g-frame with well-behaved window φ, then weighted ℓ^p-norms of $\{\|\alpha_\lambda(A^\varphi_{\chi_\Omega})\psi\|_{L^2}\}_{\lambda\in\Lambda}$ are equivalent to the norm of ψ in modulation spaces. By the properties of modulation spaces, this implies that smoothness and decay properties of ψ are captured by the coefficients $\{\|\alpha_\lambda(A^\varphi_{\chi_\Omega})\psi\|_{L^2}\}_{\lambda\in\Lambda}$. A similar result is well-known for Gabor frames [100, 101, 131], and in Corollary D.7.3.2 we extend this to a result for Gabor g-frames that includes Gabor frames and localization operators as special cases.

The fact that the results of [85, 87] can be incorporated into the theory of Gabor g-frames allows us to understand exactly how a signal ψ is recovered from its time-frequency localized components $\psi_\lambda := \alpha_\lambda(A^\varphi_{\chi_\Omega})\psi$ for $\lambda \in \Lambda$. In fact, we show that $A^\varphi_{\chi_\Omega}$ has a canonical dual operator R, such that

$$\psi = \sum_{\lambda\in\Lambda} \alpha_\lambda(R^*)\psi_\lambda \quad \text{for any } \psi \in L^2(\mathbb{R}^d).$$

This is a generalization of a well-known fact for Gabor frames to Gabor g-frames (and in particular the localization operators of [85, 87]), namely that if $\varphi \in L^2(\mathbb{R}^d)$ generates a Gabor frame, then there is a canonical dual window $\varphi' \in L^2(\mathbb{R}^d)$ with

$$\psi = \sum_{\lambda\in\Lambda} V_\varphi\psi(\lambda)\pi(\lambda)\varphi' \quad \text{for any } \psi \in L^2(\mathbb{R}^d).$$

Cohen's class and Gabor g-frames

A different perspective on Gabor g-frames uses Cohen's class of time-frequency distributions [59]. In the formalism of [204], $\|\alpha_\lambda(S)\psi\|_{L^2}^2$ equals $Q_{S^*S}(\psi)(\lambda)$, where Q_{S^*S} is the Cohen's class distribution associated with the operator S^*S as defined in [204]. Hence equation (D.1.2) states that the ℓ^1-norm of the samples $\{Q_{S^*S}(\psi)(\lambda)\}_{\lambda\in\Lambda}$ should be an equivalent norm on $L^2(\mathbb{R}^d)$. A simple example of a Cohen's class distribution is the spectrogram $|V_\varphi\psi(z)|^2$ for a window φ, which corresponds to picking rank-one S. Hence the move from Gabor frames to Gabor g-frames corresponds to replacing the spectrogram by a more general Cohen's class distribution, and we show that much of the structure of Gabor frames is preserved.

Technical tools

We give a brief overview of the non-standard technical tools needed to prove the results of the paper. We will utilize a Banach subspace $\mathcal{B}$ of the trace class operators, as studied by [62,99,102]. The space $\mathcal{B}$ consists of operators with kernel (as integral operators) in the so-called Feichtinger algebra [95], and we aim to show readers with backgrounds in other areas than time-frequency analysis the usefulness of $\mathcal{B}$. For instance, if $R \in \mathcal{B}$ the sum on the right hand side of (D.1.5) converges absolutely in the operator norm. The same will hold if we pick R from the smaller space of Schwartz operators [179], but the Schwartz operators do not form a Banach space. Hence $\mathcal{B}$ combines desirable features from the trace class operators and the Schwartz operators: it is a Banach space, yet small enough to have properties not shared by arbitrary trace class operators. A new aspect in this paper is that we also develop a theory of weighted versions of $\mathcal{B}$, and we use the projective tensor product of Banach spaces to establish a decomposition of operators in the weighted $\mathcal{B}$-spaces in terms of rank-one operators.

We will also use the dual space $\mathcal{B}'$ with its weak* topology. The sums in the Poisson summation formula (D.1.5) for trace class operators converge in this topology, but not necessarily in the weak* topology of the bounded operator $\mathcal{L}(L^2)$ – hence $\mathcal{B}'$ is necessary even for studying trace class operators.

In order to write the g-frame operator (D.1.3) as the composition of an analysis operator and a synthesis operator we will need the L^2-valued sequence spaces $\ell^p_m(\Lambda; L^2)$, consisting of sequences $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^d)$ such that

$$\sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2}^p m(\lambda)^p < \infty,$$

where m is a weight function. The use of these Banach spaces is key to reducing statements about Gabor g-frames to known results for Gabor frames in Section D.7.

Organization

We recall some definitions and results from time-frequency analysis, pseudodifferential operators and g-frames in Section D.3. Section D.4 is devoted to introducing and studying one of our main tools: Banach spaces of operators with kernels in certain weighted function spaces and their decomposition into rank-one operators. The definition and basic properties of Gabor g-frames are given in Section D.5. The theory of Fourier series of operators and its applications to Gabor g-frames, including a Janssen representation and Wexler-Raz biorthogonality for Gabor g-frames, is explored in Section D.6. Section D.7 is devoted to using Gabor g-frames to obtain equivalent norms for modulation spaces. Finally the relation of Gabor g-frames to countably generated multi-window Gabor frames using the singular value decomposition is explained in Section D.8.

D.2 Notation and conventions

By a lattice Λ we mean a full-rank lattice in $\mathbb{R}^{2d}$, i.e. $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d, \mathbb{R})$. The volume of $\Lambda = A\mathbb{Z}^{2d}$ is $|\Lambda| := \det(A)$. The Haar measure on $\mathbb{R}^{2d}/\Lambda$ will always be normalized so that $\mathbb{R}^{2d}/\Lambda$ has total measure 1.

If X is a Banach space and X' its dual space, the action of $y \in X'$ on $x \in X$ is denoted by the bracket $\langle y, x\rangle_{X',X}$, where the bracket is antilinear in the second coordinate to be compatible with the notation for inner products in Hilbert spaces. This means that we are identifying the dual space X' with *anti*linear functionals on X. For two Banach spaces X, Y we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators $S : X \to Y$, and if $X = Y$ we simply write $\mathcal{L}(X)$. The notation $X \hookrightarrow Y$ denotes a norm-continuous embedding of Banach spaces.

For $p \in [1, \infty]$, p' denotes the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The notation $P \lesssim Q$ means that there is some $C > 0$ such that $P \leq C \cdot Q$, and $P \asymp Q$ means that $Q \lesssim P$ and $P \lesssim Q$. For $\Omega \subset \mathbb{R}^{2d}$, χ_Ω is the characteristic function of Ω.

D.3 Preliminaries

D.3.1 Time-frequency analysis and modulation spaces

The fundamental operators in time-frequency analysis are the translation operators T_x and the modulation operators M_ω for $x, \omega \in \mathbb{R}^d$, defined by

$$(T_x\psi)(t) = \psi(t - x), \qquad (M_\omega\psi)(t) = e^{2\pi i\omega\cdot t}\psi(t) \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

By composing these operators, we get the time-frequency shifts $\pi(z) := M_\omega T_x$ for $z = (x, \omega) \in \mathbb{R}^{2d}$, given by

$$(\pi(z)\psi)(t) = e^{2\pi i \omega \cdot t}\psi(t - x) \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

The time-frequency shifts $\pi(z)$ are unitary operators on $L^2(\mathbb{R}^d)$, with adjoint $\pi(z)^* = e^{-2\pi i x \cdot \omega}\pi(-z)$ for $z = (x, \omega)$. For $\psi, \phi \in L^2(\mathbb{R}^d)$ we use the time-frequency shifts to define the *short-time Fourier transform* $V_\phi \psi$ of ψ with window ϕ by

$$V_\phi \psi(z) = \langle \psi, \pi(z)\phi \rangle_{L^2} \quad \text{for } z \in \mathbb{R}^{2d}. \tag{D.3.1}$$

The short-time Fourier transform satisfies an orthogonality condition, sometimes called Moyal's identity [114, 131].

Lemma D.3.1 (Moyal's identity)**.** *If $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$, then $V_{\phi_i}\psi_j \in L^2(\mathbb{R}^{2d})$ for $i, j \in \{1, 2\}$ and*

$$\left\langle V_{\phi_1}\psi_1, V_{\phi_2}\psi_2 \right\rangle_{L^2} = \langle \psi_1, \psi_2 \rangle_{L^2} \overline{\langle \phi_1, \phi_2 \rangle_{L^2}},$$

where the leftmost inner product is in $L^2(\mathbb{R}^{2d})$ and those on the right are in $L^2(\mathbb{R}^d)$.

Weight functions

To define the appropriate function spaces for our setting – the modulation spaces – we need to consider weight functions on $\mathbb{R}^{2d}$. In this paper, a *weight function* is a continuous and positive function on $\mathbb{R}^{2d}$. We will always let v denote a *submultiplicative weight function satisfying the GRS-condition.* That v is submultiplicative means that

$$v(z_1 + z_2) \leq v(z_1)v(z_2) \text{ for any } z_1, z_2 \in \mathbb{R}^{2d},$$

and the GRS-condition says that

$$\lim_{n\to\infty} (v(nz))^{1/n} = 1 \text{ for any } z \in \mathbb{R}^{2d}.$$

Furthermore, we will assume that v is symmetric in the sense that $v(x, \omega) = v(-x, \omega) = v(x, -\omega) = v(-x, -\omega)$ for any $(x, \omega) \in \mathbb{R}^{2d}$, which along with submultiplicativity implies that $v \geq 1$ [135].

By m we will always mean a weight function that is *v-moderate*; this means that

$$m(z_1 + z_2) \lesssim m(z_1)v(z_2) \text{ for any } z_1, z_2 \in \mathbb{R}^{2d}. \tag{D.3.2}$$

The interested reader is encouraged to consult the survey [135] for an excellent exposition of the reasons for making these assumptions in time-frequency analysis. The less interested reader may safely assume that all weights are polynomial weights $v_s(z) = (1 + |z|)^s$ for some $s \geq 0$.

Modulation spaces

Let φ_0 be the normalized (in L^2-norm) Gaussian $\varphi_0(x) = 2^{d/4}e^{-\pi x\cdot x}$ for $x \in \mathbb{R}^d$, and let v be a submultiplicative, symmetric GRS-weight. We first define the space $M^1_v(\mathbb{R}^d)$ to be the space of $\psi \in L^2(\mathbb{R}^d)$ such that

$$\|\psi\|_{M^1_v} := \int_{\mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)|v(z)\, dz < \infty.$$

For $p \in [1,\infty]$ and a v-moderate weight function m we then define the *modulation space* $M^p_m(\mathbb{R}^d)$ to be the set of ψ in the (antilinear) dual space $\left(M^1_v(\mathbb{R}^d)\right)'$ with

$$\|\psi\|_{M^p_m} := \left(\int_{\mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)|^p m(z)^p\, dz\right)^{1/p} < \infty, \tag{D.3.3}$$

where the integral is replaced by a supremum in the usual way when $p = \infty$. In (D.3.3), $V_{\varphi_0}\psi$ must be interpreted by (antilinear) duality, meaning that we extend the definition in equation (D.3.1) by defining

$$V_{\varphi_0}\psi(z) = \langle \psi, \pi(z)\varphi_0\rangle_{(M^1_v)',M^1_v}.$$

For $m \equiv 1$ we will write $M^p(\mathbb{R}^d) := M^p_m(\mathbb{R}^d)$. We summarize a few of the useful properties of modulation spaces in a proposition, see [131] for the proofs.

Proposition D.3.2. *Let m be a v-moderate weight and $p \in [1,\infty]$.*

(a) $M^p_m(\mathbb{R}^d)$ is a Banach space with the norm defined in (D.3.3).

(b) If we replace φ_0 with another function $0 \neq \phi \in M^1_v(\mathbb{R}^d)$ in (D.3.3), *we obtain the same space $M^p_m(\mathbb{R}^d)$ as with φ_0, with equivalent norms.*

(c) If $1 \leq p_1 \leq p_2 \leq \infty$ and $m_2 \lesssim m_1$, then $M^{p_1}_{m_1}(\mathbb{R}^d) \hookrightarrow M^{p_2}_{m_2}(\mathbb{R}^d)$.

(d) If $p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $M^{p'}_{1/m}(\mathbb{R}^d)$ is the dual space of $M^p_m(\mathbb{R}^d)$ with

$$\langle \phi, \psi\rangle_{M^{p'}_{1/m},M^p_m} = \int_{\mathbb{R}^{2d}} V_{\varphi_0}\phi(z)\overline{V_{\varphi_0}\psi(z)}\, dz. \tag{D.3.4}$$

(e) The operators $\pi(z)$ can be extended to bounded operators on $M^p_m(\mathbb{R}^d)$ with $\|\pi(z)\psi\|_{M^p_m} \lesssim v(z)\|\psi\|_{M^p_m}$ for $\psi \in M^p_m(\mathbb{R}^d)$.

(f) $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$ with equivalent norms.

(g) $M^1_v(\mathbb{R}^d)$ is dense in $M^p_m(\mathbb{R}^d)$ for $p < \infty$ and weak-dense in $M^\infty_m(\mathbb{R}^d)$.*

Remark D.1. (a) Assume that $p < \infty$. If $\phi \in L^2(\mathbb{R}^d) \cap M^{p'}_{1/m}(\mathbb{R}^d)$ and $\psi \in M^p_m(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then Moyal's identity and (D.3.4) implies that

$$\langle \phi, \psi \rangle_{M^{p'}_{1/m}, M^p_m} = \langle \phi, \psi \rangle_{L^2} .$$

We will use this fact several times in the rest of the paper.

(b) We defined modulation spaces as subspaces of $(M^1_v(\mathbb{R}^d))' = M^\infty_{1/v}(\mathbb{R}^d)$. If one restricts to weights v of at most polynomial growth, then $M^1_v(\mathbb{R}^d)$ contains the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and $M^\infty_{1/v}(\mathbb{R}^d)$ is a subspace of the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ [135].

(c) If m is v-moderate, then so is $1/m$ since we assume that v is symmetric: for $w_1, w_2 \in \mathbb{R}^{2d}$ we find by choosing $z_1 = w_1 + w_2$ and $z_2 = -w_2$ in (D.3.2) that $m(w_1) \lesssim m(w_1 + w_2)v(w_2)$, hence

$$\frac{1}{m(w_1 + w_2)} \lesssim \frac{1}{m(w_1)} v(w_2).$$

The class of modulation spaces is therefore closed under duality for $p < \infty$.

Wiener amalgam spaces and sampling estimates

Some close relatives of the modulation spaces are the *Wiener amalgam spaces*. For our purposes, these spaces are interesting because they are associated with certain sampling estimates. We first define, for $1 \le p < \infty$, any lattice Λ and weight function m, the weighted sequence spaces

$$\ell^p_m(\Lambda) = \left\{ \{c_\lambda\}_{\lambda \in \Lambda} \subset \mathbb{C} : \|c\|^p_{\ell^p_m} := \sum_{\lambda \in \Lambda} |c_\lambda|^p m(\lambda)^p < \infty \right\},$$

and $\ell^\infty_m(\Lambda)$ is defined by replacing the sum by a supremum in the usual way.

Given any function $f : \mathbb{R}^{2d} \to \mathbb{C}$ we define a sequence $\{a_{(k,l)}\}_{(k,l) \in \mathbb{Z}^{2d}}$ by

$$a_{(k,l)} = \sup_{x, \omega \in [0,1]^d} |f(x + k, \omega + l)|,$$

the Wiener amalgam space $W(L^p_m)$ on $\mathbb{R}^{2d}$ is then the Banach space of $f : \mathbb{R}^{2d} \to \mathbb{C}$ such that

$$\|f\|_{W(L^p_m)} := \|\{a_{(k,l)}\}\|_{\ell^p_m(\mathbb{Z}^{2d})} < \infty.$$

The following is Proposition 11.1.4 in [131].

Lemma D.3.3. *Let Λ be a lattice in $\mathbb{R}^{2d}$, and assume that $f \in W(L^p_m)$ is continuous. Then*

$$\|f|_\Lambda\|_{\ell^p_m} \lesssim \|f\|_{W(L^p_m)},$$

where the implicit constant may be chosen to be independent of p and m. Since $M^1(\mathbb{R}^{2d}) \hookrightarrow W(L^1_m)$ for $m \equiv 1$, it follows that $\|f|_\Lambda\|_{\ell^1} \lesssim \|f\|_{M^1}$ for $f \in M^1(\mathbb{R}^{2d})$.

By combining [63, Lem. 4.1] with Lemma D.3.3, one obtains the following result.

Lemma D.3.4. *Let Λ be a lattice, $\phi \in M^1_v(\mathbb{R}^d)$ and $\psi \in M^p_m(\mathbb{R}^d)$ where $p \in [1, \infty]$. Then*

$$\|V_\phi \psi|_\Lambda\|_{\ell^p_m(\Lambda)} \lesssim \|\phi\|_{M^1_v} \|\psi\|_{M^p_m},$$

where the implicit constant may be chosen to be independent of p and m.

The symplectic Fourier transform

As the Fourier transform of functions f on $\mathbb{R}^{2d}$, we will use the *symplectic* Fourier transform $\mathcal{F}_\sigma f$, given by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z') e^{-2\pi i \sigma(z,z')} \, dz' \quad \text{for } f \in L^1(\mathbb{R}^{2d}), z \in \mathbb{R}^{2d},$$

where σ is the standard symplectic form $\sigma((x_1, \omega_1), (x_2, \omega_2)) = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$. Then $\mathcal{F}_\sigma$ is an isomorphism on $M^1(\mathbb{R}^{2d})$, and extends to a unitary operator on $L^2(\mathbb{R}^{2d})$ and an isomorphism on $M^\infty(\mathbb{R}^{2d})$ [102, Lem. 7.6.2].

D.3.2 Trace class and Hilbert-Schmidt operators

By the singular value decomposition, see Chapter 3.2 of [54], any compact operator S on $L^2(\mathbb{R}^d)$ may be written as

$$S = \sum_{n=1}^{N_0} s_n \xi_n \otimes \varphi_n$$

for some $N_0 \in \mathbb{N} \cup \{\infty\}$, two orthonormal systems $\{\xi_n\}_{n=1}^{N_0}$, $\{\varphi_n\}_{n=1}^{N_0}$ in $L^2(\mathbb{R}^d)$ and a sequence of positive numbers $\{s_n\}_{n=1}^{N_0} \in \ell^\infty$ called the *singular values* of S. Here $\xi \otimes \varphi$ denotes the rank-one operator $\xi \otimes \varphi(\psi) = \langle \psi, \varphi \rangle_{L^2} \xi$ for $\varphi, \xi, \psi \in L^2(\mathbb{R}^d)$. We assume that $s_{n+1} \leq s_n$ for $n \in \mathbb{N}$.

Imposing summability conditions on the singular values of S allows us to define two important classes of operators. The *trace class operators* $\mathcal{S}^1$ are the operators S whose singular values satisfy $\{s_n\}_{n=1}^{N_0} \in \ell^1$. The norm $\|S\|_{\mathcal{S}^1} = \|\{s_n\}\|_{\ell^1}$ makes $\mathcal{S}^1$

into a Banach space [54]. We may define a bounded linear functional on $\mathcal{S}^1$ called the *trace* by

$$\operatorname{tr}(S) := \sum_{n\in\mathbb{N}} \langle S\eta_n, \eta_n\rangle,$$

where $\{\eta_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ – the value of $\operatorname{tr}(S)$ can be shown to be independent of the orthonormal basis used in its definition [54]. We also mention that the norm on $\mathcal{S}^1$ may be expressed by $\|S\|_{\mathcal{S}}^1 = \operatorname{tr}(|S|)$.

The *Hilbert-Schmidt operators* $\mathcal{HS}$ are the operators S where $\{s_n\}_{n=1}^{N_0} \in \ell^2$. The norm on $\mathcal{HS}$ can be expressed as the ℓ^2 norm of the singular values, but it will be more useful to note that $ST \in \mathcal{S}^1$ for any $S, T \in \mathcal{HS}$ and that $\mathcal{HS}$ becomes a Hilbert space with respect to the inner product [54]

$$\langle S, T\rangle_{\mathcal{HS}} := \operatorname{tr}(ST^*).$$

Another description of $\mathcal{HS}$ is obtained by noting that it is isomorphic to the Hilbert space tensor product $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, where the isomorphism is obtained by associating rank-one operators $\psi \otimes \varphi \in \mathcal{HS}$ with elementary tensors $\psi \otimes \varphi$ [115, Appendix 3].

D.3.3 Pseudodifferential operators

We will consider different ways to associate functions on $\mathbb{R}^{2d}$ with operators $M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$.

Integral operators

For $k \in L^2(\mathbb{R}^{2d})$, we define a necessarily bounded integral operator $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$S\psi(x) = \int_{\mathbb{R}^d} k(x,y)\psi(y)\, dy \quad \text{for } \psi \in L^2(\mathbb{R}^d). \tag{D.3.5}$$

Here $k = k_S$ is the *kernel* of S, and one can extend the definition above to $k \in M^\infty(\mathbb{R}^{2d})$ by defining $S : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ by duality:

$$\langle S\psi, \phi\rangle_{M^\infty, M^1} = \left\langle k, \phi \otimes \overline{\psi}\right\rangle_{M^\infty, M^1} \quad \text{for } \phi, \psi \in M^1(\mathbb{R}^d),$$

where $\phi \otimes \overline{\psi}(x,y) = \phi(x)\overline{\psi(y)}$. By the kernel theorem for modulation spaces [131, Thm. 14.4.1], any continuous linear operator $S : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ is induced by a unique kernel $k = k_S \in M^\infty(\mathbb{R}^{2d})$ in this way. Writing operators using a kernel k will be particularly useful for us because

$$k_{\phi\otimes\psi} = \phi \otimes \overline{\psi} \quad \text{for } \psi, \phi \in L^2(\mathbb{R}^d), \tag{D.3.6}$$

where $\phi \otimes \psi$ on the left side denotes the rank-one operator $\phi \otimes \psi(\xi) = \langle \xi, \psi \rangle_{L^2} \phi$, and on the right side the *function* $\phi \otimes \overline{\psi}(x, y) = \phi(x)\overline{\psi}(y)$. The Hilbert-Schmidt operators are precisely those operators $S : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ such that $k_S \in L^2(\mathbb{R}^{2d})$.

The Weyl calculus

For $\xi, \eta \in L^2(\mathbb{R}^d)$, the *cross-Wigner distribution* $W(\xi, \eta)$ is given by

$$W(\xi, \eta)(x, \omega) = \int_{\mathbb{R}^d} \xi\left(x + \frac{t}{2}\right) \overline{\eta\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t}\, dt \quad \text{for } (x, \omega) \in \mathbb{R}^{2d}.$$

Using the cross-Wigner distribution we introduce the *Weyl calculus*. For $f \in M^\infty(\mathbb{R}^{2d})$ and $\xi, \eta \in M^1(\mathbb{R}^d)$, we define the *Weyl transform* L_f of f to be the operator $L_f : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ given by

$$\left\langle L_f \eta, \xi \right\rangle_{M^\infty, M^1} = \langle f, W(\xi, \eta) \rangle_{M^\infty, M^1}.$$

f is called the *Weyl symbol* of the operator L_f. In general we will use a_S to denote the Weyl symbol of an operator S, in other words $L_{a_S} = S$. By the kernel theorem for modulation spaces, the Weyl transform is a bijection from $M^\infty(\mathbb{R}^{2d})$ to the continuous linear operators $M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$. As above, $\mathcal{HS}$ has a simple description in terms of the Weyl symbol: $S \in \mathcal{HS}$ if and only if $a_S \in L^2(\mathbb{R}^{2d})$.

Translation of operators

Several authors have considered the idea of translating operators by a point $z \in \mathbb{R}^{2d}$ by conjugation with $\pi(z)$ [102,188,251]: if $S : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ is a continuous operator, we define the translation of S by $z \in \mathbb{R}^{2d}$ to be

$$\alpha_z(S) = \pi(z) S \pi(z)^*.$$

This corresponds to a translation of the Weyl symbol [203, Lem. 3.2],

$$\alpha_z(S) = L_{T_z(a_S)}, \tag{D.3.7}$$

which is a major reason why the Weyl symbol is useful for us when considering Fourier series of operators in Section D.6. Since $\pi(z)$ is unitary, α also respects the product of two operators in the sense that

$$\alpha_z(ST) = \alpha_z(S)\alpha_z(T) \quad \text{for } S, T \in \mathcal{L}(L^2). \tag{D.3.8}$$

It is easily shown that α_z is an isometry on $\mathcal{S}^1, \mathcal{HS}$ and $\mathcal{L}(L^2)$ for any $z \in \mathbb{R}^{2d}$ and that applying α_z to a rank-one operator $\psi \otimes \phi$ amounts to a time-frequency shift of ψ and ϕ :

$$\alpha_z(\psi \otimes \phi) = (\pi(z)\psi) \otimes (\pi(z)\phi). \tag{D.3.9}$$

Furthermore, the map $z \mapsto \alpha_z$ is a representation of the locally compact abelian group $\mathbb{R}^{2d}$ on the space of Hilbert-Schmidt operators. In fact, if we identify the Hilbert-Schmidt operators with the Hilbert space tensor product $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, then α is the tensor product representation $\pi \otimes \overline{\pi}$ of $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, which is the notation for α used in [102].

The Fourier-Wigner transform

For a trace class operator $S \in \mathcal{S}^1$, the *Fourier-Wigner transform* $\mathcal{F}_W(S)$ of S is the function

$$\mathcal{F}_W(S)(z) = e^{-\pi i x \cdot \omega} \mathrm{tr}(\pi(-z)S) \quad \text{for } z = (x, \omega) \in \mathbb{R}^{2d}.$$

As a special case, if $\psi, \phi \in L^2(\mathbb{R}^d)$ we have [203, Lem. 6.1] that

$$\mathcal{F}_W(\phi \otimes \psi)(z) = e^{\pi i x \cdot \omega} V_\psi \phi(z) \quad \text{for } z = (x, \omega) \in \mathbb{R}^{2d}, \tag{D.3.10}$$

and we also mention the easily verified relation

$$\mathcal{F}_W(S^*)(z) = \overline{\mathcal{F}_W(S)(-z)} \quad \text{for } z = (x, \omega) \in \mathbb{R}^{2d}. \tag{D.3.11}$$

Werner [251] has shown that in many respects $\mathcal{F}_W$ behaves like a Fourier transform for operators, which is the interpretation we will often rely on. For instance, a Riemann-Lebesgue lemma holds: if $S \in \mathcal{S}^1$, then $\mathcal{F}_W(S) \in C_0(\mathbb{R}^{2d})$ and

$$\|\mathcal{F}_W(S)\|_{L^\infty} \leq \|S\|_{\mathcal{S}^1}. \tag{D.3.12}$$

The Fourier-Wigner transform and Weyl transform are related by a symplectic Fourier transform:

$$\mathcal{F}_W(S) = \mathcal{F}_\sigma(a_S), \tag{D.3.13}$$

which can be used to show that $S \in \mathcal{HS}$ if and only if $\mathcal{F}_W(S) \in L^2(\mathbb{R}^{2d})$. Finally, we remark that $\mathcal{F}_W(S)$ differs only by a phase factor $e^{-\pi i x \cdot \omega}$ from the *spreading function* of S [29, 102].

Localization operators

An important class of examples of pseudodifferential operators in this paper will be the *localization operators*. Given $\varphi \in L^2(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^{2d})$, the localization operator $A_h^\varphi \in \mathcal{L}(L^2)$ is defined by

$$A_h^\varphi \psi = \int_{\mathbb{R}^{2d}} h(z) V_\varphi \psi(z) \pi(z) \varphi \, dz \quad \text{for } \psi \in L^2(\mathbb{R}^d),$$

where the integral is an absolutely convergent Bochner integral in $L^2(\mathbb{R}^d)$. Localization operators interact nicely with the various aspects of pseudodifferential operators considered above: their Weyl symbol is given by a convolution [48]

$$a_{A_h^\varphi} = h * W(\varphi, \varphi)$$

and they satisfy the translation covariance property [203, Lem. 4.3 and Thm. 5.1]

$$\alpha_z(A_h^\varphi) = A_{T_z h}^\varphi \tag{D.3.14}$$

D.3.4 Frames and g-frames

We will briefly recall the basic definitions of frame theory in the Hilbert space $L^2(\mathbb{R}^d)$, referring the details to the monographs [57, 131, 156]. Recall that a sequence $\{\xi_i\}_{i\in I} \subset L^2(\mathbb{R}^d)$ is a *frame* for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{i\in I} |\langle \psi, \xi_i\rangle_{L^2}| \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d). \tag{D.3.15}$$

Here A and B are called the lower and upper frame bound, respectively. If (D.3.15) holds with $A = B$, we say that $\{\xi_i\}_{i\in I}$ is a *tight* frame, and if $A = B = 1$ we call $\{\xi_i\}_{i\in I}$ a *Parseval frame*. Whenever the rightmost inequality in (D.3.15) holds for some $B > 0$, $\{\xi_i\}_{i\in I}$ is a *Bessel system*.

When $\{\xi_i\}_{i\in I}$ is a Bessel system, we associated with $\{\xi_i\}_{i\in I}$ several bounded operators: the *analysis operator* $C : L^2(\mathbb{R}^d) \to \ell^2(I)$ given by

$$C\psi = \{\langle \psi, \xi_i\rangle_{L^2}\}_{i\in I} \quad \text{for } \psi \in L^2(\mathbb{R}^d),$$

the *synthesis operator* $D : \ell^2(I) \to L^2(\mathbb{R}^d)$ given by

$$D\{c_i\}_{i\in I} = \sum_{i\in I} c_i \xi_i \quad \text{for } \{c_i\}_{i\in I} \in \ell^2(I)$$

and the *frame operator* $\mathfrak{S} = DC \in \mathcal{L}(L^2)$ defined by

$$\mathfrak{S}(\psi) = \sum_{i\in I} \langle \psi, \xi_i\rangle_{L^2} \xi_i \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

In the introduction, see equation (D.1.1), we introduced a special class of frames called *Gabor frames*, which are frames of the form $\{\pi(\lambda)\varphi\}_{\lambda\in\Lambda}$ for some lattice Λ and $\varphi \in L^2(\mathbb{R}^d)$. More generally, a *multi-window Gabor frame* [257] is a frame of the form $\{\pi(\lambda)\varphi_n\}_{\lambda\in\Lambda, n=1,\dots N}$ where $\varphi_n \in L^2(\mathbb{R}^d)$ for $n = 1, \dots, N$. We call the set $\{\pi(\lambda)\varphi_n\}_{\lambda\in\Lambda, n=1,\dots N}$ the *multi-window Gabor system generated by* $\{\varphi_n\}_{n=1}^N$, even when $\{\pi(\lambda)\varphi_n\}_{\lambda\in\Lambda, n=1,\dots N}$ is not a frame.

g-frames

In [239], Sun introduced *g-frames* as a generalization of frames for Hilbert spaces. We state a special case[1] for the Hilbert space $L^2(\mathbb{R}^d)$. A sequence $\{A_i\}_{i\in I} \subset \mathcal{L}(L^2)$ is a g-frame for $L^2(\mathbb{R}^d)$ with respect to $L^2(\mathbb{R}^d)$ if there exist positive constants A, B such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{i\in I} \|A_i\psi\|_{L^2}^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d).$$

If we can choose $A = B$, we say that the g-frame is *tight*. When the above inequality holds, the g-frame operator $\mathfrak{S}$ defined by

$$\mathfrak{S}\psi = \sum_{i\in\mathbb{N}} A_i^* A_i \psi$$

is positive, bounded and invertible on $L^2(\mathbb{R}^d)$ with $A \leq \|\mathfrak{S}\|_{\mathcal{L}(L^2)} \leq B$.

D.4 The space $\mathcal{B}_{v\otimes v}$ of operators with kernel in $M^1_{v\otimes v}$

To define a suitable class of operators for our purposes, we will consider modulation spaces on $\mathbb{R}^{2d}$. The short-time Fourier transform on phase space $\mathbb{R}^{2d}$ is

$$\mathcal{V}_g f(z,\zeta) = \langle f, \pi(z)\otimes\pi(\zeta)g\rangle_{L^2} \quad \text{for } z,\zeta \in \mathbb{R}^{2d} \text{ and } f,g \in L^2(\mathbb{R}^{2d}),$$

where $\pi(z)\otimes\pi(\zeta)$ is defined by

$$\pi(z)\otimes\pi(\zeta)g = M_{(z_\omega,\zeta_\omega)}T_{(z_x,\zeta_x)}g \quad \text{for } z = (z_x, z_\omega), \zeta = (\zeta_x, \zeta_\omega).$$

Given a submultiplicative, symmetric GRS-weight v on $\mathbb{R}^{2d}$, we consider the Banach space $M^1_{v\otimes v}(\mathbb{R}^{2d})$ of $f \in L^2(\mathbb{R}^{2d})$ such that

$$\|f\|_{M^1_{v\otimes v}} = \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |\mathcal{V}_{\varphi_0\otimes\varphi_0} f(z,\zeta)| v(z)v(\zeta)\, dz d\zeta < \infty,$$

where $\varphi_0\otimes\varphi_0(x,y) = \varphi_0(x)\varphi_0(y)$. With these definitions it is easy to show that if $\phi,\psi \in M^1_v(\mathbb{R}^d)$, then $\phi\otimes\psi \in M^1_{v\otimes v}(\mathbb{R}^{2d})$ with

$$\|\psi\otimes\phi\|_{M^1_{v\otimes v}} = \|\psi\|_{M^1_v}\|\phi\|_{M^1_v}. \tag{D.4.1}$$

[1] More generally, we could consider $A_i \in \mathcal{L}(\mathcal{H}, V_i)$ where $\mathcal{H}$ is a Hilbert space and V_i is a closed subspace of another Hilbert space $\mathcal{H}'$, see [239].

In fact, $M^1_{v\otimes v}(\mathbb{R}^{2d})$ is isomorphic to $M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$ [19, Thm. 5], where $\hat{\otimes}$ denotes the projective tensor product of Banach spaces. This tensor product construction is covered in detail in [228], but for our purposes it suffices to note that

$$\begin{aligned} M^1_{v\otimes v}(\mathbb{R}^{2d}) &= M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d) \\ &= \left\{ \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)} : \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} < \infty \right\}, \end{aligned} \tag{D.4.2}$$

with an equivalent norm for $M^1_{v\otimes v}(\mathbb{R}^{2d})$ given by

$$\|f\|_{M^1_{v\otimes v}} \asymp \inf \left\{ \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} \right\}, \tag{D.4.3}$$

where the infimum is taken over all sequences $\{\phi_n^{(1)}\}_{n\in\mathbb{N}}, \{\phi_n^{(2)}\}_{n\in\mathbb{N}}$ in $M^1_v(\mathbb{R}^d)$ such that $f = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}$ and $\sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} < \infty$.

We will be particularly interested in the class of operators S whose kernel k_S belongs to $M^1_{v\otimes v}(\mathbb{R}^{2d})$, as studied by several authors [102, 168, 193] for $v \equiv 1$. We denote the class of such operators by $\mathcal{B}_{v\otimes v}$, and define the norm

$$\|S\|_{\mathcal{B}_{v\otimes v}} := \|k_S\|_{M^1_{v\otimes v}}.$$

Since $M^1_{v\otimes v}(\mathbb{R}^{2d}) \hookrightarrow L^2(\mathbb{R}^{2d})$, operators in $\mathcal{B}_{v\otimes v}$ define bounded operators on $L^2(\mathbb{R}^d)$ by (D.3.5). In fact (see [129, 138]) we have $\mathcal{B}_{v\otimes v} \hookrightarrow \mathcal{S}^1 \hookrightarrow \mathcal{L}(L^2)$, hence

$$\|S\|_{\mathcal{L}(L^2)} \leq \|S\|_{\mathcal{S}^1} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}} \quad \text{for } S \in \mathcal{B}_{v\otimes v}.$$

Now recall from (D.3.6) that the kernel of a rank-one operator $\phi\otimes\psi$ with $\psi, \phi \in M^1_v(\mathbb{R}^d)$ is the function $\phi\otimes\overline{\psi}$. By (D.4.1) we get that

$$\|\phi\otimes\psi\|_{\mathcal{B}_{v\otimes v}} = \|\phi\|_{M^1_v}\|\psi\|_{M^1_v}$$

(we have also used that $\|\overline{\psi}\|_{M^1_v} = \|\psi\|_{M^1_v}$ as v is symmetric). Equation (D.4.2) therefore has the following important consequences.

Proposition D.4.1. *Let $S \in \mathcal{B}_{v\otimes v}$.*

(a) There exist sequences $\{\phi_n^{(1)}\}_{n\in\mathbb{N}}, \{\phi_n^{(2)}\}_{n\in\mathbb{N}} \subset M^1_v(\mathbb{R}^d)$ with

$$\sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} < \infty$$

such that S can be written as a sum of rank-one operators

$$S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}. \tag{D.4.4}$$

The decomposition (D.4.4) *converges absolutely in $\mathcal{B}_{v\otimes v}$, hence in $\mathcal{S}^1$ and $\mathcal{L}(L^2)$.*

(b)

$$\|S\|_{\mathcal{B}_{v\otimes v}} \asymp \inf \left\{ \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} \right\},$$

with infimum taken over all sequences $\{\phi_n^{(1)}\}_{n\in\mathbb{N}}, \{\phi_n^{(2)}\}_{n\in\mathbb{N}}$ as in (a).

(c) Let S^* denote the Hilbert space adjoint of S when S is viewed as an operator $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Then $S^* \in \mathcal{B}_{v\otimes v}$ and S extends to a weak-to-weak*-continuous operator $S : M^\infty_{1/v}(\mathbb{R}^d) \to M^\infty_{1/v}(\mathbb{R}^d)$ by defining*

$$\langle S\phi, \psi\rangle_{M^\infty_{1/v}, M^1_v} = \langle \phi, S^*\psi\rangle_{M^\infty_{1/v}, M^1_v} \quad \text{for } \phi \in M^\infty_{1/v}(\mathbb{R}^d), \psi \in M^1_v(\mathbb{R}^d).$$

The decomposition in (D.4.4) *still holds for this extensions of S, meaning that*

$$S\psi = \sum_{n\in\mathbb{N}} \left\langle \psi, \phi_n^{(2)} \right\rangle_{M^\infty_{1/v}, M^1_v} \phi_n^{(1)} \quad \text{for } \psi \in M^\infty_{1/v}(\mathbb{R}^d)$$

with absolute convergence of the sum in the norm of $M^1_v(\mathbb{R}^d)$.

(d) The extension of S to $M^\infty_{1/v}(\mathbb{R}^d)$ is bounded from $M^\infty_{1/v}(\mathbb{R}^d)$ into $M^1_v(\mathbb{R}^d)$, and maps weak-convergent sequences in $M^\infty_{1/v}(\mathbb{R}^d)$ to norm-convergent sequences in $M^1_v(\mathbb{R}^d)$.*

Proof. (a) By (D.4.2), there exist $\{\phi_n^{(1)}\}_{n\in N}, \{\phi_n^{(2)}\}_{n\in N}$ as in the statement with

$$k_S(x, y) = \sum_{n\in\mathbb{N}} \phi_n^{(1)}(x)\overline{\phi_n^{(2)}(y)} \quad \text{for } x, y \in \mathbb{R}^d,$$

with absolute convergence of the sum in the norm of $M^1_{v\otimes v}(\mathbb{R}^{2d})$ by (D.4.1). Since the function $\phi_n^{(1)}(x)\overline{\phi_n^{(2)}(y)}$ is the kernel of the rank-one operator $\phi_n^{(1)} \otimes \phi_n^{(2)}$ by (D.3.6), the decomposition of k_S above and the definition of $\|\cdot\|_{\mathcal{B}_{v\otimes v}}$ implies that

$$S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)},$$

with absolute convergence in the norm of $\mathcal{B}_{v\otimes v}$.

(b) Follows from (D.4.3) and $\|S\|_{\mathcal{B}_{v\otimes v}} = \|k_S\|_{M^1_{v\otimes v}}$.

(c) It is well-known that the kernel of S^* is $k_{S^*}(x, y) = \overline{k_S(y, x)}$. Since $M^1_{v\otimes v}(\mathbb{R}^{2d})$ is closed under this operation – as follows from (D.4.2), for instance – we get $S^* \in \mathcal{B}_{v\otimes v}$. In particular, part (a) applied to S^* implies that S^* is bounded $M^1_v(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)$. We may therefore define an extension

$\tilde{S} : M^{\infty}_{1/v}(\mathbb{R}^d) \to M^{\infty}_{1/v}(\mathbb{R}^d)$ by defining $\tilde{S}$ to be the Banach space adjoint of S^*. By definition, this means that

$$\left\langle \tilde{S}\phi, \psi \right\rangle_{M^{\infty}_{1/v}, M^1_v} = \langle \phi, S^*\psi \rangle_{M^{\infty}_{1/v}, M^1_v} .$$

It is easy to see that $\tilde{S}$ is an extension of S: if $\phi \in L^2(\mathbb{R}^d)$, we find that

$$\begin{aligned} \left\langle \tilde{S}\phi, \psi \right\rangle_{M^{\infty}_{1/v}, M^1_v} &= \langle \phi, S^*\psi \rangle_{M^{\infty}_{1/v}, M^1_v} \\ &= \langle \phi, S^*\psi \rangle_{L^2} \\ &= \langle S\phi, \psi \rangle_{L^2} \\ &= \langle S\phi, \psi \rangle_{M^{\infty}_{1/v}, M^1_v} . \end{aligned}$$

From now on, we simply denote the extension $\tilde{S}$ by S. For the last part, note that S^* has a decomposition $S^* = \sum_{n\in\mathbb{N}} \phi_n^{(2)} \otimes \phi_n^{(1)}$ by part (a). By definition, for $\psi \in M^{\infty}_{1/v}(\mathbb{R}^d)$, we have

$$\langle S\psi, \phi \rangle_{M^{\infty}_{1/v}, M^1_v} = \langle \psi, S^*\phi \rangle_{M^{\infty}_{1/v}, M^1_v} .$$

By the decomposition above, $S^*\phi = \sum_{n=1}^{\infty} \left\langle \phi, \phi_n^{(1)} \right\rangle_{L^2} \phi_n^{(2)}$, and as this sum converges absolutely in the norm of $M^1_v(\mathbb{R}^d)$ we find

$$\begin{aligned} \langle S\psi, \phi \rangle_{M^{\infty}_{1/v}, M^1_v} &= \langle \psi, S^*\phi \rangle_{M^{\infty}_{1/v}, M^1_v} \\ &= \left\langle \psi, \sum_{n=1}^{\infty} \left\langle \phi, \phi_n^{(1)} \right\rangle_{L^2} \phi_n^{(2)} \right\rangle_{M^{\infty}_{1/v}, M^1_v} \\ &= \sum_{n=1}^{\infty} \left\langle \phi_n^{(1)}, \phi \right\rangle_{L^2} \left\langle \psi, \phi_n^{(2)} \right\rangle_{M^{\infty}_{1/v}, M^1_v} \\ &= \left\langle \sum_{n=1}^{\infty} \left\langle \psi, \phi_n^{(2)} \right\rangle_{M^{\infty}_{1/v}, M^1_v} \phi_n^{(1)}, \phi \right\rangle_{M^{\infty}_{1/v}, M^1_v} . \end{aligned}$$

The absolute convergence in the norm of $M^1_v(\mathbb{R}^d)$ follows as

$$\sum_{n=1}^{\infty} \left| \left\langle \psi, \phi_n^{(2)} \right\rangle_{M^{\infty}_{1/v}, M^1_v} \right| \|\phi_n^{(1)}\|_{M^1_v} \leq \|\psi\|_{M^{\infty}_{1/v}} \sum_{n=1}^{\infty} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} < \infty.$$

(d) The last inequality above also implies that S is bounded from $M^{\infty}_{1/v}(\mathbb{R}^d)$ to $M^1_v(\mathbb{R}^d)$, since it shows that

$$\|S\psi\|_{M^1_v} \leq \|\psi\|_{M^{\infty}_{1/v}} \sum_{n=1}^{\infty} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v}.$$

Finally, let $\{\psi_i\}_{i\in\mathbb{N}}$ be a sequence in $M^\infty_{1/v}(\mathbb{R}^d)$ that converges to $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$ in the weak* topology. Then

$$S\psi_i = \sum_{n\in\mathbb{N}} \left\langle \psi_i, \phi_n^{(2)} \right\rangle_{M^\infty_{1/v}, M^1_v} \phi_n^{(1)} \xrightarrow{i\to\infty} \sum_{n\in\mathbb{N}} \left\langle \psi, \phi_n^{(2)} \right\rangle_{M^\infty_{1/v}, M^1_v} \phi_n^{(1)} = S\psi.$$

We have used the dominated convergence theorem for Banach spaces [166, Prop. 1.2.5] to take the limit inside the sum: as $\{\psi_i\}_{i\in\mathbb{N}}$ is weak*-convergent there exists $0 < C < \infty$ such that $\|\psi_i\|_{M^\infty_{1/v}} \leq C$ for any i, so

$$\left\| \left\langle \psi_i, \phi_n^{(2)} \right\rangle_{M^\infty_{1/v}, M^1_v} \phi_n^{(1)} \right\|_{M^1_v} \leq C \|\phi_n^{(2)}\|_{M^1_v} \|\phi_n^{(1)}\|_{M^1_v}$$

for any i, and $\sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} < \infty$. □

As a first consequence, we show that $\mathcal{B}_{v\otimes v}$ is closed under composition. The proof is similar to that of [168, Cor. 3.11], where the result is proved for locally compact abelian groups with no weights.

Corollary D.4.1.1. *$\mathcal{B}_{v\otimes v}$ is closed under composition: if $S, T \in \mathcal{B}_{v\otimes v}$, then*

$$\|ST\|_{\mathcal{B}_{v\otimes v}} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}} \|T\|_{\mathcal{B}_{v\otimes v}}.$$

Proof. Let

$$S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}, \qquad T = \sum_{m\in\mathbb{N}} \psi_m^{(1)} \otimes \psi_m^{(2)}$$

be decompositions of S and T into rank-one operators as in Proposition D.4.1. A simple calculation shows that the composition ST is the operator

$$ST = \sum_{m,n\in\mathbb{N}} \left\langle \psi_m^{(1)}, \phi_n^{(2)} \right\rangle_{L^2} \phi_n^{(1)} \otimes \psi_m^{(2)}.$$

This decomposition converges absolutely in $\mathcal{B}_{v\otimes v}$, as

$$\begin{aligned}\left\| \left\langle \psi_m^{(1)}, \phi_n^{(2)} \right\rangle_{L^2} \phi_n^{(1)} \otimes \psi_m^{(2)} \right\|_{\mathcal{B}_{v\otimes v}} &\leq \left| \left\langle \psi_m^{(1)}, \phi_n^{(2)} \right\rangle_{L^2} \right| \left\| \phi_n^{(1)} \otimes \psi_m^{(2)} \right\|_{\mathcal{B}_{v\otimes v}} \\ &\leq \|\psi_m^{(1)}\|_{L^2} \|\phi_n^{(2)}\|_{L^2} \|\phi_n^{(1)}\|_{M^1_v} \|\psi_m^{(2)}\|_{M^1_v},\end{aligned}$$

so that

$$\sum_{m,n\in\mathbb{N}} \left\| \left\langle \psi_m^{(1)}, \phi_n^{(2)} \right\rangle_{L^2} \phi_n^{(1)} \otimes \psi_m^{(2)} \right\|_{\mathcal{B}_{v\otimes v}}$$

is bounded from above by

$$\sum_{m\in\mathbb{N}} \|\psi_m^{(1)}\|_{L^2}\|\psi_m^{(2)}\|_{M_v^1} \sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{L^2}\|\phi_n^{(1)}\|_{M_v^1} < \infty.$$

We have used the continuous inclusion (see Proposition D.3.2)

$$M_v^1(\mathbb{R}^d) \hookrightarrow M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$$

to obtain $\|\psi_m^{(1)}\|_{L^2} \lesssim \|\psi_m^{(1)}\|_{M_v^1}$ and $\|\phi_n^{(2)}\|_{L^2} \lesssim \|\phi_n^{(2)}\|_{M_v^1}$. Finally, the inequality $\|ST\|_{\mathcal{B}_{v\otimes v}} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}\|T\|_{\mathcal{B}_{v\otimes v}}$ follows from part (b) of Proposition D.4.1. □

Remark D.2. In [102, Thm. 7.4.1] it is claimed that $\mathcal{B}_{v\otimes v}$ for $v \equiv 1$ is even an ideal in $\mathcal{L}(L^2)$. This is not true. Consider $S = \psi \otimes \varphi_0$ and $T = \varphi_0 \otimes \varphi_0$ where $\psi \in L^2(\mathbb{R}^d) \setminus M^1(\mathbb{R}^d)$. Then $T \in \mathcal{B}_{1\otimes 1}$ and $S \in \mathcal{S}^1$, and $ST = \psi \otimes \varphi_0$. Yet $ST(\varphi_0) = \psi \notin M^1(\mathbb{R}^d)$, so part (d) of Proposition D.4.1 implies that $ST \notin \mathcal{B}_{1\otimes 1}$.

We next study a continuity property of the Fourier-Wigner transform on $\mathcal{B}_{v\otimes v}$.

Proposition D.4.2. *The Fourier-Wigner transform is bounded from $\mathcal{B}_{v\otimes v}$ into $W(L_v^1)$:*

$$\|\mathcal{F}_W(S)\|_{W(L_v^1)} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}.$$

Proof. First consider the rank-one operator $\psi \otimes \phi \in \mathcal{B}_{v\otimes v}$, with $\psi, \phi \in M_v^1(\mathbb{R}^d)$. By (D.3.10) and the proof of [131, Prop. 12.1.11], there exists $C > 0$ such that

$$\|\mathcal{F}_W(\psi \otimes \phi)\|_{W(L_v^1)} \leq C\|\psi\|_{M_v^1}\|\phi\|_{M_v^1}.$$

If we then use Proposition D.4.1 to write $S \in \mathcal{B}_{v\otimes v}$ as $S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}$, we find

$$\|\mathcal{F}_W(S)\|_{W(L_v^1)} \leq \sum_{n\in\mathbb{N}} \|\mathcal{F}_W(\phi_n^{(1)} \otimes \phi_n^{(2)})\|_{W(L_v^1)} \leq C \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M_v^1}\|\phi_n^{(2)}\|_{M_v^1}.$$

By part (b) of Proposition D.4.1 this implies that $\|\mathcal{F}_W(S)\|_{W(L_v^1)} \leq C\|S\|_{\mathcal{B}_{v\otimes v}}$. □

Remark D.3. If we consider the polynomial weights $v_s(z) = (1 + |z|)^s$ for $s \geq 0$ and $z \in \mathbb{R}^{2d}$, it is known [131, Prop. 11.3.1] that the space of Schwartz functions $\mathcal{S}(\mathbb{R}^{2d})$ is given by $\mathcal{S}(\mathbb{R}^{2d}) = \cap_{s=0}^{\infty} M^1_{v_s\otimes v_s}(\mathbb{R}^{2d})$. Therefore the space of operators with kernel in $\mathcal{S}(\mathbb{R}^{2d})$ equals $\cap_{s=0}^{\infty}\mathcal{B}_{v_s\otimes v_s}$. Such operators were recently studied in [179].

D.4.1 The space $\mathcal{B}$ and its dual

The largest of the spaces $\mathcal{B}_{v\otimes v}$ is the space $\mathcal{B} := \mathcal{B}_{1\otimes 1}$, consisting of operators S with kernel k_S in $M^1(\mathbb{R}^{2d})$. By definition the map $\kappa : \mathcal{B} \to M^1(\mathbb{R}^{2d})$ given by $\kappa(S) = k_S$ is an isometric isomorphism of Banach spaces. By [210, Thm. 3.1.18] the Banach space adjoint $(\kappa^{-1})^* : \mathcal{B}' \to M^\infty(\mathbb{R}^{2d})$ is a weak*-to-weak*-continuous isometric isomorphism, and by definition it satisfies

$$\left\langle (\kappa^{-1})^*(\tilde{A}), k_S \right\rangle_{M^\infty,M^1} = \left\langle \tilde{A}, S \right\rangle_{\mathcal{B}',\mathcal{B}} \qquad \text{for } \tilde{A} \in \mathcal{B}', S \in \mathcal{B}. \tag{D.4.5}$$

Hence, to any $\tilde{A} \in \mathcal{B}'$ we obtain a unique element $(\kappa^{-1})^*(\tilde{A}) \in M^\infty(\mathbb{R}^{2d})$, which by the kernel theorem for modulation spaces induces an operator $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ such that $k_A = (\kappa^{-1})^*(\tilde{A})$. We summarize these identifications in a simple diagram, where k.t. refers to the kernel theorem for modulation spaces:

$$\tilde{A} \in \mathcal{B}' \xleftrightarrow{(\kappa^{-1})^*} (\kappa^{-1})^*(\tilde{A}) = k_A \in M^\infty(\mathbb{R}^{2d}) \xleftrightarrow{\text{k.t.}} A \in \mathcal{L}(M^1, M^\infty). \tag{D.4.6}$$

Hereafter we will always identify $\mathcal{B}'$ with operators $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$, and use the notation A to refer to both the operator $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ and the abstract functional $\tilde{A}$, which are related by (D.4.6). Since $(\kappa^{-1})^*(\tilde{A}) = k_A$, (D.4.5) becomes

$$\langle A, S \rangle_{\mathcal{B}',\mathcal{B}} = \langle k_A, k_S \rangle_{M^\infty,M^1} \qquad \text{for } A \in \mathcal{B}', S \in \mathcal{B}. \tag{D.4.7}$$

If S is a rank-one operator $S = \phi \otimes \psi$ for $\phi, \psi \in M^1(\mathbb{R}^d)$, then $k_S = \phi \otimes \overline{\psi}$, so the equation above becomes

$$\langle A, \phi \otimes \psi \rangle_{\mathcal{B}',\mathcal{B}} = \left\langle k_A, \phi \otimes \overline{\psi} \right\rangle_{M^\infty,M^1} = \langle A\psi, \phi \rangle_{M^\infty,M^1}, \tag{D.4.8}$$

which relates the action of A as an abstract linear functional on $\mathcal{B}$ to the action of A as an operator from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$.

Lemma D.4.3. *$\mathcal{B}$ is a dense subset of $\mathcal{S}^1$ with respect to $\|\cdot\|_{\mathcal{S}^1}$.*

Proof. The rank-one operators span a dense subset of $\mathcal{S}^1$ [54, Thm. 3.11 (e)], hence it suffices to show that any $\psi \otimes \phi \in \mathcal{S}^1$ with $\psi, \phi \in L^2(\mathbb{R}^d)$ can be estimated by some $S \in \mathcal{B}$. We may safely assume that $\phi \neq 0$, otherwise $\psi \otimes \phi = 0 \in \mathcal{B}$. Let $\epsilon > 0$. Since $M^1(\mathbb{R}^d)$ is a dense subset of $L^2(\mathbb{R}^d)$ by [167, Lem. 4.19], we can find $\xi \neq 0, \eta \in M^1(\mathbb{R}^d)$ with $\|\psi - \xi\|_{L^2} < \frac{\epsilon}{2\|\phi\|_{L^2}}$ and $\|\phi - \eta\|_{L^2} < \frac{\epsilon}{2\|\xi\|_{L^2}}$. Then $\xi \otimes \phi \in \mathcal{B}$ and

$$\begin{aligned} \|\psi \otimes \phi - \xi \otimes \eta\|_{\mathcal{S}^1} &\leq \|\psi \otimes \phi - \xi \otimes \phi\|_{\mathcal{S}^1} + \|\xi \otimes \phi - \xi \otimes \eta\|_{\mathcal{S}^1} \\ &= \|\psi - \xi\|_{L^2}\|\phi\|_{L^2} + \|\xi\|_{L^2}\|\phi - \eta\|_{L^2} < \epsilon. \end{aligned}$$

□

Now recall that $\mathcal{L}(L^2)$ is the dual space of $\mathcal{S}^1$ [54, Thm. 3.13], where $A \in \mathcal{L}(L^2)$ acts on $S \in \mathcal{S}^1$ by

$$\langle A, S\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} = \operatorname{tr}(AS^*). \tag{D.4.9}$$

Since the inclusion $\mathcal{B} \hookrightarrow \mathcal{S}^1$ has dense range, [210, Thm. 3.1.17] asserts that we get a weak*-to-weak*-continuous inclusion of dual spaces $\mathcal{L}(L^2) \hookrightarrow \mathcal{B}'$ satisfying

$$\langle A, S\rangle_{\mathcal{B}',\mathcal{B}} = \langle A, S\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} = \operatorname{tr}(AS^*) \quad \text{for } A \in \mathcal{L}(L^2), S \in \mathcal{B}. \tag{D.4.10}$$

Remark D.4. Readers with little interest in these technical details need only note that we identify $\mathcal{B}'$ with operators $A \in \mathcal{L}(M^1(\mathbb{R}^d), M^\infty(\mathbb{R}^d))$, and that the action of A satisfies (D.4.7), (D.4.8) and (D.4.10).

The next result is due to Feichtinger and Kozek [102]; in their terminology the result says that $\mathcal{F}_W$ and the Weyl transform are *Gelfand triple isomorphisms.* Recall that $\mathcal{HS}$ are the Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$.

Proposition D.4.4. *The Weyl transform $S \longleftrightarrow a_s$ and Fourier-Wigner transform $S \longleftrightarrow \mathcal{F}_W(S)$ are isomorphisms $\mathcal{B} \longleftrightarrow M^1(\mathbb{R}^{2d})$, unitary maps $\mathcal{HS} \longleftrightarrow L^2(\mathbb{R}^{2d})$ and weak*-to-weak*-continuous isomorphisms $\mathcal{B}' \longleftrightarrow M^\infty(\mathbb{R}^{2d})$.*

An appropriate framework for such statements is the theory of (Banach) Gelfand triples [62, 99, 102]. In particular, that approach gives the duality bracket identity

$$\langle S, T\rangle_{\mathcal{B}',\mathcal{B}} = \langle a_S, a_T\rangle_{M^\infty,M^1}, \tag{D.4.11}$$

where a_S and a_T are the Weyl symbols of S and T, see [62, Cor. 5].

Remark D.5. We will often consider weak*-convergence of sequences in $\mathcal{B}'$. To get a better grasp of this notion of convergence, note that if a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}'$ converges in the weak* topology to $A \in \mathcal{B}'$ then (D.4.8) gives for $\psi, \phi \in M^1(\mathbb{R}^d)$

$$\langle A_n\psi, \phi\rangle_{M^\infty,M^1} \to \langle A\psi, \phi\rangle_{M^\infty,M^1}.$$

Hence: if $A_n \to A$ in the weak* topology of $\mathcal{B}'$, then $A_n\psi \to A\psi$ in the weak* topology of $M^\infty(\mathbb{R}^d)$ for any $\psi \in M^1(\mathbb{R}^d)$.

D.5 Gabor g-frames

Gabor frames, or more generally multi-window Gabor frames, have a richer structure than general frames. Since any frame is also a g-frame, we can ask whether Gabor frames belong to a certain class of g-frames, and whether this class contains other g-frames that share the rich structure of Gabor frames. This is the motivation for the following definition.

Definition D.5.1. Let Λ be a lattice in $\mathbb{R}^{2d}$ and $S \in \mathcal{L}(L^2)$. We say that S generates a *Gabor g-frame* with respect to Λ if $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a g-frame for $L^2(\mathbb{R}^d)$, i.e. if there exist positive constants $A, B > 0$ such that

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\psi\|_{L^2}^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d). \tag{D.5.1}$$

Remark D.6 (Cohen's class). This definition may also be rephrased in terms of Cohen's class of time-frequency distributions [59]. In the notation from [204] an operator $T \in \mathcal{L}(L^2)$ defines a Cohen's class distribution Q_T by

$$Q_T(\psi)(z) = \langle T\pi(z)^*\psi, \pi(z)^*\psi\rangle_{L^2} \qquad \text{for } z \in \mathbb{R}^{2d}, \psi \in L^2(\mathbb{R}^d).$$

It is straightforward to show that

$$\|\alpha_z(S)\psi\|_{L^2}^2 = Q_{S^*S}(\psi)(z),$$

hence (D.5.1) may be rephrased as

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} Q_{S^*S}(\psi)(\lambda) \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d).$$

We will soon see that (D.5.1) forces S to be a Hilbert Schmidt operator, which implies by [204, Thm. 7.6] that Q_{S^*S} is a positive Cohen's class distribution satisfying

$$\int_{\mathbb{R}^{2d}} Q_{S^*S}(\psi)(z)\, dz = \int_{\mathbb{R}^{2d}} \|\alpha_z(S)\psi\|_{L^2}^2\, dz = \|S\|_{\mathcal{HS}}^2\|\psi\|_{L^2}^2, \tag{D.5.2}$$

as recently studied in [204, 205]. This equality is a continuous version of (D.5.1), similar to how Moyal's identity is a continuous version[2] of the Gabor frame inequalities (D.1.1). The simplest example of a Cohen's class distribution of the form Q_{S^*S} is the spectrogram $Q_{S^*S}(z) = |V_\phi\psi(z)|^2$ for some $\phi \in L^2(\mathbb{R}^d)$, which corresponds to the rank-one operator $S = \frac{1}{\|\phi\|_{L^2}^2}\phi \otimes \phi$. By inserting $\|\alpha_z(S)\psi\|_{L^2}^2 = Q_{S^*S}(\psi)(z) = |V_\phi\psi(z)|^2$, (D.5.1) becomes the condition for ϕ to generate a Gabor frame. We return to this special case in Example D.5.1.

[2]In fact, Moyal's identity says that the system $\{\pi(z)\varphi\}_{z\in\mathbb{R}^{2d}}$ is a *tight continuous frame* for $L^2(\mathbb{R}^d)$ for any $0 \neq \varphi \in L^2(\mathbb{R}^d)$. See [57] for continuous frames. Similarly, (D.5.2) says that $\{\alpha_z(S)\}_{z\in\mathbb{R}^{2d}}$ is a *tight continuous g-frame* as introduced in [2].

D.5.1 The Gabor g-frame operator

By the general theory of g-frames, the g-frame operator associated to a Gabor g-frame generated by S over a lattice Λ is the operator

$$\mathfrak{S}_S = \sum_{\lambda\in\Lambda} (\alpha_\lambda(S))^*(\alpha_\lambda(S)) = \sum_{\lambda\in\Lambda} \alpha_\lambda(S^*S), \tag{D.5.3}$$

where the last equality uses (D.3.8). Furthermore, $\mathfrak{S}_S$ satisfies

$$\langle \mathfrak{S}_S\psi, \psi\rangle_{L^2} = \sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\psi\|_{L^2}^2 \quad \text{for } \psi \in L^2(\mathbb{R}^d),$$

and $\mathfrak{S}_S$ is positive, bounded and invertible on $L^2(\mathbb{R}^d)$ with $A \leq \|\mathfrak{S}_S\|_{\mathcal{L}(L^2)} \leq B$ and $\frac{1}{B} \leq \|\mathfrak{S}_S^{-1}\| \leq \frac{1}{A}$. Since we think of $\alpha_\lambda(S^*S)$ as the translation of S^*S by $\lambda \in \Lambda$, the g-frame operator $\mathfrak{S}_S$ corresponds to the *periodization* of S^*S over Λ.

D.5.2 Analysis and synthesis operators

Let $\ell^2(\Lambda; L^2(\mathbb{R}^d))$ be the Hilbert space of sequences $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^d)$ such that

$$\|\{\psi_\lambda\}\|_{\ell^2(\Lambda;L^2)} := \left(\sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2}^2\right)^{1/2} < \infty,$$

with inner product

$$\langle \{\psi_\lambda\}, \{\phi_\lambda\}\rangle_{\ell^2(\Lambda;L^2)} = \sum_{\lambda\in\Lambda} \langle \psi_\lambda, \phi_\lambda\rangle_{L^2} .$$

For $S \in \mathcal{L}(L^2)$ we define the *analysis operator* C_S by

$$C_S(\psi) = \{\alpha_\lambda(S)\psi\}_{\lambda\in\Lambda} \quad \text{for } \psi \in L^2(\mathbb{R}^d)$$

and the *synthesis operator* D_S by

$$D_S(\{\psi_\lambda\}) := \sum_{\lambda\in\Lambda} \alpha_\lambda(S^*)\psi_\lambda \quad \text{for } \{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell^2(\Lambda; L^2).$$

The upper bound in (D.5.1) is precisely the statement that $C_S : L^2(\mathbb{R}^d) \to \ell^2(\Lambda; L^2)$ is a bounded operator with operator norm $\leq \sqrt{B}$. It is not difficult to show that D_S is the Hilbert space adjoint of C_S, which implies that D_S is bounded whenever C_S is, with the same operator norm as C_S. It follows from the definitions that

$$\mathfrak{S}_S = D_S C_S.$$

Dual g-frames

If S generates a Gabor g-frame over Λ, then the theory of g-frames [239] says that the *canonical dual g-frame* is

$$\{\alpha_\lambda(S)\mathfrak{S}_S^{-1}\}_{\lambda\in\Lambda}.$$

It is clear from (D.5.3) that $\alpha_\lambda(\mathfrak{S}_S) = \mathfrak{S}_S$ for any $\lambda \in \Lambda$, and it is then easy to check that we also have $\alpha_\lambda(\mathfrak{S}_S^{-1}) = \mathfrak{S}_S^{-1}$. The canonical dual g-frame is therefore

$$\{\alpha_\lambda(S)\mathfrak{S}_S^{-1}\}_{\lambda\in\Lambda} = \{\alpha_\lambda(S)\alpha_\lambda(\mathfrak{S}_S^{-1})\}_{\lambda\in\Lambda} = \{\alpha_\lambda(S\mathfrak{S}_S^{-1})\}_{\lambda\in\Lambda}.$$

Hence the canonical dual g-frame is also a Gabor g-frame, generated by $S\mathfrak{S}_S^{-1}$. We get the reconstruction formulas

$$\psi = \mathfrak{S}_S\mathfrak{S}_S^{-1}\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(S^*S)\mathfrak{S}_S^{-1}\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(S^*)\alpha_\lambda(S\mathfrak{S}_S^{-1})\psi = D_S C_{S\mathfrak{S}_S^{-1}}\psi,$$
$$\psi = \mathfrak{S}_S^{-1}\mathfrak{S}_S\psi = \mathfrak{S}_S^{-1}\sum_{\lambda\in\Lambda}\alpha_\lambda(S^*S)\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(\mathfrak{S}_S^{-1}S^*)\alpha_\lambda(S)\psi = D_{S\mathfrak{S}_S^{-1}}C_S\psi.$$

In the very last of these equalities we have used that $\mathfrak{S}_S^{-1}$ is a positive (hence self-adjoint) operator, so $(S\mathfrak{S}_S^{-1})^* = \mathfrak{S}_S^{-1}S^*$. Inspired by these formulas and the theory of dual windows for Gabor frames [108, 131], we say that two operators $S, T \in \mathcal{L}(L^2)$ generate *dual Gabor g-frames* if S and T generate Gabor g-frames and $D_S C_T$ is the identity operator on $L^2(\mathbb{R}^d)$, i.e.

$$\sum_{\lambda\in\Lambda}\alpha_\lambda(S^*)\alpha_\lambda(T)\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(S^*T)\psi = \psi \quad \text{for any } \psi \in L^2(\mathbb{R}^d). \tag{D.5.4}$$

If D_S and C_T are bounded operators (i.e. S and T satisfy the upper g-frame bound in (D.5.1)), then (D.5.4) implies that both S and T generate Gabor g-frames. This follows from the general theory of g-frames, see [239, p. 441]: the lower bound in (D.5.1) for T follows from

$$\|\psi\|_{L^2}^2 = \|D_S C_T\psi\|_{L^2}^2 \lesssim \|C_T\psi\|_{\ell^2(\Lambda;L^2)}^2 = \sum_{\lambda\in\Lambda}\|\alpha_\lambda(T)\psi\|_{L^2}^2,$$

and the lower bound for S is similar. We state this as a proposition for later reference.

Proposition D.5.1. *Assume that $S, T \in \mathcal{L}(L^2)$ satisfy* (D.5.4) *and the upper bound in* (D.5.1)*. Then S and T generate Gabor g-frames.*

D.5.3 Two examples

We will now show that the Gabor g-frames include multi-window Gabor frames as a special case.

Example D.5.1 (Multi-window Gabor frames)**.** Consider a set of $N < \infty$ functions $\{\phi_n\}_{n=1}^N \subset L^2(\mathbb{R}^d)$. We seek an operator S such that the multi-window Gabor system generated by $\{\phi_n\}_{n=1}^N$ is captured by the system $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$. To achieve this, let $\{\xi_n\}_{n=1}^N$ be any orthonormal set in $L^2(\mathbb{R}^d)$, and consider the operator

$$S = \sum_{n=1}^N \xi_n \otimes \phi_n.$$

We start by writing out the condition (D.5.1) for S to generate a Gabor g-frame. For $\psi \in L^2(\mathbb{R}^d)$, we easily find using (D.3.9) that

$$\alpha_\lambda(S)\psi = \sum_{n=1}^N V_{\phi_n}\psi(\lambda)\pi(\lambda)\xi_n.$$

By the orthonormality of $\{\xi_n\}_{n=1}^N$ and Pythagoras' theorem for inner product spaces, this implies that $\|\alpha_\lambda(S)\psi\|_{L^2}^2 = \sum_{n=1}^N |V_{\phi_n}\psi(\lambda)|^2$. Inserting this into (D.5.1), we see that S generates a Gabor g-frame if and only if

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda}\sum_{n=1}^N |V_{\phi_n}\psi(\lambda)|^2 \leq B\|\psi\|_{L^2}^2 \quad \psi \in L^2(\mathbb{R}^d)$$

for some $A, B > 0$, which is precisely the condition that $\{\phi_n\}_{n=1}^N$ generate a multi-window Gabor frame.

We then note that $S^* = \sum_{n=1}^N \phi_n \otimes \xi_n$, and $S^*S = \sum_{n=1}^N \phi_n \otimes \phi_n$ by the orthonormality of $\{\xi_n\}_{n=1}^N$. Denote by C_{MW} and $\mathfrak{S}_{MW}$ the analysis and frame operator associated with the multi-window Gabor system generated by $\{\phi_n\}_{n=1}^N$. For $\psi \in L^2(\mathbb{R}^d)$, we find that

$$C_S(\psi) = \left\{\sum_{n=1}^N V_{\phi_n}\psi(\lambda)\pi(\lambda)\xi_n\right\}_{\lambda\in\Lambda}, \quad C_{MW}(\psi) = \{V_{\phi_n}\psi(\lambda)\}_{n\in\mathbb{Z}_N,\lambda\in\Lambda},$$

$$\mathfrak{S}_S(\psi) = \sum_{\lambda\in\Lambda}\sum_{n=1}^N V_{\phi_n}\psi(\lambda)\pi(\lambda)\phi_n, \qquad \mathfrak{S}_{MW}(\psi) = \sum_{\lambda\in\Lambda}\sum_{n=1}^N V_{\phi_n}\psi(\lambda)\pi(\lambda)\phi_n.$$

We see that the frame operators $\mathfrak{S}_S$ and $\mathfrak{S}_{MW}$ are equal. Since $\{\pi(\lambda)\xi_n\}_{n\in\mathbb{N}}$ is orthonormal for each $\lambda \in \Lambda$, we also see that $C_{MW}(\psi)$ and $C_S(\psi)$ carry exactly the same information: if we know $C_S(\psi)$, i.e. we know

$$\sum_{n=1}^{N} V_{\phi_n}\psi(\lambda)\pi(\lambda)\xi_n$$

for each $\lambda \in \Lambda$, we can find $C_{MW}(\psi)$ by

$$V_{\phi_m}\psi(\lambda) = \left\langle \sum_{n=1}^{N} V_{\phi_n}\psi(\lambda)\pi(\lambda)\xi_n, \pi(\lambda)\xi_m \right\rangle_{L^2}.$$

Hence multi-window Gabor frames are Gabor g-frames.

A less trivial example was considered in [85, 87]. Section D.7 will be dedicated to showing that the results from [87] hold for more general Gabor g-frames, and not just for the following example. The fact that the results of [85] is an example of g-frames was noted already by Sun [239].

Example D.5.2 (Localization operators)**.** Let $0 \neq \varphi \in M^1_v(\mathbb{R}^d)$, Λ a lattice and $h \in L^1_v(\mathbb{R}^{2d})$ a non-negative function. Here $h \in L^1_v(\mathbb{R}^{2d})$ means that $\|h\|_{L^1_v} := \int_{\mathbb{R}^{2d}} h(z)v(z)\, dz < \infty$. Assume further that

$$A' \leq \sum_{\lambda\in\Lambda} h(z-\lambda) \leq B' \quad \text{for all } z \in \mathbb{R}^{2d}$$

for some $A', B' > 0$. Then the localization operator A_h^{φ} generates a Gabor g-frame over Λ [85, 87, 239]. The key to connecting the summability condition on h to the Gabor g-frame condition for A_h^{φ} is equation (D.3.14). We will return to this example in Section D.7.2.

D.5.4 A trace class condition

In the definition of Gabor g-frames, we only assumed that S was a bounded linear operator on $L^2(\mathbb{R}^d)$. We will now show that S must be a Hilbert Schmidt operator. The following lemma is essentially the same as [18, Lem. 3.1].

Lemma D.5.2. *Let $T \in \mathcal{L}(L^2)$ be a positive operator. If $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ and $\{\eta_j\}_{j\in\mathbb{N}}$ is a Parseval frame, then*

$$\operatorname{tr}(T) := \sum_{n\in\mathbb{N}} \langle T\xi_n, \xi_n\rangle_{L^2} = \sum_{j\in\mathbb{N}} \langle T\eta_j, \eta_j\rangle_{L^2}.$$

Proof. Using the square root of the positive operator T, we have that

$$\left\langle T\eta_j,\eta_j\right\rangle_{L^2} = \left\langle T^{1/2}\eta_j, T^{1/2}\eta_j\right\rangle_{L^2} = \|T^{1/2}\eta_j\|_{L^2}^2.$$

Hence by Parseval's identity

$$\begin{aligned}\sum_{j\in\mathbb{N}}\left\langle T\eta_j,\eta_j\right\rangle_{L^2} &= \sum_{j\in\mathbb{N}}\|T^{1/2}\eta_j\|_{L^2}^2\\ &= \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{N}}\left|\left\langle T^{1/2}\eta_j,\xi_n\right\rangle_{L^2}\right|^2\\ &= \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}\left|\left\langle \eta_j,T^{1/2}\xi_n\right\rangle_{L^2}\right|^2\\ &= \sum_{n\in\mathbb{N}}\|T^{1/2}\xi_n\|_{L^2}^2 = \sum_{n\in\mathbb{N}}\langle T\xi_n,\xi_n\rangle_{L^2}. \end{aligned}$$

□

Proposition D.5.3. *Let Λ be any lattice and assume that $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ satisfies the upper g-frame bound in* (D.5.1). *Then S^*S is a trace class operator. Equivalently, S is a Hilbert Schmidt operator.*

Proof. The upper g-frame bound implies that $\sum_{\lambda\in\Lambda}\|\alpha_\lambda(S)\psi\|_{L^2}^2 < \infty$ for any $\psi\in L^2(\mathbb{R}^d)$. There exist $\{\varphi_n\}_{n=1}^N \subset L^2(\mathbb{R}^d)$ that generate a Parseval multi-window Gabor frame over Λ [201], i.e. $\{\pi(\lambda)\varphi_n\}_{n=1,\dots,N,\lambda\in\Lambda}$ is a Parseval frame. Then Lemma D.5.2 says that

$$\begin{aligned}\operatorname{tr}(S^*S) &= \sum_{n=1}^N\sum_{\lambda\in\Lambda}\langle S^*S\pi(\lambda)\varphi_n,\pi(\lambda)\varphi_n\rangle_{L^2}\\ &= \sum_{n=1}^N\sum_{\lambda\in\Lambda}\langle S\pi(\lambda)\varphi_n,S\pi(\lambda)\varphi_n\rangle_{L^2}\\ &= \sum_{n=1}^N\sum_{\lambda\in\Lambda}\|S\pi(\lambda)\varphi_n\|_{L^2}^2.\end{aligned}$$

By first using that $\pi(\lambda)^* = e^{-2\pi i\lambda_x\cdot\lambda_\omega}\pi(-\lambda)$ for $\lambda=(\lambda_x,\lambda_\omega)$, and then that $\pi(-\lambda)$ is a unitary operator, we see that

$$\|S\pi(\lambda)\varphi_n\|_{L^2}^2 = \|S\pi(-\lambda)^*\varphi_n\|_{L^2}^2 = \|\pi(-\lambda)S\pi(-\lambda)^*\varphi_n\|_{L^2}^2.$$

Hence

$$\operatorname{tr}(S^*S) = \sum_{n=1}^N\sum_{\lambda\in\Lambda}\|\alpha_{-\lambda}(S)\varphi_n\|_{L^2}^2 = \sum_{n=1}^N\sum_{\lambda\in\Lambda}\|\alpha_{\lambda}(S)\varphi_n\|_{L^2}^2 < \infty.$$

so S^*S is a positive trace class operator, and S is a Hilbert Schmidt operator. □

D.5.5 Periodization of operators and $\mathcal{B}$

To prepare for the next section on Fourier series of operators, we now consider the periodization of operators. The key to proving these results is [203, Thm. 8.2], which states that for $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$, the function $z \mapsto \operatorname{tr}(\alpha_z(S)T) \in M^1(\mathbb{R}^{2d})$ with

$$\|\operatorname{tr}(\alpha_z(S)T)\|_{M^1} \lesssim \|S\|_{\mathcal{B}} \|T\|_{\mathcal{S}^1} \tag{D.5.5}$$

and similarly for $S \in \mathcal{S}^1$ and $T \in \mathcal{B}$

$$\|\operatorname{tr}(\alpha_z(S)T)\|_{M^1} \lesssim \|S\|_{\mathcal{S}}^1 \|T\|_{\mathcal{B}}. \tag{D.5.6}$$

Proposition D.5.4 (Operator periodization)**.** *The periodization map given by $S \mapsto \sum_{\lambda\in\Lambda} \alpha_\lambda(S)$ is a well-defined and bounded map $\mathcal{B} \to \mathcal{L}(L^2)$:*

$$\left\| \sum_{\lambda\in\Lambda} \alpha_\lambda(S) \right\|_{\mathcal{L}(L^2)} \lesssim \|S\|_{\mathcal{B}},$$

and a well-defined and bounded map $\mathcal{S}^1 \to \mathcal{B}'$:

$$\left\| \sum_{\lambda\in\Lambda} \alpha_\lambda(S) \right\|_{\mathcal{B}'} \lesssim \|S\|_{\mathcal{S}^1}.$$

The sum $\sum_{\lambda\in\Lambda} \alpha_\lambda(S)$ converges in the weak topology of $\mathcal{L}(L^2)$ when $S \in \mathcal{B}$, and in the weak* topology of $\mathcal{B}'$ when $S \in \mathcal{S}^1$.*

Proof. Let $S \in \mathcal{B}$. Since $\mathcal{L}(L^2)$ is the dual space of $\mathcal{S}^1$ [54, Thm. 3.13], we define $\sum_{\lambda\in\Lambda} \alpha_\lambda(S) \in \mathcal{L}(L^2)$ by duality, by defining its action as an antilinear functional:

$$\left\langle \sum_{\lambda\in\Lambda} \alpha_\lambda(S), T \right\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} := \sum_{\lambda\in\Lambda} \langle \alpha_\lambda(S), T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} \quad \text{for } T \in \mathcal{S}^1.$$

To see that this defines a *bounded* antilinear functional on $\mathcal{S}^1$, we estimate that

$$\begin{aligned}
\left| \sum_{\lambda\in\Lambda} \langle \alpha_\lambda(S), T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} \right| &\leq \sum_{\lambda\in\Lambda} \left| \langle \alpha_\lambda(S), T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1} \right| \\
&= \sum_{\lambda\in\Lambda} |\operatorname{tr}(\alpha_\lambda(S)T^*)| \quad \text{by (D.4.9)} \\
&\lesssim \|\operatorname{tr}(\alpha_z(S)T^*)\|_{M^1} \quad \text{by Lemma } D.3.3 \\
&\lesssim \|S\|_{\mathcal{B}} \|T\|_{\mathcal{S}^1} \quad \text{by (D.5.5).}
\end{aligned}$$

It is clear that the partial sums converge to this element $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)$ in the weak* topology of $\mathcal{L}(L^2)$: For any finite subset $J\subset\Lambda$ we get

$$\left\langle \sum_{\lambda\in\Lambda}\alpha_\lambda(S)-\sum_{\lambda\in J}\alpha_\lambda(S),T\right\rangle_{\mathcal{L}(L^2),\mathcal{S}^1}=\sum_{\lambda\in\Lambda\setminus J}\langle\alpha_\lambda(S),T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1},$$

and we showed above that the sum $\sum_{\lambda\in\Lambda}\langle\alpha_\lambda(S),T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1}$ converges absolutely. Then let $S\in\mathcal{S}^1$. We define $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)\in\mathcal{B}'$ by duality:

$$\left\langle \sum_{\lambda\in\Lambda}\alpha_\lambda(S),T\right\rangle_{\mathcal{B}',\mathcal{B}}:=\sum_{\lambda\in\Lambda}\langle\alpha_\lambda(S),T\rangle_{\mathcal{B}',\mathcal{B}}\qquad\text{for } T\in\mathcal{B}.$$

The estimate showing that this defines a bounded antilinear functional on $\mathcal{B}$ with $\left|\sum_{\lambda\in\Lambda}\langle\alpha_\lambda(S),T\rangle_{\mathcal{B}',\mathcal{B}}\right|\lesssim\|S\|_{\mathcal{S}^1}\|T\|_{\mathcal{B}}$ is the same as above using (D.5.6), but note that we need to write $\langle\alpha_\lambda(S),T\rangle_{\mathcal{B}',\mathcal{B}}=\mathrm{tr}(\alpha_\lambda(S)T^*)$ to use (D.5.6) – this is true by (D.4.10). □

Corollary D.5.4.1. *If $S^*S\in\mathcal{B}$ then $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ satisfies the upper g-frame bound*

$$\sum_{\lambda\in\Lambda}\|\alpha_\lambda(S)\psi\|_{L^2}^2\lesssim\|S^*S\|_{\mathcal{B}}\|\psi\|_{L^2}^2\quad\text{for all } \psi\in L^2(\mathbb{R}^d).$$

In particular, this is true if $S\in\mathcal{B}$.

Proof. We observed in the proof above (now with S^*S instead of S) that

$$\sum_{\lambda\in\Lambda}\left|\langle\alpha_\lambda(S^*S),T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1}\right|\lesssim\|S^*S\|_{\mathcal{B}}\|T\|_{\mathcal{S}^1}. \tag{D.5.7}$$

If $T=\psi\otimes\psi$, it is simple to show that $\langle\alpha_\lambda(S^*S),T\rangle_{\mathcal{L}(L^2),\mathcal{S}^1}=\langle\alpha_\lambda(S^*S)\psi,\psi\rangle_{L^2}$ and $\|T\|_{\mathcal{S}^1}=\|\psi\|_{L^2}^2$. Therefore equation (D.5.7) says that

$$\sum_{\lambda\in\Lambda}|\langle\alpha_\lambda(S^*S)\psi,\psi\rangle_{L^2}|\lesssim\|S^*S\|_{\mathcal{B}}\|\psi\|_{L^2}^2.$$

As we have seen, $\langle\alpha_\lambda(S^*S)\psi,\psi\rangle_{L^2}=\|\alpha_\lambda(S)\psi\|_{L^2}^2$, which completes the proof of the first part. If $S\in\mathcal{B}$, it follows from Proposition D.4.1 and Corollary D.4.1.1 that $S^*S\in\mathcal{B}$. □

The fact that we only need $S^*S\in\mathcal{B}$ is useful in light of our treatment of multi-window Gabor frames in Example D.5.1. To a system $\{\phi_n\}_{n=1}^N\subset M^1(\mathbb{R}^d)$ we associated the operator

$$S=\sum_{n=1}^N\xi_n\otimes\phi_n,$$

where $\{\xi_n\}_{n=1}^N$ is an arbitrary orthonormal set in $L^2(\mathbb{R}^d)$. Hence we do not necessarily have $S\in\mathcal{B}$, yet $S^*S=\sum_{n=1}^N\phi_n\otimes\phi_n\in\mathcal{B}$. A version of this corollary for Gabor frames is well-known [131, Thm. 12.2.3].

D.6 Fourier series of operators: the Janssen representation

A key insight of Werner's paper [251] is that the Fourier-Wigner transform in many respects behaves as a Fourier transform for operators. Given a lattice $\Lambda \subset \mathbb{R}^{2d}$, this leads to a natural question: if an operator is in some sense Λ-periodic, can we find a Fourier series expansion of the operator? In fact, Λ-periodic operators were studied in [102], where an operator S was said to be Λ-periodic if

$$\alpha_\lambda(S) = S \quad \text{for any } \lambda \in \Lambda.$$

An important tool in [102] is the *adjoint lattice* Λ° of Λ, defined by

$$\begin{aligned}\Lambda^\circ &= \{\lambda^\circ \in \mathbb{R}^{2d} : \pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ) \text{ for any } \lambda \in \Lambda\} \\ &= \{\lambda^\circ \in \mathbb{R}^{2d} : e^{2\pi i \sigma(\lambda^\circ,\lambda)} = 1 \text{ for any } \lambda \in \Lambda\},\end{aligned}$$

where σ is the standard symplectic form. It is shown in [102] that Λ° is a lattice, and $|\Lambda^\circ| = \frac{1}{|\Lambda|}$. One can interpret Λ° using abstract harmonic analysis. Identify the dual group $\widehat{\mathbb{R}^{2d}}$ with $\mathbb{R}^{2d}$ by the bijection $\mathbb{R}^{2d} \ni z \mapsto \chi_z \in \widehat{\mathbb{R}^{2d}}$, where χ_z is the *symplectic* character $\chi_z(z') = e^{2\pi i \sigma(z,z')}$. With this identification, we see that

$$\Lambda^\circ = \{\lambda^\circ \in \mathbb{R}^{2d} : \chi_{\lambda^\circ}(\lambda) = 1 \text{ for any } \lambda \in \Lambda\}$$

Hence Λ° is the annihilator of Λ, and Λ° can therefore be identified with the dual group of $\mathbb{R}^{2d}/\Lambda$ [81, Prop. 3.6.1]. By abstract harmonic analysis, this implies that any well-behaved Λ-periodic *function* f on $\mathbb{R}^{2d}$ can be expanded in a *symplectic Fourier series*

$$f(z) = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} e^{2\pi i \sigma(\lambda^\circ, z)},$$

and we will refer to $\{c_{\lambda^\circ}\}_{\lambda^\circ \in \Lambda^\circ}$ as the *symplectic Fourier coefficients* of f.

Remark D.7. The main results of this section, namely Theorems D.6.4, D.6.5 and D.6.8, are due to Feichtinger and Kozek [102]. The spirit of our approach is also the same as in [102] – we express operators as linear combinations of time-frequency shifts by applying methods from abstract harmonic analysis to their symbol with respect to some pseudodifferential calculus. Since the results form a natural and important part of the theory of Gabor g-frames, we choose to include detailed proofs. Our proofs differ slightly from those in [102] by using the Weyl symbol (rather than Kohn-Nirenberg symbol), which makes it particularly transparent that the Janssen representation is a Fourier series of operators (see Lemma D.6.3). This fits well with our interpretation of $\mathcal{F}_W$ as a Fourier transform. We also extend the results of [102] to trace class operators.

As our function and operator spaces we will use $M^1(\mathbb{R}^{2d})$ and $\mathcal{B}$ along with their duals. In the following lemma $A(\mathbb{R}^{2d}/\Lambda)$ denotes the Λ-periodic functions $f : \mathbb{R}^{2d} \to \mathbb{C}$ with symplectic Fourier coefficients $\{c_{\lambda^\circ}\}_{\lambda^\circ \in \Lambda^\circ}$ in $\ell^1(\Lambda^\circ)$, with norm

$$\|f\|_{A(\mathbb{R}^{2d}/\Lambda)} := \|\{c_{\lambda^\circ}\}\|_{\ell^1(\Lambda^\circ)}.$$

$A'(\mathbb{R}^{2d}/\Lambda)$ denotes its dual space of distributions with symplectic Fourier coefficients in $\ell^\infty(\Lambda^\circ)$. The proofs of the two parts of the next lemma can be found in [95, Thm. 7] and [178, Prop. 13], respectively.

Lemma D.6.1. *Let Λ be a lattice and P_Λ be the periodization operator*

$$P_\Lambda f = \sum_{\lambda \in \Lambda} T_\lambda(f) \quad \text{for } f \in M^1(\mathbb{R}^{2d}).$$

(a) P_Λ is bounded and surjective from $M^1(\mathbb{R}^{2d})$ onto $A(\mathbb{R}^{2d}/\Lambda)$.

(b) The range of the Banach space adjoint operator

$$P_\Lambda^* : A'(\mathbb{R}^{2d}/\Lambda) \to M^\infty(\mathbb{R}^{2d})$$

is the set of Λ-periodic elements of $M^\infty(\mathbb{R}^{2d})$.

Remark D.8. (a) A distribution $f \in M^\infty(\mathbb{R}^{2d})$ is Λ-periodic if $T_\lambda(f) = f$ for any $\lambda \in \Lambda$, where $T_\lambda(f)$ is defined by

$$\langle T_\lambda(f), g\rangle_{M^\infty, M^1} := \langle f, T_{-\lambda}(g)\rangle_{M^\infty, M^1}$$

for $g \in M^1(\mathbb{R}^{2d})$.

(b) If $q : \mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda$ denotes the quotient map, then a simple calculation using Weil's formula [130, (6.2.11)] shows that $P_\Lambda^*(f) = \frac{1}{|\Lambda|} \cdot f \circ q$ for $f \in A(\mathbb{R}^{2d}/\Lambda)$.

Since $P_\Lambda f$ has absolutely summable symplectic Fourier coefficients when $f \in M^1(\mathbb{R}^{2d})$ by Lemma D.6.1, we can use Poisson's summation formula to find its symplectic Fourier coefficients, see [167, Example 5.11] or [81, Thm. 3.6.3] for a proof.

Proposition D.6.2 (Poisson summation formula)**.** *Let $f \in M^1(\mathbb{R}^{2d})$. The symplectic Fourier coefficients of $P_\Lambda f$ are $\{\frac{1}{|\Lambda|}\mathcal{F}_\sigma(f)(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ}$, i.e.*

$$P_\Lambda f(z) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_\sigma(f)(\lambda^\circ) e^{2\pi i \sigma(\lambda^\circ, z)}.$$

To use this to obtain Fourier series of operators, we need the following simple lemma [72, Prop. 198].

Lemma D.6.3. *For any* $z = (x, \omega) \in \mathbb{R}^{2d}$*, the Weyl symbol of* $e^{-\pi i x\cdot\omega}\pi(z)$ *is the function* $z' \mapsto e^{2\pi i \sigma(z,z')}$.

We will now consider Fourier series of operators arising as periodizations of operators in $\mathcal{B}$, in other words a Poisson summation formula for operators. The second part of the result extends Janssen's representation of multi-window Gabor frame operators to Gabor g-frame operators. As mentioned, this result is due to [102] who used it to prove the Janssen representation for multi-window Gabor frames. In this and following statements, we use the notation $\lambda^\circ = (\lambda^\circ_x, \lambda^\circ_\omega)$ to denote the elements of Λ°.

Theorem D.6.4 (Janssen's representation of Gabor g-frame operators)**.** *Let* $S \in \mathcal{B}$ *and* Λ *a lattice. Then*

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S) = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S)(\lambda^\circ)e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ).$$

In particular,

$$\mathfrak{S}_S = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S^*S)(\lambda^\circ)e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ).$$

Moreover, if $S \in \mathcal{B}_{v\otimes v}$*, then* $\{\mathcal{F}_W(S)(\lambda^\circ)\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^1_v(\Lambda^\circ)$.

Proof. Recall that α_λ corresponds to a translation of the Weyl symbol by (D.3.7). Since the map sending operators in $\mathcal{B}'$ to their Weyl symbols in $M^\infty(\mathbb{R}^{2d})$ is weak*-to-weak*-continuous by Proposition D.4.4 and $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)$ converges in the weak* topology of $\mathcal{B}'$ by Proposition D.5.4, the Weyl symbol f of $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)$ is

$$f = \sum_{\lambda\in\Lambda} T_\lambda(a_S) \in M^\infty(\mathbb{R}^{2d}),$$

where a_S is the Weyl symbol of S. Hence $f = P_\Lambda a_S$. By the Poisson summation formula the symplectic Fourier series of f is given by

$$\begin{aligned} f(z) &= \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_\sigma(a_S)(\lambda^\circ)e^{2\pi i\sigma(\lambda^\circ,z)} \\ &= \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S)(\lambda^\circ)e^{2\pi i\sigma(\lambda^\circ,z)} \quad \text{by (D.3.13).} \end{aligned}$$

By Proposition D.4.4, $\mathcal{F}_W(S) \in M^1(\mathbb{R}^{2d})$, so $\{\mathcal{F}_W(S)(\lambda^\circ)\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^1(\Lambda^\circ)$ by Lemma D.3.3 – hence the sum above converges absolutely in the norm of $M^\infty(\mathbb{R}^{2d})$. Taking the Weyl transform of this using Lemma D.6.3, we see that

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S)(\lambda^\circ) e^{-\pi i \lambda_x^\circ \cdot \lambda_\omega^\circ} \pi(\lambda^\circ).$$

For the last part, note that if $S \in \mathcal{B}_{v\otimes v}$, then $\mathcal{F}_W(S) \in W(L^1_v)$ by Proposition D.4.2, and the result follows from Lemma D.3.3. □

Example D.6.1 (Multi-window Gabor frames)**.** For $\{\phi_n\}_{n=1}^N \subset M^1(\mathbb{R}^d)$, we saw in Example D.5.1 that the frame operator of the multi-window Gabor system generated by $\{\phi_n\}_{n=1}^N$ equals $\mathfrak{S}_S$ for $S = \sum_{n=1}^N \xi_n \otimes \phi_n$, where $\{\xi_n\}_{n=1}^N$ is any orthonormal set in $L^2(\mathbb{R}^d)$. Then

$$S^*S = \sum_{n=1}^N \phi_n \otimes \phi_n \in \mathcal{B},$$

so by (D.3.10)

$$\mathcal{F}_W(S^*S)(\lambda^\circ) = \sum_{n=1}^N \mathcal{F}_W(\phi_n\otimes\phi_n)(\lambda^\circ) = \sum_{n=1}^N e^{\pi i \lambda_x^\circ\cdot\lambda_\omega^\circ} V_{\phi_n}\phi_n(\lambda^\circ).$$

Therefore Theorem D.6.4 gives that

$$\mathfrak{S}_S = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ}\sum_{n=1}^N V_{\phi_n}\phi_n(\lambda^\circ)\pi(\lambda^\circ),$$

which is the Janssen representation for multi-window Gabor frames [87, 170].

We can also prove that any periodic operator in $\mathcal{B}'$ has a Fourier series expansion. By considering Weyl symbols, this is essentially the fact that any Λ-periodic distribution $f \in M^\infty(\mathbb{R}^{2d})$ can be expanded in a symplectic Fourier series, which follows from the second part of Lemma D.6.1. The result is due to [102].

Theorem D.6.5. *Let $S \in \mathcal{B}'$ be a Λ-periodic operator. Then there exists a unique sequence $\{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^\infty(\Lambda^\circ)$ such that*

$$S = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ), \tag{D.6.1}$$

with weak convergence in $\mathcal{B}'$. Furthermore, the map*

$$\{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ} \mapsto \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ)$$

is weak-to-weak*-continuous from $\ell^\infty(\Lambda^\circ)$ to $\mathcal{B}'$.*

Proof. We first show that series of the form $\frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)$ converge in the weak* topology of $\mathcal{B}'$ when $\{c_{\lambda^\circ}\}\in\ell^\infty(\Lambda^\circ)$. For $\{c_{\lambda^\circ}\}\in\ell^\infty(\Lambda^\circ)$, we define an antilinear functional on $\mathcal{B}$ by

$$\left\langle \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ), T\right\rangle_{\mathcal{B}',\mathcal{B}} := \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\left\langle e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ), T\right\rangle_{\mathcal{B}',\mathcal{B}}.$$

To see that this is a *bounded* functional, consider

$$\begin{aligned}\left|\frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\langle\pi(\lambda^\circ), T\rangle_{\mathcal{B}',\mathcal{B}}\right| &\lesssim \sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}|\left|\left\langle e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ), T\right\rangle_{\mathcal{B}',\mathcal{B}}\right| \\ &= \sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}|\left|\mathrm{tr}(e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)T^*)\right|\end{aligned}$$

where the last step uses (D.4.9). By the definition of $\mathcal{F}_W$ and equation (D.3.11),

$$tr(e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)T^*) = \mathcal{F}_W(T^*)(-\lambda^\circ) = \overline{\mathcal{F}_W(T)(\lambda^\circ)}.$$

We may therefore continue our estimate by

$$\begin{aligned}\sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}|\left|\mathrm{tr}(e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)T^*)\right| &= \sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}||\mathcal{F}_W(T)(\lambda^\circ)| \\ &\lesssim \|\{c_{\lambda^\circ}\}\|_{\ell^\infty(\Lambda^\circ)}\|\mathcal{F}_W(T)\|_{M^1} \quad \text{by Lem. D.3.3} \\ &\lesssim \|\{c_{\lambda^\circ}\}\|_{\ell^\infty(\Lambda^\circ)}\|T\|_{\mathcal{B}} \quad \text{by Prop. D.4.4.}\end{aligned}$$

Hence $\frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)\in\mathcal{B}'$. The same calculation without absolute values shows that

$$\left\langle \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ), T\right\rangle_{\mathcal{B}',\mathcal{B}} = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\overline{\mathcal{F}_W(T)(\lambda^\circ)}, \qquad \text{(D.6.2)}$$

which implies that the map sending $\{c_{\lambda^\circ}\}$ to this functional is in fact the Banach space adjoint of the map $\mathcal{B}\to\ell^1(\Lambda^\circ)$ given by $T\mapsto\{\frac{1}{|\Lambda|}\mathcal{F}_W(T)(\lambda^\circ)\}$. In particular, the weak*-to-weak* continuity of the map

$$\{c_{\lambda^\circ}\}\mapsto\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega}\pi(\lambda^\circ)$$

follows, as does the weak* convergence of the sum.

The uniqueness also follows: the map $\mathcal{B}\to\ell^1(\Lambda^\circ)$ defined by

$$T\mapsto\left\{\frac{1}{|\Lambda|}\mathcal{F}_W(T)(\lambda^\circ)\right\}$$

is surjective by [95, Thm. 7 C)] hence its Banach space adjoint is injective. We then turn to finding $\{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ}$ such that (D.6.1) holds. Since S is a Λ-periodic operator in $\mathcal{B}'$, its Weyl symbol a_S is a Λ-periodic distribution in $M^\infty(\mathbb{R}^{2d})$. By Lemma D.6.1 there exists $\tilde{f} \in A'(\mathbb{R}^{2d}/\Lambda)$ such that $P_\Lambda^* \tilde{f} = a_S$, and we pick $\{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ}$ to be the symplectic Fourier coefficients of $\tilde{f}$. For any $T \in \mathcal{B}$ we have from (D.4.11) that

$$\begin{aligned}\langle S, T\rangle_{\mathcal{B}',\mathcal{B}} &= \langle a_S, a_T\rangle_{M^\infty,M^1} \\ &= \left\langle P_\Lambda^* \tilde{f}, a_T\right\rangle_{M^\infty,M^1} \\ &= \left\langle \tilde{f}, P_\Lambda a_T\right\rangle_{A'(\mathbb{R}^{2d}/\Lambda),A(\mathbb{R}^{2d}/\Lambda)} \\ &= \frac{1}{|\Lambda|}\left\langle \{c_\lambda^\circ\}, \{\mathcal{F}_\sigma(a_T)(\lambda^\circ)\}\right\rangle_{\ell^\infty(\Lambda^\circ),\ell^1(\Lambda^\circ)}.\end{aligned}$$

In the last equality we have used the Poisson summation formula to get that $\{\frac{1}{|\Lambda|}\mathcal{F}_\sigma(a_T)(\lambda^\circ)\}_{\lambda^\circ\in\Lambda^\circ}$ are the symplectic Fourier coefficients of $P_\Lambda a_T$. By comparing this to (D.6.2) and using $\mathcal{F}_W(T) = \mathcal{F}_\sigma(a_T)$ by (D.3.13), we have proved (D.6.1). □

Remark D.9. (a) The uniqueness part of the previous theorem amounts to a well-known fact: if $\sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\pi(\lambda^\circ) = 0$ for $c = \{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^\infty(\Lambda^\circ)$, then $c = 0$. Earlier proofs of this fact range from the rather complicated [223] to the pleasantly elementary [134]. Our proof is similar to that in [134], and comes with a simple interpretation: the Fourier coefficients of periodic operators are unique.

(b) If $S \in \mathcal{B}'$ is Λ-periodic and its Weyl symbol a_S belongs to the space $A(\mathbb{R}^{2d}/\Lambda)$ (i.e. its symplectic Fourier coefficients are absolutely summable), then there exists some $P_S \in \mathcal{B}$ such that $S = \sum_{\lambda\in\Lambda}\alpha_\lambda(P_S)$. This is [102, Thm.. 7.7.6], and follows from applying the surjectivity in part (a) of Lemma D.6.1 to a_S.

D.6.1 Poisson summation formula for trace class operators

When $S \in \mathcal{S}^1$ the periodization $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)$ converges in $\mathcal{B}'$ by Proposition D.5.4, and by Theorem D.6.5 there exists $\{c_{\lambda^\circ}\} \in \ell^\infty(\Lambda^\circ)$ such that

$$\sum_{\lambda\in\Lambda}\alpha_\lambda(S) = \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ} e^{-\pi i \lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ).$$

If $S \in \mathcal{B}$, we know from Theorem D.6.4 that c_{λ° is given by the samples of $\mathcal{F}_W(S)$. However, even if $S \in \mathcal{S}^1 \setminus \mathcal{B}$, we know from the Riemann-Lebesgue lemma (D.3.12) that $\mathcal{F}_W(S) \in C_0(\mathbb{R}^{2d})$. Hence the samples of $\mathcal{F}_W(S)$ are still well-defined, and we will use a continuity argument to show that $c_{\lambda^\circ} = \mathcal{F}_W(S)(\lambda^\circ)$ also when $S \in \mathcal{S}^1 \setminus \mathcal{B}$.

Theorem D.6.6 (Poisson summation formula for trace class operators)**.** *Let $S \in \mathcal{S}^1$. Then*

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S) = \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} \mathcal{F}_W(S)(\lambda^\circ)e^{-\pi i\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ),$$

with weak convergence of both sums in $\mathcal{B}'$.*

Proof. Let $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ be a sequence converging to S in the norm of $\mathcal{S}^1$ using Lemma D.4.3. By Theorem D.6.4, we have for each $n \in \mathbb{N}$ that

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S_n) = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S_n)(\lambda^\circ)e^{-\pi i\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ). \tag{D.6.3}$$

By Proposition D.5.4, the left hand side of (D.6.3) converges to $\sum_{\lambda\in\Lambda}\alpha_\lambda(S)$ in $\mathcal{B}'$ as $n\to\infty$. Then note that

$$\|\mathcal{F}_W(S)|_{\Lambda^\circ} - \mathcal{F}_W(S_n)|_{\Lambda^\circ}\|_{\ell^\infty(\Lambda^\circ)} = \|\mathcal{F}_W(S-S_n)|_{\Lambda^\circ}\|_{\ell^\infty(\Lambda^\circ)} \le \|S-S_n\|_{\mathcal{S}^1}$$

by (D.3.12), hence the samples $\mathcal{F}_W(S_n)|_{\Lambda^\circ}$ converge to $\mathcal{F}_W(S)|_{\Lambda^\circ}$ in $\ell^\infty(\Lambda^\circ)$ as $n\to\infty$. Combining this with the continuity statement in Theorem D.6.5, we see that the right hand side of (D.6.3) converges in the weak* topology of $\mathcal{B}'$ to $\frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ}\mathcal{F}_W(S)(\lambda^\circ)e^{-\pi i\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ)$ as $n\to\infty$. As the limits of both sides of (D.6.3) must be equal, the result follows. □

D.6.2 The twisted Wiener's lemma

The results in the previous section supplement the theory of the Fourier transform of operators, as introduced by Werner in [251], by showing that periodic operators have a Fourier series expansion. A classic result for Fourier series of functions is Wiener's lemma: if a periodic function is invertible and has an absolutely convergent Fourier series, then its inverse has an absolutely convergent Fourier series. The same holds for operators, by a result due to Gröchenig and Leinert [140]. Recall that v is a submultiplicative, symmetric GRS-weight – the GRS condition is crucial for this result.

Theorem D.6.7. *Assume that $S = \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\pi(\lambda^\circ)$ for a sequence $\{c_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^1_v(\Lambda^\circ)$ and that S is invertible on $L^2(\mathbb{R}^d)$. Then*

$$S^{-1} = \sum_{\lambda^\circ} a_{\lambda^\circ}\pi(\lambda^\circ)$$

for some sequence $\{a_{\lambda^\circ}\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^1_v(\Lambda^\circ)$.

This has consequences for Gabor g-frames generated by an operator $S \in \mathcal{B}_{v\otimes v}$.

Corollary D.6.7.1. *Assume that* $S \in \mathcal{B}_{v\otimes v}$ *generates a Gabor g-frame over a lattice* Λ. *Then* $\mathfrak{S}_S^{-1} = \sum_{\lambda^\circ \in \Lambda^\circ} a_{\lambda^\circ} \pi(\lambda^\circ)$ *for a sequence* $\{a_{\lambda^\circ}\}_{\lambda^\circ \in \Lambda^\circ} \in \ell^1_v(\Lambda^\circ)$.

Proof. $\mathfrak{S}_S$ is invertible on $L^2(\mathbb{R}^d)$ as S generates a Gabor g-frame. By the Janssen representation in Theorem D.6.4 we can apply Theorem D.6.7 to $\mathfrak{S}_S$. □

D.6.3 Wexler-Raz and some conditions for Gabor g-frames

Recall that two operators $S, T \in \mathcal{HS}$ generate *dual Gabor g-frames* if S and T generate Gabor g-frames and

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S^*T)\psi = \psi \quad \text{for any } \psi \in L^2(\mathbb{R}^d).$$

A characterization of dual Gabor g-frames is given by a version of the Wexler-Raz biorthogonality conditions from [102]. We extend the result in [102] to Hilbert Schmidt operators.

Theorem D.6.8 (Wexler-Raz biorthogonality)**.** *Let* $S, T \in \mathcal{HS}$ *such that* S *and* T *satisfy the upper g-frame bound in* (D.5.1). *Then*

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S^*T)\psi = \psi \quad \textit{for any } \psi \in L^2(\mathbb{R}^d) \tag{D.6.4}$$

if and only if

$$\mathcal{F}_W(S^*T)(\lambda^\circ) = |\Lambda|\delta_{\lambda^\circ,0} \quad \textit{for } \lambda^\circ \in \Lambda^\circ. \tag{D.6.5}$$

Proof. Our assumption on S and T ensures that $D_S C_T \psi = \sum_\lambda \alpha_\lambda(S^*T)\psi$ defines a bounded operator on $L^2(\mathbb{R}^d)$. Since $S, T \in \mathcal{HS}$, we have $S^*T \in \mathcal{S}^1$ and by Proposition D.6.6

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S^*T) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S^*T)(\lambda^\circ) e^{-i\pi\lambda_x^\circ\cdot\lambda_\omega^\circ}\pi(\lambda^\circ).$$

Equation (D.6.4) states that the left hand side is the identity operator $\pi(0)$, and the uniqueness part of Theorem D.6.5 implies that this is true if and only if (D.6.5) holds. □

Note that under the assumptions of Theorem D.6.8, both S^* and T generate Gabor g-frames by Proposition D.5.1. As first noted in [102], the theorem reproduces the familiar Wexler-Raz biorthogonality conditions for Gabor frames.

Example D.6.2. Consider two sets of N functions $\{\phi_n\}_{n=1}^N, \{\psi_n\}_{n=1}^N \subset L^2(\mathbb{R}^d)$. As in Example D.5.1, we associate an operator to each of these systems:

$$S = \sum_{n=1}^N \xi_n \otimes \phi_n, \qquad\qquad T = \sum_{n=1}^N \xi_n \otimes \psi_n,$$

where $\{\xi_n\}_{n=1}^N$ is an orthonormal system in $L^2(\mathbb{R}^d)$. Assume that the multi-window Gabor systems generated by $\{\phi_n\}_{n=1}^N$ and $\{\psi_n\}_{n=1}^N$ are Bessel systems, i.e.

$$\sum_{n=1}^N \sum_{\lambda\in\Lambda} |V_{\phi_n}\psi(\lambda)|^2 \lesssim \|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d),$$

and the same inequality for ψ_n. It is a simple exercise to show that this condition implies that S and T satisfy the upper g-frame bound, so Theorem D.6.8 applies.

Note that $S^*T = \sum_{n=1}^N \phi_n \otimes \psi_n$, and $\mathcal{F}_W(S^*T)(z) = e^{\pi i x\cdot\omega} \sum_{n=1}^N V_{\psi_n}\phi_n(z)$ by (D.3.10). We also find using (D.3.9) that

$$\sum_{\lambda\in\Lambda} \alpha_\lambda(S^*T)\eta = \sum_{\lambda\in\Lambda}\sum_{n=1}^N V_{\psi_n}\eta(\lambda)\pi(\lambda)\phi_n \quad \text{for } \eta \in L^2(\mathbb{R}^d).$$

Hence Theorem D.6.8 says that

$$\eta = \sum_{\lambda\in\Lambda}\sum_{n=1}^N V_{\psi_n}\eta(\lambda)\pi(\lambda)\phi_n \quad \text{for } \eta \in L^2(\mathbb{R}^d)$$

if and only if

$$\sum_{n=1}^N V_{\psi_n}\phi_n(\lambda^\circ) = |\Lambda|\delta_{\lambda^\circ,0} \quad \text{for } \lambda^\circ \in \Lambda^\circ.$$

This is the usual version of the Wexler-Raz biorthogonality conditions for multi-window Gabor frames.

We note some simple consequences of Theorem D.6.8.

Corollary D.6.8.1. *(a) Let $S \in \mathcal{B}$. If there exists some $T \in \mathcal{B}$ such that $\mathcal{F}_W(S^*T)(0) \neq 0$ and $\mathcal{F}_W(S^*T)(\lambda^\circ) = 0$ for $\lambda^\circ \neq 0$, then S generates a Gabor g-frame.*

(b) Let $S \in \mathcal{B}$. If there exist $\phi, \psi \in M^1(\mathbb{R}^d)$ such that

$$V_\phi(S^*\psi)(\lambda^\circ) = |\Lambda|\delta_{\lambda^\circ,0} \quad \textit{for } \lambda^\circ \in \Lambda^\circ,$$

then S generates a Gabor g-frame.

(c) If $S \in \mathcal{B}$ satisfies $\Lambda^\circ \cap \{z' - z'' : z', z'' \in supp(\mathcal{F}_W(S))\} = \{0\}$, then S generates a tight Gabor g-frame.

Proof. (a) Define $\tilde{T} = \frac{|\Lambda|}{\mathcal{F}_W(S^*T)(0)} T$. Then $S, \tilde{T} \in \mathcal{B}$, so $S, \tilde{T}$ satisfy the upper g-frame bound in (D.5.1) by Corollary D.5.4.1. Hence Theorem D.6.8 applies to give that $S, \tilde{T}$ generate dual Gabor g-frames, and the result follows from Proposition D.5.1.

(b) Let $T = \psi \otimes \phi$. Then $S^*T = (S^*\psi) \otimes \phi$. Since $\mathcal{F}_W((S^*\psi) \otimes \phi)(x, \omega) = e^{\pi i x \cdot \omega} V_\phi(S^*\psi)(x, \omega)$ by (D.3.10), the result follows from part (a).

(c) It is well-known (see [102, 203]) that

$$\mathcal{F}_W(S^*S)(z) = \int_{\mathbb{R}^{2d}} \mathcal{F}_W(S^*)(z - z')\mathcal{F}_W(S)(z')e^{\pi i \sigma(z,z')}\, dz',$$

where the right hand side is the so-called twisted convolution of $\mathcal{F}_W(S^*)$ with $\mathcal{F}_W(S)$. By (D.3.11) we get

$$\mathcal{F}_W(S^*S)(z) = \int_{\mathbb{R}^{2d}} \overline{\mathcal{F}_W(S)(z' - z)}\mathcal{F}_W(S)(z')e^{\pi i \sigma(z,z')}\, dz'.$$

One easily deduces that a necessary condition for $\mathcal{F}_W(S^*S)(z)$ to be non-zero is that $z = z' - z''$, where both $z', z'' \in \text{supp}(\mathcal{F}_W(S))$, hence the condition in the statement ensures that $\mathcal{F}_W(S^*S)(\lambda^\circ) = 0$ for $\lambda^\circ \neq 0$. In addition, $\mathcal{F}_W(S^*S)(0) = \text{tr}(S^*S) = \|S\|_{\mathcal{HS}}^2 > 0$. Therefore $\tilde{S} = \frac{\sqrt{|\Lambda|}}{\|S\|_{\mathcal{HS}}} S$ satisfies

$$\sum_{\lambda \in \Lambda} \alpha_\lambda(\tilde{S}^*\tilde{S})\psi = \psi$$

for any $\psi \in L^2(\mathbb{R}^d)$ by Theorem D.6.4, which implies that

$$\sum_{\lambda \in \Lambda} \alpha_\lambda(S^*S)\psi = \frac{\|S\|_{\mathcal{HS}}^2}{|\Lambda|}\psi.$$

□

Remark D.10. (a) The condition in part (c) above can be satisfied if S is an *underspread* operator (as defined by Kozek [189–191]), with $\text{supp}(\mathcal{F}_W(S)) \subset B_R(0)$ for some small $R > 0$, where $B_R(0) \subset \mathbb{R}^{2d}$ is the ball of radius R centered at 0. In this case $\{z' - z'' : z', z'' \in \text{supp}(\mathcal{F}_W(S))\} \subset B_{2R}(0)$, so by picking sufficiently small R the condition in the corollary can be satisfied. Such S may easily be constructed, for instance by picking a smooth bump function $f \in M^1(\mathbb{R}^{2d})$ supported in $B_R(0)$ – since $\mathcal{F}_W$ is bijective from

$\mathcal{B}$ to $M^1(\mathbb{R}^{2d})$, there exists some $S \in \mathcal{B}$ with $\mathcal{F}_W(S) = f$. By a result of Janssen [173] this simple construction will never work for Gabor frames: there is no rank-one operator $S = \psi \otimes \phi$ such that $\mathcal{F}_W(S)(x, \omega) = e^{\pi i x\cdot\omega} V_\phi \psi(x, \omega)$ has compact support.

(b) If Λ is a separable lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ for $a, b > 0$, then $\Lambda^\circ = \frac{1}{b}\mathbb{Z}^d \times \frac{1}{a}\mathbb{Z}^d$. It follows from the Janssen representation that if $\mathcal{F}_W(S^*S)(\frac{m}{b}, \frac{n}{a}) = 0$ whenever $0 \neq m \in \mathbb{Z}^d$, then the g-frame operator is simply the multiplication operator

$$\psi(t) \mapsto \left(\frac{1}{ab} \sum_{n\in\mathbb{Z}^d} \mathcal{F}_W(S^*S)\left(0, \frac{n}{a}\right) e^{2\pi i n\cdot t/a} \right) \psi(t).$$

If S is a rank-one operator $\phi \otimes \phi$, this can be achieved by picking compactly supported ϕ – this leads to the *painless nonorthogonal expansions* of [69].

The Wexler-Raz conditions sometimes allow us to deduce that S and T generate dual Gabor g-frames, or, when $S = T$, that S generates a tight Gabor g-frame. The Janssen representation also implies the following test for deciding when $S \in \mathcal{B}$ generates a (not necessarily tight) Gabor g-frame.

Proposition D.6.9. *Let $S \in \mathcal{B}$, and assume that $\sum_{0\neq\lambda^\circ\in\Lambda^\circ} |\mathcal{F}_W(S^*S)(\lambda^\circ)| < \|S\|_{\mathcal{HS}}^2$. Then S generates a Gabor g-frame.*

Proof. By the Janssen representation and the fact that $\mathcal{F}_W(S^*S)(0) = \mathrm{tr}(S^*S) = \|S\|_{\mathcal{HS}}^2 > 0$,

$$\begin{aligned} \mathfrak{S}_S &= \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(S^*S)(\lambda^\circ) e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega} \pi(\lambda^\circ) \\ &= \frac{\|S\|_{\mathcal{HS}}^2}{|\Lambda|} \underbrace{\left(I_{L^2} + \sum_{0\neq\lambda^\circ\in\Lambda^\circ} \frac{\mathcal{F}_W(S^*S)(\lambda^\circ)}{\|S\|_{\mathcal{HS}}^2} e^{-\pi i \lambda^\circ_x\cdot\lambda^\circ_\omega} \pi(\lambda^\circ) \right)}_{:=A}, \end{aligned}$$

so $\mathfrak{S}_S$ has a bounded inverse on $L^2(\mathbb{R}^d)$ if and only if A has a bounded inverse. As

$$\|A - I_{L^2}\|_{\mathcal{L}(L^2)} \leq \sum_{0\neq\lambda^\circ\in\Lambda^\circ} \frac{|\mathcal{F}_W(S^*S)(\lambda^\circ)|}{\|S\|_{\mathcal{HS}}^2} < 1,$$

by assumption, the Neumann theorem [131, Thm. A.3] implies that A has a bounded inverse on $L^2(\mathbb{R}^d)$. □

Remark D.11. When $S = \phi \otimes \phi$ for some $\phi \in M^1(\mathbb{R}^d)$, the proposition above becomes a well-known result for Gabor frames. To our knowledge the first appearance of this special case in the literature is [246, Thm. 4.1.1].

Corollary D.6.9.1. *Let $0 \neq S \in \mathcal{B}$ and Λ a lattice. There exists $N \in \mathbb{N}$ such that S generates a Gabor g-frame over the lattice $\frac{1}{N}\Lambda$.*

Proof. Since $\sum_{\lambda^\circ \in \Lambda} |\mathcal{F}_W(S^*S)(\lambda^\circ)| < \infty$ by Theorem D.6.4, there exists $K \in \mathbb{N}$ with

$$\sum_{|\lambda^\circ|>K} |\mathcal{F}_W(S^*S)(\lambda^\circ)| < \|S\|_{\mathcal{HS}}^2.$$

Let $N \in \mathbb{N}$ be the smallest integer such that $|\lambda^\circ| > K/N$ for any $0 \neq \lambda^\circ \in \Lambda^\circ$, and consider the lattice $\Gamma = \frac{1}{N}\Lambda$. Then $\Gamma^\circ = N\Lambda^\circ \subset \Lambda^\circ$. By definition, the non-zero elements $\gamma^\circ \in \Gamma^\circ$ are all of the form $\gamma^\circ = N\lambda^\circ$. In particular, they satisfy $|\gamma^\circ| > K$ and $\gamma^\circ \in \Lambda^\circ$. Therefore

$$\sum_{0\neq\gamma^\circ\in\Gamma^\circ} |\mathcal{F}_W(S^*S)(\gamma^\circ)| \leq \sum_{|\lambda^\circ|>K} |\mathcal{F}_W(S^*S)(\lambda^\circ)| < \|S\|_{\mathcal{HS}}^2,$$

hence S generates a Gabor g-frame with respect to $\Gamma = \frac{1}{N}\Lambda$ by Proposition D.6.9. □

D.7 Gabor g-frames and modulation spaces

It is a well-known fact that if a function $\phi \in M_v^1(\mathbb{R}^d)$ generates a Gabor frame, then the $\ell_m^p(\Lambda)$-norm of the coefficients $\{V_\phi\psi(\lambda)\}_{\lambda\in\Lambda}$ is an equivalent norm to $\|\psi\|_{M_m^p}$. To extend this result to Gabor g-frames, we will need to introduce some appropriate Banach spaces. Once this is done, our proofs will mainly proceed by reducing the statement for Gabor g-frames to the statement for Gabor frames, which may be found in the standard reference [131].

For $p \in [1,\infty]$ and a v-moderate weight m we define the space $\ell_m^p(\Lambda; L^2)$ to be the Banach space of sequences $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^d)$ such that

$$\|\{\psi_\lambda\}\|_{\ell_m^p(\Lambda;L^2)} := \left(\sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2}^p m(\lambda)^p\right)^{1/p} < \infty.$$

For $p = \infty$ the sum is replaced by a supremum in the usual way. For $m \equiv 1$ we write $\ell_m^p(\Lambda; L^2) = \ell^p(\Lambda; L^2)$. The dual space of $\ell_m^p(\Lambda; L^2)$ for $p < \infty$ is $\ell_{1/m}^{p'}(\Lambda; L^2)$ with

$$\langle \{\phi_\lambda\}, \{\psi_\lambda\}\rangle_{\ell_{1/m}^{p'}(\Lambda;L^2),\ell_m^p(\Lambda;L^2)} = \sum_{\lambda\in\Lambda} \langle \phi_\lambda, \psi_\lambda\rangle_{L^2} \tag{D.7.1}$$

for $\{\phi_\lambda\}_{\lambda\in\Lambda} \in \ell_{1/m}^{p'}(\Lambda; L^2)$, $\{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell_m^p(\Lambda; L^2)$. It is clear from the definitions that finite sequences $\{\psi_\lambda\}_{\lambda\in\Lambda}$ (meaning that $\psi_\lambda \neq 0$ for finitely many λ) are dense in $\ell_m^p(\Lambda; L^2)$ for $p < \infty$ and weak*-dense in $\ell_m^\infty(\Lambda; L^2)$.

Remark D.12. The norm $\|\{\psi_\lambda\}\|_{\ell^p_m(\Lambda;L^2)}$ equals $\|\{m(\lambda)\cdot\psi_\lambda\}\|_{L^p(\Lambda,L^2)}$, where $L^p(\Lambda, L^2)$ is a vector-valued L^p-space with Λ equipped with counting measure. Since $m(\lambda) > 0$ for any $\lambda \in \Lambda$, we may immediately translate results from the theory of vector-valued L^p-spaces, see Chapter 1 of [166], into statements about $\ell^p_m(\Lambda; L^2)$. In particular, they are Banach spaces and the duality (D.7.1) follows from [166, Prop. 1.3.3].

We have already met the space $\ell^2(\Lambda; L^2)$, and seen that C_S is bounded from $L^2(\mathbb{R}^d)$ into $\ell^2(\Lambda; L^2)$ when S generates a Gabor g-frame. The next result shows that this result can be generalized to other p and m when $S \in \mathcal{B}_{v\otimes v}$.

Theorem D.7.1. *If $S \in \mathcal{B}_{v\otimes v}$ and $p \in [1,\infty]$, then the analysis operator C_S is bounded from $M^p_m(\mathbb{R}^d)$ to $\ell^p_m(\Lambda; L^2)$ with operator norm $\|C_S\|_{M^p_m\to\ell^p_m(\Lambda;L^2)} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}$ where the implicit constant is independent of p and m.*

Proof. Let $S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}$, be a decomposition as in part (a) of Proposition D.4.1. Then

$$\begin{aligned}
\|\alpha_\lambda(S)\psi\|_{L^2} &= \left\| \left(\sum_{n\in\mathbb{N}} \pi(\lambda)\phi_n^{(1)} \otimes \pi(\lambda)\phi_n^{(2)} \right)\psi \right\|_{L^2} \\
&\leq \sum_{n\in\mathbb{N}} |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\pi(\lambda)\phi_n^{(1)}\|_{L^2} \\
&= \sum_{n\in\mathbb{N}} |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\phi_n^{(1)}\|_{L^2} \lesssim \sum_{n\in\mathbb{N}} |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\phi_n^{(1)}\|_{M^1_v},
\end{aligned}$$

where the last inequality uses $M^1_v(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$. Then assume that $p < \infty$, and use the inequality above and the triangle inequality for $\ell^p_m(\Lambda)$ to get

$$\begin{aligned}
&\left(\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\psi\|^p_{L^2} m(\lambda)^p \right)^{1/p} \\
&\qquad\lesssim \left(\sum_{\lambda\in\Lambda} \left(\sum_{n\in\mathbb{N}} |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\phi_n^{(1)}\|_{M^1_v} \right)^p m(\lambda)^p \right)^{1/p} \\
&\qquad= \left\| \sum_{n\in\mathbb{N}} \left\{ |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\phi_n^{(1)}\|_{M^1_v} \right\}_{\lambda\in\Lambda} \right\|_{\ell^p_m(\Lambda)} \\
&\qquad\leq \sum_{n\in\mathbb{N}} \left\| \left\{ |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot \|\phi_n^{(1)}\|_{M^1_v} \right\}_{\lambda\in\Lambda} \right\|_{\ell^p_m(\Lambda)} \\
&\qquad= \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \left\| \left\{ |V_{\phi_n^{(2)}}\psi(\lambda)| \right\}_{\lambda\in\Lambda} \right\|_{\ell^p_m(\Lambda)} \\
&\qquad\lesssim \|\psi\|_{M^p_m} \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} \qquad \text{by Lemma D.3.4.}
\end{aligned}$$

The norm inequality $\|C_S\|_{\text{op}} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}$ then follows from part (b) of Proposition D.4.1. For $p = \infty$, we use Lemma D.3.4 to find that for any $\lambda \in \Lambda$

$$\begin{aligned}\|\alpha_\lambda(S)\psi\|_{L^2} \cdot m(\lambda) &\lesssim \sum_{n\in\mathbb{N}} |V_{\phi_n^{(2)}}\psi(\lambda)| \cdot m(\lambda) \cdot \|\phi_n^{(1)}\|_{M_v^1} \\ &\leq \|\psi\|_{M_m^\infty} \sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{M_v^1} \|\phi_n^{(1)}\|_{M_v^1}. \end{aligned}$$

□

Theorem D.7.2. *If $S \in \mathcal{B}_{v\otimes v}$ and $p \in [1,\infty]$, then the synthesis operator D_S is bounded from $\ell^p_m(\Lambda; L^2)$ to $M^p_m(\mathbb{R}^d)$, with operator norm $\|D_S\|_{\ell^p_m(\Lambda;L^2)\to M^p_m} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}$ independent of p and m. For $\{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell^p_m(\Lambda; L^2)$, the expansion*

$$D_S(\{\psi_\lambda\}) = \sum_{\lambda\in\Lambda} \alpha_\lambda(S^*)\psi_\lambda$$

converges unconditionally in $M^p_m(\mathbb{R}^d)$ for $p < \infty$ and in the weak topology of $M^\infty_{1/v}(\mathbb{R}^d)$ for $p = \infty$.*

Proof. First assume that $p < \infty$, and let $\{\psi_\lambda\}_{\lambda\in\Lambda}$ be a finite sequence. Using Proposition D.4.1 we write $S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}$. Then one finds using (D.3.9) that

$$\begin{aligned} D_S(\{\psi_\lambda\}) &= \sum_{\lambda\in\Lambda}\sum_{n\in\mathbb{N}} V_{\phi_n^{(1)}}\psi_\lambda(\lambda)\pi(\lambda)\phi_n^{(2)} \\ &= \sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda} V_{\phi_n^{(1)}}\psi_\lambda(\lambda)\pi(\lambda)\phi_n^{(2)}. \end{aligned}$$

Interchanging the order of summation is allowed as the finiteness of the sum over λ implies absolute convergence in $M^p_m(\mathbb{R}^d)$: by parts (c) and (e) of Proposition D.3.2

$$\|\pi(\lambda)\phi_n^{(2)}\|_{M^p_m} \lesssim v(\lambda)\|\phi_n^{(2)}\|_{M^1_v},$$

and by Cauchy-Schwarz and $M^1_v(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$

$$|V_{\phi_n^{(1)}}\psi_\lambda(\lambda)| = \left|\left\langle \psi_\lambda, \pi(\lambda)\phi_n^{(1)}\right\rangle_{L^2}\right| \lesssim \|\psi_\lambda\|_{L^2}\|\phi_n^{(1)}\|_{M^1_v}. \tag{D.7.2}$$

Hence the absolute convergence follows by

$$\begin{aligned} &\sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda} |V_{\phi_n^{(1)}}\psi_\lambda(\lambda)| \cdot \|\pi(\lambda)\phi_n^{(2)}\|_{M^p_m} \\ &\qquad \lesssim \sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2}\|\phi_n^{(1)}\|_{M^1_v} \cdot v(\lambda) \cdot \|\phi_n^{(2)}\|_{M^1_v} \\ &\qquad = \left(\sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v}\|\phi_n^{(2)}\|_{M^1_v}\right)\left(\sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2} v(\lambda)\right) \\ &\qquad < \infty. \end{aligned}$$

Now apply the M^p_m-norm to our expression for $D_S(\{\psi_\lambda\})$. When passing to the second line, we use [131, Thm. 12.2.4], which is the Gabor frame version of the statement we are proving, and the implicit constant is independent of p and m.

$$\begin{aligned}
\|D_S(\{\psi_\lambda\})\|_{M^p_m} &\leq \sum_{n\in\mathbb{N}} \left\| \sum_{\lambda\in\Lambda} V_{\phi_n^{(1)}}\psi_\lambda(\lambda)\pi(\lambda)\phi_n^{(2)} \right\|_{M^p_m} \\
&\lesssim \sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{M^1_v} \|\{V_{\phi_n^{(1)}}\psi_\lambda\}\|_{\ell^p_m(\Lambda)} \\
&\leq \sum_{n\in\mathbb{N}} \|\phi_n^{(1)}\|_{M^1_v} \|\phi_n^{(2)}\|_{M^1_v} \|\{\|\psi_\lambda\|_{L^2}\}\|_{\ell^p_m(\Lambda)} \qquad \text{by (D.7.2)} \\
&= \|\{\psi_\lambda\}\|_{\ell^p_m(\Lambda;L^2)} \sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{M^1_v} \|\phi_n^{(1)}\|_{M^1_v}.
\end{aligned}$$

Since finite sequences are dense in $\ell^p_m(\Lambda; L^2)$, this shows that D_S extends to a bounded operator $\ell^p_m(\Lambda; L^2) \to M^p_m(\mathbb{R}^d)$ and $\|D_S\|_{\ell^p_m(\Lambda;L^2)\to M^p_m} \lesssim \|S\|_{\mathcal{B}_{v\otimes v}}$ follows from part (b) of Proposition D.4.1. The same proof works for $p = \infty$ when replacing the sum with a supremum. For the unconditional convergence for $p < \infty$, let $J \subset \Lambda$ be a finite subset and let $\{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell^p_m(\Lambda; L^2)$. Then

$$\begin{aligned}
\|D_S(\{\psi_\lambda\}) - \sum_{\lambda\in J} \alpha_\lambda(S^*)\psi_\lambda\|^p_{M^p_m(\mathbb{R}^d)} &= \|D_S(\{\psi_\lambda\}_{\lambda\in\Lambda} - \{\psi_\lambda\}_{\lambda\in J})\|^p_{M^p_m} \\
&\lesssim \|\{\psi_\lambda\}_{\lambda\in\Lambda} - \{\psi_\lambda\}_{\lambda\in J}\|^p_{\ell^p_m(\Lambda;L^2)} \\
&= \sum_{\lambda\in\Lambda\setminus J} \|\psi_\lambda\|^p_{L^2} m(\lambda)^p.
\end{aligned}$$

As the sum $\sum_{\lambda\in\Lambda} \|\psi_\lambda\|^p_{L^2} m(\lambda)^p$ converges by assumption, the estimate above shows that for any $\epsilon > 0$ we can find a finite subset $J_\epsilon \subset \Lambda$ such that $\|D_S(\{\psi_\lambda\}) - \sum_{\lambda\in J} \alpha_\lambda(S^*)\psi_\lambda\|^p_{M^p_m} < \epsilon$ whenever $J_\epsilon \subset J$. It follows that $\sum_{\lambda\in\Lambda} \alpha_\lambda(S^*)\psi_\lambda$ converges to $D_S(\{\psi_\lambda\})$ in the sense that the net of partial sums converges, which implies unconditional convergence [131, Prop. 5.3.1].
If $p = \infty$, let $\phi \in M^1_v(\mathbb{R}^d)$. Then

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} |\langle \alpha_\lambda(S^*)\psi_\lambda, \phi\rangle_{M^\infty_{1/v}, M^1_v}| \\
&\qquad = \sum_{\lambda\in\Lambda} |\langle \psi_\lambda, \alpha_\lambda(S)\phi\rangle_{L^2}| \qquad \text{by Prop. D.4.1 (c)} \\
&\qquad \leq \sum_{\lambda\in\Lambda} \|\psi_\lambda\|_{L^2} \frac{1}{v(\lambda)} \|\alpha_\lambda(S)\phi\|_{L^2} v(\lambda) \qquad \text{by Cauchy-Schwarz} \\
&\qquad \leq \|\{\psi_\lambda\}\|_{\ell^\infty_{1/v}(\Lambda;L^2)} \|C_S(\phi)\|_{\ell^1_v(\Lambda,L^2)} \\
&\qquad \lesssim \|\{\psi_\lambda\}\|_{\ell^\infty_{1/v}(\Lambda;L^2)} \|S\|_{\mathcal{B}_{v\otimes v}} \|\phi\|_{M^1_v} \qquad \text{by Theorem D.7.1.}
\end{aligned}$$

Hence the sum $\sum_{\lambda\in\Lambda} \langle \alpha_\lambda(S^*)\psi_\lambda, \phi\rangle_{M^\infty_{1/v}, M^1_v}$ converges absolutely for $\phi \in M^1_v(\mathbb{R}^d)$. □

When $p < \infty$, $\{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell^p_m(\Lambda; L^2)$ and $\phi \in M^{p'}_{1/m}(\mathbb{R}^d)$, one finds that

$$\begin{aligned}\langle \phi, D_S(\{\psi_\lambda\})\rangle_{M^{p'}_{1/m}, M^p_m} &= \left\langle \phi, \sum_{\lambda\in\Lambda} \alpha_\lambda(S^*)\psi_\lambda \right\rangle_{M^{p'}_{1/m}, M^p_m} \\ &= \sum_{\lambda\in\Lambda} \langle \phi, \alpha_\lambda(S^*)\psi_\lambda\rangle_{M^{p'}_{1/m}, M^p_m} \\ &= \sum_{\lambda\in\Lambda} \langle \alpha_\lambda(S)\phi, \psi_\lambda\rangle_{L^2} \qquad \text{by Prop. D.4.1 (c)} \\ &= \langle C_S(\phi), \{\psi_\lambda\}_{\lambda\in\Lambda}\rangle_{\ell^{p'}_{1/m}(\Lambda;L^2), \ell^p_m(\Lambda;L^2)}.\end{aligned}$$

In the same way, when $\{\psi_\lambda\}_{\lambda\in\Lambda} \in \ell^{p'}_{1/m}(\Lambda; L^2)$ and $\phi \in M^p_m(\mathbb{R}^d)$, one shows that

$$\langle D_S(\{\psi_\lambda\}), \phi\rangle_{M^{p'}_{1/m}, M^p_m} = \langle \{\psi_\lambda\}, C_S(\phi)\rangle_{\ell^{p'}_{1/m}(\Lambda;L^2), \ell^p_m(\Lambda;L^2)}.$$

These calculations and the fact that Banach space adjoints are weak*-to-weak*-continuous imply the following result.

Corollary D.7.2.1. *Let $p < \infty$. The analysis operator*

$$C_S : M^{p'}_{1/m}(\mathbb{R}^d) \to \ell^{p'}_{1/m}(\Lambda; L^2)$$

is the Banach space adjoint of the synthesis operator

$$D_S : \ell^p_m(\Lambda; L^2) \to M^p_m(\mathbb{R}^d).$$

Similarly, the synthesis operator $D_S : \ell^{p'}_{1/m}(\Lambda; L^2) \to M^{p'}_{1/m}(\mathbb{R}^d)$ is the Banach space adjoint of the analysis operator $C_S : M^p_m(\mathbb{R}^d) \to \ell^p_m(\Lambda; L^2)$. In particular, both $C_S : M^{p'}_{1/m}(\mathbb{R}^d) \to \ell^{p'}_{1/m}(\Lambda; L^2)$ and $D_S : \ell^{p'}_{1/m}(\Lambda; L^2) \to M^{p'}_{1/m}(\mathbb{R}^d)$ are weak-to-weak*-continuous.*

Using the Janssen representation, we deduced in Corollary D.6.7.1 that if $S \in \mathcal{B}_{v\otimes v}$ generates a Gabor g-frame, then $\mathfrak{S}_S^{-1}$ has a representation

$$\mathfrak{S}_S^{-1} = \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\pi(\lambda^\circ)$$

for some sequence $\{c_{\lambda^\circ}\} \in \ell^1_v(\Lambda^\circ)$. Since $\pi(\lambda^\circ)$ is bounded on any modulation space $M^p_m(\mathbb{R}^d)$ by Proposition D.3.2, we find that $\mathfrak{S}_S^{-1}$ extends to a bounded operator on any modulation space by

$$\begin{aligned}\|\mathfrak{S}_S^{-1}\psi\|_{M_m^p} &\leq \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}|\|\pi(\lambda^\circ)\psi\|_{M_m^p}\\ &\lesssim \sum_{\lambda^\circ\in\Lambda^\circ}|c_{\lambda^\circ}|v(\lambda^\circ)\|\psi\|_{M_m^p} = \|\psi\|_{M_m^p}\|\{c_{\lambda^\circ}\}\|_{\ell_v^1(\Lambda^\circ)}.\end{aligned}$$

Then recall that the canonical dual Gabor g-frame is generated by the operator $S\mathfrak{S}_S^{-1}$. The next result shows that $S\mathfrak{S}_S^{-1}$ also satisfies the assumptions of Theorems D.7.1 and D.7.2.

Proposition D.7.3. *If $S \in \mathcal{B}_{v\otimes v}$ generates a Gabor g-frame, then $S\mathfrak{S}_S^{-1} \in \mathcal{B}_{v\otimes v}$.*

Proof. Let

$$S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)},$$

be a decomposition of S from Proposition D.4.1. For $\psi \in L^2(\mathbb{R}^d)$, this implies that

$$\begin{aligned}S\mathfrak{S}_S^{-1}\psi &= \sum_{n\in\mathbb{N}}\left\langle \mathfrak{S}_S^{-1}\psi, \phi_n^{(2)}\right\rangle_{L^2}\phi_n^{(1)}\\ &= \sum_{n\in\mathbb{N}}\left\langle \psi, \mathfrak{S}_S^{-1}\phi_n^{(2)}\right\rangle_{L^2}\phi_n^{(1)},\end{aligned}$$

where we have used that $\mathfrak{S}_S^{-1}$ is positive and therefore self-adjoint. Hence

$$S\mathfrak{S}_S^{-1} = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes (\mathfrak{S}_S^{-1}\phi_n^{(2)}),$$

and this decomposition converges absolutely in $\mathcal{B}_{v\otimes v}$ since

$$\begin{aligned}\sum_{n\in\mathbb{N}}\|\phi_n^{(1)} \otimes (\mathfrak{S}_S^{-1}\phi_n^{(2)})\|_{\mathcal{B}_{v\otimes v}} &= \sum_{n\in\mathbb{N}}\|\phi_n^{(1)}\|_{M_v^1}\|\mathfrak{S}_S^{-1}\phi_n^{(2)}\|_{M_v^1}\\ &\lesssim \sum_{n\in\mathbb{N}}\|\phi_n^{(1)}\|_{M_v^1}\|\phi_n^{(2)}\|_{M_v^1} < \infty\end{aligned}$$

by the aforementioned boundedness of $\mathfrak{S}_S^{-1} : M_v^1(\mathbb{R}^d) \to M_v^1(\mathbb{R}^d)$. □

Corollary D.7.3.1. *Assume that $S \in \mathcal{B}_{v\otimes v}$ generates a Gabor g-frame. For any $\psi \in M_m^p(\mathbb{R}^d)$, the expansions*

$$\psi = D_S C_{S\mathfrak{S}_S^{-1}}\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(S^*)\alpha_\lambda(S\mathfrak{S}_S^{-1})\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(S^*S\mathfrak{S}_S^{-1})\psi,$$

$$\psi = D_{S\mathfrak{S}_S^{-1}} C_S\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda((S\mathfrak{S}_S^{-1})^*)\alpha_\lambda(S)\psi = \sum_{\lambda\in\Lambda}\alpha_\lambda(\mathfrak{S}_S^{-1}S^*S)\psi$$

converge unconditionally in $M_m^p(\mathbb{R}^d)$ for $p < \infty$ and in the weak topology of $M_{1/v}^\infty(\mathbb{R}^d)$ for $p = \infty$.*

Proof. We prove the result for $D_S C_{S\mathfrak{S}_S^{-1}}$, the same proof works for $D_{S\mathfrak{S}_S^{-1}} C_S$. From the previous proposition, we know that $S, S\mathfrak{S}_S^{-1} \in \mathcal{B}_{v\otimes v}$. In particular we know from Theorem D.7.2 that D_S is bounded from $\ell^p_m(\Lambda; L^2)$ to $M^p_m(\mathbb{R}^d)$, and that $C_{S\mathfrak{S}_S^{-1}}$ is bounded from $M^p_m(\mathbb{R}^d)$ to $\ell^p_m(\Lambda; L^2)$. Hence $D_S C_{S\mathfrak{S}_S^{-1}}$ is bounded on $M^p_m(\mathbb{R}^d)$. If $p < \infty$, then the expansions in the statement converge unconditionally by Theorem D.7.2. We know that $D_S C_{S\mathfrak{S}_S^{-1}}$ is the identity operator on $L^2(\mathbb{R}^d)$ from Section D.5.2, and as $M^1_v(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ is dense in $M^p_m(\mathbb{R}^d)$ by Proposition D.3.2 it follows that $D_S C_{\mathfrak{S}_S^{-1}}$ is the identity operator on $M^p_m(\mathbb{R}^d)$, so the expansions converge to ψ.

For $p = \infty$ the last part of the argument must be slightly modified: $M^1_v(\mathbb{R}^d)$ is only weak*-dense in $M^\infty_m(\mathbb{R}^d)$, so to conclude that $D_S C_{S\mathfrak{S}_S^{-1}}$ is the identity operator on $M^\infty_m(\mathbb{R}^d)$ we need to use that $D_S C_{S\mathfrak{S}_S^{-1}}$ is weak*-to-weak*-continuous on $M^\infty_m(\mathbb{R}^d)$ by Corollary D.7.2.1. □

We are now ready to prove one of our main results, namely that Gabor g-frames generated by $S \in \mathcal{B}_{v\otimes v}$ define equivalent norms for modulation spaces. By picking S as in Examples D.5.1 and D.5.2, we recover results for Gabor frames [100, 101, 131] and localization operators [85, 87].

Corollary D.7.3.2. *Assume that $S \in \mathcal{B}_{v\otimes v}$ generates a Gabor g-frame. There exist constants C, D depending on v and Λ such that for any $1 \leq p \leq \infty$ and v-moderate weight m we have*

$$C\|\psi\|_{M^p_m} \leq \left(\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\|^p_{L^2} m(\lambda)^p \right)^{1/p} \leq D\|\psi\|_{M^p_m},$$

and $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$ belongs to $M^p_m(\mathbb{R}^d)$ if and only if

$$\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\|^p_{L^2} m(\lambda)^p < \infty.$$

For $p = \infty$ the sum is replaced by a supremum in the usual way.

Proof. By Theorem D.7.1 we have

$$\left(\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\|^p_{L^2} m(\lambda)^p \right)^{1/p} = \|C_S\psi\|_{\ell^p_m(\Lambda;L^2)} \lesssim \|\psi\|_{M^p_m}$$

as C_S is bounded. On the other hand, Corollary D.7.3.1 says that

$$\|\psi\|_{M^p_m} = \|D_{S\mathfrak{S}_S^{-1}} C_S\psi\|_{M^p_m} \lesssim \|C_S\psi\|_{\ell^p_m(\Lambda;L^2)} = \left(\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\|^p_{L^2} m(\lambda)^p \right)^{1/p},$$

where we have used that $D_{S\mathfrak{S}_S^{-1}} : \ell^p_m(\Lambda; L^2) \to M^p_m(\mathbb{R}^d)$ is bounded by Proposition D.7.3 and Theorem D.7.2.

Finally, if $\sum_{\lambda\in\Lambda} \|\alpha_\lambda(S)\|^p_{L^2} m(\lambda)^p < \infty$, then $C_S(\psi) \in \ell^p_m(\Lambda; L^2)$. As $D_{S\mathfrak{S}_S^{-1}}$ is bounded $\ell^p_m(\Lambda; L^2) \to M^p_m(\mathbb{R}^d)$, its follows from $\psi = D_{S\mathfrak{S}_S^{-1}} C_S\psi$ that $\psi \in M^p_m(\mathbb{R}^d)$. □

Remark D.13. In this section we have assumed $S \in \mathcal{B}_{v\otimes v}$, but the result also holds for operators $S \in \mathcal{S}^1$ that can be written

$$S = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \phi_n^{(2)}$$

where $\sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{M^1_v} < \infty$ and $\{\phi_n^{(1)}\}_{n\in\mathbb{N}}$ is orthonormal in $L^2(\mathbb{R}^d)$. The proofs of Theorems D.7.1 and D.7.2 still work, with upper bound $\sum_{n\in\mathbb{N}} \|\phi_n^{(2)}\|_{M^1_v}$ for the operator norms of C_S and D_S (in the original proofs we use $\|\phi_n^{(1)}\|_{L^2} \lesssim \|\phi_n^{(1)}\|_{M^1_v}$, using $\|\phi_n^{(1)}\|_{L^2} = 1$ instead leads to this modified result). Since $S^*S = \sum_{n\in\mathbb{N}} \phi_n^{(2)} \otimes \phi_n^{(2)} \in \mathcal{B}$, we can still use the Janssen representation to get that $\mathfrak{S}_S^{-1}$ is bounded on $M^1_v(\mathbb{R}^d)$, and the proof of Proposition D.7.3 shows that

$$S\mathfrak{S}_S^{-1} = \sum_{n\in\mathbb{N}} \phi_n^{(1)} \otimes \mathfrak{S}_S^{-1}\phi_n^{(2)},$$

hence $S\mathfrak{S}_S^{-1}$ is of the same form. The proofs of the corollaries above still work without change. In particular, this shows that our treatment of multi-window Gabor frames in Example D.5.1 is compatible with the theory of this section.

D.7.1 An alternative characterization of Gabor g-frames using multi-window Gabor frames of eigenfunctions

The norm equivalences in Corollary D.7.3.2 were proved for localization operators in [85, 87]. This section is mainly a reinterpretation and slight extension of the results in [87] in terms of Gabor g-frames – the main result is Theorem D.7.6, which shows that a surprising characterization of Gabor frames from [134] holds for Gabor g-frames. We first need to understand the singular value decomposition of operators in $\mathcal{B}_{v\otimes v}$. The following is due to [87] when S is a localization operator, and our proof is a slight modification of their proof to allow general $S \in \mathcal{B}_{v\otimes v}$.

Lemma D.7.4. *Assume that $S \in \mathcal{B}_{v\otimes v}$. There exist $N_0 \in \mathbb{N} \cup \{\infty\}$, orthonormal systems $\{\xi_n\}_{n=1}^{N_0}$, $\{\varphi_n\}_{n=1}^{N_0}$ in $L^2(\mathbb{R}^d)$ and a sequence $\{s_n\}_{n=1}^{N_0} \in \ell^1$ of positive numbers with*

$$S = \sum_{n=1}^{N_0} s_n \xi_n \otimes \varphi_n \tag{D.7.3}$$

as an operator on $L^2(\mathbb{R}^d)$. Furthermore, $\varphi_n, \xi_n \in M^1_v(\mathbb{R}^d)$, and for $\lambda \in \Lambda$ the expansion

$$\alpha_\lambda(S)\psi = \sum_{n=1}^{N_0} s_n \langle \psi, \pi(\lambda)\varphi_n\rangle_{M^\infty_{1/v}, M^1_v} \pi(\lambda)\xi_n \tag{D.7.4}$$

holds even for $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$, with convergence of the sum in $L^2(\mathbb{R}^d)$.

Proof. The existence of $\{\xi_n\}_{n=1}^{N_0}$, $\{\varphi_n\}_{n=1}^{N_0}$ and $\{s_n\}_{n=1}^{N_0}$ with these properties is the singular value decomposition from Section D.3.2. To see that $\xi_n \in M^1_v(\mathbb{R}^d)$, note that Proposition D.4.1 says that $S : M^\infty_{1/v}(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)$. From (D.7.3) one obtains that $M^1_v(\mathbb{R}^d) \ni S\varphi_n = s_n\xi_n$, which forces $\xi_n \in M^1_v(\mathbb{R}^d)$ when $s_n \neq 0$. Since $S^* \in \mathcal{B}_{v\otimes v}$ by Proposition D.4.1, the same argument as above gives that $M^1_v(\mathbb{R}^d) \ni S^*\xi_n = s_n\varphi_n$, so $\varphi_n \in M^1_v(\mathbb{R}^d)$.

We prove the expansion (D.7.4) for $\lambda = 0$, without loss of generality. If $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$, we know from Proposition D.4.1 that $S\psi \in M^1_v(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Thus we may find $\gamma \in L^2(\mathbb{R}^d)$ such that

$$S\psi = \sum_{n=1}^{N_0} \langle S\psi, \xi_n\rangle_{L^2} \xi_n + \gamma,$$

where $\gamma \perp \xi_n$ for each $n \leq N_0$. The sum converges in $L^2(\mathbb{R}^d)$ as $S\psi \in L^2(\mathbb{R}^d)$ and the set $\{\xi_n\}_{n=1}^{N_0}$ is orthonormal. By Proposition D.4.1, we get

$$\langle S\psi, \xi_n\rangle_{L^2} = \langle S\psi, \xi_n\rangle_{M^\infty_{1/v}, M^1_v} = \langle \psi, S^*\xi_n\rangle_{M^\infty_{1/v}, M^1_v} = s_n \langle \psi, \varphi_n\rangle_{M^\infty_{1/v}, M^1_v},$$

hence we have shown

$$S\psi = \sum_{n=1}^{N_0} s_n \langle \psi, \varphi_n\rangle_{M^\infty_{1/v}, M^1_v} \xi_n + \gamma,$$

and it simply remains to show that $\gamma = 0$. Note that $\|\gamma\|^2_{L^2} = \langle S\psi, \gamma\rangle_{L^2}$. As is shown in the proof of [87, Cor. 7], we can pick a sequence $\{\psi_i\}_{i\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ that converges to ψ in the weak* topology of $M^\infty_{1/v}(\mathbb{R}^d)$. Then $\langle S\psi_i, \gamma\rangle_{L^2} = 0$, since (D.7.3) shows that $S\psi_i$ can be expanded in terms of the ξ_n, and γ is orthogonal to each ξ_n. However, S maps weak*-convergent sequences in $M^\infty_{1/v}(\mathbb{R}^d)$ into norm convergent sequences in $M^1_v(\mathbb{R}^d)$, hence $S\psi_i \to S\psi$ in $L^2(\mathbb{R}^d)$ and

$$0 = \langle S\psi_i, \gamma\rangle_{L^2} \to \langle S\psi, \gamma\rangle_{L^2} = \|\gamma\|_{L^2},$$

which completes the proof. □

Remark D.14. The singular value decomposition in Lemma D.7.4 should be compared to the decomposition from Proposition D.4.1. There is one clear advantage to the singular value decomposition $S = \sum_{n=1}^{N_0} s_n \xi_n \otimes \varphi_n$, namely that the systems $\{\xi_n\}_{n=1}^{N_0}$ and $\{\varphi_n\}_{n=1}^{N_0}$ are orthonormal. The disadvantage of the singular value decomposition is that, unlike the decomposition from Proposition D.4.1, it does not necessarily converge absolutely in the norm of $\mathcal{B}_{v\otimes v}$. In other words, we cannot guarantee that $\sum_{n=1}^{N_0} s_n \|\xi_n\|_{M^1_v} \|\varphi_n\|_{M^1_v} < \infty$. This was recently proved in [17], solving a problem first posed by Hans Feichtinger.

The following result is used in the proof of [87, Lem. 9] for localization operators S. Our proof is a slight modification of the proof in [87] to allow general $S \in \mathcal{B}$.

Proposition D.7.5. *Assume that $S \in \mathcal{B}$ and let $\{\varphi_n\}_{n=1}^{N_0}$ be as in Lemma D.7.4. If $C_S : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda; L^2)$ is injective, then there is some finite $N \leq N_0$ such that $\{\varphi_n\}_{n=1}^{N} \subset M^1_v(\mathbb{R}^d)$ generate a multi-window Gabor frame.*

Proof. Assume that, for any $N \leq N_0$, $\{\varphi_n\}_{n=1}^{N}$ does not generate a multi-window Gabor frame. Consider the set

$$\mathcal{W}_N = \{\eta \in M^\infty(\mathbb{R}^d) : \langle \eta, \pi(\lambda)\varphi_n \rangle_{M^\infty, M^1} = 0 \text{ for any } \lambda \in \Lambda, n = 1, ..., N.\}$$

By [87, Lem. 3], $\mathcal{W}_N$ is a non-trivial subspace of $M^\infty(\mathbb{R}^d)$, and by [87, Lem. 10], the intersection of all $\mathcal{W}_N$ for $N \leq N_0$ is a non-trivial subspace of $M^\infty(\mathbb{R}^d)$. Let η be a non-zero element from this intersection, meaning that

$$\langle \eta, \pi(\lambda)\varphi_n \rangle_{M^\infty, M^1} = 0 \text{ for any } \lambda \in \Lambda, n \leq N_0.$$

By (D.7.4), we have that $\alpha_\lambda(S)\eta = \sum_{n=1}^{N_0} s_n \langle \eta, \pi(\lambda)\varphi_n \rangle_{M^\infty, M^1} \pi(\lambda)\xi_n = 0$ for any $\lambda \in \Lambda$, since $\langle \eta, \pi(\lambda)\varphi_n \rangle_{M^\infty, M^1} = 0$ for $n \leq N_0$. This means that $C_S\eta = 0$. Thus $\eta = 0$, which contradicts our assumption. Hence there is an $N \leq N_0$ such that $\{\varphi_n\}_{n=1}^{N}$ generates a multi-window Gabor frame. □

For Gabor frames, the following theorem is one of the main results of [134], and the reader who has consulted the proof of Proposition D.7.5 may have noted that the Gabor frame-version of the statement is the key to the proof of that proposition.

Theorem D.7.6. *Let $S \in \mathcal{B}$. S generates a Gabor g-frame if and only if $C_S : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda; L^2)$ is injective.*

Proof. If S generates a Gabor g-frame, $D_{S\mathfrak{S}_S^{-1}} C_S$ is the identity operator on $M^\infty(\mathbb{R}^d)$ by Corollary D.7.3.1, hence C_S is injective. Then assume that C_S is injective. Since $S \in \mathcal{B}$, Corollary D.5.4.1 says that the upper g-frame bound in (D.5.1) is satisfied. For the lower bound, Lemma D.7.4 and Proposition D.7.5 say that $S = \sum_{n=1}^{N_0} s_n \xi_n \otimes \varphi_n$, where $\{\varphi_n\}_{n=1}^{N}$ generate a multi-window Gabor frame for some $N \leq N_0$. Note that

$$\|\alpha_\lambda(S)\psi\|_{L^2}^2 = \langle \alpha_\lambda(S)\psi, \alpha_\lambda(S)\psi \rangle_{L^2} = \langle \alpha_\lambda(S^*S)\psi, \psi \rangle_{L^2}.$$

By the decomposition $S = \sum_{n=1}^{N_0} s_n \xi_n \otimes \varphi_n$ and orthonormality of $\{\xi_n\}_{n=1}^{N_0}$, we get

$$\alpha_\lambda(S^*S) = \sum_{n=1}^{N_0} s_n^2 \pi(\lambda)\varphi_n \otimes \pi(\lambda)\varphi_n,$$

hence

$$\begin{aligned}
\sum_{\lambda \in \Lambda} \|\alpha_\lambda(S)\psi\|_{L^2}^2 &= \sum_{\lambda \in \Lambda} \left\langle \sum_{n=1}^{N_0} s_n^2 V_{\varphi_n}\psi(\lambda)\pi(\lambda)\varphi_n, \psi \right\rangle_{L^2} \\
&= \sum_{\lambda \in \Lambda} \sum_{n=1}^{N_0} s_n^2 |V_{\varphi_n}\psi(\lambda)|^2 \\
&\geq \sum_{\lambda \in \Lambda} \sum_{n=1}^{N} s_n^2 |V_{\varphi_n}\psi(\lambda)|^2 \\
&\gtrsim \|\psi\|_2^2,
\end{aligned}$$

since $\{\varphi_n\}_{n=1}^{N}$ generate a multi-window Gabor frame and $s_n > 0$ for $n \leq N$. □

D.7.2 Localization operators and time-frequency partitions

In [85, 87], the methods from the previous section were used to prove the norm equivalence in Corollary D.7.3.2 for the localization operators A_h^φ in Example D.5.2, i.e. assuming $0 \neq \varphi \in M_v^1(\mathbb{R}^d)$ and $h \in L_v^1(\mathbb{R}^{2d})$ a non-negative function satisfying

$$A' \leq \sum_{\lambda \in \Lambda} h(z - \lambda) \leq B' \quad \text{for all } z \in \mathbb{R}^{2d}$$

for some $A', B' > 0$. Their proof consists of applying Proposition D.7.5 to obtain multi-window Gabor frames of eigenfunctions of localization operators to reduce the statement to the fact that multi-window Gabor frames give equivalent norms for $M_m^p(\mathbb{R}^d)$. Since inserting $p = 2$ and $m \equiv 1$ in Corollary D.7.3.2 gives the Gabor g-frame inequality, this means in particular that these localization operators generate Gabor g-frames.

Remark D.15. Obtaining multi-window Gabor frames consisting of eigenfunctions of localization operators is itself an interesting result. Dörfler and Romero [88] use techniques from [225] to obtain frames consisting of eigenfunctions of localization operators in more general settings. If $S = A^{\varphi}_{\chi_\Omega}$, then $\alpha_\lambda(S) = A^{\varphi}_{\chi_{\Omega+\lambda}}$. In this sense, applying α_λ corresponds to covering $\mathbb{R}^{2d}$ by shifts of Ω, and the results of [88] consider much more general coverings of $\mathbb{R}^{2d}$ when S is a localization operator.

In order to apply the machinery of Section D.7 to localization operators A^{φ}_h, we need to show that $A^{\varphi}_h \in \mathcal{B}_{v\otimes v}$. The next proposition shows that this is true if we assume the stronger condition $h \in L^1_{v^2}(\mathbb{R}^{2d})$.

Proposition D.7.7. *Let $\varphi \in M^1_v(\mathbb{R}^d)$ and $h \in L^1_{v^2}(\mathbb{R}^{2d})$. Then $A^{\varphi}_h \in \mathcal{B}_{v\otimes v}$.*

Proof. It is a straightforward calculation to check that the kernel of A^{φ}_h is

$$k_{A^{\varphi}_h}(x,\omega) = \int_{\mathbb{R}^{2d}} h(t,\xi)(M_\xi T_t \varphi)(x)(M_{-\xi}T_t\overline{\varphi})(\omega)\ dtd\xi.$$

For each t and ξ, the function

$$M_\xi T_t\varphi(x) M_{-\xi}T_t\overline{\varphi}(\omega) = (\pi(t,\xi)\varphi \otimes \pi(t,-\xi)\overline{\varphi})\,(x,\omega)$$

belongs to $M^1_{v\otimes v}$ by (D.4.1) and part (e) of Proposition D.3.2, with

$$\|\pi(t,\xi)\varphi\otimes\pi(t,-\xi)\overline{\varphi}\|_{M^1_{v\otimes v}} = \|\pi(t,\xi)\varphi\|_{M^1_v}\|\pi(t,-\xi)\overline{\varphi}\|_{M^1_v} \leq v(t,\xi)^2\|\varphi\|^2_{M^1_v},$$

where we have used that v is symmetric in each coordinate. Hence

$$\int_{\mathbb{R}^{2d}} \|h(t,\xi)\ (\pi(t,\xi)\varphi\otimes\pi(t,-\xi)\overline{\varphi})\ \|_{M^1_{v\otimes v}}\ dtd\xi$$

is bounded from above by

$$\|\varphi\|^2_{M^1_v}\int_{\mathbb{R}^{2d}} |h(t,\xi)|v(t,\xi)^2\ dtd\xi,$$

and this last integral converges by assumption. It follows that the integral

$$\int_{\mathbb{R}^{2d}} h(t,\xi)\ (\pi(t,\xi)\varphi\otimes\pi(t,-\xi)\overline{\varphi})\ \ dtd\xi$$

is a convergent Bochner integral in $M^1_{v\otimes v}(\mathbb{R}^{2d})$, thus $k_{A^{\varphi}_h} \in M^1_{v\otimes v}(\mathbb{R}^{2d})$. □

The setting $S = A^{\varphi}_h$ allows us to interpret many objects and results for Gabor g-frames in a natural way, in particular when $h = \chi_\Omega \in L^1_{v^2}(\mathbb{R}^{2d})$ is the characteristic function of some compact $\Omega \subset \mathbb{R}^{2d}$. Since one has the well-known inversion formula

$$\psi = \int_{\mathbb{R}^{2d}} V_\varphi \psi(z) \pi(z) \varphi \, dz \quad \text{whenever } \|\varphi\|_{L^2} = 1,$$

one interprets

$$A^\varphi_{\chi_\Omega} \psi = \int_\Omega V_\varphi \psi(z) \pi(z) \varphi \, dz$$

as the part of ψ that "lives in Ω in the time-frequency plane" [63]. For brevity, we call $A^\varphi_{\chi_\Omega}\psi$ the Ω-component of ψ. Since $\alpha_\lambda(A^\varphi_{\chi_\Omega}) = A^\varphi_{T_\lambda(\chi_\Omega)}$, we see that $\alpha_\lambda(A^\varphi_{\chi_\Omega})\psi$ is the $\lambda + \Omega$-component of ψ, where $\lambda + \Omega = \{\lambda + z : z \in \Omega\}$. The corresponding analysis operator

$$C_{A^\varphi_{\chi_\Omega}}(\psi) = \left\{ A^\varphi_{T_\lambda(\chi_\Omega)} \psi \right\}_{\lambda \in \Lambda}$$

therefore analyzes ψ by considering its $\lambda + \Omega$-components as λ varies over Λ.

When $A^\varphi_{\chi_\Omega}$ actually generates a Gabor g-frame, Corollary D.7.3.2 says that summability conditions on the L^2-norm of the $\lambda + \Omega$-components of ψ precisely captures the modulation space norms of ψ, as first proved by [85, 87]. Furthermore, Corollary D.7.3.1 shows us *how* ψ may be reconstructed from its $\lambda + \Omega$-components. By that result, there exists some $R := A^\varphi_{\chi_\Omega} \mathfrak{S}^{-1}_{A^\varphi_{\chi_\Omega}} \in \mathcal{B}_{v \otimes v}$ such that

$$\psi = \sum_{\lambda \in \Lambda} \alpha_\lambda(R) \left(A^\varphi_{T_\lambda(\chi_\Omega)} \psi \right), \tag{D.7.5}$$

with unconditional convergence in whatever modulation space $M^p_m(\mathbb{R}^d)$, $p < \infty$, that ψ belongs to. By Remark D.6, there is also a Cohen's class distribution associated with $A^\varphi_{\chi_\Omega}$, namely

$$Q_{\left(A^\varphi_{\chi_\Omega}\right)^2}(\psi)(z) = \|A^\varphi_{T_z(\chi_\Omega)} \psi\|^2_{L^2}.$$

This Cohen's class distributions has an obvious interpretation: $\|A^\varphi_{T_z(\chi_\Omega)}\psi\|^2_{L^2}$ measures the size of the $z + \Omega$-component of ψ. By (D.5.2) one has the equality

$$\int_{\mathbb{R}^{2d}} \|A^\varphi_{T_z(\chi_\Omega)} \psi\|^2_{L^2} \, dz = \|A^\varphi_{\chi_\Omega}\|^2_{\mathcal{HS}} \|\psi\|^2_{L^2}.$$

This is a continuous version of the Gabor g-frame inequality (D.5.1) for localization operators, in the same way that Moyal's identity is the continuous version of the Gabor frame inequalities.

It should be remarked that one usually associates a different Cohen's class distribution (independently of Ω) with localization operators $A^\varphi_{\chi_\Omega}$, namely the spectrogram $|V_\varphi \psi(z)|^2$ [204, Example 8.1].

Remark D.16. (a) Let us clarify the relation between our results and those of [87]. As mentioned, Corollary D.7.3.2 was proved in [87] for localization operators A_h^φ satisfying the conditions in Example D.5.2, without the notion of Gabor g-frames. The statements in Section D.7.1 may all be deduced from proofs in [87], and we have merely reinterpreted them as natural statements about Gabor g-frames. Proposition D.7.7 says that if we assume $h \in L^1_{v^2}(\mathbb{R}^{2d})$ – a stronger condition than $h \in L^1_v(\mathbb{R}^{2d})$ as assumed in [87] – then A_h^φ satisfies the assumptions for the other results in Section D.7. In particular, we get the inversion formula (D.7.5).

(b) The discussion above generalizes without change to other Gabor g-frames $\{\alpha_\lambda S\}_{\lambda\in\Lambda}$, but the natural interpretation of $\|\alpha_\lambda(S)\|^2_{L^2}$ above does not necessarily hold when S is not a localization operator.

D.8 Singular value decomposition and multi-window Gabor frames

From the very first paper published on g-frames [239], it has been known that g-frames correspond to ordinary frames when a basis is chosen for the Hilbert spaces involved: if $\{A_i\}_{i\in I} \subset \mathcal{L}(L^2)$ and $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, then $\{A_i\}_{i\in I}$ is g-frame if and only if $\{A_i^*\xi_n\}_{i\in I,n\in\mathbb{N}}$ is a frame for $L^2(\mathbb{R}^d)$ [239, Thm. 3.1]. Gabor g-frames must therefore be related to frames in $L^2(\mathbb{R}^d)$, and we will now make this connection explicit. By the singular value decomposition, any $S \in \mathcal{HS}$ may be expanded as

$$S = \sum_{n\in\mathbb{N}} \xi_n \otimes \varphi_n,$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ and $\sum_{n\in\mathbb{N}} \|\varphi_n\|^2_{L^2} < \infty$. For $\psi \in L^2(\mathbb{R}^d)$ we find using (D.3.9) that

$$\begin{aligned}
\|\alpha_\lambda(S)\psi\|^2_{L^2} &= \left\langle \sum_{n\in\mathbb{N}} V_{\varphi_n}\psi(\lambda)\pi(\lambda)\xi_n, \sum_{m\in\mathbb{N}} V_{\varphi_m}\psi(\lambda)\pi(\lambda)\xi_m \right\rangle_{L^2} \\
&= \sum_{m,n\in\mathbb{N}} V_{\varphi_n}\psi(\lambda)\overline{V_{\varphi_m}\psi(\lambda)}\,\langle \pi(\lambda)\xi_n, \pi(\lambda)\xi_m\rangle_{L^2} \\
&= \sum_{n\in\mathbb{N}} |V_{\varphi_n}\psi(\lambda)|^2.
\end{aligned}$$

By comparing this with the definition (D.5.1) of a Gabor g-frame, we see that S generates a Gabor g-frame if and only if there exist $A, B > 0$ such that

$$A\|\psi\|^2_{L^2} \leq \sum_{\lambda\in\Lambda}\sum_{n\in\mathbb{N}} |V_{\varphi_n}\psi(\lambda)|^2 \leq B\|\psi\|^2_{L^2} \quad \text{for any } \psi \in L^2(\mathbb{R}^d),$$

in other words, if and only if the functions $\{\varphi_n\}_{n\in\mathbb{N}}$ generate a multi-window Gabor frame with countably many windows. Combining this with Proposition D.7.5, we obtain the following result on multi-window Gabor frames with countably many generators.

Theorem D.8.1. *Assume that $\{\varphi_n\}_{n\in\mathbb{N}} \subset M^1(\mathbb{R}^d)$ such that $\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1} < \infty$. If $\{\varphi_n\}_{n\in\mathbb{N}}$ generates a multi-window Gabor frame for $L^2(\mathbb{R}^d)$, i.e. there exist $A, B > 0$ such that*

$$A\|\psi\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda}\sum_{n\in\mathbb{N}} |V_{\varphi_n}\psi(\lambda)|^2 \leq B\|\psi\|_{L^2}^2 \quad \text{for any } \psi \in L^2(\mathbb{R}^d), \tag{D.8.1}$$

then there exists $N \in \mathbb{N}$ such that $\{\varphi_n\}_{n=1}^N$ generates a multi-window Gabor frame for $L^2(\mathbb{R}^d)$.

Proof. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$ such that $\|\xi_n\|_{M^1} \leq C$ for some $C > 0$ – for instance a Wilson basis [131, Prop. 12.3.8]. Then let

$$S = \sum_{n\in\mathbb{N}} \xi_n \otimes \varphi_n.$$

By our assumptions $\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1} < \infty$ and $\|\xi_n\|_{M^1} \leq C$, this sum converges absolutely in $\mathcal{B}$. Hence $S \in \mathcal{B}$. By the arguments preceding this theorem, (D.8.1) ensures that S generates a Gabor g-frame. Hence Theorem D.7.6 and Proposition D.7.5 give[3] the existence of $N \in \mathbb{N}$ such that $\{\varphi_n\}_{n=1}^N$ generates a multi-window Gabor frame for $L^2(\mathbb{R}^d)$. □

Remark D.17. The fact that Gabor g-frames correspond to multi-window Gabor frames with countably many generators, suggests that the duality theory of Gabor g-frames (in the sense of Ron-Shen duality, see [131]) is covered by the approach in [169], where multi-window Gabor frames with countably many generators are considered.

Acknowledgements

We wish to thank the author's supervisor Franz Luef for his insightful feedback on various drafts of the paper. We also thank Mads S. Jakobsen for helpful discussions. Finally, we thank Hans Feichtinger for his suggestion that the methods developed in the author's master's thesis might be applied to the theory of periodic operators.

[3]Proposition D.7.5 assumes that φ_n come from the singular value decomposition, but this is not used in the proof besides using Lemma D.7.4 to ensure that the decomposition into rank-one operators converges.

Paper E

A Wiener Tauberian Theorem for Operators and Functions

Franz Luef and Eirik Skrettingland

Published in
Journal of Functional Analysis, 2021, volume 280, issue 6.

Paper E

A Wiener Tauberian Theorem for Operators and Functions

Abstract

We prove variants of Wiener's Tauberian theorem in the framework of quantum harmonic analysis, i.e. for convolutions between an absolutely integrable function and a trace class operator, or of two trace class operators. Our results include Wiener's Tauberian theorem as a special case. Applications of our Tauberian theorems are related to localization operators, Toeplitz operators, isomorphism theorems between Bargmann-Fock spaces and quantization schemes with consequences for Shubin's pseudodifferential operator calculus and Born-Jordan quantization. Based on the links between localization operators and Tauberian theorems we note that the analogue of Pitt's Tauberian theorem in our setting implies compactness results for Toeplitz operators in terms of the Berezin transform. In addition, we extend the results on Toeplitz operators to other reproducing kernel Hilbert spaces induced by the short-time Fourier transform, known as Gabor spaces. Finally, we establish the equivalence of Wiener's Tauberian theorem and the condition in the characterization of compactness of localization operators due to Fernández and Galbis.

E.1 Introduction

In operator theory one views the space of trace class operators $\mathcal{S}^1$ as the non-commutative analogue of the space of absolutely integrable functions $L^1(\mathbb{R}^d)$ by viewing the trace of an operator as the substitute of the Lebesgue integral of a function. Over the years this point of view has led to a number of results in operator theory where one has extended concepts for functions to operators in an attempt to formulate operator-theoretic analogues of statements about functions. Guided by this meta-statement, Werner has proposed an operator-theoretic variant of harmonic analysis in [251], which originated from his work in quantum physics and is thus referred to as “quantum harmonic analysis".

In this paper we establish a version of Wiener’s Tauberian theorem in the setting of quantum harmonic analysis. Wiener’s Tauberian theorem is a cornerstone of harmonic analysis. In short, it analyses the asymptotic properties of a bounded function by testing it with convolution kernels.

Theorem (Wiener’s Tauberian Theorem). *Suppose $f \in L^\infty(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d)$ with a non-vanishing Fourier transform $\widehat{h}$. Then the following implication holds for $A \in \mathbb{C}$: if*

$$\lim_{|x|\to\infty} (h * f)(x) = A \int_{\mathbb{R}^d} h(y)\, dy,$$

then for any $g \in L^1(\mathbb{R}^d)$ we have

$$\lim_{|x|\to\infty} (g * f)(x) = A \int_{\mathbb{R}^d} g(y)\, dy.$$

Moreover, Wiener noticed that the Tauberian condition holds *only* for $h \in L^1(\mathbb{R}^d)$ satisfying the condition $\widehat{h}(\omega) \neq 0$ for any $\omega \in \mathbb{R}^d$. The key step in the proof of this equivalence is based on the following approximation theorem. For $f \in L^1(\mathbb{R}^d)$ we denote by $T_x f(t) = f(t-x)$ the translate of f by $x \in \mathbb{R}^d$.

Theorem (Wiener’s Approximation Theorem). *For $f \in L^1(\mathbb{R}^d)$ we have that $\overline{span}\{T_x f : x \in \mathbb{R}^d\} = L^1(\mathbb{R}^d)$ if and only if $\widehat{f}(\omega) \neq 0$ for any $\omega \in \mathbb{R}^d$.*

In quantum harmonic analysis one complements the convolution $f * g(x) = \int_{\mathbb{R}^d} f(t)g(x-t)\, dt$ of $f, g \in L^1(\mathbb{R}^d)$ with two new convolution operations: the convolution $f \star S$ of $f \in L^1(\mathbb{R}^d)$ and a trace class operator S, and the convolution $S \star T$ of two trace class operators S and T. This is achieved by replacing, for $z \in \mathbb{R}^{2d}$, the translation $T_z f$ of a function by the *translation* $\alpha_z(R)$ of a bounded operator R given by

$$\alpha_z(R) = \pi(z)R\pi(z)^* \quad \text{for } z \in \mathbb{R}^{2d},$$

where $(\pi(z)\psi)(t) = e^{2\pi i\omega\cdot t}\psi(x-t)$ denotes the time-frequency shift of $\psi \in L^2(\mathbb{R}^d)$ by $z = (x, \omega) \in \mathbb{R}^{2d}$.

For $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$, where $\mathcal{S}^1$ denotes the trace class operators, the convolution $f \star S \in \mathcal{S}^1$ is then defined by the Bochner integral

$$f \star S := S\star := \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\, dz,$$

which is another trace class operator. The convolution $S \star T$ of two operators $S, T \in \mathcal{S}^1$ is the function

$$S \star T(z) = \operatorname{tr}(S\alpha_z(\check{T})) \quad \text{for } z \in \mathbb{R}^{2d},$$

where $\check{T} = PTP$, with P the parity operator $P\psi(t) = \psi(-t)$.

In summary, the convolutions $f \star S$ and $S \star T$ arise as extensions of the convolution of functions where one replaces either one or both L^1-functions with trace class operators. The seminal paper [251] contains a number of operator-theoretic versions of basic results from harmonic analysis, e.g. the Riemann-Lebesgue lemma, the Hausdorff-Young theorem and Wiener's approximation theorem. The variant of Wiener's approximation theorem in [251] concerns translates of a trace class operator being dense in the space of trace class operators, and is established by defining an operator-theoretic Fourier transform, the Fourier-Wigner transform $\mathcal{F}_W(S) \in L^\infty(\mathbb{R}^{2d})$ of a trace class operator S.

The appropriate Fourier transform for functions in $L^1(\mathbb{R}^{2d})$ is the symplectic Fourier transform $\mathcal{F}_\sigma$ and the following classes of functions and operators are going to be crucial in our Tauberian theorems for quantum harmonic analysis:

$$W(\mathbb{R}^{2d}) := \{f \in L^1(\mathbb{R}^{2d}) : \mathcal{F}_\sigma(f)(z) \neq 0 \text{ for any } z \in \mathbb{R}^{2d}\},$$
$$\mathcal{W} := \{S \in \mathcal{S}^1 : \mathcal{F}_W(S)(z) \neq 0 \text{ for any } z \in \mathbb{R}^{2d}\}.$$

Our first main result is a generalization of Wiener's Tauberian Theorem for functions on $\mathbb{R}^{2d}$. Here $\mathcal{K}$ denotes the space of compact operators on $L^2(\mathbb{R}^d)$ and I_{L^2} is the identity operator.

Theorem E.4.1 (Tauberian theorem for bounded functions)**.** *Let $f \in L^\infty(\mathbb{R}^{2d})$, and assume that one of the following equivalent statements holds for some $A \in \mathbb{C}$:*

(i) There is some $S \in \mathcal{W}$ such that

$$f \star S = A \cdot \operatorname{tr}(S) \cdot I_{L^2} + K$$

for some compact operator $K \in \mathcal{K}$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ such that

$$f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h$$

for some $h \in C_0(\mathbb{R}^{2d})$.

Then both of the following statements hold:

1. *For any* $T \in \mathcal{S}^1$, $f \star T = A \cdot \mathrm{tr}(T) \cdot I_{L^2} + K_T$ *for some compact operator* $K_T \in \mathcal{K}$.
2. *For any* $g \in L^1(\mathbb{R}^{2d})$, $f * g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\,dz + h_g$ *for some* $h_g \in C_0(\mathbb{R}^{2d})$.

We note that the equivalence $(ii) \iff (2)$ is Wiener's original Tauberian theorem. Similarly to Wiener's Tauberian theorem, this theorem concerns the asymptotic properties of the operator R when we use the common intuition that asymptotic properties of an operator are properties that are invariant under compact perturbations, see [15, Chap.3]. There is also a version of the preceding theorem for bounded operators:

Theorem E.5.1 (Tauberian theorem for bounded operators)**.** *Let* $R \in \mathcal{L}(L^2)$, *and assume that one of the following equivalent statements holds for some* $A \in \mathbb{C}$:

(i) There is some $S \in \mathcal{W}$ *such that*

$$R \star S = A \cdot \mathrm{tr}(S) + h$$

for some $h \in C_0(\mathbb{R}^{2d})$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ *such that*

$$R \star a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\,dz \cdot I_{L^2} + K$$

for some compact operator $K \in \mathcal{K}$.

Then both of the following statements hold:

1. *For any* $T \in \mathcal{S}^1$, $R \star T = A \cdot \mathrm{tr}(T) + h_T$ *for some* $h_T \in C_0(\mathbb{R}^{2d})$.
2. *For any* $g \in L^1(\mathbb{R}^{2d})$, $R \star g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\,dz \cdot I_{L^2} + K_g$ *for some compact operator* $K_g \in \mathcal{K}$.

These Tauberian theorems have numerous applications to localization operators, Toeplitz operators and quantization schemes. The link to localization operators allows us to add another equivalent assumption to Theorem E.4.1, formulated in terms of the short-time Fourier transform. Recall that the short-time Fourier transform $V_\phi \psi$ of ψ for the window ϕ is given by $V_\phi \psi(z) = \langle \psi, \pi(z)\phi \rangle_{L^2}$.

Proposition E.4.3. *Let* $A \in \mathbb{C}$. *Then* $f \in L^\infty(\mathbb{R}^{2d})$ *satisfies the equivalent conditions* (i) *and* (ii) *in Theorem E.4.1 if and only if*

(iii) There is some non-zero Schwartz function Φ *on* $\mathbb{R}^{2d}$ *such that for every* $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi(f - A)(x,\omega)| = 0.$$

As condition (*ii*) in Theorem E.4.1 is the condition from Wiener's classical Tauberian theorem, condition (*iii*) above, which first appeared in the context of localization operators in [109], is a new characterization of the functions to which Wiener's classical Tauberian theorem applies.

To be precise, the localization operator $\mathcal{A}_f^{\varphi_1,\varphi_2}$ with *mask* $f \in L^\infty(\mathbb{R}^{2d})$ and *windows* $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, is defined by

$$\mathcal{A}_f^{\varphi_1,\varphi_2}(\psi) = \int_{\mathbb{R}^{2d}} f(z) V_{\varphi_1}\psi(z)\pi(z)\varphi_2 \, dz.$$

The link from localization operators to Theorem E.4.1 is then the simple relation $\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1)$, where $\varphi_2 \otimes \varphi_1(\psi) = \langle \psi, \varphi_1 \rangle_{L^2} \varphi_2$. Localization operators are further linked to Toeplitz operators on *Gabor spaces* $V_\varphi(L^2)$ – which contain the Bargmann-Fock space as a special case – this allows the study of Toeplitz operators using Theorem E.4.1.

The *Gabor space* associated with φ with $\|\varphi\|_{L^2} = 1$ is the closed subspace of $L^2(\mathbb{R}^{2d})$ given by $V_\varphi(L^2) := V_\varphi(L^2(\mathbb{R}^d))$. The Gabor space $V_\varphi(L^2)$ is a reproducing kernel Hilbert space with reproducing kernel

$$k_z^\varphi(z') = \langle \pi(z)\varphi, \pi(z')\varphi \rangle_{L^2} = V_\varphi(\pi(z)\varphi)(z'),$$

for any $\psi \in L^2(\mathbb{R}^d)$. We will show that the intersection of different Gabor spaces is trivial whenever the windows are not scalar multiples of each other. Every $f \in L^\infty(\mathbb{R}^{2d})$ then defines a *Gabor Toeplitz* operator $T_f^\varphi : V_\varphi(L^2) \to V_\varphi(L^2)$ by

$$T_f^\varphi(V_\varphi\psi) = \mathcal{P}_{V_\varphi(L^2)}(f \cdot V_\varphi\psi),$$

where $\mathcal{P}_{V_\varphi(L^2)} : L^2(\mathbb{R}^{2d}) \to V_\varphi(L^2)$ is the orthogonal projection. It is well-known that T_f^φ and $\mathcal{A}_f^{\varphi,\varphi}$ are unitarily equivalent.

If the window function φ is the Gaussian $\varphi_0(x) = 2^{d/4}e^{-\pi x^2}$, then $V_{\varphi_0}(L^2)$ is, up to a simple unitary transformation, the space of entire functions on $\mathbb{C}^d$ known as the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$, For every $F \in L^\infty(\mathbb{C}^d)$ one defines the *Bargmann-Fock Toeplitz operator* $T_F^{\mathcal{F}^2}$ on $\mathcal{F}^2(\mathbb{C}^d)$ by

$$T_F^{\mathcal{F}^2}(H) = \mathcal{P}_{\mathcal{F}^2}(F \cdot H)$$

for any $H \in \mathcal{F}^2(\mathbb{C}^d)$. One has that if $f \in L^\infty(\mathbb{R}^{2d})$ and $F \in L^\infty(\mathbb{C}^d)$ are related by $F(x + i\omega) = f(x, -\omega)$ the the following operators are unitarily equvialent:

1. The localization operator $\mathcal{A}_f^{\varphi_0,\varphi_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

2. The Gabor Toeplitz operator $T_f^{\varphi_0} : V_{\varphi_0}(L^2) \to V_{\varphi_0}(L^2)$.

3. The Bargmann-Fock Toeplitz operator $T_F^{\mathcal{F}^2} : \mathcal{F}^2(\mathbb{C}^d) \to \mathcal{F}^2(\mathbb{C}^d)$.

Since $\mathcal{A}_f^{\varphi_0,\varphi_0} = f \star (\varphi_0 \otimes \varphi_0)$, the equivalences above allow us to translate statements from convolutions of operators to Toeplitz operators. One of the results we translate to Toeplitz operators follows by noting that the Tauberian theorems concern compact perturbations of a scaling of the identity, i.e. operators $A \cdot I_{L^2} + K$ for $0 \neq A \in \mathbb{C}$ and $K \in \mathcal{K}$. Inspired by this – without using the Tauberian theorem itself – we apply Riesz' theory of such operators to obtain sufficient conditions for localization operators to be isomorphisms:

Proposition E.4.10. *Let $0 \neq M \in \mathbb{R}$, $a \in L^\infty(\mathbb{R}^{2d})$ and $\Delta \subset \mathbb{R}^{2d}$ a set of finite Lebesgue measure. Assume that the following assumptions hold:*

(i) $a(z) \geq -M$ for a.e. $z \in \mathbb{R}^{2d}$,

(ii) $a(z) > -M$ for $z \notin \Delta$,

(iii) a satisfies assumption (i) or (ii) in Theorem E.4.1 with $A = 0$.

Let $f = M + a$. Then $\mathcal{A}_f^{\varphi,\varphi}$ is an isomorphism on $L^2(\mathbb{R}^d)$ for any $0 \neq \varphi \in L^2(\mathbb{R}^d)$.

We translate these results to the polyanalytic Bargmann-Fock space $\mathcal{F}_n^2(\mathbb{C}^d)$ for $n \in \mathbb{N}^d$ – in particular $\mathcal{F}_0^2(\mathbb{C}^d)$ is the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$.

Proposition E.4.12. *1. If $\Omega \subset \mathbb{C}^d$ satisfies that Ω^c has finite Lebesgue measure, then $T_{\chi_\Omega}^{\mathcal{F}_n^2}$ is an isomorphism on $\mathcal{F}_n^2(\mathbb{C}^d)$.*

2. There is a real-valued, continuous $F \in L^\infty(\mathbb{C}^d)$ such that $\lim_{|z|\to\infty} |F(z)|$ does not exist, yet $T_F^{\mathcal{F}_n^2}$ is an isomorphism on $\mathcal{F}_n^2(\mathbb{C}^d)$.

Another class of our results concerns the *Berezin transform*. For the Gabor space $V_\varphi(L^2)$ we can express the Berezin transform $\mathfrak{B}^\varphi : V_\varphi(L^2) \to L^\infty(\mathbb{R}^{2d})$ as a convolution of operators. In particular, the Berezin transform of the Gabor Toeplitz operator T_f^φ is simply a convolution of functions:

$$\mathfrak{B}^\varphi T_f^\varphi(z) = \left(f * |V_\varphi \varphi|^2\right)(z).$$

Pitt's classical theorem gives a condition on $f \in L^\infty(\mathbb{R}^{2d})$ that ensures that $f * g \in C_0(\mathbb{R}^{2d})$ for $g \in W(\mathbb{R}^{2d})$ implies $f \in C_0(\mathbb{R}^{2d})$. In particular, this holds for uniformly continuous f. A natural analogue of uniformly continuous functions for operators is the set

$$\mathcal{C}_1 := \{R \in \mathcal{L}(L^2) : z \mapsto \alpha_z(R) \text{ is continuous from } \mathbb{R}^{2d} \text{ to } \mathcal{L}(L^2)\},$$

see [28, 251]. Werner has obtained the following result in [251] which in light of our Tauberian theorem is an analogue of Pitt's theorem for operators.

Theorem E.5.2. *Let $R \in \mathcal{C}_1$. The following are equivalent.*

- $R \in \mathcal{K}$.
- $R \star S \in C_0(\mathbb{R}^{2d})$ *for some* $S \in \mathcal{W}$.
- $R \star f \in \mathcal{K}$ *for some* $f \in W(\mathbb{R}^{2d})$.

Fulsche [117] has recently noted that the preceding theorem implies a result in [24] for the Bargmann-Fock space. We show that the result holds for any Gabor space $V_\varphi(L^2)$ under certain conditions on φ. We would like to stress that it is a Pitt-type theorem for the Tauberian theorem for operators.

Theorem E.5.4. *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$ satisfy that $V_\varphi\varphi$ has no zeros, and let $\mathcal{T}^\varphi$ be the Banach algebra generated by Toeplitz operators T_f^φ in $\mathcal{L}(V_\varphi(L^2))$ for $f \in L^\infty(\mathbb{R}^{2d})$. Then the following are equivalent for $\tilde{T} \in \mathcal{T}^\varphi$.*

- $\tilde{T}$ *is a compact operator on* $V_\varphi(L^2)$.
- $\mathfrak{B}^\varphi\tilde{T} \in C_0(\mathbb{R}^{2d})$.

Furthermore, if $\tilde{T} = T_f^\varphi$ for some slowly oscillating $f \in L^\infty(\mathbb{R}^{2d})$, then the conditions above are equivalent to $\lim_{|z|\to\infty} |f(z)| = 0$.

Examples of φ satisfying that $V_\varphi\varphi$ has no zeros were recently investigated in [139], for example the one-sided exponential. Hence these φ's give different reproducing kernel Hilbert spaces $V_\varphi(L^2)$ such that Toeplitz operators are compact if and only if their Berezin transform vanishes at infinity.

The main result in [24] follows in particular, as shown in [117]. We have added a statement on slowly oscillating functions that follows from the original version of Pitt's theorem.

Theorem E.5.5 (Bauer, Isralowitz)**.** *Let $\mathcal{T}^{\mathcal{F}^2}$ be the Banach algebra generated by the Toeplitz operators $T_F^{\mathcal{F}^2}$ for $F \in L^\infty(\mathbb{C}^d)$. The following are equivalent for $\tilde{T} \in \mathcal{T}^{\mathcal{F}^2}$.*

- $\tilde{T}$ *is a compact operator on* $\mathcal{F}^2(\mathbb{C}^d)$.
- $\mathfrak{B}^{\mathcal{F}^2}\tilde{T} \in C_0(\mathbb{C}^d)$.

If $\tilde{T} = T_F^{\mathcal{F}^2}$ for a slowly oscillating $F \in L^\infty(\mathbb{C}^d)$, then the conditions above are equivalent to $\lim_{|z|\to\infty} F(z) = 0$.

As a consequence we state a compactness result for Toeplitz operators.

Corollary E.5.5.1. *A Toeplitz operator $T_F^{\mathcal{F}^2}$ for $F \in L^\infty(\mathbb{C}^d)$ is a compact operator on $\mathcal{F}^2(\mathbb{C}^d)$ if and only if*

$$f * |V_{\varphi_0}\varphi_0|^2 \in C_0(\mathbb{R}^{2d}),$$

where $f(x,\omega) = F(x - i\omega)$ for $x, \omega \in \mathbb{R}^d$ and $|V_{\varphi_0}\varphi_0(z)|^2 = e^{-\pi|z|^2}$.

Finally, Theorem E.5.2 gives a simple condition for compactness of localization operators in terms of the Gaussian φ_0.

Proposition E.5.6. *Let $f \in L^\infty(\mathbb{R}^{2d})$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$. The localization operator $\mathcal{A}_f^{\psi_1,\psi_2}$ is compact if and only if*

$$f * (V_{\varphi_0}\psi_2\overline{V_{\varphi_0}\psi_1}) \in C_0(\mathbb{R}^{2d}).$$

Finally we recall from [204] that any $R \in \mathcal{L}(L^2)$ defines a quantization scheme given by $f \mapsto f \star R$ for $f \in L^1(\mathbb{R}^{2d})$ and a time-frequency distribution Q_R, given by sending $\psi \in L^2(\mathbb{R}^d)$ to $Q_R(\psi)(z) = (\psi \otimes \psi) \star \check{R}(z)$ for $z \in \mathbb{R}^{2d}$. The distribution Q_R is of Cohen's class since we have $Q_R(\psi) = a_{\check{R}} * W(\psi, \psi)$, where $a_{\check{R}}$ is the Weyl symbol of $\check{R}$ and $W(\psi,\psi)$ the Wigner distribution of ψ.

In the final section we deduce a statement relating compactness properties of the quantization scheme of $f \mapsto f \star R$ to properties of $Q_R(\psi)$.

Proposition E.6.1. *Let $R \in \mathcal{L}(L^2)$. The following are equivalent.*

(i) $Q_R(\varphi) \in C_0(\mathbb{R}^{2d})$ *for some* $\varphi \in L^2(\mathbb{R}^d)$ *such that* $V_\varphi\varphi$ *has no zeros.*

(ii) $g \star R \in \mathcal{K}$ *for some* $g \in W(\mathbb{R}^{2d})$.

(iii) $Q_R(\psi) \in C_0(\mathbb{R}^{2d})$ *for all* $\psi \in L^2(\mathbb{R}^d)$.

(iv) $f \star R \in \mathcal{K}$ *for all* $f \in L^1(\mathbb{R}^{2d})$.

Hence if one takes the Gaussian φ_0 for (i), then checking if $Q_R(\varphi_0) \in C_0(\mathbb{R}^{2d})$ provides a simple test for checking whether Conditions (iii) and (iv) hold. We apply this result to Shubin's τ-quantization scheme and Born-Jordan quantization.

E.1.1 Notations and conventions

For topological vector spaces X, Y, we denote by $\mathcal{L}(X,Y)$ the set of continuous, linear operators from X to Y. If $X = Y$ we write $\mathcal{L}(X) = \mathcal{L}(X,X)$. The space of compact operators on $L^2(\mathbb{R}^d)$ is denoted by $\mathcal{K}$. For $1 \leq p < \infty$ we let $\mathcal{S}^p$ denote the Schatten p-class of compact operators with singular values in ℓ^p, and we use the convention that $\mathcal{S}^\infty = \mathcal{L}(L^2)$. In particular, $\mathcal{S}^1$ denotes the space of trace class operators on $L^2(\mathbb{R}^d)$, and the trace of a trace class operator $T \in \mathcal{S}^1$ is denoted by $\mathrm{tr}(T)$. Also, $\mathcal{S}^2$ is the space of Hilbert-Schmidt operators, which form a Hilbert space with respect to the inner product $\langle S, T\rangle_{\mathcal{S}^2} = \mathrm{tr}(ST^*)$.

Given a topological vector space X and its continuous dual X', the action of $x^* \in X'$ on $y \in x$ is denoted by $\langle x^*, y\rangle_{X',X}$. To agree with the Hilbert space inner product we use the convention that the duality bracket is linear in the first coordinate and antilinear in the second coordinate. The Schwartz functions on $\mathbb{R}^d$ are denoted by $\mathscr{S}(\mathbb{R}^d)$.

The Euclidean norm on $\mathbb{R}^d$ or $\mathbb{C}^d$ will be denoted by $|\cdot|$. For $\Omega \subset \mathbb{R}^d$, χ_Ω denotes the characteristic function of Ω. As usual, $C_0(\mathbb{R}^d)$ denotes the continuous functions on $\mathbb{R}^d$ vanishing at infinity, and we use $L^0(\mathbb{R}^d)$ to denote the space of measurable, bounded functions f on $\mathbb{R}^d$ such that $\lim_{|z|\to\infty} f(z) = 0$, i.e. for every $\epsilon > 0$ there is $R > 0$ such that $|f(z)| < \epsilon$ for a.e. $|z| > R$. We will refer to L^p-spaces on $\mathbb{R}^d, \mathbb{R}^{2d}$ and $\mathbb{C}^d$, and sometimes we will omit explicit reference to the underlying space when it is clear from the context, for instance by writing $\mathcal{L}(L^2)$ for $\mathcal{L}(L^2(\mathbb{R}^d))$. In all statements, measurability and "almost everywhere" properties will refer to Lebesgue measure.

E.2 Preliminaries

E.2.1 Concepts from time-frequency analysis

The mathematical theory of time-frequency analysis will provide the setup and many of the tools we use in this paper. We therefore introduce the *time-frequency shifts* $\pi(z) \in \mathcal{L}(L^2)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$, given by

$$(\pi(z)\psi)(t) = e^{2\pi i \omega\cdot t}\psi(t-x) \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

The time-frequency shift $\pi(z)$ is clearly given as a composition $\pi(z) = M_\omega T_x$ of a *modulation operator* $M_\omega\psi(t) = e^{2\pi i\omega\cdot t}\psi(t)$ and a *translation operator* $T_x\psi(t) = \psi(t-x)$. Given $\psi, \phi \in L^2(\mathbb{R}^d)$, the *short-time Fourier transform* $V_\phi\psi$ of ψ with window ϕ is the function on $\mathbb{R}^{2d}$ defined by

$$V_\phi\psi(z) = \langle \psi, \pi(z)\phi\rangle_{L^2} \quad \text{for } z \in \mathbb{R}^{2d}.$$

The short-time Fourier transform satisfies the important orthogonality relation

$$\int_{\mathbb{R}^{2d}} V_{\phi_1}\psi_1(z)\overline{V_{\phi_2}\psi_2(z)}\, dz = \langle \psi_1, \psi_2\rangle_{L^2} \langle \phi_2, \phi_1\rangle_{L^2}, \tag{E.2.1}$$

see [114, 131], sometimes called Moyal's identity. Throughout this paper we will use φ_0 to denote the normalized Gaussian

$$\varphi_0(t) = 2^{d/4}e^{-\pi t^2} \quad \text{for } t \in \mathbb{R}^d,$$

and we will often refer to its short-time Fourier transform, which by [131, Lem. 1.5.2] is given by

$$V_{\varphi_0}\varphi_0(z) = e^{-\pi i x\cdot\omega}e^{-\pi|z|^2/2} \quad \text{for } z=(x,\omega); \tag{E.2.2}$$

the reader should note already at this point that $V_{\varphi_0}\varphi_0$ has no zeros.

Wigner functions and the Weyl transform

Given $\phi, \psi \in L^2(\mathbb{R}^d)$, a close relative of the short-time Fourier transform $V_\phi\psi$ is the *cross-Wigner distribution* $W(\psi,\phi)$ defined by

$$W(\psi,\phi)(x,\omega) = \int_{\mathbb{R}^d} \psi(x+t/2)\overline{\phi(x-t/2)}e^{-2\pi i\omega\cdot t}\,dt \quad \text{for } (x,\omega)\in\mathbb{R}^{2d}.$$

The cross-Wigner distribution is the main tool needed to introduce the *Weyl transform*, which associates to any $f \in \mathcal{S}'(\mathbb{R}^{2d})$ an operator

$$L_f \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$$

defined by requiring

$$\left\langle L_f(\psi),\phi\right\rangle_{\mathcal{S}'(\mathbb{R}^d),\mathcal{S}(\mathbb{R}^d)} = \langle f, W(\phi,\psi)\rangle_{\mathcal{S}'(\mathbb{R}^{2d}),\mathcal{S}(\mathbb{R}^{2d})} \quad \text{for all } \phi,\psi\in\mathcal{S}(\mathbb{R}^d). \tag{E.2.3}$$

By the Schwartz kernel theorem [159], any $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ is the Weyl transform L_f for some unique $f \in \mathcal{S}'(\mathbb{R}^{2d})$. We denote this f by a_S, and call it the *Weyl symbol* of S. In other words, $S = L_{a_S}$. Note that there is no relationship between boundedness of the function f and boundedness of the operator L_f on $L^2(\mathbb{R}^d)$: there is $f \in L^\infty(\mathbb{R}^{2d})$ such that $L_f \notin \mathcal{L}(L^2)$, and there is $S \in \mathcal{L}(L^2)$ such that $a_S \notin L^\infty(\mathbb{R}^{2d})$. See Remark E.20 for examples.

Example E.2.1 (Rank-one operators)**.** Given $\psi, \phi \in L^2(\mathbb{R}^d)$, the rank-one operator $\psi\otimes\phi \in \mathcal{L}(L^2)$ is defined by

$$(\psi\otimes\phi)(\xi) = \langle \xi,\phi\rangle_{L^2}\,\psi \quad \text{for } \xi\in L^2(\mathbb{R}^d).$$

It is well-known that the Weyl symbol of $\psi\otimes\phi$ is $W(\psi,\phi)$.

Localization operators

For a *mask* $f \in L^\infty(\mathbb{R}^{2d})$ and a pair of *windows* $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, we define the *localization operator* $\mathcal{A}_f^{\varphi_1,\varphi_2}(\psi) \in \mathcal{L}(L^2)$ by

$$\mathcal{A}_f^{\varphi_1,\varphi_2}(\psi) = \int_{\mathbb{R}^{2d}} f(z)V_{\varphi_1}\psi(z)\pi(z)\varphi_2\,dz,$$

where the integral is interpreted weakly in the sense that we require

$$\left\langle \mathcal{A}_f^{\varphi_1,\varphi_2}(\psi), \phi \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle f, V_{\varphi_2}\phi \overline{V_{\varphi_1}\psi} \right\rangle_{L^2(\mathbb{R}^{2d})} \qquad \text{for any } \psi, \phi \in L^2(\mathbb{R}^d). \tag{E.2.4}$$

It is well-known that $\mathcal{A}_f^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ for $f \in L^\infty(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ [63], but one may also define localization operators for other Banach function spaces of masks f and windows φ_1, φ_2 by interpreting the brackets in (E.2.4) as duality brackets, see [63]. We postpone this discussion until we have a more suitable framework, which we now introduce.

E.2.2 Quantum harmonic analysis: convolutions of operators and functions

In this section we introduce the quantum harmonic analysis developed by Werner in [251], the main concepts of which are convolutions of operators and functions and a Fourier transform of operators. For a more detailed introduction in our terminology we refer to [203]. Given any $z \in \mathbb{R}^{2d}$ and an operator $R \in \mathcal{L}(L^2)$, we define the *translation* $\alpha_z(R)$ of R by z to be the operator

$$\alpha_z(R) = \pi(z) R \pi(z)^*.$$

At the level of Weyl symbols, we have that

$$\alpha_z(R) = L_{T_z(a_R)},$$

hence α_z corresponds to a translation of the Weyl symbol. For $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$ we then define the convolution $f \star S \in \mathcal{S}^1$ by the Bochner integral

$$f \star S := S \star f := \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S)\, dz. \tag{E.2.5}$$

Hence the convolution of a function with an operator is a new operator. The convolution $S \star T$ of two operators $S, T \in \mathcal{S}^1$ is the *function*

$$S \star T(z) = \operatorname{tr}(S \alpha_z(\check{T})) \qquad \text{for } z \in \mathbb{R}^{2d}. \tag{E.2.6}$$

Here $\check{T} = PTP$, with P the parity operator $P\psi(t) = \psi(-t)$. Then $S \star T \in L^1(\mathbb{R}^{2d})$ with $\int_{\mathbb{R}^{2d}} S \star T(z)\, dz = \operatorname{tr}(S)\operatorname{tr}(T)$ and $S \star T = T \star S$ [251]. Taking convolutions with a fixed operator or function is easily seen to be a linear map.

One of the most important properties of the convolutions (E.2.5) and (E.2.6) is that they interact nicely with each other and with the usual convolution $f * g(x) = \int_{\mathbb{R}^d} f(t) g(x-t)\, dt$ of functions, as is most strikingly shown by their associativity [203, 251].

Proposition E.2.1. *The convolutions* (E.2.5) *and* (E.2.6) *are associative. Written out in detail, this means that for $S, T, R \in \mathcal{S}^1$ and $f, g \in L^1(\mathbb{R}^{2d})$ we have*

$$(R \star S) \star T = R \star (S \star T)$$
$$f * (R \star S) = (f \star R) \star T$$
$$(f * g) \star R = f \star (g \star R).$$

Remark E.1. Special cases of this associativity have appeared several times in the literature, typically with less transparent formulations and proofs than those allowed by the convolution formalism. See for instance [109, Prop. 3.10].

The convolutions also have an interesting interpretation in terms of the Weyl symbol, as we have that

$$S \star T(z) = a_S * a_T(z) \tag{E.2.7}$$
$$a_{f\star S}(z) = f * a_S(z).$$

As is shown in detail in [203], one can extend the domains of the convolutions by duality. For instance, the convolution $f \star S \in \mathcal{L}(L^2)$ of $S \in \mathcal{S}^1$ and $f \in L^\infty(\mathbb{R}^{2d})$ is defined by

$$\langle f \star S, T \rangle_{\mathcal{L}(L^2),\mathcal{S}^1} = \left\langle f, \check{S}^* \star T \right\rangle_{L^\infty, L^1}.$$

Combining this with a complex interpolation argument gives a version of Young's inequality [203, 251]. Recall our convention that $\mathcal{S}^\infty = \mathcal{L}(L^2)$.

Proposition E.2.2 (Young's inequality)**.** *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^{2d}), S \in \mathcal{S}^p$ and $T \in \mathcal{S}^q$, then $f \star T \in \mathcal{S}^r$ and $S \star T \in L^r(\mathbb{R}^{2d})$ may be defined and satisfy the norm estimates*

$$\|f \star T\|_{\mathcal{S}^r} \leq \|f\|_{L^p} \|T\|_{\mathcal{S}^q},$$
$$\|S \star T\|_{L^r} \leq \|S\|_{\mathcal{S}^p} \|T\|_{\mathcal{S}^q}.$$

Remark E.2. It is worth noting that if $S \in \mathcal{S}^1$ and $T \in \mathcal{L}(L^2)$, then $S \star T$ is still given by (E.2.6), which can be interpreted pointwise, so that $S \star T$ is a continuous, bounded function.

Young's inequality above shows that the convolutions interact in a predictable way with $L^p(\mathbb{R}^{2d})$ and $\mathcal{S}^q$. We now show that the same is true for functions vanishing at infinity and compact operators. Recall that $L^0(\mathbb{R}^{2d})$ denotes the Banach subspace of $L^\infty(\mathbb{R}^{2d})$ consisting of $f \in L^\infty(\mathbb{R}^{2d})$ that vanish at infinity. The following result shows that convolutions with trace class operators interchange $L^0(\mathbb{R}^{2d})$ and $\mathcal{K}$, which is the basis for our main theorems. These results are known, in particular we mention that part (ii) was proved for rank-one operators S in [45] using essentially the same proof.

Lemma E.2.3. *Let $R \in \mathcal{K}$ and $f \in L^0(\mathbb{R}^{2d})$. If $S \in \mathcal{S}^1$, then*

(i) $R \star S \in C_0(\mathbb{R}^{2d})$,

(ii) $f \star S \in \mathcal{K}$,

and if $a \in L^1(\mathbb{R}^{2d})$ then

(iii) $R \star a \in \mathcal{K}$,

(iv) $f * a \in C_0(\mathbb{R}^{2d})$.

Proof. Part (i) is [203, Prop. 4.6]. For (ii) and (iv), note that any $f \in L^0(\mathbb{R}^{2d})$ is the limit in the norm topology of $L^\infty(\mathbb{R}^{2d})$ of a sequence of compactly supported functions f_n – simply pick $f_n = f \cdot \chi_{B_n(0)}$, where $B_n(0) = \{z \in \mathbb{R}^{2d} : |z| < n\}$. Clearly $f_n \in L^1(\mathbb{R}^{2d})$, hence $f_n \star S \in \mathcal{S}^1 \subset \mathcal{K}$. We therefore have by Young's inequality (recall that $\mathcal{S}^\infty = \mathcal{L}(L^2)$):

$$\|f \star S - f_n \star S\|_{\mathcal{L}(L^2)} = \|(f - f_n) \star S\|_{\mathcal{L}(L^2)} \leq \|f - f_n\|_{L^\infty} \|S\|_{\mathcal{S}^1} \to 0 \text{ as } n \to \infty,$$

so $f \star S$ is the limit in the operator norm of compact operators, hence itself compact. Similarly, $f_n * a \in C_0(\mathbb{R}^{2d})$ and $f_n * a$ converges uniformly to $f * a$ by Young's inequality $\|(f - f_n) * a\|_{L^\infty} \leq \|f - f_n\|_{L^\infty} \|a\|_{L^1}$, so that $f * a \in C_0(\mathbb{R}^{2d})$. Finally, (iii) follows by noting that any $R \in \mathcal{K}$ is the limit in the operator norm of a sequence $R_n \in \mathcal{S}^1$ of finite-rank operators. Then $R_n \star a \in \mathcal{S}^1$ is compact, so it follows by $\|(R - R_n) \star a\|_{\mathcal{L}(L^2)} \leq \|R - R_n\|_{\mathcal{L}(L^2)} \|a\|_{L^1}$ that $R \star a$ is the limit in the operator norm of a sequence of compact operators, hence itself compact. □

Remark E.3. In combination with Proposition E.2.2 and the fact that $\mathcal{S}^p \subset \mathcal{K}$ for $p < \infty$, we see that $L^p(\mathbb{R}^{2d}) \star \mathcal{S}^1 \subset \mathcal{K}$ for $p = 0$ and $1 \leq p < \infty$.

Finally, the convolutions preserve identity elements [251, Prop. 3.2 (3)]. Here $I_{L^2} \in \mathcal{L}(L^2)$ is the identity operator and $1 \in L^\infty(\mathbb{R}^{2d})$ is given by $1(z) = z$.

Lemma E.2.4. *Let $S \in \mathcal{S}^1$ and $f \in L^1(\mathbb{R}^{2d})$. Then*

$$\begin{aligned} S \star I_{L^2} &= \operatorname{tr}(S) \cdot 1, \\ S \star 1 &= \operatorname{tr}(S) \cdot I_{L^2}, \\ f \star I_{L^2} &= \int_{\mathbb{R}^{2d}} f(z)\, dz \cdot I_{L^2}, \\ f * 1 &= \int_{\mathbb{R}^{2d}} f(z)\, dz \cdot 1. \end{aligned}$$

Fourier transforms of functions and operators

As our Fourier transform of functions on $\mathbb{R}^{2d}$ we will use the symplectic Fourier transform $\mathcal{F}_\sigma$, given, for $f \in L^1(\mathbb{R}^{2d})$, by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i \sigma(z,z')}\, dz' \quad \text{for } z \in \mathbb{R}^{2d},$$

where σ is the standard symplectic form $\sigma((x_1,\omega_1),(x_2,\omega_2)) = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$. Clearly $\mathcal{F}_\sigma$ is related to the usual Fourier transform $\widehat{f}(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i z\cdot z'}\, dz'$ by

$$\mathcal{F}_\sigma(f)(x,\omega) = \widehat{f}(\omega,-x),$$

so $\mathcal{F}_\sigma$ shares most properties with $\widehat{f}$: it extends to a unitary operator on $L^2(\mathbb{R}^{2d})$ and to a bijection on $\mathcal{S}'(\mathbb{R}^{2d})$ – see [72]. In addition, $\mathcal{F}_\sigma$ is its own inverse: $\mathcal{F}_\sigma \circ \mathcal{F}_\sigma = I_{L^2}$.

We will also use a Fourier transform of operators, namely the Fourier-Wigner transform $\mathcal{F}_W$ introduced by Werner [251] (Werner calls it the Fourier-Weyl transform, our usage of Fourier-Wigner agrees with [114]). When $S \in \mathcal{S}^1$, $\mathcal{F}_W(S)$ is the function

$$\mathcal{F}_W(S)(z) = e^{-\pi i x\cdot\omega}\mathrm{tr}(\pi(-z)S) \quad \text{for } z = (x,\omega) \in \mathbb{R}^{2d}. \tag{E.2.8}$$

As is shown in [204, 251], $\mathcal{F}_W$ extends to a unitary mapping $\mathcal{F}_W : \mathcal{S}^2 \to L^2(\mathbb{R}^{2d})$ and a bijection onto $\mathcal{S}'(\mathbb{R}^{2d})$ from $\mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$.

The Fourier transforms interact in the expected way with convolutions [251]: if $S, T \in \mathcal{S}^1$ and $f \in L^1(\mathbb{R}^{2d})$, then

$$\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S) \cdot \mathcal{F}_W(T), \tag{E.2.9}$$

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S). \tag{E.2.10}$$

We may also connect $\mathcal{F}_W$ and $\mathcal{F}_\sigma$ by the Weyl transform. In fact, we have by [204, Prop. 3.16] that

$$\mathcal{F}_W(L_f) = \mathcal{F}_\sigma(f) \text{ for } f \in \mathcal{S}'(\mathbb{R}^{2d}). \tag{E.2.11}$$

A main concern for this paper will be functions and operators satisfying that the appropriate Fourier transform never vanishes. Following the notation of [187] for the function case, we introduce the following notation:

$$W(\mathbb{R}^{2d}) := \{f \in L^1(\mathbb{R}^{2d}) : \mathcal{F}_\sigma(f)(z) \neq 0 \text{ for any } z \in \mathbb{R}^{2d}\},$$
$$\mathcal{W} := \{S \in \mathcal{S}^1 : \mathcal{F}_W(S)(z) \neq 0 \text{ for any } z \in \mathbb{R}^{2d}\}.$$

The key tool for proving the Tauberian theorem for operators is the following generalization of Wiener's approximation theorem, originally proved by Werner [251]. See also [182, 203] for more general statements.

Theorem E.2.5 (Werner)**.** *Let $S \in \mathcal{S}^1$. The following are equivalent.*

1. *The linear span of the translates $\{\alpha_z(S)\}_{z\in\mathbb{R}^{2d}}$ is dense in $\mathcal{S}^1$.*
2. *$S \in \mathcal{W}$.*
3. *The set $L^1(\mathbb{R}^{2d}) \star S = \{f \star S : f \in L^1(\mathbb{R}^{2d})\}$ is dense in $\mathcal{S}^1$.*
4. *The map $T \mapsto S \star T$ is injective from $\mathcal{L}(L^2)$ to $L^\infty(\mathbb{R}^{2d})$.*
5. *The set $\mathcal{S}^1 \star S = \{T \star S : T \in \mathcal{S}^1\}$ is dense in $L^1(\mathbb{R}^{2d})$.*
6. *The map $f \mapsto f \star S$ is injective from $L^\infty(\mathbb{R}^{2d})$ to $\mathcal{L}(L^2)$.*

The special case of rank-one operators

When $S \in \mathcal{S}^1$ is a rank-one operator $\psi \otimes \phi$ for $\psi, \phi \in L^2(\mathbb{R}^d)$, then many of the concepts introduced above are familiar concepts from time-frequency analysis. First we note that by [203, Thm. 5.1], localization operators $\mathcal{A}_f^{\varphi_1,\varphi_2}$ can be described as convolutions by

$$\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1). \tag{E.2.12}$$

Other convolutions and Fourier-Wigner transforms of rank-one operators are summarized in the next lemma. See [203, Thm. 5.1 and Lem. 6.1] for proofs. Here $\check{\varphi}(t) := (P\varphi)(t) = \varphi(-t)$.

Lemma E.2.6. *Let $\varphi_1, \varphi_2, \xi_1, \xi_2 \in L^2(\mathbb{R}^d)$ and $S \in \mathcal{L}(L^2)$. Then, for $(x,\omega) \in \mathbb{R}^{2d}$,*

1. *$\mathcal{F}_W(\varphi_1 \otimes \varphi_2)(x,\omega) = e^{i\pi x\cdot\omega} V_{\varphi_2}\varphi_1(x,\omega)$.*
2. *$S \star (\varphi_1 \otimes \varphi_2)(z) = \langle S\pi(z)\check{\varphi}_1, \pi(z)\check{\varphi}_2\rangle_{L^2}$.*
3. *$(\xi_1 \otimes \xi_2) \star (\check{\varphi}_1 \otimes \check{\varphi}_2)(x,\omega) = V_{\varphi_2}\xi_1(x,\omega)\overline{V_{\varphi_1}\xi_2(x,\omega)}$.*

In particular, for $\xi, \varphi \in L^2(\mathbb{R}^d)$

$$(\xi \otimes \xi) \star (\check{\varphi} \otimes \check{\varphi})(z) = |V_\varphi \xi(z)|^2.$$

Example E.2.2 (Standard Gaussian)**.** By (E.2.2), $\mathcal{F}_W(\varphi_0 \otimes \varphi_0)(z) = e^{-\pi|z|^2/2}$. We point out this simple case as it shows that $\varphi_0 \otimes \varphi_0 \in \mathcal{W}$. In particular, $\mathcal{W}$ is non-empty.

E.3 Toeplitz operators and Berezin transforms

In this section we will introduce some families of reproducing kernel Hilbert spaces and the corresponding Toeplitz operators and Berezin transforms. We will relate these spaces and operators to the convolutions introduced in Section E.2.2, which will later allow us to deduce results for reproducing kernel Hilbert spaces from the main results of this paper. By far the most studied of the spaces we consider is the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$, and we will later investigate whether some well-known result for $\mathcal{F}^2(\mathbb{C}^d)$ can hold for other of the reproducing kernel Hilbert spaces we consider.

E.3.1 Gabor spaces $V_\varphi(L^2)$

Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. By (E.2.1), the short-time Fourier transform

$$V_\varphi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$$

is an isometry, and one easily confirms that its adjoint operator is

$$V_\varphi^* F = \int_{\mathbb{R}^{2d}} F(z)\pi(z)\varphi \, dz \qquad \text{for } F \in L^2(\mathbb{R}^{2d}),$$

where the vector-valued integral is interpreted in a weak sense, see [131, Sec. 3.2] for details. The *Gabor space* associated with φ is then the image $V_\varphi(L^2(\mathbb{R}^d)) \subset L^2(\mathbb{R}^{2d})$, which we denote by $V_\varphi(L^2)$ for brevity. One can show using (E.2.1) that

$$\begin{aligned} V_\varphi^* V_\varphi &= I_{L^2(\mathbb{R}^d)}, \\ V_\varphi V_\varphi^* &= \mathcal{P}_{V_\varphi(L^2)}, \end{aligned} \tag{E.3.1}$$

where $\mathcal{P}_{V_\varphi(L^2)}$ denotes the orthogonal projection onto the subspace $V_\varphi(L^2)$ of $L^2(\mathbb{R}^{2d})$. This means that V_φ is a unitary operator from $L^2(\mathbb{R}^d)$ to $V_\varphi(L^2)$, with inverse $V_\varphi^*|_{V_\varphi(L^2)}$. By writing out the operators in (E.3.1) one deduces that $V_\varphi(L^2)$ is a reproducing kernel Hilbert space with reproducing kernel

$$k_z^\varphi(z') = \langle \pi(z)\varphi, \pi(z')\varphi \rangle_{L^2} = V_\varphi(\pi(z)\varphi)(z'), \tag{E.3.2}$$

meaning that we have the reproducing formula

$$V_\varphi \psi(z) = \left\langle V_\varphi \psi, k_z^\varphi \right\rangle_{L^2(\mathbb{R}^{2d})}$$

for any $\psi \in L^2(\mathbb{R}^d)$. Every $f \in L^\infty(\mathbb{R}^{2d})$ then defines a *Gabor Toeplitz* operator $T_f^\varphi : V_\varphi(L^2) \to V_\varphi(L^2)$ by

$$T_f^\varphi(V_\varphi \psi) = \mathcal{P}_{V_\varphi(L^2)}(f \cdot V_\varphi \psi).$$

To study such Toeplitz operators in this paper, we will use the map

$$\begin{aligned}\Theta^\varphi &: \mathcal{L}(V_\varphi(L^2)) \to \mathcal{L}(L^2)\\ \Theta^\varphi(\tilde{T}) &:= V_\varphi^*|_{V_\varphi(L^2)}\tilde{T}V_\varphi \quad \text{for } \tilde{T} \in \mathcal{L}(V_\varphi(L^2)).\end{aligned} \tag{E.3.3}$$

As $V_\varphi : L^2(\mathbb{R}^d) \to V_\varphi(L^2)$ is unitary, Θ^φ encodes a unitary equivalence, and is easily seen to be a linear, multiplicative and isometric isomorphism. We obtain the following well-known and easily verified result.

Proposition E.3.1. *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$ and $f \in L^\infty(\mathbb{R}^{2d})$. Then*

$$\mathcal{A}_f^{\varphi,\varphi} = \Theta^\varphi(T_f^\varphi).$$

In particular, T_f^φ and $\mathcal{A}_f^{\varphi,\varphi}$ are unitarily equivalent.

Now recall that in a reproducing kernel Hilbert space $\mathcal{H}$ consisting of functions on $\mathbb{R}^{2d}$ with normalized reproducing kernel k_z for $z \in \mathbb{R}^{2d}$, the *Berezin transform* $\mathfrak{B}\tilde{T}$ of a bounded operator $\tilde{T} \in \mathcal{L}(\mathcal{H})$ is the function $\mathbb{R}^{2d} \to \mathbb{C}$ defined by

$$\mathfrak{B}\tilde{T}(z) = \left\langle \tilde{T}k_z, k_z \right\rangle_{\mathcal{H}}.$$

For the Gabor space $V_\varphi(L^2)$ we can express the Berezin transform $\mathfrak{B}^\varphi : V_\varphi(L^2) \to L^\infty(\mathbb{R}^{2d})$ as a convolution of operators.

Lemma E.3.2. *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$, and let $\tilde{T} \in \mathcal{L}(V_\phi(L^2))$. Then*

$$\mathfrak{B}^\varphi\tilde{T}(z) = \Theta^\varphi(T) \star (\check{\varphi} \otimes \check{\varphi})(z).$$

In particular the Berezin transform of the Gabor Toeplitz operator T_f^φ is

$$\mathfrak{B}^\varphi T_f^\varphi(z) = \left(f * |V_\varphi\varphi|^2\right)(z).$$

Proof. Since $k_z^\varphi(z') = V_\varphi(\pi(z)\varphi)(z')$ by (E.3.2), we have

$$\begin{aligned}\Theta^\varphi(\tilde{T}) \star (\check{\varphi} \otimes \check{\varphi})(z) &= \left\langle \Theta^\varphi(\tilde{T})\pi(z)\varphi, \pi(z)\varphi \right\rangle_{L^2(\mathbb{R}^d)} \quad \text{by Lemma E.2.6}\\ &= \left\langle V_\varphi^*\tilde{T}V_\varphi(\pi(z)\varphi), \pi(z)\varphi \right\rangle_{L^2(\mathbb{R}^d)} \quad \text{by (E.3.3)}\\ &= \left\langle \tilde{T}V_\varphi(\pi(z)\varphi), V_\varphi(\pi(z)\varphi) \right\rangle_{L^2(\mathbb{R}^{2d})}\\ &= \mathfrak{B}^\varphi\tilde{T}(z).\end{aligned}$$

Since Proposition E.3.1 and (E.2.12) give that

$$f \star (\varphi \otimes \varphi) = \mathcal{A}_f^{\varphi,\varphi} = \Theta^\varphi(T_f^\varphi),$$

we get from the first part and associativity of convolutions that

$$\mathfrak{B}^\varphi T_f^\varphi = [f \star (\varphi \otimes \varphi)] \star (\check{\varphi} \otimes \check{\varphi}) = f * |V_\varphi\varphi|^2 \quad \text{by Lemma E.2.6.}$$ □

Remark E.4. Gabor spaces and their relation to localization operators has been discussed in [165], with emphasis on $f \in L^\infty(\mathbb{R}^{2d})$ depending only on x. The reproducing kernel k_z^φ has also been studied as the kernel of determinantal point processes called *Weyl-Heisenberg ensembles* [7, 10].

Gabor spaces with different windows

Having introduced the Gabor spaces $V_\varphi(L^2)$, we naturally ask whether the properties of $V_\varphi(L^2)$ as a reproducing kernel Hilbert space depend on the window φ in an essential way. As a first result in this direction, we note that the intersection of different Gabor spaces is trivial whenever the windows are not scalar multiples of each other, first proved with different methods in [126, Thm. 4.2].

Lemma E.3.3. *Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ with $\|\varphi_1\|_{L^2} = \|\varphi_2\|_{L^2} = 1$. If there exists $c \in \mathbb{C}$ such that $\varphi_1 = c\varphi_2$, then $V_{\varphi_1}(L^2) = V_{\varphi_2}(L^2)$. Otherwise $V_{\varphi_1}(L^2) \cap V_{\varphi_2}(L^2) = \{0\}$.*

Proof. If $\varphi_1 = c\varphi_2$, then $V_{\varphi_1}\xi = V_{\varphi_2}(\overline{c}\xi)$, which implies the first part. Then assume that $0 \neq V_{\varphi_1}\xi = V_{\varphi_2}\psi$ for $\xi, \psi \in L^2(\mathbb{R}^d)$. It follows by Lemma E.2.6 that

$$\xi \otimes \varphi_1 = \psi \otimes \varphi_2,$$

as $\mathcal{F}_W$ is a bijection from $\mathcal{S}^2$ to $L^2(\mathbb{R}^{2d})$. Taking adjoints, we get

$$\varphi_1 \otimes \xi = \varphi_2 \otimes \psi. \tag{E.3.4}$$

If we apply (E.3.4) to ξ, we obtain

$$\varphi_1 = \frac{\langle \xi, \psi \rangle_{L^2}}{\|\xi\|_{L^2}^2} \varphi_2.$$

Note that dividing by $\|\xi\|_{L^2}^2$ is allowed, as we assumed $V_{\varphi_1}\xi \neq 0$ which by (E.2.1) implies $\xi \neq 0$. □

Even though the result above shows that Gabor spaces with different windows φ_1 and φ_2 usually have trivial intersection, there is always an obvious Hilbert space isomorphism $\Psi : V_{\varphi_1}(L^2) \to V_{\varphi_2}(L^2)$ given by $\Psi = V_{\varphi_2} V_{\varphi_1}^* |_{V_{\varphi_1}(L^2)}$. However, this does not preserve the reproducing kernels: $k_z^{\varphi_1} = V_{\varphi_1}(\pi(z)\varphi_1)$ by (E.3.2), so clearly $\Psi(k_z^{\varphi_1}) = V_{\varphi_2}(\pi(z)\varphi_1)$. By the injectivity of V_{φ_2}, the only way $\Psi(k_z^{\varphi_1}) = V_{\varphi_2}(\pi(z)\varphi_1)$ can equal $k_z^{\varphi_2} = V_{\varphi_2}(\pi(z)\varphi_2)$ is if $\varphi_1 = \varphi_2$.

If we use Proposition E.3.1 and Lemma E.3.2 to translate parts of Theorem E.2.5 into a result on Toeplitz operators, we clearly see that the properties of the window φ must be taken into account when studying Toeplitz operators on $V_\varphi(L^2)$.

Proposition E.3.4. *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. The following are equivalent.*

1. *$V_\varphi \varphi$ has no zeros.*
2. *The Berezin transform $\mathfrak{B}^\varphi$ is injective on $\mathcal{L}(V_\varphi(L^2))$.*
3. *The map $f \mapsto T_f^\varphi$ is injective from $L^\infty(\mathbb{R}^{2d})$ to $\mathcal{L}(V_\varphi(L^2))$.*

Proof. The result will follow from Theorem E.2.5 once we have shown that each statement is equivalent to a statement in that theorem with $S = \varphi \otimes \varphi$. As $\mathcal{F}_W(S)(x,\omega) = e^{i\pi x\cdot\omega} V_\varphi\varphi(x,\omega)$ by Lemma E.2.6, (1) states that $S \in \mathcal{W}$. Since Proposition E.3.1 gives that T_f^φ is unitarily equivalent with $\mathcal{A}_f^{\varphi,\varphi} = f \star S$, the map $f \mapsto T_f^\varphi$ is injective if and only if the map $f \mapsto f \star S$ is injective. Similarly, since Lemma E.3.2 gives that

$$\mathfrak{B}^\varphi \tilde{T}(z) = \Theta^\varphi(\tilde{T}) \star \check{S}$$

and $\Theta^\varphi : \mathcal{L}(V_\varphi(L^2)) \to \mathcal{L}(L^2)$ is a bijection, we get that $\mathfrak{B}^\varphi$ is injective if and only if $T \mapsto T \star \check{S}$ is injective. It is simple to check that the last condition is equivalent to $T \mapsto T \star S$ being injective, as a calculation shows that $T \star \check{S}(z) = \check{T} \star S(-z)$. □

Remark E.5. The other parts of Theorem E.2.5 could also be translated into statements on $V_\varphi(L^2)$, and one could obtain other equivalences by imposing weaker requirements on the set of zeros of $V_\varphi\varphi$, see [182, 203].

E.3.2 Toeplitz operators on Bargmann-Fock space

For the Gaussian φ_0, the Gabor space $V_{\varphi_0}(L^2)$ is closely related to another much-studied reproducing kernel Hilbert space: the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$, consisting of all analytic functions F on $\mathbb{C}^d$ such that $\|F\|_{\mathcal{F}^2} < \infty$, where $\|F\|_{\mathcal{F}^2}$ is the norm induced by the inner product

$$\langle F, G\rangle_{\mathcal{F}^2} = \int_{\mathbb{C}^d} F(z)\overline{G(z)} e^{-\pi|z|^2}\, dz.$$

An important tool in the study of $\mathcal{F}^2(\mathbb{C}^d)$ is the *Bargmann transform*, which is the unitary mapping $\mathcal{B} : L^2(\mathbb{R}^d) \to \mathcal{F}^2(\mathbb{C}^d)$ defined by

$$\mathcal{B} = \mathcal{A} \circ V_{\varphi_0}, \tag{E.3.5}$$

where $\mathcal{A} : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{C}^d, e^{-\pi|z|^2} dz)$ is a unitary operator given by

$$\mathcal{A}(f)(x + i\omega) = e^{-\pi i x\cdot\omega} e^{\frac{\pi}{2}|z|^2} f(x, -\omega) \quad \text{ for } z = (x,\omega) \in \mathbb{R}^{2d}.$$

The restriction $\mathcal{A}|_{V_{\varphi_0}(L^2)}$ is unitary from $V_{\varphi_0}(L^2)$ to $\mathcal{F}^2(\mathbb{C}^d)$, as it may be written as the composition $\mathcal{B} \circ V_{\varphi_0}^*|_{V_{\varphi_0}(L^2)}$ of unitary operators. Hence $\mathcal{A}$ allows us to relate the spaces $V_{\varphi_0}(L^2)$ and $\mathcal{F}^2(\mathbb{C}^d)$.

The orthogonal projection from $L^2(\mathbb{C}^d, e^{-\pi|z|^2}dz)$ to $\mathcal{F}^2(\mathbb{C}^d)$ is given by

$$\mathcal{P}_{\mathcal{F}^2} = \mathcal{B}\mathcal{B}^* = \mathcal{A}V_{\varphi_0}V_{\varphi_0}^*\mathcal{A}^* = \mathcal{A}\mathcal{P}_{V_{\varphi_0}(L^2)}\mathcal{A}^*, \tag{E.3.6}$$

and the non-normalized reproducing kernel of $\mathcal{F}^2(\mathbb{C}^d)$ is

$$K_z(z') = e^{\pi \overline{z}\cdot z'} \quad \text{for } z, z' \in \mathbb{C}^d.$$

For our purposes it is convenient to note that we can use the reproducing kernel $k_{(x,\omega)}^{\varphi_0}$ for $V_{\varphi_0}(L^2)$ to express K_z for $z = x + i\omega$ by

$$K_z(x' + i\omega') = e^{i\pi x\cdot\omega} e^{\pi|z|^2/2} \left[\mathcal{A}k_{(x,-\omega)}^{\varphi_0}\right](x' + i\omega'), \tag{E.3.7}$$

as follows from the calculation

$$\begin{aligned}
\left\langle \mathcal{B}(\psi), e^{i\pi x\cdot\omega} e^{\pi|z|^2/2}\mathcal{A}k_{(x,-\omega)}^{\varphi_0}\right\rangle_{\mathcal{F}^2} &= e^{-\pi i x\cdot\omega} e^{\pi|z|^2/2}\left\langle \mathcal{A}V_{\varphi_0}\psi, \mathcal{A}k_{(x,-\omega)}^{\varphi_0}\right\rangle_{\mathcal{F}^2} \\
&= e^{-\pi i x\cdot\omega} e^{\pi|z|^2/2}\left\langle V_{\varphi_0}\psi, k_{(x,-\omega)}^{\varphi_0}\right\rangle_{L^2(\mathbb{R}^{2d})} \\
&= e^{-\pi i x\cdot\omega} e^{\pi|z|^2/2} V_{\varphi_0}\psi(x,-\omega) \\
&= \mathcal{B}(\psi)(x + i\omega).
\end{aligned}$$

For every $F \in L^\infty(\mathbb{C}^d)$ one defines the *Bargmann-Fock Toeplitz operator* $T_F^{\mathcal{F}^2}$ on $\mathcal{F}^2(\mathbb{C}^d)$ by

$$T_F^{\mathcal{F}^2}(H) = \mathcal{P}_{\mathcal{F}^2}(F \cdot H)$$

for any $H \in \mathcal{F}^2(\mathbb{C}^d)$. Using (E.3.6) and the unitarity of $\mathcal{A}$, one can calculate that if $f \in L^\infty(\mathbb{R}^{2d})$ and $F \in L^\infty(\mathbb{C}^d)$ are related by

$$F(x + i\omega) = f(x, -\omega) \quad \text{for } x, \omega \in \mathbb{R}^{2d}, \tag{E.3.8}$$

then

$$T_f^{\varphi_0} = \mathcal{A}^* T_F^{\mathcal{F}^2}\mathcal{A}. \tag{E.3.9}$$

In combination with Proposition E.3.1 this gives the following result.

Proposition E.3.5. *Let* $f \in L^\infty(\mathbb{R}^{2d})$ *and* $F \in L^\infty(\mathbb{C}^d)$ *be related by* (E.3.8). *Then the following operators are unitarily equivalent.*

1. *The localization operator* $\mathcal{A}_f^{\varphi_0,\varphi_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

2. *The Gabor Toeplitz operator* $T_f^{\varphi_0} : V_{\varphi_0}(L^2) \to V_{\varphi_0}(L^2)$.

3. *The Bargmann-Fock Toeplitz operator* $T_F^{\mathcal{F}^2} : \mathcal{F}^2(\mathbb{C}^d) \to \mathcal{F}^2(\mathbb{C}^d)$.

Remark E.6. The simple result above is far from new, going back to at least [58]. A related and more complicated question that appears in the literature is to relate $\mathcal{A}_f^{\varphi,\varphi}$, where φ needs no longer be Gaussian, to a Bargmann-Fock Toeplitz operator $T^{\mathcal{F}^2}_{(I+D)F}$, where D is some differential operator [5, 58, 92].

The Berezin transform can also be defined on $\mathcal{F}^2(\mathbb{C}^d)$. Since $\mathcal{A} : V_\varphi(L^2) \to \mathcal{F}^2(\mathbb{C}^d)$ is unitary, one easily checks using (E.3.7) that the *normalized* reproducing kernel $\tilde{k}_z$ on $\mathcal{F}^2(\mathbb{C}^d)$ is

$$\tilde{k}_z(z') = e^{i\pi x\cdot\omega}\left[\mathcal{A}k^{\varphi_0}_{(x,-\omega)}\right](x'+i\omega') \quad \text{for } z = x+i\omega, z' = x'+i\omega'.$$

This implies the following result on the Berezin transform $\mathfrak{B}^{\mathcal{F}^2}$ on $\mathcal{F}^2(\mathbb{C}^d)$.

Lemma E.3.6. *Let $\tilde{T} \in \mathcal{L}(\mathcal{F}^2(\mathbb{C}^d))$. Then*

$$\begin{aligned}\mathfrak{B}^{\mathcal{F}^2}\tilde{T}(x+i\omega) &= \mathfrak{B}^{\varphi_0}[\mathcal{A}^*\tilde{T}\mathcal{A}](x,-\omega)\\ &= (\mathcal{B}^*\tilde{T}\mathcal{B}) \star (\varphi_0\otimes\varphi_0)(x,-\omega).\end{aligned}$$

In particular, if $F \in L^\infty(\mathbb{C}^d)$, then

$$\mathfrak{B}^{\mathcal{F}^2}T_F^{\mathcal{F}^2}(x+i\omega) = \left(f * |V_{\varphi_0}\varphi_0|^2\right)(x,-\omega),$$

where $f \in L^\infty(\mathbb{R}^{2d})$ is given by $f(x,\omega) = F(x-i\omega)$ and $|V_{\varphi_0}\varphi_0(z)|^2 = e^{-\pi|z|^2}$.

Proof. By definition,

$$\begin{aligned}\mathfrak{B}^{\mathcal{F}^2}\tilde{T}(x+i\omega) &= \langle \tilde{T}\tilde{k}_{x+i\omega}, \tilde{k}_{x+i\omega}\rangle_{\mathcal{F}^2}\\ &= \left\langle \tilde{T}\mathcal{A}k^{\varphi_0}_{(x,-\omega)}, \mathcal{A}k^{\varphi_0}_{(x,-\omega)}\right\rangle_{\mathcal{F}^2}\\ &= \left\langle \mathcal{A}^*\tilde{T}\mathcal{A}k^{\varphi_0}_{(x,-\omega)}, k^{\varphi_0}_{(x,-\omega)}\right\rangle_{L^2(\mathbb{R}^{2d})}\\ &= \mathfrak{B}^{\varphi_0}[\mathcal{A}^*\tilde{T}\mathcal{A}](x,-\omega).\end{aligned}$$

That this last expression equals $(\mathcal{B}^*\tilde{T}\mathcal{B}) \star (\varphi_0\otimes\varphi_0)(x,-\omega)$ follows from Lemma E.3.2, since $\mathcal{B}^*\tilde{T}\mathcal{B} = V^*_{\varphi_0}[\mathcal{A}^*\tilde{T}\mathcal{A}]V_{\varphi_0}$. For the formula for Toeplitz operators, combine the first part with (E.3.9) and the final part of Lemma E.3.2. ◻

The results above show the intimate connection between $\mathcal{F}^2(\mathbb{C}^d)$ and the Gabor space $V_{\varphi_0}(L^2)$. Many of the results known for $\mathcal{F}^2(\mathbb{C}^d)$ can easily be translated into results for $V_{\varphi_0}(L^2)$, and we will later investigate certain conditions on φ that allow us to generalize these results to other Gabor spaces $V_\varphi(L^2)$.

E.3.3 Polyanalytic Bargmann-Fock spaces

By (E.3.5), we may identify $V_{\varphi_0}(L^2)$ and the Bargmann-Fock space by the operator $\mathcal{A} : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{C}^d, e^{-\pi|z|^2}dz)$. If the Gaussian φ_0 is replaced by another Hermite function φ_n for $n \in \mathbb{N}^d$, and we define the *polyanalytic Bargmann transform* $\mathcal{B}_n : L^2(\mathbb{R}^d) \to L^2(\mathbb{C}^d, e^{-\pi|z|^2}dz)$ by

$$\mathcal{B}_n = \mathcal{A} \circ V_{\varphi_n},$$

then the image of $\mathcal{B}_n$, which we denote by $\mathcal{F}^2_n$, is again a reproducing kernel Hilbert space with reproducing kernel $K^{\varphi_n}_z$ for $z = x + i\omega$ given by

$$K^{\varphi_n}_z(x' + i\omega') = e^{i\pi x\cdot\omega} e^{\pi|z|^2/2} \left[\mathcal{A}k^{\varphi_n}_{(x,-\omega)}\right](x' + i\omega').$$

Unlike the Bargmann-Fock space $\mathcal{F}^2 = \mathcal{F}^2_0$, $\mathcal{F}^2_n$ does not in general consist of analytic functions, but rather of so-called polyanalytic functions. For this reason $\mathcal{F}^2_n$ is sometimes called the *true polyanalytic Fock space of degree n* [3, 6, 22]. Following [177, 226] we define, given $F \in L^\infty(\mathbb{C}^d)$, the *polyanalytic Toeplitz operator* $T^{\mathcal{F}^2_n}_F : \mathcal{F}^2_n \to \mathcal{F}^2_n$ by

$$T^{\mathcal{F}^2_n}_F(H) = \mathcal{P}_{\mathcal{F}^2_n}(F \cdot H)$$

for $H \in \mathcal{F}^2_n$. Similarly to Bargmann-Fock space the orthogonal projection $\mathcal{P}_{\mathcal{F}^2_n}$ from $L^2(\mathbb{C}^d, e^{-\pi|z|^2}dz)$ to $\mathcal{F}^2_n$ is given by

$$\mathcal{P}_{\mathcal{F}^2_n} = \mathcal{B}\mathcal{B}^* = \mathcal{A}V_{\varphi_n}V^*_{\varphi_n}\mathcal{A}^*.$$

If $f \in L^\infty(\mathbb{R}^{2d})$ and $F \in L^\infty(\mathbb{C}^d)$ are related as in (E.3.8), one can show that $T^{\varphi_n}_f = \mathcal{A}^* T^{\mathcal{F}^2_n}_F \mathcal{A}$. Hence we obtain the following result.

Proposition E.3.7. *Let $f \in L^\infty(\mathbb{R}^{2d})$ and $F \in \mathbb{C}^d$ be related as in* (E.3.8). *For $n \in \mathbb{N}^d$, the following operators are unitarily equivalent.*

1. *The localization operator $\mathcal{A}^{\varphi_n,\varphi_n}_f : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.*
2. *The Gabor Toeplitz operator $T^{\varphi_n}_f : V_{\varphi_n}L^2 \to V_{\varphi_n}L^2$.*
3. *The polyanalytic Toeplitz operator $T^{\mathcal{F}^2_n}_F : \mathcal{F}^2_n(\mathbb{C}^d) \to \mathcal{F}^2_n(\mathbb{C}^d)$.*

We have related polyanalytic Toeplitz operators to Gabor Toeplitz operators on $V_{\varphi_n}(L^2)$. By [172, (4.16)], $V_{\varphi_n}\varphi_n$ has zeros if and only if $n \neq 0$. An easy argument using the previous proposition then translates Proposition E.3.4 into the following statement. A version of this is also discussed with different tools in [226, Sec. 5.1.2].

Proposition E.3.8. *Let $n \in \mathbb{N}^d$. The map $F \mapsto T_F^{\mathcal{F}_n^2}$ is injective from $L^\infty(\mathbb{C}^d)$ if and only if $n = 0$. In other words, assigning a bounded function to a Toeplitz operator is only injective on the Bargmann-Fock space.*

E.4 A Tauberian theorem for bounded functions

As our first main result we present a generalization of Wiener's classical Tauberian theorem that applies to bounded functions and convolutions with integrable functions and trace class operators. The key tool is Werner's generalization of Wiener's approximation theorem from Theorem E.2.5.

Theorem E.4.1 (Tauberian theorem for bounded functions)**.** *Let $f \in L^\infty(\mathbb{R}^{2d})$, and assume that one of the following equivalent statements holds for some $A \in \mathbb{C}$:*

(i) There is some $S \in \mathcal{W}$ such that

$$f \star S = A \cdot \mathrm{tr}(S) \cdot I_{L^2} + K$$

for some compact operator $K \in \mathcal{K}$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ such that

$$f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h$$

for some $h \in C_0(\mathbb{R}^{2d})$.

Then both of the following statements hold:

1. *For any $T \in \mathcal{S}^1$, $f \star T = A \cdot \mathrm{tr}(T) \cdot I_{L^2} + K_T$ for some compact operator $K_T \in \mathcal{K}$.*
2. *For any $g \in L^1(\mathbb{R}^{2d})$, $f * g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\, dz + h_g$ for some $h_g \in C_0(\mathbb{R}^{2d})$.*

Proof. We start by proving that (i) and (ii) are equivalent. Assume (i), and consider $a = S \star S \in L^1(\mathbb{R}^{2d})$. Since $\mathcal{F}_\sigma(S \star S)(z) = \mathcal{F}_W(S)(z)^2$ for any $z \in \mathbb{R}^{2d}$ by (E.2.9), we obtain both that $\mathcal{F}_\sigma(a)$ has no zeros and (by evaluating the relation at $z = 0$) that

$$\int_{\mathbb{R}^{2d}} a(z)\, dz = \mathrm{tr}(S) \cdot \mathrm{tr}(S).$$

Then observe using associativity of the convolutions that

$$\begin{aligned} f * a &= f * (S \star S) \\ &= (f \star S) \star S \\ &= (A \cdot \operatorname{tr}(S) \cdot I_{L^2} + K) \star S \\ &= A \cdot \operatorname{tr}(S) \cdot \operatorname{tr}(S) + K \star S \quad \text{by Lemma E.2.4} \\ &= A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + K \star S, \end{aligned}$$

and $K \star S \in C_0(\mathbb{R}^{2d})$ by Lemma E.2.3. The proof that (ii) implies (i) is similar by picking $S = a \star T$, where $T \in \mathcal{S}^1$ is any operator in $\mathcal{W}$. Then $\mathcal{F}_W(S)(z) = \mathcal{F}_\sigma(a)(z)\mathcal{F}_W(T)(z)$ by (E.2.10), so $\mathcal{F}_W(S)$ has no zeros and $\operatorname{tr}(S) = \int_{\mathbb{R}^{2d}} a(z)\, dz \cdot \operatorname{tr}(T)$ by evaluating the relation at $z = 0$. Furthermore, associativity of convolutions gives

$$\begin{aligned} f \star S &= f \star (a \star T) \\ &= (f * a) \star T \\ &= \left(A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h \right) \star T \\ &= A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz \cdot \operatorname{tr}(T) \cdot I_{L^2} + h \star T \quad \text{by Lemma E.2.4} \\ &= A \cdot \operatorname{tr}(S) \cdot I_{L^2} + h \star T, \end{aligned}$$

and $h \star T \in \mathcal{K}$ by Lemma E.2.3. Hence (i) and (ii) are equivalent.
The fact that (ii) implies (2) is Wiener's classical Tauberian theorem. The proof will therefore be completed if we can show $(i) \implies (1)$, so assume that S satisfies (i), and for now assume $A = 0$. In short, we assume $f \star S \in \mathcal{K}$. We need to show that $f \star T \in \mathcal{K}$ for any $T \in \mathcal{S}^1$. Part (3) of Theorem E.2.5 implies that T is the limit in the norm of $\mathcal{S}^1$ of a sequence $r_n \star S$ for $r_n \in L^1(\mathbb{R}^{2d})$. By commutativity and associativity of the convolutions,

$$f \star (r_n \star S) = r_n \star (f \star S) \in \mathcal{K} \quad \text{by Lemma E.2.3.}$$

Proposition E.2.2 then gives that

$$\|f \star T - f \star (r_n \star S)\|_{\mathcal{L}(L^2)} \leq \|f\|_{L^\infty} \|T - r_n \star S\|_{\mathcal{S}^1} \to 0 \quad \text{as } n \to \infty.$$

Hence $f \star T$ is the limit in the operator norm of compact operators, thus compact. Finally, assume that $A \neq 0$. Then $(f - A) \star S \in \mathcal{K}$ by Lemma E.2.4, so the result for $A = 0$ implies that $(f - A) \star T \in \mathcal{K}$ for any $T \in \mathcal{S}^1$, and applying Lemma E.2.4 again we see that this is equivalent to (1). □

The case $A = 0$ is particularly interesting, as it concerns the compactness of operators of the form $f \star T$ for $T \in \mathcal{S}^1$. We will return to this special case on several occasions.

Remark E.7.

1. Note that the convolution of a bounded and an integrable function is continuous, so we lose no generality by assuming that h and h_g belong to $C_0(\mathbb{R}^{2d})$ rather than merely assuming that they belong to $L^0(\mathbb{R}^{2d})$.

2. As already mentioned in the proof, the classical Tauberian theorem of Wiener is the implication $(ii) \implies (2)$.

3. The conditions on the Fourier transforms of S in (i) are necessary to imply (1) and (2). To see this, assume that $S \in \mathcal{S}^1$ satisfies $\mathcal{F}_W(S)(z_0) = 0$ for some $z_0 = (x_0, \omega_0) \in \mathbb{R}^{2d}$. Then consider the function $f_{z_0}(z) = e^{2\pi i \sigma(z_0, z)} \in L^\infty(\mathbb{R}^{2d})$. One can show that for any $T \in \mathcal{S}^1$ we have
$$f_{z_0} \star T = \mathcal{F}_W(T)(z_0) e^{-\pi i x_0 \cdot \omega_0} \pi(z_0).$$
In particular, $f_{z_0} \star S = 0 \in \mathcal{K}$ since $\mathcal{F}_W(S)(z_0) = 0$, so apart from the condition on $\mathcal{F}_W(S)$ we see that S satisfies (i) with $A = 0$. However, $f_{z_0} \star T = \mathcal{F}_W(T)(z_0) e^{-\pi i x_0 \cdot \omega_0} \pi(z_0)$ is not compact if $\mathcal{F}_W(T)(z_0) \neq 0$, hence (1) is not true for f_{z_0}. A similar argument with the same functions f_{z_0} shows that the condition on a in (ii) is also necessary.

E.4.1 A result by Fernández and Galbis

In [109], Fernández and Galbis proved the following result on compactness of localization operators.

Theorem E.4.2 (Fernández and Galbis)**.** *Let $f \in L^\infty(\mathbb{R}^{2d})$. Then $\mathcal{A}_f^{\varphi_1,\varphi_2}$ is compact for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ if and only if there is a non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$*
$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi f(x,\omega)| = 0. \tag{E.4.1}$$

Remark E.8.

1. This requirement is weaker than both $f \in L^0(\mathbb{R}^{2d})$ and $V_\Phi f \in C_0(\mathbb{R}^{4d})$ for some non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$. Proving that either of these two statements implies compactness of $\mathcal{A}_f^{\varphi_1,\varphi_2}$ requires far less advanced tools than (E.4.1), see [109].

2. The theorem holds for $f \in M^\infty(\mathbb{R}^{2d})$, where $M^\infty(\mathbb{R}^{2d})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $V_{\varphi_0} f \in L^\infty(\mathbb{R}^{4d})$. The space $M^\infty(\mathbb{R}^{2d})$ contains $L^\infty(\mathbb{R}^{2d})$ and certain distributions such as Dirac's delta distribution, see [131].

This allows us to add another equivalent assumption to Theorem E.4.1, formulated in terms of the short-time Fourier transform of f.

Proposition E.4.3. *Let $A \in \mathbb{C}$. Then $f \in L^\infty(\mathbb{R}^{2d})$ satisfies the equivalent conditions (i) and (ii) in Theorem E.4.1 if and only if*

(iii) There is some non-zero Schwartz function Φ on $\mathbb{R}^{2d}$ such that for every $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi(f-A)(x,\omega)| = 0.$$

Proof. Consider the operator $S = \varphi_0 \otimes \varphi_0$. Then $S \in \mathcal{W}$ by (E.2.2) and $f \star S = \mathcal{A}_f^{\varphi_0,\varphi_0}$ by (E.2.12). If (iii) is satisfied, Theorem E.4.2 implies using Lemma E.2.4 that

$$\mathcal{A}_{f-A}^{\varphi_0,\varphi_0} = (f-A) \star S = f \star S - A \cdot \operatorname{tr}(S) \cdot I_{L^2}$$

is compact, hence (i) holds. If (i) holds, then Theorem E.4.1 (1) implies that

$$f \star (\varphi_2 \otimes \varphi_1) - A \cdot \operatorname{tr}(\varphi_2 \otimes \varphi_1) \cdot I_{L^2} = (f-A) \star (\varphi_2 \otimes \varphi_1) = \mathcal{A}_{f-A}^{\varphi_1,\varphi_2}$$

is compact for any $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, so Theorem E.4.2 implies that (iii) holds. □

Remark E.9. One may easily calculate that

$$V_\Phi(f-A)(x,\omega) = V_\Phi f(x,\omega) - A \cdot e^{-2\pi i x\cdot\omega}\overline{\widehat{\Phi}(-\omega)}.$$

Condition (iii) therefore says that for any $R > 0$, if fixed x is picked with $|x|$ sufficiently large, then $V_\Phi f(x,\omega)$ should uniformly approximate $A \cdot e^{-2\pi i x\cdot\omega}\overline{\widehat{\Phi}(-\omega)}$ for $|\omega| \leq R$.

Theorem E.4.2 is a theorem concerning compactness of operators – its proof in [109] relies on results on relatively compact subsets of $\mathcal{K}$. However, Theorem E.4.1 along with Proposition E.4.3 allows us to translate the result to functions on $\mathbb{R}^{2d}$. In fact, it leads to a characterization in terms of the short-time Fourier transform of those $f \in L^\infty(\mathbb{R}^{2d})$ satisfying the assumptions of Wiener's classical Tauberian theorem. To our knowledge this result is new, so we formulate it as a separate statement.

Theorem E.4.4. *Let $A \in \mathbb{C}$ and $f \in L^\infty(\mathbb{R}^{2d})$ be given. The following are equivalent.*

- *There is some non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$*

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi(f-A)(x,\omega)| = 0.$$

- *There is $a \in W(\mathbb{R}^{2d})$ and $h \in C_0(\mathbb{R}^{2d})$ such that*

$$f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h.$$

- *For any $g \in L^1(\mathbb{R}^{2d})$ there is $h_g \in C_0(\mathbb{R}^{2d})$ such that*

$$f * g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\, dz + h_g.$$

Remark E.10. One might naturally ask if this result holds for $\mathbb{R}^d$ for any $d \geq 1$, and not just for even d. Our proof exploits Theorem E.4.2, which has no analogue for odd d. We can therefore not extend the proof to the general case.

E.4.2 A closer look at the two assumptions of Theorem E.4.1

By Remark E.3 and Lemma E.2.4, $f \in L^\infty(\mathbb{R}^{2d})$ trivially satisfies the assumptions (and conclusions) in Theorem E.4.1 if $f = A + h$ for some $A \in \mathbb{C}$ and $h \in L^p(\mathbb{R}^{2d})$ for $1 \leq p < \infty$ or $p = 0$. We will now see examples that do not satisfy these conditions.

Example E.4.1.

1. In [110, Prop. 4.1], Galbis and Fernández show that the function $f(x,\omega) = e^{i\pi|z|^2}$ satisfies condition (iii) from Proposition E.4.3, hence it satisfies (i) and (ii) in Theorem E.4.1. Clearly $f \notin L^p(\mathbb{R}^{2d})$ for $p = 0$ or $1 \leq p < \infty$.

2. Given $\tau \in (0,1)\setminus\{1/2\}$, the function $a_\tau(x,\omega) = \frac{2^d}{|2\tau-1|^d} \cdot e^{2\pi i \frac{2}{2\tau-1} x\cdot\omega}$ satisfies assumption (i) in Theorem E.4.1 with $A = 0$, as we prove in Proposition E.6.3. Again, we see that $a_\tau \notin L^p(\mathbb{R}^{2d})$ for $p = 0$ or $1 \leq p < \infty$.

3. If $f \in L^\infty(\mathbb{R}^{2d})$ is a so-called *pseudomeasure*, meaning that $\mathcal{F}_\sigma(f) \in L^\infty(\mathbb{R}^{2d})$, then f satisfies (ii) with $A = 0$. To see this, let $a(z) = e^{-\pi|z|^2}$. Then $\mathcal{F}_\sigma(a) = a$ has no zeros, and

$$f * a = \mathcal{F}_\sigma \mathcal{F}_\sigma(f * a) = \mathcal{F}_\sigma(\mathcal{F}_\sigma(f) \cdot a),$$

and since $\mathcal{F}_\sigma(f) \in L^\infty(\mathbb{R}^{2d})$ we have $\mathcal{F}_\sigma(f) \cdot a \in L^1(\mathbb{R}^{2d})$. Hence $f * a \in C_0(\mathbb{R}^{2d})$ by the Riemann-Lebesgue lemma.
Rather surprisingly, we may prove (1) directly in this case by considering the operator side of our setup. For any $T \in \mathcal{S}^1$, we obtain that $\mathcal{F}_W(T) \in L^2(\mathbb{R}^{2d})$ since $\mathcal{S}^1 \subset \mathcal{S}^2$ and $\mathcal{F}_W : \mathcal{S}^2 \to L^2(\mathbb{R}^{2d})$ is a unitary operator. By our assumption on f, it follows that $\mathcal{F}_W(f \star T) = \mathcal{F}_\sigma(f)\mathcal{F}_W(T) \in L^2(\mathbb{R}^{2d})$, hence $f \star T \in \mathcal{S}^2 \subset \mathcal{K}$. The key to this calculation is the inclusion $L^\infty(\mathbb{R}^{2d}) \cdot \mathcal{F}_W(\mathcal{S}^1) \subset L^2(\mathbb{R}^{2d})$ – the corresponding function result $L^\infty \cdot \mathcal{F}_\sigma(L^1) \subset L^2$ is not true by the results in [53].

The examples above show that it is not necessary to have $\lim_{|z|\to\infty} f(z) = 0$ in order to satisfy assumptions (i) and (ii) with $A = 0$. A well-known result in the Tauberian theory of functions due to Pitt [214] says that if we assume that f is *slowly oscillating*, then $\lim_{|z|\to\infty} f(z) = 0$ is necessary for f to satisfy (ii).

Recall that f is slowly oscillating on $\mathbb{R}^{2d}$ if for every $\epsilon > 0$ there is $\delta > 0$ and $K > 0$ such that $|f(z) - f(z - z')| < \epsilon$ for $|z'| < \delta$ and $|z| > K$. We refer to [115, Thm. 4.74] for a formulation of Pitt's result that applies to $\mathbb{R}^{2d}$.

Theorem E.4.5 (Pitt)**.** *If $f \in L^\infty(\mathbb{R}^{2d})$ is slowly oscillating and satisfies either assumption (i) or (ii) in Theorem E.4.1 or (iii) from Proposition E.4.3 with $A = 0$, then $f \in L^0(\mathbb{R}^{2d})$.*

Remark E.11. Any uniformly continuous $f \in L^\infty(\mathbb{R}^{2d})$ is slowly oscillating, hence if such f satisfies (i) with $A = 0$, then $f \in C_0(\mathbb{R}^{2d})$. This weaker statement actually follows from the correspondence theory introduced by Werner in [251], more precisely by [251, Thm. 4.1 (3)]. In Werner's terminology $C_0(\mathbb{R}^{2d})$ and $\mathcal{K}$ are corresponding subspaces, since convolutions with trace class operators interchanges these two spaces by Lemma E.2.3. We will see the operator-analogue of this result in Section E.5.1

Consequences for Toeplitz operators

We now formulate a version of the Tauberian theorem for (polyanalytic) Bargmann-Fock Toeplitz operators. As a preliminary observation, let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. If $S \in \mathcal{L}(\mathcal{H}_1)$ and $T \in \mathcal{L}(\mathcal{H}_2)$ are unitarily equivalent, i.e. there is unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $S = U^*TU$, then one easily checks that $S = A \cdot I_{\mathcal{H}_1} + K_1$ for $A \in \mathbb{C}$ and compact $K_1 \in \mathcal{L}(\mathcal{H}_1)$ if and only if $T = A \cdot I_{\mathcal{H}_2} + K_2$ for compact $K_2 \in \mathcal{L}(\mathcal{H}_2)$.

Proposition E.4.6. *Let $F \in L^\infty(\mathbb{C}^d)$ and $A \in \mathbb{C}$. Define $f \in L^\infty(\mathbb{R}^{2d})$ by $f(x,\omega) = F(x - i\omega)$. The following are equivalent:*

(i) $T_F^{\mathcal{F}^2} = A \cdot I_{\mathcal{F}^2} + \tilde{K}_0$ for some compact operator $\tilde{K}_0$ on $\mathcal{F}^2(\mathbb{C}^d)$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ such that

$$f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h_a$$

for some $h_a \in C_0(\mathbb{R}^{2d})$.

(iii) There is some non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\omega|\leq R} |V_\Phi(f - A)(x,\omega)| = 0.$$

Furthermore, if any of the equivalent conditions above holds, then for any $n \in \mathbb{N}^d$ the polyanalytic Toeplitz operator $T_F^{\mathcal{F}_n^2}$ is of the form

$$T_F^{\mathcal{F}_n^2} = A \cdot I_{\mathcal{F}_n^2} + \tilde{K}_n,$$

where $\tilde{K}_n$ is a compact operator on $\mathcal{F}_n^2(\mathbb{C}^d)$.

Proof. By Proposition E.3.5, $T_F^{\mathcal{F}^2}$ is unitarily equivalent to $\mathcal{A}_f^{\varphi_0,\varphi_0} = f \star (\varphi_0 \otimes \varphi_0)$. By the remark above, part (i) holds if and only if $f \star (\varphi_0 \otimes \varphi_0) = A \cdot I_{L^2} + K_0$ for some compact operator K_0 on $L^2(\mathbb{R}^d)$. Since $\varphi_0 \otimes \varphi_0 \in \mathcal{W}$ by (E.2.2), the fact that (i), (ii) and (iii) are equivalent follows from Proposition E.4.3.

As we have seen that (i) implies that $f \star (\varphi_0 \otimes \varphi_0) = A \cdot I_{L^2} + K_0$ and that $\varphi_0 \otimes \varphi_0 \in \mathcal{W}$, Theorem E.4.1 implies that for every n there is a compact K_n with

$$f \star (\varphi_n \otimes \varphi_n) = A \cdot I_{L^2} \cdot \operatorname{tr}(\varphi_n \otimes \varphi_n) + K_n = A \cdot I_{L^2} + K_n.$$

The last statement then follows as $T_F^{\mathcal{F}_n^2}$ is unitarily equivalent to $\mathcal{A}_f^{\varphi_n,\varphi_n} = f \star (\varphi_n \otimes \varphi_n)$ by Proposition E.3.7. □

Remark E.12. The equivalence of (i) and (ii) when a is fixed to be the Gaussian $a(x,\omega) = e^{-\pi(x^2+\omega^2)}$ is due to Engliš, see the equivalence of (a) and (c) in [91, Thm. B]. Note that Engliš also considers products of Toeplitz operators, which is a setting we will return to in Section E.5.1.

The same reasoning gives the following Tauberian theorem for Toeplitz operators on Gabor spaces using Proposition E.3.1.

Proposition E.4.7. *Let $f \in L^\infty(\mathbb{R}^{2d})$ and $A \in \mathbb{C}$. The following are equivalent.*

(i) There is some $\varphi \in L^2(\mathbb{R}^d)$ such that $V_\varphi\varphi$ has no zeros and $T_f^\varphi = A \cdot I_{V_\varphi(L^2)} + K$ for some compact operator $K \in \mathcal{L}(V_\varphi(L^2))$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ such that $f * a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz + h$ for some $h \in C_0(\mathbb{R}^{2d})$.

(iii) There is some non-zero $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ such that for every $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\omega|\le R} |V_\Phi(f - A)(x,\omega)| = 0.$$

Furthermore, if any of the equivalent conditions above holds, then for every normalized $\varphi' \in L^2(\mathbb{R}^d)$ we have that $T_f^{\varphi'}$ is of the form $A \cdot I_{V_{\varphi'}(L^2)} + K_{\varphi'}$ for some compact operator $K_{\varphi'} \in \mathcal{L}(V_{\varphi'}(L^2))$.

E.4.3 Injectivity of localization operators and Riesz theory of compact operators

We will now let the kind of operators appearing in Theorem E.4.1 inspire a slight detour that does not explicitly build on the Tauberian theorems. Theorem E.4.1 gives conditions to ensure that a localization operator $\mathcal{A}_f^{\varphi,\varphi}$ is a compact perturbation of a scaling of the identity, i.e. of the form $A \cdot I_{L^2} + K$ for $0 \neq A \in \mathbb{C}$ and $K \in \mathcal{K}$. The theory of such operators, sometimes referred to as Riesz theory due to the seminal work of F. Riesz [224], contains several powerful results similar to those that hold for matrices. We will use the following result, see [52, Lem. 6.30 & Thm. 6.33] for proofs.

Proposition E.4.8. *Assume that $T \in \mathcal{L}(L^2)$ is of the form $A \cdot I_{L^2} + K$ for $A \neq 0$ and $K \in \mathcal{K}$. Then T has closed range and* $\dim(\ker T) = \dim(\operatorname{coker}(T)) < \infty$. *In particular, T is injective if and only if T is surjective.*

As an obvious consequence, we note that if $\mathcal{A}_f^{\varphi,\varphi} = A \cdot I_{L^2} + K$ for $A \neq 0$ and $K \in \mathcal{K}$ and $\mathcal{A}_f^{\varphi,\varphi}$ is injective, then $\mathcal{A}_f^{\varphi,\varphi}$ is an isomorphism on $L^2(\mathbb{R}^d)$. Inspired by this, we investigate conditions ensuring that localization operators are injective. The proof of the next result is similar to that of [50, Lem. 1.4].

Lemma E.4.9. *Assume that $f \in L^\infty(\mathbb{R}^{2d})$ such that $f(z) \geq 0$ for a.e. $z \in \mathbb{R}^{2d}$.*

1. *If $0 \neq \varphi \in L^2(\mathbb{R}^{2d})$ and there is $\Delta \subset \mathbb{R}^{2d}$ of finite Lebesgue measure with*

$$f(z) > 0 \quad \text{for a.e. } z \notin \Delta,$$

then the localization operator $\mathcal{A}_f^{\varphi,\varphi}$ is injective.

2. *If there is some open subset $\Omega \subset \mathbb{R}^{2d}$ such that*

$$f(z) > 0 \quad \text{for a.e. } z \in \Omega,$$

then the localization operator $\mathcal{A}_f^{\varphi_0,\varphi_0}$ is injective.

Proof. We first prove (1). Assume that $\mathcal{A}_f^{\varphi,\varphi}(\psi) = 0$. This implies by (E.2.4) that

$$\left\langle \mathcal{A}_f^{\varphi,\varphi}(\psi), \psi \right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} f(z) |V_\varphi \psi(z)|^2 \, dz = 0.$$

Since we assume that f is non-negative for a.e. z, this further implies that

$$\int_{\mathbb{R}^{2d} \setminus \Delta} f(z) |V_\varphi \psi(z)|^2 \, dz = 0.$$

This implies that $V_\varphi\psi(z) = 0$ for a.e. $z \notin \Delta$. Hence $\psi = 0$, as the main result of [173] says that $V_\varphi\psi(z)$ cannot be supported on a set of finite Lebesgue measure unless $\psi = 0$ or $\varphi = 0$.

To prove (2), a similar argument as above shows that $\mathcal{A}_f^{\varphi_0,\varphi_0}(\psi) = 0$ implies that $V_{\varphi_0}\psi(z) = 0$ for a.e. $z \in \Omega$. Continuity gives that $V_{\varphi_0}\psi(z) = 0$ for all $z \in \Omega$. The analytic function $\mathcal{B}(\psi)(x + i\omega) = e^{-\pi i x\cdot\omega}e^{-\frac{\pi}{2}(x^2+\omega^2)}V_{\varphi_0}\psi(x, -\omega)$ therefore vanishes on an open subset of $\mathbb{C}^d$, hence $\mathcal{B}(\psi) = 0$ by uniqueness of analytic continuation. Thus $\psi = 0$ as $\mathcal{B}$ is injective. □

We deduce sufficient conditions for localization operators to be isomorphisms.

Proposition E.4.10. *Let $0 \neq M \in \mathbb{R}$, $a \in L^\infty(\mathbb{R}^{2d})$ and $\Delta \subset \mathbb{R}^{2d}$ a set of finite Lebesgue measure. Assume that the following assumptions hold:*

(i) *$a(z) \geq -M$ for a.e. $z \in \mathbb{R}^{2d}$,*

(ii) *$a(z) > -M$ for $z \notin \Delta$,*

(iii) *a satisfies assumption (i) or (ii) in Theorem E.4.1 with $A = 0$.*

Let $f = M + a$. Then $\mathcal{A}_f^{\varphi,\varphi}$ is an isomorphism on $L^2(\mathbb{R}^d)$ for any $0 \neq \varphi \in L^2(\mathbb{R}^d)$.

Proof. By Lemma E.4.9 part (1), $\mathcal{A}_f^{\varphi,\varphi}$ is injective. By assumption (iii), Theorem E.4.1 gives that $a \star (\varphi \otimes \varphi) \in \mathcal{K}$, so that

$$\mathcal{A}_f^{\varphi,\varphi} = (M + a) \star (\varphi \otimes \varphi) = M \cdot \|\varphi\|_{L^2}^2 \cdot I_{L^2} + a \star (\varphi \otimes \varphi)$$

is a compact perturbation of a scaling of the identity. Hence Proposition E.4.8 implies that $\mathcal{A}_f^{\varphi,\varphi}$ is also surjective. □

Remark E.13.

1. Finding specific examples of a satisfying the assumptions above is not difficult, but it is worth noting that a need not vanish at infinity. For instance, a standard construction gives continuous $a \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ such that

$$0 \leq a \leq 1, \quad \limsup_{|z|\to\infty} |a(z)| = 1, \quad \liminf_{|z|\to\infty} |a(z)| = 0.$$

Then a satisfies all three conditions above for $M > 0$, even though $f = M + a$ has no limit as $|z| \to \infty$. Of course, if we add the condition that a is slowly oscillating, then a must vanish at infinity by Theorem E.4.5.

2. Other isomorphism theorems for localization operators may be found in [50, 143, 144].

We state a special case of Proposition E.4.10 as a theorem, namely the case where $f = \chi_\Omega$ such that Ω^c has finite measure. We find that as long as Ω^c has finite measure, the values of $V_\varphi\psi(z)$ for $z \in \Omega^c$ are not needed to reconstruct ψ – independently of the geometry of Ω and the window φ.

Theorem E.4.11. *Assume that $\Omega \subset \mathbb{R}^{2d}$ satisfies that Ω^c has finite Lebesgue measure, and that $0 \neq \varphi \in L^2(\mathbb{R}^d)$. Then the localization operator $\mathcal{A}_{\chi_\Omega}^{\varphi,\varphi}$ is an isomorphism on $L^2(\mathbb{R}^d)$. In particular, any $0 \neq \psi \in L^2(\mathbb{R}^d)$ is uniquely determined by the values of $V_\varphi\psi(z)$ for $z \in \Omega$ and there exist constants $C, D > 0$ independent of ψ such that*

$$C \cdot \|\psi\|_{L^2} \leq \left\| \int_\Omega V_\varphi\psi(z)\pi(z)\varphi \, dz \right\|_{L^2} \leq D \cdot \|\psi\|_{L^2}.$$

Proof. This is a special case of Proposition E.4.10 with $M = 1$ and $a = -\chi_{\Omega^c}$. Then $f = 1 - \chi_{\Omega^c} = \chi_\Omega$, and one easily checks that the conditions in the proposition are satisfied with $\Delta = \Omega^c$, in particular (iii) follows as $\chi_{\Omega^c} \in L^1(\mathbb{R}^{2d})$. □

Remark E.14. 1. After submitting this paper for publication, we were made aware that stronger versions of this result by different approaches exist in the literature, see [111] and references therein. To our knowledge the strongest of these results is [111], where it is shown that Theorem E.4.11 holds if χ_{Ω^c} satisfies assumption (i) or (ii) of Theorem E.4.1 with $A = 0$. It follows that this assumption is sufficient in part (1) of Proposition E.4.12 as well.

2. Theorem E.4.11 is an example of turning uncertainty principles into signal recovery results, as proposed by Donoho and Stark [84]. An alternative proof more in line with the methods of [84] could be obtained by showing that $\|\mathcal{A}_{\chi_{\Omega^c}}^{\varphi,\varphi}\|_{\mathcal{L}(L^2)} < 1$ using [173], and using a Neumann series argument to deduce the invertibility of $\mathcal{A}_{\chi_\Omega}^{\varphi,\varphi} = I_{L^2} - \mathcal{A}_{\chi_{\Omega^c}}^{\varphi,\varphi}$.

3. If φ belongs to *Feichtinger's algebra* $M^1(\mathbb{R}^d)$ [95, 131], then invertibility of $\mathcal{A}_f^{\varphi,\varphi}$ on $L^2(\mathbb{R}^d)$ implies that $\mathcal{A}_f^{\varphi,\varphi}$ is also invertible on all *modulation spaces* $M^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$ (see [131] for an introduction to modulation spaces). This follows by combining [63, Thm. 3.2] and [133, Cor. 4.7].

Isomorphism results for $\mathcal{F}_n^2(\mathbb{C}^d)$

Any Toeplitz operator $T_F^{\mathcal{F}_n^2}$ on polyanalytic Bargmann-Fock space is unitarily equivalent to a localization operator $\mathcal{A}_f^{\varphi_n,\varphi_n}$ by Proposition E.3.7, where $f \in L^\infty(\mathbb{R}^{2d})$ and $F \in L^\infty(\mathbb{C}^d)$ are related by

$$F(x + i\omega) = f(x, -\omega).$$

Hence the results of this section may be translated into results for Toeplitz operators on $\mathcal{F}_n^2(\mathbb{C}^d)$. We include a couple of such results in the next statement. One may of course obtain isomorphism results for Gabor spaces in the same way by using Proposition E.3.1.

Proposition E.4.12. *1. If $\Omega \subset \mathbb{C}^d$ satisfies that Ω^c has finite Lebesgue measure, then $T_{\chi_\Omega}^{\mathcal{F}_n^2}$ is an isomorphism on $\mathcal{F}_n^2(\mathbb{C}^d)$.*

2. There is a real-valued, continuous $F \in L^\infty(\mathbb{C}^d)$ such that $\lim_{|z|\to\infty} |F(z)|$ does not exist, yet $T_F^{\mathcal{F}_n^2}$ is an isomorphism on $\mathcal{F}_n^2(\mathbb{C}^d)$.

Proof. In light of Proposition E.3.7, the first part follows from Theorem E.4.11 and the second from Remark E.13. □

E.5 A Tauberian theorem for bounded operators

A guiding principle in the theory of quantum harmonic analysis is that the role of functions and operators may often be interchanged in theorems. It should therefore come as no surprise that we can prove a Tauberian theorem where the bounded function f from Theorem E.4.1 is replaced by a bounded operator R, with just a few modifications of the proof.

Theorem E.5.1 (Tauberian theorem for bounded operators)**.** *Let $R \in \mathcal{L}(L^2)$, and assume that one of the following equivalent statements holds for some $A \in \mathbb{C}$:*

(i) There is some $S \in \mathcal{W}$ such that

$$R \star S = A \cdot \operatorname{tr}(S) + h$$

for some $h \in C_0(\mathbb{R}^{2d})$.

(ii) There is some $a \in W(\mathbb{R}^{2d})$ such that

$$R \star a = A \cdot \int_{\mathbb{R}^{2d}} a(z)\, dz \cdot I_{L^2} + K$$

for some compact operator $K \in \mathcal{K}$.

Then both of the following statements hold:

1. *For any $T \in \mathcal{S}^1$, $R \star T = A \cdot \operatorname{tr}(T) + h_T$ for some $h_T \in C_0(\mathbb{R}^{2d})$.*
2. *For any $g \in L^1(\mathbb{R}^{2d})$, $R \star g = A \cdot \int_{\mathbb{R}^{2d}} g(z)\, dz \cdot I_{L^2} + K_g$ for some compact operator $K_g \in \mathcal{K}$.*

Proof. The equivalence of the assumptions is proved in a similar way as for Theorem E.4.1: for $(i) \implies (ii)$ pick $a = S \star S$, and for $(ii) \implies (i)$ pick $S = a \star T$ for any $T \in \mathcal{W}$.

Then assume that (i) holds with $A = 0$, the extension to $A \neq 0$ is done as in the proof of Theorem E.4.1. To show (1), one proceeds as in the proof of Theorem E.4.1 by first showing that $S \star T \in C_0(\mathbb{R}^{2d})$ if $T = r \star S$ for some $r \in L^1(\mathbb{R}^{2d})$. Using Theorem E.2.5 one has that any $T \in \mathcal{S}^1$ is the limit in the norm of $\mathcal{S}^1$ of a sequence $r_n \star S$ with $r_n \in L^1(\mathbb{R}^{2d})$. The proof is completed by showing that the sequence $R \star (r_n \star S)$ – which is a sequence of functions in $C_0(\mathbb{R}^{2d})$ – converges uniformly to $R \star T$. Since $C_0(\mathbb{R}^{2d})$ is closed under uniform limits, this implies (1).

The proof that (i) implies (2) follows the same pattern. First show it for $g = T \star S$ for some $T \in \mathcal{S}^1$, then extend to all g by density, since Theorem E.2.5 implies that any $g \in L^1(\mathbb{R}^{2d})$ is the limit of a sequence $T_n \star S$ for $T_n \in \mathcal{S}^1$. □

Remark E.15. The conditions on the Fourier transforms of S and a in (i) and (ii) are necessary to imply (1) and (2), as can be shown by picking $R = \pi(z_0)$ for $z_0 = (x_0, \omega_0) \in \mathbb{R}^{2d}$. A calculation from the definitions (E.2.6) and (E.2.8) shows that

$$[\pi(z_0) \star S]\,(z) = e^{2\pi i \sigma(z_0,z)} e^{\pi i x_0 \cdot \omega_0} \mathcal{F}_W(S)(z_0).$$

So if $\mathcal{F}_W(S)(z_0) = 0$, we get that $\pi(z_0) \star S = 0 \in C_0(\mathbb{R}^{2d})$. On the other hand we may consider $\varphi_0 \otimes \varphi_0$. By Example E.2.2, we get that

$$[\pi(z_0) \star (\varphi_0 \otimes \varphi)]\,(z) = e^{2\pi i \sigma(z_0,z)} e^{\pi i x_0 \cdot \omega_0} e^{-\pi z_0^2} \notin C_0(\mathbb{R}^{2d}).$$

Hence the condition in (i) is necessary. To show that the condition on a in (ii) is necessary one uses a similar argument and the fact that

$$\pi(z_0) \star a = \mathcal{F}_\sigma(a)(z_0)\pi(z_0),$$

as a calculation shows.

From Lemmas E.2.3 and E.2.4 it is clear that (i) and (ii) are satisfied if $R = A \cdot I_{L^2} + K$ for some compact operator K. However, these are not the only examples.

Example E.5.1. If $R \in \mathcal{L}(L^2)$ satisfies that $\mathcal{F}_W(R) \in L^\infty(\mathbb{R}^{2d})$, then R satisfies assumption (ii) of Theorem E.5.1 with $A = 0$ – such R are the operator-analogues of the pseudomeasures considered in Example E.4.1. To prove this, let $S = \varphi_0 \otimes \varphi_0$. Then $\mathcal{F}_W(S)(z) = e^{-\pi |z|^2}$, so $S \in \mathcal{W}$, and

$$\mathcal{F}_\sigma(R \star S) = \mathcal{F}_W(R) \cdot \mathcal{F}_W(S) \in L^1(\mathbb{R}^{2d}).$$

By Fourier inversion we have

$$R \star S = \mathcal{F}_\sigma(\mathcal{F}_W(R) \cdot \mathcal{F}_W(S)),$$

which belongs to $C_0(\mathbb{R}^{2d})$ by the Riemann-Lebesgue lemma.

An example of such R is $R = P$, the parity operator. One can show that $\mathcal{F}_W(P)(z) = 2^d$ for any $z \in \mathbb{R}^{2d}$, hence P is a non-compact operator satisfying assumption (ii) of Theorem E.5.1 with $A = 0$. We will return to this and other examples below.

E.5.1 Pitt improvements, compactness and the Berezin transform

As we saw in Theorem E.4.5, Pitt's classical theorem gives a condition on $f \in L^\infty(\mathbb{R}^{2d})$ that ensures that

$$f * g \in C_0(\mathbb{R}^{2d}) \text{ for } g \in W(\mathbb{R}^{2d}) \implies f \in C_0(\mathbb{R}^{2d}).$$

In particular, we noted that this is true if f is uniformly continuous. To generalize this statement to operators $R \in \mathcal{L}(L^2)$, recall that $f \in L^\infty(\mathbb{R}^{2d})$ is uniformly continuous if and only if $z \mapsto T_z(f)$ is continuous map from $\mathbb{R}^{2d}$ to $L^\infty(\mathbb{R}^{2d})$. Hence a natural analogue of the uniformly continuous functions is the set

$$\mathcal{C}_1 := \{R \in \mathcal{L}(L^2) : z \mapsto \alpha_z(R) \text{ is continuous from } \mathbb{R}^{2d} \text{ to } \mathcal{L}(L^2)\};$$

this heuristic was also followed by Werner [251] and Bekka [28]. With this in mind, the following result from [251] is an analogue of Pitt's theorem for operators.

Theorem E.5.2. *Let $R \in \mathcal{C}_1$. The following are equivalent.*

- *$R \in \mathcal{K}$.*
- *$R \star S \in C_0(\mathbb{R}^{2d})$ for some $S \in \mathcal{W}$.*
- *$R \star f \in \mathcal{K}$ for some $f \in W(\mathbb{R}^{2d})$.*

Proof. That the first statement implies the other two is Lemma E.2.3. That the other statements imply the first follows from the theory of *corresponding subspaces* developed by Werner in [251], more precisely from [251, Thm. 4.1 (3)]. In the notation of [251] we have picked $\mathcal{D}_0 = C_0(\mathbb{R}^{2d})$ and $\mathcal{D}_1 = \mathcal{K}$. □

We then try to gain a better understanding of the elements of $\mathcal{C}_1$.

Lemma E.5.3. *The following set inclusion and equality hold:*

$$L^\infty(\mathbb{R}^{2d}) \star \mathcal{S}^1 \subset L^1(\mathbb{R}^{2d}) \star \mathcal{L}(L^2) = \mathcal{C}_1. \tag{E.5.1}$$

Proof. The equality $\mathcal{C}_1 = L^1(\mathbb{R}^{2d}) \star \mathcal{L}(L^2)$ is [203, Prop. 4.5]. Then assume $R = f \star S$ for $f \in L^\infty(\mathbb{R}^{2d})$ and $S \in \mathcal{S}^1$. By [203, Prop. 7.4] there must exist $g \in L^1(\mathbb{R}^{2d})$ and $T \in \mathcal{S}^1$ such that $S = g \star T$. It follows by associativity and commutativity of convolutions that we have $R = f \star (g \star T) = g \star (f \star T)$. Since $f \star T \in \mathcal{L}(L^2)$ by Proposition E.2.2, it follows that $R \in L^1(\mathbb{R}^{2d}) \star \mathcal{L}(L^2)$. □

Furthermore, it is not difficult to see that $\mathcal{C}_1$ equipped with the operator norm is a Banach algebra. Hence it must contain the Banach algebra generated by elements of the form $f \star T$ for $f \in L^\infty(\mathbb{R}^{2d})$ and $T \in \mathcal{S}^1$, and Proposition E.5.2 applies to operators in this Banach algebra.

This allows us to apply the results above to characterizing compactness of Toeplitz operators by their Berezin transform, a much-studied question going back to results of Axler and Zheng [16] for the so-called Bergman space, and soon after Engliš [91] for the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^d)$. The central question is whether a Toeplitz operator on a reproducing kernel Hilbert space must be compact if its Berezin transform vanishes at infinity – see Section 4 of [23] for an overview over results of this nature in the literature. We will use Proposition E.5.2 to reprove the main result of [24] for $\mathcal{F}^2(\mathbb{C}^d)$ and extend it to a class of Gabor spaces, but we hasten to add that the method of proving the results of [24] using the results of [251] was already noted recently by Fulsche [117]. Before the proof, recall the linear and multiplicative isometric isomorphism $\Theta^\varphi : \mathcal{L}(V_\varphi(L^2)) \to \mathcal{L}(L^2)$ from (E.3.3), which satisfies that $\Theta^\varphi(T_f^\varphi) = \mathcal{A}_f^{\varphi,\varphi}$ and $\mathfrak{B}^\varphi \tilde{T} = \Theta^\varphi(\tilde{T}) \star (\check{\varphi} \otimes \check{\varphi})$.

Theorem E.5.4. *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$ satisfy that $V_\varphi\varphi$ has no zeros, and let $\mathcal{T}^\varphi$ be the Banach algebra generated by Toeplitz operators T_f^φ in $\mathcal{L}(V_\varphi(L^2))$ for $f \in L^\infty(\mathbb{R}^{2d})$. Then the following are equivalent for $\tilde{T} \in \mathcal{T}^\varphi$.*

- *$\tilde{T}$ is a compact operator on $V_\varphi(L^2)$.*
- *$\mathfrak{B}^\varphi \tilde{T} \in C_0(\mathbb{R}^{2d})$.*

Furthermore, if $\tilde{T} = T_f^\varphi$ for some slowly oscillating $f \in L^\infty(\mathbb{R}^{2d})$, then the conditions above are equivalent to $\lim_{|z|\to\infty} |f(z)| = 0$.

Proof. First note that the assumption on $V_\varphi\varphi$ means that $\varphi \otimes \varphi \in \mathcal{W}$ by Lemma E.2.6, and as a simple calculation shows that $\mathcal{F}_W(\check{\varphi} \otimes \check{\varphi})(z) = \mathcal{F}_W(\varphi \otimes \varphi)(-z)$ it also means that $\check{\varphi} \otimes \check{\varphi} \in \mathcal{W}$. To see that the first statement implies the second, note that $\Theta^\varphi(\tilde{T})$ is compact if and only if $\tilde{T}$ is, so

$$\mathfrak{B}^\varphi \tilde{T} = \Theta^\varphi(\tilde{T}) \star (\check{\varphi} \otimes \check{\varphi}) \in C_0(\mathbb{R}^{2d})$$

by Lemma E.2.3. For the other direction, it is clear by the properties of Θ^φ that it maps $\mathcal{T}^\varphi$ into the Banach algebra generated by localization operators $\mathcal{A}_f^{\varphi,\varphi} = f \star (\varphi \otimes \varphi)$

for $f \in L^\infty(\mathbb{R}^{2d})$. In particular, $\Theta^\varphi(\mathcal{T}^\varphi) \subset \mathcal{C}_1$ by (E.5.1) as $\mathcal{C}_1$ is a Banach algebra containing $\mathcal{A}_f^{\varphi,\varphi}$ for all $f \in L^\infty(\mathbb{R}^{2d})$. Since $\mathfrak{B}^\varphi \tilde{T} = \Theta^\varphi(\tilde{T}) \star (\check{\varphi} \otimes \check{\varphi}) \in C_0(\mathbb{R}^{2d})$ and $\check{\varphi} \otimes \check{\varphi} \in \mathcal{W}$ by assumption, Proposition E.5.2 gives that $\Theta^\varphi(\tilde{T})$ is compact, hence $\tilde{T}$ is compact as Θ^φ is a unitary equivalence by definition.

The last statement follows from Theorem E.4.5, as T_f^φ is compact if and only if $\Theta^\varphi(T_f^\varphi) = \mathcal{A}_f^{\varphi,\varphi} = f \star (\varphi \otimes \varphi)$ is compact, and $\varphi \otimes \varphi \in \mathcal{W}$. □

Remark E.16. Similar techniques have also recently been used by Hagger [149] to give a characterization of some generalizations of $\mathcal{T}^\varphi$.

There are several examples of φ satisfying that $V_\varphi \varphi$ has no zeros, which by the proposition gives examples of reproducing kernel Hilbert spaces $V_\varphi(L^2)$ such that Toeplitz operators are compact if and only if their Berezin transform vanishes at infinity. One example is the one-sided exponential $\varphi(t) = \chi_{[0,\infty)}(t)e^{-t}$ for $t \in \mathbb{R}$ considered by Janssen [171], and new examples were recently explored in [139].

Essentially the same argument as for Theorem E.5.4, only replacing Θ^φ by the map $\Theta^{\mathcal{F}^2} : \mathcal{L}(\mathcal{F}^2(\mathbb{C}^d)) \to \mathcal{L}(L^2)$ defined by $\Theta^{\mathcal{F}^2}(\tilde{T}) = \mathcal{B}^*\tilde{T}\mathcal{B}$, gives a Bargmann-Fock space result from [24]. For this to work, it is important that $\varphi_0 \otimes \varphi_0 \in \mathcal{W}$, since Proposition E.3.5 and Lemma E.3.6 relate the Bargmann-Fock setting to convolutions with $\varphi_0 \otimes \varphi_0$. The definition of slowly oscillating functions on $\mathbb{R}^{2d}$ given after that theorem is adapted to $\mathbb{C}^d$ in an obvious way.

Theorem E.5.5 (Bauer, Isralowitz). *Let $\mathcal{T}^{\mathcal{F}^2}$ be the Banach algebra generated by the Toeplitz operators $T_F^{\mathcal{F}^2}$ for $F \in L^\infty(\mathbb{C}^d)$. The following are equivalent for $\tilde{T} \in \mathcal{T}^{\mathcal{F}^2}$.*

- *$\tilde{T}$ is a compact operator on $\mathcal{F}^2(\mathbb{C}^d)$.*
- *$\mathfrak{B}^{\mathcal{F}^2}\tilde{T} \in C_0(\mathbb{C}^d)$.*

If $\tilde{T} = T_F^{\mathcal{F}^2}$ for a slowly oscillating $F \in L^\infty(\mathbb{C}^d)$, then the conditions above are equivalent to $\lim_{|z|\to\infty} F(z) = 0$.

Remark E.17. The last remark on slowly oscillating functions is, to our knowledge, a new contribution, and follows from Theorem E.4.5. However, we mention that there exist other results relating the behaviour of F and $\mathfrak{B}^{\mathcal{F}^2}T_F^{\mathcal{F}^2}$ to the essential spectrum and Fredholmness of $T_F^{\mathcal{F}^2}$, also for classes of F defined in terms of the oscillation [11, 40, 118, 237]. For instance, [118, Thm. 33] implies that slow oscillation could be replaced by *vanishing oscillation* (see [118] for the definition) in the theorem above, which is weaker as functions of vanishing oscillation are bounded and uniformly continuous.

By Lemma E.3.6 we immediately obtain the following compactness criterion.

Corollary E.5.5.1. *A Toeplitz operator $T_F^{\mathcal{F}^2}$ for $F \in L^\infty(\mathbb{C}^d)$ is a compact operator on $\mathcal{F}^2(\mathbb{C}^d)$ if and only if*

$$f * |V_{\varphi_0}\varphi_0|^2 \in C_0(\mathbb{R}^{2d}),$$

where $f(x,\omega) = F(x - i\omega)$ for $x,\omega \in \mathbb{R}^d$ and $|V_{\varphi_0}\varphi_0(z)|^2 = e^{-\pi|z|^2}$.

Remark E.18. One could also define the Berezin transform for Toeplitz operators on polyanalytic Bargmann-Fock spaces and relate it to convolutions with $\varphi_n \otimes \varphi_n$. However, we would not be able to apply Proposition E.5.2 to this case, as $V_{\varphi_n}\varphi_n$ always has zeros for $n \neq 0$.

Finally, we note that Theorem E.5.2 gives a simple condition for compactness of localization operators in terms of the Gaussian φ_0.

Proposition E.5.6. *Let $f \in L^\infty(\mathbb{R}^{2d})$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$. The localization operator $\mathcal{A}_f^{\psi_1,\psi_2}$ is compact if and only if*

$$f * (V_{\varphi_0}\psi_2 \overline{V_{\varphi_0}\psi_1}) \in C_0(\mathbb{R}^{2d}).$$

Proof. Recall that $\mathcal{A}_f^{\psi_1,\psi_2} = f \star (\psi_2 \otimes \psi_1)$, so $\mathcal{A}_f^{\psi_1,\psi_2} \in \mathcal{C}_1$ by (E.5.1). Since $\varphi_0 \otimes \varphi_0 \in \mathcal{W}$ by Example E.2.2, Proposition E.5.2 gives that $f \star (\psi_2 \otimes \psi_1)$ is compact if and only if $[f \star (\psi_2 \otimes \psi_1)] \star (\varphi_0 \otimes \varphi_0) \in C_0(\mathbb{R}^{2d})$. The result therefore follows (using $\check{\varphi}_0 = \varphi_0$) by

$$\begin{aligned}[f \star (\psi_2 \otimes \psi_1)] \star (\varphi_0 \otimes \varphi_0) &= f * [(\psi_2 \otimes \psi_1) \star (\varphi_0 \otimes \varphi_0)] \quad \text{by associativity} \\ &= f * (V_{\varphi_0}\psi_2 \overline{V_{\varphi_0}\psi_1}) \quad \text{by Lemma E.2.6.}\end{aligned}$$

□

In a sense, this result complements Theorem E.4.2. Theorem E.4.2 characterized those f such that $\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1)$ is compact for all non-zero windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Proposition E.5.6 gives a characterization of compactness of $\mathcal{A}_f^{\psi_1,\psi_2}$ for a particular pair of windows ψ_1, ψ_2. Of course, when

$$\mathcal{F}_W(\psi_2 \otimes \psi_1)(x,\omega) = e^{i\pi x\cdot\omega} V_{\psi_1}\psi_2(x,\omega)$$

has no zeros, compactness of $\mathcal{A}_f^{\psi_1,\psi_2}$ implies compactness of $\mathcal{A}_f^{\varphi_1,\varphi_2}$ for all windows φ_1, φ_2 by picking $S = \psi_2 \otimes \psi_1$ and $A = 0$ in Theorem E.4.1.

E.6 Quantization schemes and Cohen's class

The perspective of [204] is that any $R \in \mathcal{L}(L^2)$ defines both a quantization scheme and a time-frequency distribution. The quantization scheme associated with R – by which we simply mean a map sending functions on phase space $\mathbb{R}^{2d}$ to operators on $L^2(\mathbb{R}^d)$ – is given by

$$f \mapsto f \star R \quad \text{for } f \in L^1(\mathbb{R}^{2d}).$$

The time-frequency distribution Q_R associated with R is given by sending $\psi \in L^2(\mathbb{R}^d)$ to its time-frequency distribution

$$Q_R(\psi)(z) = [(\psi \otimes \psi) \star \check{R}](z) \quad \text{for } z \in \mathbb{R}^{2d}.$$

Recall that a quadratic time-frequency distribution Q is said to be of Cohen's class if there is some $a \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$Q(\psi) = a * W(\psi, \psi) \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^d). \tag{E.6.1}$$

The distribution Q_R is of Cohen's class as (E.2.7) implies that

$$Q_R(\psi) = a_{\check{R}} * W(\psi, \psi), \tag{E.6.2}$$

where $a_{\check{R}}$ is the Weyl symbol of $\check{R}$. Using Theorem E.5.1, we deduce the following result relating compactness of the quantization scheme of R to $C_0(\mathbb{R}^{2d})$ membership of Q_R.

Proposition E.6.1. *Let $R \in \mathcal{L}(L^2)$. The following are equivalent.*

(i) $Q_R(\varphi) \in C_0(\mathbb{R}^{2d})$ *for some* $\varphi \in L^2(\mathbb{R}^d)$ *such that* $V_\varphi \varphi$ *has no zeros.*

(ii) $g \star R \in \mathcal{K}$ *for some* $g \in W(\mathbb{R}^{2d})$.

(iii) $Q_R(\psi) \in C_0(\mathbb{R}^{2d})$ *for all* $\psi \in L^2(\mathbb{R}^d)$.

(iv) $f \star R \in \mathcal{K}$ *for all* $f \in L^1(\mathbb{R}^{2d})$.

Proof. Since $Q_R(\psi)(z) = \check{R} \star (\psi \otimes \psi)$ and $\mathcal{F}_W(\varphi \otimes \varphi)(x, \omega) = e^{i\pi x \cdot \omega} V_\varphi \varphi(x, \omega)$, it follows from Theorem E.5.1 with $A = 0$ that $(i) \iff (iii)$ and $(ii) \iff (iv)$. A short calculation shows that $R \star (\psi \otimes \psi)(z) = Q_R(\check{\psi})(-z)$. Since $\psi \mapsto \check{\psi}$ is a bijection on $L^2(\mathbb{R}^d)$, it follows that (iii) is equivalent to

$$(iii') \; R \star (\psi \otimes \psi) \in C_0(\mathbb{R}^{2d}) \; \text{ for all } \psi \in L^2(\mathbb{R}^d).$$

By Theorem E.5.1, $(iii') \iff (iv)$, which finishes the proof. □

Remark E.19. 1. By the remark following Theorem E.5.1, the conditions on φ in (i) and g in (ii) are also necessary to imply (iii) and (iv).

2. One advantage of using the operator convolutions to describe Cohen's class is that $\psi \otimes \psi \in \mathcal{S}^1$ for any $\psi \in L^2(\mathbb{R}^d)$, so as long as R is a bounded operator we may exploit results on $\mathcal{L}(L^2) \star \mathcal{S}^1$ to study $Q_R(\psi) = \check{R} \star (\psi \otimes \psi)$. If we had used the description of Cohen's class using functions in (E.6.1), one could similarly hope that $W(\psi, \psi) \in L^1(\mathbb{R}^{2d})$, so that picking $a \in L^\infty(\mathbb{R}^{2d})$ allows us to study $Q(\psi) = a * W(\psi, \psi)$ as convolutions of bounded and integrable functions. Unfortunately, $W(\psi, \psi) \in L^1(\mathbb{R}^{2d})$ if and only if ψ belongs to a proper subspace of $L^2(\mathbb{R}^d)$ called *Feichtinger's algebra* [95]. Hence this approach fails in general.

The gist of the above proposition is that (i) provides a simple test for checking whether (iii) and (iv) hold. A typical choice for φ in (i) would be the Gaussian $\varphi = \varphi_0$, then $Q_R(\varphi_0)$ is the so-called *Husimi function* of R. Hence the quantization $f \star R$ of any $f \in L^1(\mathbb{R}^{2d})$ is compact and $Q_R(\psi) \in C_0(\mathbb{R}^{2d})$ for any $\psi \in L^2(\mathbb{R}^d)$ if and only if the Husimi function of R belongs to $C_0(\mathbb{R}^{2d})$.

τ-Wigner distributions

For $\tau \in [0, 1]$, define

$$a_\tau(x, \omega) = \begin{cases} \frac{2^d}{|2\tau-1|^d} \cdot e^{2\pi i \frac{2}{2\tau-1} x \cdot \omega} & \text{if } \tau \neq \frac{1}{2}, \\ \delta_0 & \text{if } \tau = \frac{1}{2}, \end{cases}$$

where δ_0 is Dirac's delta distribution. A slightly tedious calculation using the definition (E.2.3) shows that the Weyl transform S_τ of a_τ is given for $\psi \in \mathcal{S}(\mathbb{R}^d)$ by

$$S_\tau(\psi)(t) = \begin{cases} \frac{1}{(1-\tau)^d} \psi\left(\frac{\tau}{\tau-1} \cdot t\right) & \text{if } \tau \in (0, 1), \\ \psi(0) & \text{if } \tau = 0, \\ \int_{\mathbb{R}^d} \psi(t)\, dt \cdot \delta_0 & \text{if } \tau = 1, \end{cases}$$

as already noted for $d = 1$ in [186, Thm. 7.2]. If $\tau \in (0, 1)$, it is easy to check that S_τ is bounded on $L^2(\mathbb{R}^d)$ with $\|S_\tau\|_{\mathcal{L}(L^2)} = \frac{1}{(1-\tau)^{d/2}\tau^{d/2}}$, that $S_\tau^* = S_{1-\tau}$, $\check{S}_\tau = S_\tau$ and the inverse of S_τ is $\tau^d(1-\tau)^d S_{1-\tau}$. In particular, S_τ is not compact.

In light of (E.6.2), [49, Prop. 5.6] states that $Q_{S_\tau}(\psi)$ is the τ-Wigner distribution $W_\tau(\psi)$ introduced in [49], given explicitly by

$$Q_{S_\tau}(\psi)(z) = W_\tau(\psi)(z) := \int_{\mathbb{R}^d} e^{-2\pi i t \cdot \omega} \psi(x + \tau t) \overline{\psi(x - (1-\tau)t)} dt.$$

On the other hand, we easily find for $f \in L^1(\mathbb{R}^{2d})$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\begin{aligned}\langle (f \star S_{1-\tau})\psi, \psi\rangle_{L^2} &= [(f \star S_{1-\tau}) \star (\psi \otimes \psi)]\,(0)\\ &= [f * (S_{1-\tau} \star (\psi \otimes \psi))]\,(0)\\ &= \int_{\mathbb{R}^{2d}} f(z) S_{1-\tau} \star (\psi \otimes \psi)(z)\, dz\\ &= \int_{\mathbb{R}^{2d}} f(z) W_{1-\tau}(\psi)(z)\, dz\\ &= \langle f, W_\tau(\psi)\rangle_{L^2(\mathbb{R}^{2d})}\,.\end{aligned}$$

In the last line we use that $\overline{Q_S(\psi)} = Q_{S^*}(\psi)$ for $S \in \mathcal{L}(L^2)$, and $S_\tau^* = S_{1-\tau}$. This shows precisely that $f \star S_{1-\tau}$ satisfies the definition of the τ-Weyl quantization of f introduced by Shubin [231] – in the notation of [49] we have that

$$f \star S_{1-\tau} = W_\tau^f\,.$$

The case $\tau = 1/2$ is of particular interest, as $S_{1/2} = S_{1-1/2} = 2^d P$ – a scalar multiple of the parity operator. This case corresponds to the Weyl calculus, in the sense that $Q_{2^d P}(\psi) = W(\psi, \psi)$ for $\psi \in L^2(\mathbb{R}^d)$ and $f \star (2^d P)$ is the Weyl transform of f for $f \in L^1(\mathbb{R}^{2d})$.

We can now show that the τ-Wigner theory and the non-compact operators S_τ give a family of non-trivial examples to Theorem E.5.1. The compactness part of the next result was also noted using different methods in [49, Thm. 6.9].

Proposition E.6.2. *Let $\tau \in (0,1)$. Then S_τ satisfies condition (i) of Proposition E.6.1, hence*

1. *$W_\tau(\psi) = Q_{S_\tau}(\psi) \in C_0(\mathbb{R}^{2d})$ for any $\psi \in L^2(\mathbb{R}^d)$.*
2. *$W_\tau^f = f \star S_{1-\tau}$ is a compact operator on $L^2(\mathbb{R}^d)$ for any $f \in L^1(\mathbb{R}^{2d})$.*

Proof. Recall from Example E.2.2 that $V_{\varphi_0}\varphi_0$ has no zeros. By [49, Prop. 4.4],

$$Q_{S_\tau}(\varphi_0) = W_\tau(\varphi_0) \in C_0(\mathbb{R}^{2d})$$

for any $\tau \in [0,1]$. Hence (i) in Proposition E.6.1 is satisfied, and the result follows by (iii) and (iv) of the same proposition. □

In fact, the same proof shows that the functions $a_\tau \in L^\infty(\mathbb{R}^{2d})$ for $\tau \neq 1/2$ are non-trivial examples of Theorem E.4.1, where non-trivial refers to the fact $a_\tau \notin L^p(\mathbb{R}^{2d})$ for $p = 0$ or $1 \leq p < \infty$.

Proposition E.6.3. *For $\tau \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, a_τ satisfies the assumptions of Theorem E.4.1 with $A = 0$.*

Proof. Recall that $W_\tau(\varphi_0) = S_\tau \star (\varphi_0 \otimes \varphi_0) = a_\tau * W(\varphi_0, \varphi_0)$ by (E.6.2). As a special case of (E.2.11) one gets that $\mathcal{F}_\sigma(W(\varphi_0, \varphi_0)) = \mathcal{F}_W(\varphi_0 \otimes \varphi_0)$, hence $W(\varphi_0, \varphi_0) \in W(\mathbb{R}^{2d})$ by Example E.2.2. The previous proof showed that $W_\tau(\varphi_0) \in C_0(\mathbb{R}^{2d})$, so f satisfies assumption (ii) of Theorem E.4.1. □

Remark E.20. The operators S_0 and S_1 are clearly not bounded on $L^2(\mathbb{R}^d)$, even though $a_0, a_1 \in L^\infty(\mathbb{R}^{2d})$. Hence a_0 and a_1 are examples of bounded functions with unbounded Weyl transform. Similarly, $S_{1/2}$ is a bounded operator with unbounded Weyl symbol.

We end by considering the example of Born-Jordan quantization.

Example E.6.1 (Born-Jordan distribution)**.** The *Born-Jordan distribution* $Q_{BJ}(\psi)$ of $\psi \in L^2(\mathbb{R}^d)$ is given by

$$Q_{BJ}(\psi)(z) = \int_0^1 W_\tau(\psi)(z)\, d\tau = \int_0^1 Q_{S_\tau}(\psi)(z)\, d\tau,$$

see [49,73]. It is well-known that Q_{BJ} is of Cohen's class, and from [49, Prop. 5.8] it follows that $Q_{BJ} = Q_{S_{BJ}}$ where $S_{BJ} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ is defined by

$$\mathcal{F}_W(S_{BJ})(x, \omega) = \operatorname{sinc}(\pi x \cdot \omega). \tag{E.6.3}$$

The associated quantization scheme $f \mapsto f \star S_{BJ}$ is then the *Born-Jordan quantization* [73].

For $d = 1$ it was shown in [186, Prop. 2] that $S_{BJ} \in \mathcal{L}(L^2)$. Since (E.6.3) shows that $\mathcal{F}_W(S_{BJ}) \in L^\infty(\mathbb{R}^{2d})$, combining Example E.5.1 and Proposition E.6.1 we may conclude that the Born-Jordan quantization of any $f \in L^1(\mathbb{R}^2)$ is compact, and that the Born-Jordan distribution of any $\psi \in L^2(\mathbb{R})$ belongs to $C_0(\mathbb{R}^2)$.

E.6.1 Counterexample to Schatten class results

For the special case $A = 0$, Theorem E.5.1 states that if $R \star a \in \mathcal{K}$ for some $a \in W(\mathbb{R}^{2d})$, then $R \star g \in \mathcal{K}$ for all $g \in L^1(\mathbb{R}^{2d})$. An obvious generalization is to replace $\mathcal{K}$ by a Schatten class $\mathcal{S}^p$ for some $1 \le p < \infty$. Is it true that $R \star a \in \mathcal{S}^p$ for $a \in W(\mathbb{R}^{2d})$ implies that $R \star g \in \mathcal{S}^p$ for all $g \in L^1(\mathbb{R}^{2d})$? A simple counterexample is provided by the Weyl calculus.

Example E.6.2. Recall that $f \star S_{1/2}$ is the Weyl transform of $f \in L^1(\mathbb{R}^{2d})$. If we let $a(z) = 2^d e^{-\pi|z|^2}$, then $a \in W(\mathbb{R}^{2d})$ and it is well-known that the Weyl transform $a \star S_{1/2}$ of a is the rank-one operator $\varphi_0 \otimes \varphi_0$. In particular, $a \star S_{1/2} \in \mathcal{S}^1 \subset \mathcal{S}^p$ for any $1 \le p \le \infty$. However, if we pick $f \in L^1(\mathbb{R}^{2d}) \setminus L^2(\mathbb{R}^{2d})$, then $f \star S_{1/2} \notin \mathcal{S}^p$ for any $1 \le p \le 2$, since the Weyl transform is a unitary mapping from $L^2(\mathbb{R}^{2d})$ to $\mathcal{S}^2$, and $\mathcal{S}^p \subset \mathcal{S}^2$ for $1 \le p \le 2$. Hence we cannot conclude from $a \star S_{1/2} \in \mathcal{S}^p$ for $a \in W(\mathbb{R}^{2d})$ that $f \star S_{1/2} \in \mathcal{S}^p$ for all $f \in L^1(\mathbb{R}^{2d})$, at least for $1 \le p \le 2$.

Acknowledgements

We wish to thank Alexander Bufetov for asking one of the authors about the existence of Tauberian theorems in quantum harmonic analysis, which prompted the writing of this paper. We also thank Hans Feichtinger for pointing out the result from reference [53] mentioned in Example E.4.1, and Eirik Berge for helpful discussion on Gabor spaces.

Paper F

Equivalent Norms for Modulation Spaces from Positive Cohen's Class Distributions

Eirik Skrettingland

Preprint

Paper F

Equivalent Norms for Modulation Spaces from Positive Cohen's Class Distributions

Abstract

We give a new class of equivalent norms for modulation spaces by replacing the window of the short-time Fourier transform by a Hilbert-Schmidt operator. The main result is applied to Cohen's class of time-frequency distributions, Weyl operators and localization operators. In particular, any positive Cohen's class distribution with Schwartz kernel can be used to give an equivalent norm for modulation spaces. We also obtain a description of modulation spaces as time-frequency Wiener amalgam spaces. The Hilbert-Schmidt operator must satisfy a nuclearity condition for these results to hold, and we investigate this condition in detail.

F.1 Introduction

The modulation spaces introduced by Hans Feichtinger [96] have long been recognized as suitable function spaces for various problems in time-frequency analysis [101, 131], PDEs [35, 249], pseudodifferential operators [34, 63, 132, 243] and others areas – comprehensive lists of references can be found in [98] and the recent monograph [36]. Perhaps the most common definition of the modulation spaces nowadays uses the language of *time-frequency analysis*. To motivate the definition, we consider a function ψ on $\mathbb{R}^d$ and its Fourier transform

$$\hat{\psi}(\omega) = \int_{\mathbb{R}^d} \psi(t) e^{-2\pi i \omega \cdot t}\, dt \qquad \text{for } \omega \in \mathbb{R}^d.$$

Together, ψ and $\hat{\psi}$ describe the behaviour of ψ as a function of time and frequency, respectively, and give us different approaches to study properties of ψ. For instance, smoothness of ψ is related to decay of $\hat{\psi}$. But although $\hat{\psi}$ shows which frequencies ω contribute to ψ – those such that $|\hat{\psi}(\omega)|$ is large – it does not indicate *when*, i.e. for which $t \in \mathbb{R}^d$, the frequency contributes to ψ. In time-frequency analysis one therefore looks for *time-frequency distributions* $Q(\psi)$, which should be a function on $\mathbb{R}^{2d}$ such that the size of $Q(\psi)(x,\omega)$ describes the contribution of frequency ω at time x in ψ.

The existence of an ideal time-frequency distribution Q is prohibited by various uncertainty principles, but a common choice in time-frequency analysis is the *short-time Fourier transform* (STFT)

$$V_\varphi \psi(z) = \langle \psi, \pi(z)\varphi \rangle_{L^2} \text{ for } z \in \mathbb{R}^{2d},$$

where the *window* φ is a function on $\mathbb{R}^d$ well-localized in time and frequency, and $\pi(z)$ denotes the *time-frequency shift* for $z = (x, \omega)$ given by

$$\pi(z)\varphi(t) = e^{2\pi i \omega \cdot t} \varphi(t - x).$$

The modulation spaces $M^{p,q}_m(\mathbb{R}^d)$ are then defined, for $1 \leq p, q \leq \infty$ and a weight function m on $\mathbb{R}^{2d}$, by the norm

$$\|\psi\|_{M^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_{\varphi_0}\psi(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{\frac{1}{q}}, \tag{F.1.1}$$

where $\varphi_0(t) = 2^{d/4} e^{-\pi |t|^2}$ and the integrals are replaced by supremums for $p, q = \infty$. By our interpretation of $V_{\varphi_0}\psi(x,\omega)$ as a time-frequency distribution, we see that $\|\psi\|_{M^{p,q}_m}$ measures how localized ψ is in the time-frequency plane. More precisely, L^p measures the decay of ψ in time, and L^q the decay of ψ in frequency – i.e. the decay of $\hat{\psi}$, or the smoothness of ψ. The fact that $\|\psi\|_{M^{p,q}_m}$ is finite is therefore a statement on the decay and smoothness of ψ.

A useful result on modulation spaces from [96] is that replacing the window φ_0 in (F.1.1) by another window φ with good time-frequency localization, we obtain an equivalent norm on $M^{p,q}_m(\mathbb{R}^d)$:

$$\|\psi\|_{M^{p,q}_m} \asymp \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_{\varphi}\psi(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{\frac{1}{q}}. \tag{F.1.2}$$

The main result of this contribution is an extension of this fact: we show that the window can even be replaced by a Hilbert-Schmidt operator S on $L^2(\mathbb{R}^d)$. To explain this transition from function-windows to operator-windows, we fix an arbitrary $\xi \in L^2(\mathbb{R}^d)$ with $\|\xi\|_{L^2} = 1$ and consider the rank-one operator $S = \xi \otimes \varphi$ defined by

$$S(\psi) = \xi \otimes \varphi(\psi) = \langle \psi, \varphi \rangle_{L^2} \xi. \tag{F.1.3}$$

It is easy to see that $\|S\pi(z)^*\psi\|_{L^2} = |V_\varphi \psi(z)|$, hence we may reformulate (F.1.2) as

$$\|\psi\|_{M^{p,q}_m} \asymp \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \|S\pi(z)^*\psi\|^p_{L^2} m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{\frac{1}{q}}. \tag{F.1.4}$$

Our main result in Theorem F.5.1 states that this holds not only for rank-one S as in (F.1.3), but for all Hilbert-Schmidt operators S having good time-frequency localization – a statement that itself will need elaboration. By choosing different S we will see that we obtain equivalent norms for the modulation spaces that express quite different properties from those expressed in (F.1.1), hence giving new insights into the structure of modulation spaces.

Comparing (F.1.1) and (F.1.4), we see that the STFT $|V_\varphi \psi(z)|$ is replaced by $\|S\pi(z)^*\psi\|_{L^2}$. This suggests that we replace the STFT by the function $\mathfrak{V}_S : \mathbb{R}^{2d} \to L^2(\mathbb{R}^d)$ given by

$$\mathfrak{V}_S(\psi)(z) = S\pi(z)^*\psi.$$

In Section F.4 we show that $\mathfrak{V}_S$ actually behaves like the usual STFT V_φ, by showing that it satisfies an isometry property and an inversion formula. This insight allows us to prove (F.1.4) in Section F.5 using methods similar to those used to prove that the modulation spaces are independent of the window function in [131].

Sections F.6, F.7 and F.8 are then devoted to examples and reinterpretations of the main result. First we consider Weyl operators in Section F.6. The reformulation of (F.1.4) in Theorem F.6.1 generalizes a result by Gröchenig and Toft [143] that identifies certain modulation spaces with function spaces introduced by Bony and Chemin [51].

In Section 7 we turn our attention to Cohen's class of time-frequency distributions. As there is no ideal time-frequency distribution, Cohen's class was introduced by Cohen in [59] as the time-frequency distributions Q_a given by

$$Q_a(\psi)(z) = a * W(\psi) \quad \text{for } z \in \mathbb{R}^{2d},$$

where a is some function (or distribution) on $\mathbb{R}^{2d}$ and $W(\psi)$ is the Wigner-distribution, see (F.6.1) for its definition. By varying a one obtains time-frequency distributions with different properties. An important example of a Cohen's class distribution is the *spectrogram* $Q(\psi)(z) = |V_{\varphi_0}\psi(z)|^2$. Then (F.1.1) shows that the modulation space norm of ψ is given by the $L^{p,q}_m$-norm of (the square root of) $Q(\psi)$. We might therefore ask whether this is true if we replace the spectrogram by another Cohen's class distributions Q_a. Using a description of Cohen's class in terms of bounded operators given in [204] together with (F.1.4), we are able to give in Theorem F.7.1 a set of Cohen's class distributions whose $L^{p,q}_m$ norms define the modulation space norms. The question of characterizing these Cohen class distributions Q_a in terms of a seems to be a difficult problem in general. However, using a result from [179] we are able to prove the following in Theorem F.7.2:

> Let $1 \leq p, q \leq \infty$ and assume that the weight m grows at most polynomially. If a is a Schwartz function on $\mathbb{R}^{2d}$ and $Q_a(\psi)$ is a positive function for each $\psi \in L^2(\mathbb{R}^d)$, then
>
> $$\|\psi\|_{M^{p,q}_m} \asymp \left\| \sqrt{Q_a(\psi)} \right\|_{L^{p,q}_m(\mathbb{R}^{2d})}.$$

Finally, we let S in (F.1.4) be a *localization operator* in Section 8. This leads to a characterization of modulation spaces as time-frequency Wiener amalgam spaces in Theorem F.8.1, which is a continuous version of results by Dörfler, Feichtinger and Gröchenig [85, 87], see also [88, 225], much like the fact that the standard Wiener amalgam spaces have both a continuous and discrete description. We mention that [1, 50, 143, 144] also use localization operators to get equivalent norms for modulation spaces, but their approach and results are different from those we consider.

Before ending this introduction, we wish to point out that sufficient conditions on S for (F.1.4) to hold will be a recurring theme throughout the paper. The most general sufficient condition on S is that its Hilbert space adjoint must be a nuclear operator from $L^2(\mathbb{R}^d)$ to $M^1_v(\mathbb{R}^d)$. In some ways this a very natural condition: if applied to the rank-one operator in (F.1.3) it means that $\varphi \in M^1_v(\mathbb{R}^d)$, which is the standard condition for windows for modulation spaces. As we see in Section F.3, this nuclearity condition is also easy to handle when working with localization operators. From other perspectives, such as the Weyl calculus, the condition is more mysterious, and we will therefore also study stronger sufficient conditions on S for (F.1.4) to hold.

Notation and conventions

If X is a Banach space, we denote by X' its dual space and the action of $y \in X'$ on $x \in X$ is denoted by the bracket $\langle y, x \rangle_{X',X}$, where the bracket is antilinear in the second coordinate to be compatible with the notation for inner products in Hilbert spaces. This means that we are identifying the dual space X' with *anti*linear functionals on X. For two Banach spaces X, Y we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators $S : X \to Y$, and if $X = Y$ we simply write $\mathcal{L}(X)$. For brevity we often write $\mathcal{L}(L^2)$ for $\mathcal{L}(L^2(\mathbb{R}^d))$. For topological spaces X, Y we write $X \hookrightarrow Y$ to denote that there is a continuous inclusion of X into Y.

For $p \in [1,\infty]$, p' denotes the conjugate exponent, i.e. $\frac{1}{p}+\frac{1}{p'}=1$. The notation $P \lesssim Q$ means that there is some $C>0$ such that $P \leq C \cdot Q$, and $P \asymp Q$ means that $Q \lesssim P$ and $P \lesssim Q$. For $\Omega \subset \mathbb{R}^{2d}$, χ_Ω is the characteristic function of Ω. $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz space, and $\mathscr{S}'(\mathbb{R}^d)$ its dual space of tempered distributions.

F.2 Time-frequency analysis

As we have seen in the introduction, our main results are phrased in terms of the *time-frequency shifts* $\pi(z) \in \mathcal{L}(L^2)$ for $z=(x,\omega) \in \mathbb{R}^{2d}$, defined by

$$\pi(z)\psi(t) = e^{2\pi i \omega \cdot t}\psi(t-x) \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

The time-frequency shifts are unitary on $L^2(\mathbb{R}^d)$, and they satisfy

$$\pi(x,\omega)\pi(x',\omega') = e^{-2\pi i \omega' \cdot x}\pi(x+x',\omega+\omega') \tag{F.2.1}$$

$$\pi(x,\omega)^* = e^{-2\pi i x\cdot\omega}\pi(-x,-\omega) \tag{F.2.2}$$

for $x, x', \omega, \omega' \in \mathbb{R}^d$. Closely related to the time-frequency shifts is the short-time Fourier transform (STFT) $V_\varphi \psi \in L^2(\mathbb{R}^{2d})$, given by

$$V_\varphi\psi(z) = \langle \psi, \pi(z)\varphi\rangle_{L^2} \quad \text{for } \psi, \phi \in L^2(\mathbb{R}^d), z \in \mathbb{R}^{2d}. \tag{F.2.3}$$

The function φ is often referred to as the *window* of the STFT $V_\varphi\psi$. An important property of the STFT is *Moyal's identity* [131, Thm. 3.2.1].

Lemma F.2.1 (Moyal's identity)**.** *Given $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$, then $V_{\phi_i}\psi_j \in L^2(\mathbb{R}^{2d})$ for $i, j \in \{1,2\}$ and*

$$\int_{\mathbb{R}^{2d}} V_{\phi_1}\psi(z)\overline{V_{\phi_2}\psi(z)}\, dz = \langle \psi_1, \psi_2\rangle_{L^2} \overline{\langle \phi_1, \phi_2\rangle}_{L^2}.$$

In particular, we see that for fixed window φ with $\|\varphi\|_2 = 1$ the map $\psi \mapsto V_\varphi\psi$ is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.

F.2.1 Admissible weight functions and weighted, mixed L^p spaces

A *submultiplicative* weight function v on $\mathbb{R}^{2d}$ is a non-negative function $v : \mathbb{R}^{2d} \to \mathbb{R}$ such that $v(z_1+z_2) \leq v(z_1)v(z_2)$ for $z_1, z_2 \in \mathbb{R}^{2d}$. Whenever we refer to a submultiplicative weight function v we will assume that v is continuous and satisfies $v(x,\omega) = v(-x,\omega) = v(x,-\omega) = v(-x,-\omega)$; these assumptions do not lead to a loss of generality as any submultiplicative weight function is equivalent in a natural sense to a weight satisfying these assumptions, see [131, 135]. Furthermore, these assumptions imply that if v is not identically 0, then $v(z) \geq 1$ for all $z \in \mathbb{R}^{2d}$. The assumptions above are satisfied by standard examples such as the polynomial weights

$$v_s(z) = (1 + |z|^2)^{s/2} \quad s \geq 0,$$

but also by the exponential weights $v_a(z) = e^{a|z|}$ for $a \geq 0$. A non-negative weight function m on $\mathbb{R}^{2d}$ is said to be v-*moderate* if v is a submultiplicative weight function and there exists some constant $C_v^m > 0$ such that

$$m(z_1 + z_2) \leq C_v^m v(z_1) m(z_2).$$

We refer the reader to the survey [135] for more examples and motivation for these assumptions. For any v-moderate weight m and $1 \leq p, q \leq \infty$ we may define the Banach space $L_m^{p,q}(\mathbb{R}^{2d})$ to be the equivalence classes of Lebesgue measurable functions $F : \mathbb{R}^{2d} \to \mathbb{C}$ such that

$$\|F\|_{L_m^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{\frac{1}{q}} < \infty.$$

If $p < \infty$ or $q < \infty$, the corresponding integral is replaced by an essential supremum.

F.2.2 Modulation spaces

Throughout the rest of the paper, we will let $\varphi_0 \in L^2(\mathbb{R}^d)$ denote the normalized Gaussian, i.e.

$$\varphi_0(t) = 2^{d/4} e^{-\pi |t|^2} \quad \text{for } t \in \mathbb{R}^d.$$

For a submultiplicative weight v, we define the space $M_v^1(\mathbb{R}^d)$ to be the Banach space of those $\psi \in L^2(\mathbb{R}^d)$ such that

$$\|\psi\|_{M_v^1} := \int_{\mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)| v(z) \, dz < \infty.$$

This will serve as our space of test functions. It is always non-empty as it contains φ_0, and for weights v of polynomial growth it contains the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ [131, Prop. 11.3.4]. For more general weights $M_v^1(\mathbb{R}^d)$ will not necessarily contain $\mathcal{S}(\mathbb{R}^d)$ and might be quite small. The time-frequency shifts $\pi(z)$ are bounded on $M_v^1(\mathbb{R}^d)$ [131, Thm. 11.3.5] with

$$\|\pi(z)\psi\|_{M_v^1} \leq v(z) \|\psi\|_{M_v^1}, \tag{F.2.4}$$

and hence the STFT $V_\phi \psi(z)$ for $\phi \in M_v^1(\mathbb{R}^d)$ and $\psi \in (M_v^1(\mathbb{R}^d))'$ can be defined by modifying the inner product in the definition (F.2.3) to a duality bracket: $V_\phi \psi(z) = \langle \psi, \pi(z)\phi \rangle_{(M_v^1)', M_v^1}$.

For any v-moderate weight m and $1 \leq p, q \leq \infty$, we then define the modulation space $M_m^{p,q}(\mathbb{R}^d)$ to consist of those $\psi \in (M_v^1(\mathbb{R}^d))'$ such that

$$\|\psi\|_{M_m^{p,q}} := \|V_{\varphi_0}\psi\|_{L_m^{p,q}} < \infty.$$

When $p = q$ we will write $M_m^p(\mathbb{R}^d)$ for $M_m^{p,p}(\mathbb{R}^d)$, and when $m \equiv 1$ we write $M^{p,q}(\mathbb{R}^d)$. Some properties of the modulation spaces are summarized below, proofs may be found in the monograph [131].

Proposition F.2.2. *Let m be a v-moderate weight and $1 \leq p, q \leq \infty$.*

(a) $M_m^{p,q}(\mathbb{R}^d)$ *is a Banach space with the norm* $\|\cdot\|_{M_m^{p,q}}$.

(b) If $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ *and* $m_2 \lesssim m_1$, *then* $M_{m_1}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{m_2}^{p_2,q_2}(\mathbb{R}^d)$.

(c) If $p, q < \infty$, *then* $M_{1/m}^{p',q'}(\mathbb{R}^d)$ *is the dual space of* $M_m^{p,q}(\mathbb{R}^d)$ *with*

$$\langle \phi, \psi \rangle_{M_{1/m}^{p',q'}, M_m^{p,q}} = \int_{\mathbb{R}^{2d}} V_{\varphi_0}\psi(z)\overline{V_{\varphi_0}\phi(z)}\, dz.$$

(d) $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$.

Remark F.1. (a) As a particular case of part c), we may identify $(M_v^1(\mathbb{R}^d))'$ with $M_{1/v}^{\infty}(\mathbb{R}^d)$, which we will do for the rest of the paper. The reader should also note that the duality extends the inner product on $L^2(\mathbb{R}^d)$, since if $\psi \in L^2(\mathbb{R}^d) \cap M_{1/v}^{\infty}(\mathbb{R}^d)$ and $\phi \in M_v^1(\mathbb{R}^d)$, we find by Moyal's identity that

$$\langle \psi, \phi \rangle_{M_{1/v}^{\infty}, M_v^1} = \int_{\mathbb{R}^{2d}} V_{\varphi_0}\phi(z)\overline{V_{\varphi_0}\psi(z)}\, dz = \langle \psi, \phi \rangle_{L^2}.$$

(b) A simple calculation using our assumption that $v(-z) = v(z)$ gives that if m is v-moderate, then so is $1/m$.

(c) As mentioned, $\mathcal{S}(\mathbb{R}^d)$ embeds continuously into $M_v^1(\mathbb{R}^d)$ when v grows polynomially, so in this case we may identify $M_{1/v}^{\infty}(\mathbb{R}^d)$ with a subspace of the tempered distributions. This is not true for more general weights, hence we need to work with the abstract space $M_{1/v}^{\infty}(\mathbb{R}^d)$ defined as the dual space of our test functions $M_v^1(\mathbb{R}^d)$.

The property of modulation spaces that is our main focus is the fact that changing the window for the STFT leads to an equivalent norm.

Theorem F.2.3. *Let m be a v-moderate weight function and let $0 \neq \phi \in M^1_v(\mathbb{R}^d)$. Then $\|V_\phi\psi\|_{L^{p,q}_m}$ defines an equivalent norm on $M^{p,q}_m(\mathbb{R}^d)$: for $\psi \in M^{p,q}_m(\mathbb{R}^d)$ we have*

$$\|V_\phi\psi\|_{L^{p,q}_m} \asymp \|\psi\|_{M^{p,q}_m}.$$

Our main result is that we also obtain equivalent norms for $M^{p,q}_m(\mathbb{R}^d)$ when ϕ is replaced by an operator S satisfying certain conditions, after modifying the definition of the STFT correspondingly. To prove this, we will use the precise statement of the upper bound $\|V_\phi\psi\|_{L^{p,q}_m} \lesssim \|\psi\|_{M^{p,q}_m}$; it follows from equation (11.33) in [131]. Recall that C^m_v is the constant from $m(z_1+z_2) \leq C^m_v v(z_1)m(z_2)$.

Proposition F.2.4. *Let m be a v-moderate weight function and let $\phi \in M^1_v(\mathbb{R}^d)$. The map $\psi \mapsto V_\phi\psi$ is bounded from $M^{p,q}_m(\mathbb{R}^d)$ to $L^{p,q}_m(\mathbb{R}^{2d})$ with $\|V_\phi\psi\|_{L^{p,q}_m} \leq C^m_v\|\phi\|_{M^1_v}\|\psi\|_{M^{p,q}_m}$.*

F.3 Classes of operators for time-frequency analysis

Our main result rests upon properties of certain classes of operators, all of which may be described as integral operators.

F.3.1 Hilbert-Schmidt operators

Given a function $k \in L^2(\mathbb{R}^{2d})$, we define the bounded integral operator $T_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$T_k(\psi)(x) = \int_{\mathbb{R}^d} k(x,y)\psi(y)\, dy \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

We call k the integral kernel of the operator T_k. When equipped with the inner product

$$\left\langle T_{k_1}, T_{k_2}\right\rangle_{\mathcal{HS}} := \langle k_1, k_2\rangle_{L^2},$$

the set of integral operators T_k with integral kernels $k \in L^2(\mathbb{R}^{2d})$ forms a Hilbert space of compact operators called the *Hilbert-Schmidt operators*, which we will denote by $\mathcal{HS}$. Given $T \in \mathcal{HS}$, we will sometimes denote its integral kernel by k_T, which means that $T = T_{k_T}$. An important subspace of $\mathcal{HS}$ is the space $\mathcal{S}^1$ of *trace class operators*, consisting of those $T \in \mathcal{HS}$ such that

$$\sum_{n=1}^{\infty} \langle |T|e_n, e_n\rangle_{L^2} < \infty,$$

where $\{e_n\}_{n=1}^{\infty}$ is any orthonormal basis of $L^2(\mathbb{R}^d)$ and $|T|$ is the positive part in the polar decomposition of T. If T is a trace class operator, we may therefore define its *trace* $\mathrm{tr}(T)$ by

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle_{L^2},$$

which can be shown to be independent of the orthonormal basis. For our part, we will need that if $S, T \in \mathcal{HS}$, then ST is a trace class operator. In particular, this allows us to express the inner product on $\mathcal{HS}$ without reference to their kernels as integral operators, as one may show (see [72, Thm. 269]) that

$$\langle S, T \rangle_{\mathcal{HS}} = \mathrm{tr}(ST^*).$$

F.3.2 A space of nuclear operators

Both Hilbert-Schmidt and trace class operators will often be too large spaces for our purposes. We therefore introduce a Banach subspace of $\mathcal{HS}$ more adapted to the needs of time-frequency analysis. Let v be a submultiplicative weight function. The space we will need is the space $\mathcal{N}(L^2; M^1_v)$ consisting of all *nuclear* operators $T : L^2(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)$. An operator $T : L^2(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)$ is said to be nuclear [228] if it has an expansion of the form

$$T = \sum_{n=1}^{\infty} \phi_n \otimes \xi_n, \tag{F.3.1}$$

where $\phi \otimes \psi$ denotes the rank-one operator

$$\phi \otimes \psi(\xi) = \langle \xi, \psi \rangle_{L^2} \phi$$

and $\sum_{n=1}^{\infty} \|\phi_n\|_{M^1_v} \|\xi_n\|_{L^2} < \infty$. The space $\mathcal{N}(L^2; M^1_v)$ becomes a Banach space with norm given by

$$\|T\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|\phi_n\|_{M^1_v} \|\xi_n\|_{L^2} \right\}, \tag{F.3.2}$$

where the infimum is taken over all decompositions as in (F.3.1). It can be shown that if $\phi \in M^1_v(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)$, then

$$\|\phi \otimes \psi\|_{\mathcal{N}} = \|\phi\|_{M^1_v} \|\psi\|_{L^2}, \tag{F.3.3}$$

hence the expansion in (F.3.1) converges absolutely in $\mathcal{N}(L^2, M^1_v)$. Using the expansion in (F.3.1) it is straightforward to check that the inclusion of $\mathcal{N}(L^2, M^1_v)$ into $\mathcal{L}(L^2; M^1_v)$ is continuous, i.e.

$$\|T\|_{\mathcal{L}(L^2; M^1_v)} \leq \|T\|_{\mathcal{N}}, \tag{F.3.4}$$

and that if $S \in \mathcal{N}(L^2, M^1_v)$, $T \in \mathcal{L}(L^2)$ and $R \in \mathcal{L}(M^1_v)$, then $RST \in \mathcal{N}(L^2, M^1_v)$.

We will need the following simple property.

Lemma F.3.1. *Let $T \in \mathcal{N}(L^2, M^1_v)$ for a submultiplicative weight function v and let $z \in \mathbb{R}^{2d}$. Then $\pi(z)T\pi(z)^* \in \mathcal{N}(L^2, M^1_v)$ with $\|\pi(z)T\pi(z)^*\|_{\mathcal{N}} \leq v(z)\|T\|_{\mathcal{N}}$.*

Proof. If T has an expansion

$$T = \sum_{n=1}^{\infty} \phi_n \otimes \xi_n$$

where $\sum_{n=1}^{\infty} \|\phi_n\|_{M^1_v} \|\xi_n\|_{L^2} < \infty$, then

$$\pi(z)T\pi(z)^* = \sum_{n=1}^{\infty} \pi(z)\phi_n \otimes \pi(z)\xi_n,$$

and

$$\sum_{n=1}^{\infty} \|\pi(z)\phi_n\|_{M^1_v} \|\pi(z)\xi_n\|_{L^2} \leq v(z) \sum_{n=1}^{\infty} \|\phi_n\|_{M^1_v} \|\xi_n\|_{L^2}$$

by (F.2.4) and the fact that $\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$. The norm inequality then follows from the definition (F.3.2) of the nuclear norm. □

To be more precise, the class of operators we will be interested in are those $S \in \mathcal{L}(L^2)$ such that $S^* \in \mathcal{N}(L^2, M^1_v)$, where S^* is the Hilbert space adjoint of S. We can give a much more concrete description of this condition by noting that if $\sum_{n=1}^{\infty} \|\xi_n\|_{L^2} \|\phi_n\|_{M^1_v} < \infty$, then $S^* = \sum_{n=1}^{\infty} \xi_n \otimes \phi_n$ if and only if $S = \sum_{n=1}^{\infty} \phi_n \otimes \xi_n$. Hence (F.3.1) gives that $S^* \in \mathcal{N}(L^2, M^1_v)$ if and only if

$$S = \sum_{n=1}^{\infty} \xi_n \otimes \phi_n \tag{F.3.5}$$

with $\sum_{n=1}^{\infty} \|\xi_n\|_{L^2} \|\phi_n\|_{M^1_v} < \infty$. Abusing notation slightly, we will write $S^* \in \mathcal{N}(L^2, M^1_v)$ to denote that $S \in \mathcal{L}(L^2)$ and $S^* \in \mathcal{N}(L^2, M^1_v)$.

Lemma F.3.2. *Let $S^* \in \mathcal{N}(L^2, M^1_v)$ for a submultiplicative weight function v. Then S extends to a bounded operator $\tilde{S} : M^{\infty}_{1/v}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|\tilde{S}\|_{\mathcal{L}(M^{\infty}_{1/v}, L^2)} \leq \|S^*\|_{\mathcal{N}(L^2, M^1_v)}$ by defining*

$$\left\langle \tilde{S}\psi, \phi \right\rangle_{L^2} := \langle \psi, S^*\phi \rangle_{M^{\infty}_{1/v}, M^1_v} \quad \text{for } \psi \in M^{\infty}_{1/v}(\mathbb{R}^d), \phi \in L^2(\mathbb{R}^d). \tag{F.3.6}$$

Furthermore, given an expansion of S of the form (F.3.5)*, this extension satisfies*

$$\tilde{S}(\psi) = \sum_{n=1}^{\infty} \langle \psi, \phi_n \rangle_{M^{\infty}_{1/v}, M^1_v} \xi_n, \tag{F.3.7}$$

where the sum converges absolutely in $L^2(\mathbb{R}^d)$.

Proof. The definition (F.3.6) simply means that $\tilde{S}$ is the Banach space adjoint of $S^* : L^2(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)$, hence $\tilde{S}$ is well-defined. Since for $\psi \in L^2(\mathbb{R}^d)$ we have

$$\langle S\psi, \phi\rangle_{L^2} = \langle \psi, S^*\phi\rangle_{L^2} = \langle \psi, S^*\phi\rangle_{M^\infty_{1/v}, M^1_v},$$

we see that $\tilde{S}$ extends S. The absolute convergence of the sum in (F.3.7) follows directly from (F.3.5). To show that the decomposition into rank-one operators still holds for $\tilde{S}$, we need to show that for $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$ and $\phi \in L^2(\mathbb{R}^d)$ we have

$$\left\langle \sum_{n=1}^\infty \langle \psi, \phi_n\rangle_{M^\infty_{1/v}, M^1_v} \xi_n, \phi \right\rangle_{L^2} = \langle \psi, S^*\phi\rangle_{M^\infty_{1/v}, M^1_v},$$

which is a straightforward calculation using the expansion of S^* in (F.3.1) and the fact that all expansions converge absolutely in an appropriate Banach space, so that we may take the duality brackets inside the sum. The details are left for the reader. □

In what follows we will simply denote the extension $\tilde{S}$ by S. Note that if $S^* \in \mathcal{N}(L^2, M^1_v)$, $R \in \mathcal{L}(L^2)$ and $T^* \in \mathcal{L}(M^1_v)$, then $(RST)^* \in \mathcal{N}(L^2, M^1_v)$, as follows from using (F.3.5).

The fact that we use the Hilbert space $L^2(\mathbb{R}^d)$ is not strictly necessary. We could have considered any separable Hilbert space $\mathcal{H}$, and required that $S \in \mathcal{L}(L^2, \mathcal{H})$ with $S^* \in \mathcal{N}(\mathcal{H}, M^1_v)$. The result above would still hold, as would the main result of this paper. Our reason for considering $\mathcal{H} = L^2(\mathbb{R}^d)$ is that it gives us easier access to non-trivial examples, as it allows us to formulate our results in terms of integral operators as we explain in detail in the next subsection.

The projective tensor product

The theory of nuclear operators is closely related to the projective tensor product of Banach spaces, as explained for instance in [228], which leads to a useful connection to integral operators. Abstractly, the projective tensor product $X\hat{\otimes}Y$ of two Banach spaces X, Y is the completion of the algebraic tensor product $X \otimes Y$ with respect to the norm

$$\|u\|_{X\hat{\otimes}Y} = \inf\left\{ \sum_{n=1}^N \|x_n\|_X \|y_n\|_Y : u = \sum_{k=n}^N x_n \otimes y_n \right\}.$$

One can show (see [228, Prop. 2.8]) that $X\hat{\otimes}Y$ consists precisely of elements $\sum_{n=1}^\infty x_n \otimes y_n$ such that $\sum_{n=1}^\infty \|x_n\|_X \|y_n\|_Y < \infty$.

When X and Y are function spaces on $\mathbb{R}^d$, which is the case we will consider, we identify the elementary tensors $x \otimes y$ for $x \in X$ and $y \in Y$ with the function

$x \otimes y(s,t) = x(s)y(t)$. For instance, we identify $L^2(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ with all functions $\Psi \in L^2(\mathbb{R}^{2d})$ such that $\Psi(s,t) = \sum_{n=1}^\infty \xi_n(s)\psi_n(t)$ with $\sum_{n=1}^\infty \|\xi_n\|_{L^2}\|\psi_n\|_{L^2} < \infty$.

Now assume that the integral kernel k_T of $T \in \mathcal{HS}$ belongs to $X\hat{\otimes}Y$ for Banach function spaces $X, Y \subset L^2(\mathbb{R}^d)$. By definition, this means that we have a decomposition

$$k_T(s,t) = \sum_{n=1}^\infty x_n(s)y_n(t)$$

with $\sum_{n=1}^\infty \|x_n\|_X\|y_n\|_Y < \infty$. A simple calculation then shows that

$$T = \sum_{n=1}^\infty x_n \otimes \overline{y_n},$$

where $x_n \otimes \overline{y_n}$ now denotes a rank-one operator. Hence if we apply this to $X = M^1_v(\mathbb{R}^d)$ and $Y = L^2(\mathbb{R}^d)$ (since all function spaces we consider are invariant under complex conjugation, we need not pay any attention to the fact that $\overline{y_n}$ appears in place of y_n), we see that $k_T \in M^1(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ is equivalent to T having an expansion of the form (F.3.1) – i.e. $k_T \in M^1(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ if and only if $T \in \mathcal{N}(L^2, M^1_v)$.

Remark F.2. The map $k_T \mapsto T$ is in fact a Banach space isomorphism from $M^1(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ to $\mathcal{N}(L^2, M^1_v)$. Surjectivity and boundedness follow from above. Injectivity is not too difficult to show in this case, but for more general Banach spaces X and Y the injectivity of the natural map from $X\hat{\otimes}Y^*$ onto $\mathcal{N}(Y, X)$ boils down to the approximation property for Banach spaces [228, Cor. 4.8].

The slightly awkward condition $T^* \in \mathcal{N}(L^2, M^1_v)$ may similarly be reformulated as requiring $k_T \in L^2(\mathbb{R}^d)\hat{\otimes}M^1(\mathbb{R}^d)$, this is essentially the content of (F.3.7). This condition cannot be reformulated as nuclearity of T, which is why we have opted for phrasing it as $T^* \in \mathcal{N}(L^2, M^1_v)$. We also mention that there is a natural isomorphism $L^2(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d) \cong M^1_v(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ extending the map $\xi \otimes \phi \mapsto \phi \otimes \xi$ for $\xi \in L^2(\mathbb{R}^d), \phi \in M^1_v(\mathbb{R}^d)$.

Formulating our assumption on T by requiring k_T to belong to some projective tensor product makes it possible to relate $T^* \in \mathcal{N}(L^2, M^1_v)$ to other spaces of operators. For instance, we may identify the trace class operators $\mathcal{S}^1$ as the operators $S \in \mathcal{HS}$ such that k_S belongs to the projective tensor product $L^2(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$, which clearly contains $M^1_v(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ as a subset since $M^1_v(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$.

Finally, the operators $T \in \mathcal{HS}$ with kernel k_T in the subspace $M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$ of $M^1(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d)$ have also been studied recently in [236], where this space of operators is denoted by $\mathcal{B}_{v\otimes v}$. It follows by [19, Thm. 5] that

$$M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d) = M^1_{v\tilde{\otimes}v}(\mathbb{R}^{2d}), \tag{F.3.8}$$

with equivalent norms, where $v\tilde{\otimes}v(x_1, x_2, \omega_1, \omega_2) = v(x_1, \omega_1) \cdot v(x_2, \omega_2)$. The particular case $\mathcal{B} := \mathcal{B}_{1\otimes 1}$ corresponding to $v \equiv 1$ has been studied in several other sources, see for instance [102, 168].

We summarize this discussion, which essentially amounts to prodding the definitions in various ways, in a proposition.

Proposition F.3.3. *Given $T \in \mathcal{HS}$ and a submultiplicative weight function v, then*

$$\begin{aligned} T \in \mathcal{N}(L^2, M^1_v) &\iff k_T \in M^1_v(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d), \\ T^* \in \mathcal{N}(L^2, M^1_v) &\iff k_T \in L^2(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d). \end{aligned}$$

At the level of k_T we have the inclusions

$$\begin{array}{ccccc} & & L^2(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d) & & \\ & \nearrow & & \searrow & \\ M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d) & & & & L^2(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^{2d}) \\ & \searrow & & \nearrow & \\ & & M^1_v(\mathbb{R}^d)\hat{\otimes}L^2(\mathbb{R}^d) & & \end{array}$$

which at the operator level leads to the inclusions

$$\mathcal{B}_{v\otimes v} \subset \mathcal{N}(L^2, M^1_v) \subset \mathcal{S}^1 \subset \mathcal{HS}.$$

The same inclusion holds when $\mathcal{N}(L^2, M^1_v)$ is replaced by the set of operators T such that $T^* \in \mathcal{N}(L^2, M^1_v)$.

F.3.3 Examples of nuclear operators

The connection to the projective tensor product allows us to write down some examples of $S^* \in \mathcal{N}(L^2, M^1_v)$.

Example F.3.1. The *Schwartz operators* $\mathfrak{S}$ are those integral operator T_k on $L^2(\mathbb{R}^d)$ such that $k \in \mathcal{S}(\mathbb{R}^{2d})$ [179]. If the submultiplicative weight v grows at most polynomially, then so does the weight function $v\tilde{\otimes}v(x_1, x_2, \omega_1, \omega_2) = v(x_1, \omega_1) \cdot v(x_2, \omega_2)$ on $\mathbb{R}^{4d}$, hence we know that $\mathcal{S}(\mathbb{R}^{2d}) \hookrightarrow M^1_{v\tilde{\otimes}v}(\mathbb{R}^{2d}) \cong M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$. It follows that $\mathfrak{S} \subset \mathcal{B}_{v\otimes v} \subset \mathcal{N}(L^2, M^1_v)$. It is also straightforward to check that $S \in \mathfrak{S} \iff S^* \in \mathfrak{S}$.

Example F.3.2 (The Feichtinger algebra and the inner kernel theorem)**.** By Proposition F.3.3 we know that $\mathcal{B}_{v\otimes v} \subset \mathcal{N}(L^2, M^1_v)$, where $T \in \mathcal{B}_{v\otimes v}$ if

$$k_T \in M^1_{v\tilde{\otimes}v}(\mathbb{R}^{2d}) \cong M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d).$$

This class of operators was recently studied in [236], where the reader may find a proof that T belongs to this space if and only if its Hilbert space adjoint T^* does.

The unweighted case $k_T \in M^1(\mathbb{R}^d)\hat{\otimes}M^1(\mathbb{R}^d) = M^1(\mathbb{R}^{2d})$ has been studied by several sources [102, 168, 193, 235]. We mention in particular that [102, 168] give a characterization of such operators that is independent of their kernel as an integral operator: Given $T \in \mathcal{HS}$, $k_T \in M^1(\mathbb{R}^{2d})$ if and only if T extends to a bounded map $M^\infty(\mathbb{R}^d) \to M^1(\mathbb{R}^d)$ sending weak* convergent sequences to norm-convergent sequences.

We now consider finite rank operators. By choosing S of the form in this example, we will be able to recover Theorem F.2.3 from our main result.

Example F.3.3 (Finite rank operators)**.** For $N \in \mathbb{N}$, consider $\{\phi_n\}_{n=1}^N \subset M^1_v(\mathbb{R}^d)$. Let $\{\xi_n\}_{n=1}^N$ be an orthonormal set in $L^2(\mathbb{R}^d)$. If we define $S = \sum_{n=1}^N \xi_n \otimes \phi_n$, we clearly have $S^* \in \mathcal{N}(L^2, M^1_v)$. This S is just a convenient way of storing the functions ϕ_n in an operator – by applying S to ξ_m for $1 \leq m \leq N$ we recover ϕ_m.

Localization operators

We also have some methods for producing new examples of operators in $\mathcal{N}(L^2, M^1_v)$ from known examples. As $\mathcal{N}(L^2, M^1_v)$ is a normed space we may of course take linear combinations, but a more interesting method is to use the quantum convolutions introduced by Werner [251]. Given $f \in L^1(\mathbb{R}^{2d})$ and a trace class operator $S \in \mathcal{S}^1$, the *convolution* of f with S is defined to be the trace class operator $f \star S$ given by the Bochner integral

$$f \star S := \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^*\, dz. \tag{F.3.9}$$

In particular, if we pick S to be a rank-one operator $\varphi_2 \otimes \varphi_1$ for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, we find that

$$f \star (\varphi_2 \otimes \varphi_1) = \mathcal{A}_f^{\varphi_1,\varphi_2},$$

where $\mathcal{A}_f^{\varphi_1,\varphi_2}$ is the *time-frequency localization operator* [63, 67] given by

$$\mathcal{A}_f^{\varphi_1,\varphi_2}(\psi) = \int_{\mathbb{R}^{2d}} f(z)V_{\varphi_1}\psi(z)\pi(z)\varphi_2\, dz \quad \text{for } \psi \in L^2(\mathbb{R}^d).$$

Proposition F.3.4. *If $S \in \mathcal{N}(L^2, M^1_v)$ and $f \in L^1_v(\mathbb{R}^{2d})$, then $f \star S \in \mathcal{N}(L^2, M^1_v)$ with $\|f \star S\|_{\mathcal{N}} \leq \|f\|_{L^1_v}\|S\|_{\mathcal{N}}$. In particular, if $\varphi_1 \in L^2(\mathbb{R}^d)$ and $\varphi_2 \in M^1_v(\mathbb{R}^d)$, then $\mathcal{A}_f^{\varphi_1,\varphi_2} \in \mathcal{N}(L^2, M^1_v)$, with $\|\mathcal{A}_f^{\varphi_1,\varphi_2}\|_{\mathcal{N}} \leq \|f\|_{L^1_v}\|\varphi_1\|_{L^2}\|\varphi_2\|_{M^1_v}$.*

Proof. By definition,

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^* \, dz.$$

This integral converges as a Bochner integral in $\mathcal{N}(L^2, M^1_v)$, as Lemma F.3.1 gives

$$\int_{\mathbb{R}^{2d}} \|f(z)\pi(z)S\pi(z)^*\|_{\mathcal{N}} \, dz \leq \int_{\mathbb{R}^{2d}} |f(z)|v(z)\|S\|_{\mathcal{N}} \, dz = \|S\|_{\mathcal{N}}\|f\|_{L^1_v}.$$

The result for $\mathcal{A}_f^{\varphi_1,\varphi_2}$ follows by $\mathcal{A}_f^{\varphi_1,\varphi_2} = f \star (\varphi_2 \otimes \varphi_1)$ and (F.3.3). □

It is easy to check that the Hilbert space adjoint of $f \star S$ is $\overline{f} \star S^*$. Hence we immediately obtain the following.

Corollary F.3.4.1. *If $S^* \in \mathcal{N}(L^2, M^1_v)$ and $f \in L^1_v(\mathbb{R}^{2d})$, then $(f \star S)^* \in \mathcal{N}(L^2, M^1_v)$ with $\|(f \star S)^*\|_{\mathcal{N}} \leq \|f\|_{L^1_v}\|S^*\|_{\mathcal{N}}$. In particular, if $\varphi_1 \in M^1_v(\mathbb{R}^d)$ and $\varphi_2 \in L^2(\mathbb{R}^d)$, then $\left(\mathcal{A}_f^{\varphi_1,\varphi_2}\right)^* \in \mathcal{N}(L^2, M^1_v)$, with*

$$\left\|\left(\mathcal{A}_f^{\varphi_1,\varphi_2}\right)^*\right\|_{\mathcal{N}} \leq \|f\|_{L^1_v}\|\varphi_1\|_{M^1_v}\|\varphi_2\|_{L^2}.$$

Underspread operators

When operators between function spaces are used to model communication channels, the resulting operators will typically be (at least approximately) *underspread* [238]. An underspread operator $T \in \mathcal{HS}$ is of the form

$$T = \int_{\mathbb{R}^{2d}} F(x,\omega)e^{-i\pi x\cdot\omega}\pi(x,\omega) \, dxd\omega, \tag{F.3.10}$$

where the support of F is contained in $[-\tau,\tau]^d \times [-\nu,\nu]^d$ for $4\tau\nu < 1$. The function $F(x,\omega)e^{-i\pi x\cdot\omega}$ is called the spreading function of T, and one can show that any $T \in \mathcal{HS}$ has a spreading function in $L^2(\mathbb{R}^{2d})$, as long as the integral in (F.3.10) is interpreted appropriately [102]. In quantum harmonic analysis the spreading function is considered a Fourier transform of the operator [251]. The next lemma shows that underspread *trace class* operators belong to $\mathcal{B}$, i.e. have integral kernel in $M^1(\mathbb{R}^{2d}) \subset \mathcal{N}(L^2; M^1)$. This is an operator-version of the well-known fact that band-limited integrable functions belong to $M^1(\mathbb{R}^d)$ [108, Cor. 3.2.7]. The proof is moved to an appendix, as it requires the introduction of several results from quantum harmonic analysis that will not be needed later in the paper.

Proposition F.3.5. *If the spreading function of $T \in \mathcal{S}^1$ has compact support, then $T \in \mathcal{B} \subset \mathcal{N}(L^2; M^1)$.*

F.4 Time-frequency analysis with operators as windows

A fundamental object in time-frequency analysis is the short-time Fourier transform (STFT) $V_\phi \psi$ with window ϕ. The goal of this section is the define an STFT where the window ϕ is replaced by an operator S, and to show that the basic properties of the STFT remain true for this generalized STFT.

As a first step, we will need the Hilbert space $L^2(\mathbb{R}^{2d}; L^2)$ of equivalence classes of strongly Lebesgue measurable $\Psi : \mathbb{R}^{2d} \to L^2(\mathbb{R}^d)$ such that

$$\|\Psi\|_{L^2(\mathbb{R}^{2d};L^2)} := \left(\int_{\mathbb{R}^{2d}} \|\Psi(z)\|_{L^2}^2 \, dz \right)^{1/2} < \infty,$$

with inner product

$$\langle \Psi, \Phi \rangle_{L^2(\mathbb{R}^{2d};L^2)} = \int_{\mathbb{R}^{2d}} \langle \Psi(z), \Phi(z) \rangle_{L^2} \, dz.$$

The equivalence relation on $L^2(\mathbb{R}^{2d}; L^2)$ is that $\Psi \sim \Phi$ if $\Psi(z) = \Phi(z)$ as elements of $L^2(\mathbb{R}^d)$ for a.e. $z \in \mathbb{R}^{2d}$.

We then define a version of the short-time Fourier transform with operators as windows. For $S \in \mathcal{HS}$ and $\psi \in L^2(\mathbb{R}^d)$ we let

$$\mathfrak{V}_S(\psi)(z) = S\pi(z)^* \psi \quad \text{for } z \in \mathbb{R}^{2d}.$$

Remark F.3. When S is a localization operator $\mathcal{A}_f^{\varphi,\varphi}$, the short-time Fourier transform above is closely related to the vector-valued analysis operator introduced by Romero in [225] to obtain equivalent norms for modulation spaces (and several other spaces) from certain discrete expressions. See (F.8.4) for the precise expression.

We obtain a generalization of Moyal's identity. It shows that $\mathfrak{V}_S$ is a linear isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d}; L^2)$.

Lemma F.4.1. *Let $S \in \mathcal{HS}$ and $\psi \in L^2(\mathbb{R}^d)$. Then $\mathfrak{V}_S(\psi) \in L^2(\mathbb{R}^{2d}; L^2)$ and*

$$\|\mathfrak{V}_S(\psi)\|_{L^2(\mathbb{R}^{2d},L^2)}^2 = \int_{\mathbb{R}^{2d}} \|S\pi(z)^*\psi\|_{L^2}^2 \, dz = \|S\|_{\mathcal{HS}}^2 \|\psi\|_{L^2}^2.$$

Proof. We may rewrite

$$\|S\pi(z)^*\psi\|_{L^2}^2 = \langle \pi(z) S^* S \pi(z)^* \psi, \psi \rangle_{L^2} = \operatorname{tr}(\pi(z) S^* S \pi(z)^* (\psi \otimes \psi)).$$

The result therefore follows by [203, Lem. 4.1], which states that

$$\int_{\mathbb{R}^{2d}} \operatorname{tr}(\pi(z) R \pi(z)^* T) \, dz = \operatorname{tr}(R) \operatorname{tr}(T)$$

for trace class operators R, T on $L^2(\mathbb{R}^d)$; pick $R = S^*S$ and $T = \psi \otimes \psi$ and note that $\operatorname{tr}(S^*S) = \|S\|_{\mathcal{HS}}^2$ and $\operatorname{tr}(\psi \otimes \psi) = \|\psi\|_{L^2}^2$ □

Example F.4.1. To see that $\mathfrak{V}_S$ actually generalizes the usual STFT, consider $\phi \in L^2(\mathbb{R}^d)$ and let $\xi \in L^2(\mathbb{R}^d)$ be any function satisfying $\|\xi\|_{L^2} = 1$. Then let $S = \xi \otimes \phi$. For any $\psi \in L^2(\mathbb{R}^d)$ we then have

$$\mathfrak{V}_S(\psi)(z) = S\pi(z)^*\psi = \langle \pi(z)^*\psi, \phi\rangle_{L^2}\, \xi = V_\phi\psi(z) \cdot \xi,$$

which contains precisely the same information as $V_\phi\psi(z)$ given that we know ξ. In particular, it is easy to show that $\|S\|_{\mathcal{HS}}^2 = \|\phi\|_{L^2}^2$ and $\|\mathfrak{V}_S(\psi)\|_{L^2(\mathbb{R}^{2d};L^2)} = \|V_\phi\psi\|_{L^2(\mathbb{R}^{2d})}$, so Lemma F.4.1 reduces to Moyal's identity in this case.

Remark F.4. Strong measurability of $\mathfrak{V}_S(\psi)$ is always satisfied: since $L^2(\mathbb{R}^d)$ is separable, the Pettis measurability theorem [166, Thm. 1.1.20] ensures that strong measurability follows from weak measurability. Weak measurability means that for each $\phi \in L^2(\mathbb{R}^d)$, the map

$$z \mapsto \langle \mathfrak{V}_S(\psi)(z), \phi\rangle_{L^2}$$

is Lebesgue measurable. We may rewrite $\langle \mathfrak{V}_S(\psi)(z), \phi\rangle_{L^2} = \langle S\pi(z)^*\psi, \phi\rangle_{L^2} = V_{S^*\phi}\psi(z)$. It is well-known that the STFT $z \mapsto V_\xi\psi(z)$ is continuous for any $\xi \in L^2(\mathbb{R}^d)$, in particular for $\xi = S^*\phi$, hence the map is Lebesgue measurable.

We then define for $\Psi \in L^2(\mathbb{R}^{2d}; L^2)$ a function $\mathfrak{V}_S^*(\Psi)$ on $\mathbb{R}^d$ by the $L^2(\mathbb{R}^d)$-valued integral

$$\mathfrak{V}_S^*(\Psi) = \int_{\mathbb{R}^{2d}} \pi(z)S^*\Psi(z)\, dz. \tag{F.4.1}$$

The integral (F.4.1) is interpreted in a weak sense: we will see that

$$\left| \int_{\mathbb{R}^{2d}} \langle \pi(z)S^*\Psi(z), \phi\rangle_{L^2}\, dz \right| \lesssim \|\phi\|_{L^2} \quad \text{for any } \phi \in L^2(\mathbb{R}^d), \tag{F.4.2}$$

so it follows from the Riesz representation theorem for Hilbert spaces that there must exist an element in $L^2(\mathbb{R}^d)$, which we denote by $\int_{\mathbb{R}^{2d}} \pi(z)S^*\Psi(z)\, dz$, such that for any $\phi \in L^2(\mathbb{R}^d)$

$$\left\langle \int_{\mathbb{R}^{2d}} \pi(z)S^*\Psi(z)\, dz, \phi \right\rangle_{L^2} = \int_{\mathbb{R}^{2d}} \langle \pi(z)S^*\Psi(z), \phi\rangle_{L^2}\, dz. \tag{F.4.3}$$

The next lemma shows that the integral in (F.4.1) is well-defined in this sense.

Lemma F.4.2. *Let $S \in \mathcal{HS}$.*

(a) *Equation* (F.4.3) *defines $\mathfrak{V}_S^*(\psi) = \int_{\mathbb{R}^{2d}} \pi(z)S^*\Psi(z)\, dz$ as an element of $L^2(\mathbb{R}^d)$, and $\mathfrak{V}_S^* : L^2(\mathbb{R}^{2d}; L^2) \to L^2(\mathbb{R}^d)$ defines a bounded operator that is the Hilbert space adjoint of $\mathfrak{V}_S$.*

(b) The composition $\mathfrak{V}_S^*\mathfrak{V}_S$ is $\|S\|_{\mathcal{HS}}^2$ times the identity operator on $L^2(\mathbb{R}^d)$.

Proof. Let $\Psi \in L^2(\mathbb{R}^{2d}; L^2)$ and let $\phi \in L^2(\mathbb{R}^d)$. We need to show (F.4.2), as mentioned (F.4.3) then defines an element $\int_{\mathbb{R}^{2d}} \pi(z)S^*\Psi(z)\,dz$ of $L^2(\mathbb{R}^d)$ by Riesz' representation theorem. We find that

$$\begin{aligned}\left|\int_{\mathbb{R}^{2d}} \langle \pi(z)S^*\Psi(z), \phi\rangle_{L^2}\ dz\right| &= \left|\int_{\mathbb{R}^{2d}} \langle \Psi(z), S\pi(z)^*\phi\rangle_{L^2}\ dz\right| \\ &= |\langle \Psi, \mathfrak{V}_S(\phi)\rangle_{L^2(\mathbb{R}^d;L^2)}| \\ &\leq \|\Psi\|_{L^2(\mathbb{R}^{2d};L^2)}\|\mathfrak{V}_S(\phi)\|_{L^2(\mathbb{R}^{2d};L^2)} \\ &= \|\Psi\|_{L^2(\mathbb{R}^{2d};L^2)}\|S\|_{\mathcal{HS}}\|\phi\|_{L^2}\end{aligned}$$

by Lemma F.4.1. It is clear that $\mathfrak{V}_S^*$ is linear, and the estimate also shows that it is bounded from $L^2(\mathbb{R}^d; L^2(\mathbb{R}^d))$ to $L^2(\mathbb{R}^d)$. A simple calculation shows that it is the adjoint of $\mathfrak{V}_S$. The second part states that

$$\int_{\mathbb{R}^{2d}} \pi(z)S^*S\pi(z)^*\psi\ dz = \|S\|_{\mathcal{HS}}^2\psi \quad \text{for any } \psi \in L^2(\mathbb{R}^d),$$

which is part (c) of [251, Prop. 3.3]. □

F.5 Equivalent norms for modulation spaces

The generalized Moyal identity in Lemma F.4.1 shows that the norm of $\mathfrak{V}_S(\psi)$ in $L^2(\mathbb{R}^{2d}; L^2)$ is equivalent to the norm of ψ in $L^2(\mathbb{R}^d)$. We will now generalize Theorem F.2.3 by showing that if S satisfies some extra assumptions, the same is true if $L^2(\mathbb{R}^d)$ is replaced by $M_m^{p,q}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{2d}; L^2)$ is replaced by $L_m^{p,q}(\mathbb{R}^{2d}; L^2)$, where $1 \leq p, q \leq \infty$ and m is some v-moderate weight. As before, v always denotes a submultiplicative weight function on $\mathbb{R}^{2d}$.

We start by defining $L_m^{p,q}(\mathbb{R}^{2d}; L^2)$. For $1 \leq p, q \leq \infty$ and any v-moderate weight m, the Banach space $L_m^{p,q}(\mathbb{R}^{2d}; L^2)$ consists of the equivalence classes of strongly Lebesgue measurable functions $\Psi : \mathbb{R}^{2d} \to L^2(\mathbb{R}^d)$ such that

$$\|\Psi\|_{L_m^{p,q}(\mathbb{R}^{2d};L^2)} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \|\Psi(x,\omega)\|_{L^2}^p m(x,\omega)^p\ dx\right)^{q/p} d\omega\right)^{\frac{1}{q}} < \infty,$$

where $\Phi \sim \Psi$ if $\Psi(z) = \Phi(z)$ for a.e. $z \in \mathbb{R}^{2d}$. When $p = \infty$ or $q = \infty$ the definition is modified in the usual way by replacing integrals by essential supremums.

With this definition in place, we are ready to state our main result.

Theorem F.5.1. *Let* $0 \neq S \in \mathcal{HS}$ *such that* $S^* \in \mathcal{N}(L^2, M^1_v)$*. For any* $1 \leq p, q \leq \infty$ *and* v*-moderate weight* m*, we have*

$$C_{\text{lower}} \cdot \|\psi\|_{M^{p,q}_m} \leq \|\mathfrak{B}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \leq C_{\text{upper}} \cdot \|\psi\|_{M^{p,q}_m}$$

with

$$C_{\text{lower}} = \|S\|^2_{\mathcal{HS}} \cdot \left(C^m_v \cdot \|S^*\|_{\mathcal{N}} \cdot \|\varphi_0\|_{M^1_v}\right)^{-1},$$
$$C_{\text{upper}} = C^m_v \cdot \|S^*\|_{\mathcal{N}}.$$

Our proof will follow the same structure as the usual proof that $M^{p,q}_m$ is independent of the window function [131]: we will show that $\mathfrak{B}_S$ is bounded from $M^{p,q}_m(\mathbb{R}^d)$ to $L^{p,q}_m(\mathbb{R}^{2d};L^2)$ and that $\mathfrak{B}^*_S$ is bounded from $L^{p,q}_m(\mathbb{R}^{2d};L^2)$ to $M^{p,q}_m(\mathbb{R}^d)$.

Before we start, we make sure that there is no ambiguity in interpreting

$$\mathfrak{B}_S(\psi)(z) = S\pi(z)^*\psi$$

even when $\psi \in M^\infty_{1/v}(\mathbb{R}^{2d})$. First note that as $\pi(z)$ is bounded on $M^1_v(\mathbb{R}^d)$ by (F.2.4), we may extend $\pi(z)$ to a bounded operator on $M^\infty_{1/v}(\mathbb{R}^d)$ by defining

$$\langle \pi(z)\psi, \phi\rangle_{M^\infty_{1/v},M^1_v} := \langle \psi, \pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v} \qquad \text{for } \psi \in M^\infty_{1/v}(\mathbb{R}^d), \phi \in M^1_v(\mathbb{R}^d). \tag{F.5.1}$$

As $\pi(x,\omega)^* = e^{-2\pi i x\cdot\omega}\pi(-x,\omega)$, $\pi(z)^*$ is also bounded on $M^\infty_{1/v}(\mathbb{R}^d)$. Therefore

$$\mathfrak{B}_S(\psi)(z) = S\pi(z)^*\psi$$

makes sense by Lemma F.3.2, as S extends to a bounded operator from $M^\infty_{1/v}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ – hence $S\pi(z)^*\psi$ is a well-defined element of $L^2(\mathbb{R}^d)$.

Lemma F.5.2. *Let* m *be a* v*-moderate weight. For any* $1 \leq p, q \leq \infty$*,* $\mathfrak{B}_S$ *is a bounded, linear map from* $M^{p,q}_m(\mathbb{R}^d)$ *to* $L^{p,q}_m(\mathbb{R}^{2d};L^2)$ *with* $\|\mathfrak{B}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \leq C^m_v \cdot \|S^*\|_{\mathcal{N}} \cdot \|\psi\|_{M^{p,q}_m}$*.*

Proof. Throughout the proof we will use the expansion in (F.3.7) to write

$$S = \sum_{n=1}^\infty \xi_n \otimes \phi_n$$

with $\sum_{n=1}^\infty \|\xi_n\|_{L^2}\|\phi_n\|_{M^1_v} < \infty$. Assume that $\psi \in M^{p,q}_m(\mathbb{R}^d)$. Then

$$S\pi(z)^*\psi = \sum_{n=1}^\infty \langle \pi(z)^*\psi, \phi\rangle_{M^\infty_{1/v},M^1_v}\xi_n = \sum_{n=1}^\infty V_{\phi_n}\psi(z)\xi_n.$$

This implies that

$$\|S\pi(z)^*\psi\|_{L^2} \leq \sum_{n=1}^{\infty} |V_{\phi_n}\psi(z)| \cdot \|\xi_n\|_{L^2},$$

hence the triangle inequality for $L^{p,q}_m(\mathbb{R}^{2d})$ gives

$$\begin{aligned}\|\mathfrak{B}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} &\leq \left\| \sum_{n=1}^{\infty} |V_{\phi_n}\psi(-)| \cdot \|\xi_n\|_{L^2} \right\|_{L^{p,q}_m(\mathbb{R}^{2d})} \\ &\leq \sum_{n=1}^{\infty} \|\xi_n\|_{L^2} \left\|V_{\phi_n}\psi\right\|_{L^{p,q}_m(\mathbb{R}^{2d})}\end{aligned}$$

We then apply Proposition F.2.4 to get

$$\|\mathfrak{B}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \leq C_v^m \|\psi\|_{M^{p,q}_m} \sum_{n=1}^{\infty} \|\xi_n\|_{L^2} \|\phi_n\|_{M^1_v}.$$

Using the definition of $\|S^*\|_{\mathcal{N}}$ from (F.3.2) we get that

$$\|\mathfrak{B}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \leq C_v^m \|S^*\|_{\mathcal{N}} \|\psi\|_{M^{p,q}_m}. \qquad \square$$

In order to give a sensible definition of $\mathfrak{B}_S^*(\Psi)$ for $\Psi \in L^{p,q}_m(\mathbb{R}^{2d}; L^2)$, we will need Hölder's inequality for the mixed-norm spaces $L^{p,q}_m(\mathbb{R}^{2d})$ [30, 131]: given $F \in L^{p,q}_m(\mathbb{R}^{2d})$ and $G \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$ for $1 \leq p, q \leq \infty$, then $F \cdot G \in L^1(\mathbb{R}^{2d})$ with

$$\int_{\mathbb{R}^{2d}} |F(z)G(z)|\, dz \leq \|F\|_{L^{p,q}_m} \|G\|_{L^{p',q'}_{1/m}}. \tag{F.5.2}$$

For any $\Psi \in L^{p,q}_m(\mathbb{R}^{2d}; L^2)$ we then define $\mathfrak{B}_S^*(\Psi)$ as an element of $M^\infty_{1/v}(\mathbb{R}^d)$ by duality:

$$\left\langle \mathfrak{B}_S^*(\Psi), \phi \right\rangle_{M^\infty_{1/v}, M^1_v} := \int_{\mathbb{R}^{2d}} \langle \Psi(z), \mathfrak{B}_S(\phi)(z) \rangle_{L^2}\ dz \quad \text{ for all } \phi \in M^1_v(\mathbb{R}^d).$$

To see that this actually defines a bounded linear functional on $M^1_v(\mathbb{R}^d)$, we can use Cauchy-Schwartz, (F.5.2) and Lemma F.5.2 to find that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} | \langle \Psi(z), \mathfrak{B}_S(\phi)(z) \rangle_{L^2} |\, dz &\leq \int_{\mathbb{R}^{2d}} \|\Psi(z)\|_{L^2} \|\mathfrak{B}_S(\phi)(z)\|_{L^2}\, dz \\ &\leq \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \|\mathfrak{B}_S(\phi)\|_{L^{p',q'}_{1/m}(\mathbb{R}^{2d};L^2)} \\ &\leq \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} C_v^m \|S^*\|_{\mathcal{N}} \|\phi\|_{M^{p',q'}_{1/m}} \\ &\lesssim \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} C_v^m \|S^*\|_{\mathcal{N}} \|\phi\|_{M^1_v},\end{aligned}$$

where the last inequality uses that $M^1_v(\mathbb{R}^d) \hookrightarrow M^{p,q}_m(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$ and all v-moderate weights m. The reader should observe that this definition agrees with our original definition (F.4.1) when $\Psi \in L^2(\mathbb{R}^{2d}; L^2)$.

Lemma F.5.3. *Let m be a v-moderate weight. For any $1 \le p, q \le \infty$, the map $\mathfrak{V}^*_S$ is a bounded, linear map from $L^{p,q}_m(\mathbb{R}^{2d}; L^2)$ to $M^{p,q}_m(\mathbb{R}^d)$, with*

$$\|\mathfrak{V}^*_S(\Psi)\|_{M^{p,q}_m} \le \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \cdot C^m_v \cdot \|S^*\|_{\mathcal{N}} \cdot \|\varphi_0\|_{M^1_v}.$$

Proof. As a short preparation, we consider $\mathfrak{V}_S(\pi(z)\phi)$ for $\phi \in L^2(\mathbb{R}^d)$. By definition

$$\mathfrak{V}_S(\pi(z)\phi)(z') = S\pi(z')^*\pi(z)\phi = S[\pi(z)^*\pi(z')]^*\phi.$$

With $z = (x, \omega)$ and $z' = (x', \omega')$, we find using (F.2.1) and (F.2.2) that

$$\mathfrak{V}_S(\pi(z)\phi)(z') = S[e^{2\pi i x\cdot(\omega'-\omega)}\pi(z'-z)]^*\phi = e^{2\pi i x\cdot(\omega-\omega')}\mathfrak{V}_S(\phi)(z'-z). \tag{F.5.3}$$

Recall that φ_0 is the L^2-normalized Gaussian on $\mathbb{R}^d$, and that the norm on $M^{p,q}_m(\mathbb{R}^d)$ is given by $\|\psi\|_{M^{p,q}_m} = \|V_{\varphi_0}\psi\|_{L^{p,q}_m}$. We therefore calculate that

$$\begin{aligned}
|V_{\varphi_0}(\mathfrak{V}^*_S(\Psi))(z)| &= |\left\langle \mathfrak{V}^*_S(\Psi), \pi(z)\varphi_0 \right\rangle_{M^\infty_{1/v}, M^1_v}| \\
&= \left| \int_{\mathbb{R}^{2d}} \langle \Psi(z'), \mathfrak{V}_S(\pi(z)\varphi_0)(z')\rangle_{L^2}\, dz' \right| \\
&\le \int_{\mathbb{R}^{2d}} |\langle \Psi(z'), \mathfrak{V}_S(\pi(z)\varphi_0)(z')\rangle_{L^2}|\, dz' \\
&\le \int_{\mathbb{R}^{2d}} \|\Psi(z')\|_{L^2} \|\mathfrak{V}_S(\pi(z)\varphi_0)(z')\|_{L^2}\, dz' \\
&= \int_{\mathbb{R}^{2d}} \|\Psi(z')\|_{L^2} \|\mathfrak{V}_S(\varphi_0)(z'-z)\|_{L^2}\, dz' \quad \text{by (F.5.3)}.
\end{aligned}$$

By [131, Prop. 11.1.3] the space $L^{p,q}_m(\mathbb{R}^{2d})$ satisfies the convolution relation

$$\|F * G\|_{L^{p,q}_m} \le \|F\|_{L^{p,q}_m} \|G\|_{L^1_v} \tag{F.5.4}$$

for $F \in L^{p,q}_m(\mathbb{R}^{2d})$ and $G \in L^1_v(\mathbb{R}^{2d})$. With $F(z) = \|\Psi(z)\|_{L^2}$ and $G(z) = \|\mathfrak{V}_S(\varphi_0)(-z)\|_{L^2}$ the calculation above states that

$$|V_{\varphi_0}(\mathfrak{V}^*_S(\Psi))(z)| \le F * G(z),$$

which in light of (F.5.4) gives

$$\begin{aligned}
\|V_{\varphi_0}(\mathfrak{V}^*_S\Psi)\|_{L^{p,q}_m} &\le \|F\|_{L^{p,q}_m} \|G\|_{L^1_v} \\
&= \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \|\mathfrak{V}_S(\varphi_0)\|_{L^1_v(\mathbb{R}^{2d};L^2)} \\
&\le \|\Psi\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} C^m_v \|S^*\|_{\mathcal{N}} \|\varphi_0\|_{M^1_v},
\end{aligned}$$

where we have used Lemma F.5.2 in the last step. The reader should also note that $\|\mathfrak{V}_S(\varphi_0)\|_{L^1_v(\mathbb{R}^{2d};L^2)} = \|G\|_{L^1_v}$ is a straightforward computation, but relies on our assumption that $v(-z) = v(z)$. □

Finally, we also need that the inversion formula $\mathfrak{V}_S^*\mathfrak{V}_S\psi = \|S\|_{\mathcal{HS}}^2\psi$ from Lemma F.4.2 remains valid on the other modulation spaces.

Lemma F.5.4. *Let $\psi \in M^\infty_{1/v}(\mathbb{R}^{2d})$. Then $\|S\|_{\mathcal{HS}}^2 \cdot \psi = \mathfrak{V}_S^*\mathfrak{V}_S(\psi)$.*

Proof. We need to show that for $\phi \in M^1_v(\mathbb{R}^d)$ we have

$$\left\langle \mathfrak{V}_S^*\mathfrak{V}_S\psi, \phi \right\rangle_{M^\infty_{1/v},M^1_v} = \|S\|_{\mathcal{HS}}^2 \langle \psi, \phi\rangle_{M^\infty_{1/v},M^1_v}.$$

As a preliminary step, we rewrite the left hand side of this expression in a way that involves explictly the action of ψ as a functional:

$$\begin{aligned}
\left\langle \mathfrak{V}_S^*\mathfrak{V}_S(\psi), \phi \right\rangle_{M^\infty_{1/v},M^1_v} &= \int_{\mathbb{R}^{2d}} \langle \mathfrak{V}_S(\psi)(z), \mathfrak{V}_S(\phi)(z)\rangle_{L^2}\ dz \\
&= \int_{\mathbb{R}^{2d}} \langle S\pi(z)^*\psi, S\pi(z)^*\phi\rangle_{L^2}\ dz \\
&= \int_{\mathbb{R}^{2d}} \langle \pi(z)^*\psi, S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v}\ dz \quad \text{by (F.3.6)} \\
&= \int_{\mathbb{R}^{2d}} \langle \psi, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v}\ dz \quad \text{by (F.5.1).}
\end{aligned}$$

Hence it suffices to show that

$$\int_{\mathbb{R}^{2d}} \langle \psi, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v}\ dz = \|S\|_{\mathcal{HS}}^2 \langle \psi, \phi\rangle_{M^\infty_{1/v},M^1_v}.$$

When $\psi \in L^2(\mathbb{R}^d) \subset M^\infty_{1/v}(\mathbb{R}^d)$, this holds by Lemma F.4.2. To proceed, we will use that for any $\psi \in M^\infty_{1/v}(\mathbb{R}^d)$ there exists a sequence $\{\psi_n\}_{n=1}^\infty$ in $L^2(\mathbb{R}^d)$ with $\|\psi_n\|_{M^\infty_{1/v}} \lesssim \|\psi\|_{M^\infty_{1/v}}$ such that ψ_n converges to ψ in the weak* topology of $M^\infty_{1/v}(\mathbb{R}^d)$ as $n \to \infty$; a construction of such a sequence may be found in the proof of [87, Cor. 7]. Let us define

$$\begin{aligned}
\Xi_n &:= \|S\|_{\mathcal{HS}}^2 \langle \psi_n, \phi\rangle_{M^\infty_{1/v},M^1_v} \\
&= \int_{\mathbb{R}^{2d}} \langle \psi_n, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v}\ dz.
\end{aligned}$$

Using the upper expression for Ξ_n above, we have that $\Xi_n \to \|S\|_{\mathcal{HS}}^2 \langle \psi, \phi\rangle_{M^\infty_{1/v},M^1_v}$ as $n \to \infty$ by the weak* convergence of ψ_n to ψ. Using the lower expression, we find – assuming for now that the limit may be taken inside the integral – that

$$\begin{aligned}\lim_{n\to\infty} \Xi_n &= \lim_{n\to\infty} \int_{\mathbb{R}^{2d}} \langle \psi_n, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v} \, dz \\ &= \int_{\mathbb{R}^{2d}} \lim_{n\to\infty} \langle \psi_n, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v} \, dz \\ &= \int_{\mathbb{R}^{2d}} \langle \psi, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v} \, dz.\end{aligned}$$

Hence we have shown that

$$\int_{\mathbb{R}^{2d}} \langle \psi, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v} \, dz = \lim_{n\to\infty} \Xi_n = \|S\|^2_{\mathcal{HS}} \langle \psi, \phi\rangle_{M^\infty_{1/v},M^1_v} ,$$

which means that we are done once the interchange of the limit and integral has been justified. For each n we may bound the integrand by

$$\begin{aligned}|\langle \psi_n, \pi(z)S^*S\pi(z)^*\phi\rangle_{M^\infty_{1/v},M^1_v}| &\le \|\psi_n\|_{M^\infty_{1/v}} \cdot \|\pi(z)S^*S\pi(z)^*\phi\|_{M^1_v} \\ &\lesssim \|\psi\|_{M^\infty_{1/v}} \cdot v(z) \cdot \|S^*\|_{\mathcal{N}} \cdot \|S\pi(z)^*\phi\|_{L^2} \\ &= \|\psi\|_{M^\infty_{1/v}} \cdot \|S^*\|_{\mathcal{N}} \cdot v(z) \cdot \|\mathfrak{V}_S(\phi)(z)\|_{L^2},\end{aligned}$$

where we use (F.2.4) and (F.3.4) to move to the second line. Since $\phi \in M^1_v(\mathbb{R}^d)$, it follows by Lemma F.5.2 that $z \mapsto v(z) \cdot \|\mathfrak{V}_S(\phi)(z)\|_{L^2}$ is an integrable function. Hence we may apply the dominated convergence theorem. □

The proof of Theorem F.5.1 is now straightforward.

Proof of Theorem F.5.1. The upper bound

$$\|\mathfrak{V}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)} \le C^m_v \|S^*\|_{\mathcal{N}} \cdot \|\psi\|_{M^{p,q}_m}$$

is the content of Lemma F.5.2. By the inversion formula and Lemma F.5.3 we obtain

$$\|\psi\|_{M^{p,q}_m} = \frac{1}{\|S\|^2_{\mathcal{HS}}} \|\mathfrak{V}_S^*\mathfrak{V}_S\psi\|_{M^{p,q}_m} \le \frac{C^m_v \|S^*\|_{\mathcal{N}} \cdot \|\varphi_0\|_{M^1_v}}{\|S\|^2_{\mathcal{HS}}} \|\mathfrak{V}_S(\psi)\|_{L^{p,q}_m(\mathbb{R}^{2d};L^2)},$$

which implies the lower bound. □

Remark F.5. A different proof of a lower bound, more in line with the arguments in the proof of [143, Prop. 2.2] (see Section F.6 for more on this result), is to use that S has a singular value decomposition

$$S = \sum_{n=1}^{\infty} \lambda_n \eta_n \otimes \xi_n,$$

where λ_n is a summable sequence of non-negative numbers and $\{\eta\}_{n=1}^\infty$, $\{\xi_n\}_{n=1}^\infty$ are orthonormal sequences in $L^2(\mathbb{R}^d)$. It is easy to check that since S^* is bounded from $L^2(\mathbb{R}^d)$ to $M_v^1(\mathbb{R}^d)$, we must have $\xi_n \in M_v^1(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Then we find that

$$\begin{aligned}\|S\pi(z)^*\psi\|_{L^2}^2 &= \left\|\sum_{n=1}^\infty \lambda_n V_{\xi_n}\psi(z)\phi_n\right\|_{L^2}^2 \\ &= \sum_{n=1}^\infty \lambda_n^2 |V_{\xi_n}\psi(z)|^2.\end{aligned}$$

Hence $\|S\pi(z)^*\psi\|_{L^2} \geq \lambda_1 |V_{\xi_1}\psi(z)|$, which leads to a lower bound by Theorem F.2.3. We have chosen to prove the lower bound in terms of $\mathfrak{V}_S^*$ to emphasize the interpretation of our results as an STFT with operators as windows.

As a first example we make sure that our result includes the well-known window independence from Theorem F.2.3 as a special case.

Example F.5.1. As in Example F.3.3, we consider $\{\phi_n\}_{n=1}^N \subset M_v^1(\mathbb{R}^d)$, let $\{\xi_n\}_{n=1}^N$ be an orthonormal set in $L^2(\mathbb{R}^d)$ and define $S = \sum_{n=1}^N \xi_n \otimes \phi_n$. For $\psi \in M_{1/v}^\infty(\mathbb{R}^d)$ we then have

$$\mathfrak{V}_S(\psi)(z) = \sum_{n=1}^N V_{\phi_n}\psi(z)\xi_n.$$

By the orthonormality of the ξ_n's we therefore have

$$\|\mathfrak{V}_S(\psi)(z)\|_{L^2}^2 = \sum_{n=1}^N |V_{\phi_n}\psi(z)|^2.$$

It follows by Theorem F.5.1 that

$$C_{\text{lower}} \cdot \|\psi\|_{M_m^{p,q}} \leq \left\|\sqrt{\sum_{n=1}^N |V_{\phi_n}\psi|^2}\right\|_{L_m^{p,q}} \leq C_{\text{upper}} \cdot \|\psi\|_{M_m^{p,q}}.$$

In particular, if $N = 1$ we recover Theorem F.2.3 in the form

$$C_{\text{lower}} \cdot \|\psi\|_{M_m^{p,q}} \leq \left\|V_{\phi_1}\psi\right\|_{L_m^{p,q}} \leq C_{\text{upper}} \cdot \|\psi\|_{M_m^{p,q}},$$

and it is easy to show that in this case

$$\begin{aligned}C_{\text{upper}} &= C_v^m \cdot \|\phi_1\|_{M_v^1} \\ C_{\text{lower}} &= \|\phi_1\|_{L^2}^2 \cdot (C_v^m \cdot \|\phi_1\|_{M_v^1} \cdot \|\varphi_0\|_{M_v^1})^{-1}.\end{aligned}$$

F.6 The Weyl calculus and Bony-Chemin spaces

In Section F.3.1 we defined Hilbert-Schmidt operators as integral operators, but any Hilbert-Schmidt operator can also be described as a *Weyl operator.* To define Weyl operators, we first introduce the *cross-Wigner distribution* of $\phi, \psi \in L^2(\mathbb{R}^d)$, which is the function

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi(x + t/2)\overline{\phi(x - t/2)}e^{-2\pi i\omega\cdot t}\, dt \quad \text{for } x, \omega \in \mathbb{R}^d. \quad \text{(F.6.1)}$$

When $\psi = \phi$ we write $W(\psi) = W(\psi, \psi)$. Given $a \in L^2(\mathbb{R}^{2d})$, we can define the Weyl operator $L_a \in \mathcal{HS}$ by requiring that

$$\langle L_a\phi, \psi\rangle_{L^2} = \langle a, W(\psi, \phi)\rangle_{L^2} \quad \text{for all } \psi, \phi \in L^2(\mathbb{R}^d).$$

The operator L_a is called the *Weyl transform* of a, and a is the *Weyl symbol* of L_a. It is well-known that the Weyl transform $a \mapsto L_a$ is unitary from $L^2(\mathbb{R}^{2d})$ to $\mathcal{HS}$. In particular, every $T \in \mathcal{HS}$ has a unique Weyl symbol $a \in L^2(\mathbb{R}^{2d})$ such that $T = L_a$.

An interesting property of the Weyl symbol is its interaction with the time-frequency shifts. In fact, we have by [203, Lem. 3.2] that

$$\pi(z)L_a\pi(z)^* = L_{T_z(a)},$$

where $T_z f(z') = f(z' - z)$ for functions f on $\mathbb{R}^{2d}$. Since $\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$, this means that for $a \in L^2(\mathbb{R}^{2d})$ we have

$$\|\mathfrak{V}_{L_a}\psi(z)\|_{L^2} = \|\pi(z)L_a\pi(z)^*\psi\|_{L^2} = \|L_{T_z(a)}\psi\|_{L^2}.$$

We may therefore reformulate Theorem F.5.1 in terms of the Weyl transform.

Theorem F.6.1. *Let* $0 \neq a \in L^2(\mathbb{R}^{2d})$ *such that* $(L_a)^* \in \mathcal{N}(L^2, M^1_v)$. *For any* $1 \leq p, q \leq \infty$ *and* v*-moderate weight* m*, we have*

$$\|\psi\|_{M^{p,q}_m} \asymp \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \|L_{T_{(x,\omega)}(a)}\psi\|^p_{L^2} m(x, \omega)^p\, dx \right)^{q/p} d\omega \right)^{\frac{1}{q}}.$$

The above theorem generalizes a result by Gröchenig and Toft in [143, Prop. 2.2], who showed that the the middle expression above defines an equivalent norm on $M^2_m(\mathbb{R}^d)$, i.e.

$$\|\psi\|^2_{M^2_m} \asymp \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|L_{T_{(x,\omega)}(a)}\psi\|^2_{L^2} m(x, \omega)^2\, dx d\omega, \quad \text{(F.6.2)}$$

under the assumptions that m is of polynomial growth and a is a Schwartz function (stronger conditions are stated in [143], but their proof uses only that $a \in \mathcal{S}(\mathbb{R}^d)$).

In fact, it is shown in [143] that the space of $\psi \in \mathcal{S}'(\mathbb{R}^d)$ such that the right hand side of (F.6.2) is finite coincides with a space $H(m,g)$ introduced by Bony and Chemin [51, Def. 5.1] when g is the standard Euclidean metric on $\mathbb{R}^{2d}$. Hence (F.6.2) states that $H(m,g) = M^2_m(\mathbb{R}^d)$ with equivalent norms.

Theorem F.6.1 extends (F.6.2) in several directions. It extends from $p = q = 2$ to any $1 \leq p, q \leq \infty$ and from polynomial weights to general v-moderate weights. Our requirements on the Weyl symbol a are also weaker, although this is slightly obscured by the mysterious requirement that $(L_a)^* \in \mathcal{N}(L^2, M^1_v)$. By Proposition F.3.3 the condition $S^* \in \mathcal{N}(L^2, M^1_v)$ means that the integral kernel k_S belongs to the projective tensor product $L^2(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$, and the Weyl symbol a and k_S are related by [154]

$$k_S(x,y) = \int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, \omega\right) e^{2\pi i \omega\cdot(x-y)}\, d\omega. \tag{F.6.3}$$

Understanding the condition $(L_a)^* \in \mathcal{N}(L^2, M^1_v)$ thus boils down to understanding what assumptions we need on a to ensure that the kernel k_S in (F.6.3) belongs to $L^2(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$.

F.6.1 Polynomial weights

By restricting our attention to polynomial weights $v_s(z) = (1+|z|^2)^{s/2}$ for $s \geq 0$, we obtain some sufficient conditions for $(L_a)^* \in \mathcal{N}(L^2, M^1_{v_s})$, so that Theorem F.6.1 holds.

Example F.6.1 (Schwartz symbols)**.** If $v = v_s$ for $s \geq 0$, we know from Example F.3.1 that the Schwartz operators $\mathfrak{S}$, i.e. operators T with $k_T \in \mathcal{S}(\mathbb{R}^{2d})$, form a subspace of $\mathcal{N}(L^2, M^1_v)$. Furthermore, the space $\mathfrak{S}$ is closed under taking adjoints, and may equivalently be described as the Weyl operators L_a with $a \in \mathcal{S}(\mathbb{R}^{2d})$ [179]. Taken together, this means that $a \in \mathfrak{S}$ implies $(L_a)^* \in \mathfrak{S} \subset \mathcal{N}(L^2, M^1_{v_s})$. Thus Theorem F.6.1 applies for all Schwartz functions a.

We then prove a slightly more refined result. Below we denote by v_s^{4d} the weight function on $\mathbb{R}^{4d}$ given by $v_s^{4d}(z,\zeta) = (1+|z|^2+|\zeta|^2)^{s/2}$.

Proposition F.6.2. *If $a \in M^1_{v^{4d}_{2s}}(\mathbb{R}^{2d})$ for $s \geq 0$, then $(L_a)^* \in \mathcal{N}(L^2; M^1_{v_s})$. Hence Theorem F.6.1 applies with $v = v_s$.*

Proof. Recall from equation (F.3.8) that with $v_s\tilde{\otimes}v_s(x_1,x_2,\omega_1,\omega_2) = v_s(x_1,\omega_1)\cdot v_s(x_2,\omega_2)$, we have the equality $M^1_{v_s\tilde{\otimes}v_s}(\mathbb{R}^{2d}) = M^1_{v_s}(\mathbb{R}^d)\hat{\otimes}M^1_{v_s}(\mathbb{R}^d)$. One easily checks that $v_s\tilde{\otimes}v_s \lesssim v^{4d}_{2s}$, which implies by part b) of Proposition F.2.2 and Proposition F.3.3 that

$$M^1_{v^{4d}_{2s}}(\mathbb{R}^{2d}) \hookrightarrow M^1_{v_s\tilde{\otimes}v_s}(\mathbb{R}^{2d}) = M^1_{v_s}(\mathbb{R}^d)\hat{\otimes}M^1_{v_s}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)\hat{\otimes}M^1_{v_s}(\mathbb{R}^d).$$

By [154, Prop. 7.4.1], if $a \in M^1_{v_{2s}^{4d}}(\mathbb{R}^{2d})$ then the integral kernel k_{L_a} also satisfies $k_{L_a} \in M^1_{v_{2s}^{4d}}(\mathbb{R}^{2d})$. By the chain of inclusions above, it follows that $k_{L_a} \in L^2(\mathbb{R}^d)\hat{\otimes} M^1_{v_s}(\mathbb{R}^d)$, which implies $(L_a)^* \in \mathcal{N}(L^2, M^1_{v_s})$ by Proposition F.3.3. □

When $s = 0$ the condition above is rather weak, as $M^1(\mathbb{R}^{2d})$ even contains non-differentiable functions.

F.7 Cohen's class

Another interesting interpretation of Theorem F.5.1 is in terms of Cohen's class of time-frequency distributions introduced by Cohen in [59]. Typically the definition of the Cohen's class distribution Q_a associated with $a \in \mathcal{S}'(\mathbb{R}^{2d})$ is that [131]

$$Q_a(\psi) = a * W(\psi) \quad \text{for any } \psi \in \mathcal{S}(\mathbb{R}^d). \tag{F.7.1}$$

One can show that $\psi \in \mathcal{S}(\mathbb{R}^d)$ implies that $W(\psi) \in \mathcal{S}(\mathbb{R}^{2d})$, so (F.7.1) is well-defined as the convolution of a tempered distribution with a Schwartz function. All our examples will satisfy $a \in L^2(\mathbb{R}^{2d})$, and in this case $Q_a(\psi)$ is defined by (F.7.1) for any $\psi \in L^2(\mathbb{R}^d)$, as a slight modification of Moyal's identity gives that $W(\psi) \in L^2(\mathbb{R}^{2d})$, so (F.7.1) is well-defined by Young's inequality.

In [204] we have given an alternative description of Cohen's class. Given a Hilbert-Schmidt operator $T \in \mathcal{HS}$, we define the Cohen's class distribution Q_T associated with T by

$$Q_T(\psi)(z) = \langle T\pi(z)^*\psi, \pi(z)^*\psi\rangle_{L^2}. \tag{F.7.2}$$

Any Cohen class distribution Q_a for $a \in L^2(\mathbb{R}^{2d})$ can equivalently be described using (F.7.2), since it follows from [204, Prop. 7.1] that

$$Q_a(\psi) = Q_{L_{\check{a}}}(\psi),$$

where L denotes the Weyl transform and $\check{a}(z) = a(-z)$. From now on we will therefore write Cohen's class distributions in the form Q_T for $T \in \mathcal{HS}$ rather than using (F.7.1).

In light of (F.7.2) we clearly have the relation

$$\|\mathfrak{V}_S(\psi)(z)\|^2_{L^2} = \langle S\pi(z)^*\psi, S\pi(z)^*\psi\rangle_{L^2} = Q_{S^*S}(\psi)(z). \tag{F.7.3}$$

Hence $\|\mathfrak{V}_S(\psi)(z)\|^2_{L^2} = \sqrt{Q_{S^*S}(\psi)(z)}$, and we see that another reinterpretation of Theorem F.5.1 is the following.

Theorem F.7.1. *Let* $0 \neq S \in \mathcal{HS}$ *such that* $S^* \in \mathcal{N}(L^2, M^1_v)$*. For any* $1 \leq p, q \leq \infty$ *and v-moderate weight m, we have*

$$\|\psi\|_{M^{p,q}_m} \asymp \left\| \sqrt{Q_{S^*S}(\psi)} \right\|_{L^{p,q}_m(\mathbb{R}^{2d})}.$$

Example F.7.1 (Spectrograms)**.** To see why the square root appears in Theorem F.7.1, it is worth recalling the simple case of $S = \xi \otimes \phi$ for some $0 \neq \phi \in M^1_v(\mathbb{R}^d)$ and $\|\xi\|_{L^2} = 1$. Then $S^*S = \phi \otimes \phi$, and one may check that

$$Q_{S^*S}(\psi)(z) = |V_\phi \psi(z)|^2.$$

This is the so-called *spectrogram* of ψ with window ϕ, and we know from Theorem F.2.3 that $\|\psi\|_{M^{p,q}_m} \asymp \|V_\phi \psi\|_{L^{p,q}_m}$, hence we need the square root in Theorem F.7.1.

Remark F.6. We have skipped one technical detail in the Theorem F.7.1 above, namely how to interpret $Q_T(\psi)$ for $\psi \in M^\infty_{1/v}(\mathbb{R}^{2d})$. This is certainly not immediately covered by (F.7.1) or (F.7.2). We solve this issue by rewriting $Q_T(\psi)$ to

$$Q_T(\psi) = \langle \pi(z)^*\psi, T^*\pi(z)^*\psi \rangle_{L^2}$$

and then replacing the bracket by duality:

$$Q_T(\psi) := \langle \pi(z)^*\psi, T^*\pi(z)^*\psi \rangle_{M^\infty_{1/v}, M^1_v}. \tag{F.7.4}$$

This defines $Q_T(\psi)$ for $\psi \in M^\infty_{1/v}(\mathbb{R}^{2d})$ whenever T^* maps $M^\infty_{1/v}(\mathbb{R}^d)$ into $M^1_v(\mathbb{R}^d)$, which is true if $T = S^*S$ for $S^* \in \mathcal{N}(L^2, M^1_v)$ or if $k_T \in M^1_{v\otimes v}(\mathbb{R}^{2d})$, see [236, Prop. 4.1] for a proof. It is straightforward to check that (F.7.2) and (F.7.4) agree when $\psi \in L^2(\mathbb{R}^d)$, and that $Q_T(\psi)(z) = \|\mathfrak{B}_S(\psi)(z)\|^2_{L^2}$ when $T = S^*S$.

F.7.1 On positive Cohen class distributions

The reader will not fail to notice that the Cohen class distributions for which Theorem F.7.1 applies are of a particular kind, namely of the form Q_T for $T = S^*S$ with $S^* \in \mathcal{N}(L^2, M^1_v)$.

The condition $S^* \in \mathcal{N}(L^2, M^1_v)$ may be interpreted as requiring a certain time-frequency localization for Q_{S^*S}, as one can show that $S^* \in \mathcal{N}(L^2, M^1_v)$ implies that the integral kernel k_{S^*S} belongs to $M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$. If $S = \xi \otimes \phi$ for $\|\xi\|_{L^2} = 1$, which we know from Example F.5.1 corresponds to choosing the window ϕ for the modulation spaces, then $S^*S = \phi\otimes\phi$, which has integral kernel in $M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$ precisely when $\phi \in M^1_v(\mathbb{R}^d)$. Hence requiring $S^* \in \mathcal{N}(L^2, M^1_v)$ seems like a natural generalization of the assumption in Theorem F.2.3 that windows ϕ for modulation spaces need to satisfy $\phi \in M^1_v(\mathbb{R}^d)$.

In addition, the fact that $T = S^*S$ means that T is a positive operator. By [204, Prop. 7.3], this is equivalent to $Q_T(\psi)$ being a positive function for each $\psi \in L^2(\mathbb{R}^d)$. This assumption cannot simply be replaced by considering $|Q_T(\psi)|$, as the following example shows.

Example F.7.2. Let ϕ_1 and ϕ_2 be compactly supported functions in $\mathcal{S}(\mathbb{R}^d)$ such that their supports do not overlap. Define $T = \phi_1 \otimes \phi_2$. Then the integral kernel (or equivalently the Weyl symbol) of T belongs to $\mathcal{S}(\mathbb{R}^{2d})$, and has good time-frequency localization in this sense. However, T is not a positive operator as $\phi_1 \neq \phi_2$, and Theorem F.7.1 fails when replacing Q_{S^*S} by $|Q_T|$: for instance, one easily finds using that (F.7.4) that when δ is the Dirac delta distribution

$$Q_T(\delta)(z) = \phi_1(x)\overline{\phi_2(x)} \equiv 0.$$

An obvious question is whether the positivity and good time-frequency properties exhibited by Q_{S^*S} when $S^* \in \mathcal{N}(L^2, M^1_v)$ are sufficient for Theorem F.7.1 to hold:

> If $T \in \mathcal{HS}$ has integral kernel in $M^1_v(\mathbb{R}^d)\hat{\otimes}M^1_v(\mathbb{R}^d)$ and is a positive operator on $L^2(\mathbb{R}^d)$, does a version of Theorem F.7.1 hold with Q_{S^*S} replaced by Q_T?

As a first step in this direction, we note that the statement is true if $T \in \mathfrak{S}$, i.e. if $k_T \in \mathcal{S}(\mathbb{R}^{2d})$, as [179, Prop. 3.15] states that if $T \in \mathfrak{S}$ is positive, then $\sqrt{T} \in \mathfrak{S}$.

Theorem F.7.2. *Let $T \in \mathfrak{S}$ be a positive operator, and assume that v grows at most polynomially. Then, for any $1 \leq p, q \leq \infty$ and v-moderate weight m, we have*

$$\|\psi\|_{M^{p,q}_m} \asymp \left\| \sqrt{Q_T(\psi)} \right\|_{L^{p,q}_m}.$$

Proof. As noted, $S := \sqrt{T} \in \mathfrak{S}$. Then $T = S^*S$, and we saw in Example F.3.1 that $S \in \mathfrak{S}$ implies that $S \in \mathcal{N}(L^2, M^1_v)$ under the assumption that v grows at most polynomially. The result therefore follows by Theorem F.7.1. □

This theorem can also be formulated using the classic definition (F.7.1) of Cohen's class. In this formulation it states that if $a \in \mathcal{S}(\mathbb{R}^{2d})$ and $Q_a(\psi)$ is a positive function for each $\psi \in L^2(\mathbb{R}^d)$, then $\|\psi\|_{M^{p,q}_m} \asymp \|\sqrt{Q_a(\psi)}\|_{L^{p,q}_m}$.

A question for further research is then if the same holds for $M^1_{v\otimes v}(\mathbb{R}^{2d})$: if T is a positive operator with $k_T \in M^1_{v\otimes v}(\mathbb{R}^{2d})$, what can we say about $k_{\sqrt{T}}$?

F.8 Examples: Localization operators

We now return to the localization operators considered in Section F.3.3 by choosing $S = \mathcal{A}^{\varphi_1,\varphi_2}_f$ with $\varphi_1 \in M^1_v(\mathbb{R}^d)$, $\varphi_2 \in L^2(\mathbb{R}^d)$ and $f \in L^1_v(\mathbb{R}^{2d})$. Then $S^* \in$

$\mathcal{N}(L^2, M^1_v)$ by Corollary F.3.4.1. To apply Theorem F.5.1 to this example, we first note that a calculation gives

$$\pi(z)\mathcal{A}_f^{\varphi_1,\varphi_2}\pi(z)^* = \mathcal{A}_{T_z(f)}^{\varphi_1,\varphi_2},$$

i.e. conjugating the localization operator by $\pi(z)$ amounts to translating f by z. As we saw in Section F.6 we also have by the unitarity of $\pi(z)$ that

$$\|\mathfrak{V}_{\mathcal{A}_f^{\varphi_1,\varphi_2}}\psi(z)\|_{L^2} = \|\pi(z)\mathcal{A}_f^{\varphi_1,\varphi_2}\pi(z)^*\psi\|_{L^2} = \|\mathcal{A}_{T_z(f)}^{\varphi_1,\varphi_2}(\psi)\|_{L^2}, \tag{F.8.1}$$

hence we obtain the following from Theorem F.5.1.

Theorem F.8.1. *Assume that $\varphi_1 \in M^1_v(\mathbb{R}^d)$, $\varphi_2 \in L^2(\mathbb{R}^d)$ and $f \in L^1_v(\mathbb{R}^{2d})$. For any $1 \leq p, q \leq \infty$ and v-moderate weight m, we have*

$$\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left\|\mathcal{A}_{T_{(x,\omega)}(f)}^{\varphi_1,\varphi_2}(\psi)\right\|_{L^2}^p m(x,\omega)^p\,dx\right)^{q/p}\,d\omega\right)^{1/q} \asymp \|\psi\|_{M^{p,q}_m},$$

where the integrals are replaced by supremums if $p = \infty$ or $q = \infty$.

In light of (F.7.3) and (F.8.1), we know that

$$\left\|\mathcal{A}_{T_{(x,\omega)}(f)}^{\varphi_1,\varphi_2}(\psi)\right\|_{L^2}^2 = Q_T(\psi)$$

where $T = \left(\mathcal{A}_f^{\varphi_1,\varphi_2}\right)^* \mathcal{A}_f^{\varphi_1,\varphi_2} = \mathcal{A}_{\overline{f}}^{\varphi_2,\varphi_1}\mathcal{A}_f^{\varphi_1,\varphi_2}$. In this sense Theorem F.8.1 concerns the study of a particular kind of Cohen's class distribution.

Remark F.7. We mention that there is another line of research that leads to equivalent norms for modulation spaces in terms of localization operators, see [1, 50, 143, 144]. In this approach one considers weights function m, m_0 and shows that under various conditions on m the localization operator $\mathcal{A}_m^{\varphi,\varphi}$ is an isomorphism from $M^{p,q}_{m_0}(\mathbb{R}^d)$ to $M^{p,q}_{m_0/m}(\mathbb{R}^d)$. This implies a norm equivalence $\|\psi\|_{M^{p,q}_{m_0}} \asymp \|\mathcal{A}_m^{\varphi,\varphi}\psi\|_{M^{p,q}_{m_0/m}}$, which is of a different nature than the one we consider.

F.8.1 Modulation spaces as time-frequency Wiener amalgam spaces

A consequence of Theorem F.8.1 is that we may interpret modulation spaces as a time-frequency version of the so-called *Wiener amalgam spaces* [96]; a class of function function spaces that have been closely tied to the development of modulation spaces since the inception of the latter in [96]. To explain this interpretation, we start by considering for $\varphi \in M^1_v(\mathbb{R}^d)$ and $f \in L^1_v(\mathbb{R}^{2d})$ the localization operator

$$\mathcal{A}_f^{\varphi,\varphi}(\psi) = \int_{\mathbb{R}^{2d}} f(z)V_\varphi\psi(z)\pi(z)\varphi\,dz. \tag{F.8.2}$$

In time-frequency analysis, when φ is well-localized in time and frequency such as the Gaussian, the size of $|V_\varphi\psi(x,\omega)|$ is interpreted as a measure of the contribution of the frequency ω at time x of the signal ψ. By the reconstruction formula

$$\psi = \frac{1}{\|\varphi\|_{L^2}^2} \int_{\mathbb{R}^{2d}} V_\varphi\psi(z)\pi(z)\varphi \, dz \tag{F.8.3}$$

we can recover ψ from $V_\varphi\psi$, and (F.8.2) finds a natural interpretation as a multiplication operator in the time-frequency plane: we represent ψ in the time-frequency plane by forming $V_\varphi\psi$, but before we reconstruct ψ from $V_\varphi\psi$ we multiply it by $f(z)$. A particular choice of f is to let f be the characteristic function χ_Ω for some compact subset Ω. Then

$$\mathcal{A}^{\varphi,\varphi}_{T_z(\chi_\Omega)}(\psi) = \int_{\mathbb{R}^{2d}} \chi_{z+\Omega}(z')V_\varphi\psi(z')\pi(z')\psi \, dz',$$

which in light of (F.8.3) may be interpreted as saying that $\mathcal{A}^{\varphi,\varphi}_{T_z(\chi_\Omega)}$ picks out the component of ψ localized in $z+\Omega := \{z+z' : z' \in \Omega\}$ in the time-frequency plane. Theorem F.8.1 says that an equivalent norm on $M^{p,q}_m(\mathbb{R}^d)$ is given by first measuring the *local* size of ψ near z in the time-frequency plane by $\|\mathcal{A}^{\varphi,\varphi}_{T_z(\chi_\Omega)}\psi\|_{L^2}$, and then measuring the *global* properties of ψ by taking the $L^{p,q}_m$ norm.

When $p=q$, this parallels the definition of the Wiener amalgam space $W(L^2, L^p_w)$ with local component L^2 and global component L^p_w. For a fixed, compact domain $Q \subset \mathbb{R}^d$, the Wiener amalgam space $W(L^2, L^p_w)$ for $1 \le p \le \infty$ and a weight function w on $\mathbb{R}^d$ consists of all functions $\psi : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|\psi\|_{W(L^2,L^{p,q}_m)} := \left(\int_{\mathbb{R}^d} \left\|\chi_{x+Q}\cdot\psi\right\|^p_{L^2} w(x)^p \, dx\right)^{1/p}.$$

Since we interpret $\mathcal{A}^{\varphi,\varphi}_{T_z(\chi_\Omega)}(\psi)$ as ψ localized to $z+\Omega$ in the time-frequency plane and $\chi_{x+Q}\cdot\psi$ is the localization of ψ to $x+Q$ in time, Theorem F.8.1 says that modulation spaces are the natural analogues of Wiener amalgam spaces when we localize ψ in the time-frequency plane using $\mathcal{A}^{\varphi,\varphi}_{\chi_\Omega}$, not just in time by multiplying with χ_Q. We have merely scratched the surface of Wiener amalgam spaces, and the interested reader should consult the survey [153]. However, it is worth noting that when the cutoff-function χ_Q is replaced by a smooth cutoff-function ϕ satisfying $\sum_{\ell\in\mathbb{Z}^d} T_\ell(\phi) \equiv 1$, then an equivalent norm on $W(L^2, L^p_m)$ is given by

$$\left(\sum_{\ell\in\mathbb{Z}^d} \|T_\ell(\phi)\cdot\psi\|^p_{L^2} w(\ell)^p\right)^{1/p}.$$

The fact that modulation spaces have a similar discrete description has already been shown by Dörfler, Feichtinger and Gröchenig in [85, 87] (also more generally by

Romero [225]): if $f \in L^1_v(\mathbb{R}^{2d})$ satisfies

$$\sum_{(j,k)\in\mathbb{Z}^{2d}} T_{(j,k)}(f) \asymp 1,$$

then an equivalent norm on $M^{p,q}_m(\mathbb{R}^d)$ is given by

$$\left(\sum_{k\in\mathbb{Z}^d}\left(\sum_{j\in\mathbb{Z}^d}\|\mathcal{A}^{\varphi,\varphi}_{T_{(j,k)}(f)}(\psi)\|^p_{L^2} m(j,k)^p\right)^{p/q}\right)^{1/q}. \tag{F.8.4}$$

The interpretation of modulation spaces as Wiener amalgam spaces in the time-frequency plane does of course also follow from these earlier results, but we include it here as the author was not able to locate an explicit formulation of this insight in the literature. Finally, we remark that the local component L^2 in $W(L^2, L^p_w)$ can be replaced by several other function spaces X to obtain new Wiener amalgam spaces $W(X, L^p_m)$. One might therefore naturally replace the L^2-norm in Theorem F.8.1 or Theorem F.5.1 by another function space norm and investigate the resulting function spaces, and Romero [225, Thm. 7] has shown that (F.8.4) still defines an equivalent norm on $M^{p,q}_m(\mathbb{R}^d)$ when the L^2-norm is replaced by the norm of any unweighted modulation space $M^{p_0,q_0}(\mathbb{R}^d)$.

F.8.2 Smoothing spectrograms

So far in this section we have picked S to be a localization operator $\mathcal{A}^{\varphi_1,\varphi_2}_f$, which corresponds to studying the Cohen's class distribution Q_T for $T = \mathcal{A}^{\varphi_2,\varphi_1}_{\overline{f}}\mathcal{A}^{\varphi_1,\varphi_2}_f$. However, we may also proceed as in Section F.7.1 and study the Cohen class distribution Q_T for $T = \mathcal{A}^{\varphi,\varphi}_f$, where $f \in L^1_v(\mathbb{R}^{2d})$ is a non-negative function and $\psi \in M^1_v(\mathbb{R}^d)$. The fact that f is non-negative implies that T is a positive operator, and it is not difficult to show that

$$Q_{\mathcal{A}^{\varphi,\varphi}_f}(\psi)(z) = f * |V_\varphi\psi(z)|^2(z),$$

i.e. the Cohen class of $\mathcal{A}^{\varphi,\varphi}_f$ is a smoothed spectrogram. Theorem F.7.1 says that if $\mathcal{A}^{\varphi,\varphi}_f = S^*S$ for some $S^* \in \mathcal{N}(L^2, M^1_v)$, then

$$\|\psi\|_{M^{p,q}_m} \asymp \left\|\sqrt{f * |V_\varphi\psi|^2}\right\|_{L^{p,q}_m(\mathbb{R}^{2d})}.$$

As we discussed in Section F.7.1, the existence of such S is not clear in general, but if $\mathcal{A}^{\varphi,\varphi}_f \in \mathfrak{S}$ we can use Theorem F.7.2 to deduce the following result.

Proposition F.8.2. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and let $f \in L^1(\mathbb{R}^{2d})$ be a positive function of compact support. If v grows at most polynomially and m is v-moderate, then*

$$\|\psi\|_{M_m^{p,q}} \asymp \left\| \sqrt{f * |V_\varphi \psi|^2} \right\|_{L_m^{p,q}(\mathbb{R}^{2d})}.$$

Proof. The Weyl symbol of $\mathcal{A}_f^{\varphi,\varphi}$ is the function $f * W(\varphi)$, see for instance [48, Lem. 2.4]. As $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it follows by [131, Lem. 14.5.1] that $W(\phi) \in \mathcal{S}(\mathbb{R}^{2d})$. Hence the assumptions on f imply that $f * W(\varphi) \in \mathcal{S}(\mathbb{R}^{2d})$, which means that $\mathcal{A}_f^{\varphi,\varphi} \in \mathfrak{S}$. The result therefore follows by Theorem F.7.2. □

F.9 Appendix: Proof of Proposition F.3.5

Proof of Proposition F.3.5. First recall from Section F.6 that $\mathcal{B}$ consists precisely of those $T \in \mathcal{HS}$ such that the Weyl symbol a_T belongs to $M^1(\mathbb{R}^{2d})$. Then recall that we assume

$$T = \int_{\mathbb{R}^{2d}} F(x,\omega) e^{-i\pi x \cdot \omega} \pi(x,\omega)\, dx d\omega,$$

where $F(x,\omega) \in L^2(\mathbb{R}^{2d})$ has compact support, say $\operatorname{supp}(F) \subset K$ for $K \subset \mathbb{R}^{2d}$ compact. As in [203], we denote the function F by $\mathcal{F}_W(T)$ – it plays the role of a Fourier transform of the operator T in quantum harmonic analysis.

One can show that

$$\mathcal{F}_W(T) = \mathcal{F}_\sigma(a_T),$$

where $\mathcal{F}_\sigma(f)$ is the *symplectic Fourier transform* of $f \in L^1(\mathbb{R}^{2d})$ given by

$$\mathcal{F}_\sigma f(x,\omega) = \int_{\mathbb{R}^{2d}} f(x',\omega') e^{-2\pi i (x' \cdot \omega - x \cdot \omega')}\, dx' d\omega' \quad \text{for } x, x', \omega, \omega' \in \mathbb{R}^d.$$

Then fix some $R \in \mathcal{B}$ such that $\mathcal{F}_W(R)$ has no zeros, an explicit example is $R = \varphi_0 \otimes \varphi_0$ [203, Ex. 6.1]. As $R \in \mathcal{B}$, we have $a_R = \mathcal{F}_\sigma \mathcal{F}_W(R) \in M^1(\mathbb{R}^{2d})$.

Since $a_R \in L^1(\mathbb{R}^{2d})$ and $\mathcal{F}_\sigma(a_R) = \mathcal{F}_W(R)$ never vanishes, the Wiener-Lévy theorem [221, Thm. 3.1] implies the existence of some $h \in L^1(\mathbb{R}^{2d})$ such that

$$\mathcal{F}_\sigma(h)(z) = \frac{1}{\mathcal{F}_\sigma(a_R)} = \frac{1}{\mathcal{F}_W(R)(z)} \quad \text{for } z \in K.$$

Then define the operator

$$T' = (h * a_R) \star T,$$

where $\star$ is the operation from (F.3.9). The "Fourier transform" $\mathcal{F}_W$ interacts with the convolutions in the expected way [203, Prop. 6.4]; more precisely, we have that

$$\mathcal{F}_W(T') = \mathcal{F}_\sigma(h) \mathcal{F}_W(R) \mathcal{F}_W(T) = \mathcal{F}_W(T)$$

by construction of h. As $\mathcal{F}_W$ is injective, see [102, Cor. 7.6.3], it follows that $T = T'$.

On the other hand, the function $b := h * a_R$ belongs to $M^1(\mathbb{R}^{2d})$ since $L^1(\mathbb{R}^{2d}) * M^1(\mathbb{R}^{2d}) \subset M^1(\mathbb{R}^{2d})$ by [131, Prop. 12.1.7]. The Weyl symbol of $T = T' = b \star T$ is given by $a_T = b * a_T$ [204, Prop. 5.2]. Since $b \in M^1(\mathbb{R}^{2d})$ and $T \in \mathcal{S}^1$, [203, Thm. 8.1] implies[1] that $a_T = b * a_T \in M^1(\mathbb{R}^{2d})$, hence $T \in \mathcal{B}$. □

Acknowledgements

The author acknowledges helpful feedback from Franz Luef on various drafts of the paper.

[1]The theorem states that if $S \in \mathcal{B}$ and $T \in \mathcal{S}^1$, then $S \star T \in M^1(\mathbb{R}^{2d})$. $S \star T$ is just another notation for $a_S * a_T$, so our result follows by picking $S = L_b$.

Paper G

Affine Quantum Harmonic Analysis

Eirik Berge, Stine Marie Berge, Franz Luef and Eirik Skrettingland

Preprint

Paper G

Affine Quantum Harmonic Analysis

Abstract

We develop a quantum harmonic analysis framework for the affine group. This encapsulates several examples in the literature such as affine localization operators, covariant integral quantizations, and affine quadratic time-frequency representations. In the process, we develop a notion of admissibility for operators and extend well known results to the operator setting. A major theme of the paper is the interaction between operator convolutions, affine Weyl quantization, and admissibility.

G.1 Introduction

The affine group and the Heisenberg group play prominent roles in wavelet theory and Gabor analysis, respectively. As is well-known, the representation theory of the Heisenberg group is intrinsically linked to quantization on phase space $\mathbb{R}^{2n}$. Similarly, the relation between quantization schemes on the affine group and its representation theory has received some attention and several schemes have been proposed, e.g. [39, 122, 125]. However, there are still many open questions awaiting a definite answer in the case of the affine group.

As has been shown by two of the authors in [203], the theory of *quantum harmonic analysis on phase space* introduced by Werner [251] provides a coherent framework for many aspects of quantization and Gabor analysis associated with the Heisenberg group. Based on this connection, advances in the understanding of time-frequency analysis have been made [204–206]. In this paper we aim to develop a variant of Werner's quantum harmonic analysis in [251] for time-scale analysis. This is based on unitary representations of the affine group in a similar way to the Schrödinger representation of the Heisenberg group being used in Werner's framework. We will refer to this theory on the affine group as *affine quantum harmonic analysis*.

Affine Operator Convolutions

In Werner's quantum harmonic analysis on phase space, a crucial component is extending convolutions to operators. Recall that the affine group Aff has the underlying set $\mathbb{R} \times \mathbb{R}_+$ and group operation modeling composition of affine transformations. A key feature of this group is that the left Haar measure $a^{-2} dx\, da$ and the right Haar measure $a^{-1} dx\, da$ are not equal, making the group *non-unimodular*. Both measures play a role in affine quantum harmonic analysis, making the theory more involved than the case of the Heisenberg group. In addition to the standard function (right-)convolution on the affine group

$$f *_{\mathrm{Aff}} g(x,a) := \int_{\mathrm{Aff}} f(y,b) g((x,a) \cdot (y,b)^{-1}) \, \frac{dy\, db}{b},$$

we introduce the following *operator convolutions* for operators on $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\, dr)$ in Section G.3:

- Let $f \in L^1_r(\mathrm{Aff}) := L^1(\mathrm{Aff}, a^{-1} dx\, da)$ and let S be a trace-class operator on $L^2(\mathbb{R}_+)$. We define the *convolution* $f \star_{\mathrm{Aff}} S$ between f and S to be the operator on $L^2(\mathbb{R}_+)$ given by

$$f \star_{\mathrm{Aff}} S := \int_{\mathrm{Aff}} f(x,a) U(-x,a)^* S U(-x,a) \, \frac{dx\, da}{a},$$

  where U is the unitary representation of Aff on $L^2(\mathbb{R}_+)$ given by

$$U(x,a)\psi(r) := e^{2\pi i x r} \psi(ar).$$

- Let S be a trace-class operator and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we define the *convolution* $S \star_{\mathrm{Aff}} T$ between S and T to be the function on Aff given by

$$S \star_{\mathrm{Aff}} T(x,a) := \mathrm{tr}(S U(-x,a)^* T U(-x,a)).$$

The three convolutions are compatible in the following sense: Let $f, g \in L^1_r(\mathrm{Aff})$ and denote by S a trace-class operator and by T a bounded operator, both on $L^2(\mathbb{R}_+)$. Then

$$(f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T = f *_{\mathrm{Aff}} (S \star_{\mathrm{Aff}} T),$$
$$f \star_{\mathrm{Aff}} (g \star_{\mathrm{Aff}} S) = (f *_{\mathrm{Aff}} g) \star_{\mathrm{Aff}} S.$$

Interplay Between Affine Weyl Quantization and Convolutions

Integral to the theory in this paper is the affine Wigner distribution and the associated affine Weyl quantization. The *affine (cross-)Wigner distribution* $W_{\mathrm{Aff}}^{\psi,\phi}$ of $\phi, \psi \in L^2(\mathbb{R}_+)$ is the function on Aff given by

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \int_{-\infty}^{\infty} \psi\left(\frac{aue^u}{e^u - 1}\right)\overline{\phi\left(\frac{au}{e^u - 1}\right)} e^{-2\pi i x u}\, du. \tag{G.1.1}$$

Although at first glance the definition (G.1.1) might look unnatural, it can be motivated through the representation theory of the affine group as illustrated in [14]. We will elaborate on this viewpoint in Section G.5. One defines the *affine Weyl quantization* of $f \in L_r^2(\mathrm{Aff}) := L^2(\mathrm{Aff}, a^{-1} dx\, da)$ as the operator A_f given by

$$\left\langle A_f \phi, \psi \right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\psi,\phi} \right\rangle_{L_r^2(\mathrm{Aff})}, \qquad \text{for all } \phi, \psi \in L^2(\mathbb{R}_+).$$

We will explore the intimate relation between the convolutions and the affine Weyl quantization. The following theorem, being a combination of Proposition G.3.4 and Proposition G.3.5, highlights this relation.

Theorem 1. *Let $f, g \in L_r^2(\mathrm{Aff})$, where g is additionally in $L_r^1(\mathrm{Aff})$ and square integrable with respect to the left Haar measure. Then*

$$\begin{aligned} g \star_{\mathrm{Aff}} A_f &= A_{g *_{\mathrm{Aff}} f}, \\ A_g \star_{\mathrm{Aff}} A_f &= f *_{\mathrm{Aff}} \check{g}, \end{aligned}$$

where $\check{g}(x,a) := g((x,a)^{-1})$.

We will exploit the previous theorem to define the affine Weyl quantization of tempered distributions in Section G.3.3. To do this rigorously, we will utilize a Schwartz space $\mathcal{S}(\mathrm{Aff})$ on the affine group introduced in [39]. An important example we prove in Theorem G.3.7 is the affine Weyl quantization of the coordinate functions:

Theorem 2. *Let $f_x(x,a) := x$ and $f_a(x,a) := a$ be the coordinate functions on* Aff. *The affine Weyl quantizations A_{f_x} and A_{f_a} satisfy the commutation relation*

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i} A_{f_a}.$$

This is, up to re-normalization, the infinitesimal structure of the affine group.

We define *affine parity operator* P_{Aff} as

$$P_{\mathrm{Aff}} = A_{\delta_{(0,1)}},$$

where $\delta_{(0,1)}$ denotes the Dirac distribution at the identity element $(0,1) \in \mathrm{Aff}$. The following result, which will be rigorously stated in Section G.3.5, builds on these definitions.

Theorem 3. *The affine Weyl quantization A_g of $g \in \mathcal{S}(\mathrm{Aff})$ can be written as*

$$A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Moreover, for ϕ, ψ such that $\phi(e^x), \psi(e^x) \in \mathcal{S}(\mathbb{R})$, the affine Weyl symbol $W_{\mathrm{Aff}}^{\psi,\phi}$ of the rank-one operator $\psi \otimes \phi$ can be written as

$$W_{\mathrm{Aff}}^{\psi,\phi} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Operator Admissibility

One of the key features of representations of non-unimodular groups is the concept of admissibility. Recall that the *Duflo-Moore operator* $\mathcal{D}^{-1}$ corresponding to the representation U is the densely defined positive operator on $L^2(\mathbb{R}_+)$ given by $\mathcal{D}^{-1}\psi(r) = r^{-1/2}\psi(r)$. We will often use that $\mathcal{D}^{-1}$ has a densely defined inverse given by $\mathcal{D}\psi(r) = r^{1/2}\psi(r)$. A function ψ is said to be an *admissible wavelet* if $\psi \in \mathrm{dom}(\mathcal{D}^{-1})$. It is well known [90] that admissible wavelets satisfy the orthogonality relation

$$\int_{\mathrm{Aff}} |\langle \phi, U(-x,a)^*\psi \rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\,da}{a} = \|\phi\|_{L^2(\mathbb{R}_+)}^2 \|\mathcal{D}^{-1}\psi\|_{L^2(\mathbb{R}_+)}^2. \tag{G.1.2}$$

We extend the definition of admissibility to operators as follows:

Definition. Let S be a non-zero bounded operator on $L^2(\mathbb{R}_+)$ that maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$. We say that S is *admissible* if the composition $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is bounded on $\mathrm{dom}(\mathcal{D}^{-1})$ and extends to a trace-class operator $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ on $L^2(\mathbb{R}_+)$.

Note that the rank-one operator $S = \psi \otimes \psi$ for $\psi \in L^2(\mathbb{R}_+)$ is admissible precisely when ψ is an admissible wavelet. In Section G.4.2 we show that a large class of admissible operators can be constructed from Laguerre bases. The following result, which we prove in Corollary G.4.2.1, is motivated by [251, Lemma 3.1] and extends (G.1.2) to the operator setting.

Theorem 4. *Let S be an admissible operator on $L^2(\mathbb{R}_+)$. For any trace-class operator T on $L^2(\mathbb{R}_+)$, we have that $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$ with*

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \mathrm{tr}(T)\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

Determining whether an operator is admissible or not can be a daunting task. We managed in Corollary G.4.3.1 to find an elegant characterization in terms of operator convolutions of admissible operators that are additionally positive trace-class operators.

Theorem 5. *Let S be a non-zero, positive trace-class operator. Then S is admissible if and only if $S \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$.*

The following result is derived in Section G.4.4 and uses the affine Weyl quantization to show that admissibility is an operator manifestation of the non-unimodularity of the affine group.

Theorem 6.

- *Let $f \in L^1_r(\mathrm{Aff})$ be such that A_f is a trace-class operator on $L^2(\mathbb{R}_+)$. Then*

$$\mathrm{tr}(A_f) = \int_{\mathrm{Aff}} f(x,a)\,\frac{dx\,da}{a}.$$

- *Let $g \in L^1_l(\mathrm{Aff}) := L^1(\mathrm{Aff}, a^{-2}dx\,da)$ such that A_g is an admissible Hilbert-Schmidt operator. Then*

$$\mathrm{tr}\left(\mathcal{D}^{-1}A_g\mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} g(x,a)\,\frac{dx\,da}{a^2}.$$

Relationship with Fourier Transforms

For completeness, we will also investigate how notions of Fourier transforms on the affine group fit into the theory, and use known results from abstract harmonic analysis to explore the relationship between affine Weyl quantization and affine Fourier transforms. Recall that the *integrated representation* $U(f)$ of $f \in L^1_l(\mathrm{Aff})$ is the operator on $L^2(\mathbb{R}_+)$ given by

$$U(f)\psi := \int_{\mathrm{Aff}} f(x,a)U(x,a)\psi\,\frac{dx\,da}{a^2}, \qquad \psi \in L^2(\mathbb{R}_+).$$

We define the following operator Fourier transform in the affine setting.

Definition. The *affine Fourier-Wigner transform* is the isometry $\mathcal{F}_W$ sending a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ to a function in $L^2_r(\mathrm{Aff})$ such that

$$\mathcal{F}_W^{-1}(f) = U(\check{f}) \circ \mathcal{D}, \qquad f \in \mathrm{Im}(\mathcal{F}_W) \cap L^1_r(\mathrm{Aff}).$$

The following result is proved in Proposition G.5.3 and provides a connection between the affine Fourier-Wigner transform and admissibility.

Theorem 7. *Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. The following are equivalent:*

1. $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$.

2. $A\mathcal{D}^{-1}$ *extends from* $\mathrm{dom}(\mathcal{D}^{-1})$ *to a Hilbert-Schmidt operator on* $L^2(\mathbb{R}_+)$.

3. A^*A *is admissible.*

Another Fourier transform of interest is the (modified) *Fourier-Kirillov transform* on the affine group $\mathcal{F}_{\mathrm{KO}}$ given by

$$(\mathcal{F}_{\mathrm{KO}}f)(x,a) = \sqrt{a}\int_{\mathbb{R}^2} f\left(\frac{v}{\lambda(-u)}, e^u\right) e^{-2\pi i(xu+av)}\,\frac{du\,dv}{\sqrt{\lambda(-u)}}, \qquad f \in \mathrm{Im}(\mathcal{F}_W).$$

As in quantum harmonic analysis on phase space, we have that the affine Weyl quantization is the composition of these Fourier transforms, see Proposition G.5.4. In the affine setting we have in general that

$$\mathcal{F}_W(f \star_{\mathrm{Aff}} S) \neq \mathcal{F}_{KO}(f)\mathcal{F}_W(S), \qquad \mathcal{F}_{KO}(S \star_{\mathrm{Aff}} T) \neq \mathcal{F}_W(S)\mathcal{F}_W(T).$$

This contrasts the analogous result in Werner's original quantum harmonic analysis, see (G.5.6). In spite of this, not all properties typically associated with the Fourier transform are lost: In Section G.5.2 we prove a quantum Bochner theorem in the affine setting.

Main Applications

In Section G.6 we show that affine quantum harmonic analysis provides a conceptual framework for the study of *covariant integral quantizations* and a version of the *Cohen class* for the affine group. In addition, we show in Section G.6.1 that if S is a rank-one operator, then the study of operators $f \star_{\mathrm{Aff}} S$ for functions f on Aff reduces to the study of time-scale localization operators [71].

We have seen that affine Weyl quantization is given by $f \mapsto f \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ for $f \in \mathcal{S}(\mathrm{Aff})$. Inspired by this, we consider a whole class of quantization procedures: For any suitably nice operator S on $L^2(\mathbb{R}_+)$ we define a quantization procedure Γ_S for functions f on Aff by

$$\Gamma_S(f) := f \star_{\mathrm{Aff}} S.$$

This class of quantization procedures coincides with the *covariant integral quantizations* studied by Gazeau and his collaborators motivated by applications in physics, see e.g. [123–125]. Our results on affine quantum harmonic analysis are therefore also results on covariant integral quantizations. In particular, the abstract notion of admissibility of an operator S implies that Γ_S satisfies the simple property

$$\Gamma_S(1) = c \cdot I_{L^2(\mathbb{R}_+)},$$

where c is some constant, $I_{L^2(\mathbb{R}_+)}$ is the identity operator on $L^2(\mathbb{R}_+)$ and $1(x,a) = 1$ for all $(x,a) \in \mathrm{Aff}$.

As the name suggests, covariant integral quantizations Γ_S satisfy a *covariance* property, namely

$$U(-x,a)^*\Gamma_S(f)U(-x,a) = \Gamma_S(R_{(x,a)^{-1}}f),$$

where R denotes right translations of functions on Aff. In Theorem G.6.3 we point out that, by a known result on covariant positive operator valued measures [55, 181], this covariance assumption together with other mild assumptions completely characterize the covariant integral quantizations.

We have also seen that the affine cross-Wigner distribution is given for sufficiently nice ψ, ϕ by $W_{\mathrm{Aff}}^{\psi,\phi} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$. Inspired by this and the description in [204] of the Cohen class on $\mathbb{R}^{2n}$, we make the following definition.

Definition. A bilinear map $Q : L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ belongs to the *affine Cohen class* if $Q = Q_S$ for some operator S on $L^2(\mathbb{R}_+)$, where

$$Q_S(\psi,\phi)(x,a) \coloneqq (\psi \otimes \phi) \star_{\mathrm{Aff}} S(x,a) = \langle SU(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)}.$$

We will show how properties of S (such as admissibility) influence properties of Q_S, and obtain an abstract characterization of the affine Cohen class. Readers familiar with the Cohen class on $\mathbb{R}^{2n}$ [59] will know that it is defined by convolutions with the Wigner function. In the affine setting, we have the analogous result

$$Q_{A_f}(\psi,\phi) = W_{\mathrm{Aff}}^{\psi,\phi} *_{\mathrm{Aff}} \check{f}.$$

As we explain in Proposition G.6.8, the *affine class of quadratic time-frequency representations* in [212] may be identified with a subclass of the affine Cohen class.

Structure of the Paper

In Section G.2 we recall necessary background material for completeness. In particular, Section G.2.2 should serve as a brief reference for quantum harmonic analysis on phase space. We define affine operator convolution in Section G.3.1 and show the relationship with the affine Weyl quantization in Section G.3.2. The affine parity operator will be introduced in Section G.3.4, and its relationship to affine Weyl quantization will be explored in Section G.3.5. We have dedicated the entirety of Section G.4 to operator admissibility. Section G.5 discusses affine Weyl quantization from the viewpoint of representation theory. In particular, in Section G.5.2 we derive a Bochner type theorem for our setting. In Section G.6.1 and Section G.6.2 we relate our work to time-scale localization operators and covariant integral quantizations, respectively. Finally, in Section G.6.3 we define the affine Cohen class and derive some basic properties.

G.2 Preliminaries

Notation: Given a Hilbert space $\mathcal{H}$ we let $\mathcal{L}(\mathcal{H})$ denote the bounded operators on $\mathcal{H}$. The notation $\mathcal{S}_p(\mathcal{H})$ for $1 \leq p < \infty$ will be used for the *Schatten-p class operators* on $\mathcal{H}$. We remark that $\mathcal{S}_1(\mathcal{H})$ and $\mathcal{S}_2(\mathcal{H})$ are respectively the trace-class operators and the Hilbert-Schmidt operators on $\mathcal{H}$. The space $\mathcal{S}_\infty(\mathcal{H})$ is by definition $\mathcal{L}(\mathcal{H})$ for duality reasons. When the Hilbert space in question is $\mathcal{H} = L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\, dr)$, we will simplify the notation to $\mathcal{S}_p := \mathcal{S}_p(L^2(\mathbb{R}_+))$ for readability. We will denote by $\mathscr{S}(\mathbb{R}^n)$ the space of Schwartz functions on $\mathbb{R}^n$. For a function f on a group G, $\check{f}$ is defined by $\check{f}(g) = f(g^{-1})$ for $g \in G$.

G.2.1 Basic Constructions on the Affine Group

We begin by giving a brief introduction to the affine group and relevant constructions on it. The *(reduced) affine group* $(\mathrm{Aff}, \cdot_{\mathrm{Aff}})$ is the Lie group whose underlying set is the upper half plane $\mathrm{Aff} := \mathbb{R} \times \mathbb{R}_+ := \mathbb{R} \times (0, \infty)$, while the group operation is given by

$$(x, a) \cdot_{\mathrm{Aff}} (y, b) := (ay + x, ab), \qquad (x, a), (y, b) \in \mathrm{Aff}.$$

We will often neglect the subscript in the group operation to improve readability. Moreover, the notation $L_{(x,a)}$ and $R_{(x,a)}$ denotes respectively the *left-translation* and *right-translation* by $(x, a) \in \mathrm{Aff}$, acting on functions $f : \mathrm{Aff} \to \mathbb{C}$ by

$$\begin{aligned}\left(L_{(x,a)} f\right)(y, b) &:= f((x, a)^{-1} \cdot_{\mathrm{Aff}} (y, b)),\\ \left(R_{(x,a)} f\right)(y, b) &:= f((y, b) \cdot_{\mathrm{Aff}} (x, a)).\end{aligned}$$

Recall that the *translation operator* T_x and the *dilation operator* D_a are given by

$$T_x f(y) := f(y - x), \qquad D_a f(y) := \frac{1}{\sqrt{a}} f\left(\frac{y}{a}\right), \qquad x, y \in \mathbb{R},\ a \in \mathbb{R}_+. \quad \text{(G.2.1)}$$

The following computation motivates the group operation on the affine group:

$$(T_x D_a)(T_y D_b) = T_x T_{ay} D_a D_b = T_{x+ay} D_{ab}.$$

We can represent the affine group Aff and its Lie algebra $\mathfrak{aff}$ in matrix form

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \,\middle|\, a > 0,\, x \in \mathbb{R} \right\}, \quad \mathfrak{aff} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \,\middle|\, u, v \in \mathbb{R} \right\}.$$

The Lie algebra structure of $\mathfrak{aff}$ is completely determined by

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{G.2.2}$$

An important feature of the affine group is that it is non-unimodular; the left and right Haar measures are respectively given by

$$\mu_L(x,a) = \frac{dx\, da}{a^2}, \qquad \mu_R(x,a) = \frac{dx\, da}{a}.$$

As such, the modular function on the affine group is given by $\Delta(x,a) = a^{-1}$. The affine group is exponential, meaning that the exponential map $\exp : \mathfrak{aff} \to \mathrm{Aff}$ given by

$$\exp \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^u & \frac{v(e^u - 1)}{u} \\ 0 & 1 \end{pmatrix}$$

is a global diffeomorphism. Hence we can write the left and right Haar measures in exponential coordinates by the formulas

$$\mu_L(x,a) = \frac{du\, dv}{\lambda(u)}, \qquad \mu_R(x,a) = \frac{du\, dv}{\lambda(-u)}, \qquad \lambda(u) := \frac{ue^u}{e^u - 1}. \tag{G.2.3}$$

Throughout the paper, we will heavily use the spaces $L^p_l(\mathrm{Aff}) := L^p(\mathrm{Aff}, \mu_L)$ and $L^p_r(\mathrm{Aff}) := L^p(\mathrm{Aff}, \mu_R)$ for $1 \leq p \leq \infty$.

G.2.2 Quantum Harmonic Analysis on the Heisenberg Group

Before delving into quantum harmonic analysis on the affine group, it is advantageous to review the Heisenberg setting, originally introduced by Werner [251]. There are three primary constructions that appear: *a)* A quantization scheme, *b)* an integrated representation, and *c)* a way to define convolution that incorporates operators. We give a brief overview of these three constructions and refer the reader to [131, 203, 251] for more details.

Weyl Quantization

The *cross-Wigner distribution* of $\phi, \psi \in L^2(\mathbb{R}^n)$ is given by

$$W(\phi,\psi)(x,\omega) := \int_{\mathbb{R}^n} \phi\left(x + \frac{t}{2}\right) \overline{\psi\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega t}\, dt, \quad (x,\omega) \in \mathbb{R}^{2n}.$$

When $\phi = \psi$ we refer to $W\phi := W(\phi, \phi)$ as the *Wigner distribution* of $\phi \in L^2(\mathbb{R}^n)$. The cross-Wigner distribution satisfies the orthogonality relation

$$\langle W(\phi_1, \psi_1), W(\phi_2, \psi_2)\rangle_{L^2(\mathbb{R}^{2n})} = \langle \phi_1, \phi_2\rangle_{L^2(\mathbb{R}^n)} \overline{\langle \psi_1, \psi_2\rangle}_{L^2(\mathbb{R}^n)},$$

for $\phi_1, \phi_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^n)$. Moreover, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ the Wigner distribution satisfies the marginal properties

$$\int_{\mathbb{R}^n} W\phi(x, \omega)\, d\omega = |\phi(x)|^2, \qquad \int_{\mathbb{R}^n} W\phi(x, \omega)\, dx = |\hat{\phi}(x)|^2,$$

Our primary interest in the cross-Wigner distribution stems from the following connection: For each $f \in L^2(\mathbb{R}^{2n})$ we define the operator $L_f : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the formula

$$\langle L_f \phi, \psi\rangle_{L^2(\mathbb{R}^n)} = \langle f, W(\psi, \phi)\rangle_{L^2(\mathbb{R}^{2n})}, \qquad \phi, \psi \in L^2(\mathbb{R}^n).$$

Then L_f is the *Weyl quantization* of f, see [131, Ch. 14] for details. It is a non-trivial fact, see [215], that the Weyl quantization gives a well-defined isomorphism between $L^2(\mathbb{R}^{2n})$ and $\mathcal{S}_2(L^2(\mathbb{R}^n))$, the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$.

Integrated Schrödinger Representation

Recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and with the group multiplication

$$(x, \omega, t) \cdot (x', \omega', t') := \left(x + x', \omega + \omega', t + t' + \frac{1}{2}(x'\omega - x\omega')\right).$$

The Heisenberg group is omnipresent in modern mathematics and theoretical physics, see [160]. For a Hilbert space $\mathcal{H}$ we let $\mathcal{U}(\mathcal{H})$ denote the unitary operators on $\mathcal{H}$. The most important representation of the Heisenberg group is the *Schrödinger representation* $\rho : \mathbb{H}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ given by

$$\rho(x, \omega, t)\phi(y) := e^{2\pi i t} e^{-\pi i x\omega} M_\omega T_x \phi(y), \tag{G.2.4}$$

where T_x is the n-dimensional analogue of the translation operator defined in (G.2.1) and M_ω is the *modulation operator* given by

$$M_\omega \phi(y) := e^{2\pi i \omega y}\phi(y), \qquad \phi \in L^2(\mathbb{R}^n).$$

The Schrödinger representation is irreducible and unitary. Let us use the notation $z := (x, \omega) \in \mathbb{R}^{2n}$ and $\pi(z) = M_\omega T_x$. Ignoring the central variable t, we consider the *integrated Schrödinger representation* $\rho : L^1(\mathbb{R}^{2n}) \to \mathcal{L}(L^2(\mathbb{R}^n))$ given by

$$\rho(f) = \int_{\mathbb{R}^{2n}} f(z) e^{-\pi i x\omega}\pi(z)\, dz, \tag{G.2.5}$$

where $\mathcal{L}(L^2(\mathbb{R}^n))$ denotes the bounded linear operators on $L^2(\mathbb{R}^n)$. We remark that the integral in (G.2.5) is defined weakly. It turns out, see [114, Thm. 1.30], that the integrated representation ρ extends from $L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$ to a unitary map $\rho : L^2(\mathbb{R}^{2n}) \to \mathcal{S}_2(L^2(\mathbb{R}^n))$.

Operator Convolution

Given a function $f \in L^1(\mathbb{R}^{2n})$ and a trace-class operator $S \in \mathcal{S}_1(L^2(\mathbb{R}^n))$, their convolution is the trace-class operator on $L^2(\mathbb{R}^n)$ defined by

$$f \star S := \int_{\mathbb{R}^{2n}} f(z)\pi(z)S\pi(z)^*\, dz.$$

The convolution $f \star S$ satisfies the estimate $\|f \star S\|_{\mathcal{S}_1} \leq \|f\|_{L^1}\|S\|_{\mathcal{S}_1}$.

One can also define the convolution between two operators: For two trace-class operators $S, T \in \mathcal{S}_1(L^2(\mathbb{R}^n))$ we define their convolution to be the function on $\mathbb{R}^{2n}$ given by

$$S \star T(z) := \mathrm{tr}(S\pi(z)PTP\pi(z)^*),$$

where $P\psi(t) := \psi(-t)$ is the *parity operator*. The convolution $S \star T$ satisfies the estimate $\|S \star T\|_{L^1} \leq \|S\|_{\mathcal{S}_1}\|T\|_{\mathcal{S}_1}$, and the important integral relation [251, Lem. 3.1]

$$\int_{\mathbb{R}^{2n}} S \star T(z)\, dz = \mathrm{tr}(S)\mathrm{tr}(T). \tag{G.2.6}$$

To see the connection with the Wigner distribution, we note that the cross-Wigner distribution of $\psi, \phi \in L^2(\mathbb{R}^n)$ can be written as

$$W(\psi, \phi) = \psi \otimes \phi \star P, \tag{G.2.7}$$

where $\psi \otimes \phi$ denotes the rank-one operator on $L^2(\mathbb{R}^n)$ given by

$$(\psi \otimes \phi)(\xi) := \langle \xi, \phi \rangle_{L^2(\mathbb{R}^n)}\psi \qquad \text{for } \xi \in L^2(\mathbb{R}^n).$$

Similarly, the Weyl quantization of $f \in L^1(\mathbb{R}^{2n})$ may be expressed in terms of operator convolutions:

$$L_f = f \star P. \tag{G.2.8}$$

Hence convolution with the parity operator P gives a convenient way to represent the Wigner distribution and the Weyl quantization.

Finally, there is a Fourier transform for operators: Given a trace-class operator $S \in \mathcal{S}_1(L^2(\mathbb{R}^n))$ we define the *Fourier-Wigner transform* $\mathcal{F}_W(S)$ of S to be the function on $\mathbb{R}^{2n}$ given by

$$\mathcal{F}_W(S)(z) := e^{i\pi x\omega}\mathrm{tr}(S\pi(z)^*), \qquad z \in \mathbb{R}^{2n}. \tag{G.2.9}$$

The Fourier-Wigner transform extends to a unitary map $\mathcal{F}_W : \mathcal{S}_2(L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}^{2n})$, where it turns out the to be inverse of the integrated Schrödinger representation given in (G.2.5). By [114, Prop. 2.5] it is related to the Weyl transform by the elegant formula

$$f = \mathcal{F}_\sigma(\mathcal{F}_W(L_f)), \tag{G.2.10}$$

where $\mathcal{F}_\sigma$ denotes the *symplectic Fourier transform*.

G.2.3 Affine Weyl Quantization

We briefly describe affine Weyl quantization and how this gives rise to the affine Wigner distribution. There is a unitary representation π of the affine group Aff on $L^2(\mathbb{R}_+, r^{-1}\,dr)$ given by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar) = \frac{1}{\sqrt{a}} M_x D_{\frac{1}{a}}\psi(r), \qquad \psi \in L^2(\mathbb{R}_+, r^{-1}\,dr). \tag{G.2.11}$$

Since $r^{-1}\,dr$ is the Haar measure on $\mathbb{R}_+$ we will write $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\,dr)$.

To define the quantization scheme we will utilize the *Stratonovich-Weyl operator* on $L^2(\mathbb{R}_+)$ given by

$$\Omega(x,a)\psi(r) := a\int_{\mathbb{R}^2} e^{-2\pi i(xu+av)} U\left(\frac{ve^u}{\lambda(u)}, e^u\right)\psi(r)\,du\,dv. \tag{G.2.12}$$

The following result was shown in [122] and provides us with an affine analogue of Weyl quantization.

Proposition G.2.1 ([122]). *There is a norm-preserving isomorphism between $L^2_r(\mathrm{Aff})$ and the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$. The isomorphism sends $f \in L^2_r(\mathrm{Aff})$ to the operator A_f on $L^2(\mathbb{R}_+)$ defined weakly by*

$$A_f\psi(r) := \int_{-\infty}^{\infty}\int_0^{\infty} f(x,a)\Omega(x,a)\psi(r)\,\frac{da\,dx}{a}, \qquad \psi \in L^2(\mathbb{R}_+).$$

We will refer to the association $f \mapsto A_f$ as *affine Weyl quantization*, while f is called the *affine (Weyl) symbol* of A_f. To emphasize the correspondence between a Hilbert-Schmidt operator A and its affine symbol f we use the notation $f_A := f$. The affine Weyl symbol of an operator A is explicitly given by

$$f_A(x,a) = \int_{-\infty}^{\infty} A_K\left(a\lambda(u), a\lambda(-u)\right) e^{-2\pi i x u}\,du, \tag{G.2.13}$$

where $A_K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ is the integral kernel of A defined by

$$A\psi(r) = \int_0^{\infty} A_K(r,s)\psi(s)\,\frac{ds}{s}, \qquad \psi \in L^2(\mathbb{R}_+).$$

Taking the affine Weyl symbol of the rank-one operator $\psi \otimes \phi$ on $L^2(\mathbb{R}_+)$ given by $\psi \otimes \phi(\xi) = \langle \xi, \phi \rangle_{L^2(\mathbb{R}_+)} \psi$ for $\psi, \phi, \xi \in L^2(\mathbb{R}_+)$, we obtain the following definition.

Definition G.2.1. For $\phi, \psi \in L^2(\mathbb{R}_+)$ we define the *affine (cross-)Wigner distribution* $W_{\text{Aff}}^{\psi,\phi}$ to be the function on Aff given for $(x, a) \in \text{Aff}$ by

$$\begin{aligned} W_{\text{Aff}}^{\psi,\phi}(x,a) &:= \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\,du \\ &= \int_{-\infty}^{\infty} \psi\left(\frac{aue^u}{e^u-1}\right)\overline{\phi\left(\frac{au}{e^u-1}\right)}e^{-2\pi i x u}\,du. \end{aligned}$$

When $\phi = \psi$ we refer to $W_{\text{Aff}}^{\psi} := W_{\text{Aff}}^{\psi,\psi}$ as the *affine Wigner distribution* of ψ. The weak interpretation of the integral defining A_f means that we have the relation

$$\left\langle A_f \phi, \psi \right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\text{Aff}}^{\psi,\phi} \right\rangle_{L^2_r(\text{Aff})}, \tag{G.2.14}$$

for $f \in L^2_r(\text{Aff})$ and $\phi, \psi \in L^2(\mathbb{R}_+)$. The affine Wigner distribution satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\text{Aff}}^{\psi_1,\psi_2}(x,a)\overline{W_{\text{Aff}}^{\phi_1,\phi_2}(x,a)}\,\frac{da\,dx}{a} = \langle \psi_1, \phi_1\rangle_{L^2(\mathbb{R}_+)}\overline{\langle \psi_2, \phi_2\rangle}_{L^2(\mathbb{R}_+)}, \tag{G.2.15}$$

for $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}_+)$. Moreover, the affine Wigner distribution also satisfies the marginal property

$$\int_{-\infty}^{\infty} W_{\text{Aff}}^{\psi}(x,a)\,dx = |\psi(a)|^2, \quad (x,a) \in \text{Aff}, \tag{G.2.16}$$

for all rapidly decaying smooth functions ψ on $\mathbb{R}_+$. We remark that a *rapidly decaying smooth function* (also called a *Schwartz function*) $\psi : \mathbb{R}_+ \to \mathbb{C}$ is by definition a smooth function such that $x \mapsto \psi(e^x)$ is a rapidly decaying function on $\mathbb{R}$. The space of all rapidly decaying smooth functions on $\mathbb{R}_+$ will be denoted by $\mathcal{S}(\mathbb{R}_+)$. We will later also need the space $\mathcal{S}'(\mathbb{R}_+)$ of bounded, anti-linear functionals on $\mathcal{S}(\mathbb{R}_+)$ called the *tempered distributions* on $\mathbb{R}_+$. For more information regarding the affine Wigner distribution the reader is referred to [39].

G.3 Affine Operator Convolutions

In this part we introduce operator convolutions in the affine setting. We show that this notion is intimately related to affine Weyl quantization in Section G.3.2. In Section G.3.4 we will introduce the affine Grossmann-Royer operator, which will be essential in Section G.3.5 where we prove the main connection between the affine Weyl quantization and the operator convolutions in Theorem G.3.13.

G.3.1 Definitions and Basic Properties

We begin by defining operator convolutions in the affine setting and derive basic properties. Recall that the usual convolution on the affine group with respect to the right Haar measure is given by

$$f *_{\mathrm{Aff}} g(x,a) := \int_{\mathrm{Aff}} f(y,b)g((x,a)\cdot(y,b)^{-1})\,\frac{dy\,db}{b}.$$

Remark G.1. Other sources, e.g. [115], use the left Haar measure and define a convolution $*_{\mathrm{Aff}_L}$ related to $*_{\mathrm{Aff}}$ by

$$f *_{\mathrm{Aff}_L} g((x,a)) = \check{f} *_{\mathrm{Aff}} \check{g}((x,a)^{-1}),$$

where $\check{f}(x,a) := f((x,a)^{-1})$. We will mainly work with the right Haar measure, and our definition ensures that

$$\|f *_{\mathrm{Aff}} g\|_{L^1_r(\mathrm{Aff})} \le \|f\|_{L^1_r(\mathrm{Aff})}\|g\|_{L^1_r(\mathrm{Aff})}.$$

Additionally, we have that

$$R_{(x,a)}(f *_{\mathrm{Aff}} g) = (R_{(x,a)}f) *_{\mathrm{Aff}} g. \tag{G.3.1}$$

Definition G.3.1. Let $f \in L^1_r(\mathrm{Aff})$ and let S be a trace-class operator on $L^2(\mathbb{R}_+)$. The *convolution* $f \star_{\mathrm{Aff}} S$ between f and S is the operator on $L^2(\mathbb{R}_+)$ given by

$$f \star_{\mathrm{Aff}} S := \int_{\mathrm{Aff}} f(x,a)U(-x,a)^* S U(-x,a)\,\frac{dx\,da}{a},$$

where U is the unitary representation given in (G.2.11). The integral is a convergent Bochner integral in the space of trace-class operators.

Remark G.2.

1. As we will see later, using $U(-x,a)$ instead of $U(x,a)$ in Definition G.3.1 ensures that the convolution is compatible with the following covariance property of the affine Wigner distribution:

$$W_{\mathrm{Aff}}^{U(-x,a)\phi,U(-x,a)\psi}(y,b) = W_{\mathrm{Aff}}^{\phi,\psi}((y,b)\cdot(x,a)). \tag{G.3.2}$$

2. The notation $\star$ has a different meaning in [122], where it is used to denote the so-called Moyal product of two functions defined on Aff.

Definition G.3.2. Let S be a trace-class operator and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we define the *convolution* $S \star_{\mathrm{Aff}} T$ between S and T to be the function on Aff given by

$$S \star_{\mathrm{Aff}} T(x,a) := \mathrm{tr}(SU(-x,a)^* TU(-x,a)).$$

Remark G.3. Recently, [56] defined another notion of convolution of trace-class operators. Unlike our definition, this convolution produces a new trace-class operator, with the aim of interpreting the trace-class operators as an analogue of the Fourier algebra.

It is straightforward to check that if f is a positive function and S, T are positive operators, then $f \star_{\mathrm{Aff}} S$ is a positive operator and $S \star_{\mathrm{Aff}} T$ is a positive function. Moreover, we have the elementary estimate

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{S}_1} \tag{G.3.3}$$

and

$$\|S \star_{\mathrm{Aff}} T\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{L}(L^2(\mathbb{R}_+))}. \tag{G.3.4}$$

The following result is proved by a simple computation.

Lemma G.3.1. *For $\psi, \phi \in L^2(\mathbb{R}_+)$ and $S \in \mathcal{L}(L^2(\mathbb{R}_+))$, we have*

$$(\psi \otimes \phi) \star_{\mathrm{Aff}} S(x,a) = \langle SU(-x,a)\psi, U(-x,a)\phi \rangle_{L^2(\mathbb{R}_+)}.$$

In particular, for $\eta, \xi \in L^2(\mathbb{R}_+)$ we have

$$(\psi \otimes \phi) \star_{\mathrm{Aff}} (\eta \otimes \xi)(x,a) = \langle \psi, U(-x,a)^* \xi \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi, U(-x,a)^* \eta \rangle_{L^2(\mathbb{R}_+)}},$$

and

$$(\psi \otimes \psi) \star_{\mathrm{Aff}} (\xi \otimes \xi)(x,a) = |\langle \psi, U(-x,a)^* \xi \rangle_{L^2(\mathbb{R}_+)}|^2.$$

A natural question to ask is whether the three different notions of convolution we have introduced are compatible. The following proposition gives an affirmative answer to this question.

Proposition G.3.2. *Let $f, g \in L^1_r(\mathrm{Aff})$, $S \in \mathcal{S}_1$, and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we have the compatibility equations*

$$\begin{aligned}(f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T &= f *_{\mathrm{Aff}} (S \star_{\mathrm{Aff}} T), \\ f \star_{\mathrm{Aff}} (g \star_{\mathrm{Aff}} S) &= (f *_{\mathrm{Aff}} g) \star_{\mathrm{Aff}} S.\end{aligned}$$

Proof. The first equality follows from the computation

$$\begin{aligned}
&(f *_{\mathrm{Aff}} (S \star_{\mathrm{Aff}} T))\,(x,a) \\
&\qquad = \int_{\mathrm{Aff}} f(y,b)\operatorname{tr}(SU(-y,b)U(-x,a)^*TU(-x,a)U(-y,b)^*)\,\frac{dy\,db}{b} \\
&\qquad = \int_{\mathrm{Aff}} f(y,b)\operatorname{tr}(U(-y,b)^*SU(-y,b)U(-x,a)^*TU(-x,a))\,\frac{dy\,db}{b} \\
&\qquad = \operatorname{tr}\left(\int_{\mathrm{Aff}} f(y,b)U(-y,b)^*SU(-y,b)\,\frac{dy\,db}{b}U(-x,a)^*TU(-x,a)\right) \\
&\qquad = ((f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T)\,(x,a).
\end{aligned}$$

We are allowed to take the trace outside the integral since the second to last line is essentially the duality action of the bounded operator $U(-x,a)^*TU(-x,a)$ on a convergent Bochner integral in the space of trace-class operators.

For the second equality, use change of variables to rewrite $(f *_{\mathrm{Aff}} g) \star_{\mathrm{Aff}} S$ as

$$\begin{aligned}
&\int_{\mathrm{Aff}}\int_{\mathrm{Aff}} f(x,a)g((z,c)\cdot(x,a)^{-1})U(-z,c)^*SU(-z,c)\,\frac{dx\,da}{a}\,\frac{dz\,dc}{c} \\
&= \int_{\mathrm{Aff}}\int_{\mathrm{Aff}} f(x,a)g(y,b)U(-x,a)^*U(-y,b)^*SU(-y,b)U(-x,a)\,\frac{dy\,db}{b}\,\frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} f(x,a)U(-x,a)^*\int_{\mathrm{Aff}} g(y,b)U(-y,b)^*SU(-y,b)\,\frac{dydb}{b}U(-x,a)\,\frac{dx\,da}{a} \\
&= f \star_{\mathrm{Aff}} (g \star_{\mathrm{Aff}} S).
\end{aligned}$$

Changing the order of integration above is allowed by Fubini's theorem for Bochner integrals [166, Prop. 1.2.7]. Fubini's theorem is applicable since

$$\int_{\mathrm{Aff}}\int_{\mathrm{Aff}} |f(x,a)|\cdot|g((z,c)\cdot(x,a)^{-1})|\cdot\|U(-z,c)^*SU(-z,c)\|_{\mathcal{S}_1}\,\frac{dx\,da}{a}\,\frac{dz\,dc}{c}$$

is bounded from above by

$$\|S\|_{\mathcal{S}_1}\int_{\mathrm{Aff}} |f(x,a)|\,\frac{dx\,da}{a}\int_{\mathrm{Aff}} |g(z,c)|\,\frac{dz\,dc}{c} < \infty. \qquad \square$$

G.3.2 Relationship With Affine Weyl Quantization

The goal of this section is to connect the affine Weyl quantization described in Section G.2.3 with the convolutions defined in Section G.3.1. We first establish a preliminary result describing how right multiplication on the affine group affects the affine Weyl quantization.

Lemma G.3.3. *Let $A_f \in \mathcal{S}_2$ with affine Weyl symbol $f \in L^2_r(\mathrm{Aff})$. For $(x,a) \in \mathrm{Aff}$, the affine Weyl symbol of $U(-x,a)^* A_f U(-x,a)$ is $R_{(x,a)^{-1}} f$.*

Proof. The result follows from (G.2.14) and the computation

$$\begin{aligned}\langle U(-x,a)^* A_f U(-x,a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} &= \langle A_f U(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \langle f, W_{\mathrm{Aff}}^{U(-x,a)\phi, U(-x,a)\psi}\rangle_{L^2_r(\mathrm{Aff})} \\ &= \langle f, R_{(x,a)} W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{L^2_r(\mathrm{Aff})} \\ &= \langle R_{(x,a)^{-1}} f, W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{L^2_r(\mathrm{Aff})}. \qquad \square\end{aligned}$$

We are now ready to prove the first result showing the connection between convolution and affine Weyl quantization.

Proposition G.3.4. *Assume that $A_f \in \mathcal{S}_2$ with affine Weyl symbol $f \in L^2_r(\mathrm{Aff})$, and let $g \in L^1_r(\mathrm{Aff})$. Then the affine Weyl symbol of $g \star_{\mathrm{Aff}} A_f$ is $g *_{\mathrm{Aff}} f$, that is,*

$$g \star_{\mathrm{Aff}} A_f = A_{g *_{\mathrm{Aff}} f}.$$

Proof. The operator $g \star_{\mathrm{Aff}} A_f$ is defined as the $\mathcal{S}_2$-convergent Bochner integral

$$g \star_{\mathrm{Aff}} A_f = \int_{\mathrm{Aff}} g(x,a) U(-x,a)^* A_f U(-x,a) \, \frac{dx\,da}{a}.$$

By Proposition G.2.1, the map $\mathfrak{W} : \mathcal{S}_2 \to L^2_r(\mathrm{Aff})$ given by $\mathfrak{W}(A_f) = f$ is unitary. Since bounded operators commute with convergent Bochner integrals, we have using Lemma G.3.3 that

$$\begin{aligned}\mathfrak{W}\left(g \star_{\mathrm{Aff}} A_f\right) &= \int_{\mathrm{Aff}} g(x,a) \mathfrak{W}\left(U(-x,a)^* A_f U(-x,a)\right) \frac{dx\,da}{a} \\ &= \int_{\mathrm{Aff}} g(x,a) R_{(x,a)^{-1}} \mathfrak{W}\left(A_f\right) \frac{dx\,da}{a} \\ &= g *_{\mathrm{Aff}} f. \qquad \square\end{aligned}$$

We can also express the convolution of two operators in terms of their affine Weyl symbols.

Proposition G.3.5. *Let $A_f, A_g \in \mathcal{S}_2$ with affine Weyl symbols $f, g \in L^2_r(\mathrm{Aff})$. If additionally $g \in L^2_l(\mathrm{Aff})$, then we have*

$$A_f \star_{\mathrm{Aff}} A_g = f *_{\mathrm{Aff}} \check{g},$$

where $\check{g}(x,a) = g((x,a)^{-1})$ for $(x,a) \in \mathrm{Aff}$.

Proof. Using Proposition G.2.1 and Lemma G.3.3 we compute that

$$\begin{aligned}(A_f \star_{\mathrm{Aff}} A_g)(x,a) &= \operatorname{tr}(A_f U(-x,a)^* A_g U(-x,a)) \\ &= \langle A_f, U(-x,a)^* A_g^* U(-x,a)\rangle_{\mathcal{S}_2} \\ &= \langle f, R_{(x,a)^{-1}}\overline{g}\rangle_{L^2_r(\mathrm{Aff})} \\ &= \int_{\mathrm{Aff}} f(y,b) g((y,b)\cdot(x,a)^{-1})\,\frac{dy\,db}{b} \\ &= \int_{\mathrm{Aff}} f(y,b)\check{g}((x,a)\cdot(y,b)^{-1})\,\frac{dy\,db}{b} \\ &= f *_{\mathrm{Aff}} \check{g}(x,a).\end{aligned}$$

The result follows as $\check{g} \in L^2_r(\mathrm{Aff})$ if and only if $g \in L^2_l(\mathrm{Aff})$. □

G.3.3 Affine Weyl Quantization of Coordinate Functions

Of particular interest is the affine Weyl quantization of the coordinate functions $f_x(x,a) := x$ and $f_a(x,a) := a$ for $(x,a) \in \mathrm{Aff}$. Due to the fact that the coordinate functions are not in $L^2_r(\mathrm{Aff})$, we first need to interpret the quantizations A_{f_x} and A_{f_a} in a rigorous manner. We begin this task by defining rapidly decaying smooth function and tempered distributions on the affine group.

Definition G.3.3. Let $\mathcal{S}(\mathrm{Aff})$ denote the smooth functions $f : \mathrm{Aff} \to \mathbb{C}$ such that

$$(x,\omega) \longmapsto f(x,e^{\omega}) \in \mathcal{S}(\mathbb{R}^2).$$

We refer to $\mathcal{S}(\mathrm{Aff})$ as the space of *rapidly decaying smooth functions* (or *Schwartz functions*) on the affine group.

There is a natural topology on $\mathcal{S}(\mathrm{Aff})$ induced by the semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x,\,\omega\in\mathbb{R}} |x|^{\alpha_1}|\omega|^{\alpha_2}\left|\partial_x^{\beta_1}\partial_\omega^{\beta_2} f(x,e^{\omega})\right|, \qquad \text{(G.3.5)}$$

for $\alpha = (\alpha_1,\alpha_2)$ and $\beta = (\beta_1,\beta_2)$ in $\mathbb{N}_0 \times \mathbb{N}_0$. With these semi-norms, the space $\mathcal{S}(\mathrm{Aff})$ becomes a Fréchet space. The space of bounded, anti-linear functionals on $\mathcal{S}(\mathrm{Aff})$ is denoted by $\mathcal{S}'(\mathrm{Aff})$ and called the space of *tempered distributions* on Aff.

Lemma G.3.6. *For any $f \in \mathcal{S}'(\mathrm{Aff})$ we can define A_f as the map $A_f : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+)$ defined by the relation*

$$\langle A_f\psi,\phi\rangle_{\mathcal{S}',\mathcal{S}} = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi}\right\rangle_{\mathcal{S}',\mathcal{S}}, \qquad \psi,\phi \in \mathcal{S}(\mathbb{R}_+).$$

Additionally, the map $f \mapsto A_f$ is injective.

Proof. It was shown in [39, Cor. 6.6] that for any $\phi, \psi \in \mathcal{S}(\mathbb{R}_+)$ then $W_{\text{Aff}}^{\phi,\psi} \in \mathcal{S}(\text{Aff})$. Hence the pairing $\left\langle f, W_{\text{Aff}}^{\phi,\psi} \right\rangle_{\mathcal{S}',\mathcal{S}}$ is well defined.

For the injectivity it suffices to show that $A_f = 0$ implies that $f = 0$. Let us first reformulate this slightly: If $A_f = 0$, then we have that

$$\langle A_f \psi, \phi \rangle_{\mathcal{S}',\mathcal{S}} = \left\langle f, W_{\text{Aff}}^{\phi,\psi} \right\rangle_{\mathcal{S}',\mathcal{S}} = 0$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$. We could conclude that $f = 0$ if we knew that any $g \in \mathcal{S}(\text{Aff})$ could be approximated (in the Fréchet topology) by linear combinations of elements on the form $W_{\text{Aff}}^{\phi,\psi}$ for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$. To see that this is the case, we translate the problem to the Heisenberg setting.

The *Mellin transform* $\mathcal{M}$ is given by

$$\mathcal{M}(\phi)(x) = \mathcal{M}_r(\phi)(x) := \int_0^\infty \phi(r) r^{-2\pi i x} \, \frac{dr}{r}.$$

Define the functions Ψ and Φ to be $\Psi(x) := \psi(e^x)$ and $\Phi(x) := \phi(e^x)$ for $\psi, \phi \in L^2(\mathbb{R}_+)$. A reformulation of [39, Lem. 6.4] shows that we have the relation

$$W_{\text{Aff}}^{\psi,\phi}(x,a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\left(\frac{\sqrt{b}\log(b)}{b-1} \right)^{2\pi i y} \mathcal{F}_\sigma W(\Psi,\Phi)\big(\log(b), y\big) \right](x,a),$$

where W is the cross-Wigner distribution. The correspondence preserves Schwartz functions, due to the term

$$\left(\frac{\sqrt{b}\log(b)}{b-1} \right)^{2\pi i y}$$

being smooth with polynomially bounded derivatives. This gives a bijective correspondence between $W_{\text{Aff}}^{\psi,\phi} \in \mathcal{S}(\text{Aff})$ and $W(\Psi,\Phi) \in \mathcal{S}(\mathbb{R}^2)$. As such, the injectivity question is reduced to asking whether the linear span of elements on the form $W(f,g)$ for $f, g \in \mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R}^2)$. One way to verify this well-known fact is to note that the map $f \otimes g \mapsto W(f,g)$, where $f \otimes g(x,y) = f(x)g(y)$, extends to a topological isomorphism on $\mathcal{S}(\mathbb{R}^2)$, see for instance [131, (14.21)] for the formula of this isomorphism. The density of elements on the form $W(f,g)$ for $f, g \in \mathcal{S}(\mathbb{R})$ therefore follows as the functions $h_m \otimes h_n$, where $\{h_n\}_{n=0}^\infty$ are the Hermite functions, span a dense subspace of $\mathcal{S}(\mathbb{R}^2)$ by [220, Thm. V.13]. □

Example G.3.1. Consider the constant function on the affine group given by $1(x,a) = 1$ for all $(x,a) \in \text{Aff}$. Then the quantization A_1 is the identity operator since for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$

$$\begin{aligned}\langle A_1\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \langle 1, W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \int_{\mathrm{Aff}} \overline{W_{\mathrm{Aff}}^{\phi,\psi}(x,a)}\,\frac{da\,dx}{a} \\ &= \int_0^\infty \psi(a)\overline{\phi(a)}\,\frac{da}{a} \\ &= \langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Notice that we used a straightforward generalization of the marginal property of the affine Wigner distribution given in (G.2.16), see the proof of [39, Prop. 3.4] for details.

To motivate the next result, consider the coordinate functions $\sigma_x(x,\omega) := x$ and $\sigma_\omega(x,\omega) := \omega$ for $(x,\omega) \in \mathbb{R}^{2n}$. The Weyl quantizations L_{σ_x} and L_{σ_ω} are the well-known *position operator* and *momentum operator* in quantum mechanics. In particular, the commutator

$$\left[L_{\sigma_x}, L_{\sigma_\omega}\right] := L_{\sigma_x} \circ L_{\sigma_\omega} - L_{\sigma_\omega} \circ L_{\sigma_x}$$

is a constant times the identity by [150, Prop. 3.8]. This is precisely the relation for the Lie algebra of the Heisenberg group. In light of this, the following proposition shows that the affine Weyl quantization has the expected expression for the coordinate functions.

Theorem G.3.7. *Let f_x and f_a be the coordinate functions on the affine group. The affine Weyl quantizations A_{f_x} and A_{f_a} are well-defined as maps from $\mathcal{S}(\mathbb{R}_+)$ to $\mathcal{S}'(\mathbb{R}_+)$ and are explicitly given by*

$$A_{f_x}\psi(r) = \frac{1}{2\pi i} r\psi'(r), \qquad A_{f_a}\psi(r) = r\psi(r), \qquad \psi \in \mathcal{S}(\mathbb{R}_+).$$

In particular, we have the commutation relation

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i} A_{f_a}.$$

This is, up to re-normalization, precisely the Lie algebra structure of $\mathfrak{aff}$ given in (G.2.2).

Proof. Let us begin by computing A_{f_x}. We can change the order of integrating by

Fubini's theorem and obtain for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ that

$$\begin{aligned}
\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \left\langle f_x, W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} x \overline{\int_{-\infty}^{\infty} \phi(a\lambda(u))\overline{\psi(a\lambda(-u))}e^{-2\pi i x u}\, du}\, \frac{da\, dx}{a} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\int_{-\infty}^{\infty} x e^{2\pi i x u}\, dx\right) \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\, \frac{da\, du}{a}.
\end{aligned}$$

Notice that the inner integral is equal to

$$\int_{-\infty}^{\infty} x e^{2\pi i x u}\, dx = \frac{1}{2\pi i}\delta_0'(u),$$

where

$$\int_{-\infty}^{\infty} \delta_0'(u)\psi(u)\, du = \psi'(0).$$

Hence we have the relation

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{2\pi i}\int_0^{\infty} \frac{\partial}{\partial u}\left(\psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\right)\Big|_{u=0}\, \frac{da}{a}.$$

By using the formulas $\lambda(0) = 1$ and $\lambda'(0) = 1/2$ we can simplify and obtain

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{4\pi i}\int_0^{\infty} a \cdot \left(\psi'(a)\overline{\phi(a)} - \psi(a)\overline{\phi'(a)}\right)\, \frac{da}{a}.$$

Using integration by parts we obtain the claim since

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \int_0^{\infty} \left[\frac{1}{2\pi i} a\psi'(a)\right]\overline{\phi(a)}\, \frac{da}{a}.$$

For A_{f_a} we have by similar calculations as above that

$$\begin{aligned}
\langle A_{f_a}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\int_{-\infty}^{\infty} 1 \cdot e^{2\pi i x u}\, dx\right) a \cdot \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\, \frac{da\, du}{a} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} \delta_0(u)\left(a \cdot \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\right)\, \frac{da\, du}{a} \\
&= \int_0^{\infty} a\psi(a)\overline{\phi(a)}\, \frac{da}{a}.
\end{aligned}$$

The commutation relation follows from straightforward computation. □

G.3.4 The Affine Grossmann-Royer Operator

In this section we introduce the affine Grossmann-Royer operator with the aim of obtaining an affine parity operator analogous to the (Heisenberg) parity operator P in Section G.2.2. The main reason for this is to obtain affine version of the formulas (G.2.7) and (G.2.8) so that we can describe the affine Weyl quantization through convolution. Recall that the *(Heisenberg) Grossmann-Royer operator* $R(x,\omega)$ for $(x,\omega) \in \mathbb{R}^{2n}$ is defined by the relation

$$W(f,g)(x,\omega) = \langle R(x,\omega)f, g\rangle_{L^2(\mathbb{R}^n)}, \qquad f,g \in L^2(\mathbb{R}^n).$$

Analogously, we have the following definition.

Definition G.3.4. We define the *affine Grossmann-Royer operator* $R_{\mathrm{Aff}}(x,a)$ for $(x,a) \in \mathrm{Aff}$ by the relation

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \langle R_{\mathrm{Aff}}(x,a)\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}}, \qquad \psi,\phi \in \mathcal{S}(\mathbb{R}_+).$$

We restrict our attention to Schwartz functions for convenience since then $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathcal{S}(\mathrm{Aff})$ by [39, Cor. 6.6], and hence have well-defined point values. The Grossmann-Royer operator $R_{\mathrm{Aff}}(x,a)$ is precisely the affine Weyl quantization of the point mass $\delta_{\mathrm{Aff}}(x,a) \in \mathcal{S}'(\mathrm{Aff})$ for $(x,a) \in \mathrm{Aff}$ defined by

$$\langle \delta_{\mathrm{Aff}}(x,a), f\rangle_{\mathcal{S}',\mathcal{S}} := \overline{f(x,a)}, \qquad f \in \mathcal{S}(\mathrm{Aff}).$$

Since this is also true for the Stratonovich-Weyl operator $\Omega(x,a)$ given in (G.2.12), it follows that $R_{\mathrm{Aff}}(x,a) = \Omega(x,a)$ for all $(x,a) \in \mathrm{Aff}$. From [122, p. 12] it follows that we have the *affine covariance relation*

$$U(-x,a)^* R_{\mathrm{Aff}}(0,1) U(-x,a) = R_{\mathrm{Aff}}(x,a). \tag{G.3.6}$$

The following result, which is a straightforward computation, shows that $R_{\mathrm{Aff}}(x,a)$ is an unbounded and densely defined operator on $L^2(\mathbb{R}_+)$.

Lemma G.3.8. *Fix $\psi \in \mathcal{S}(\mathbb{R}_+)$ and $(x,a) \in \mathrm{Aff}$. The affine Grossmann-Royer operator $R_{\mathrm{Aff}}(x,a)$ has the explicit form*

$$R_{\mathrm{Aff}}(x,a)\psi(r) = \frac{e^{2\pi i x \lambda^{-1}\left(\frac{r}{a}\right)} \lambda^{-1}\left(\frac{r}{a}\right)\left(1 - e^{\lambda^{-1}\left(\frac{r}{a}\right)}\right)}{1 + \lambda^{-1}\left(\frac{r}{a}\right) - e^{\lambda^{-1}\left(\frac{r}{a}\right)}} \cdot \psi\left(r e^{-\lambda^{-1}\left(\frac{r}{a}\right)}\right),$$

where λ is the function given in (G.2.3).

We will be particularly interested in the *affine parity operator* P_{Aff} given by the affine Grossmann-Royer operator at the identity element, that is,

$$P_{\text{Aff}}(\psi)(r) := R_{\text{Aff}}(0,1)\psi(r) = \frac{\lambda^{-1}(r)(1-e^{\lambda^{-1}(r)})}{1+\lambda^{-1}(r)-e^{\lambda^{-1}(r)}}\psi\left(re^{-\lambda^{-1}(r)}\right),$$

for $\psi \in \mathcal{S}(\mathbb{R}_+)$. The affine parity operator P_{Aff} is symmetric as an unbounded operator on $L^2(\mathbb{R}_+)$. Moreover, we see from the relation

$$e^{\lambda^{-1}(r)} - 1 = \frac{\lambda^{-1}(r)e^{\lambda^{-1}(r)}}{r}$$

that we have the alternative formula

$$P_{\text{Aff}}(\psi)(r) = \frac{\lambda^{-1}(r)}{1-re^{-\lambda^{-1}(r)}}\psi\left(re^{-\lambda^{-1}(r)}\right). \tag{G.3.7}$$

An important commutation relation for the (Heisenberg) Grossman-Royer operator $R(x,\omega)$ for $(x,\omega) \in \mathbb{R}^{2n}$ is given by

$$P \circ R(x,\omega) = R(-x,-\omega) \circ P. \tag{G.3.8}$$

The following proposition shows that the analogue of (G.3.8) breaks down in the affine setting due to Aff being non-unimodular. As the proof is a straightforward computation, we leave the details to the reader.

Proposition G.3.9. *The commutation relation*

$$P_{\text{Aff}} \circ R_{\text{Aff}}(x,a) = R_{\text{Aff}}\left((x,a)^{-1}\right) \circ P_{\text{Aff}}$$

holds precisely for those $(x,a) \in \text{Aff}$ *such that* $\Delta(x,a) = \frac{1}{a} = 1$.

We will now show that both the function λ in (G.2.3) and the affine parity operator P_{Aff} are related to the Lambert W function. Recall that the *(real) Lambert W function* is the multivalued function defined to be the inverse relation of the function $f(x) = xe^x$ for $x \in \mathbb{R}$. The function $f(x)$ for $x < 0$ is not injective. There exist for each $y \in (-1/e, 0)$ precisely two values $x_1, x_2 \in (-\infty, 0)$ such that

$$x_1 e^{x_1} = x_2 e^{x_2} = y.$$

As the solutions appear in pairs, we can define σ to be the function that permutes these solutions, that is, $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$. For $y = -1/e$ there is only one solution to the equation $xe^x = y$, namely $x = -1$. Hence we define $\sigma(-1) = -1$. We can represent the function σ as

$$\sigma(x) = \begin{cases} W_0(xe^x), & x < -1 \\ -1, & x = -1 \\ W_{-1}(xe^x), & -1 < x < 0 \end{cases},$$

where W_0, W_{-1} are the two branches of the Lambert W function satisfying

$$W_0(xe^x) = x, \qquad \text{for } x \geq -1$$

and

$$W_{-1}(xe^x) = x, \qquad \text{for } x \leq -1.$$

Lemma G.3.10. *The inverse of λ is given by*

$$\lambda^{-1}(r) = \log\left(\frac{-r}{\sigma(-r)}\right) = \sigma(-r) + r, \qquad r > 0.$$

Proof. To find the inverse of λ we solve the equation

$$r = \lambda(u) = \frac{ue^u}{e^u - 1} = \frac{-u}{e^{-u} - 1}.$$

A simple computation shows that $-r = -u - re^{-u}$. Making the substitution $v = e^{-u}$ together with straightforward manipulations shows that

$$-re^{-r} = -rve^{-rv}. \tag{G.3.9}$$

The trivial solution to (G.3.9) is given by solving the equation $-r = -rv$. Checking with the original equation, this can not give the inverse of λ. We get the first equality from the definition of σ together with recalling that $u = -\log(v)$. The final equality follows from

$$\log\left(\frac{-r}{\sigma(-r)}\right) = \log\left(\frac{-r}{\sigma(-r)} \frac{\sigma(-r)e^{\sigma(-r)}}{-re^{-r}}\right) = \sigma(-r) + r. \qquad \square$$

Remark G.4. A minor variation of the function σ appeared in [122, Section 3] where it was defined by the relation in Lemma G.3.10. The advantage of understanding the connection to the Lambert W function is that properties such as $\sigma(\sigma(x)) = x$ for every $x < 0$ become trivial in this description.

Corollary G.3.10.1. *The affine parity operator P_{Aff} can be written as*

$$P_{\mathrm{Aff}}(\psi)(r) = \frac{\sigma(-r) + r}{\sigma(-r) + 1}\psi(-\sigma(-r)), \qquad \psi \in \mathcal{S}(\mathbb{R}_+).$$

In particular, we have $P_{\mathrm{Aff}}(\psi)(1) = 2\psi(1)$.

Proof. The formula for $P_{\mathrm{Aff}}(\psi)$ is obtained from Lemma G.3.10 together with (G.3.7). To find the value $P_{\mathrm{Aff}}(\psi)(1)$, we use (G.3.7) and the fact that

$$\psi\left(re^{-\lambda^{-1}(r)}\right)\Big|_{r=1} = \psi(1).$$

Hence the claim follows from L'Hopital's rule since

$$\lim_{r\to 1}\frac{\lambda^{-1}(r)}{\lambda^{-1}(r)+1-r} = \frac{(\lambda^{-1})'(1)}{(\lambda^{-1})'(1)-1} = 2. \qquad \square$$

G.3.5 Operator Convolution for Tempered Distributions

This section is all about expressing the affine Weyl quantization of a function $f \in \mathcal{S}(\mathrm{Aff})$ by using affine convolution. To be able to do this, we will first define what it means for A_f to be a Schwartz operator.

Definition G.3.5. We say that a Hilbert-Schmidt operator $A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a *Schwartz operator* if the integral kernel A_K of A satisfies $A_K \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, that is, if

$$(x,\omega) \longmapsto A_K(e^x, e^\omega) \in \mathcal{S}(\mathbb{R}^2).$$

Proposition G.3.11. *A Hilbert-Schmidt operator $A \in \mathcal{S}_2$ is a Schwartz operator if and only if $A = A_f$ for some $f \in \mathcal{S}(\mathrm{Aff})$.*

Proof. Assume that A is a Schwartz operator. In [122, Equation (4.8)] it is shown that the integral kernel A_K of A is related to the affine Weyl symbol f_A of A by the formula

$$A_K(r,s) = \int_{-\infty}^{\infty} f_A\left(x, \frac{r-s}{\log(r/s)}\right) e^{2\pi i x\log(r/s)}\,dx.$$

Since the inverse-Fourier transform preserves Schwartz functions, together with the definition of $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, we have that

$$(r,s) \longmapsto f_A\left(\log(r/s), \frac{r-s}{\log(r/s)}\right) \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+).$$

By performing the change of variable $x = \log(r/s)$ and $s = e^\omega$ for $\omega \in \mathbb{R}$ we obtain

$$(x,\omega) \longmapsto f_A\left(x, e^\omega \frac{e^x - 1}{x}\right) \in \mathcal{S}(\mathbb{R}^2).$$

Finally, by letting $u = \log((e^x - 1)/x) + \omega$ we see that

$$(x,u) \longmapsto f_A\left(x, e^u\right) \in \mathcal{S}(\mathbb{R}^2),$$

due to the fact that $x \mapsto \log((e^x - 1)/x)$ has polynomial growth.

Conversely, assume that $A = A_f$ for $f \in \mathcal{S}(\mathrm{Aff})$. The integral kernel A_K is then given by

$$A_K(r, s) = \mathcal{F}_1^{-1}(f)\left(\log(r/s), \frac{r-s}{\log(r/s)}\right).$$

By using that the inverse-Fourier transform $\mathcal{F}_1^{-1}$ in the first component preserves $\mathcal{S}(\mathrm{Aff})$ together with similar substitutions as previously, we have that $A_K \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$. □

We will use the notation $\mathcal{S}(L^2(\mathbb{R}_+))$ for all Schwartz operators on $L^2(\mathbb{R}_+)$. There is a natural topology on $\mathcal{S}(L^2(\mathbb{R}_+))$ induced by the semi-norms $\|A_f\|_{\alpha,\beta} := \|f\|_{\alpha,\beta}$ where $\|\cdot\|_{\alpha,\beta}$ are the semi-norms on $\mathcal{S}(\mathrm{Aff})$ given in (G.3.5).

Proposition G.3.12. *The affine convolution gives a well-defined map*

$$\mathcal{S}(\mathrm{Aff}) \star_{\mathrm{Aff}} \mathcal{S}(L^2(\mathbb{R}_+)) \to \mathcal{S}(L^2(\mathbb{R}_+)).$$

Moreover, for fixed $A \in \mathcal{S}(L^2(\mathbb{R}_+))$ the map

$$\mathcal{S}(\mathrm{Aff}) \ni f \longmapsto f \star_{\mathrm{Aff}} A \in \mathcal{S}(L^2(\mathbb{R}_+))$$

is continuous.

Proof. Let $f \in \mathcal{S}(\mathrm{Aff})$ and $A \in \mathcal{S}(L^2(\mathbb{R}_+))$. Then $A = A_g$ for some $g \in \mathcal{S}(\mathrm{Aff})$ and we have by Proposition G.3.4 that

$$f \star_{\mathrm{Aff}} A = f \star_{\mathrm{Aff}} A_g = A_{f *_{\mathrm{Aff}} g}. \tag{G.3.10}$$

Hence the first statement reduces to showing that the usual affine group convolution is a well-defined map

$$\mathcal{S}(\mathrm{Aff}) *_{\mathrm{Aff}} \mathcal{S}(\mathrm{Aff}) \to \mathcal{S}(\mathrm{Aff}).$$

After a change of variables, the question becomes whether the map

$$(x, u) \longmapsto (f *_{\mathrm{Aff}} g)(x, e^u) = \int_{\mathbb{R}^2} f(y, e^z) g(x - ye^{u-z}, e^{u-z})\, dy\, dz \tag{G.3.11}$$

is an element in $\mathcal{S}(\mathbb{R}^2)$. It is straightforward to check that (G.3.11) is a smooth function. Moreover, since f and g are both in $\mathcal{S}(\mathrm{Aff})$, it suffices to show that (G.3.11) decays faster than any polynomial towards infinity; we can then iterate the argument to obtain the required decay statements for the derivatives.

We claim that

$$\sup_{x,u} |x|^k |u|^l |g(x - ye^{u-z}, e^{u-z})| \leq A^g_{k,l}\,(1 + |y|)^k\,(1 + |z|)^l, \tag{G.3.12}$$

where $A^g_{k,l}$ is a constant that depends only on the indices $k, l \in \mathbb{N}_0$ and $g \in \mathcal{S}(\mathrm{Aff})$. To show this, we need to individually consider three cases:

- Assume that we only take the supremum over x and u satisfying $2|z| \geq |u|$ and $2|y| \geq |x|$. Then clearly (G.3.12) is satisfied with $A^g_{k,l} = 2^{k+l} \max |g|$.
- Assume that we only take the supremum over u satisfying $2|z| \leq |u|$ and let $x \in \mathbb{R}$ be arbitrary. Then e^{u-z} is outside the interval $[e^{-|u|/2}, e^{|u|/2}]$. Since $g \in \mathscr{S}(\mathrm{Aff})$ the left-hand side of (G.3.12) will eventually decrease when increasing u. When $y \leq 0$ the left hand-side of (G.3.12) will also obviously eventually decrease by increasing x. When $y > 0$ then any increase of x would necessitate an increase of u on the scale of $u \sim \ln(x)$ to compensate so that the first coordinate in g does not blow up. However, this again forces the second coordinate to grow on the scale of x and we would again, due to $g \in \mathscr{S}(\mathrm{Aff})$, have that the left hand-side of (G.3.12) would eventually decrease.
- Finally, we can consider taking the supremum over x and u satisfying $2|z| \geq |u|$ and $2|y| \leq |x|$. As this case uses similar arguments as above, we leave the straightforward verification to the reader.

Using (G.3.12) we have that

$$\sup_{x,u} |x^k u^l (f *_{\mathrm{Aff}} g)(x, e^u)| \leq A^g_{k,l} \int_{\mathbb{R}^2} |f(y, e^z)| \, (1 + |y|)^k \, (1 + |z|)^l \, dy \, dz < \infty, \tag{G.3.13}$$

where the last inequality follows from that $f \in \mathscr{S}(\mathrm{Aff})$. Finally, the continuity of the map $f \mapsto f \star_{\mathrm{Aff}} A$ follows from (G.3.10) and (G.3.13). □

Remark G.5. Notice that the proof of Proposition G.3.12 shows that affine convolution between $f, g \in \mathscr{S}(\mathrm{Aff})$ satisfies $f *_{\mathrm{Aff}} g \in \mathscr{S}(\mathrm{Aff})$. This fact, together with Proposition G.3.11, strengthens the claim that $\mathscr{S}(\mathrm{Aff})$ is the correct definition for Schwartz functions on the group Aff.

The main result in this section is Theorem G.3.13 presented below. To state the result rigorously, we first need to make sense of the convolution between Schwartz functions $g \in \mathscr{S}(\mathrm{Aff})$ and the affine parity operator P_{Aff}. As motivation for our definition we will use the following computation: Let $S, T \in \mathcal{S}_2$ with affine Weyl symbols $f_S, f_T \in L^2_r(\mathrm{Aff})$. Fix $g \in \mathscr{S}(\mathrm{Aff})$ and consider the affine Weyl symbol $f_{g \star_{\mathrm{Aff}} S}$ corresponding to the convolution $g \star_{\mathrm{Aff}} S$. Then

$$\begin{aligned}
\langle f_{g \star_{\mathrm{Aff}} S}, f_T \rangle_{L^2_r(\mathrm{Aff})} &= \langle g \star_{\mathrm{Aff}} S, T \rangle_{\mathcal{S}_2} \\
&= \left\langle S, \int_{\mathrm{Aff}} \overline{g(x,a)} U(-x,a) T U(-x,a)^* \, \frac{dx \, da}{a} \right\rangle_{\mathcal{S}_2} \\
&= \left\langle f_S, \int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} f_T \, \frac{dx \, da}{a} \right\rangle_{L^2_r(\mathrm{Aff})}.
\end{aligned}$$

With this motivation in mind we get the following definition.

Definition G.3.6. Let $S : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+)$ be the operator with affine Weyl symbol $f_S \in \mathcal{S}'(\mathrm{Aff})$ and let $g \in \mathcal{S}(\mathrm{Aff})$. Then $g \star_{\mathrm{Aff}} S$ is defined by its Weyl symbol $f_{g\star_{\mathrm{Aff}}S} \in \mathcal{S}'(\mathrm{Aff})$ satisfying

$$\langle f_{g\star_{\mathrm{Aff}}S}, h\rangle_{\mathcal{S}',\mathcal{S}} := \left\langle f_S, \int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} h \,\frac{dx\,da}{a}\right\rangle_{\mathcal{S}',\mathcal{S}},$$

for all $h \in \mathcal{S}(\mathrm{Aff})$.

Recall that the injectivity in Lemma G.3.6 ensures that the operator S in Definition G.3.6 is well-defined. The argument to show $f_{g\star_{\mathrm{Aff}}S} \in \mathcal{S}'(\mathrm{Aff})$ is similar to the one presented in Proposition G.3.12. Hence $g \star_{\mathrm{Aff}} S$ is well-defined.

Remark G.6. We could similarly have defined $S \star_{\mathrm{Aff}} A_f$ for $S \in \mathcal{S}(L^2(\mathbb{R}_+))$ and $f \in \mathcal{S}'(\mathrm{Aff})$ by using Proposition G.3.5. For brevity, we restrict ourselves in the next theorem to the case where $S = \phi \otimes \psi$ for $\psi, \phi \in \mathcal{S}(\mathrm{Aff})$. In this case, we can extend Lemma G.3.1 and define

$$(\phi \otimes \psi) \star_{\mathrm{Aff}} A_f := \langle A_f U(-x,a)\psi, U(-x,a)\phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

We can now finally state the main theorem in this section.

Theorem G.3.13. *The affine Weyl quantization A_g of $g \in \mathcal{S}(\mathrm{Aff})$ can be written as*

$$A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}},$$

where P_{Aff} is the affine parity operator. Moreover, for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ we have that the affine Weyl symbol $W_{\mathrm{Aff}}^{\psi,\phi}$ of the rank-one operator $\psi \otimes \phi$ can be written as

$$W_{\mathrm{Aff}}^{\psi,\phi} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Proof. Recall that the affine parity operator P_{Aff} is the affine Weyl quantization of the point measure $\delta_{(0,1)} \in \mathcal{S}'(\mathrm{Aff})$. As such, the convolution $g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ is well-defined with the interpretation given in Definition G.3.6. The affine Weyl symbol $f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}}$ of $g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ is acting on $h \in \mathcal{S}(\mathrm{Aff})$ by

$$\begin{aligned}
\langle f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}}, h\rangle_{\mathcal{S}',\mathcal{S}} &:= \overline{\left\langle \delta_{(0,1)}, \int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} h \,\frac{dx\,da}{a}\right\rangle_{\mathcal{S}',\mathcal{S}}} \\
&= \int_{\mathrm{Aff}} \overline{\overline{g(x,a)} h((0,1)\cdot(x,a))} \,\frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} g(x,a)\overline{h(x,a)} \,\frac{dx\,da}{a} \\
&= \langle g, h\rangle_{L^2_r(\mathrm{Aff})}.
\end{aligned}$$

Since $\mathcal{S}(\mathrm{Aff}) \subset L^2_r(\mathrm{Aff})$ is dense, we can conclude that $f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}} = g$ and thus $A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$. For the second statement, we get

$$\begin{aligned}((\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}})\,(x,a) &= \langle P_{\mathrm{Aff}} U(-x,a)\psi, U(-x,a)\phi\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \langle R_{\mathrm{Aff}}(x,a)\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} \\ &= W^{\psi,\phi}_{\mathrm{Aff}}(x,a). \qquad \square\end{aligned}$$

G.4 Operator Admissibility

For operator convolutions on the Heisenberg group, we have from (G.2.6) the important integral relation

$$\int_{\mathbb{R}^{2n}} S \star T(z)\, dz = \mathrm{tr}(S)\mathrm{tr}(T).$$

A similar formula for the integral of operator convolutions will not hold generally in the affine setting. We therefore search for a class of operators where such a relation does hold: the *admissible operators*. As a first step, we recall the notion of *admissible functions*.

Definition G.4.1. We say that $\psi \in L^2(\mathbb{R}_+)$ is *admissible* if

$$\int_0^\infty \frac{|\psi(r)|^2}{r}\,\frac{dr}{r} < \infty.$$

This definition of admissibility is motivated by the theorem of Duflo and Moore [90], see also [147]. The *Duflo-Moore operator* $\mathcal{D}^{-1}$ in our setting is formally given by

$$\mathcal{D}^{-1}\psi(r) := r^{-1/2}\psi(r).$$

It is clear that the Duflo-Moore operator $\mathcal{D}^{-1}$ is a densely defined, self-adjoint positive operator on $L^2(\mathbb{R}_+)$ with a densely defined inverse, namely

$$\mathcal{D}\psi(r) := \sqrt{r}\psi(r).$$

Clearly a function $\psi \in L^2(\mathbb{R}_+)$ is admissible if and only if $\mathcal{D}^{-1}\psi \in L^2(\mathbb{R}_+)$. We will on several occasions use the commutation relations

$$\mathcal{D}U(x,a) = \sqrt{\frac{1}{a}}U(x,a)\mathcal{D}, \quad U(x,a)^*\mathcal{D}^{-1} = \sqrt{a}\mathcal{D}^{-1}U(x,a)^*, \quad (x,a) \in \mathrm{Aff}. \tag{G.4.1}$$

The following orthogonality relation is a trivial reformulation of the classic orthogonality relations for wavelets, see for instance [148].

Proposition G.4.1. *Let $\phi, \psi, \xi, \eta \in L^2(\mathbb{R}_+)$ and assume that ψ and η are admissible. Then, with the abbreviation $\langle \cdot, \cdot \rangle \coloneqq \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)}$,*

$$\int_{\mathrm{Aff}} \langle \phi, U(-x,a)^*\psi\rangle \overline{\langle \xi, U(-x,a)^*\eta\rangle} \, \frac{dx\, da}{a} = \langle \phi, \xi\rangle \langle \mathcal{D}^{-1}\eta, \mathcal{D}^{-1}\psi\rangle.$$

In particular, we have

$$\int_{\mathrm{Aff}} \langle \phi, U(-x,a)^*\psi\rangle \overline{\langle \xi, U(-x,a)^*\psi\rangle} \, \frac{dx\, da}{a} = \langle \phi, \xi\rangle \|\mathcal{D}^{-1}\psi\|^2_{L^2(\mathbb{R}_+)}.$$

Remark G.7. By Proposition G.4.1, admissibility of $\psi \in L^2(\mathbb{R}_+)$ is equivalent to the condition

$$\int_{\mathrm{Aff}} |\langle \psi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\, da}{a} < \infty.$$

G.4.1 Admissibility for Operators

Our goal is now to extend the notion of admissibility to bounded operators on $L^2(\mathbb{R}_+)$, with the aim of obtaining a class of operators where a formula for the integral of operator convolutions similar to (G.2.6) holds. We will often use that any compact operator S on $L^2(\mathbb{R}_+)$ has a *singular value decomposition*

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \eta_n, \qquad N \in \mathbb{N} \cup \{\infty\}, \tag{G.4.2}$$

where $\{\xi_n\}_{n=1}^N$ and $\{\eta_n\}_{n=1}^N$ are orthonormal sets in $L^2(\mathbb{R}_+)$. The *singular values* $\{s_n\}_{n=1}^N$ with $s_n > 0$ will converge to zero when $N = \infty$. If S is a trace-class operator we have $\{s_n\}_{n=1}^N \in \ell^1(\mathbb{N})$ with $\|S\|_{\mathcal{S}_1} = \|s_n\|_{\ell^1}$. Since the admissible functions in $L^2(\mathbb{R}_+)$ form a dense subspace, we can always find an orthonormal basis consisting of admissible functions.

The next result concerns bounded operators $\mathcal{D}S\mathcal{D}$ for a trace-class operator S. To be precise, this means that we assume that S maps $\mathrm{dom}(\mathcal{D}^{-1})$ into $\mathrm{dom}(\mathcal{D})$, and that the operator $\mathcal{D}S\mathcal{D}$ defined on $\mathrm{dom}(\mathcal{D})$ extends to a bounded operator.

Theorem G.4.2. *Let $S \in \mathcal{S}_1$ satisfy that $\mathcal{D}S\mathcal{D} \in \mathcal{L}(L^2(\mathbb{R}_+))$. For any $T \in \mathcal{S}_1$ we have that $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} \in L^1_r(\mathrm{Aff})$ with*

$$\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{S}_1},$$

and

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a) \, \frac{dx\, da}{a} = \mathrm{tr}(T)\mathrm{tr}(S). \tag{G.4.3}$$

Proof. We divide the proof into three steps.
Step 1: We first assume that $T = \psi \otimes \phi$ for $\psi, \phi \in \mathrm{dom}(\mathcal{D})$. Recall that S can be written in the form (G.4.2). From Lemma G.3.1 and (G.4.1) we find that

$$\begin{aligned} T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a) &= \langle S\mathcal{D}U(-x,a)\psi, \mathcal{D}U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \frac{1}{a}\langle SU(-x,a)\mathcal{D}\psi, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \sum_{n=1}^{N} s_n \frac{1}{a}\langle U(-x,a)\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Integrating with respect to the right Haar measure and using that $(x,a) \mapsto (x,a)^{-1}$ interchanges left and right Haar measure, we get

$$\begin{aligned} &\int_{\mathrm{Aff}} |\langle U(-x,a)\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}| \frac{1}{a}\, \frac{dx\, da}{a} \\ &= \int_{\mathrm{Aff}} |\langle U(-x,a)^*\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)^*\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}| \, \frac{dx\, da}{a} \\ &\le \left(\int_{\mathrm{Aff}} |\langle U(-x,a)^*\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\, da}{a} \right)^{1/2} \\ &\quad \cdot \left(\int_{\mathrm{Aff}} |\langle \xi_n, U(-x,a)^*\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\, da}{a} \right)^{1/2} \\ &= \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}, \end{aligned}$$

where the last line uses Proposition G.4.1. It follows that the sum in the expression for $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a)$ converges absolutely in $L^1_r(\mathrm{Aff})$ with

$$\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \le \left(\sum_{n=1}^{N} s_n \right) \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)} = \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{S}_1}.$$

Equation (G.4.3) follows in a similar way by integrating the sum expressing $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ and using Proposition G.4.1.
Step 2: We now assume that $T = \psi \otimes \phi$ for arbitrary $\psi, \phi \in L^2(\mathbb{R}_+)$. Pick sequences $\{\psi_n\}_{n=1}^{\infty}, \{\phi_n\}_{n=1}^{\infty}$ in $\mathrm{dom}(\mathcal{D})$ converging to ψ and ϕ, respectively, and let $T_n = \psi_n \otimes \phi_n$. It is straightforward to check that T_n converges to T in $\mathcal{S}_1$. By (G.3.4) this implies that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges uniformly to $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$. On the other hand, $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ is a Cauchy sequence in $L^1_r(\mathrm{Aff})$: for $m, n \in \mathbb{N}$ we find by

Step 1 that

$$\begin{aligned}\|T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} - T_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} &\\ &\leq \|\psi_n \otimes \phi_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} - \psi_m \otimes \phi_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\ &\quad + \|\psi_m \otimes \phi_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} - \psi_m \otimes \phi_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\ &= \|(\psi_n - \psi_m) \otimes \phi_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\ &\quad + \|\psi_m \otimes (\phi_n - \phi_m) \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\ &\leq \|S\|_{\mathcal{S}_1} \|\psi_n - \psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_n\|_{L^2(\mathbb{R}_+)} \\ &\quad + \|S\|_{\mathcal{S}_1} \|\psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_m - \phi_n\|_{L^2(\mathbb{R}_+)}\end{aligned}$$

which clearly goes to zero as $m, n \to \infty$. This means that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges in $L^1_r(\mathrm{Aff})$, and the limit must be $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ as we already know that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges uniformly to this function. In particular, this implies

$$\begin{aligned}\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} &= \lim_{n\to\infty} \|T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\ &\leq \lim_{n\to\infty} \|\psi_n\|_{L^2(\mathbb{R}_+)} \|\phi_n\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1} \\ &= \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1}.\end{aligned}$$

Equation (G.4.3) also follows by taking the limit of $\int_{\mathrm{Aff}} T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a)\, \frac{dx\,da}{a}$.
Step 3: We now assume that $T \in \mathcal{S}_1$. Consider the singular value decomposition of T given by

$$T = \sum_{m=1}^{M} t_m \psi_m \otimes \phi_m$$

for $M \in \mathbb{N} \cup \{\infty\}$. By (G.3.4) we have, with uniform convergence of the sum, that

$$T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} = \sum_{m=1}^{M} t_m \psi_m \otimes \phi_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}. \tag{G.4.4}$$

Notice that Step 2 implies that the convergence is also in $L^1_r(\mathrm{Aff})$, since

$$\begin{aligned}\sum_{m=1}^{M} t_m \|\psi_m \otimes \phi_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} &\leq \sum_{m=1}^{M} t_m \|\psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_m\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1} \\ &= \|T\|_{\mathcal{S}_1} \|S\|_{\mathcal{S}_1}.\end{aligned}$$

In particular, $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} \in L^1_r(\mathrm{Aff})$. Finally, (G.4.3) follows by integrating (G.4.4) and using that the sum converges in $L^1_r(\mathrm{Aff})$ and Step 2. □

The integral relation (G.4.3) is somewhat artificial in the sense that it introduces $\mathcal{D}$ in the integrand. We will typically be interested in the integral of $T \star_{\text{Aff}} S$, not of $T \star_{\text{Aff}} \mathcal{D}S\mathcal{D}$. This motivates the following definition.

Definition G.4.2. Let $S \neq 0$ be a bounded operator on $L^2(\mathbb{R}_+)$ that maps $\text{dom}(\mathcal{D})$ into $\text{dom}(\mathcal{D}^{-1})$. We say that S is *admissible* if the composition $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is bounded on $\text{dom}(\mathcal{D}^{-1})$ and extends to a trace-class operator $\mathcal{D}^{-1}S\mathcal{D}^{-1} \in \mathcal{S}_1$.

Assume now that S is admissible, and define $R := \mathcal{D}^{-1}S\mathcal{D}^{-1}$. Clearly R maps $\text{dom}(\mathcal{D}^{-1})$ into $\text{dom}(\mathcal{D})$ as we assume that S maps $\text{dom}(\mathcal{D})$ into $\text{dom}(\mathcal{D}^{-1})$. The following corollary is therefore immediate from Theorem G.4.2. We also note that it extends [181, Cor. 1] to non-positive, non-compact operators.

Corollary G.4.2.1. *Let $S \in \mathcal{L}(L^2(\mathbb{R}_+))$ be an admissible operator. For any $T \in \mathcal{S}_1$ we have that $T \star_{\text{Aff}} S \in L^1_r(\text{Aff})$ with*

$$\|T \star_{\text{Aff}} S\|_{L^1_r(\text{Aff})} \leq \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}\|T\|_{\mathcal{S}_1},$$

and

$$\int_{\text{Aff}} T \star_{\text{Aff}} S(x,a)\,\frac{dx\,da}{a} = \text{tr}(T)\text{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

Example G.4.1. A rank-one operator $S = \eta \otimes \xi$ for non-zero η, ξ is an admissible operator if and only if $\eta, \xi \in L^2(\mathbb{R}_+)$ are admissible functions. Requiring that S maps $\text{dom}(\mathcal{D})$ into $\text{dom}(\mathcal{D}^{-1})$ clearly implies that $\eta \in \text{dom}(\mathcal{D}^{-1})$, i.e. η is admissible. For $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ to be trace-class, the map

$$\psi \mapsto \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\psi\|_{L^2(\mathbb{R}_+)} = |\langle \mathcal{D}^{-1}\psi, \xi\rangle_{L^2(\mathbb{R}_+)}| \cdot \|\mathcal{D}^{-1}\eta\|_{L^2(\mathbb{R}_+)}, \quad \psi \in \text{dom}(\mathcal{D}^{-1}),$$

must at least be bounded for $\|\psi\|_{L^2(\mathbb{R}_+)} \leq 1$. This is bounded if and only if

$$\psi \mapsto \langle \mathcal{D}^{-1}\psi, \xi\rangle_{L^2(\mathbb{R}_+)}$$

is bounded, which is precisely the condition that $\xi \in \text{dom}\left(\left(\mathcal{D}^{-1}\right)^*\right) = \text{dom}(\mathcal{D}^{-1})$. Hence our notion of admissibility for operators naturally extends the classical function admissibility. In the case of rank-one operators, it follows from Lemma G.3.1 and the computation

$$\text{tr}(\mathcal{D}^{-1}(\eta \otimes \xi)\mathcal{D}^{-1}) = \langle \mathcal{D}^{-1}\eta, \mathcal{D}^{-1}\xi\rangle_{L^2(\mathbb{R}_+)}$$

that Corollary G.4.2.1 reduces to Proposition G.4.1.

When both S and T are admissible trace-class operators, their convolution $T \star_{\text{Aff}} S$ behaves well with respect to both the left and right Haar measures.

Corollary G.4.2.2. *Let S and T be admissible trace-class operators on $L^2(\mathbb{R}_+)$. Then the convolution $T \star_{\mathrm{Aff}} S$ satisfies $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff}) \cap L^1_l(\mathrm{Aff})$ and*

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \mathrm{tr}(T)\mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}),$$
$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a^2} = \mathrm{tr}(S)\mathrm{tr}(\mathcal{D}^{-1} T \mathcal{D}^{-1}).$$

Proof. The first equation and the claim that $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$ is Corollary G.4.2.1. The second equation and the claim that $T \star_{\mathrm{Aff}} S \in L^1_l(\mathrm{Aff})$ follows since

$$T \star_{\mathrm{Aff}} S(x,a) = S \star_{\mathrm{Aff}} T((x,a)^{-1}).$$ □

We now turn to the case where S is a positive compact operator. We first note that admissibility in this case becomes a statement about the eigenvectors and eigenvalues of S.

Proposition G.4.3. *Let S be a non-zero positive compact operator with spectral decomposition*

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \xi_n$$

for $N \in \mathbb{N} \cup \{\infty\}$. Then S is admissible if and only each ξ_n is admissible and

$$\sum_{n=1}^{N} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)} < \infty.$$

Proof. We first assume that S is admissible. By linearity and Lemma G.3.1 we get for $\xi \in L^2(\mathbb{R}_+)$ with $\|\xi\|_{L^2(\mathbb{R}_+)} = 1$ that

$$\xi \otimes \xi \star_{\mathrm{Aff}} S(x,a) = \sum_{n=1}^{N} s_n |\langle \xi, U(-x,a)^* \xi_n \rangle_{L^2(\mathbb{R}_+)}|^2. \tag{G.4.5}$$

Integrating (G.4.5) using the monotone convergence theorem and Proposition G.4.1, we obtain

$$\int_{\mathrm{Aff}} \xi \otimes \xi \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \sum_{n=1}^{N} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)}.$$

The claim now follows from Corollary G.4.2.1.

For the converse, it is clear by the assumption that the operator

$$\sum_{n=1}^{N} s_n (\mathcal{D}^{-1}\xi_n) \otimes (\mathcal{D}^{-1}\xi_n) \tag{G.4.6}$$

is a trace-class operator. It only remains to show that S maps $\operatorname{dom}(\mathcal{D})$ into $\operatorname{dom}(\mathcal{D}^{-1})$ and that $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is given by (G.4.6). This is easily shown when N is finite, so we do the proof for $N = \infty$.

The partial sums for $\psi \in L^2(\mathbb{R}_+)$ are denoted by

$$(S\psi)_M \coloneqq \sum_{n=1}^{M} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \xi_n,$$

and converge in the sense that $(S\psi)_M \to S\psi$ as $M \to \infty$. Furthermore, it is clear that $(S\psi)_M$ is in the domain of $\mathcal{D}^{-1}$ for each M as each ξ_n is admissible. We also have that

$$\mathcal{D}^{-1}(S\psi)_M = \sum_{n=1}^{M} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n.$$

The sequence of partial sums $\mathcal{D}^{-1}(S\psi)_M$ also converges in $L^2(\mathbb{R}_+)$, since by using Hölder's inequality and Bessel's inequality we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} s_n |\langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)}| \cdot \|\mathcal{D}^{-1}\xi_n\|_{L^2(\mathbb{R}_+)} \\
&\qquad \le \left(\sum_{n=1}^{\infty} |\langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)}|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} s_n^2 \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)} \right)^{1/2} \\
&\qquad \lesssim \|\psi\|_{L^2(\mathbb{R}_+)} \left(\sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)} \right)^{1/2}.
\end{aligned}$$

Since $\mathcal{D}^{-1}$ is a closed operator, we get that $S\psi$ belongs to the domain of $\mathcal{D}^{-1}$ and

$$\mathcal{D}^{-1}S\psi = \sum_{n=1}^{\infty} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n.$$

For any $\phi \in \operatorname{dom}(\mathcal{D}^{-1})$, we have that

$$\mathcal{D}^{-1}S\mathcal{D}^{-1}\phi = \sum_{n=1}^{\infty} s_n \langle \mathcal{D}^{-1}\phi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n = \sum_{n=1}^{\infty} s_n \langle \phi, \mathcal{D}^{-1}\xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n,$$

so $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ agrees with (G.4.6) on this dense subspace. In fact, they agree on all of $L^2(\mathbb{R}_+)$ since

$$\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\phi\|_{L^2(\mathbb{R}_+)} \le \|\phi\|_{L^2(\mathbb{R}_+)} \sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)},$$

shows that $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ extends to a bounded operator. □

As a consequence of Proposition G.4.3, we obtain a compact reformulation of admissibility for positive trace-class operators.

Corollary G.4.3.1. *Let T be a non-zero positive trace-class operator on $L^2(\mathbb{R}_+)$, and let S be a non-zero positive compact operator. If*

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\, \frac{dx\, da}{a} < \infty,$$

then S is admissible with

$$\mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}) = \frac{1}{\mathrm{tr}(T)} \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\, \frac{dx\, da}{a}.$$

In particular, if S is a non-zero, positive trace-class operator, then S is admissible if and only if $S \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$.

Proof. Let

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \xi_n$$

be the spectral decomposition of S. An argument similar to the one giving in the proof of Proposition G.4.3 shows that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\, \frac{dx\, da}{a} = \mathrm{tr}(T) \sum_{n=1}^{N} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)}.$$

The claims now follow immediately from Proposition G.4.3. □

G.4.2 Admissible Operators from Laguerre Functions

Although we derived several basic properties of admissible operators in Section G.4.1, we have not given any way to construct such operators in practice. Our construction is based on the following observation: From Proposition G.4.3 we know that if

$$S = \sum_{n=1}^{\infty} s_n \varphi_n \otimes \varphi_n$$

is a non-zero positive compact operator with

$$\sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1}\varphi_n\|^2_{L^2(\mathbb{R}_+)} < \infty,$$

then S is admissible. So if we can find an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of admissible functions such that we can control the terms $\|\mathcal{D}^{-1}\varphi_n\|_{L^2(\mathbb{R}_+)}$, then we can construct admissible operators as infinite linear combinations of rank-one operators. It turns out that the Laguerre basis works extremely well in this regard.

Definition G.4.3. For fixed $\alpha \in \mathbb{R}_+$ we define the *Laguerre basis* $\left\{\mathcal{L}_n^{(\alpha)}\right\}_{n=0}^{\infty}$ for $L^2(\mathbb{R}_+)$ by

$$\mathcal{L}_n^{(\alpha)}(r) := \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\, r^{\frac{\alpha+1}{2}} e^{-\frac{r}{2}} L_n^{(\alpha)}(r), \qquad n \in \mathbb{N}_0,\ r \in \mathbb{R}_+, \tag{G.4.7}$$

where Γ denotes the gamma function and $L_n^{(\alpha)}$ denotes the *generalized Laguerre polynomials* given by

$$L_n^{(\alpha)}(r) := \frac{r^{-\alpha} e^r}{n!} \frac{d^n}{dr^n} \left(e^{-r} r^{n+\alpha}\right) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{r^k}{k!}.$$

The classical orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \tag{G.4.8}$$

for the generalized Laguerre polynomials ensures that the Laguerre bases are orthonormal bases for $L^2(\mathbb{R}_+)$ for any fixed $\alpha \in \mathbb{R}_+$. The following result shows that the Laguerre basis is especially compatible with the Duflo-Moore operator $\mathcal{D}^{-1}$.

Proposition G.4.4. *For any $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{N}_0$ we have*

$$\left\|\mathcal{D}^{-1} \mathcal{L}_n^{(\alpha)}\right\|_{L^2(\mathbb{R}_+)}^2 = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-r} r^{\alpha-1} \left(L_n^{(\alpha)}(r)\right)^2 dr = \frac{1}{\alpha}. \tag{G.4.9}$$

Proof. The first equality in (G.4.9) follows from unwinding the definitions. For the second equality in (G.4.9), we will use the well-known identity

$$L_n^{(\alpha)}(r) = \sum_{j=0}^{n} L_j^{(\alpha-1)}(r)$$

together with the orthogonality relation (G.4.8). This gives

$$\begin{aligned}
\int_0^\infty e^{-r} r^{\alpha-1} \left(L_n^{(\alpha)}(r)\right)^2 dr &= \sum_{i,j=0}^{n} \int_0^\infty e^{-r} r^{\alpha-1} L_i^{(\alpha-1)}(r) L_j^{(\alpha-1)}(r)\, dr \\
&= \sum_{i=0}^{n} \frac{\Gamma(i+\alpha)}{i!} \\
&= \frac{1}{\alpha} \frac{\Gamma(n+\alpha+1)}{n!},
\end{aligned}$$

where the last equality follows from a straightforward induction argument. □

The following consequence from Proposition G.4.3 shows that we can explicitly construct admissible operators by using the Laguerre basis.

Corollary G.4.4.1. *Let* $\{s_n\}_{n=0}^{\infty} \in \ell^1(\mathbb{N})$ *be a sequence of non-negative numbers and let* $\alpha \in \mathbb{R}_+$*. Then*

$$S := \sum_{n=0}^{\infty} s_n \mathcal{L}_n^{(\alpha)} \otimes \mathcal{L}_n^{(\alpha)} \tag{G.4.10}$$

is an admissible operator with

$$\operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}) = \frac{1}{\alpha} \sum_{n=0}^{\infty} s_n.$$

Remark G.8. The corollary may be considered a reformulation with slightly different proof of the calculations in [125, Section 3.3], where a resolution of the identity operator is constructed from thermal states that are diagonal in the Laguerre basis. We will return to resolutions of the identity operator and the relation to admissibility in Section G.6.2.

G.4.3 Connection with Convolutions and Quantizations

We will now see how admissibility relates to the convolution of a function with an operator. The following result shows that we can use convolutions to generate new admissible operators from a given admissible operator.

Proposition G.4.5. *Let* $f \in L^1_l(\mathrm{Aff}) \cap L^1_r(\mathrm{Aff})$ *be a non-zero positive function. If* S *is a positive, admissible trace-class operator on* $L^2(\mathbb{R}_+)$*, then so is* $f \star_{\mathrm{Aff}} S$ *with*

$$\operatorname{tr}\left(\mathcal{D}^{-1}(f \star_{\mathrm{Aff}} S)\mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} f(x,a) \,\frac{dx\,da}{a^2} \operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

Proof. It is clear from (G.3.3) that $f \star_{\mathrm{Aff}} S$ is a trace-class operator, and positivity follows from the definition of the convolution $f \star_{\mathrm{Aff}} S$. Let T be a non-zero positive trace-class operator on $L^2(\mathbb{R}_+)$. It suffices by Corollary G.4.3.1 to show that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b) \,\frac{dy\,db}{b} = \operatorname{tr}(T) \int_{\mathrm{Aff}} f(x,a) \,\frac{dx\,da}{a^2} \operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

We have that

$$\begin{aligned}
&T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b) \\
&\qquad = \operatorname{tr}\left(TU(-y,b)^* \int_{\mathrm{Aff}} f(x,a) U(-x,a)^* S U(-x,a) \,\frac{dx\,da}{a} U(-y,b)\right) \\
&\qquad = \int_{\mathrm{Aff}} f(x,a) \operatorname{tr}(TU((-x,a)\cdot(-y,b))^* S U((-x,a)\cdot(-y,b)) \,\frac{dx\,da}{a} \\
&\qquad = \int_{\mathrm{Aff}} f(x,a) T \star_{\mathrm{Aff}} S((x,a)\cdot(y,b)) \,\frac{dx\,da}{a}.
\end{aligned}$$

We use Fubini's theorem, which applies by our assumptions on f and S, to get

$$\begin{aligned}
&\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b) \, \frac{dy\, db}{b} \\
&\qquad = \int_{\mathrm{Aff}} f(x,a) \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S((x,a)\cdot(y,b)) \, \frac{dy\, db}{b} \, \frac{dx\, da}{a} \\
&\qquad = \int_{\mathrm{Aff}} f(x,a) \, \frac{dx\, da}{a} \Delta(x,a) \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(y,b) \, \frac{dy\, db}{b} \\
&\qquad = \int_{\mathrm{Aff}} f(x,a) \, \frac{dx\, da}{a^2} \mathrm{tr}(T) \mathrm{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right),
\end{aligned}$$

where we used the admissibility of S and Theorem G.4.2.1 in the last line. □

Remark G.9. We can give a simple heuristic argument for Proposition G.4.5 by ignoring that $\mathcal{D}^{-1}$ is unbounded as follows: We have by using (G.4.1) that

$$\begin{aligned}
\mathcal{D}^{-1}(f \star_{\mathrm{Aff}} S)\mathcal{D}^{-1} &= \int_{\mathrm{Aff}} f(x,a)\mathcal{D}^{-1} U(-x,a)^* S U(-x,a) \mathcal{D}^{-1} \, \frac{dx\, da}{a} \\
&= \int_{\mathrm{Aff}} f(x,a) U(-x,a)^* \mathcal{D}^{-1} S \mathcal{D}^{-1} U(-x,a) \, \frac{dx\, da}{a^2}.
\end{aligned}$$

Since $\mathcal{D}^{-1} S \mathcal{D}^{-1}$ is a trace-class operator, the integral above is a convergent Bochner integral and we obtain the desired equality.

G.4.4 Admissibility as a Measure of Non-Unimodularity

In this section we will delve more into how the non-unimodularity of the affine group affects the affine Weyl quantization. As we will see, both the left and right Haar measures take on an active role in this picture.

Proposition G.4.6. *Let S be an admissible Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ such that its affine Weyl symbol f_S satisfies $f_S \in L^1_l(\mathrm{Aff})$. Then*

$$\mathrm{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} f_S(x,a) \, \frac{dx\, da}{a^2}.$$

Proof. Let $T = \varphi \otimes \varphi$ for some non-zero $\varphi \in \mathcal{S}(\mathbb{R}_+)$. Then the affine Weyl symbol of T is $f_T = W^{\varphi}_{\mathrm{Aff}} \in \mathcal{S}(\mathrm{Aff})$. We know by Corollary G.4.2.1 that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\, da}{a} = \mathrm{tr}(T) \mathrm{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right).$$

On the other hand, Fubini's theorem together with Proposition G.3.5 gives

$$\begin{aligned}\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\,\frac{dx\,da}{a} &= \int_{\mathrm{Aff}} f_T *_{\mathrm{Aff}} \check{f}_S(x,a)\,\frac{dx\,da}{a}\\ &= \int_{\mathrm{Aff}} f_T(y,b)\int_{\mathrm{Aff}} f_S((y,b)(x,a)^{-1})\,\frac{dx\,da}{a}\,\frac{dy\,db}{b}\\ &= \int_{\mathrm{Aff}} f_T(y,b)\,\frac{dy\,db}{b}\int_{\mathrm{Aff}} f_S(x,a)\,\frac{dx\,da}{a^2}.\end{aligned}$$

The marginal properties of the affine Wigner distribution (G.2.16) show that

$$\int_{\mathrm{Aff}} f_T(y,b)\,\frac{dy\,db}{b} = \|\varphi\|^2_{L^2(\mathbb{R}_+)} = \mathrm{tr}(T).$$

The claim now follows from combining the calculations we have done. □

Remark G.10. Assuming that T is a trace-class operator we have that

$$\mathrm{tr}(T) = \int_{\mathrm{Aff}} f_T(x,a)\,\frac{dx\,da}{a},$$

which follows from a similar proof to the one in Proposition G.4.6. This gives the interesting heuristic interpretation that taking $\mathcal{D}^{-1}T\mathcal{D}^{-1}$ of an operator T coincides with multiplying f_T by $\frac{1}{a}$.

The following result shows that the affine Wigner distribution satisfies both left and right integrability when more is assumed of the input. This should be compared with the Heisenberg case where the Heisenberg group $\mathbb{H}^n$ is unimodular.

Theorem G.4.7. *Assume that $\phi, \psi, \mathcal{D}\phi, \mathcal{D}\psi \in L^2(\mathbb{R}_+)$. Then the affine Wigner distribution satisfies*

$$W^{\phi,\psi}_{\mathrm{Aff}} \in L^2_r(\mathrm{Aff}) \cap L^2_l(\mathrm{Aff}).$$

Proof. We already know that $W^{\phi,\psi}_{\mathrm{Aff}}$ is in $L^2_r(\mathrm{Aff})$ by the orthogonality relations (G.2.15). Using the definition of the affine Wigner distribution and Plancherel's theorem, we have that

$$\begin{aligned}\|W^{\phi,\psi}_{\mathrm{Aff}}\|_{L^2_l(\mathrm{Aff})} &= \int_{\mathrm{Aff}} \left|\phi(a\lambda(x))|^2|\psi(a\lambda(-x))\right|^2\,\frac{dx\,da}{a^2}\\ &= \int_0^\infty\int_0^\infty |\phi(v)|^2|\psi(w)|^2\frac{v-w}{\log(v/w)}\frac{dw\,dv}{vw},\end{aligned}$$

where we used the change of variables $v = a\lambda(x)$ and $w = a\lambda(-x)$ in the last line. By our assumptions on ϕ and ψ, it will suffice to show that for all $v, w \in \mathbb{R}_+$ we have the upper bound

$$\frac{v-w}{vw\log(v/w)} \leq 2\cdot\max\left\{1, \frac{1}{v}, \frac{1}{w}, \frac{1}{vw}\right\}.$$

It will be enough by symmetry to consider $\Lambda = \{(v, w) \in \mathbb{R}_+ \times \mathbb{R}_+ \,:\, v > w\}$. We have the decomposition $\Lambda = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, where

$$\mathcal{C}_1 \coloneqq \left\{(v, w) \in \Lambda \,:\, w \le -2\sigma(-v/2)\right\},$$
$$\mathcal{C}_2 \coloneqq \left\{(v, w) \in \Lambda \,:\, w \ge \frac{-1}{\sigma(-1/v)}\right\},$$
$$\mathcal{C}_3 \coloneqq \left\{(v, w) \in \Lambda \,:\, -2\sigma(-v/2) \le w \le \frac{-1}{\sigma(-1/v)}\right\},$$

where σ is the function appearing in Lemma G.3.10.

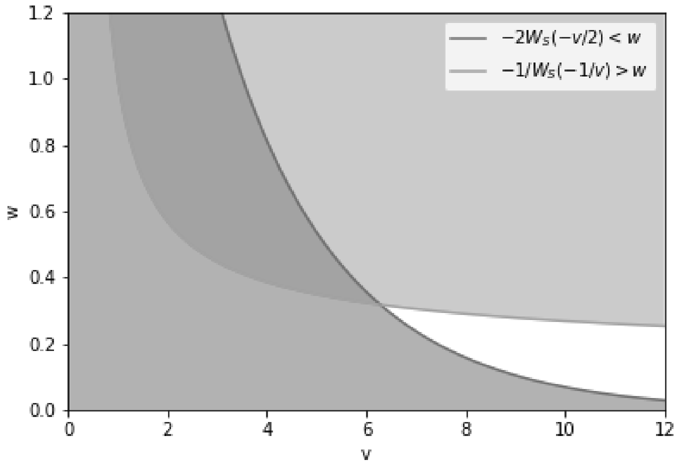

Figure G.1: A drawing marking the beginning and end of the different domains.

- The level surface $g(v, w) = (v - w)/\log(v/w) = C$ for $C > 0$ is given by the equation
$$w = -C\sigma\left(-\frac{v}{C}\right). \tag{G.4.11}$$
On $\mathcal{C}_1$ we are below the level surface (G.4.11) with $C = 2$. Notice that $(1, 0.5) \in \mathcal{C}_1$ with $g(1, 0.5) = \log(\sqrt{2}) < 2$. The continuity of g forces the inequality $g(v, w) \le 2$ for all $(v, w) \in \mathcal{C}_1$. Hence
$$\frac{v - w}{vw \log(v/w)} \le \frac{2}{vw}.$$

- Notice that
$$\frac{v - w}{vw \log(v/w)} = \frac{\frac{1}{v} - \frac{1}{w}}{\log((1/v)/(1/w))}.$$

Hence the case of $\mathcal{C}_2$ follows from the previous the argument for $\mathcal{C}_1$ by considering the level surface of $g(1/v, 1/w) = 1$.

- It is straightforward to verify that $v > 2$ and $w < 1$ when $(v, w) \in \mathcal{C}_3$. Hence we obtain for any $(v, w) \in \mathcal{C}_3$ that

$$\frac{v - w}{wv \log(v/w)} \leq \frac{v}{wv \log(2)} \leq 2/w. \qquad \square$$

Remark G.11. The connection from this result to admissibility is that the assumptions boil down to $S = \mathcal{D}\psi \otimes \mathcal{D}\phi$ being an admissible operator.

Remark G.12. Let A be a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ with integral kernel A_K. One can gauge from the proof of Theorem G.4.7 that the affine Weyl symbol f_A satisfies $f_A \in L^2_r(\mathrm{Aff}) \cap L^2_l(\mathrm{Aff})$ if and only if the integral kernel A_K satisfies

$$A_K \in L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, \frac{s-t}{st \log(s/t)} dt\, ds\right) \cap L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, \frac{1}{st} dt\, ds\right).$$

G.4.5 Extending the Setting

Except for Section G.3.5, we have so far considered convolutions between rather well-behaved functions and operators and obtained norm estimates for the norms of $L^1_r(\mathrm{Aff})$, $L^\infty(\mathrm{Aff})$, $\mathcal{S}_1$ and $\mathcal{L}(L^2(\mathbb{R}_+))$. We have seen that

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{S}_1},$$
$$\|T \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|T\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|S\|_{\mathcal{S}_1}.$$

This section generalizes these inequalities to other Schatten classes and L^p spaces.

Proposition G.4.8. *Let $1 \leq p \leq \infty$ and let q be its conjugate exponent given by $p^{-1} + q^{-1} = 1$. If $S \in \mathcal{S}_p, T \in \mathcal{S}_q$, and $f \in L^1_r(\mathrm{Aff})$, then the following hold:*

1. *$f \star_{\mathrm{Aff}} S \in \mathcal{S}_p$ with $\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_p} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{S}_p}$.*
2. *$T \star_{\mathrm{Aff}} S \in L^\infty(\mathrm{Aff})$ with $\|T \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_p} \|T\|_{\mathcal{S}_q}$.*

Proof. For $p < \infty$, we can clearly interpret the definition of $f \star_{\mathrm{Aff}} S$ as a convergent Bochner integral in $\mathcal{S}_p$. Hence the first inequality follows from [166, Prop. 1.2.2]. For $p = \infty$, we avoid the unpleasantness of Bochner integration in non-separable Banach spaces by interpreting $f \star_{\mathrm{Aff}} S$ weakly by

$$\langle f \star_{\mathrm{Aff}} S\psi, \phi \rangle_{L^2(\mathbb{R}_+)} = \int_{\mathrm{Aff}} f(x, a) \langle SU(-x, a)\psi, U(-x, a)\phi \rangle_{L^2(\mathbb{R}_+)} \frac{dx\, da}{a},$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$. By a standard argument, $f \star_{\mathrm{Aff}} S$ is a bounded operator with

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{L}(L^2(\mathbb{R}_+))}.$$

Inequality 2. follows from the Hölder type inequality [232, Thm. 2.8]. □

We have already seen in Section G.4.1 that we can say more about operator convolutions when one of the operators is admissible. As the next lemma shows, admissibility is also the correct condition to ensure that $f \star_{\mathrm{Aff}} S$ defines a bounded operator for all $f \in L^\infty(\mathrm{Aff})$.

Lemma G.4.9. *Let $S \in \mathcal{S}_1$ and $f \in L^\infty(\mathrm{Aff})$. Define the operator $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ weakly for $\psi, \phi \in \mathrm{Dom}(\mathcal{D})$ by*

$$\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)} = \int_{\mathrm{Aff}} f(x,a) \langle S\mathcal{D}U(-x,a)\psi, \mathcal{D}U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a}. \tag{G.4.12}$$

Then $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ uniquely extends to a bounded linear operator on $L^2(\mathbb{R}_+)$ satisfying

$$\|f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^\infty(\mathrm{Aff})} \|S\|_{\mathcal{S}_1}.$$

In particular, if R is an admissible operator, then $f \star_{\mathrm{Aff}} R \in \mathcal{L}(L^2(\mathbb{R}_+))$ with

$$\|f \star_{\mathrm{Aff}} R\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^\infty(\mathrm{Aff})} \|\mathcal{D}^{-1} R \mathcal{D}^{-1}\|_{\mathcal{S}_1}.$$

Proof. By using (G.4.1) and abbreviating $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)}$ we get that

$$\begin{aligned}
\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle &= \int_{\mathrm{Aff}} f(x,a) \langle SU(-x,a)\mathcal{D}\psi, U(-x,a)\mathcal{D}\phi\rangle \frac{dx\,da}{a^2} \\
&= \int_{\mathrm{Aff}} \check{f}(x,a) \langle SU(-x,a)^*\mathcal{D}\psi, U(-x,a)^*\mathcal{D}\phi\rangle \frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} \check{f}(x,a) (S \star_{\mathrm{Aff}} (\mathcal{D}\psi \otimes \mathcal{D}\phi))(x,a) \frac{dx\,da}{a}.
\end{aligned}$$

Clearly $\mathcal{D}\psi \otimes \mathcal{D}\phi$ is an admissible operator with

$$|\mathrm{tr}(\mathcal{D}^{-1}(\mathcal{D}\psi \otimes \mathcal{D}\phi)\mathcal{D}^{-1})| = |\langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)}| \leq \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$

By Corollary G.4.2.1 we therefore get

$$\left|\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)}\right| \leq \|f\|_{L^\infty(\mathrm{Aff})} \|S\|_{\mathcal{S}_1} \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$

The density of $\mathrm{dom}(\mathcal{D})$ implies that $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ extends to a bounded operator on $L^2(\mathbb{R}_+)$. □

Armed with Lemma G.4.9 and Corollary G.4.2.1, we prove the following result describing L^p and $\mathcal{S}_p$ properties of convolutions with admissible operators. The proof is essentially an application of complex interpolation; we refer to [232, Thm. 2.10] and [43, Thm. 5.1.1] for the interpolation theory of $\mathcal{S}_p$ and $L^p_r(\mathrm{Aff})$.

Proposition G.4.10. *Let $1 \leq p \leq \infty$ and q its conjugate exponent. If $R \in \mathcal{S}_p$, $g \in L^p_r(\mathrm{Aff})$, and S is an admissible trace-class operator, then:*

1. $g \star_{\mathrm{Aff}} S \in \mathcal{S}_p$ *with* $\|g \star_{\mathrm{Aff}} S\|_{\mathcal{S}_p} \leq \|S\|_{\mathcal{S}_1}^{1/p} \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}_1}^{1/q} \|g\|_{L^p_r(\mathrm{Aff})}$.

2. $R \star_{\mathrm{Aff}} S \in L^p_r(\mathrm{Aff})$ *with* $\|R \star_{\mathrm{Aff}} S\|_{L^p_r(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1}^{1/q} \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}_1}^{1/p} \|R\|_{\mathcal{S}_p}$.

Proof. For $g \in L^1_r(\mathrm{Aff}) \cap L^\infty(\mathrm{Aff})$, we have for $p = \infty$ that Lemma G.4.9 gives

$$\|g \star_{\mathrm{Aff}} S\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}_1} \|g\|_{L^\infty(\mathrm{Aff})}.$$

Since we also have $\|g \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|g\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{S}_1}$, the first result follows by complex interpolation. For the second claim, if $R \in \mathcal{S}_1$ we know from Corollary G.4.2.1 that

$$\|R \star_{\mathrm{Aff}} S\|_{L^1_r(\mathrm{Aff})} \leq \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}_1} \|R\|_{\mathcal{S}_1}.$$

The result follows by complex interpolation since

$$\|R \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1} \|R\|_{\mathcal{L}(L^2(\mathbb{R}_+))}.$$ □

G.5 From the Viewpoint of Representation Theory

We will for completeness investigate how various notions of affine Fourier transforms fit into our framework. As we will see, known results from abstract wavelet analysis give connections between affine Weyl quantization, affine Fourier transforms, and admissibility for operators.

G.5.1 Affine Fourier Transforms

Definition G.5.1. For $f \in L^1_l(\mathrm{Aff})$ we define the *(left) integrated representation* $U(f)$ to be the operator on $L^2(\mathbb{R}_+)$ given by

$$U(f)\psi := \int_{\mathrm{Aff}} f(x,a) U(x,a)\psi \, \frac{dx\, da}{a^2}, \qquad \psi \in L^2(\mathbb{R}_+).$$

The *inverse affine Fourier-Wigner transform* $\mathcal{F}_W^{-1}(f)$ of $f \in L^1_r(\mathrm{Aff})$ is given by

$$\mathcal{F}_W^{-1}(f) := U(\check{f}) \circ \mathcal{D}.$$

The inverse affine Fourier-Wigner transform $\mathcal{F}_W^{-1}(f)$ of $f \in L_r^1(\mathrm{Aff})$ is explicitly given by

$$\mathcal{F}_W^{-1}(f)\psi(s) = \int_0^\infty \sqrt{r}\mathcal{F}_1(f)(r, s/r)\psi(r)\,\frac{dr}{r},$$

where $\mathcal{F}_1$ denotes the Fourier transform in the first coordinate and $\psi \in L^2(\mathbb{R}^+)$. Hence the integral kernel of $\mathcal{F}_W^{-1}(f)$ is given by

$$K_f(s,r) = \sqrt{r}(\mathcal{F}_1 f)(r, s/r), \qquad s, r \in \mathbb{R}_+. \tag{G.5.1}$$

It is straightforward to verify that we have the estimate

$$\|\mathcal{F}_W^{-1}(f)\|_{\mathcal{S}_2} \leq \|f\|_{L_r^2(\mathrm{Aff})},$$

for every $f \in L_r^1(\mathrm{Aff}) \cap L_r^2(\mathrm{Aff})$. Hence we can extend $\mathcal{F}_W^{-1}$ to be defined on $L_r^2(\mathrm{Aff})$ and we have that $\mathcal{F}_W^{-1}(f) \in \mathcal{S}_2$ for any $f \in L_r^2(\mathrm{Aff})$.

Proposition G.5.1. *The inverse affine Fourier-Wigner transform is a unitary transformation $\mathcal{F}_W^{-1} : \mathcal{Q}_1 \to \mathcal{S}_2$, where*

$$\mathcal{Q}_1 := \{f \in L_r^2(\mathrm{Aff}) \mid \operatorname{ess\,supp}(\mathcal{F}_1(f)) \subset \mathbb{R}_+ \times \mathbb{R}_+\}.$$

Proof. Any function $K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ can be written uniquely on the form K_f in (G.5.1) for some $f \in \mathcal{Q}_1$. Moreover, we have

$$\|K_f\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+)} = \sqrt{\int_0^\infty \int_0^\infty |\mathcal{F}_1 f(r, s/r)|^2\, dr\, \frac{ds}{s}} = \|f\|_{L_r^2(\mathrm{Aff})}.$$

Since there is a norm-preserving correspondence between integral kernels in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ and Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$, the claim follows. □

It is straightforward to check that the inverse affine Fourier-Wigner transform $\mathcal{F}_W^{-1}$ satisfies for $f, g \in \mathcal{Q}_1$ the properties

- $\mathcal{F}_W^{-1}(f)^* = \mathcal{F}_W^{-1}(\Delta^{1/2} f^*), \qquad f^*(x,a) := \overline{f((x,a)^{-1})}$;
- $\mathcal{F}_W^{-1}(f *_{\mathrm{Aff}} g) = \mathcal{F}_W^{-1}(f) \circ \mathcal{D}^{-1} \circ \mathcal{F}_W^{-1}(g) = U(\check{f}) \circ \mathcal{F}_W^{-1}(g)$;
- $U(x,a) \circ \mathcal{F}_W^{-1}(f) = \mathcal{F}_W^{-1}(R_{(x,a)}(f))$;
- $\mathcal{F}_W^{-1}(f) \circ U(x,a) = \mathcal{F}_W^{-1}\left(\sqrt{a} L_{(x,a)^{-1}}(f)\right)$.

Definition G.5.2. The *affine Fourier-Wigner transform* $\mathcal{F}_W : \mathcal{S}_2 \to \mathcal{Q}_1$ is defined to be the inverse of $\mathcal{F}_W^{-1}|_{\mathcal{Q}_1}$.

Remark G.13.

- To avoid overly cluttered notation, we have used the symbol $\mathcal{F}_W$ for both the classical Fourier-Wigner transform in Section G.2.2, and the affine Fourier-Wigner transform. It should be clear from the context which operator we are referring to.

- Recall that the right multiplication R acts on elements in $L_r^2(\mathrm{Aff})$ by

$$R_{(y,b)}f(x,a) = f((x,a)(y,b))$$

for $(x,a),(y,b) \in \mathrm{Aff}$. For a closed subspace $\mathcal{H} \subset L_r^2(\mathrm{Aff})$ invariant under R, we write $R|_{\mathcal{H}} \cong U$ if there exists a unitary map $T : \mathcal{H} \to L^2(\mathbb{R}_+)$ satisfying

$$T \circ R(x,a)f = U(x,a) \circ Tf,$$

for all $f \in \mathcal{H}$ and $(x,a) \in \mathrm{Aff}$. Define

$$L_U^2(\mathrm{Aff}) := \overline{\mathrm{span}}\{\mathcal{H} \subset L_r^2(\mathrm{Aff}) : R|_{\mathcal{H}} \cong U\}.$$

From [90, Lem. 3] we deduce that

$$L_U^2(\mathrm{Aff}) = \mathcal{Q}_1,$$

as both spaces are the image of the Hilbert-Schmidt operators under the Fourier-Wigner transform. Note that [90] uses left Haar measure, but translating to right Haar measure is an easy exercise using that $f \mapsto \check{f}$ is a unitary equivalence from the left regular representation on $L_l^2(\mathrm{Aff})$ to the right regular representation on $L_r^2(\mathrm{Aff})$.

Example G.5.1. Let $\phi, \psi \in L^2(\mathbb{R}_+)$ with $\psi \in \mathrm{dom}(\mathcal{D})$. If

$$f(x,a) = \langle \phi, U(x,a)^*\mathcal{D}\psi \rangle_{L^2(\mathbb{R}_+)},$$

one finds using Proposition G.4.1 that $f \in L_r^2(\mathrm{Aff})$ and

$$\langle \mathcal{F}_W^{-1}(f)\xi, \eta \rangle_{L^2(\mathbb{R}_+)} = \langle (\phi \otimes \psi)\xi, \eta \rangle_{L^2(\mathbb{R}_+)}$$

for $\eta \in L^2(\mathbb{R}_+)$ and $\xi \in \mathrm{dom}(\mathcal{D})$. This implies that $\mathcal{F}_W^{-1}(f) = \phi \otimes \psi$, in other words for $(x,a) \in \mathrm{Aff}$ that

$$\mathcal{F}_W(\phi \otimes \psi)(x,a) = \langle \phi, U(x,a)^*\mathcal{D}\psi \rangle_{L^2(\mathbb{R}_+)}.$$

For the Heisenberg group, the Fourier-Wigner transform has a very convenient expression for trace-class operators, see (G.2.9). The corresponding expression on the affine group is $\mathcal{F}_W(A)(x,a) = \mathrm{tr}(A\mathcal{D}U(x,a))$, and the next result shows that it holds as long as the objects in the formula are well-defined. The result is due to Führ in this generality [116, Thm. 4.15], and builds on an earlier result due to Duflo and Moore [90, Cor. 2].

Proposition G.5.2 (Führ; Duflo and Moore)**.** *Let $A \in \mathcal{S}_1$ be such that $A\mathcal{D}^{-1}$ extends to a Hilbert-Schmidt operator. Then*

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x,a) = \mathrm{tr}(AU(x,a)). \tag{G.5.2}$$

Proof. To see how the result follows from [116, Thm. 4.15], we need some terminology regarding direct integrals, see [116, Section 3.3]. Recall that the Plancherel theorem [116, Thm. 3.48] supplies a measurable field of Hilbert spaces indexed by the dual group $\{\mathcal{H}_\pi\}_{[\pi]\in\hat{G}}$. For the affine group $G = \mathrm{Aff}$, the Plancherel measure is counting measure supported on the two irreducible representations $\pi_1(x,a) = U(x,a)$ on $L^2(\mathbb{R}_+)$ and $\pi_2(x,a) = U(x,a)$ on $L^2(\mathbb{R}_-) := L^2(\mathbb{R}_-, r^{-1}\,dr)$. So we can construct an element $\{A_{[\pi]}\}_{[\pi]\in\hat{G}}$ of the direct integral

$$\int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\,d\hat{\mu}([\pi])$$

by choosing $A_{[\pi_1]} = A\mathcal{D}^{-1}$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$. Inserting this measurable field of trace-class operators into [116, Thm. 4.15] then gives the conclusion. □

For $f,g \in L^2(\mathbb{R})$ we denote by $\mathrm{SCAL}_g f$ the *scalogram* of f with respect to g given by $\mathrm{SCAL}_g f(x,a) := |\mathcal{W}_g f(x,a)|^2$ where $\mathcal{W}_g f$ is the continuous wavelet transform

$$\mathcal{W}_g f(x,a) := \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(t)\overline{g\left(\frac{t-x}{a}\right)}\,dt.$$

The following result, which follows from Lemma G.3.1 and Example G.5.1, gives a connection between the affine Fourier-Wigner transform, affine convolutions, and the scalogram.

Corollary G.5.2.1. *Let $f,g \in L^2(\mathbb{R})$ such that $\psi := \hat{f}$ and $\phi := \hat{g}$ are supported in $\mathbb{R}_+$ and are in $L^2(\mathbb{R}_+)$. If ψ is admissible then*

$$|\mathcal{F}_W(\phi\otimes\mathcal{D}^{-1}\psi)(x,a)|^2 = (\phi\otimes\phi)\star_{\mathrm{Aff}}(\psi\otimes\psi)(-x,a) = \frac{1}{a}\mathrm{SCAL}_g f(x,a). \tag{G.5.3}$$

Remark G.14. The condition that ψ is admissible in Corollary G.5.2.1 is only necessary for the first equality in (G.5.3). Recall that the affine Wigner distribution W^{ψ}_{Aff} is the affine Weyl symbol of the rank-one operator $\psi\otimes\psi$. If we use Proposition G.3.5 together with Corollary G.5.2.1, then we recover [39, Thm. 5.1].

Corollary G.5.2.1 shows that we have the simple relation

$$|\mathcal{F}_W(A\mathcal{D}^{-1})(x,a)|^2 = A \star_{\mathrm{Aff}} A(-x,a) \tag{G.5.4}$$

for positive rank-one operators A. By Corollary G.4.3.1, admissibility therefore means that $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$ in this case. For more general operators, (G.5.4) will no longer hold. However, we still obtain a result relating admissibility to the Fourier-Wigner transform.

Note that in the first statement in Proposition G.5.3 if $A \in \mathcal{S}_1$ we interpret $\mathcal{F}_W(A\mathcal{D}^{-1}) := \mathrm{tr}(AU(x,a))$ if we do not know that $A\mathcal{D}^{-1}$ extends to a Hilbert-Schmidt operator.

Proposition G.5.3. *Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. Then the following are equivalent:*

1. $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$.
2. $A\mathcal{D}^{-1}$ *extends from* $\mathrm{dom}(\mathcal{D}^{-1})$ *to a Hilbert-Schmidt operator on* $L^2(\mathbb{R}_+)$.
3. A^*A *is admissible.*

Proof. The equivalence of 1 and 2 follows from [116, Thm. 4.15], by applying that theorem to the element $\{A_{[\pi]}\}_{[\pi]\in\hat{G}}$ of the direct integral (see proof of Proposition G.5.2)

$$\int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\,d\hat{\mu}([\pi])$$

given by choosing $A_{[\pi_1]} = A$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$.

The equivalence of 2 and 3 is clear apart from technicalities resulting from the unboundedness of $\mathcal{D}^{-1}$. If we assume 2, then [227, Thm. 13.2] gives that $(A\mathcal{D}^{-1})^* = \mathcal{D}^{-1}A^*$, where the equality includes equality of domains. As the domain of the left term is all of $L^2(\mathbb{R}_+)$ by assumption, this means that the range of A^* is contained in $\mathrm{dom}(\mathcal{D}^{-1})$. In particular, A^*A maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$, and as we also have $\mathcal{D}^{-1}A^*A\mathcal{D}^{-1} = (A\mathcal{D}^{-1})^*A\mathcal{D}^{-1}$ where $A\mathcal{D}^{-1}$ is Hilbert-Schmidt, A^*A satisfies all requirements for being admissible.

Conversely, if A^*A is admissible, then we have for $\psi \in \mathrm{dom}(\mathcal{D}^{-1})$

$$\|A\mathcal{D}^{-1}\psi\|^2_{L^2(\mathbb{R}_+)} = \langle \mathcal{D}^{-1}A^*A\mathcal{D}^{-1}\psi, \psi\rangle_{L^2(\mathbb{R}_+)} \leq \|\mathcal{D}^{-1}A^*A\mathcal{D}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}_+))}\|\psi\|^2_{L^2(\mathbb{R}_+)}.$$

So $A\mathcal{D}^{-1}$ extends to a bounded operator, and as this operator satisfies that

$$(A\mathcal{D}^{-1})^*A\mathcal{D}^{-1} = \mathcal{D}^{-1}A^*A\mathcal{D}^{-1}$$

is trace-class, $A\mathcal{D}^{-1}$ is a Hilbert-Schmidt operator. □

Remark G.15. Recall that we consider $\mathcal{F}_W$ a Fourier transform of operators. The inequality $\|\mathcal{F}_W(A\mathcal{D}^{-1})\|_{L^\infty(\mathrm{Aff})} \leq \|A\|_{\mathcal{S}_1}$ and the equality $\|A\|_{\mathcal{S}_2} = \|\mathcal{F}_W(A)\|_{L^2_r(\mathrm{Aff})}$ might therefore be interpreted as the endpoints $p = \infty$ and $p = 2$ of a Hausdorff-Young inequality, where the appearance of $\mathcal{D}^{-1}$ suggests that the definition of the Fourier-Wigner transform must depend on p. In fact, a Hausdorff-Young inequality of this kind—formulated in the other direction, i.e. for maps from functions on Aff to operators—was shown in [94, Thm. 1.41] for $1 \leq p \leq 2$.

There is a second Fourier transform related to the affine group that comes from representation theory. We define the *affine Fourier-Kirillov transform* as the map $\mathcal{F}_{\mathrm{KO}} : \mathcal{Q}_1 \to L^2_r(\mathrm{Aff})$ given by

$$(\mathcal{F}_{\mathrm{KO}}f)(x,a) = \sqrt{a}\int_{\mathbb{R}^2} f\left(\frac{v}{\lambda(-u)}, e^u\right) e^{-2\pi i(xu+av)}\,\frac{du\,dv}{\sqrt{\lambda(-u)}}, \qquad (x,a) \in \mathrm{Aff}.$$

More information about the Fourier-Kirillov transform can be found in [180]. The following result, which is motivated by (G.2.10) and is a slight generalization of [14, Section VIII.6], shows that the affine Weyl quantization is intrinsically linked with the Fourier transforms on the affine group.

Proposition G.5.4. *Let A_f be a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ with affine symbol $f \in L^2_r(\mathrm{Aff})$. Then the following diagram commutes:*

$$\begin{array}{ccc} & \mathcal{S}_2 & \\ {}^{\mathcal{F}_W}\swarrow & & \searrow^{A_f \longmapsto f} \\ \mathcal{Q}_1 & \xrightarrow{\ \mathcal{F}_{\mathrm{KO}}\ } & L^2_r(\mathrm{Aff}) \end{array}$$

Proof. Recall from (G.5.1) that the integral kernel of $\mathcal{F}_W^{-1}(g)$ for $g \in \mathcal{Q}_1$ is given by

$$K_g(s,r) = \sqrt{r}(\mathcal{F}_1 g)(r, s/r), \qquad s, r \in \mathbb{R}_+.$$

Hence by using (G.2.13) and a change of variables, we see that the affine Weyl symbol of $\mathcal{F}_W^{-1}(g)$ is given at the point $(x,a) \in \mathrm{Aff}$ by

$$\begin{aligned}\int_{-\infty}^{\infty} &\sqrt{a\lambda(-u)}\mathcal{F}_1(g)(a\lambda(-u), e^u)e^{-2\pi i xu}\,du \\ &= \int_{\mathbb{R}^2} \sqrt{a\lambda(-u)}g(v, e^u)e^{-2\pi i(xu+av\lambda(-u))}\,du\,dv \\ &= \sqrt{a}\int_{\mathbb{R}^2} g\left(\frac{v}{\lambda(-u)}, e^u\right)e^{-2\pi i(xu+av)}\,\frac{du\,dv}{\sqrt{\lambda(-u)}} \\ &= (\mathcal{F}_{\mathrm{KO}}g)(x,a).\end{aligned}$$

□

Remark G.16.

- In [208] the authors define an alternative quantization scheme on general type 1 groups. Their quantization scheme together with the affine Weyl quantization is used in [208] to define a quantization scheme on the cotangent bundle $T^*\mathrm{Aff}$.

- Consider A_f for some $f \in L^2_r(\mathrm{Aff})$. Inserting $f = \mathcal{F}_{KO}\mathcal{F}_W(A_f)$ into Proposition G.4.6 allows us to obtain a formal expression for $\mathrm{tr}(\mathcal{D}^{-1}A_f\mathcal{D}^{-1})$ in terms of $\mathcal{F}_W(A_f)$: a formal calculation gives that for sufficiently nice operators A_f we have

$$\mathrm{tr}(\mathcal{D}^{-1}A_f\mathcal{D}^{-1}) = \int_0^\infty [\mathcal{F}_1\mathcal{F}_W(A_f)](a,1)\,\frac{da}{a^{3/2}}, \tag{G.5.5}$$

where $\mathcal{F}_1$ is the Fourier transform in the first coordinate. This is similar to a condition in [125, Cor. 5.2], where finiteness of (G.5.5) is used as a necessary condition for $1 \star_{\mathrm{Aff}} A_f = I_{L^2(\mathbb{R}_+)}$ to hold, where $1(x,a) = 1$ for all $(x,a) \in \mathrm{Aff}$. We will see in Section G.6.2 that this is closely related to admissibility of A_f. Unfortunately, the formal calculation leading to (G.5.5) does not give clear conditions on A_f for the equality to hold.

G.5.2 Affine Quantum Bochner Theorem

On the Heisenberg group, the Fourier-Wigner transform behaves in many ways like the Fourier transform on functions. In particular, for $f \in L^1(\mathbb{R}^{2n})$ and $S,T \in \mathcal{S}_1(L^2(\mathbb{R}^n))$ we get the *decoupling equations*

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S), \qquad \mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T), \tag{G.5.6}$$

where $\mathcal{F}_\sigma$ denotes the symplectic Fourier transform and $\mathcal{F}_W$ denotes the classical Fourier-Wigner transform introduced in Section G.2.2. Although the affine version of (G.5.6) does not hold, one can develop as a special case of [116, Thm. 4.12] a version of Bochner's theorem for the affine Fourier-Wigner transform. This is analogous to the *quantum Bochner theorem* [251, Prop. 3.2] for the Heisenberg group.

Bochner's classical theorem [115, Thm. 4.19] characterizes functions that are Fourier transforms of positive measures. The Bochner theorem for the affine Fourier-Wigner transform answers the following question: Which functions on Aff are of the form $\mathcal{F}_W(S)$, where S is a positive trace-class operator? As in Bochner's classical theorem, it turns out that the correct notion to consider is functions of positive type. Recall that a function $f : \mathrm{Aff} \to \mathbb{C}$ is a *function of positive type* if for any finite selection of points $\Omega \coloneqq \{(x_1, a_1), \ldots, (x_n, a_n)\} \subset \mathrm{Aff}$ the matrix A_Ω with entries

$$(A_\Omega)_{i,j} \coloneqq f((x_i, a_i)^{-1}(x_j, a_j))$$

is positive semi-definite. Before stating the general result we consider an illuminating special case.

Example G.5.2. Assume that $A = \phi \otimes \psi$ is a rank-one operator where $\phi, \psi \in L^2(\mathbb{R}_+)$. We will show that

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x, a) = \langle U(x, a)\phi, \psi\rangle_{L^2(\mathbb{R}_+)} \tag{G.5.7}$$

is a function of positive type on Aff if and only if A is a positive operator. If A is positive, then a standard fact [115, Prop. 3.15] shows that (G.5.7) is a function of positive type. Conversely, we have from [115, Cor. 3.22] that

$$\mathcal{F}_W(\phi \otimes \psi\mathcal{D}^{-1})((x, a)^{-1}) = \overline{\mathcal{F}_W(\psi \otimes \phi\mathcal{D}^{-1})(x, a)} = \overline{\mathcal{F}_W(\phi \otimes \psi\mathcal{D}^{-1})(x, a)}.$$

Hence $\langle U(x, a)\phi, \psi\rangle_{L^2(\mathbb{R}_+)} = \langle U(x, a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)}$; it follows from [126, Thm. 4.2] that $\phi = c \cdot \psi$ for some $c \in \mathbb{C}$. We can conclude from [115, Cor. 3.22] that $c \geq 0$ since

$$\mathcal{F}_W(c\psi \otimes \psi\mathcal{D}^{-1})(0, 1) = c \cdot \|\psi\|_{L^2(\mathbb{R}_+)} \geq 0.$$

We are now ready to state the main result regarding positivity. This result is actually, when interpreted correctly, a special case of the general result [116, Thm. 4.12].

Theorem G.5.5. *Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. Then A is a positive operator if and only if the function*

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x, a) = \mathrm{tr}(AU(x, a))$$

is of positive type on Aff.

Proof. We use the same notation as in the proof of Proposition G.5.2. For $G = \mathrm{Aff}$, the abstract result in [116] says that if

$$\{A_{[\pi]}\}_{[\pi]\in\hat{G}} \in \int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\, d\hat{\mu}([\pi]) \tag{G.5.8}$$

consists of trace-class operators, then $A_{[\pi]}$ is positive a.e. with respect to $\hat{\mu}$ if and only if the function $\int_{\hat{G}} \mathrm{tr}(A_{[\pi]}\pi(g)^*)\, d\hat{\mu}([\pi])$ is of positive type.

As in the proof of Proposition G.5.3, we pick $A_{[\pi_1]} = A$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$. The resulting section consists of positive operators for a.e. $[\pi]$ if and only if A is positive. By the abstract result in [116], this happens if and only if

$$\int_{\hat{G}} \mathrm{tr}(A_{[\pi]}\pi(g)^*)\,d\hat{\mu}([\pi]) = \mathrm{tr}(AU(x,a)^*)$$

is a function of positive type. The definition of functions of positive type gives that this is equivalent to $\mathrm{tr}(AU(x,a))$ being of positive type. □

G.6 Examples

In this section, we show how the theory developed in this paper provides a common framework for various operators and functions studied by other authors. We also introduce an analogue of the Cohen class of time-frequency distributions for the affine group, and deduce its relation to the previously studied *affine quadratic time-frequency representations*.

G.6.1 Affine Localization Operators

There is no general consensus of a localization operator in the affine setting. We will use the following definition based on the convolution framework.

Definition G.6.1. Let $f \in L^1_r(\mathrm{Aff})$ and $\varphi \in L^2(\mathbb{R}_+)$. We say that

$$A = f \star_{\mathrm{Aff}} (\varphi \otimes \varphi)$$

is an *affine localization operator* on $L^2(\mathbb{R}_+)$.

Inequality (G.3.3) shows that an affine localization operator A is a trace-class operator on $L^2(\mathbb{R}_+)$ with

$$\|A\|_{\mathcal{S}_1} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|\varphi\|^2_{L^2(\mathbb{R}_+)}.$$

Moreover, Proposition G.4.5 implies that A is admissible whenever φ is admissible and $f \in L^1_l(\mathrm{Aff}) \cap L^1_r(\mathrm{Aff})$.

We will now see that the affine localization operators are naturally unitarily equivalent to the more commonly defined localization operators on the Hardy space $H^2_+(\mathbb{R})$. Recall that the space $H^2_+(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of elements ψ whose Fourier transform $\mathcal{F}\psi$ is supported on $\mathbb{R}_+$. Note that the composition $\mathcal{DF}$ is a unitary map from $H^2_+(\mathbb{R})$ to $L^2(\mathbb{R}_+)$. An *admissible wavelet* $\xi \in H^2_+(\mathbb{R})$ satisfies by definition that

$$c_\xi := \int_0^\infty \frac{|\mathcal{F}(\xi)(\omega)|^2}{\omega}\,d\omega < \infty.$$

In other words, $\mathcal{DF}\xi \in L^2(\mathbb{R}_+)$ is an admissible function in the sense of Definition G.4.1. In [256, Thm. 18.13] the localization operator A_f^ξ on $H_+^2(\mathbb{R})$, given an admissible wavelet $\xi \in H_+^2(\mathbb{R})$ and $f \in L_l^1(\mathrm{Aff})$, is defined by

$$A_f^\xi \psi = c_\xi \int_{\mathrm{Aff}} f(x,a) \langle \xi, \pi(x,a)\xi \rangle_{H_+^2(\mathbb{R})} \pi(x,a)\xi \, \frac{dx\,da}{a^2}, \quad \xi \in H_+^2(\mathbb{R}),$$

where π acts on $H_+^2(\mathbb{R})$ by

$$\pi(x,a)\xi(t) = \frac{1}{\sqrt{a}} \xi\left(\frac{t-x}{a}\right), \qquad \psi \in H_+^2(\mathbb{R}). \tag{G.6.1}$$

The next proposition is straightforward and relates operators on the form A_f^ξ with affine localization operators.

Proposition G.6.1. *Consider $f \in L_l^1(\mathrm{Aff})$ and an admissible wavelet $\xi \in H_+^2(\mathbb{R})$. Then*

$$(\mathcal{DF}) A_f^\xi (\mathcal{DF})^* = c_\xi \cdot \check{f} \star_{\mathrm{Aff}} (\mathcal{DF}\xi \otimes \mathcal{DF}\xi).$$

Remark G.17.

1. From Proposition G.6.1 it follows that Proposition G.4.10 is a generalization of the result [256, Thm. 18.13].

2. In [71], Daubechies and Paul define localization operators in the same way as in [256], except that they use $\pi(-x,a)$ instead of $\pi(x,a)$ in (G.6.1) and consider symbols f on the full affine group $\mathrm{Aff}_F = \mathbb{R} \times \mathbb{R}^*$. The eigenfunctions and eigenvalues of the resulting localization operators acting on $L^2(\mathbb{R})$ are studied in detail in [71] when the window is related to the first Laguerre function, and $f = \chi_{\Omega_C}$ where

$$\Omega_C := \{(x,a) \in \mathrm{Aff} \, : \, |(x,a) - (0,C)|^2 \leq (C^2 - 1)\}.$$

   The corresponding inverse problem, i.e. conditions on the eigenfunctions of the localization operator that imply that $\Omega = \Omega_C$, is studied in [4].

3. Localization operators with windows related to Laguerre functions have also been extensively studied by Hutník, see for instance [162–164], with particular emphasis on symbols f depending only on either x or a. When $f(x,a) = f(a)$, it is shown that the resulting localization operator is unitarily equivalent to multiplication with some function γ_f. This correspondence allows properties of the localization operator to be deduced from properties of γ_f.

G.6.2 Covariant Integral Quantizations

Operators of the form $f \star_{\text{Aff}} S$ form the basis of the study of covariant integral quantizations by Gazeau and his collaborators in [13, 41, 42, 123–125]. Apart from differing conventions that we clarify at the end of this section, *covariant integral quantizations* on Aff are maps Γ_S sending functions on Aff to operators given by

$$\Gamma_S(f) = f \star_{\text{Aff}} S,$$

for some fixed operator S. By varying S we obtain several quantization maps Γ with properties depending on the properties of S. Examples of such quantization procedures with a different parametrization of Aff are studied in [42, 125]. Their approach is to define S either by $\mathcal{F}_W(S)$ or by its kernel as an integral operator, and deduce conditions on this function that ensures the condition

$$1 \star_{\text{Aff}} S = I_{L^2(\mathbb{R}_+)}.$$

Example G.6.1. The affine Weyl quantization is an example of a covariant integral quantization Γ_S, where S is not a bounded operator. It corresponds to choosing $S = P_{\text{Aff}}$ by Theorem G.3.13.

Remark G.18. The example above leads to a natural question: could there be other operators P such that $f \star_{\text{Aff}} P$ behaves as an affine analogue of Weyl quantization? Since Weyl quantization on $\mathbb{R}^{2n}$ is given by convolving with the parity operator, a natural guess is

$$P\psi(r) = \psi(1/r), \qquad \psi \in L^2(\mathbb{R}_+).$$

The resulting quantization $\Gamma_P(f) = f \star_{\text{Aff}} P$ has been studied by Gazeau and Murenzi in [125, Sec. 7]. It has the advantage that P is a bounded operator, but unfortunately by [125, Prop. 7.5] it does not satisfy the natural dequantization rule

$$f = \Gamma_P(f) \star_{\text{Aff}} P.$$

We also mention that Gazeau and Bergeron have shown that this choice of P is merely a special case corresponding to $\nu = -1/2$ of a class P_ν of operators defining possible affine versions of the Weyl quantization [42, Sec. 4.5].

In quantization theory one typically wishes that the domain of Γ_S contains $L^\infty(\text{Aff})$. This, by Lemma G.4.9, leads us to chose $S = \mathcal{D}T\mathcal{D}$ for some trace-class operator T. In particular, one requires that $\Gamma_S(1) = I_{L^2(\mathbb{R}_+)}$, which can be easily satisfied as the following proposition shows.

Proposition G.6.2. *Let T be a trace-class operator on $L^2(\mathbb{R}_+)$. Then*

$$1 \star_{\text{Aff}} \mathcal{D}T\mathcal{D} = \operatorname{tr}(T) I_{L^2(\mathbb{R}_+)}.$$

Proof. Let $\psi, \phi \in \mathrm{dom}(D)$. We have by (G.4.12) that

$$\begin{aligned}\langle 1 \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} \langle U(-x,a)^*\mathcal{D}T\mathcal{D}U(-x,a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a} \\ &= \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (\mathcal{D}\psi \otimes \mathcal{D}\phi) \frac{dx\,da}{a} \\ &= \mathrm{tr}(T)\langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)},\end{aligned}$$

where the last equality uses Theorem G.4.2. ◻

Following the terminology used by Gazeau et al., we have a *resolution of the identity operator* of the form

$$I_{L^2(\mathbb{R}_+)} = \Gamma_{\mathcal{D}T\mathcal{D}}(1) = \int_{\mathrm{Aff}} U(-x,a)^*\mathcal{D}T\mathcal{D}U(-x,a) \frac{dx\,da}{a},$$

where $\mathrm{tr}(T) = 1$ and the integral has the usual weak interpretation.

Given a positive trace-class operator T with $\mathrm{tr}(T) = 1$, we know that

$$\Gamma_{\mathcal{D}T\mathcal{D}}(f) = f \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}$$

defines a bounded map $\Gamma_{\mathcal{D}T\mathcal{D}} : L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$ with $\Gamma_{\mathcal{D}T\mathcal{D}}(1) = I_{L^2(\mathbb{R}_+)}$. Moreover, $\Gamma_{\mathcal{D}T\mathcal{D}}$ maps positive functions to positive operators and by a variation of Lemma G.3.3 satisfies the covariance property

$$U(-x,a)^*\Gamma_{\mathcal{D}T\mathcal{D}}(f)U(-x,a) = \Gamma(R_{(x,a)^{-1}}f).$$

The following result, which is a modification of the remark given at the end of [181], shows a remarkable converse to these observations.

Theorem G.6.3. *Let* $\Gamma : L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$ *be a linear map satisfying*

1. Γ *sends positive functions to positive operators,*
2. $\Gamma(1) = I_{L^2(\mathbb{R}_+)}$,
3. Γ *is continuous from the weak* topology on* $L^\infty(\mathrm{Aff})$ *(as the dual space of* $L^1_r(\mathrm{Aff})$*) to the weak* topology on* $\mathcal{L}(L^2(\mathbb{R}_+))$,
4. $U(-x,a)^*\Gamma(f)U(-x,a) = \Gamma(R_{(x,a)^{-1}}f)$.

Then there exists a unique positive trace-class operator T *with* $\mathrm{tr}(T) = 1$ *such that*

$$\Gamma(f) = f \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}.$$

Proof. The map $\Gamma \mapsto \Gamma_l$ where $\Gamma_l(f) = \Gamma(\check{f})$ is a bijection from maps Γ satisfying the four assumptions to maps Γ_l satisfying

i) Γ_l sends positive functions to positive operators,

ii) $\Gamma_l(1) = I_{L^2(\mathbb{R}_+)}$,

iii) Γ_l is continuous from the weak* topology on $L^\infty(\mathrm{Aff})$ (as the dual space of $L^1_l(\mathrm{Aff})$) to the weak* topology on $\mathcal{L}(L^2(\mathbb{R}_+))$,

iv) $U(-x,a)^*\Gamma_l(f)U(-x,a) = \Gamma_l(L_{(x,a)^{-1}}f)$.

The remark in [181] applied to $G = \mathrm{Aff}$ and $U(-x,a)$ says that if a map Γ_l satisfies i)-iv) then it must be given for $\psi, \phi \in \mathrm{dom}(\mathcal{D})$ by

$$\langle \Gamma_l(f)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} = \int_{\mathrm{Aff}} f(x,a)\langle U(-x,a)TU(-x,a)^*\mathcal{D}\psi, \mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a},$$

for some trace-class operator T as in the theorem. The relation (G.4.1) gives that

$$\begin{aligned}\langle \Gamma_l(f)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} f(x,a)\langle U(-x,a)\mathcal{D}T\mathcal{D}U(-x,a)^*\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a^2} \\ &= \int_{\mathrm{Aff}} \check{f}(x,a)\langle U(-x,a)^*\mathcal{D}T\mathcal{D}U(-x,a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a}.\end{aligned}$$

Hence $\Gamma_l(f) = \check{f} \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}$ and the result follows. ◻

Quantization using admissible trace-class operators

As we have mentioned, the properties of the quantization map $\Gamma(f) = f \star_{\mathrm{Aff}} S$ depend on the properties of S. From Lemma G.4.9 we know that if S is admissible, i.e. we can write $S = \mathcal{D}T\mathcal{D}$ for some trace-class operator T, then $\Gamma_S : L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$ is bounded. If we further assume that S is a trace-class operator, then Proposition G.4.10 shows that Γ_S is bounded from $L^p_r(\mathrm{Aff})$ to $\mathcal{S}_p$ for all $1 \leq p \leq \infty$. In this sense, the ideal class of covariant integral quantizations Γ_S are those given by *admissible trace-class operators.*

Example G.6.2. If $\varphi \in L^2(\mathbb{R}_+)$ is an admissible function, then $\varphi \otimes \varphi$ is an admissible operator. The resulting quantization $\Gamma_{\varphi\otimes\varphi}$ is then a special case of the quantization procedures introduced by Berezin [38]; Berezin calls f the *contravariant symbol* of $\Gamma_{\varphi\otimes\varphi}(f)$. In this sense, the quantization procedures Γ_S for admissible S generalize Berezin's contravariant symbols.

Relation to the Conventions of Gazeau and Murenzi

Gazeau and Murenzi [125] work with another parametrization of the affine group, namely $\Pi_+ \coloneqq \mathbb{R}_+ \times \mathbb{R}$ where the group operation between $(q_1, p_1), (q_2, p_2) \in \Pi_+$ is given by

$$(q_1, p_1) \cdot (q_2, p_2) \coloneqq (q_1 q_2, p_2/q_1 + p_1).$$

There is a unitary representation $U_G : \Pi_+ \to \mathcal{U}(L^2(\mathbb{R}_+, dr))$ given by

$$U_G(q, p)\psi(r) = \sqrt{\frac{1}{q}} e^{ipr} \psi(r/q) = \sqrt{\frac{1}{q}} U(p/2\pi, 1/q)\psi(r).$$

Given a function $\tilde{f}$ on Π_+ and an operator S on $L^2(\mathbb{R}_+, dr)$, Gazeau and Murenzi define (note that the adjoint is now with respect to $L^2(\mathbb{R}_+, dr)$, not $L^2(\mathbb{R}_+)$)

$$A^S_{\tilde{f}} \coloneqq \frac{1}{C_S} \int_{-\infty}^{\infty} \int_0^{\infty} \tilde{f}(q, p) U_G(q, p) S U_G(q, p)^* \, dq \, dp,$$

where we assume that S satisfies

$$\int_{-\infty}^{\infty} \int_0^{\infty} U_G(q, p) S U_G(q, p)^* \, dq \, dp = C_S \cdot I_{L^2(\mathbb{R}_+, dr)}.$$

The next proposition is straightforward and shows that Gazeau and Murenzi's framework is easily related to our affine operator convolutions.

Proposition G.6.4. *Let S be an operator on $L^2(\mathbb{R}_+, dr)$. Then $\mathcal{D}^{-1} S \mathcal{D}$ is an operator on $L^2(\mathbb{R}_+, r^{-1} dr)$ and*

$$\mathcal{D} A^S_{\tilde{f}} \mathcal{D}^{-1} = \frac{2\pi}{C_S} f \star_{\mathrm{Aff}} (\mathcal{D} S \mathcal{D}^{-1}),$$

where $f(x, a) = \tilde{f}(a, \frac{2\pi x}{a})$ for $(x, a) \in \mathrm{Aff}$.

G.6.3 Affine Cohen Class Distributions

The cross-Wigner distribution $W(\psi, \phi)$ of $\psi, \phi \in L^2(\mathbb{R}^n)$ is known to have certain undesirable properties. A typical example is that one would like to interpret $W(\psi, \phi)$ as a probability distribution, but $W(\psi, \phi)$ is seldom a non-negative function as shown by Hudson in [161]. To remedy this, Cohen introduced in [59] a new class of time-frequency distributions Q_f given by

$$Q_f(\psi, \phi) \coloneqq W(\psi, \phi) * f, \tag{G.6.2}$$

where f is a tempered distribution on $\mathbb{R}^{2n}$. In light of our setup, it is natural to investigate the affine analogue of the Cohen class.

Definition G.6.2. We say that a bilinear map $Q : L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ belongs to the *affine Cohen class* if $Q = Q_S$ for some $S \in \mathcal{L}(L^2(\mathbb{R}_+))$, where

$$Q_S(\psi, \phi)(x, a) \coloneqq (\psi \otimes \phi) \star_{\mathrm{Aff}} S(x, a) = \langle SU(-x, a)\psi, U(-x, a)\phi\rangle_{L^2(\mathbb{R}_+)}.$$

We will write $Q_S(\psi) \coloneqq Q_S(\psi, \psi)$. By Proposition G.3.5 we get for $S = A_f$ that

$$Q_S(\psi, \phi) = W^{\psi,\phi}_{\mathrm{Aff}} *_{\mathrm{Aff}} \check{f}, \tag{G.6.3}$$

which shows that our definition of the affine Cohen class is a natural analogue of (G.6.2). It is straightforward to verify that $Q_S(\psi, \phi)$ is a continuous function on Aff for all $\psi, \phi \in L^2(\mathbb{R}_+)$ and $S \in \mathcal{L}(L^2(\mathbb{R}_+))$. Since the affine Cohen class is defined in terms of the operator convolutions, we get some simple properties: The statements 1 and 2 in Proposition G.6.5 follow from Proposition G.4.10 and Corollary G.4.2.1. Statement 3 is a simple calculation and the last statement follows from a short polarization argument.

Proposition G.6.5. *Let $S \in \mathcal{L}(L^2(\mathbb{R}_+))$. Then for $\psi, \phi \in L^2(\mathbb{R}_+)$ we have the following properties:*

1. *The function $Q_S(\psi, \phi)$ satisfies*

$$\|Q_S(\psi, \phi)\|_{L^\infty(\mathrm{Aff})} \le \|S\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$

2. *If S is admissible, then $Q_S(\psi, \phi) \in L^1_r(\mathrm{Aff})$ and*

$$\int_{\mathrm{Aff}} Q_S(\psi, \phi)(x, a)\, \frac{dx\, da}{a} = \langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)} \mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

3. *We have the covariance property*

$$Q_S(U(-x, a)\psi, U(-x, a)\phi)(y, b) = Q_S(\psi, \phi)((y, b) \cdot (x, a)) \tag{G.6.4}$$

for all $(x, a), (y, b) \in \mathrm{Aff}$.

4. *The function $Q_S(\psi, \psi)$ is (real-valued) positive for all $\psi \in L^2(\mathbb{R}_+)$ if and only if S is (self-adjoint) positive.*

Example G.6.3.

1. For $\psi, \phi \in L^2(\mathbb{R}_+)$ we have

$$Q_{\phi\otimes\phi}(\psi)(x, a) = |\langle \psi, U(-x, a)^* \phi\rangle_{L^2(\mathbb{R}_+)}|^2,$$

which by Corollary G.5.2.1 is simply a Fourier transform away from being a scalogram.

2. If we relax the requirement that S is bounded in Definition G.6.2, then it follows from Theorem G.3.13 that
$$Q_{P_{\mathrm{Aff}}}(\psi) = W^{\psi}_{\mathrm{Aff}}$$
for $\psi \in \mathcal{S}(\mathbb{R}_+)$. Hence the affine Wigner distribution can be represented as a (generalized) affine Cohen class operator. If we define an alternative affine Weyl quantization using an operator P as in Section G.6.2, then it is clear that Q_P gives an alternative Wigner function. See [125, Sec. 7.2] for the case of $P\psi(r) = \psi(1/r)$.

The covariance property (G.6.4) and some rather weak continuity conditions completely characterize the affine Cohen class, as is shown in the following result.

Proposition G.6.6. *Let $Q : L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ be a bilinear map. Assume that for all $\psi, \phi \in L^2(\mathbb{R}_+)$ we know that $Q(\psi,\phi)$ is a continuous function on* Aff *that satisfies* (G.6.4) *and the estimate*
$$|Q(\psi,\phi)(0,1)| \lesssim \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$
Then there exists a unique $S \in \mathcal{L}(L^2(\mathbb{R}_+))$ such that $Q = Q_S$.

Proof. By assumption, the map $(\psi,\phi) \mapsto Q(\psi,\phi)(0,1)$ is a bounded bilinear form. Hence there exists a unique bounded operator S such that
$$\langle S\psi, \phi\rangle_{L^2(\mathbb{R}_+)} = Q(\psi,\phi)(0,1).$$
To see that $Q = Q_S$, note that we have
$$\begin{aligned} Q(\psi,\phi)(x,a) &= Q(U(-x,a)\psi, U(-x,a)\phi)(0,1) \\ &= \langle SU(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= Q_S(\psi,\phi)(x,a). \end{aligned}$$
□

At this point we have seen that operators S define a quantization procedure $\Gamma_S(f) = f \star_{\mathrm{Aff}} S$ as in Section G.6.2, and an affine Cohen class distribution Q_S. The connection between these concepts is provided by the next proposition.

Proposition G.6.7. *Let S be a positive, compact operator on $L^2(\mathbb{R}_+)$ and let $f \in L^1_r(\mathrm{Aff})$ be a positive function. Then $f \star_{\mathrm{Aff}} S$ is a positive, compact operator. Denote by $\{\lambda_n\}_{n=1}^\infty$ its eigenvalues in non-increasing order with associated orthogonal eigenvectors $\{\phi_n\}_{n=1}^\infty$. Then*
$$\lambda_n = \max_{\|\psi\|=1} \left\{ \int_{\mathrm{Aff}} f(x,a) Q_S(\psi,\psi)(x,a)\, \frac{dx\,da}{a} \;:\; \psi \perp \phi_k \text{ for } k = 1,\dots,n-1 \right\}.$$

Proof. The integral defining $f \star_{\mathrm{Aff}} S$ is a Bochner integral of compact operators converging in the operator norm, hence it defines a compact operator. It is straightforward to check that $f \star_{\mathrm{Aff}} S$ is also a positive operator. Furthermore, for $\psi \in L^2(\mathbb{R}_+)$ we have

$$\begin{aligned}\langle f \star_{\mathrm{Aff}} S\psi, \psi\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} f(x,a)\langle SU(-x,a)\psi, U(-x,a)\psi\rangle_{L^2(\mathbb{R}_+)} \, \frac{dx\,da}{a} \\ &= \int_{\mathrm{Aff}} f(x,a) Q_S(\psi,\psi)(x,a) \, \frac{dx\,da}{a}.\end{aligned}$$

The result therefore follows from Courant's minimax theorem [194, Thm. 28.4]. □

Example G.6.4. Let us consider a localization operator $\chi_\Omega \star_{\mathrm{Aff}} \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}_+)$ and a compact subset $\Omega \subset \mathrm{Aff}$. The first eigenfunction ϕ_0 of this operator maximizes the quantity

$$\langle \chi_\Omega \star_{\mathrm{Aff}} (\varphi \otimes \varphi)\phi_0, \phi_0\rangle_{L^2(\mathbb{R}_+)} = \int_\Omega |\langle \varphi_0, U(-x,a)^* \varphi\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\,da}{a}.$$

Hence in this sense, the eigenfunctions are the best localized functions in Ω, which explains the terminology of localization operators.

Relation to the Affine Quadratic Time-Frequency Representations

The signal processing literature contains a wealth of two-dimensional representations of signals. Among them we find the *affine class of quadratic time-frequency representations*, see [212]. A member of the affine class of quadratic time-frequency representations is a map sending functions ψ on $\mathbb{R}$ to a function $Q_\Phi^A(\psi)$ on $\mathbb{R}^2$ given by

$$Q_\Phi^A(\psi)(x,a) = \frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(t/a, s/a) e^{2\pi i x(t-s)} \psi(t)\overline{\psi(s)} \, dt\, ds$$

for some kernel function Φ on $\mathbb{R}^2$. There are clearly a few differences between our setup and the affine class of quadratic time-frequency representations. The domain of the affine class consists of functions on $\mathbb{R}$, whereas the affine Cohen class acts on functions on $\mathbb{R}_+$. For a function ψ on $\mathbb{R}_+$ we therefore define

$$\psi_0(t) = \begin{cases} \psi(t) & t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we recall that a function K_S defined on $\mathbb{R}_+ \times \mathbb{R}_+$ defines an integral operator S with respect to the measure $\frac{dt}{t}$ by

$$S\psi(s) = \int_0^\infty K_S(s,t)\psi(t) \, \frac{dt}{t}.$$

The following formal result is straightforward to verify.

Proposition G.6.8. *Let S be an integral operator with kernel K_S and define*

$$\Phi_S(s,t) = \begin{cases} \frac{K_S(t,s)}{\sqrt{st}} & \textit{if } s,t > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

For $x > 0$ and ψ defined on $\mathbb{R}_+$, we have

$$Q_S(\mathcal{D}\psi, \mathcal{D}\psi)(x,a) = Q^A_{\Phi_S}(\psi_0)(-x/a, a).$$

Bibliography

[1] A. Abdeljawad, S. Coriasco, and J. Toft. Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey-type pseudo-differential operators. *Anal. Appl. (Singap.)*, 18(4):523–583, 2020.

[2] M. R. Abdollahpour and M. H. Faroughi. Continuous G-frames in Hilbert spaces. *Southeast Asian Bull. Math.*, 32(1):1–19, 2008.

[3] L. D. Abreu. Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions. *Appl. Comput. Harmon. Anal.*, 29(3):287–302, 2010.

[4] L. D. Abreu and M. Dörfler. An inverse problem for localization operators. *Inverse Problems*, 28(11):115001, 16, 2012.

[5] L. D. Abreu and N. Faustino. On Toeplitz operators and localization operators. *Proc. Amer. Math. Soc.*, 143(10):4317–4323, 2015.

[6] L. D. Abreu and H. G. Feichtinger. Function spaces of polyanalytic functions. In *Harmonic and Complex Analysis and its Applications*, Trends Math., pages 1–38. Birkhäuser/Springer, Cham, 2014.

[7] L. D. Abreu, K. Gröchenig, and J. Romero. Harmonic analysis in phase space and finite Weyl-Heisenberg ensembles. *J. Stat. Phys.*, 174(5):1104–1136, 2019.

[8] L. D. Abreu, K. Gröchenig, and J. L. Romero. On accumulated spectrograms. *Trans. Amer. Math. Soc.*, 368(5):3629 – 3649, 2016.

[9] L. D. Abreu, J. Pereira, and J. Romero. Sharp rates of convergence for accumulated spectrograms. *Inverse Problems*, 33(11):115008, 12, 2017.

[10] L. D. Abreu, J. M Pereira, J. Romero, and S. Torquato. The Weyl-Heisenberg ensemble: hyperuniformity and higher Landau levels. *J. Stat. Mech. Theory Exp.*, 2017(4):043103, 2017.

[11] A. Al-Qabani and J. A. Virtanen. Fredholm theory of Toeplitz operators on standard weighted Fock spaces. *Ann. Acad. Sci. Fenn. Math.*, 43(2):769–783, 2018.

[12] R. Alaifari and M. Wellershoff. Phase retrieval from sampled Gabor transform magnitudes: Counterexamples, 2020. Preprint, arXiv:2010.01078.

[13] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. *Coherent States, Wavelets, and their Generalizations*. Theoret. and Math. Phys. Springer, New York, second edition, 2014.

[14] S. T. Ali, N. M. Atakishiyev, S. M. Chumakov, and K. B. Wolf. The Wigner function for general Lie groups and the wavelet transform. *Ann. Henri Poincaré*, 1:685–714, 2000.

[15] W. B. Arveson. *A Short Course on Spectral Theory*, volume 209 of *Grad. Texts in Math.* Springer-Verlag, New York, 2002.

[16] S. Axler and D. Zheng. Compact operators via the Berezin transform. *Indiana Univ. Math. J.*, 47(2):387–400, 1998.

[17] R. Balan, K. A. Okoudjou, and A. Poria. On a problem by Hans Feichtinger. *Oper. Matrices*, 12(3):881–891, 2018.

[18] P. Balazs. Hilbert-Schmidt operators and frames - classification, best approximation by multipliers and algorithms. *Int. J. Wavelets Multiresolut. Inf. Process.*, 6(2):315 – 330, 2008.

[19] P. Balazs, K. Gröchenig, and M. Speckbacher. Kernel theorems in coorbit theory. *Trans. Amer. Math. Soc. Ser. B*, 6:346–364, 2019.

[20] P. Balazs, B. Laback, G. Eckel, and W. A. Deutsch. Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking. *IEEE/ACM Trans. Audio, Speech, Language Process.*, 18(1):34–49, 2010.

[21] P. Balazs and D. T. Stoeva. Representation of the inverse of a frame multiplier. *J. Math. Anal. Appl.*, 422(2):981–994, 2015.

[22] M. B. Balk. *Polyanalytic Functions*, volume 63 of *Mathematical Research*. Akademie-Verlag, Berlin, 1991.

[23] W. Bauer and R. Fulsche. Berger-Coburn theorem, localized operators, and the Toeplitz algebra. In *Operator Algebras, Toeplitz Operators and Related Topics*, pages 53–77. Springer International Publishing, Cham, 2020.

[24] W. Bauer and J. Isralowitz. Compactness characterization of operators in the Toeplitz algebra of the Fock space F^p_α. *J. Funct. Anal.*, 263(5):1323–1355, 2012.

[25] D. Bayer. *Bilinear Time-Frequency Distributions and Pseudodifferential Operators*. PhD thesis, University of Vienna, Austria, 2010.

[26] D. Bayer and K. Gröchenig. Time-frequency localization operators and a Berezin transform. *Integr. Equ. Oper. Theory*, 82(1):95 – 117, 2015.

[27] B. Bekka. Square integrable representations, von Neumann algebras and an application to Gabor analysis. *J. Fourier Anal. Appl.*, 10(4):325–349, 2004.

[28] M. E. B. Bekka. Amenable unitary representations of locally compact groups. *Invent. Math.*, 100(1):383–401, 1990.

[29] P. A. Bello. Characterization of randomly time-variant linear channels. *IEEE Trans. Comm.*, 11:360–393, 1963.

[30] A. Benedek and R. Panzone. The space L^p, with mixed norm. *Duke Math. J.*, 28:301–324, 1961.

[31] J. J. Benedetto and S. Li. The theory of multiresolution analysis frames and applications to filter banks. *Appl. Comput. Harmon. Anal.*, 5(4):389–427, 1998.

[32] J. J. Benedetto and G. E. Pfander. Frame expansions for Gabor multipliers. *Appl. Comput. Harmon. Anal.*, 20(1):26–40, 2006.

[33] R. Beneduci. *Mathematical Structure of Positive Operator Valued Measures and Applications*. PhD thesis, University of Debrecen, Hungary, 2014.

[34] Á. Bényi, K. Gröchenig, C. Heil, and K. Okoudjou. Modulation spaces and a class of bounded multilinear pseudodifferential operators. *J. Operator Theory*, 54(2):387–399, 2005.

[35] Á. Bényi, K. Gröchenig, K. A. Okoudjou, and L. G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366–384, 2007.

[36] Á. Bényi and K. A Okoudjou. *Modulation Spaces*. Birkhäuser/Springer, New York, 2020.

[37] S. K. Berberian. *Notes on Spectral Theory*. Van Nostrand Mathematical Studies, No. 5. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[38] F. A. Berezin. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:1134–1167, 1972.

[39] E. Berge, S. M. Berge, and F. Luef. The affine Wigner distribution, 2019. Preprint, arXiv:1912.03112.

[40] C. A. Berger and L. A. Coburn. Toeplitz operators on the Segal-Bargmann space. *Trans. Amer. Math. Soc.*, 301(2):813–829, 1987.

[41] H. Bergeron and J.-P. Gazeau. Integral quantizations with two basic examples. *Ann. Physics*, 344:43–68, 2014.

[42] H. Bergeron and J.-P. Gazeau. Variations à la Fourier-Weyl-Wigner on quantizations of the plane and the half-plane. *Entropy*, 20(10):Paper No. 787, 16, 2018.

[43] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*. Number 223 in Grundlehren Math. Wiss. Springer, 1976.

[44] S. Bochner and R. Phillips. Absolutely convergent Fourier expansions for non-commutative normed rings. *Ann. of Math. (2)*, 43:409–418, 1942.

[45] P. Boggiatto. Localization operators with L^p symbols on modulation spaces. In *Advances in Pseudo-differential Operators*, volume 155 of *Oper. Theory Adv. Appl.*, pages 149–163. Birkhäuser, Basel, 2004.

[46] P. Boggiatto, E. Carypis, and A. Oliaro. Cohen operators associated with signal representations. *AIP Conference Proceedings*, 1907(1):030055, 2017.

[47] P. Boggiatto, E. Carypis, and A. Oliaro. Cohen class of time-frequency representations and operators: boundedness and uncertainty principles. *J. Math. Anal. Appl.*, 461(1):304–318, 2018.

[48] P. Boggiatto, E. Cordero, and K. Gröchenig. Generalized anti-Wick operators with symbols in distributional Sobolev spaces. *Integr. Equ. Oper. Theory*, 48(4):427–442, 2004.

[49] P. Boggiatto, G. De Donno, and A. Oliaro. Time-frequency representations of Wigner type and pseudo-differential operators. *Trans. Amer. Math. Soc.*, 362(9):4955–4981, 2010.

[50] P. Boggiatto and J. Toft. Embeddings and compactness for generalized Sobolev-Shubin spaces and modulation spaces. *Appl. Anal.*, 84(3):269–282, 2005.

[51] J.-M. Bony and J.-Y. Chemin. Functional spaces associated with the Weyl-Hörmander calculus. (Espaces fonctionnels associés au calcul de Weyl-Hörmander.). *Bull. Soc. Math. France*, 122(1):77–118, 1994.

[52] A. Bowers and N. Kalton. *An Introductory Course in Functional Analysis*. New York, NY: Springer, 2014.

[53] W. Braun and H G. Feichtinger. Banach spaces of distributions having two module structures. *J. Funct. Anal.*, 51(2):174–212, 1983.

[54] P. Busch, P. Lahti, J.-P. Pellonpää, and K. Ylinen. *Quantum Measurement*. Theoret. and Math. Phys. Springer, 2016.

[55] G. Cassinelli, E. De Vito, and A. Toigo. Positive operator valued measures covariant with respect to an irreducible representation. *J. Math. Phys.*, 44(10):4768–4775, 2003.

[56] Y. Choi and M. Ghandehari. Dual convolution for the affine group of the real line, 2020. Preprint, arXiv:2009.05497.

[57] O. Christensen. *An Introduction to Frames and Riesz Bases*. Appl. Numer. Harmon. Anal. Birkhäuser Basel, Second edition, 2016.

[58] L. A. Coburn. The Bargmann isometry and Gabor-Daubechies wavelet localization operators. In *Systems, Approximation, Singular Integral Operators, and Related Topics (Bordeaux, 2000)*, volume 129 of *Oper. Theory Adv. Appl.*, pages 169–178. Birkhäuser, Basel, 2001.

[59] L. Cohen. Generalized phase-space distribution functions. *J. Math. Phys.*, 7(5):781–786, 1966.

[60] E. Cordero, M. de Gosson, M. Dörfler, and F. Nicola. On the symplectic covariance and interferences of time-frequency distributions. *SIAM J. Math. Anal.*, 50(2):2178–2193, 2018.

[61] E. Cordero, M. de Gosson, and F. Nicola. On the positivity of trace class operators. *Adv. Theor. Math. Phys.*, 23(8):2061–2091, 2019.

[62] E. Cordero, H. G. Feichtinger, and F. Luef. Banach Gelfand triples for Gabor analysis. In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Math.*, pages 1–33. Springer, 2008.

[63] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[64] E. Cordero and K. Gröchenig. Necessary conditions for Schatten class localization operators. *Proc. Amer. Math. Soc.*, 133(12):3573–3579, 2005.

[65] E. Cordero and F. Nicola. Sharp integral bounds for Wigner distributions. *Int. Math. Res. Not.*, 2018(6):1779–1807, 2018.

[66] M. G. Cowling, A. Martini, D. Müller, and J. Parcet. The Hausdorff-Young inequality on Lie groups. *Math. Ann.*, 375(1-2):93–131, 2019.

[67] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.

[68] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.

[69] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, 1986.

[70] I. Daubechies, H. J. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *J. Fourier Anal. Appl.*, 1(4):437–478, 1995.

[71] I. Daubechies and T. Paul. Time-frequency localisation operators - a geometric phase space approach: II. The use of dilations. *Inverse Problems*, 4(3):661–680, 1988.

[72] M. de Gosson. *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, volume 7 of *Pseudo Diff. Oper.* Birkhäuser/Springer Basel AG, Basel, 2011.

[73] M. de Gosson. *Born-Jordan Quantization*, volume 182 of *Fundam. Theor. Phys.* Springer, [Cham], 2016.

[74] M. de Gosson. Quantum harmonic analysis of the density matrix. *Quanta*, 7(1):74–110, 2018.

[75] M. de Gosson and F. Luef. Symplectic capacities and the geometry of uncertainty: the irruption of symplectic topology in classical and quantum mechanics. *Phys. Rep.*, 484(5):131–179, 2009.

[76] M. de Gosson and F. Luef. Preferred quantization rules: Born–Jordan vs. Weyl: the pseudo-differential point of view. *J. Pseudo-Differ. Oper. Appl.*, 2(1):115–139, 2011.

[77] K de Leeuw. Fourier series of operators and an extension of the F. and M. Riesz theorem. *Bull. Amer. Math. Soc.*, 79:342–344, 1973.

[78] K. de Leeuw. A harmonic analysis for operators. I: Formal properties. *Ill. J. Math.*, 19:593–606, 1975.

[79] K. de Leeuw. An harmonic analysis for operators. II: Operators on Hilbert space and analytic operators. *Ill. J. Math.*, 21:164–175, 1977.

[80] F. De Mari, H. G. Feichtinger, and K. Nowak. Uniform eigenvalue estimates for time-frequency localization operators. *J. London Math. Soc.*, 65(3):720–732, 2002.

[81] A. Deitmar and S. Echterhoff. *Principles of Harmonic Analysis*. Universitext. Springer, second edition, 2014.

[82] J. Deshpande et al. Time-frequency toolbox. `https://github.com/scikit-signal/tftb`.

[83] J. Diestel and J. J. Uhl, Jr. *Vector Measures*. Amer. Math. Soc., Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[84] D. L. Donoho and P. B. Stark. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, 48(3):906–931, 1989.

[85] M. Dörfler, H. G. Feichtinger, and K. Gröchenig. Time-frequency partitions for the Gelfand triple $(S_0, L^2, S_0{}')$. *Math. Scand.*, 98(1):81–96, 2006.

[86] M. Dörfler, T. Grill, R. Bammer, and A. Flexer. Basic filters for convolutional neural networks applied to music: Training or design? *Neural Comput. Appl.*, 32(4):941–945, 2018.

[87] M. Dörfler and K. Gröchenig. Time-frequency partitions and characterizations of modulations spaces with localization operators. *J. Funct. Anal.*, 260(7):1903 – 1924, 2011.

[88] M. Dörfler and J. L. Romero. Frames adapted to a phase-space cover. *Constr. Approx.*, 39(3):445–484, 2014.

[89] M. Dörfler and B. Torresani. Representation of operators in the time-frequency domain and generalized Gabor multipliers. *J. Fourier Anal. Appl.*, 16(2):261–293, 2010.

[90] M. Duflo and C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Funct. Anal.*, 21(2):209–243, 1976.

[91] M. Engliš. Compact Toeplitz operators via the Berezin transform on bounded symmetric domains. *Integr. Equ. Oper. Theory*, 33(4):426–455, 1999.

[92] M. Engliš. Toeplitz operators and localization operators. *Trans. Amer. Math. Soc.*, 361(2):1039–1052, 2009.

[93] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics. CRC Press, 1992.

[94] P. Eymard and M. Terp. La transformation de Fourier et son inverse sur le groupe des $ax + b$ d'un corps local. In *Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg 1976–1978), II*, volume 739 of *Lecture Notes in Math.*, pages 207–248. Springer, Berlin, 1979.

[95] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

[96] H. G. Feichtinger. Modulation spaces on locally compact Abelian groups. Technical report, 1983.

[97] H. G. Feichtinger. Spline-type spaces in Gabor analysis. In D. X. Zhou, editor, *Wavelet Analysis: Twenty Years Developments Proceedings of the International Conference of Computational Harmonic Analysis, Hong Kong, China, June 4–8, 2001*, volume 1 of *Ser. Anal.*, pages 100–122. World Sci.Pub., 2002.

[98] H. G. Feichtinger. Modulation spaces: looking back and ahead. *Sampl. Theory Signal Image Process.*, 5(2):109–140, 2006.

[99] H. G. Feichtinger. Banach Gelfand triples for applications in physics and engineering. volume 1146 of *AIP Conf. Proc.*, pages 189–228. Amer. Inst. Phys., 2009.

[100] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.

[101] H. G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Funct. Anal.*, 146(2):464–495, 1997.

[102] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser Boston, 1998.

[103] H. G. Feichtinger and F. Luef. Wiener amalgam spaces for the fundamental identity of Gabor analysis. *Collect. Math.*, 57(Extra Volume (2006)):233–253, 2006.

[104] H. G. Feichtinger and K. Nowak. A Szegö-type theorem for Gabor-Toeplitz localization operators. *Michigan Math. J.*, 49(1):13–21, 2001.

[105] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

[106] H. G. Feichtinger, K. Nowak, and M. Pap. Spectral properties of Toeplitz operators acting on Gabor type reproducing kernel Hilbert spaces. In *Mathematics without Boundaries: Surveys in Pure Mathematics*. Springer, 2014.

[107] H. G. Feichtinger, K. Nowak, and M. Pap. Asymptotic boundary forms for tight Gabor frames and lattice localization domains. *J. Appl. Math. Phys.*, 3:1316–1342, 2015.

[108] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmon. Anal., pages 123–170. Birkhäuser Boston, 1998.

[109] C. Fernández and A. Galbis. Compactness of time-frequency localization operators on $L^2(\mathbb{R}^d)$. *J. Funct. Anal.*, 233(2):335–350, 2006.

[110] C. Fernández and A. Galbis. Some remarks on compact Weyl operators. *Integral Transforms Spec. Funct.*, 18(7-8):599–607, 2007.

[111] C. Fernández and A. Galbis. Annihilating sets for the short time Fourier transform. *Adv. Math.*, 224(5):1904–1926, 2010.

[112] P. Flandrin. Maximum signal energy concentration in a time-frequency domain. In *ICASSP-88., International Conference on Acoustics, Speech, and Signal Processing*, pages 2176–2179 vol.4, 1988.

[113] P. Flandrin. *Explorations in Time-Frequency Analysis*. Cambridge University Press, 2018.

[114] G. B. Folland. *Harmonic Analysis in Phase Space*. Princeton University Press, 1989.

[115] G. B. Folland. *A Course in Abstract Harmonic Analysis*. Textb. Math. CRC Press, Boca Raton, FL, second edition, 2016.

[116] H. Führ. *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, volume 1863 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[117] R. Fulsche. Correspondence theory on p-Fock spaces with applications to Toeplitz algebras. *J. Funct. Anal.*, 279(7):108661, 41, 2020.

[118] R. Fulsche and R. Hagger. Fredholmness of Toeplitz operators on the Fock space. *Complex Anal. Oper. Theory*, 13(2):375–403, 2019.

[119] D. Gabor. Theory of communication. *J. IEE*, 93(26):429–457, 1946.

[120] A. Galbis. Norm estimates for selfadjoint Toeplitz operators on the Fock space, 2021. Preprint, arXiv:2102.02031.

[121] A. G. García. Average sampling in certain subspaces of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$, 2020. Preprint, arXiv:2011.05871.

[122] V. Gayral, J. M. Gracia-Bondía, and J. C. Várilly. Fourier analysis on the affine group, quantization and noncompact Connes geometries. *J. Noncommut. Geom.*, 2:215–261, 2008.

[123] J.-P Gazeau and C. Habonimana. Signal analysis and quantum formalism: Quantizations with no Planck constant. In P. Boggiatto et al., editors, *Landscapes of Time-Frequency Analysis: ATFA 2019*, pages 135–155. Springer International Publishing, Cham, 2020.

[124] J.-P. Gazeau, T. Koide, and R. Murenzi. 2-D covariant affine integral quantization(s). *Adv. Oper. Theory*, 5(3):901–935, 2020.

[125] J.-P. Gazeau and R. Murenzi. Covariant affine integral quantization(s). *J. Math. Phys.*, 57(5):052102, 2016.

[126] M. Ghandehari and K. F. Taylor. Images of the continuous wavelet transform. In *Operator Methods in Wavelets, Tilings, and Frames*, volume 626 of *Contemp. Math.*, pages 55–65. Amer. Math. Soc., Providence, RI, 2014.

[127] J. M. Gracia-Bondía and J. C. Várilly. Nonnegative mixed states in Weyl-Wigner-Moyal theory. *Phys. Lett. A*, 128(1-2):20–24, 1988.

[128] A. W. M. Graven. *Banach Modules over Banach Algebras*. PhD thesis, Katholieke Universiteit Nijmegen, the Netherlands, 1974.

[129] K. Gröchenig. An uncertainty principle related to the Poisson summation formula. *Studia Math.*, 121(1):87–104, 1996.

[130] K. Gröchenig. Aspects of Gabor analysis on locally compact abelian groups. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 211–231. Birkhäuser Boston, 1998.

[131] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser, 2001.

[132] K. Gröchenig. A pedestrian's approach to pseudodifferential operators. In *Harmonic analysis and applications*, Appl. Numer. Harmon. Anal., pages 139–169. Birkhäuser Boston, Boston, MA, 2006.

[133] K. Gröchenig. Time-Frequency analysis of Sjöstrand's class. *Rev. Mat. Iberoam.*, 22(2):703–724, 2006.

[134] K. Gröchenig. Gabor frames without inequalities. *Int. Math. Res. Not. IMRN*, (23):Art. ID rnm111, 21, 2007.

[135] K. Gröchenig. Weight functions in time-frequency analysis. In L. Rodino et al., editors, *Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis*, volume 52 of *Fields Inst. Commun.*, pages 343–366. Amer. Math. Soc., 2007.

[136] K. Gröchenig. Wiener's lemma: Theme and variations. An introduction to spectral invariance and its applications. In K. Gröchenig, editor, *Four Short Courses on Harmonic Analysis. Wavelets, Frames, Time-frequency Methods, and Applications to Signal and Image Analysis*, Appl. Numer. Harmon. Anal., pages 175–244. Birkhäuser, 2010.

[137] K. Gröchenig. Representation and approximation of pseudodifferential operators by sums of Gabor multipliers. *Appl. Anal.*, 90(3-4):385–401, 2011.

[138] K. Gröchenig and C. Heil. Modulation spaces and pseudodifferential operators. *Integr. Equ. Oper. Theory*, 34(4):439–457, 1999.

[139] K. Gröchenig, P. Jaming, and E. Malinnikova. Zeros of the Wigner distribution and the short-time Fourier transform. *Rev. Mat. Complut.*, (3):723–744, 2019.

[140] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17:1–18, 2004.

[141] K. Gröchenig and E. Pauwels. Uniqueness and reconstruction theorems for pseudodifferential operators with a bandlimited Kohn-Nirenberg symbol. *Adv. Comput. Math.*, 40:49–63, 2014.

[142] K. Gröchenig and J. Stöckler. Gabor frames and totally positive functions. *Duke Math. J.*, 162(6):1003–1031, 2013.

[143] K. Gröchenig and J. Toft. Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces. *J. Anal. Math.*, 114:255–283, 2011.

[144] K. Gröchenig and J. Toft. The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces. *Trans. Amer. Math. Soc.*, 365:4475–4496, 2013.

[145] P. Grohs and M. Rathmair. Stable Gabor phase retrieval and spectral clustering. *Comm. Pure Appl. Math.*, 72(5):981–1043, 2019.

[146] A. Grossmann. Parity operator and quantization of δ-functions. *Comm. Math. Phys.*, 48(3):191–194, 1976.

[147] A. Grossmann, J. Morlet, and T. Paul. Transforms associated to square integrable group representations. I: General results. *J. Math. Phys.*, 26:2473–2479, 1985.

[148] A. Grossmann, J. Morlet, and T. Paul. Transforms associated to square integrable group representations. II. Examples. *Ann. Inst. H. Poincaré Phys. Théor.*, 45(3):293–309, 1986.

[149] R. Hagger. Essential commutants and characterizations of the Toeplitz algebra, 2020. Preprint, arXiv:2002.02344.

[150] B. Hall. *Quantum Theory for Mathematicians*, volume 267 of *Grad. Texts in Math.* Springer, New York, 2013.

[151] D. Han, D. R. Larson, B. Liu, and R. Liu. Dilations for systems of imprimitivity acting on Banach spaces. *J. Funct. Anal.*, 266(12):6914–6937, 2014.

[152] D. Han, D. R. Larson, B. Liu, and R. Liu. Operator-valued measures, dilations, and the theory of frames. *Mem. Amer. Math. Soc.*, 229(1075):viii+84, 2014.

[153] C. Heil. An introduction to weighted Wiener amalgams. In M. Krishna, R. Radha, and S. Thangavelu, editors, *Wavelets and their Applications (Chennai, January 2002)*, pages 183–216. Allied Publishers, 2003.

[154] C. Heil. Integral operators, pseudodifferential operators, and Gabor frames. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 153–169. Birkhäuser, 2003.

[155] C. Heil. History and evolution of the density theorem for Gabor frames. *J. Fourier Anal. Appl.*, 13(2):113–166, 2007.

[156] C. Heil. *A Basis Theory Primer. Expanded ed.* Appl. Numer. Harmon. Anal. Basel: Birkhäuser, 2011.

[157] F. Hlawatsch and W. Kozek. Time-frequency projection filters and TF signal expansions. *IEEE Trans. Signal Process.*, 42:3321–3334, 1994.

[158] A. S. Holevo. Covariant measurements and uncertainty relations. *Rep. Math. Phys.*, 16(3):385–400, 1979.

[159] L. Hörmander. *The Analysis of Linear Partial Differential Operators. I*, volume 256 of *Grundlehren Math. Wiss.* Springer-Verlag, Berlin, 1983.

[160] R. Howe. On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc.*, 3:821–843, 1980.

[161] R. L. Hudson. When is the Wigner quasi-probability density non-negative? *Rep. Mathematical Phys.*, 6(2):249–252, 1974.

[162] O. Hutník. On Toeplitz-type operators related to wavelets. *Integral Equations Operator Theory*, 63(1):29–46, 2009.

[163] O. Hutník. On boundedness of Calderón-Toeplitz operators. *Integral Equations Operator Theory*, 70(4):583–600, 2011.

[164] O. Hutník. Wavelets from Laguerre polynomials and Toeplitz-type operators. *Integral Equations Operator Theory*, 71(3):357–388, 2011.

[165] M. Hutníková and O. Hutník. An alternative description of Gabor spaces and Gabor-Toeplitz operators. *Rep. Math. Phys.*, 66(2):237–250, 2010.

[166] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces. Volume I. Martingales and Littlewood-Paley theory.*, volume 63 of *Ergeb. Math. Grenzgeb., 3. Folge*. Cham: Springer, 2016.

[167] M. S. Jakobsen. On a (no longer) new Segal algebra: A review of the Feichtinger algebra. *J. Fourier Anal. Appl.*, 24(6):1579–1660, 2018.

[168] M. S. Jakobsen and H. G. Feichtinger. The inner kernel theorem for a certain Segal algebra, 2018. Preprint, arXiv:1806.06307.

[169] M. S. Jakobsen and F. Luef. Duality of Gabor frames and Heisenberg modules. *J. Noncommut. Geom.*, 14(4):1445–1500, 2020.

[170] A. J. E. M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Fourier Anal. Appl.*, 1(4):403–436, 1995.

[171] A. J. E. M. Janssen. Some Weyl-Heisenberg frame bound calculations. *Indag. Math. (N.S.)*, 7(2):165–183, 1996.

[172] A. J. E. M. Janssen. Positivity and spread of bilinear time-frequency distributions. In *The Wigner Distribution*, pages 1–58. Elsevier Sci. B. V., Amsterdam, 1997.

[173] A. J. E. M. Janssen. Proof of a conjecture on the supports of Wigner distributions. *J. Fourier Anal. Appl.*, 4(6):723–726, 1998.

[174] A. J. E. M. Janssen. Zak transforms with few zeros and the tie. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 31–70. Birkhäuser Boston, Boston, MA, 2003.

[175] D. Kastler. The C^*-algebras of a free Boson field. *Comm. Math. Phys.*, 1:14–48, 1965.

[176] J. Keller. The spectrogram expansion of Wigner functions. *Appl. Comput. Harmon. Anal.*, 47(1):172 – 189, 2019.

[177] J. Keller and F. Luef. Polyanalytic Toeplitz operators: Isomorphisms, symbolic calculus and approximation of Weyl operators, 2019. Preprint, arXiv:1905.07741.

[178] B. Keville. *Multidimensional Second Order Generalised Stochastic Processes on Locally Compact Abelian Groups*. PhD thesis, University of Dublin, 2003.

[179] M. Keyl, J. Kiukas, and R. F. Werner. Schwartz operators. *Rev. Math. Phys.*, 28(03):1630001, 2016.

[180] A. A. Kirillov. *Lectures on the Orbit Method*. Amer. Math. Soc., 2004.

[181] J. Kiukas. Covariant observables on a nonunimodular group. *J. Math. Anal. Appl.*, 324(1):491–503, 2006.

[182] J. Kiukas, P. Lahti, J. Schultz, and R. F. Werner. Characterization of informational completeness for covariant phase space observables. *J. Math. Phys.*, 53(10):102103, 2012.

[183] J. Kiukas, P. Lahti, and K. Ylinen. Normal covariant quantization maps. *J. Math. Anal. Appl.*, 319(2):783–801, 2006.

[184] J. R. Klauder and B.-S. Skagerstam. Generalized phase-space representation of operators. *J. Phys. A, Math. Theor.*, 40(9):2093–2105, 2007.

[185] J. R. Klauder and B.-S. Skagerstam. Extension of Berezin-Lieb inequalities. In *Excursions in Harmonic Analysis. Volume 2*, Appl. Numer. Harmon. Anal., pages 251–266. Birkhäuser/Springer, New York, 2013.

[186] B. Koczor, F. vom Ende, M. de Gosson, S. J. Glaser, and R. Zeier. Phase spaces, parity operators, and the Born-Jordan distribution, 2018. Preprint, arXiv:1811.05872.

[187] J. Korevaar. *Tauberian Theory. A Century of Developments.* Springer, 2004.

[188] W. Kozek. On the generalized Weyl correspondence and its application to time-frequency analysis of linear time-varying systems. In *IEEE Int. Symp. on Time–Frequency and Time–Scale Analysis*, pages 167–170, 1992.

[189] W. Kozek. *Matched Weyl-Heisenberg Expansions of Nonstationary Environments*. PhD thesis, Vienna University of Technology, 1997.

[190] W. Kozek. On the transfer function calculus for underspread LTV channels. *IEEE Trans. Signal Process.*, 45(1):219–223, 1997.

[191] W. Kozek. Adaptation of Weyl-Heisenberg frames to underspread environments. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 323–352. Birkhäuser Boston, 1998.

[192] W. Kozek and F. Hlawatsch. A comparative study of linear and nonlinear time–frequency filters. In *IEEE Int. Symp. on Time–Frequency and Time–Scale Analysis*, pages 163–166, 1992.

[193] W. Kozek and G. E. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2006.

[194] P. D. Lax. *Functional Analysis*. Pure Appl. Math. (New York). Wiley-Interscience, New York, 2002.

[195] J. M. Lee. *Introduction to Smooth Manifolds*. Grad. Texts in Math. Springer, 2003.

[196] M. Lesch and H. Moscovici. Modular curvature and Morita equivalence. *Geom. Funct. Anal.*, 26(3):818–873, 2016.

[197] E. H. Lieb. Integral bounds for radar ambiguity functions and Wigner distributions. *J. Math. Phys.*, 31(3):594–599, 1990.

[198] E. H. Lieb and Y. Ostrover. Localization of multidimensional Wigner distributions. *J. Math. Phys.*, 51(10):102101, 6, 2010.

[199] G. Loupias and S. Miracle-Sole. C^*-algèbres des systèmes canoniques. I. *Comm. Math. Phys.*, 2:31–48, 1966.

[200] G. Loupias and S. Miracle-Sole. C^*-algèbres des systèmes canoniques. II. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 6:39–58, 1967.

[201] F. Luef. Projective modules over non-commutative tori are multi-window Gabor frames for modulation spaces. *J. Funct. Anal.*, 257(6):1921–1946, 2009.

[202] F. Luef and E. Skrettingland. Convolutions for Berezin quantization and Berezin-Lieb inequalities. *J. Math. Phys.*, 59(2):023502, 2018.

[203] F. Luef and E. Skrettingland. Convolutions for localization operators. *J. Math. Pures Appl. (9)*, 118:288–316, 2018.

[204] F. Luef and E. Skrettingland. Mixed-state localization operators: Cohen's class and trace class operators. *J. Fourier Anal. Appl.*, 25(4):2064–2108, 2019.

[205] F. Luef and E. Skrettingland. On accumulated Cohen's class distributions and mixed-state localization operators. *Constr. Approx.*, 52(1):31–64, 2020.

[206] F. Luef and E. Skrettingland. A Wiener Tauberian theorem for operators and functions. *J. Funct. Anal.*, 280(6):108883, 44, 2021.

[207] Yu. I. Lyubarskiĭ. Frames in the Bargmann space of entire functions. In *Entire and Subharmonic Functions*, volume 11 of *Adv. Soviet Math.*, pages 167–180. Amer. Math. Soc., Providence, RI, 1992.

[208] M. Măntoiu and M. Sandoval. Pseudo-differential operators associated to general type I locally compact groups. In *Analysis and Partial Differential Equations: Perspectives from Developing Countries*, pages 172–190. Springer International Publishing, 2019.

[209] G. Matz and F. Hlawatsch. Linear time-frequency filters: Online algorithms and applications. In A. Papandreou Suppappola, editor, *Applications in Time-Frequency Signal Processing*, pages 205–272. CRC Press, 2002.

[210] R. Megginson. *An Introduction to Banach Space Theory*, volume 183 of *Grad. Texts in Math.* Springer-Verlag, New York, 1998.

[211] B. Moran, S. Howard, and D. Cochran. Positive-operator-valued measures: a general setting for frames. In *Excursions in Harmonic Analysis. Volume 2*, Appl. Numer. Harmon. Anal., pages 49–64. Birkhäuser/Springer, New York, 2013.

[212] A. Papandreou-Suppappola, F. Hlawatsch, and G. F. Boudreaux-Bartels. Quadratic time-frequency representations with scale covariance and generalized time-shift covariance: A unified framework for the affine, hyperbolic, and power classes. *Digit. Signal Process.*, 8(1):3–48, 1998.

[213] E. Pauwels. *Pseudodifferential Operators, Wireless Communications and Sampling Theorems*. PhD thesis, University of Vienna, 2011.

[214] H. R. Pitt. General Tauberian theorems. *Proc. London Math. Soc. (2)*, 44(4):243–288, 1938.

[215] J. C. T. Pool. Mathematical aspects of the Weyl correspondence. *J. Math. Phys.*, 7:66–76, 1966.

[216] S. Rajbamshi, G. Tauböck, P. Balazs, and L. D. Abreu. Random Gabor multipliers for compressive sensing: A simulation study. In *2019 27th European Signal Processing Conference (EUSIPCO)*, pages 1–5, 2019.

[217] J. Ramanathan and P. Topiwala. Time-frequency localization via the Weyl correspondence. *SIAM J. Math. Anal.*, 24(5):1378–1393, 1993.

[218] J. Ramanathan and P. Topiwala. Time-frequency localization and the spectrogram. *Appl. Comput. Harmon. Anal.*, 1(2):209–215, 1994.

[219] J. Ramanathan and P. Topiwala. Time-frequency localization operators of Cohen's class. In J. S. Byrnes et al., editors, *Wavelets and Their Applications*, pages 313–324. Springer Netherlands, Dordrecht, 1994.

[220] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I: Functional Analysis. Rev. and enl. ed.* Academic Press, 1980.

[221] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups*, volume 22 of *London Math. Soc. Monogr. Ser.* Clarendon Press, Oxford, second edition, 2000.

[222] M. A. Rieffel. C^*-algebras associated with irrational rotations. *Pacific J. Math.*, 93(2):415–429, 1981.

[223] M. A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40(2):257–338, 1988.

[224] F. Riesz. Über lineare Funktionalgleichungen. *Acta Math.*, 41(1):71–98, 1916.

[225] J. L. Romero. Characterization of coorbit spaces with phase-space covers. *J. Funct. Anal.*, 262(1):59 – 93, 2012.

[226] G. Rozenblum and N. Vasilevski. Toeplitz operators in polyanalytic Bergman type spaces. In *Functional Analysis and Geometry: Selim Grigorievich Krein Centennial*, volume 733 of *Contemp. Math.*, pages 273–290. Amer. Math. Soc., Providence, RI, 2019.

[227] W. Rudin. *Functional Analysis 2nd ed.* International Series in Pure and Applied Mathematics. McGraw-Hill, 1991.

[228] R. A. Ryan. *Introduction to Tensor Products of Banach Spaces.* Springer Monogr. Math. Springer, 2002.

[229] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. *Bull. Amer. Math. Soc. (N.S.)*, 26(2):322–328, 1992.

[230] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[231] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987. Translated from the Russian by Stig I. Andersson.

[232] B. Simon. *Trace Ideals and their Applications.* Cambridge University Press, 1979.

[233] R. Simon, E. C. G. Sudarshan, and N. Mukunda. Gaussian Wigner distributions: a complete characterization. *Phys. Lett. A*, 124(4-5):223–228, 1987.

[234] E. Skrettingland. Convolutions for localization operators. Master's thesis, Norwegian University of Science and Technology, Norway, 2017.

[235] E. Skrettingland. Quantum harmonic analysis on lattices and Gabor multipliers. *J. Fourier Anal. Appl.*, 26(3), 2020.

[236] E. Skrettingland. On Gabor g-frames and Fourier series of operators. *Studia Math.*, 259(1), 2021.

[237] K. Stroethoff. Hankel and Toeplitz operators on the Fock space. *Michigan Math. J.*, 39(1):3–16, 1992.

[238] T. Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. *Appl. Comput. Harmon. Anal.*, 20(2):237–249, 2006.

[239] W. Sun. G-frames and g-Riesz bases. *J. Math. Anal. Appl.*, 322:437–452, 2006.

[240] G. Tauböck, S. Rajbamshi, P. Balazs, and L. D. Abreu. Random Gabor multipliers and compressive sensing. In *2019 13th International Conference on Sampling Theory and Applications (SampTA)*, pages 1–4, 2019.

[241] N. Teofanov. Gelfand-Shilov spaces and localization operators. *Funct. Anal. Approx. Comput.*, 7(2):135–158, 2015.

[242] N. Teofanov. Continuity and Schatten–von Neumann properties for localization operators on modulation spaces. *Mediterr. J. Math.*, 13(2):745–758, 2016.

[243] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.*, 207(2):399–429, 2004.

[244] J. Toft. Hudson's theorem and rank one operators in Weyl calculus. In *Pseudo-differential Operators and Related Topics*, volume 164 of *Oper. Theory Adv. Appl.*, pages 153–159. Birkhäuser, Basel, 2006.

[245] R. Tolimieri and R. S. Orr. Poisson summation, the ambiguity function, and the theory of Weyl-Heisenberg frames. *J. Fourier Anal. Appl.*, 1(3):233–247, 1994.

[246] T. Tschurtschenthaler. The Gabor Frame Operator (its Structure and Numerical Consequences). Master's thesis, University of Vienna, Austria, 2000.

[247] J. Ville. Théorie et applications de la notion de signal analytique. *Câbles et Transmissions*, 2A:61–74, 1948.

[248] J. von Neumann. *Mathematical Foundations of Quantum Mechanics.*, volume 2 of *Investigations in Physics*. Princeton University Press, Translated from the German edition by R. T. Beyer. edition, 1955.

[249] B. Wang and H. Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. *J. Differential Equations*, 232(1):36–73, 2007.

[250] A. Weil. *L'intégration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940.

[251] R. F. Werner. Quantum harmonic analysis on phase space. *J. Math. Phys.*, 25(5):1404–1411, 1984.

[252] J. Wexler and S. Raz. Discrete Gabor expansions. *Signal Proc.*, 21(3):207 – 220, 1990.

[253] N. Wiener. Tauberian theorems. *Ann. of Math. (2)*, 33(1):1–100, 1932.

[254] E. P. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev., II. Ser.*, 40:749–759, 1932.

[255] E. P. Wigner. Quantum-mechanical distribution functions revisited. In A. S. Wightman, editor, *Part I: Physical Chemistry. Part II: Solid State Physics*, pages 251–262. Springer Berlin Heidelberg, 1997.

[256] M.-W. Wong. *Wavelet Transforms and Localization Operators.* Operator Theory: Advances and Applications. 136. Basel: Birkhäuser, 2002.

[257] M. Zibulski and Y. Y. Zeevi. Analysis of multiwindow Gabor-type schemes by frame methods. *Appl. Comput. Harmon. Anal.*, 4(2):188–221, 1997.

ISBN 978-82-326-5647-9 (printed ver.)
ISBN 978-82-326-6684-3 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

NTNU

Norwegian University of
Science and Technology