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Doctoral thesis

Are Austad

Twisted Convolution Algebras and Applications to Gabor Analysis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Abstract

This thesis concerns several aspects of twisted convolution algebras, with a particular focus on problems arising in Gabor analysis. A significant portion of the thesis is dedicated to the study of Hilbert C^*-modules known as Heisenberg modules and how they relate to Gabor frame theory. This relation showcases the link between finite Hilbert C^*-module frames and Gabor frames. Further, the thesis concerns certain properties of twisted convolution algebras of locally compact groups, in particular spectral invariance and C^*-uniqueness, and we find use for both these properties in Gabor analysis. The problem of C^*-uniqueness is also considered for the case of twisted convolution algebras of second-countable locally compact Hausdorff étale groupoids.

Sammendrag

Denne avhandlingen omfatter flere aspekter ved tvistede konvolusjonsalgebraer, med spesielt fokus på problemer som oppstår i Gaboranalyse. En stor del av avhandlingen er dedikert til studiet av Hilbert C^*-moduler kjent som Heisenbergmoduler og hvordan disse relateres til teorien om Gaborrammer. Denne relasjonen viser sammenhengen mellom endelige Hilbert C^*-modulrammer og Gaborrammer. Videre omfatter avhandlingen enkelte egenskaper ved tvistede konvolusjonsalgebraer, spesielt spektralinvarians og C^*-entydighet, og vi finner anvendelser for begge disse konseptene i Gaboranalyse. Spørsmålet om C^*-entydighet blir også bektraktet for tvistede konvolusjonsalgebraer relatert til annentellbare lokalkompakte Hausdorff étalegruppoider.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) in Mathematical Sciences at the Norwegian University of Science and Technology (NTNU). The research presented here was conducted at the Department of Mathematical Sciences at NTNU, under the supervision of Professor Franz Luef and Associate Professor Eduard Ortega.

The thesis consists of a collection of four research papers and an introductory part that provides background and motivation for the work. The introductory part concludes with a summary of each individual paper, which relates them together and puts them into context. There is a single bibliography at the end of thesis which serves both the introductory part and the research papers.

Acknowledgements

First and foremost I would like to thank my main supervisor Franz Luef. I have benefitted greatly from both working with him on projects, as well as mathematical discussions all throughout my time as both a master's student and a PhD student. I also wish to extend my gratitude to my cosupervisor Eduard Ortega who encouraged me to broaden my mathematical horizons to include groupoids, the results of which are present in this thesis.

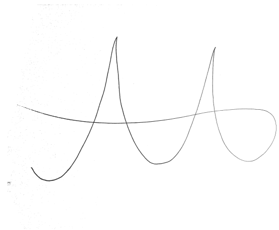
Furthermore, I would like to thank Mads S. Jakobsen for his help in making me better acquainted with time-frequency analysis on locally compact abelian groups, something which has broadened the scope and application of my work significantly.

Throughout my time as a PhD student I have engaged in quite a bit of travel, and I would like to thank Ulrik Enstad for being my travel companion to several conferences, as well as for our many mathematical discussions, which have birthed a plethora of ideas, and some papers.

I would be remiss not to mention my office mates of four years, namely Stine Berge, Eirik Berge, and Eirik Skrettingland. You guys made the office a fun place to be, and I will miss our discussions, both mathematical and non-mathematical.

My time at NTNU has been made immensely better by extracurricular activities, and I would like to thank Magnus Eggen, Fredrik Høeg, Petter Nyland, and Paul Trygsland for our adventures. I would especially like to thank Petter for our many lengthy coffee breaks and accompanying good conversations.

Last, but certainly not least, I want to thank my mother Eva and my brother Tord without whose love and support this would not be possible.



Are Austad
Trondheim, January 2021

Contents

Part I

Introduction

Chapter 1

From locally compact groups and groupoids to twisted convolution algebras

All four papers constituting the thesis in some way or another concern twisted convolution algebras related to locally compact groups or groupoids. Hence this chapter of the introduction presents the constructions and themes concerning this used in the thesis at large. Although any group is a groupoid, we will only consider étale groupoids for the purposes of this thesis. As such, it is easier to present the relevant constructions first in the case of locally compact groups, then afterwards in the case of étale groupoids. This chapter does not aim to fix notation used in the four papers of the thesis as this varied slightly due to stylistic preferences of different coauthors.

1.1 Locally compact groups, representations, and associated convolution algebras

1.1.1 Fundamentals on locally compact groups

For a reference for the material of this section and Section 1.1.2, we refer the reader to [33]. Throughout the entirety of the thesis, we will understand a *locally compact group* G to be a group which is also a locally compact topological space such that both multiplication and inversion are homeomorphisms of the space. It will always be implied that the topology is Hausdorff. When the underlying group G is abelian, G is known as a *locally compact abelian group*, or *LCA group* for short.

Any locally compact group may be equipped with a non-zero left-invariant outer Radon measure with respect to its Borel σ-algebra generated by the open sets, and it is unique up to multiplication by a positive scalar. Any such measure is known as a *Haar measure* on the group. As a special example we mention that whenever the locally compact group is discrete, the Haar measure is (a positive scalar multiple of) the counting measure.

Note that we require our Haar measures to be left-invariant, that is, for any Haar measure μ on the locally compact group G, any $y \in G$ and any measurable subset $M \subseteq G$, we have $\mu(yM) = \mu(M)$. Now let $x \in G$ and define $\mu_x(M) = \mu(Mx)$. The translation by x is done from the right, and we do not assume μ to be right-invariant. However, one can verify that μ_x defines a left-invariant Haar measure on G, and by uniqueness of Haar measure on G, there is a number $m(x)$ such that $\mu_x = m(x)\mu$. This gives rise to the *modular function* m for the group G. Groups for which $m \equiv 1$ are known as *unimodular groups*. As examples of unimodular groups we mention compact groups and LCA groups.

A Haar measure on a locally compact group G also gives rise to an integral, so we may consider L^p-spaces over G for various values of p. Denote the Haar measure on G by $\mathrm{d}x$. For any measurable function f on G and any $p \in [1, \infty)$, we then define

$$\|f\|_{L^p(G)} = \left(\int_G |f(x)|^p \,\mathrm{d}x \right)^{1/p}.$$

Using this, the definition of $L^p(G)$ for $p \in [1, \infty)$ is

$$L^p(G) = \{\text{measurable functions } f \text{ on } G \text{ such that } \|f\|_{L^p(G)} < \infty\}.$$

The compactly supported continuous functions on G, denoted by $C_c(G)$, are dense in $L^p(G)$ for all $p \in [1, \infty)$. Moreover, we may for any measurable function f on G define

$$\|f\|_{L^\infty(G)} = \operatorname*{ess\,sup}_{x \in G} |f(x)|,$$

and set

$$L^\infty(G) = \{\text{measurable functions } f \text{ on } G \text{ such that } \|f\|_{L^\infty(G)} < \infty\}.$$

Note that $C_c(G)$ is in general not dense in $L^\infty(G)$.

In the sequel we will repeatedly make use of projective unitary representations of groups. Let G be a locally compact group. A *projective unitary representation* of G is a continuous map $\pi\colon G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the unitary operators on a Hilbert space $\mathcal{H}$ and is given the strong topology, for which there exists a continuous map $c\colon G \times G \to \mathbb{T}$ such that

$$\pi(x)\pi(y) = c(x, y)\pi(xy)$$

for all $x, y \in G$. To emphasize the role of c in the projective unitary representation we may also call π a *c-projective unitary representation of G*. By associativity we deduce

$$c(x_1, x_2)c(x_1x_2, x_3) = c(x_1, x_2x_3)c(x_2, x_3)$$

for all $x_1, x_2, x_3 \in G$, and by requiring $\pi(e) = \mathrm{Id}_{\mathcal{H}}$ we also find that

$$c(x, e) = c(e, x) = 1$$

for all $x \in G$. Any continuous map c satisfying these conditions is known as a *continuous 2-cocycle for G*. For any locally compact group G and any continuous 2-cocycle c for G there is a canonical c-projective unitary representation of G: The *c-twisted left regular representation* $L^c : G \to \mathcal{U}(L^2(G))$ is defined by

$$L_y^c f(x) = c(y, y^{-1}x)f(y^{-1}x) \tag{1.1.1}$$

for $x, y \in G$ and $f \in L^2(G)$. The assignment $y \mapsto L_y^c$ is then a c-projective unitary representation of G, and it plays a major role in the representation theory for locally compact groups.

Closely related to c-projective unitary representations of a locally compact group G is the *Mackey obstruction group*, or just *Mackey group*, associated to the locally compact group G and continuous 2-cocycle c. We will denote this locally compact group by G_c. As a topological space it is just the product $G \times \mathbb{T}$, its Haar measure is the product measure of the Haar measure on G with the Lebesgue measure on $\mathbb{T}$, but the product is given by

$$(x_1, \tau_1)(x_2, \tau_2) = (x_1x_2, \tau_1\tau_2\overline{c}(x_1, x_2)).$$

One of the primary reasons for looking at G_c is that the theory of c-projective unitary representations of a locally compact group G can be related to unitary representations of the "c-twisted" group G_c. In other words, instead of having our representations be "twisted" by a cocycle, we "twist" the entire group and look at the usual unitary representations of the resulting group.

Example 1.1.1. To illustrate the role of the Mackey group in representation theory we present a locally compact group closely related to the polarized Heisenberg group and how it relates to time-frequency analysis, a very central theme in this thesis. We consider the locally compact group $\mathbb{R}$ with its natural group structure, topology, and Lebesgue measure, and look at a projective representation of $\mathbb{R}^2$ given by

$$\begin{aligned} \pi \colon \mathbb{R}^2 &\to \mathcal{U}(L^2(\mathbb{R})) \\ (x, \omega) &\mapsto M_\omega T_x. \end{aligned}$$

Here $T_x\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the *translation operator* and $M_\omega\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the *modulation operator*, and they are given by

$$T_x f(t) = f(t-x), \quad M_\omega f(t) = e^{2\pi i \omega t} f(t),$$

for $f \in L^2(\mathbb{R})$. Both operators are clearly unitary. The operators T_x and M_ω do not in general commute. Indeed, we have

$$T_x M_\omega = e^{-2\pi i x \omega} M_\omega T_x.$$

Given $\xi_1 = (x_1, \omega_1), \xi_2 = (x_2, \omega_2) \in \mathbb{R}^2$ the *Heisenberg* 2*-cocycle* can be defined by

$$c(\xi_1, \xi_2) = e^{2\pi i x_2 \omega_1}.$$

The assignment $\pi\colon (x,\omega) \mapsto M_\omega T_x$ is thus a c-projective unitary representation of the locally compact group $\mathbb{R}^2$.

The associated Mackey group $\mathbb{R}^2_c$ is the topological space $\mathbb{R}^2 \times \mathbb{T}$, where $\mathbb{T}$ denotes the circle group, with the product topology and product measure, and multiplication given by

$$(x_1, \omega_1, \tau_1)(x_2, \omega_2, \tau_2) = (x_1 + x_2, \omega_1 + \omega_2, \tau_1 \tau_2 e^{-2\pi i x_2 \omega_1}).$$

This group is sometimes referred to as the *(reduced) polarized Heisenberg group*. The c-projective unitary representation π of $\mathbb{R}^2$ can be extended to a unitary representation $\overline{\pi}\colon \mathbb{R}^2_c \to \mathcal{U}(L^2(G))$ by setting $\overline{\pi}(x,\omega,\tau) f(t) = \tau \cdot (M_\omega T_x f(t))$.

1.1.2 Locally compact abelian groups

Three out of the four papers of the thesis concern Gabor analysis on locally compact abelian (LCA) groups. As such, we expand on some constructions and results specific to these groups.

Let G be an LCA group. By a *character for* G we mean a continuous group homomorphism $\tau\colon G \to \mathbb{T}$. We denote the set of all characters by $\widehat{G}$. With pointwise multiplication as binary operation and complex conjugation as inversion, $\widehat{G}$ becomes a group in itself. Equipping $\widehat{G}$ with the compact-open topology, it indeed becomes a locally compact group, and as it is clearly abelian, $\widehat{G}$ is an LCA group as well, known as the *dual group of* G.

For any function $f \in L^1(G)$ it is then possible to define its Fourier transform, denoted $\widehat{f}$, by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,\mathrm{d}x$$

for $\chi \in \widehat{G}$. As in the case for $\mathbb{R}$, whenever $f \in L^1(G)$, $\widehat{f}$ is a continuous function on $\widehat{G}$ vanishing at infinity.

Having fixed a Haar measure on the LCA group G, there is a uniquely determined Haar measure on $\widehat{G}$ such that the Plancherel theorem holds, that is, such that the Fourier transform extends to implement a unitary equivalence between $L^2(G)$ and $L^2(\widehat{G})$. In particular, under this extension we have $\|f\|_{L^2(G)} = \|\widehat{f}\|_{L^2(\widehat{G})}$ for all $f \in L^2(G)$. This is known as the *Plancherel identity*, and the unique measure on $\widehat{G}$ such that the Plancherel identity holds is known as the *Plancherel measure on* $\widehat{G}$.

As $\widehat{G}$ is itself an LCA group whenever G is an LCA group, we could consider the set of unitary characters on $\widehat{G}$ and construct the dual group of $\widehat{G}$, denoted $\widehat{\widehat{G}}$. $\widehat{\widehat{G}}$ turns out to be canonically isomorphic to G again as LCA groups through the map $d\colon G \to \widehat{\widehat{G}}$, $x \mapsto d_x$, where $d_x(\chi) = \chi(x)$ for all $\chi \in \widehat{G}$. The identification $G \cong \widehat{\widehat{G}}$ through this map is known as *Pontryagin duality*.

Lastly, we want to present Weil's formula for LCA groups. Technically, Weil's formula holds for more general locally compact groups under certain assumptions on the modular functions of the groups involved, but we shall not have need for it outside LCA groups. Let G be an LCA group and let H be a closed subgroup of G. Then there is a unique choice of Haar measure on the quotient group G/H such that for all $f \in L^1(G)$ we have

$$\int_G f(x)\,\mathrm{d}x = \int_{G/H}\int_H f(yh)\,\mathrm{d}h\,\mathrm{d}y. \tag{1.1.2}$$

The measure on G/H such that Weil's formula holds is known as the *quotient measure on* G/H.

1.1.3 Twisted convolution algebras from locally compact groups

As a reference on the twisted convolution algebras treated in this section we mention [35].

The spaces $L^1(G)$ and $C_c(G)$ defined earlier for a locally compact group G with a fixed Haar measure can be made into $*$-algebras. In fact, specifying a continuous 2-cocycle for the group G, we may associate to it the c-twisted convolution algebras $L^1(G,c)$ and $C_c(G,c)$. We do this for the latter, but note that the formulas are identical for $L^1(G,c)$.

Fix a locally compact group G and a continuous 2-cocycle c for G. We make the $*$-algebra $C_c(G,c)$ in the following way. As a set, $C_c(G,c) = C_c(G)$. Then, for $f, g \in C_c(G,c)$, we define the *c-twisted convolution of f and g* by

$$f *_c g(x) = \int_G f(y)g(y^{-1}x)c(y, y^{-1}x)\,\mathrm{d}y$$

for all $x \in G$, and we define the *c-twisted involution in $C_c(G,c)$* by

$$f^{*_c}(x) = m(x^{-1})\overline{c(x^{-1}, x)f(x^{-1})}$$

for $f \in C_c(G)$ and $x \in G$, and where m is the modular function of the group G. We will sometimes suppress the c in the notation for both the twisted convolution and the twisted involution. If we complete $C_c(G, c)$ in the $L^1(G)$-norm, we obtain the c-twisted convolution algebra $L^1(G, c)$. With the $L^1(G)$-norm $L^1(G, c)$ becomes a Banach $*$-algebra.

For a c-projective unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ we may induce a $*$-representation of $L^1(G, c)$ by way of integrated representations. The integrated representation will also be denoted by π. For $f \in L^1(G, c)$ we define

$$\pi(f)\xi = \int_G f(x)\pi(x)\xi \,\mathrm{d}x$$

for $\xi \in \mathcal{H}$. We interpret the integral weakly in $\mathcal{H}$. By this expression, $\pi(f)$ defines a bounded linear operator on $\mathcal{H}$, that is, $\pi(f) \in \mathbb{B}(\mathcal{H})$. The assignment $f \mapsto \pi(f)$ defines a $*$-representation of $L^1(G, c)$.

Note that even if $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is faithful, its integrated representation may not be a faithful representation of the Banach $*$-algebra $L^1(G, c)$. Indeed, consider the unitary representation $\eta\colon \mathbb{Z}/2\mathbb{Z} \to \mathcal{U}(\mathbb{C})$ given by $\eta(0) = \mathrm{Id}_{\mathbb{C}}$, $\eta(1) = -\mathrm{Id}_{\mathbb{C}}$. This unitary representation is faithful by inspection, but the integrated representation of $\ell^1(\mathbb{Z}/2\mathbb{Z})$ is clearly not faithful. For any locally compact group G and any 2-cocycle c for G there is however always one faithful c-projective unitary representation of G such that its integrated representation is a faithful representation of $L^1(G, c)$. The representation in question is the integrated representation of the c-twisted left regular representation, see (1.1.1).

Any faithful $*$-representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ realizes $L^1(G, c)$ as bounded operators on a Hilbert space $\mathcal{H}$. By taking the norm closure of $\pi(L^1(G, c))$ in $\mathbb{B}(\mathcal{H})$ we obtain a C^*-algebra, which we denote by $C^*_\pi(G, c)$. There are two canonical C^*-completions of $L^1(G, c)$. The first is known as the *c-twisted reduced C^*-algebra of G*, and is the completion of $L^1(G, c)$ with respect to the norm coming from the integrated representation of L^c, the c-twisted left regular representation of G. We denote this completion by $C^*_r(G, c)$. The other canonical completion is the *full c-twisted C^*-algebra of G*. It is the completion of $L^1(G, c)$ with respect to the norm

$$\|f\|_{\max} = \sup\{\|\pi(f)\|_{\mathbb{B}(\mathcal{H}_\pi)} \mid \pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H}_\pi) \text{ is a } *\text{-representation}\}.$$

We denote this completion by $C^*(G, c)$. If $C^*_r(G, c) \cong C^*(G, c)$, we say that the group G is *amenable*. The standard way to introduce amenability for groups is by ways of existence of a left-invariant mean on the group in question. However, we shall only need the equivalent condition that the full and reduced (twisted) C^*-algebras coincide up to isomorphism.

As noted above, any faithful $*$-representation of $L^1(G, c)$ gives rise to a C^*-completion of $L^1(G, c)$, which we opted to denote by $C^*_\pi(G, c)$. Even if the group is amenable, there could be C^*-completions of $L^1(G, c)$ that are not isomorphic to $C^*(G, c)$, see e.g. [22, 92]. If the Banach $*$-algebra $L^1(G, c)$ has a unique C^*-completion up to isomorphism, we say that $L^1(G, c)$ is C^**-unique*. This leads us to one of the questions considered in this thesis.

Problem. When is $L^1(G, c)$ C^*-unique?

Part of Paper C consists of finding sufficient conditions for $L^1(G, c)$ to be C^*-unique. It turns out that imposing C^*-uniqueness conditions on $L^1(G_c)$ is very useful. In particular, if $L^1(G_c)$ is C^*-unique, so is $L^1(G, c)$. C^*-uniqueness of $L^1(G_c)$ is a question of C^*-uniqueness of a convolution algebra (no 2-cocycle twist), and this has been studied before, see e.g. [21, 22].

Realizing $L^1(G, c)$ as bounded operators on a Hilbert space $\mathcal{H}$ through a faithful $*$-representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$, we may also ask if the spectrum of elements of $L^1(G, c)$ is preserved.

Problem. For a faithful $*$-representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$, when is $L^1(G, c)$ spectrally invariant in $\mathbb{B}(\mathcal{H})$? In other words, when is it true that $\sigma_{L^1(G,c)}(f) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(f))$? (Here $\sigma_A(a)$ denotes the spectrum of a in the algebra A).

This problem occupies a large part of Paper C, where we find sufficient conditions for spectral invariance of $L^1(G, c)$ in terms of C^*-uniqueness and symmetry of $L^1(G_c)$.

1.2 Étale groupoids, representations, and associated convolution algebras

At the end of the last section we presented one of the problems considered in Paper C, namely finding conditions guaranteeing the C^*-uniqueness of $L^1(G, c)$. In Paper D we consider the question of C^*-uniqueness for $L^1(\mathcal{G}, c)$, where $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid and c is a 2-cocycle for $\mathcal{G}$. Hence the following section is dedicated to introducing relevant notions and results from the theory of (étale) groupoids. A nice reference for the material in this section is [104].

Although we will exclusively only have need for étale groupoids, we begin by defining the notion of a groupoid in its full generality.

Definition 1.2.1. A groupoid is a set $\mathcal{G}$ together with a distinguished set $\mathcal{G}^{(2)} \subseteq \mathcal{G}\times\mathcal{G}$ equipped with a binary operation $\mathcal{G}^{(2)} \to \mathcal{G}$, denoted $(\gamma, \mu) \mapsto \gamma\mu$, and a unary operation $\mathcal{G} \to \mathcal{G}$, denoted $\gamma \mapsto \gamma^{-1}$, such that the following axioms are satisfied:

1) If $(\gamma, \mu), (\mu, \nu) \in \mathcal{G}^{(2)}$, then $(\gamma\mu, \nu), (\gamma, \mu\nu) \in \mathcal{G}^{(2)}$ and $(\gamma\mu)\nu = \gamma(\mu\nu)$.

2) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$.

3) For every $\gamma \in \mathcal{G}$, $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$, and whenever $(\gamma, \mu) \in \mathcal{G}^{(2)}$ we have $\gamma\mu\mu^{-1} = \gamma$ and $\gamma^{-1}\gamma\mu = \mu$.

The first axiom tells us that the binary operation is associative. We will refer to the binary operation as multiplication. From the third axiom we deduce that $\gamma\gamma^{-1}$ acts as a right identity for all elements μ such that $(\mu, \gamma) \in \mathcal{G}^{(2)}$. Likewise, $\gamma^{-1}\gamma$ acts as a left identity on all elements μ such that $(\gamma, \mu) \in \mathcal{G}^{(2)}$. Based on this, we refer to the set

$$\mathcal{G}^{(0)} = \{\gamma^{-1}\gamma \mid \gamma \in \mathcal{G}\} = \{\gamma\gamma^{-1} \mid \gamma \in \mathcal{G}\}$$

as the *unit space* of $\mathcal{G}$. The elements of $\mathcal{G}^{(0)}$ are often referred to as *units*. The second axiom above can now be interpreted as every element of $\mathcal{G}$ having an inverse. It is not difficult to see that any group is a groupoid with unit space equal to the one-point space consisting of the group identity. Indeed, a groupoid is a group if and only if its unit space is equal to the one-point space.

We may now define two maps $r, s \colon \mathcal{G} \to \mathcal{G}^{(0)}$ by

$$r(\gamma) = \gamma\gamma^{-1} \quad \text{and} \quad s(\gamma) = \gamma^{-1}\gamma$$

for $\gamma \in \mathcal{G}$. The maps are sometimes known as the *range map* and the *source map*, respectively. Now $(\gamma, \mu) \in \mathcal{G}^{(2)}$ if and only if $r(\mu) = s(\gamma)$.

For $x \in \mathcal{G}^{(0)}$ we will write $\mathcal{G}_x = \{\gamma \in \mathcal{G} \mid s(\gamma) = x\}$, $\mathcal{G}^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = x\}$, and $\mathcal{G}_x^y = \mathcal{G}^y \cap \mathcal{G}_x$. The *isotropy subgroupoid* of $\mathcal{G}$ is then

$$\operatorname{Iso}(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x.$$

$\operatorname{Iso}(\mathcal{G})$ is a subset of $\mathcal{G}$ closed under inversion and multiplication, that is, it is a subgroupoid of $\mathcal{G}$.

Definition 1.2.2. Let $\mathcal{G}$ be a groupoid. We say $\mathcal{G}$ is a Hausdorff *topological groupoid* if it is equipped with a locally compact topology such that $\mathcal{G}^{(0)}$ is Hausdorff in its relative topology, the inversion is continuous, and the multiplication is continuous with respect to the relative topology on $\mathcal{G}^{(2)}$ as a subset of $\mathcal{G} \times \mathcal{G}$.

Remark 1.2.3. We continue to specify that the groupoid $\mathcal{G}$ is Hausdorff as the study of locally compact but non-Hausdorff groupoids is an active research field.

It follows from the definition that the source map s and the range map r are continuous. Moreover, it is well known that $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ itself is Hausdorff.

Definition 1.2.4. Let $\mathcal{G}$ be a topological groupoid. We say $\mathcal{G}$ is an étale groupoid if the range map $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

Since inversion $\gamma \mapsto \gamma^{-1}$ on $\mathcal{G}$ is continuous and its own inverse, $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism if and only if $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

Let now $\mathcal{G}$ be an étale groupoid. It is then well-known that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. In particular, if $\mathcal{G}$ is also Hausdorff, $\mathcal{G}^{(0)}$ is both closed and open in $\mathcal{G}$.

Definition 1.2.5. Let $\mathcal{G}$ be an étale groupoid. We say a subset $B \subseteq \mathcal{G}$ is a *bisection* if there is an open set U containing B for which r and s are injective when restricted to U.

Whenever $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid, the case which we will be concerned with in Paper D, the topology of $\mathcal{G}$ has a very useful base. Indeed, $\mathcal{G}$ has a countable base of open bisections. Moreover, whenever $\mathcal{G}$ is Hausdorff and étale, both $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete in the relative topology for all $x \in \mathcal{G}^{(0)}$. In particular, $\mathcal{G}_x^x$ is discrete in its relative topology for all $x \in \mathcal{G}^{(0)}$.

We are going to consider convolution algebras of second-countable locally compact Hausdorff étale groupoids and their various C^*-completions. As such, we will also need to consider $*$-representations for these convolution algebras. However, unlike what we did in the case of locally compact groups, we will not consider projective unitary representations of groupoids directly. This is done in order not to discuss unnecessary technicalities never explicitly needed in Paper D.

A *normalized continuous* 2*-cocycle* for a topological groupoid is a continuous map $\sigma\colon \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying

$$\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$$

for all $\gamma \in \mathcal{G}$, and

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$$

whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$. We will refer to normalized continuous 2-cocycles as just 2-cocycles in this section.

In order to construct the convolution algebras of interest, let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and let σ be a 2-cocycle for $\mathcal{G}$. Recall that both $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete for all $x \in \mathcal{G}^{(0)}$. Analogously to the case of locally compact groups, we then equip $C_c(\mathcal{G})$ with σ-twisted convolution

$$(f *_\sigma g)(\gamma) = \sum_{\mu \in \mathcal{G}_{s(\gamma)}} f(\gamma\mu^{-1})g(\mu)\sigma(\gamma\mu^{-1}, \mu), \quad f, g \in C_c(\mathcal{G}), \gamma \in \mathcal{G},$$

and σ-twisted involution

$$f^{*_\sigma}(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma) f(\gamma^{-1})}, \quad f \in C_c(\mathcal{G}), \gamma \in \mathcal{G}.$$

We denote $C_c(\mathcal{G})$ equipped with this convolution and involution by $C_c(\mathcal{G},\sigma)$.

For a locally compact group G we may view $L^1(G)$ as the completion of $C_c(G)$ under the L^1-norm. The analogous notion for groupoids is that of the I-norm, which for étale groupoids is given by

$$\|f\|_I = \sup_{x\in\mathcal{G}^{(0)}} \max\Big\{ \sum_{\gamma\in\mathcal{G}_x} |f(\gamma)|, \sum_{\gamma\in\mathcal{G}^x} |f(\gamma)|\Big\}$$

for $f \in C_c(\mathcal{G})$. Inspecting the expression we see that the I-norm can be regarded as a "fiberwise L^1-norm". We denote the corresponding completion of $C_c(\mathcal{G})$ by $\ell^1(\mathcal{G})$. The reason for the choice of ℓ^1 instead of L^1 is to reflect the discreteness of the fibers, even if the groupoid $\mathcal{G}$ as a whole is not equipped with the discrete topology. The expressions for the c-twisted convolution and c-twisted involution still makes sense on $\ell^1(\mathcal{G})$, and we denote the resulting $*$-algebra by $\ell^1(\mathcal{G},\sigma)$. Indeed, this becomes a Banach $*$-algebra when equipped with the I-norm. It is even a reduced Banach $*$-algebra as there is a canonical faithful $*$-representation, namely the σ-twisted left regular representation. To construct this, let $x \in \mathcal{G}^{(0)}$ and consider the $*$-representation $L^{\sigma,x}\colon C_c(\mathcal{G},\sigma) \to \mathbb{B}(\ell^2(\mathcal{G}_x))$ given by

$$L^{\sigma,x}(f)\delta_\gamma = \sum_{\mu\in\mathcal{G}_{r(\gamma)}} \sigma(\mu,\mu^{-1}\gamma)f(\mu)\delta_{\mu\gamma}, \quad \text{for } f \in C_c(\mathcal{G},\sigma) \text{ and } \gamma \in \mathcal{G}_x.$$

Here δ_γ is the function taking the value 1 in γ and 0 elsewhere. We then obtain a faithful I-norm bounded $*$-representation of $C_c(\mathcal{G},c)$ given by

$$\bigoplus_{x\in\mathcal{G}^{(0)}} L^{\sigma,x}\colon C_c(\mathcal{G},\sigma) \to \bigoplus_{x\in\mathcal{G}^{(0)}} \mathbb{B}(\ell^2(\mathcal{G}_x)) \subseteq \mathbb{B}\Big(\bigoplus_{x\in\mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)\Big), \tag{1.2.1}$$

and we denote the completion of $C_c(\mathcal{G},\sigma)$ in the induced C^*-norm by $C_r^*(\mathcal{G},\sigma)$. This is the σ-twisted reduced C^*-algebra of $\mathcal{G}$. As the $*$-representation is I-norm bounded, $C_r^*(\mathcal{G},\sigma)$ is also the completion of $\ell^1(\mathcal{G},\sigma)$ with respect to the same induced norm. For any reduced Banach $*$-algebra we may also consider the maximal C^*-completion, or the C^*-envelope. For $\ell^1(\mathcal{G},\sigma)$ this is the completion in the norm

$$\|f\|_{\max} = \sup\{\|\pi(f)\| \mid \pi \text{ is a } *\text{-representation of } \ell^1(\mathcal{G},\sigma)\},$$

for $f \in \ell^1(\mathcal{G},\sigma)$. We denote the resulting C^*-algebra by $C^*(\mathcal{G},\sigma)$. When $C_r^*(\mathcal{G},\sigma) \cong C^*(\mathcal{G},\sigma)$ we say that $\mathcal{G}$ has the *weak containment property with respect to σ*. In the case of locally compact groups, we said that a locally compact group G is amenable if the (twisted) full and (twisted) reduced group C^*-algebras are isomorphic. However, there is a notion of amenability of groupoids, and it is not equivalent to the coincidence of the (twisted) full and reduced groupoid C^*-algebras [108].

As for any reduced Banach $*$-algebra we may now consider various faithful $*$-representations and look at their corresponding C^*-completions. For a second-countable Hausdorff étale groupoid $\mathcal{G}$ with 2-cocycle σ we may then ask the analogous question we had asked for locally compact groups and their 2-cocycles.

Problem. When does $\ell^1(\mathcal{G}, \sigma)$ have a unique C^*-norm?

This is the primary focus of Paper D. It will turn out that we can find sufficient conditions for C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ by looking at the question of C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, where $\mathrm{Iso}(\mathcal{G})^\circ$ is the interior of the isotropy subgroupoid of $\mathcal{G}$, and σ is restricted to this subgroupoid. Moreover, we find sufficient conditions for C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ by looking at C^*-uniqueness for $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$, for $x \in \mathcal{G}^{(0)}$. Here $\mathrm{Iso}(\mathcal{G})^\circ_x = (\mathrm{Iso}(\mathcal{G})^\circ)^x \cap (\mathrm{Iso}(\mathcal{G})^\circ)_x$, and σ_x is the restriction of σ onto the fiber x. But $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ is the twisted convolution algebra of a discrete group. Thus sufficient conditions for C^*-uniqueness of a twisted groupoid convolution algebra $\ell^1(\mathcal{G}, \sigma)$ can be deduced by C^*-uniqueness of twisted group convolution algebras studied in Paper C.

Chapter 2

Frames and convolution algebras

The first three papers of the thesis directly concern the theory of frames and Gabor frames, and so this part of the introduction will be dedicated to explaining these concepts and how they relate to the twisted convolution algebras of the previous chapter, as well as the construction of, and relevance of, Heisenberg modules. As in the previous chapter, this chapter does not aim to fix notation used in the four papers of the thesis.

2.1 Frames in Hilbert C^*-modules

This section serves only to introduce the concept of frames for Hilbert C^*-modules. No central questions considered in the thesis will be presented here, but the section is important in order to present core questions of the thesis in the subsequent sections. We refer the reader to [47], where Hilbert C^*-module frames were defined.

Throughout this section let A denote a C^*-algebra, and let E denote a left Hilbert C^*-module. We will denote the A-valued inner product on E by ${}_{\bullet}\langle \cdot, \cdot \rangle$. Moreover, we will consider A as a left Hilbert C^*-module over itself. For two left Hilbert C^*-modules E and F, we denote the Banach space of A-adjointable maps from E to F by $\mathcal{L}(E, F)$, and as is customary we write $\mathcal{L}(E) := \mathcal{L}(E, E)$. Lastly before defining a frame we introduce a Hilbert C^*-module important to several operators related to frames in the sequel. For any (at most) countable index set J we denote by $\ell^2(J, A)$ the set of sequences $(a_j)_{j \in J} \subseteq A$ for which the sum $\sum_{j \in J} a_j a_j^*$ converges in A-norm. The set $\ell^2(J, A)$ becomes a left Hilbert A-module with left module action

$$a \cdot (a_j)_{j \in J} = (a \cdot a_j)_{j \in J} \quad \text{for } a \in A \text{ and } (a_j)_{j \in J} \in \ell^2(J, A),$$

and

$$_\bullet\langle (a_j)_{j\in J}, (b_j)_{j\in J}\rangle = \sum_{j\in J} a_j b_j^* \quad \text{for } (a_j)_{j\in J}, (b_j)_{j\in J} \in \ell^2(J, A).$$

We will never consider frames over two different index sets simultaneously, so we may for ease of notation sometimes leave the index set implied.

Definition 2.1.1. Let A be a C^*-algebra and let E be a left Hilbert A-module. Moreover, fix a (at most) countable index set J. We say that a sequence $(x_j)_j \subseteq E$ is a *(module) frame* for E if there are positive real numbers $C, D > 0$ such that

$$C\, {}_\bullet\langle x, x\rangle \le \sum_{j\in J} {}_\bullet\langle x, x_j\rangle\, {}_\bullet\langle x_j, x\rangle \le D\, {}_\bullet\langle x, x\rangle \tag{2.1.1}$$

for all $x \in E$, and where the middle sum converges in norm. The constants C and D are known as the *lower frame bound* and *upper frame bound*, respectively. If we may choose $C = D$ we say $(x_j)_j$ is a *tight* frame for E, and if $C = D = 1$ we say $(x_j)_j$ is a *Parseval* frame E. Moreover, if the upper inequality holds, we say that $(x_j)_j$ is a *Bessel sequence*. In particular, every frame is a Bessel sequence.

We include some examples to illustrate the notion of frames.

Example 2.1.2. Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{H}$ is a Hilbert C^*-module over the C^*-algebra $\mathbb{C}$. Any orthonormal basis $\{h_1, h_2, \ldots\}$ for $\mathcal{H}$ is then clearly a frame for $\mathcal{H}$. Moreover, we can then set $C = D = 1$, so this becomes a Parseval frame for $\mathcal{H}$.

However, uniformly norm-bounded bases are not the only examples of frames. Indeed, the notion of a frame may be regarded as a relaxation of the concept of a basis, while preserving some key desirable properties. One of these key properties is that of a reconstruction formula, which we will explore below. For instance, if we consider the same orthonormal basis $\{h_1, h_2, \ldots\}$ as in Example 2.1.2, then $\{h_1, h_1, h_2, h_3, \ldots\}$ is not a basis for $\mathcal{H}$ due to linear dependence, but it is a frame for $\mathcal{H}$ with $C = 1$ and $D = 2$.

Example 2.1.3. Suppose A is a unital C^*-algebra, and regard A as a left Hilbert A-module over itself. Then $\{1_A\}$ is a Parseval frame for A.

We proceed to introduce some operators related to frames. As above, let E be a left Hilbert A-module, let J be a (at most) countable index set, and suppose $(x_j)_{j\in J}$ is a Bessel sequence. The *analysis operator* $\Phi_{(x_j)_j}$ is defined as

$$\begin{aligned} \Phi_{(x_j)_j} : E &\to \ell^2(J, A) \\ y &\mapsto ({}_\bullet\langle y, x_j\rangle)_{j\in J} \end{aligned}$$

for $y \in E$. We have $\Phi_{(x_j)_j} \in \mathcal{L}(E, \ell^2(J,A))$. Its adjoint $\Psi_{(x_j)_j} := \Phi^*_{(x_j)_j}$ is known as the *synthesis operator* and is given by

$$\begin{aligned}\Psi_{(x_j)_j} : \ell^2(J,A) &\to E \\ (a_j)_j &\mapsto \sum_{j \in J} a_j x_j,\end{aligned}$$

for $(a_j)_j \in \ell^2(J,A)$. Combining the two operators we obtain the *frame operator* $\Theta_{(x_j)_j} := \Psi_{(x_j)_j} \circ \Phi_{(x_j)_j}$, and it is explicitly given by

$$\begin{aligned}\Theta_{(x_j)_j} : E &\to E \\ y &\mapsto \sum_{j \in J} {}_\bullet\langle y, x_j \rangle x_j,\end{aligned}$$

for $y \in E$. As the frame operator $\Theta_{(x_j)_j}$ is the composition of an adjointable operator and its adjoint, it is an adjointable positive operator. If $(x_j)_{j\in J}$, in addition to being a Bessel sequence, is also a frame for E as a Hilbert A-module, then by (2.1.1) we see that $\Theta_{(x_j)_j}$ is invertible. We then have

$$y = \Theta_{(x_j)_j} \Theta^{-1}_{(x_j)_j} y = \sum_{j \in J} {}_\bullet\langle y, \Theta^{-1}_{(x_j)_j} x_j \rangle x_j, \tag{2.1.2}$$

for all $y \in E$. In other words, we may reconstruct any $y \in E$ in terms of an A-linear combination of the elements $(x_j)_{j\in J}$. This is one of the major features of frames; they allow for reconstruction formulas. Note that the coefficients ${}_\bullet\langle y, \Theta^{-1}_{(x_j)_j} x_j \rangle$ are in general not unique. The sequence $(\Theta^{-1}_{(x_j)_j} x_j)_{j\in J} \subseteq E$ is known as the *canonical dual frame of* $(x_j)_{j\in J}$. We also have

$$y = \Theta^{-1}_{(x_j)_j} \Theta_{(x_j)_j} y = \sum_{j \in J} {}_\bullet\langle y, \Theta_{(x_j)_j} x_j \rangle \Theta^{-1}_{(x_j)_j} x_j, \tag{2.1.3}$$

for $y \in E$.

For the papers of the thesis we can make certain simplifications. Unless we are working with frames for Hilbert $\mathbb{C}$-modules, that is, Hilbert spaces, we will always work with finitely generated projective Hilbert C^*-modules, and thus all frames will have a finite number of elements. Any sequence $(x_j)_{j=1}^n$ trivially satisfies the upper inequality of (2.1.1) by the Cauchy-Schwarz inequality, and so is automatically a Bessel sequence. Even when we consider the C^*-algebra $\mathbb{C}$ and frames for Hilbert spaces, they will have a very specific form. The only types of frames for Hilbert spaces considered in this thesis are that of Gabor frames, which we introduce in the next section. Note however that when considering frames for Hilbert $\mathbb{C}$-modules in the papers of the thesis we sometimes consider uncountable index sets in the form of so-called continuous frames. This will become clearer in the next section.

2.2 Time-frequency analysis and Gabor frames

Much of the thesis concerns the interplay between operator algebras and time-frequency analysis, specifically Gabor frames. We therefore dedicate this section to explaining what Gabor frames are. However, we have to wait until we introduce Heisenberg modules in the next section to properly outline most of the remaining core questions considered in the papers of the thesis. Introductions to time-frequency analysis are found in [53] for the case of lattices in $\mathbb{R}^{2d}$, and in [52] for the case of locally compact abelian groups.

We have already considered the two main operators of time-frequency analysis, the time-shift operator and the frequency-shift operator (or modulation operator) in Example 1.1.1, albeit in the specific case of $\mathbb{R}^2$. We shall however need to consider more general phase spaces than just $\mathbb{R}^2 \cong \mathbb{R} \times \widehat{\mathbb{R}}$.

Let G be a second-countable LCA group with a fixed Haar measure, and let $\widehat{G}$ be its dual group. We equip $\widehat{G}$ with the corresponding Plancherel measure. The *phase space of* G is the product space $G \times \widehat{G}$ with product topology and product measure. Moreover, let Δ be a closed cocompact subgroup of $G \times \widehat{G}$. We fix a Haar measure on Δ, and will always equip the quotient group $(G \times \widehat{G})/\Delta$ with the quotient measure, i.e. the unique measure such that (1.1.2) holds. Denoting the Haar measure on $(G \times \widehat{G})/\Delta$ by μ, we can associate to $(G \times \widehat{G})/\Delta$ the *size of* Δ, which we denote by $s(\Delta) = \mu((G \times \widehat{G})/\Delta)$. This is finite as Δ is cocompact.

Having fixed conventions for Haar measures, we may begin to introduce time-frequency analysis on (second-countable) LCA groups. Given $x \in G$ and $\omega \in \widehat{G}$, define the translation operator T_x and the modulation operator M_ω on $L^2(G)$ by

$$T_x f(t) = f(x^{-1}t), \quad M_\omega f(t) = \omega(t) f(t),$$

for $f \in L^2(G)$ and $t \in G$. They are both clearly unitary operators on $L^2(G)$.

The assignment $\pi\colon (x,\omega) \mapsto M_\omega T_x$ becomes a projective representation of $G \times \widehat{G}$ as unitary operators on $L^2(G)$. The projectivity is governed by the *Heisenberg 2-cocycle* c, defined by

$$c(\chi_1, \chi_2) = \overline{\omega_2(x_1)}, \tag{2.2.1}$$

for $\chi_1 = (x_1, \omega_1), \chi_2 = (x_2, \omega_2) \in G \times \widehat{G}$. So, to be more precise

$$\begin{aligned} \pi\colon G \times \widehat{G} &\to \mathcal{U}(L^2(G)) \\ (x,\omega) &\mapsto M_\omega T_x, \end{aligned} \tag{2.2.2}$$

is a c-projective unitary representation.

Associated to a closed subgroup $\Delta \subseteq G \times \widehat{G}$ is the *adjoint subgroup* Δ°. We define it as

$$\Delta^\circ := \{\chi \in G \times \widehat{G} \mid \pi(\chi)\pi(\lambda) = \pi(\lambda)\pi(\chi) \text{ for all } \lambda \in \Delta\}. \tag{2.2.3}$$

It is well known that the adjoint subgroup Δ° is isomorphic to the annihilator $\Delta^\perp$ through a measure-preserving topological isomorphism. From this we may deduce that when Δ is cocompact, Δ° is discrete, a fact which is used time and time again in the thesis.

So far we have introduced quite a few of the fundamental concepts relevant to the time-frequency analysis used in this thesis. However, we have not really discussed functions. We will not simply be content with looking at functions in $L^2(G)$. Instead, we want to look at functions with good decay in both time and frequency in some technical sense. To make sense of this, we consider the *short-time Fourier transform*. More specifically, fix $g \in L^2(G)$. The short-time Fourier transform with respect to g is the operator $V_g \colon L^2(G) \to L^2(G \times \widehat{G})$ defined by

$$V_g f(\chi) = \langle f, \pi(\chi) g\rangle_{L^2(G)}. \tag{2.2.4}$$

While an interesting and well-studied operator in its own right, we use the short-time Fourier transform to define the following important function space known as the *Feichtinger algebra*. We denote the Feichtinger algebra on G by $S_0(G)$, and it is given by

$$S_0(G) := \{ g \in L^2(G) \mid V_g g \in L^1(G \times \widehat{G})\}. \tag{2.2.5}$$

A modern survey on the Feichtinger algebra is found in [64]. It is well known that elements of $S_0(G)$ are continuous, indeed, they are absolutely continuous. For any $g \in S_0(G) \setminus \{0\}$ we can define a norm on $S_0(G)$ by

$$\|f\|_{S_0(G)} = \|V_g f\|_{L^1(G\times\widehat{G})}$$

for $f \in S_0(G)$. All elements of $S_0(G) \setminus \{0\}$ induce equivalent norms on $S_0(G)$, and $S_0(G)$ becomes a Banach space equipped with any one of these equivalent norms. In particular, if G is discrete, $S_0(G) \cong \ell^1(G)$. Moreover, $S_0(G)$ is in general dense in both $L^1(G)$ and $L^2(G)$. The Feichtinger algebra plays a major role in the first three papers of the thesis. For the time being, however, we use it to make precise the phrase “good decay in time and frequency”. Indeed, our notion of $f \in L^2(G)$ having good decay in time and frequency will simply be that $f \in S_0(G)$. We see from the defining relation (2.2.5) that $\langle f, \pi(\chi) f\rangle_{L^2(G)}$ decays in an L^1-sense as χ varies over $G \times \widehat{G}$, that is, over both time and frequency.

The Feichtinger algebra is also of importance as the fundamental identity of Gabor analysis (or FIGA for short) holds for functions in $S_0(G)$. To be more precise, let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup and let Δ° be the corresponding adjoint subgroup. Then we have

$$\int_\Delta \langle f, \pi(z) g\rangle_{L^2(G)} \pi(z) h \,\mathrm{d}z = s(\Delta)^{-1} \sum_{z^\circ \in \Delta^\circ} \langle h, \pi(z^\circ) g\rangle_{L^2(G)} \pi(z^\circ) f \tag{2.2.6}$$

for $f, g, h \in S_0(G)$, where the integral and sum are interpreted weakly in an L^2-sense. This identity comes into play in a major way in Section 2.3.

We return to the theme of Section 2.1 in that we need to introduce a certain type of frame important to the first three papers of the thesis, namely *Gabor frames*. Let Λ be a lattice (i.e. a discrete and cocompact subgroup) in $G \times \widehat{G}$, and let $g \in L^2(G)$. We may then consider a *Gabor system* $\mathcal{G}(g; \Lambda)$ defined by

$$\mathcal{G}(g; \Lambda) = (\pi(\lambda)g)_{\lambda \in \Lambda}.$$

We see that Λ plays the role of the index set J in Section 2.1. We say that $\mathcal{G}(g; \Lambda)$ is a *Gabor frame* for $L^2(G)$ if it is a frame for $L^2(G)$. The function g is sometimes called a *window* or an *atom*. Translating (2.1.1) to our current setting, we see that $\mathcal{G}(g; \Lambda)$ is a Gabor frame exactly when there are positive real numbers $C, D > 0$ such that

$$C\|f\|_{L^2(G)}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle_{L^2(G)}|^2 \leq D\|f\|_{L^2(G)}^2,$$

for all $f \in L^2(G)$. As for module frames we may consider the associated frame operator. We denote the frame operator associated to $\mathcal{G}(g; \Lambda)$ by $S_{g,\Lambda}$. If $g \in S_0(G)$, then it is well known that $S_{g,\Lambda} g \in S_0(G)$ as well. Moreover, if $\mathcal{G}(g; \Lambda)$ is a Gabor frame, then $S_{g,\Lambda}$ is invertible, and it is even true that if $g \in S_0(G)$, then the canonical dual atom $S_{g,\Lambda}^{-1} g \in S_0(G)$ when Λ is a lattice. This was shown for specific lattices in $\mathbb{R}^{2d}$ in [58], and it was claimed to hold for lattices in phase spaces of arbitrary LCA groups in the same paper. This claim has been accepted as true in the mathematical community. Their proofs make heavy use of techniques specific to Gabor analysis.

We could also consider *continuous Gabor frames*. For this, let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup and let $g \in L^2(G)$. We say $\mathcal{G}(g; \Delta) = (\pi(z)g)_{z \in \Delta}$ is a continuous Gabor frame for $L^2(G)$ if it is weakly measurable and there exist positive real constants $C, D > 0$ such that

$$C\|f\|_{L^2(G)}^2 \leq \int_\Delta |\langle f, \pi(z)g\rangle_{L^2(G)}|^2 \,\mathrm{d}z \leq D\|f\|_{L^2(G)}^2,$$

for all $f \in L^2(G)$. If the upper inequality is satisfied we say $\mathcal{G}(g; \Delta)$ is a *Bessel system*. Further, if $\mathcal{G}(g; \Delta)$ is a continuous Gabor frame, then the corresponding Gabor frame operator

$$S_{g,\Delta} f = \int_\Delta \langle f, \pi(z)g\rangle_{L^2(G)} \pi(z)g \,\mathrm{d}z, \tag{2.2.7}$$

with the integral interpreted weakly in $L^2(G)$, is invertible. The motivating problem for Paper C was the following.

Problem. Suppose $\mathcal{G}(g;\Delta)$ is a Gabor frame for $L^2(G)$, potentially continuous, and suppose $g \in S_0(G)$. Is it then true that $S_{g,\Delta}^{-1} g \in S_0(G)$, too? And can this be proved without heavy use of Gabor analytic techniques?

The phrase "heavy use of Gabor analytic techniques" is not precise, but should be understood in a sense that we wish to use operator algebraic techniques to prove as much as possible so that it is potentially possible to extend the proofs to other representations of more general locally compact groups. The answer to the problem is affirmative, and turns out to be related to spectral invariance and C^*-uniqueness of the twisted convolution algebra $\ell^1(\Delta^\circ, \overline{c})$, where c is the Heisenberg 2-cocycle.

To round off this section we present two cornerstone results of Gabor analysis that become relevant in Paper B in the thesis. First we present the Wexler-Raz biorthogonality relations. Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup and let $g \in L^2(G)$ be such that $\mathcal{G}(g;\Delta)$ is a Gabor frame for $L^2(G)$. Suppose there is $h \in L^2(G)$ such that

$$f = \int_G \langle f, \pi(z)g \rangle \pi(z) h \,\mathrm{d}z,$$

for all $f \in L^2(G)$, where we interpret the integral weakly in $L^2(G)$. Then $\mathcal{G}(h;\Delta)$ is also a Gabor frame for $L^2(G)$, and is known as a *dual frame* of $\mathcal{G}(g;\Delta)$. This is symmetric in the sense that $\mathcal{G}(g;\Delta)$ is also a dual frame of $\mathcal{G}(h;\Delta)$.

Proposition 2.2.1 (Wexler-Raz biorthogonality relations)**.** *Let $\Delta \subseteq G \times \widehat{G}$ be a closed and cocompact subgroup, and suppose $g, h \in L^2(G)$. Then the following are equivalent:*

i) $\mathcal{G}(g;\Delta)$ and $\mathcal{G}(h;\Delta)$ are dual frames for $L^2(G)$.

ii) $\langle g, \pi(z^\circ)h \rangle_{L^2(G)} = s(\Delta)\delta_{0,z^\circ}$ for all $z^\circ \in \Delta^\circ$, and where $s(\Delta)$ is the size of Δ and δ_{0,z° is the Kronecker delta.

To present the duality principle we need to consider Gabor systems not only over Δ, but also over Δ°. If $\mathcal{G}(g;\Delta^\circ)$ is a Gabor system, we denote the analysis operator by C_{g,Δ°. The duality principle then says the following.

Proposition 2.2.2 (Duality principle)**.** *Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup and let $g \in L^2(G)$. Then the following are equivalent:*

i) $\mathcal{G}(g;\Delta)$ is a Gabor frame for $L^2(G)$.

ii) The composition $C_{g,\Delta^\circ} \circ C_{g,\Delta^\circ}^* \colon \ell^2(\Delta^\circ) \to \ell^2(\Delta^\circ)$ is an isomorphism of Hilbert spaces.

We show in Paper B in the thesis that the biorthogonality relations, duality principle, and several other important results of Gabor analysis can be deduced from an operator algebraic approach by studying Heisenberg modules, which we introduce in the next section.

2.3 Heisenberg modules

At last we may set the stage for most of the operator algebraic reformulations of Gabor analysis needed in the sequel. This is also the section where we get to combine some of the concepts from earlier sections and formulate most of the key problems considered in the papers of the thesis. To do this properly we need to introduce Heisenberg modules. They were extensively studied in [97] in the case of the Schwarz-Bruhat space, and the study was extended to the case of the Feichtinger algebra in [82]. As such, these make for nice references for the following material.

First, fix a second-countable LCA group G and a closed cocompact subgroup Δ of the phase space $G \times \widehat{G}$. As in Section 2.2 we choose the corresponding Plancherel measure on $\widehat{G}$ and the quotient measure on $(G \times \widehat{G})/\Delta$ such that Weil's formula holds.

Let c denote the Heisenberg 2-cocycle on $G \times \widehat{G}$, and let $\pi \colon G \times \widehat{G} \to \mathcal{U}(L^2(G))$ be the c-projective unitary representation given by time-frequency shifts, see (2.2.1) and (2.2.2). Denote by c and π also the restrictions to $\Delta \subseteq G \times \widehat{G}$. Equipping $S_0(\Delta)$ with c-twisted convolution and c-twisted involution as we did for $L^1(G, c)$, we obtain a twisted convolution algebra $S_0(\Delta, c)$ which is dense in $L^1(G, c)$. We equip it with any of the equivalent norms coming from $g \in S_0(\Delta) \setminus \{0\}$, making it a Banach $*$-algebra. Then we may turn $S_0(G)$ into a left $S_0(\Delta, c)$-inner product module in the following way. Let $a \in S_0(\Delta, c)$ and $f \in S_0(G)$. Then the left module action of $S_0(\Delta, c)$ on $S_0(G)$ is given by

$$(a \cdot f)(x) = \int_{\Delta} a(z)\pi(z)f(x)\, \mathrm{d}z, \tag{2.3.1}$$

for $x \in G$, and this action is continuous. Moreover, if $f, g \in S_0(G)$, we obtain an $S_0(\Delta, c)$-valued inner product ${}_{\bullet}\langle \cdot, \cdot \rangle \colon S_0(G) \times S_0(G) \to S_0(\Delta, c)$ by

$${}_{\bullet}\langle f, g\rangle(z) = \langle f, \pi(z)g\rangle_{L^2(G)} \tag{2.3.2}$$

for $z \in \Delta$. While this in itself might be useful to look at, the true machinery being utilized in the thesis comes from the fact that there is a corresponding right inner product module structure that interacts nicely with the left one. Just as we constructed $S_0(\Delta, c)$ we may also construct $S_0(\Delta^\circ, \overline{c})$, where Δ° is given by (2.2.3). Note the conjugation $\overline{c}$ of the Heisenberg 2-cocycle. As Δ is closed and cocompact, Δ° is discrete, hence $S_0(\Delta^\circ, \overline{c}) \cong \ell^1(\Delta^\circ, \overline{c})$. Let $b \in \ell^1(\Delta^\circ, \overline{c})$ and $f \in S_0(G)$. We then obtain a right module action of $\ell^1(\Delta^\circ, \overline{c})$ on $S_0(G)$ by

$$(f \cdot b)(x) = s(\Delta)^{-1} \sum_{z^\circ \in \Delta^\circ} b(z^\circ)\pi(z^\circ)^* f(x) \tag{2.3.3}$$

for $x \in G$, and this action is continuous. Also, if $f, g \in S_0(G)$ we obtain an $\ell^1(\Delta^\circ, \overline{c})$-valued inner product $\langle \cdot, \cdot \rangle_\bullet \colon S_0(G) \times S_0(G) \to \ell^1(\Delta^\circ, \overline{c})$ by

$$\langle f, g \rangle_\bullet(z^\circ) = \langle \pi(z^\circ)g, f \rangle, \tag{2.3.4}$$

for $z^\circ \in \Delta^\circ$. Hence $S_0(G)$ is both a left and a right inner product module. The left and right structures are also compatible in the sense that $S_0(G)$ becomes an $S_0(\Delta, c)$-$\ell^1(\Delta^\circ, \overline{c})$-pre-equivalence module. We postpone the technicalities of this concept to the papers of the thesis. For the purposes of this introduction it suffices to know that one of the consequences of being an $S_0(\Delta, c)$-$\ell^1(\Delta^\circ, \overline{c})$-pre-equivalence module is that

$${}_\bullet\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_\bullet \tag{2.3.5}$$

for $f, g, h \in S_0(G)$. Hence we get a glimpse into what is meant by the structures being compatible, namely that there is an intimate relation between the left inner product and left action with the right inner product and right action. Indeed, if we write out (2.3.5) in terms of the above defined module actions and inner products we see that it is nothing more than the FIGA for $S_0(G)$, see (2.2.6).

Now recall from Section 1.1 that we may complete $L^1(\Delta, c)$ to a C^*-algebra through faithful $*$-representations. As $S_0(\Delta, c)$ is dense in $L^1(\Delta, c)$, they have the same enveloping C^*-algebra. It turns out that the integrated representation of π yields a faithful $*$-representation $\pi \colon S_0(\Delta, c) \to \mathbb{B}(L^2(G))$. It is well known that the resulting C^*-completion is isomorphic to the enveloping C^*-algebra of $L^1(G, c)$, denoted $C^*(\Delta, c)$. The same can be done for $\ell^1(\Delta^\circ, \overline{c})$ to obtain $C^*(\Delta^\circ, \overline{c})$. Then $S_0(G)$ may be completed in the norm

$$\|f\| := \| {}_\bullet\langle f, f \rangle \|_{C^*(\Delta, c)}^{1/2}, \quad f \in S_0(G). \tag{2.3.6}$$

We denote the completion by $E_\Delta(G)$. This becomes a left Hilbert $C^*(\Delta, c)$-module. However, as $S_0(G)$ is an $S_0(\Delta, c)$-$\ell^1(\Delta^\circ, \overline{c})$-pre-equivalence module, the norm defined by (2.3.6) would be the same if we defined it in terms of the $C^*(\Delta^\circ, \overline{c})$-valued inner product, and it turns out $E_\Delta(G)$ is also a right Hilbert $C^*(\Delta^\circ, \overline{c})$-module. Indeed, $E_\Delta(G)$ is an instance of a $C^*(\Delta, c)$-$C^*(\Delta^\circ, \overline{c})$-equivalence bimodule. Once again we postpone the technicalities of this concept to the papers of the thesis, but note that (2.3.5) can be extended to

$${}_\bullet\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_\bullet, \tag{2.3.7}$$

for all $f, g, h \in E_\Delta(G)$. Moreover, note that $S_0(G) \subseteq E_\Delta(G)$, and that the inclusion is strict for most examples of interest.

When Δ is closed and cocompact in $G \times \widehat{G}$, Δ° is discrete. Hence there is a faithful finite trace $\mathrm{tr}_{\Delta^\circ}$ on $C^*(\Delta^\circ, \overline{c})$ given by the extension of

$$\mathrm{tr}_{\Delta^\circ}(b) = b(0) \tag{2.3.8}$$

for $b \in \ell^1(\Delta^\circ, \overline{c})$, and where 0 is the identity of Δ°. Using this we may induce a $\mathbb{C}$-valued inner product on $E_\Delta(G)$ by setting

$$\langle f, g \rangle_{E_\Delta(G)} = \langle f, g \rangle_\bullet(0). \tag{2.3.9}$$

Writing this out using (2.3.4) we obtain

$$\langle f, g \rangle_{E_\Delta(G)} = \langle f, g \rangle_\bullet(0) = \langle \pi(0)g, f \rangle_{L^2(G)} = \langle g, f \rangle_{L^2(G)},$$

so the induced inner product on $E_\Delta(G)$ is just the inner product on $L^2(G)$. Not only can this be used to realize $E_\Delta(G) \subseteq L^2(G)$, but it also lays the groundwork for the main question of the first paper of the thesis. To expand on this, let $f, g \in S_0(G)$. Using (2.3.1), (2.3.2) and (2.3.7) we then obtain

$$f\langle g, g \rangle_\bullet = \int_\Delta \langle f, \pi(z)g \rangle_{L^2(G)} \pi(z)g \,\mathrm{d}z = S_{g,\Delta} f, \tag{2.3.10}$$

where $S_{g,\Delta}$ is the Gabor frame operator corresponding to the Gabor system $\mathcal{G}(g; \Delta)$, see (2.2.7). As the trace $\mathrm{tr}_{\Delta^\circ}$ we use to induce an inner product on $E_\Delta(G)$ is continuous, and the induced inner product is the inner product on $L^2(G)$, we can extend by continuity so that (2.3.10) is true for all $f \in L^2(G)$. Note that we specified $g \in S_0(G)$, and we know that elements of $S_0(G)$ are Bessel vectors, i.e. for $g \in S_0(G)$, $\mathcal{G}(g; \Delta)$ is a Bessel system. However, for general closed cocompact $\Delta \subseteq G \times \widehat{G}$ it is known that not every L^2-function is going to be a Bessel vector, see e.g. [53, Proposition 6.2.6]. The following natural question arises.

Problem. Suppose Δ is a closed cocompact subgroup of $G \times \widehat{G}$, and let $g \in E_\Delta(G)$. Is g a Bessel vector? Equivalently, is $S_{g,\Delta} \colon L^2(G) \to L^2(G)$ a bounded linear operator?

This is the main problem of Paper A in the thesis. The question has an affirmative answer, so elements of the Heisenberg module $E_\Delta(G)$ are always going to be Bessel vectors.

Knowing that elements of the Heisenberg module $E_\Delta(G)$ can be realized as functions in $L^2(G)$, and are even Bessel vectors, we may ask if (2.3.7) can actually be written out in terms of (2.2.6) in any meaningful way.

Problem. Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup. Do elements of $E_\Delta(G)$ satisfy the FIGA? In other words, is it true that

$$\int_\Delta \langle f, \pi(z)g \rangle_{L^2(G)} \pi(z)h \,\mathrm{d}z = s(\Delta)^{-1} \sum_{z^\circ \in \Delta^\circ} \langle h, \pi(z^\circ)g \rangle_{L^2(G)} \pi(z^\circ) f \tag{2.3.11}$$

for $f, g, h \in E_\Delta(G)$?

We attempt to answer this question in Paper A. The answer is positive if we make an extra restriction. Due to the technical setup in Paper A we find need for Δ to be a lattice. In that case (2.3.11) is true for $f, g, h \in E_\Delta(G)$.

The identity (2.3.10) can also be rewritten

$$\Theta_g f = S_{g,\Delta} f$$

for all $f, g \in S_0(G)$. Luef observed in [82] that if $g \in S_0(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is a lattice, then $\mathcal{G}(g;\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if $\{g\}$ is a module frame for $E_\Lambda(\mathbb{R}^d)$. Indeed, this was proved for so-called *multi-window Gabor frames*, meaning it was shown for the Gabor system $\mathcal{G}(g_1, \dots, g_k; \Lambda) := \mathcal{G}(g_1;\Lambda) \cup \cdots \cup \mathcal{G}(g_k;\Lambda)$ with $g_1, \dots, g_k \in S_0(G)$. The following natural question arises.

Problem. Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup and let $g_1, \dots, g_k \in E_\Delta(G)$. Is it then true that $(g_i)_{i=1}^k$ is a module frame for $E_\Delta(G)$ if and only if $\mathcal{G}(g_1, \dots, g_k; \Delta)$ is a (multi-window) Gabor frame for $L^2(G)$?

This problem also has an affirmative answer. Indeed, the difficult part of the problem is showing that the frame operator is bounded when the elements are in $E_\Delta(G)$ rather than in $S_0(G)$. But we already know this as we know elements of $E_\Delta(G)$ are Bessel vectors. As a consequence we get that finite module frames for the Heisenberg module $E_\Delta(G)$ are exactly multi-window Gabor frames for $L^2(G)$ with windows in the Heisenberg module. This is a powerful link between two concepts of frames that a priori describe reconstruction properties on Hilbert C^*-modules over very different C^*-algebras.

Lastly, we describe some of the key problems considered in Paper B in the thesis. Let $\mathbb{Z}_m$ denote the group $\mathbb{Z}/(m\mathbb{Z})$. We still consider a closed cocompact subgroup $\Delta \subseteq G \times \widehat{G}$, but we will also regard Δ as sitting inside $G \times \mathbb{Z}_n \times \mathbb{Z}_d \times \widehat{G} \times \widehat{\mathbb{Z}_n} \times \widehat{\mathbb{Z}_d}$ for some $n, d \in \mathbb{N}$. In the sequel we will write $f_{i,j}$ instead of $f(\cdot, i, j)$ for $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. Fixing $n, d \in \mathbb{N}$ we introduce in Paper B a new type of frames for $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ called *matrix Gabor frames*, or, if we want to specify the dependence on n and d, *(n,d)-matrix Gabor frames*. The bookkeeping can get a bit involved, but in short, a function $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ generates an (n,d)-matrix Gabor frame with respect to Δ if the collection of time-frequency shifts

$$\mathcal{G}(g;\Delta) := \{\pi(z) g_{i,j} \mid z \in \Delta\}_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d} \tag{2.3.12}$$

is a frame for $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. Here $\pi\colon \Delta \to \mathcal{U}(L^2(G))$ is the usual projective unitary representation by time-frequency shifts. It is shown in Paper B that these generalize multi-window super Gabor frames known from earlier literature, see e.g. [12, 13, 59, 61, 66]. We note that super Gabor frames are frames of the form

(2.3.12) where we use $n = 1$ to make $\mathbb{Z}_n$ into the trivial group. There are definitely more motivated and verbose ways of introducing super Gabor frames, but they will play a very minor role in this thesis.

The reason for the name matrix Gabor frame is due to them arising naturally from a module frame perspective when lifting Morita equivalence bimodules to matrix modules over matrix algebras over C^*-algebras. More precisely, we consider $E_\Delta(G)$ as a $C^*(\Delta, c)$-$C^*(\Delta^\circ, \overline{c})$-equivalence bimodule as before, and we simply lift this to regard $M_{n,d}(E_\Delta(G))$ as an $M_n(C^*(\Delta, c))$-$M_d(C^*(\Delta^\circ, \overline{c}))$-equivalence bimodule. So from an operator algebraic standpoint this is quite the simple construction. However, using the machinery from Paper A we do as mentioned obtain a new type of Gabor frame generalizing previously considered notions of Gabor frames.

However, the introduction of matrix Gabor frames is merely a corollary of the results of Paper A as well as the following problem considered in the first half of Paper B.

Problem. Is it possible to formulate analogues of cornerstone results of Gabor analysis such as the Wexler-Raz biorthogonality relations, the duality principle and others for Morita equivalence bimodules?

We do show that this is indeed possible when the equivalence bimodule is finitely generated and projective over at least one of the C^*-algebras, and the reformulations are almost trivial in the Hilbert C^*-module setting. In the case of Heisenberg modules we know we can pass from module frames to Gabor frames for the Hilbert space induced by the canonical trace, and this does indeed still hold true for matrix Gabor frames. As matrix Gabor frames generalize previously considered types of Gabor frames in the literature it becomes natural to ask the following questions.

Problem. Do matrix Gabor frames satisfy an analogue of the Wexler-Raz biorthogonality relations? Is there a duality principle for matrix Gabor frames? Do other cornerstone results regarding Gabor frames carry over?

The answers to these questions are once again positive in the sense that there is both an analogue of the Wexler-Raz biorthogonality relations, a duality principle, as well as the fact that some other known results for Gabor frames carry over. What is new about the approach in Paper B is that the Gabor frames considered are introduced purely from an operator algebraic point of view. Indeed, most proofs in the paper make heavy use of the compatibility between the left and right inner products (2.3.7). Utilizing (2.3.7) makes the proofs of e.g. an analogue of the Wexler-Raz biorthogonality relations and a duality principle very short when used in conjunction with main results from Paper A.

Chapter 3

Summary of papers

Paper A: Heisenberg modules as function spaces

The first paper builds on the observation that the Heisenberg module $E_\Delta(G)$ can be realized as a subspace of $L^2(G)$, and in the paper we attempt to describe the Heisenberg module as a function space. We find that when Δ is closed and cocompact in $G \times \widehat{G}$, Heisenberg modules $E_\Delta(G)$ are function spaces eligible for time-frequency analysis as we prove that their elements are Bessel vectors for Δ, i.e. that the frame operators with respect to Δ are bounded operators. Moreover, if Δ is a lattice, elements of the Heisenberg module satisfy the fundamental identity of Gabor analysis.

In this paper we also establish the result that a sequence $(g_i)_{i=1}^k$ with $g_i \in E_\Delta(G)$ generates a module frame for $E_\Delta(G)$ if and only if the corresponding (multi-window) Gabor system $\mathcal{G}(g_1, \ldots, g_k; \Delta)$ is a Gabor frame for $L^2(G)$. This extends previous results in [82] where this was established in the case of $g_1, \ldots, g_k$ in the Feichtinger algebra $S_0(G)$.

Paper B: Gabor duality theory for Morita equivalent C^*-algebras

The second paper builds upon Paper A in a very direct way. Indeed, knowing that elements of the Heisenberg module $E_\Delta(G)$ are Bessel vectors for Δ when Δ is closed and cocompact in $G \times \widehat{G}$ allows us to do the constructions of this paper for general elements of $E_\Delta(G)$ rather than for the dense subspace $S_0(G)$. In this paper we first establish analogues of the Wexler-Raz biorthogonality relations, the duality principle of Gabor analysis, as well as density theorems of Gabor analysis for certain Morita equivalence bimodules. Using the machinery from Paper A we can transfer the results to corresponding results for $L^2(G)$. Moreover, from the point of view of Morita equivalence bimodules it becomes very natural to consider frames for matrix modules over matrix algebras over C^*-algebras. Applying the machinery of Paper A to these matrix modules we obtain a new type of Gabor frames called matrix Gabor frames, and they generalize the multi-window super Gabor frames considered in the literature previously. The Wexler-Raz biorthogonality relations, the duality principle and the density theorems for Morita equivalence bimodules carry over almost directly to establish the corresponding results for matrix Gabor frames, yielding these results with very little use of standard Gabor analytic techniques.

Paper C: Spectral invariance of $*$-representations of twisted convolution algebras with applications in Gabor analysis

The paper's main result concerns spectral invariance of twisted convolution algebras when realizing them as subalgebras of bounded operators on Hilbert spaces through faithful $*$-representations. We find that for a locally compact group G with a continuous 2-cocycle c, we can guarantee spectral invariance of $L^1(G, c)$ in $\mathbb{B}(\mathcal{H})$ for any faithful $*$-representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ if $L^1(G_c)$ is symmetric and C^*-unique. This is a useful result as symmetry and C^*-uniqueness of (untwisted) convolution algebras, such as $L^1(G_c)$, have been studied in the literature before. As part of proving this result we obtain the result that $L^1(G, c)$ is C^*-unique if $L^1(G_c)$ is C^*-unique, which is of independent interest as there are very few results in the available literature concerning C^*-uniqueness of twisted convolution algebras. These results allow us to prove a result in Gabor analysis concerning the regularity of the canonical dual atom and the canonical tight atom related to a Gabor atom in Feichtinger's algebra. More precisely, if G is a locally compact abelian group, Δ is a closed cocompact subgroup of $G \times \widehat{G}$, and $g \in S_0(G)$ generates a Gabor frame $\mathcal{G}(g; \Delta)$ for $L^2(G)$, is it true that the canonical dual atom $S^{-1}g$ and the canonical tight atom $S^{-1/2}g$ are also in $S_0(G)$? Here S denotes the frame operator associated to $\mathcal{G}(g; \Delta)$. Using the previously proved results of this paper we show that the answer to this question is affirmative. Moreover, we do so without the use of certain techniques specific to the setting of Gabor analysis. In doing so, the approach used may be adaptable to $*$-representations of other (twisted) convolution algebras.

Paper D: C^*-uniqueness results for groupoids

Building on some results of Paper C, we describe a sufficient condition for C^*-uniqueness for a second-countable locally compact Hausdorff étale groupoid. More precisely, we find a condition guaranteeing the C^*-uniqueness of the I-norm completion of $C_c(\mathcal{G}, \sigma)$, denoted by $\ell^1(\mathcal{G}, \sigma)$, where $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid and σ is a continuous 2-cocycle for $\mathcal{G}$. Using recently proved results in the theory of étale groupoids we obtain sufficient conditions for this by posing the analogous question for $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, where $\mathrm{Iso}(\mathcal{G})^\circ$ is the interior of the isotropy subgroupoid of $\mathcal{G}$. We further find sufficient conditions for C^*-uniqueness for $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ by instead considering C^*-uniqueness of the twisted convolution algebras of the individual fibers of $\mathrm{Iso}(\mathcal{G})^\circ$. All in all we obtain a result yielding C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ in terms of twisted convolution algebras of certain associated discrete groups. This allows us to describe some classes of C^*-unique groupoids by applying results from Paper C. Moreover, we find an example of a non-amenable groupoid which is C^*-unique, as well as deducing C^*-uniqueness of a group appearing as a wreath product by reformulating the question in terms of C^*-uniqueness of a groupoid.

Part II

Research Papers

Paper A

Heisenberg modules as function spaces

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Paper A

Heisenberg modules as function spaces

Abstract

Let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$, where G is a second countable, locally compact abelian group. Using localization of Hilbert C^*-modules, we show that the Heisenberg module $\mathcal{E}_\Delta(G)$ over the twisted group C^*-algebra $C^*(\Delta, c)$ due to Rieffel can be continuously and densely embedded into the Hilbert space $L^2(G)$. This allows us to characterize a finite set of generators for $\mathcal{E}_\Delta(G)$ as exactly the generators of multi-window (continuous) Gabor frames over Δ, a result which was previously known only for a dense subspace of $\mathcal{E}_\Delta(G)$. We show that $\mathcal{E}_\Delta(G)$ as a function space satisfies two properties that make it eligible for time-frequency analysis: Its elements satisfy the fundamental identity of Gabor analysis if Δ is a lattice, and their associated frame operators corresponding to Δ are bounded.

A.1 Introduction

Gabor analysis concerns sets of time-frequency shifts of functions. The field has its roots in a paper by the electrical engineer and physicist Dennis Gabor [48]. In this paper, the author made the claim that one could obtain basis-like representations of functions in $L^2(\mathbb{R})$ in terms of the set $\{e^{2\pi i l x}\phi(x-k) : k, l \in \mathbb{Z}\}$, where ϕ denotes a Gaussian. Today, one of the central problems of the field remains understanding the spanning and basis-like properties of sets of the form $\{e^{2\pi i \beta l x}\eta(x-\alpha k) : k, l \in \mathbb{Z}\}$ for a given $\eta \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$.

Although Gabor analysis is usually carried out for functions of one or several real variables, the nature of time-frequency shifts makes it possible to generalize many aspects of the theory to the setting of a locally compact abelian group G [52]. In this setting, elements of G represent time, while elements of the Pontryagin dual

$\widehat{G}$ represent frequency. If $\eta \in L^2(G)$, then a time-frequency shift of η is a function of the form $\pi(x,\omega)\eta(t) = \omega(t)\eta(t-x)$ for $t, x \in G$ and $\omega \in \widehat{G}$. A Gabor system with generator η will in general be any collection of time-frequency shifts of η. In this paper, we will allow continuous Gabor systems over any closed subgroup Δ of the time-frequency plane $G \times \widehat{G}$, which will be of the form $(\pi(z)\eta)_{z\in\Delta}$. We say that such a system forms a Gabor frame if it is a continuous frame for $L^2(G)$, which means that there exist $C, D > 0$ such that

$$C\|\xi\|_2^2 \leq \int_\Delta |\langle \xi, \pi(z)\eta \rangle|^2 \, \mathrm{d}z \leq D\|\xi\|_2^2$$

for every $\xi \in L^2(G)$. Here, we integrate with respect to a fixed Haar measure on Δ. More generally, if $\eta_1, \ldots, \eta_k \in L^2(G)$, one calls $(\pi(z)\eta_j)_{z\in\Delta, 1\leq j\leq k}$ a multi-window Gabor frame if there exist $C, D > 0$ such that

$$C\|\xi\|_2^2 \leq \sum_{j=1}^{k} \int_\Delta |\langle \xi, \pi(z)\eta_j \rangle|^2 \, \mathrm{d}z \leq D\|\xi\|_2^2$$

for all $\xi \in L^2(G)$. If Δ is a discrete subgroup of $G\times\widehat{G}$, one recovers the usual notion of a (discrete) regular Gabor frame. Here, regular means that the discrete subset Δ of $G \times \widehat{G}$ has the structure of a subgroup. A basic fact of Gabor frame theory is that $(\pi(z)\eta)_{z\in\Delta}$ is a Gabor frame if and only if the associated frame operator $S_\eta \colon L^2(G) \to L^2(G)$ is invertible. The operator is given weakly by

$$S_\eta \xi = \int_\Delta \langle \xi, \pi(z)\eta \rangle \pi(z)\eta \, \mathrm{d}z$$

for $\xi \in L^2(G)$.

In [66, 82, 83], Luef and later Jakobsen and Luef discovered that the duality theory of regular Gabor frames is closely related to a class of imprimitivity bimodules constructed by Rieffel [97]. These imprimitivity bimodules are known as *Heisenberg modules*. In general, a Hilbert C^*-module over a C^*-algebra A can be thought of as a generalized Hilbert space where the field of scalars $\mathbb{C}$ is replaced with A, and where the inner product takes values in A rather than $\mathbb{C}$. Hilbert C^*-modules were introduced by Kaplansky in [70], and have since become essential in many parts of operator algebras and noncommutative geometry [29]. An imprimitivity A-B-bimodule is both a left Hilbert C^*-module over A and a right Hilbert C^*-module over B, with compatibility conditions on the left and right structures. If there exists an imprimitivity A-B-bimodule, then the C^*-algebras A and B are called Morita equivalent, a notion first described by Rieffel in [95, 96]. Morita equivalent C^*-algebras share many important properties, such as representation theory and ideal structure.

For a closed subgroup Δ of $G \times \widehat{G}$, the Heisenberg module $\mathcal{E}_\Delta(G)$ can be constructed as a norm completion of the Feichtinger algebra $S_0(G)$ [82]. The latter is an important space of functions in time-frequency analysis [39]. The Heisenberg module implements the Morita equivalence between the twisted group C^*-algebras $C^*(\Delta, c)$ and $C^*(\Delta^\circ, \overline{c})$. Here, Δ° denotes the adjoint subgroup of Δ, which consists of all points $w \in G\times\widehat{G}$ for which $\pi(w)$ commutes with $\pi(z)$ for every $z \in \Delta$. Readers familiar with Gabor analysis know that the adjoint subgroup plays a central role in results such as the fundamental identity of Gabor analysis, and this result can indeed be inferred directly from the structure of the Heisenberg modules. An important class of examples come from when $G = \mathbb{R}^n$ and Δ is a lattice in $G\times\widehat{G} \cong \mathbb{R}^{2n}$, in which case the twisted group C^*-algebras $C^*(\Delta, c)$ and $C^*(\Delta^\circ, \overline{c})$ are both noncommutative $2n$-tori. Indeed, these examples were the original motivation for the construction of Heisenberg modules in [97]. However, the construction has also been applied in other contexts, such as in the construction of finitely generated projective modules over noncommutative solenoids [37, 75, 76].

For a general left Hilbert C^*-module $\mathcal{E}$ over a C^*-algebra A, one defines rank-one operators in analogy with the Hilbert space case. Specifically, if $\eta, \gamma \in \mathcal{E}$, the rank-one operator $\Theta_{\eta,\gamma} \colon \mathcal{E} \to \mathcal{E}$ is given by

$$\Theta_{\eta,\gamma}\xi = {}_{\bullet}\langle \xi, \eta \rangle\, \gamma$$

for $\xi \in \mathcal{E}$. Here, ${}_{\bullet}\langle \cdot, \cdot \rangle$ denotes the A-valued inner product on $\mathcal{E}$. A central observation in [82] is that for $\eta \in S_0(G)$, the rank-one operator $\Theta_{\eta,\eta}$ associated to the Heisenberg module $\mathcal{E}_\Delta(G)$ agrees with the Gabor frame operator S_η on a dense subspace of $\mathcal{E}_\Delta(G)$, namely the Feichtinger algebra $S_0(G)$. This observation has an important consequence: It allows a finite generating set of the Heisenberg module coming from the dense subspace $S_0(G)$ to be characterized exactly as the generators of a multi-window Gabor frame over Δ [66, p. 14]. Moreover, such a finite generating set exists (that is, $\mathcal{E}_\Delta(G)$ is finitely generated) if and only if Δ is cocompact in $G \times \widehat{G}$ [66, Theorem 3.9]. However, since $\mathcal{E}_\Delta(G)$ is an abstract completion of $S_0(G)$, its elements can a priori not be interpreted as functions in any sense. Therefore, it is not straightforward to obtain a similar characterization for generators of $\mathcal{E}_\Delta(G)$ not necessarily in $S_0(G)$.

Nonetheless, it was recently remarked in [11] that $\mathcal{E}_\Delta(G)$ can be continuously embedded into $L^2(G)$. In the present paper, we elaborate on this embedding, and show how it arises naturally from the notion of localization of Hilbert C^*-modules as discussed in [74]. The important extra structure on the Heisenberg module when localizing is a faithful trace on the C^*-algebra $C^*(\Delta, c)$. In the case that Δ is a lattice in $G \times \widehat{G}$, we use the canonical tracial state on $C^*(\Delta, c)$ (see e.g. [19, p. 951]). If

Δ is only cocompact, we have to work a bit more, see Proposition A.3.1. It was already observed in [82] that this trace plays an important role when connecting Heisenberg modules and Gabor frames. However, the consequence that the trace makes it possible to embed $\mathcal{E}_\Delta(G)$ continuously into $L^2(G)$ was first observed in [11].

Furthermore, in the language of localization, the rank-one operator $\Theta_{\eta,\eta}$ for $\eta \in \mathcal{E}_\Delta(G)$ extends uniquely to a bounded linear operator on $L^2(G)$, and we show in this paper that the extension is exactly the Gabor frame operator S_η (Theorem A.3.15). As a consequence, we generalize the equivalence between generators of Heisenberg modules and generators of multi-window Gabor frames to the case when the generators belong to $\mathcal{E}_\Delta(G)$ (Theorem A.3.16). We summarize some of our main results in the following.

Theorem A (cf. Proposition A.3.12, Theorem A.3.15, Theorem A.3.16)**.** Let G be a second countable, locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. Denote by $B_\Delta(G)$ the subspace of $L^2(G)$ consisting of those $\eta \in L^2(G)$ for which $(\pi(z)\eta)_{z\in\Delta}$ is a Bessel family for $L^2(G)$, that is,

$$\int_\Delta |\langle \xi, \pi(z)\eta\rangle|^2 \, \mathrm{d}z < \infty$$

for every $\xi \in L^2(G)$. This is a Banach space with respect to the norm

$$\|\eta\|_{B_\Delta(G)} = \|S_\eta\|^{1/2} = \sup_{\|\xi\|_2=1} \left(\int_\Delta |\langle \xi, \pi(z)\eta\rangle|^2 \, \mathrm{d}z \right)^{1/2}.$$

The following hold:

1. The Heisenberg module $\mathcal{E}_\Delta(G)$ has a concrete description as the completion of $S_0(G)$ in the Banach space $B_\Delta(G)$. The actions are given in Proposition A.3.12.

2. For $\eta \in \mathcal{E}_\Delta(G)$, the Heisenberg module rank-one operator $\Theta_\eta \colon \mathcal{E}_\Delta(G) \to \mathcal{E}_\Delta(G)$ extends to the Gabor frame operator $S_\eta \colon L^2(G) \to L^2(G)$.

3. Let $\eta_1, \ldots, \eta_k \in \mathcal{E}_\Delta(G)$. Then $\{\eta_1, \ldots, \eta_k\}$ is a generating set for $\mathcal{E}_\Delta(G)$ as a left $C^*(\Delta, c)$-module if and only if $(\pi(z)\eta_j)_{z\in\Delta, 1\leq j\leq k}$ is a multi-window Gabor frame for $L^2(G)$.

Part (iii) of Theorem A gives a complete description of finite generating sets of the Heisenberg modules due to Rieffel, showing that they are the generators of a multi-window Gabor frame. Conversely, multi-window Gabor frames over Δ with generators in $\mathcal{E}_\Delta(G)$ give rise to finite generating sets for $\mathcal{E}_\Delta(G)$.

Note also that part (i) of Theorem A implies that $(\pi(z)\eta)_{z\in\Delta}$ is a Bessel family for $L^2(G)$ whenever $\eta \in \mathcal{E}_\Delta(G)$. Consequently, the Gabor analysis operator $C_\eta \colon L^2(G) \to L^2(\Delta)$, synthesis operator $D_\eta \colon L^2(\Delta) \to L^2(G)$, and frame operator $S_\eta \colon L^2(G) \to L^2(G)$ associated to η over Δ are all bounded linear operators. This is an attractive property of $\mathcal{E}_\Delta(G)$ as a function space in time-frequency analysis, at least when focusing on the subgroup Δ. We also show that elements of the Heisenberg module satisfy the fundamental identity of Gabor analysis over the subgroup Δ when it is a lattice (Proposition A.3.18).

We also comment on the assumption in Theorem A that Δ is cocompact. This is necessary for our localization techniques to work, see Proposition A.3.1. However, as shown in [65, Theorem 5.1], the existence of a multi-window Gabor frame over Δ implies that the quotient $(G \times \widehat{G})/\Delta$ is compact, i.e. Δ is a cocompact subgroup of $G \times \widehat{G}$. The assumption is therefore mild.

The paper is structured as follows: In Section A.2, we cover the necessary background material on frames in Hilbert C^*-modules, continuous Gabor frames and Heisenberg modules. In Section A.3, we introduce the notion of the localization of a Hilbert C^*-module with respect to a (possibly unbounded) trace on the coefficient algebra, and compute the localization of the Heisenberg module. We then give applications to Gabor analysis.

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A.2 Preliminaries

A.2.1 Frames in Hilbert C^*-modules

In the interest of brevity, we will assume basic knowledge about C^*-algebras, Hilbert C^*-modules, imprimitivity bimodules and adjointable operators between such modules. We mention [74, 93] as references. Instead, we dedicate this section to introduce module frames.

The A-valued inner product of a left Hilbert A-module will in general be denoted by ${}_\bullet\langle \cdot, \cdot \rangle$, while the A-valued inner product of a right Hilbert A-module will be denoted by $\langle \cdot, \cdot \rangle_\bullet$. We often refer to A as the *coefficient algebra* of $\mathcal{E}$. If $\mathcal{E}$ and $\mathcal{F}$ are left Hilbert A-modules, we use $\mathcal{L}_A(\mathcal{E}, \mathcal{F})$ to denote the Banach space of adjointable operators $\mathcal{E} \to \mathcal{F}$, or just $\mathcal{L}(\mathcal{E}, \mathcal{F})$ when there is no chance of confusion. As is standard, we write $\mathcal{L}(\mathcal{E}) = \mathcal{L}_A(\mathcal{E})$ for the C^*-algebra $\mathcal{L}_A(\mathcal{E}, \mathcal{E})$, and $\mathcal{K}(\mathcal{E}) = \mathcal{K}_A(\mathcal{E})$ for the (generalized) compact operators on $\mathcal{E}$.

For an (at most) countable index set J, we denote by $\ell^2(J, A)$ the left Hilbert A-module of all sequences $(a_j)_{j\in J}$ in A for which the sum $\sum_{j\in J} a_j a_j^*$ converges in A-norm, with A-valued inner product

$$ {}_{\bullet}\langle (a_j)_{j\in J}, (b_j)_{j\in J} \rangle = \sum_{j\in J} a_j b_j^*. $$

There is an analogous way to make $\ell^2(J, A)$ into a *right* Hilbert A-module, by replacing $a_j b_j^*$ with $a_j^* b_j$ in the definition. We will work with left modules throughout this section, but obvious modifications can be made for the case of right modules as well.

We now define module frames in Hilbert A-modules, introduced in [47] in the case where A is unital. For a treatment of the possibly non-unital case, see [6].

Definition A.2.1. Let A be a C^*-algebra and $\mathcal{E}$ be a left Hilbert A-module. Furthermore, let J be some countable index set and let $(\eta_j)_{j\in J}$ be a sequence in $\mathcal{E}$. We say $(\eta_j)_{j\in J}$ is a *module frame for* $\mathcal{E}$ if there exist constants $C, D > 0$ such that

$$ C{}_{\bullet}\langle \xi, \xi \rangle \leq \sum_{j\in J} {}_{\bullet}\langle \xi, \eta_j \rangle \, {}_{\bullet}\langle \eta_j, \xi \rangle \leq D{}_{\bullet}\langle \xi, \xi \rangle \tag{A.2.1} $$

for all $\xi \in \mathcal{E}$, and the middle sum converges in norm. The constants C and D are called lower and upper frame bounds, respectively.

Remark A.2.2. If $A = \mathbb{C}$ in the above definition then $\mathcal{E}$ is a Hilbert space, and we recover the definition of frames in Hilbert spaces due to Duffin and Schaeffer [34].

Remark A.2.3. We will never treat frames over different index sets simultaneously, so to ease notation we will sometimes leave the index set implied.

Let $(\eta_j)_{j\in J}$ be a sequence in $\mathcal{E}$ that satisfies the upper frame bound condition in Definition A.2.1 but not necessarily the lower frame bound condition. Such a sequence is called a *Bessel sequence* and every constant $D > 0$ for which (A.2.1) is true is called a *Bessel bound* for $(\eta_j)_{j\in J}$. To a Bessel sequence $(\eta_j)_{j\in J}$ we associate the *module analysis operator* $\Phi = \Phi_{(\eta_j)_j} : \mathcal{E} \to \ell^2(J, A)$ given by

$$ \Phi\xi = ({}_{\bullet}\langle \xi, \eta_j \rangle)_{j\in J} \tag{A.2.2} $$

for $\xi \in \mathcal{E}$. It is an adjointable A-linear operator, and its adjoint $\Psi = \Psi_{(\eta_j)_j}$ is known as the *module synthesis operator*, and is given by

$$ \Psi((a_j)_j) = \sum_{j\in J} a_j \cdot \eta_j, \tag{A.2.3} $$

for $(a_j)_j \in \ell^2(J, A)$. Now let $(\gamma_j)_{j\in J}$ be another Bessel sequence. We then define the *module frame-like operator* $\Theta \in \mathcal{L}_A(\mathcal{E})$ by $\Theta = \Theta_{(\eta_j)_j,(\gamma_j)_j} := \Psi_{(\gamma_j)_j}\Phi_{(\eta_j)_j}$. That is, for all $\xi \in \mathcal{E}$ we have

$$\Theta\xi = \sum_{j\in J} {}_\bullet\langle \xi, \eta_j \rangle \cdot \gamma_j. \tag{A.2.4}$$

In case $(\eta_j)_j = (\gamma_j)_j$ we write $\Theta_{(\eta_j)_j} := \Theta_{(\eta_j)_j,(\eta_j)_j}$ and call it the *module frame operator* (associated to $(\eta_j)_j$). Since $\Theta_{(\eta_j)_j} = \Phi^*_{(\eta_j)_j}\Phi_{(\eta_j)_j}$, we see that $\Theta_{(\eta_j)_j}$ is always a positive operator.

A special case of the above situation is when we consider a sequence (η) consisting of a single element $\eta \in \mathcal{E}$, i.e. $|J| = 1$. It follows by the Cauchy-Schwarz inequality for Hilbert C^*-modules that such a sequence is automatically a Bessel sequence. We write $\Phi_\eta = \Phi_{(\eta)}$, $\Psi_\eta = \Psi_{(\eta)}$, $\Theta_{\eta,\gamma} = \Theta_{(\eta),(\gamma)}$ for another sequence (γ) where $\gamma \in \mathcal{E}$, and $\Theta_\eta = \Theta_{(\eta)}$. Note that in this case, $\Phi_\eta \in \mathcal{L}_A(\mathcal{E}, A)$, $\Psi_\eta \in \mathcal{L}_A(A, \mathcal{E})$ and $\Theta_{\eta,\gamma} \in \mathcal{L}_A(\mathcal{E}, \mathcal{E})$ are given by

$$\begin{aligned} \Phi_\eta\xi &= {}_\bullet\langle \xi, \eta \rangle \\ \Psi_\eta a &= a \cdot \eta \\ \Theta_{\eta,\gamma}\xi &= {}_\bullet\langle \xi, \eta \rangle \cdot \gamma \end{aligned}$$

for $\xi \in \mathcal{E}$, $a \in A$. Also, for a finite Bessel sequence $(\eta_1, \ldots, \eta_k)$, we have that $\Phi_{(\eta_j)_{j=1}^k} = \sum_{j=1}^k \Phi_{\eta_j}$, and similar equalities for the synthesis and frame-like operators. The operator $\Theta_{\eta,\gamma}$ is often called a *rank-one operator*, and we have the following proposition, which is immediate by [93, Lemma 2.30, Proposition 3.8].

Proposition A.2.4. *Let η be an element of a full left Hilbert A-module $\mathcal{E}$. Then*

$$\|\eta\|_{\mathcal{E}} = \|\Theta_\eta\|_{\mathcal{L}_A(\mathcal{E})}.$$

More generally, if $\mathcal{E}$ is an imprimitivity A-B-bimodule, then

$$\| {}_\bullet\langle \xi, \eta \rangle \|_A = \| \langle \eta, \xi \rangle_\bullet \|_B$$

for every $\xi, \eta \in \mathcal{E}$. Hence, the norm of $\mathcal{E}$ as a left Hilbert A-module coincides with the norm of $\mathcal{E}$ as a right Hilbert B-module.

The frame property of a Bessel sequence $(\eta_j)_{j\in J}$ can be characterized in terms of the invertibility of the associated frame operator $\Theta_{(\eta_j)_j}$. For a proof, see [6, Theorem 1.2].

Proposition A.2.5. *Let $(\eta_j)_{j\in J}$ be a Bessel sequence in $\mathcal{E}$. Then the frame operator $\Theta_{(\eta_j)_j}$ associated to $(\eta_j)_j$ is invertible if and only if $(\eta_j)_j$ is a module frame for $\mathcal{E}$.*

The following proposition shows that finite module frames are nothing more than (algebraic) generating sets, and conversely.

Proposition A.2.6. *Let $\mathcal{E}$ be a left Hilbert A-module, and let $\eta_1, \ldots, \eta_k \in \mathcal{E}$. Then $(\eta_1, \ldots, \eta_k)$ is a module frame for $\mathcal{E}$ if and only if it is a generating set for $\mathcal{E}$, i.e. for every $\xi \in \mathcal{E}$ there exist coefficients $a_1, \ldots, a_k \in A$ such that*

$$\xi = \sum_{j=1}^{k} a_j \cdot \eta_j.$$

Proof. Let Θ be the module frame operator corresponding to $(\eta_j)_j$. If $(\eta_j)_j$ is a frame for $\mathcal{E}$, then by [6, Theorem 1.2] one has the expansion $\xi = \sum_{j=1}^{k} {}_{\bullet}\langle \xi, \Theta^{-1}\eta_j \rangle \cdot \eta_j$ for every $\xi \in \mathcal{E}$. This shows that $(\eta_j)_j$ is a generating set for $\mathcal{E}$.

We now prove the converse. Denote by $\Phi : \mathcal{E} \to A^k$ the map $\Phi\xi = ({}_{\bullet}\langle \xi, \eta_j \rangle)_{j=1}^{k}$. This is an adjointable A-module map, with $\Phi^*(a_j)_{j=1}^{k} = \sum_{j=1}^{k} a_j \eta_j$. By assumption Φ^* is a surjection. [74, Theorem 3.2] then gives that the image of Φ is a complementable submodule of A^k. The usual Hilbert space argument then gives that $\Phi^*\Phi : \mathcal{E} \to \mathcal{E}$ is invertible, and it follows from Proposition A.2.5 that $(\eta_1, \ldots, \eta_k)$ is a module frame for $\mathcal{E}$.

■

A.2.2 Gabor analysis on locally compact abelian groups

For the rest of the paper (unless stated otherwise), G will denote a second countable, locally compact abelian group with group operation written additively and with identity $0 \in G$, and Δ will denote a closed subgroup of the time-frequency plane $G \times \widehat{G}$. We fix a Haar measure on G and equip $\widehat{G}$ with the dual measure [46, Theorem 4.21]. Furthermore, we pick a Haar measure on Δ, and let $(G \times \widehat{G})/\Delta$ have the unique measure such that Weil's formula holds [65, equation (2.4)]. We can then associate to Δ the quantity $s(\Delta) = \mu((G \times \widehat{G})/\Delta)$, known as the *size of* Δ [65, p. 235]. Here μ denotes the chosen Haar measure. The size of Δ is finite precisely when $(G \times \widehat{G})/\Delta$ is compact, that is, Δ is cocompact in $G \times \widehat{G}$.

Given $x \in G$ and $\omega \in \widehat{G}$, we define the translation operator T_x and modulation operator M_ω on $L^2(G)$ by

$$(T_x\xi)(t) = \xi(t - x), \qquad (M_\omega\xi)(t) = \omega(t)\xi(t)$$

for $\xi \in L^2(G)$ and $t \in G$. The translation and modulation operators are unitary linear operators on $L^2(G)$. Moreover, a time-frequency shift is an operator of the form $\pi(x, \omega) = M_\omega T_x$ for $x \in G$ and $\omega \in \widehat{G}$.

The *adjoint subgroup* of Δ, denoted by Δ°, is the closed subgroup of $G \times \widehat{G}$ given by

$$\Delta^\circ = \{w \in G \times \widehat{G} : \pi(z)\pi(w) = \pi(w)\pi(z) \text{ for all } z \in \Delta\}.$$

We use the identification of Δ° with $((G \times \widehat{G})/\Delta)\widehat{\ }$ in [65, p. 234] to pick the dual measure on Δ° corresponding to the measure on $(G\times\widehat{G})/\Delta$ induced from the chosen measure on Δ. If Δ is cocompact in $G \times \widehat{G}$, then Δ° is discrete, and the induced measure on Δ° will be the counting measure scaled by the constant $s(\Delta)^{-1}$ [66, equation (13)].

We consider the two following important examples:

Example A.2.7. Suppose Δ is a lattice in $G \times \widehat{G}$, namely a discrete, cocompact subgroup of $G \times \widehat{G}$. Then Δ° is also a lattice in $G \times \widehat{G}$ [97, Lemma 3.1]. In this situation, we will usually equip Δ with the counting measure. The size of Δ is then the measure of any fundamental domain for Δ in $G \times \widehat{G}$ [65, Remark 1]. Since Δ in particular is cocompact, the measure on Δ° will not be the counting measure in general, but rather the counting measure scaled by $s(\Delta)^{-1}$.

Example A.2.8. Let $\Delta = G \times \widehat{G}$. Δ is then cocompact in $G \times \widehat{G}$, since $(G \times \widehat{G})/\Delta$ is trivial. The natural choice of measure on Δ in this situation is the product measure coming from the chosen measure on G and the dual measure on $\widehat{G}$. The induced measure on $\Delta^\circ = \{0\}$ is then the normalized measure assigning the value 1 to $\{0\}$.

A.2.3 Gabor frames.

We will need a continuous version of Gabor frames, and so we cannot treat our Gabor frames as a special case of Definition A.2.1. However, note the similarities between the definitions and results given here and in Section A.2.1.

Given $\eta \in L^2(G)$, the family $\mathcal{G}(\eta; \Delta) = (\pi(z)\eta)_{z\in\Delta}$ is called a *Gabor system* over Δ with generator η. More generally, given $\eta_1, \dots, \eta_k \in L^2(G)$, the family $\mathcal{G}(\eta_1, \dots, \eta_k; \Delta) = (\pi(z)\eta_j)_{z\in\Delta, 1\leq j\leq k}$ is called a *multi-window Gabor system* over Δ with generators $\eta_1, \dots, \eta_k$.

The multi-window Gabor system $\mathcal{G}(\eta_1, \dots, \eta_k; \Delta)$ is called a *multi-window Gabor frame* if it is a *(continuous) frame* [4, 65, 69] for $L^2(G)$ in the sense that both of the following hold:

1. The family $\mathcal{G}(\eta_1, \dots, \eta_k; \Delta)$ is weakly measurable, that is, for every $\xi \in L^2(G)$ and each $1 \leq j \leq k$, the map $z \mapsto \langle \xi, \pi(z)\eta_j \rangle$ is measurable.

2. There exist positive constants $C, D > 0$ such that for all $\xi \in L^2(G)$ we have that

$$C\|\xi\|_2^2 \leq \sum_{j=1}^{k} \int_\Delta |\langle \xi, \pi(z)\eta_j \rangle|^2 \,\mathrm{d}z \leq D\|\xi\|_2^2. \tag{A.2.5}$$

The numbers C and D are called lower and upper frame bounds respectively. We may also refer to the upper frame bound as a *Bessel bound* in analogy with Section A.2. If the family $\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta)$ is weakly measurable and has an upper frame bound but not necessarily a lower frame bound, we call it a *Bessel family*. A (single-window) Gabor system which is a frame is called a *Gabor frame*.

The *analysis operator* associated to a Bessel family $(\pi(z)\eta)_{z\in\Delta}$ is the bounded linear operator $C_\eta \colon L^2(G) \to L^2(\Delta)$ given by

$$C_\eta \xi = (\langle \xi, \pi(z)\eta \rangle)_{z\in\Delta} \tag{A.2.6}$$

for $\xi \in L^2(G)$. Its adjoint $D_\eta \colon L^2(\Delta) \to L^2(G)$ is called the *synthesis operator* and is given weakly by

$$D_\eta (c_z)_{z\in\Delta} = \int_\Delta c_z \pi(z)\eta \; \mathrm{d}z \tag{A.2.7}$$

for $(c_z)_{z\in\Delta} \in L^2(\Delta)$. The *frame-like operator* associated to two Bessel families $\mathcal{G}(\eta; \Delta)$ and $\mathcal{G}(\gamma; \Delta)$ is the operator $S_{\eta,\gamma} = D_\gamma C_\eta$ which is given weakly by

$$S_{\eta,\gamma}\xi = \int_\Delta \langle \xi, \pi(z)\eta \rangle \pi(z)\gamma \; \mathrm{d}z \tag{A.2.8}$$

for $\xi \in L^2(G)$. In particular, the *frame operator* associated to the Bessel family $\mathcal{G}(\eta; \Delta)$ is the operator $S_\eta := S_{\eta,\eta}$. This is a positive operator.

If $\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta)$ is a multi-window Gabor Bessel family, then its analysis, synthesis and frame operators are given respectively by $C = \sum_{j=1}^k C_{\eta_j}$, $D = \sum_{j=1}^k D_{\eta_j}$ and $S = \sum_{j=1}^k S_{\eta_j}$.

Note how the following proposition is analogous to Proposition A.2.5. The result is well-known in frame theory.

Proposition A.2.9. *Let $\eta_1, \ldots, \eta_k \in L^2(G)$ be such that $\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta)$ is a Bessel family for $L^2(G)$. Then $\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta)$ is a multi-window Gabor frame if and only if the associated frame operator $S = \sum_{j=1}^k S_{\eta_j}$ is invertible on $L^2(G)$.*

The *Feichtinger algebra* $S_0(G)$ is the set of $\xi \in L^2(G)$ for which

$$\int_{G\times\widehat{G}} |\langle \xi, \pi(z)\xi \rangle| \; \mathrm{d}z < \infty. \tag{A.2.9}$$

See [64] for a comprehensive introduction to $S_0(G)$. For us, the Feichtinger algebra will play a key role in the construction of Heisenberg modules as in [82], see Proposition A.2.12. Note that in the original paper [97], the Schwartz-Bruhat space $\mathcal{S}(G)$ was used instead. The Schwartz-Bruhat space has a more technical definition. Although it will not be important to us, we mention that the Feichtinger algebra has a natural Banach space structure [39, Theorem 1].

Proposition A.2.10. *The following properties hold for the Feichtinger algebra:*

1. *If $\eta \in S_0(G)$, then $\mathcal{G}(\eta;\Delta)$ is a Bessel family for $L^2(G)$.*

2. *If G is discrete, then $S_0(G) = \ell^1(G)$.*

For a proof of these results, see [65, Corollary A.5] and [64, Lemma 4.11].

A.2.4 Twisted group C^*-algebras and Heisenberg modules

For the moment, let Δ be a general second countable, locally compact abelian group. A *(normalized) continuous 2-cocycle* on Δ is a continuous map $c\colon \Delta \times \Delta \to \mathbb{T}$ that satisfies the following two identities:

1. For every $z_1, z_2, z_3 \in \Delta$ we have that

$$c(z_1, z_2)c(z_1 + z_2, z_3) = c(z_1, z_2 + z_3)c(z_2, z_3). \tag{A.2.10}$$

2. If 0 denotes the identity element of Δ, then

$$c(0, 0) = 1. \tag{A.2.11}$$

Note that if c is a continuous 2-cocycle, then its pointwise complex conjugate $\overline{c}$ is a continuous 2-cocycle as well.

Given a continuous 2-cocycle c on Δ, one can equip the Feichtinger algebra $S_0(\Delta)$ with a multiplication and involution as follows: For $a, b \in S_0(\Delta)$ and $z \in \Delta$, one defines

$$a * b(z) = \int_\Delta c(w, z - w)a(w)b(z - w) \,\mathrm{d}w \tag{A.2.12}$$

$$a^*(z) = \overline{c(z, -z)a(-z)}. \tag{A.2.13}$$

The C^*-enveloping algebra of $S_0(\Delta, c)$ is called the *c-twisted group C^*-algebra of* Δ and is denoted by $C^*(\Delta, c)$. Note that this definition is equivalent to the usual definition of $C^*(\Delta, c)$ as the C^*-enveloping algebra of $L^1(\Delta, c)$, as $S_0(\Delta, c)$ is dense in $L^1(\Delta, c)$ and the L^1-norm dominates the universal C^*-norm on $L^1(\Delta, c)$.

Let H be a Hilbert space, and denote by $\mathcal{U}(H)$ the unitary operators on H. A map $\pi\colon \Delta \to \mathcal{U}(H)$ is called a *c-projective unitary representation of* Δ *on* H if the following two properties hold:

1. π is strongly continuous, i.e. for every $\xi \in H$, the map $\Delta \to H$, $z \mapsto \pi(z)\xi$ is continuous.

2. For every $z, w \in \Delta$, we have that

$$\pi(z)\pi(w) = c(z, w)\pi(z + w). \tag{A.2.14}$$

The twisted group C^*-algebra $C^*(\Delta, c)$ captures the c-projective unitary representation theory of Δ in the following sense: For every c-projective unitary representation $\pi\colon \Delta \to \mathcal{U}(H)$ on a Hilbert space H, there is a nondegenerate $*$-representation $\overline{\pi}\colon C^*(\Delta, c) \to \mathcal{L}(H)$ which for $a \in L^1(\Delta, c)$ is given weakly by

$$\overline{\pi}(a) = \int_{\Delta} a(z)\pi(z)\,\mathrm{d}z. \tag{A.2.15}$$

The above representation is called the *integrated* representation of π. Conversely, if $\Pi : C^*(\Delta, c) \to \mathcal{L}(H)$ is any nondegenerate $*$-representation of $C^*(\Delta, c)$ on H, then there exists a unique c-projective unitary representation $\pi\colon \Delta \to \mathcal{U}(H)$ such that $\overline{\pi} = \Pi$. This correspondence can be seen as a consequence of e.g. [88, Proposition 2.7].

Note also that if π is a c-projective unitary representation, then π^* defined by $\pi^*(z) = \pi(z)^*$ is $\overline{c}$-projective. This follows from taking the adjoint of both sides of (A.2.14) (it is essential that we are working with abelian groups in this situation).

When Δ is discrete, we have by Proposition A.2.10 2 that $S_0(\Delta, c) \cong \ell^1(\Delta, c)$. If we equip Δ with the counting measure, there is a canonical tracial state on $C^*(\Delta, c)$ [19, p. 951]. On the dense $*$-subalgebra $\ell^1(\Delta, c)$, it is given by

$$\mathrm{tr}(a) = a(0) \tag{A.2.16}$$

for $a \in \ell^1(\Delta, c)$.

We now return to the situation where G is a second countable, locally compact abelian group, and Δ is a closed subgroup of $G \times \widehat{G}$. The map $c\colon \Delta \times \Delta \to \mathbb{T}$ given by

$$c((x, \omega), (y, \tau)) = \overline{\tau(x)} \tag{A.2.17}$$

for $(x, \omega), (y, \tau) \in \Delta$ is a continuous 2-cocycle on Δ called the *Heisenberg 2-cocycle* [97, p. 263]. Moreover, the time-frequency shifts $\pi(x, \omega) = M_\omega T_x$ define a c-projective unitary representation of $G \times \widehat{G}$ on $L^2(G)$, and so we have that

$$\pi(x, \omega)\pi(y, \tau) = \overline{\tau(x)}\pi(x + y, \omega\tau).$$

This representation is often called the *Heisenberg representation*. Restricting to the closed subgroup Δ of $G \times \widehat{G}$, we obtain a c-projective unitary representation of Δ on $L^2(G)$. We denote the restriction by π_Δ. This representation then induces a $*$-representation of $C^*(\Delta, c)$ on $L^2(G)$, which we also (by slight abuse of notation) denote by π_Δ. We have the following result, see [97, Proposition 2.2].

Proposition A.2.11. *The integrated representation* $\pi_\Delta \colon C^*(\Delta, c) \to \mathcal{L}(L^2(G))$ *is faithful, i.e.* $\pi_\Delta(a) = 0$ *implies* $a = 0$ *for all* $a \in C^*(\Delta, c)$.

In the following proposition, we give the definition of Heisenberg modules. For a proof, see the proof of [66, Theorem 3.4] or Rieffel's arguments from [97] which are similar.

Proposition A.2.12. *Let* G *be a locally compact abelian group, and let* Δ *be a closed subgroup of* $G \times \widehat{G}$*, both with chosen Haar measures. Equip* Δ° *with the Haar measure determined as in Section A.2.2. The* Heisenberg module $\mathcal{E}_\Delta(G)$ *is an imprimitivity* $C^*(\Delta, c)$*-*$C^*(\Delta^\circ, \overline{c})$*-module obtained as a completion of the Feichtinger algebra* $S_0(G)$*. The actions and inner products are given densely as follows:*

1. *If* $a \in S_0(\Delta, c)$*,* $b \in S_0(\Delta^\circ, \overline{c})$ *and* $\xi \in S_0(G)$*, then* $a \cdot \xi, \xi \cdot b \in S_0(G)$*, with*

$$a \cdot \xi = \int_\Delta a(z)\pi(z)\xi \,\mathrm{d}z, \qquad \xi \cdot b = \int_{\Delta^\circ} b(w)\pi(w)^*\xi \,\mathrm{d}w. \tag{A.2.18}$$

2. *If* $\xi, \eta \in S_0(G)$*, then* ${}_\bullet\langle \xi, \eta \rangle \in S_0(\Delta, c)$ *and* $\langle \xi, \eta \rangle_\bullet \in S_0(\Delta^\circ, \overline{c})$*, with*

$${}_\bullet\langle \xi, \eta \rangle (z) = \langle \xi, \pi(z)\eta \rangle, \qquad \langle \xi, \eta \rangle_\bullet (w) = \langle \pi(w)\eta, \xi \rangle \tag{A.2.19}$$

for $z \in \Delta$ *and* $w \in \Delta^\circ$.

We can rewrite the left and right actions of Proposition A.2.12 as follows: Since $\pi \colon G \times \widehat{G} \to \mathcal{L}(L^2(G))$ is a c-projective unitary representation, it follows that π^* is $\overline{c}$-projective. We restrict π and π^* to Δ and Δ° respectively and obtain the representations π_Δ and $\pi^*_{\Delta^\circ}$. Passing to the integrated representations, we obtain $*$-representations of $C^*(\Delta, c)$ and $C^*(\Delta^\circ, \overline{c})$ which we also denote by π_Δ and $\pi^*_{\Delta^\circ}$ respectively. We can then write the left and right module actions given in (A.2.18) as

$$a \cdot \xi = \pi_\Delta(a)\xi, \qquad \xi \cdot b = \pi^*_{\Delta^\circ}(b)\xi \tag{A.2.20}$$

for $\xi \in S_0(G)$, $a \in S_0(\Delta, c)$ and $b \in S_0(\Delta^\circ, \overline{c})$.

A.3 Results

A.3.1 Localization of Hilbert C^*-modules.

We will use localization of Hilbert C^*-modules with respect to positive linear functionals as defined in [74, p. 7]. Localization is a technique reminiscent of the

GNS construction. It uses a positive linear functional on the coefficient algebra of a Hilbert C^*-module to embed the module continuously into a Hilbert space. The authors are not aware of many uses of localization in the literature, but an example is found in [68]. We will focus exclusively on the case of faithful traces, but we will need a version for (possibly) unbounded traces, which we develop after reviewing the case of finite faithful traces.

Let $\mathrm{tr} : A \to \mathbb{C}$ denote a finite trace on A, i.e. a positive linear functional on A that satisfies $\mathrm{tr}(a^*a) = \mathrm{tr}(aa^*)$ for all $a \in A$. Assume also that tr is faithful, that is, $\mathrm{tr}(a^*a) = 0$ implies $a = 0$ for all $a \in A$. If $\mathcal{E}$ is a left Hilbert A-module, it is easily verified that

$$\langle \xi, \eta \rangle_{\mathrm{tr}} = \mathrm{tr}({}_{\bullet}\langle \xi, \eta \rangle) \tag{A.3.1}$$

for $\xi, \eta \in \mathcal{E}$ defines a ($\mathbb{C}$-valued) inner product on $\mathcal{E}$, and we denote the Hilbert space completion of $\mathcal{E}$ in the norm $\| \cdot \|_{H_{\mathcal{E}}}$ coming from $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ by $H_{\mathcal{E}}$. For $\xi \in \mathcal{E}$, the chain of inequalities

$$\|\xi\|_{H_{\mathcal{E}}}^2 = \mathrm{tr}({}_{\bullet}\langle \xi, \xi \rangle) \leq \| \mathrm{tr} \| \| {}_{\bullet}\langle \xi, \xi \rangle \|_A = \| \mathrm{tr} \| \|\xi\|_{\mathcal{E}}^2$$

shows that the embedding $\mathcal{E} \hookrightarrow H_{\mathcal{E}}$ is continuous. Moreover, if tr is a state, that is, $\| \mathrm{tr} \| = 1$, then the embedding is norm-decreasing. The Hilbert space $H_{\mathcal{E}}$ is called the *localization* of $\mathcal{E}$ with respect to tr.

If $\mathcal{E}$ and $\mathcal{F}$ are left Hilbert A-modules, we obtain localizations $H_{\mathcal{E}}$ and $H_{\mathcal{F}}$ with respect to tr. Let $T \colon \mathcal{E} \to \mathcal{F}$ be an adjointable linear operator. Then in particular, T is a bounded linear operator when viewing the Hilbert C^*-modules as Banach spaces, and we denote its norm by $\|T\|$. For all $\xi \in \mathcal{E}$ we have that ${}_{\bullet}\langle T\xi, T\xi \rangle \leq \|T\|^2 {}_{\bullet}\langle \xi, \xi \rangle$ [93, Corollary 2.22]. Applying tr on both sides, we obtain

$$\|T\xi\|_{H_{\mathcal{F}}}^2 \leq \|T\|^2 \|\xi\|_{H_{\mathcal{E}}}^2, \tag{A.3.2}$$

which shows that T extends to a bounded linear operator of Hilbert spaces $\overline{T} \colon H_{\mathcal{E}} \to H_{\mathcal{F}}$. If $\|\overline{T}\|_h$ denotes the norm of $\overline{T}$ as a Hilbert space operator, then (A.3.2) also shows that $\|\overline{T}\|_h \leq \|T\|$. Hence we have a norm-decreasing (hence continuous) inclusion of Banach spaces $\mathcal{L}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{L}(H_{\mathcal{E}}, H_{\mathcal{F}})$. If $\mathcal{E} = \mathcal{F}$, then more is true: We obtain an injective $*$-homomorphism of C^*-algebras [74, p. 58] $\mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{L}(H_{\mathcal{E}})$. Since injective $*$-homomorphisms of C^*-algebras are necessarily isometries [86, Theorem 3.1.5], we deduce that $\mathcal{L}(\mathcal{E}) \to \mathcal{L}(H_{\mathcal{E}})$ is an isometry. Hence in this case we have

$$\|\overline{T}\|_h = \|T\| \tag{A.3.3}$$

for all $T \in \mathcal{L}(\mathcal{E})$.

We can define the localization of a *right* Hilbert A-module $\mathcal{E}$ at a faithful trace tr similarly, except in this situation we have to set the inner product to be

$\langle \xi, \eta \rangle_{\mathrm{tr}} = \mathrm{tr}(\langle \eta, \xi \rangle_{\bullet})$ for $\xi, \eta \in \mathcal{E}$ to get linearity in the first argument instead of the second. Just as with left modules, we obtain a Hilbert space $H_{\mathcal{E}}$ together with an injective bounded linear map $\mathcal{E} \hookrightarrow H_{\mathcal{E}}$.

In the following, we develop a version of localization with respect to a possibly unbounded trace that works for our purposes. Denote by A_+ the positive elements of the C^*-algebra A. By a *weight* on A, we will mean a function $\phi : A_+ \longrightarrow [0, \infty]$ that satisfies $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in A_+$, $\phi(\lambda a) = \lambda\phi(a)$ for all $a \in A_+$ and $\lambda > 0$, and $\phi(0) = 0$. The weight ϕ is *lower semi-continuous* if whenever $(a_\alpha)_\alpha$ is a net in A_+ converging to a, then $\phi(a) \leq \liminf_\alpha \phi(a_\alpha)$. A weight ϕ on A is a *trace* if $\phi(a^*a) = \phi(aa^*)$ for all $a \in A$, and is *faithful* if $\phi(a) = 0$ implies $a = 0$ for every $a \in A_+$.

For a weight ϕ on A, let $A_+^\phi = \{a \in A_+ : \phi(a) < \infty\}$. The weight ϕ is called *densely defined* if A_+^ϕ is dense in A_+ (in the norm topology). Moreover, let $A^\phi = \operatorname{span} A_+^\phi$. By [91, Lemma 5.1.2], ϕ has a unique extension to a positive linear functional on A^ϕ, and ϕ is densely defined if and only if A^ϕ is dense in A. A weight ϕ on A is called *finite* if $A_+^\phi = A_+$. In that case, ϕ extends uniquely to a positive linear functional on $A^\phi = \operatorname{span} A_+^\phi = \operatorname{span} A_+ = A$, and so we obtain a positive linear functional on the whole of A. Conversely, any positive linear functional on A restricts to a finite weight on A_+. If A is a unital C^*-algebra, then ϕ is finite if and only if $1 \in A_+^\phi$ if and only if ϕ is densely defined.

Now let $\mathcal{E}$ be a left Hilbert A-module, and tr a (possibly unbounded) trace on A. There are two problems with localizing $\mathcal{E}$ with respect to A: The first one is that $\mathrm{tr}({}_{\bullet}\langle \xi, \eta \rangle)$ might not be finite for $\xi, \eta \in \mathcal{E}$, which means that we do not get a well-defined inner product by setting $\langle \xi, \eta \rangle = \mathrm{tr}({}_{\bullet}\langle \xi, \eta \rangle)$. The other problem is that we might not get a continuous embedding $\mathcal{E} \to H_{\mathcal{E}}$ even if the inner product is well-defined. However, the following set-up is sufficient for our purposes, and solves the aforementioned problems. The essential ingredient in the proof is a result due to Combes and Zettl [27].

Proposition A.3.1. *Let A and B be C^*-algebras, and suppose tr_B is a faithful finite trace on B. Then the following hold:*

1. *If $\mathcal{E}$ is an imprimitivity A-B-bimodule, then there exists a unique lower semi-continuous trace tr_A such that*

$$\mathrm{tr}_A({}_{\bullet}\langle \xi, \xi \rangle) = \mathrm{tr}_B(\langle \xi, \xi \rangle_{\bullet}) \tag{A.3.4}$$

for all $\xi \in \mathcal{E}$. Moreover, tr_A is faithful and densely defined, with $\operatorname{span}\{{}_{\bullet}\langle \xi, \eta \rangle : \xi, \eta \in \mathcal{E}\} \subseteq A^{\mathrm{tr}_A}$, and setting

$$\langle \xi, \eta \rangle_{\mathrm{tr}_A} = \mathrm{tr}_A({}_{\bullet}\langle \xi, \eta \rangle) \tag{A.3.5}$$

for $\xi, \eta \in \mathcal{E}$ defines an inner product on $\mathcal{E}$, with $\langle \xi, \eta \rangle_{\mathrm{tr}_A} = \langle \xi, \eta \rangle_{\mathrm{tr}_B}$ for all $\xi, \eta \in \mathcal{E}$. Consequently, the Hilbert space obtained by completing $\mathcal{E}$ in the norm $\|\xi\|' = \mathrm{tr}_A({}_\bullet\langle \xi, \xi \rangle)^{1/2}$ is just the localization of $\mathcal{E}$ with respect to tr_B.

2. *If $\mathcal{E}$ and $\mathcal{F}$ are imprimitivity A-B-bimodules, then every adjointable A-linear operator $\mathcal{E} \to \mathcal{F}$ has a unique extension to a bounded linear operator $H_{\mathcal{E}} \to H_{\mathcal{F}}$. Furthermore, the map $\mathcal{L}_A(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(H_{\mathcal{E}}, H_{\mathcal{F}})$ given by sending T to its unique extension is a norm-decreasing linear map of Banach spaces. Finally, if $\mathcal{E} = \mathcal{F}$, the map $\mathcal{L}_A(\mathcal{E}) \to \mathcal{L}(H_{\mathcal{E}})$ is an isometric $*$-homomorphism of C^*-algebras.*

Proof. Suppose $\mathcal{E}$ is an imprimitivity A-B-bimodule. By [27, Proposition 2.2], there is a unique lower semi-continuous trace tr_A on A such that the relation in equation (A.3.4) holds for all $\xi \in \mathcal{E}$. Since tr_B is finite, it is densely defined, and so tr_A is densely defined by the same proposition. The same goes for faithfulness. Since $\mathrm{tr}_A({}_\bullet\langle \xi, \xi \rangle) = \mathrm{tr}_B(\langle \xi, \xi \rangle_\bullet) < \infty$, we have that $\operatorname{span}\{{}_\bullet\langle \xi, \xi \rangle : \xi \in \mathcal{E}\} \subseteq \operatorname{span} A_+^{\mathrm{tr}_A} = A^{\mathrm{tr}_A}$. By the polarization identity for Hilbert C^*-modules, elements of the form ${}_\bullet\langle \xi, \eta \rangle$ are in $\operatorname{span}\{{}_\bullet\langle \xi, \xi \rangle : \xi \in \mathcal{E}\}$, and so the unique extension of tr_A to a positive linear functional on A^{tr_A} is defined on all elements of the form ${}_\bullet\langle \xi, \eta \rangle$ with $\xi, \eta \in \mathcal{E}$. Thus, in this situation the inner product proposed in (A.3.5) is well-defined. Again by the polarization identity, the relation in (A.3.4) implies that $\mathrm{tr}_A({}_\bullet\langle \xi, \eta \rangle) = \mathrm{tr}_B(\langle \eta, \xi \rangle_\bullet)$ for all $\xi, \eta \in \mathcal{E}$, and so $\langle \xi, \eta \rangle_{\mathrm{tr}_A} = \langle \xi, \eta \rangle_{\mathrm{tr}_B}$.

If $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, then we have that ${}_\bullet\langle T\xi, T\xi \rangle \leq \|T\| {}_\bullet\langle \xi, \xi \rangle$ for every $\xi \in \mathcal{E}$. Taking the trace tr_A, we obtain that $\|T\xi\|_{H_{\mathcal{E}}} \leq \|T\| \|\xi\|_{H_{\mathcal{F}}}$, just as in the discussion of localization with respect to finite traces. This shows that T extends to a bounded linear map $H_{\mathcal{E}} \to H_{\mathcal{F}}$, and that the inclusion $\mathcal{L}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(H_{\mathcal{E}}, H_{\mathcal{F}})$ is norm-decreasing. In particular, if $\mathcal{E} = \mathcal{F}$, it becomes an isometric $*$-homomorphism of C^*-algebras. ∎

We will refer to the localization of $\mathcal{E}$ with respect to tr_B in Proposition A.3.1 above also as the localization of $\mathcal{E}$ with respect to tr_A.

Remark A.3.2. If both A and B are unital in Proposition A.3.1, then tr_A, being a densely defined trace on a unital C^*-algebra, has to be finite. In that case, we can localize $\mathcal{E}$ as a left A-module with respect to tr_A in the usual fashion, and then Proposition A.3.1 tells us that the localization is exactly the same as when done with respect to tr_B.

A.3.2 Localization of the twisted group C^*-algebra

The following proposition shows that for a discrete group Δ with a 2-cocycle c, the localization of $C^*(\Delta, c)$ as a left Hilbert module over itself with respect to the canonical trace can be identified in a natural way with $\ell^2(\Delta)$.

Proposition A.3.3. *Let Δ be a discrete group equipped with the counting measure and a 2-cocycle c. Denote by H the localization of $C^*(\Delta, c)$ as a left module over itself with respect to its canonical faithful tracial state. Then H can be identified with $\ell^2(\Delta)$ in such a way that the following diagram of inclusions commutes:*

$$\begin{tikzcd}
\ell^1(\Delta) \arrow[r, hook] \arrow[d, hook] & C^*(\Delta, c) \arrow[d, hook] \\
\ell^2(\Delta) \arrow[r, "\cong"] & H
\end{tikzcd}$$

Moreover, the inclusion map $C^*(\Delta, c) \to \ell^2(\Delta)$ is norm-decreasing, that is, for all $a \in C^*(\Delta, c)$ we have that

$$\|a\|_{\ell^2(\Delta)} \le \|a\|_{C^*(\Delta,c)}.$$

Proof. We have that $C^*(\Delta, c)$ is dense in H in the Hilbert space norm on H, and that $\ell^1(\Delta)$ is dense in $C^*(\Delta, c)$ in the C^*-norm on $C^*(\Delta, c)$. Since the C^*-norm on $C^*(\Delta, c)$ dominates the Hilbert space norm of H, we get that $\ell^1(\Delta)$ is also dense in H in the Hilbert space norm. Moreover $\ell^1(\Delta)$ is also dense in $\ell^2(\Delta)$ in the ℓ^2-norm.

Denote by $\langle\cdot,\cdot\rangle$ the inner product on $\ell^2(\Delta)$. The $C^*(\Delta, c)$-valued inner product on $C^*(\Delta, c)$ as a left Hilbert C^*-module over itself is given by ${}_\bullet\langle a, b\rangle = ab^*$ for $a, b \in C^*(\Delta, c)$, and so the inner product with respect to tr is given by $\langle a, b\rangle_{\mathrm{tr}} = \mathrm{tr}(ab^*)$. If $a, b \in \ell^1(\Delta, c)$, then

$$\begin{aligned}
\langle a, b\rangle_{\mathrm{tr}} = \mathrm{tr}(ab^*) = (ab^*)(0) &= \sum_{z\in\Delta} c(w, 0-w)a(w)b^*(0-w) \\
&= \sum_{z\in\Delta} c(w,-w)a(w)\overline{c(-w,w)b(w)} = \sum_{z\in\Delta} a(w)\overline{b(w)} = \langle a, b\rangle.
\end{aligned}$$

This shows that $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ and $\langle\cdot,\cdot\rangle$ agree on the subspace $\ell^1(\Delta, c)$ which is dense in both of the Hilbert spaces as argued. It follows that H can be identified with $\ell^2(\Delta)$ in such a way that the inclusions of $\ell^1(\Delta)$ into $\ell^2(\Delta)$ and $C^*(\Delta, c)$ are preserved. Moreover, since tr is a state, we have that the inclusion $C^*(\Delta, c) \hookrightarrow \ell^2(G)$ is norm-decreasing. ∎

Remark A.3.4. In the sequel the following situation will be relevant: Let Δ be a discrete group, and denote by μ the counting measure on Δ. Let $k > 0$ be a constant. Then we can consider the C^*-algebra $C^*(\Delta, c)$ defined with respect to the measure $k\mu$ rather than μ, and so all sums involved in formulas for convolutions and norms will have a factor of k in front. In this situation there is still a faithful trace tr on $C^*(\Delta, c)$ given by $\mathrm{tr}(a) = a(0)$ for $a \in \ell^1(\Delta, c)$. However, note that this is not a state when $k \neq 1$. Indeed, the multiplicative identity of $C^*(\Delta, c)$ is $k^{-1}\delta_0$ rather than δ_0, and so

$$\mathrm{tr}(1) = \mathrm{tr}(k^{-1}\delta_0) = k^{-1}\delta_0(0) = k^{-1}.$$

If we rescale tr by k, we obtain a state.

A.3.3 Localization of the Heisenberg module

We will need a trace on the left C^*-algebra $A = C^*(\Delta, c)$ of the Heisenberg module in Proposition A.2.12. When Δ is a lattice in $G\times\widehat{G}$, we will just consider the canonical faithful trace tr_A on $C^*(\Delta, c)$. Note that by Proposition A.3.1 and Remark A.3.2, there exists a finite faithful trace on the right C^*-algebra $B = C^*(\Delta^\circ, \overline{c})$ such that $\mathrm{tr}_A({}_\bullet\langle \xi, \eta\rangle) = \mathrm{tr}_B(\langle \eta, \xi\rangle_\bullet)$ for all $\xi, \eta \in \mathcal{E}_\Delta(G)$. If $\xi, \eta \in S_0(G)$, then

$$\langle \xi, \eta\rangle = \langle \xi, \pi(0)\eta\rangle = {}_\bullet\langle \xi, \eta\rangle\,(0) = \mathrm{tr}_A({}_\bullet\langle \xi, \eta\rangle) = \mathrm{tr}_B(\langle \eta, \xi\rangle_\bullet).$$

But there is a canonical trace tr'_B on B such that $\mathrm{tr}'_B(b) = b(0)$ whenever $b \in \ell^1(\Delta^\circ, \overline{c})$. Since $\mathrm{tr}'_B(\langle \eta, \xi\rangle_\bullet) = \langle \xi, \eta\rangle_\bullet(0) = \langle \pi(0)\xi, \eta\rangle = \langle \xi, \eta\rangle$, this shows that tr_B and tr'_B agree on $\mathrm{span}\{\langle \xi, \eta\rangle_\bullet : \xi, \eta \in S_0(G)\}$. Since the latter is dense in B, we conclude that $\mathrm{tr}_B = \mathrm{tr}'_B$. Note however by Remark A.3.4 that the faithful trace tr_B which satisfies (A.3.5) is not a state unless $s(\Delta) = 1$.

In the case when Δ is only cocompact and not necessarily discrete, Δ° is discrete, and we obtain a (possibly unbounded) trace on $C^*(\Delta, c)$ by the following proposition. Note that we use the measures as chosen in the beginning of this section, and that B is equipped with the canonical trace that is not a state in general.

Proposition A.3.5. *Let G be a second countable, locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. Let $A = C^*(\Delta, c)$ and $B = C^*(\Delta^\circ, \overline{c})$. Denote by tr_B the canonical faithful trace on B as in Remark A.3.4. Then the induced trace tr_A on A via the Heisenberg module $\mathcal{E}_\Delta(G)$ as in Proposition A.3.1 is given by*

$$\mathrm{tr}_A({}_\bullet\langle \xi, \eta\rangle) = \langle \xi, \eta\rangle$$

for $\xi, \eta \in S_0(G)$. In particular, if Δ is a lattice in $G \times \widehat{G}$, then tr_A is the canonical faithful tracial state on $C^*(\Delta, c)$.

Proof. By Proposition A.3.1, the induced trace tr_A satisfies

$$\mathrm{tr}_A({}_\bullet\langle \xi, \eta\rangle) = \mathrm{tr}_B(\langle \eta, \xi\rangle_\bullet)$$

for all $\xi, \eta \in \mathcal{E}_\Delta(G)$. If $\xi, \eta \in S_0(G)$, then $\langle \eta, \xi\rangle_\bullet \in S_0(\Delta^\circ, \overline{c}) = \ell^1(\Delta^\circ, \overline{c})$ by Proposition A.2.12 and Proposition A.2.10 2, and so

$$\mathrm{tr}_B(\langle \eta, \xi\rangle_\bullet) = \langle \eta, \xi\rangle_\bullet(0) = \langle \pi(0)\xi, \eta\rangle = \langle \xi, \eta\rangle.$$

If Δ is a lattice, then A is the twisted group C^*-algebra of a discrete group, and in this case we know that the canonical faithful tracial state tr on $C^*(\Delta, c)$ is given by $\mathrm{tr}(a) = a(0)$ for $a \in \ell^1(\Delta, c) = S_0(\Delta, c)$. In particular, $\mathrm{tr}({}_\bullet\langle \xi, \eta \rangle) = \langle \xi, \eta \rangle$. By fullness of $\mathcal{E}$ as a left Hilbert A-module, it follows that tr and tr_A agree on a dense subspace of A, hence on all of A. This shows that tr_A is indeed the faithful canonical tracial state on A. ∎

Based on the above proposition, we make the following convention for the rest of the paper:

Convention A.3.6. We fix a second countable, locally compact abelian group G, and a closed, cocompact subgroup Δ of $G \times \widehat{G}$. We fix Haar measures on G and Δ. If Δ is a lattice in $G \times \widehat{G}$, we assume the counting measure on Δ. From these measures, we obtain measures on $\widehat{G}$, $G \times \widehat{G}$ and Δ° as in Section A.2.2. Note that the measure on Δ° will be the counting measure scaled by a factor of $s(\Delta)^{-1}$. Let $A = C^*(\Delta, c)$ and $B = C^*(\Delta^\circ, \overline{c})$, so that the Heisenberg module $\mathcal{E}_\Delta(G)$ is an imprimitivity A-B-bimodule. We assume the canonical faithful trace tr_B on B given by $\mathrm{tr}_B(b) = b(0)$ for $b \in \ell^1(\Delta^\circ, c)$. We equip A with the possibly unbounded trace tr_A induced from tr_B as in Proposition A.3.5. In particular, if Δ is a lattice, then tr_A is the canonical faithful tracial state on A.

In the following proposition, we compute the localization of the Heisenberg module associated to a cocompact subgroup $\Delta \subseteq G \times \widehat{G}$.

Proposition A.3.7. *Let G denote a second countable locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. Then the localization H of the Heisenberg module $\mathcal{E}_\Delta(G)$ with respect to either of the traces on $C^*(\Delta, c)$ and $C^*(\Delta^\circ, \overline{c})$ can be identified with $L^2(G)$ in such a way that the diagram of inclusions commutes:*

$$\begin{tikzcd} S_0(G) \arrow[r, hook] \arrow[d, hook] & \mathcal{E}_\Delta(G) \arrow[d, hook] \\ L^2(G) \arrow[r, "\cong"] & H \end{tikzcd}$$

Thus, the Heisenberg module can be continuously embedded into $L^2(G)$, with

$$\|\eta\|_2 \leq s(\Delta)^{1/2} \|\eta\|_{\mathcal{E}_\Delta(G)} \tag{A.3.6}$$

for all $\eta \in \mathcal{E}_\Delta(G)$. In particular, if $(\eta_n)_n$ is a sequence in $\mathcal{E}_\Delta(G)$ that converges to an element $\eta \in \mathcal{E}_\Delta(G)$ in the $\mathcal{E}_\Delta(G)$-norm, then $(\eta_n)_n$ also converges to η in the $L^2(G)$-norm.

Proof. Let $\xi, \eta \in S_0(G)$. Then ${}_\bullet\langle \xi, \eta \rangle \in S_0(\Delta, c)$ by Proposition A.2.12, and so by (A.2.19) and Proposition A.3.5 we obtain

$$\langle \xi, \eta \rangle_{\mathrm{tr}_A} = \mathrm{tr}_A({}_\bullet\langle \xi, \eta \rangle) = {}_\bullet\langle \xi, \eta \rangle (0) = \langle \xi, \pi(0)\eta \rangle = \langle \xi, \eta \rangle.$$

This shows that $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ and $\langle \cdot, \cdot \rangle$ agree on the dense subspace $S_0(G)$ of H. Hence, the localization H can be identified with $L^2(G)$ in such a way that the above diagram commutes. Moreover, since $\| \mathrm{tr}_B \| = \mathrm{tr}_B(1_B) = s(\Delta)$, see A.3.4, we have

$$\|\eta\|_2^2 = \langle \eta, \eta \rangle = \mathrm{tr}_B(\langle \eta, \eta \rangle_\bullet) \leq \| \mathrm{tr}_B \| \| \langle \eta, \eta \rangle_\bullet \| = s(\Delta) \|\eta\|_{\mathcal{E}}^2.$$

This implies (A.3.6). ∎

Proposition A.3.7 embeds the Heisenberg module as a dense subspace of $L^2(G)$, and allows us to think of $\mathcal{E}_\Delta(G)$ as a function space.

A.3.4 Applications to Gabor analysis

In light of Proposition A.3.7 and Proposition A.3.1, it follows that every adjointable $C^*(\Delta, c)$-module operator $\mathcal{E}_\Delta(G) \to \mathcal{E}_\Delta(G)$ has a unique extension to a bounded linear map $L^2(G) \to L^2(G)$. The following lemma states that when $\eta, \gamma \in S_0(G)$, the extension of the adjointable operator $\Theta_{\eta,\gamma}$ on $\mathcal{E}_\Delta(G)$ to a bounded linear operator on $L^2(G)$ is equal to $S_{\eta,\gamma}$. This will be generalized to functions $\eta, \gamma \in \mathcal{E}_\Delta(G)$ in Theorem A.3.15. The lemma was observed in [82] in the case of $G = \mathbb{R}^d$, but without using the language of localization. It was also covered in greater generality in [66, Theorem 3.14].

Lemma A.3.8. *Let $\eta, \gamma \in S_0(G)$. The module frame-like operator $\Theta_{\eta,\gamma} \colon \mathcal{E}_\Delta(G) \to \mathcal{E}_\Delta(G)$ then extends to the Gabor frame-like operator $S_{\eta,\gamma} \colon L^2(G) \to L^2(G)$.*

Proof. Suppose $\eta, \gamma \in S_0(G)$. To begin with, let $\xi \in S_0(G)$. Then by Proposition A.2.12, ${}_\bullet\langle \xi, \eta \rangle \in S_0(\Delta, c)$, and consequently ${}_\bullet\langle \xi, \eta \rangle \cdot \gamma \in S_0(G)$. Moreover, equations (A.2.19) and (A.2.18) give that

$$\Theta_{\eta,\gamma}\xi = {}_\bullet\langle \xi, \eta \rangle \gamma = \int_\Delta {}_\bullet\langle \xi, \eta \rangle (z) \pi(z)\gamma \, \mathrm{d}z = \int_\Delta \langle \xi, \pi(z)\eta \rangle \pi(z)\gamma \, \mathrm{d}z = S_{\eta,\gamma}\xi.$$

Now let $\xi \in \mathcal{E}_\Delta(G)$, and suppose $(\xi_n)_n$ is a sequence in $S_0(G)$ that converges to ξ in the $\mathcal{E}_\Delta(G)$-norm. Then by continuity, $\Theta_{\eta,\gamma}\xi = \lim_n \Theta_{\eta,\gamma}\xi_n$ in the $\mathcal{E}_\Delta(G)$-norm. By A.3.7, the sequence $(\xi_n)_n$ also converges to ξ in the $L^2(G)$-norm, and so by continuity, $S_{\eta,\gamma}\xi = \lim_n S_{\eta,\gamma}\xi_n$ in the $L^2(G)$-norm. From what we already proved for functions in $S_0(G)$, we obtain that $\Theta_{\eta,\gamma}\xi = S_{\eta,\gamma}\xi$ (as elements of $L^2(G)$).

But this shows that $S_{\eta,\gamma}|_{\mathcal{E}_\Delta(G)} = \Theta_{\eta,\gamma}$, and since the extension of $S_{\eta,\gamma}|_{\mathcal{E}_\Delta(G)}$ to $L^2(G)$ is $S_{\eta,\gamma}$, we conclude that the extension of $\Theta_{\eta,\gamma}$ to $L^2(G)$ is $S_{\eta,\gamma}$. ∎

The following lemma was also noted in [66, Lemma 3.6]. We give a different proof here which uses localization.

Lemma A.3.9. *Let $\eta \in S_0(G)$. Then the Heisenberg module norm of η can be expressed in the following ways:*

$$\begin{aligned} \|\eta\|_{\mathcal{E}_\Delta(G)} &= \|C_\eta\| && \text{(A.3.7)} \\ &= \|S_\eta\|^{1/2} && \text{(A.3.8)} \\ &= \sup_{\|\xi\|_2=1} \left(\int_\Delta |\langle \xi, \pi(z)\eta\rangle|^2 \,\mathrm{d}z \right)^{1/2} && \text{(A.3.9)} \\ &= \inf\{ D^{1/2} : \ D \text{ is a Bessel bound for } \mathcal{G}(\eta;\Delta) \}. && \text{(A.3.10)} \end{aligned}$$

Proof. By Proposition A.2.4, the Heisenberg module norm of η is given by $\|\eta\|_{\mathcal{E}_\Delta(G)} = \|\Theta_\eta\|^{1/2}_{\mathcal{L}(\mathcal{E}_\Delta(G))}$. Since $\eta \in S_0(G)$, we get from Lemma A.3.8 and (A.3.3) that

$$\|\eta\|_{\mathcal{E}_\Delta(G)} = \|S_\eta\|^{1/2}_{\mathcal{L}(L^2(G))}.$$

Now from the equality $S_\eta = C_\eta^* C_\eta$ it follows that $\|S_\eta\|^{1/2} = \|C_\eta\|$. This takes care of (A.3.7) and (A.3.8). The expressions in (A.3.9) and (A.3.10) are well-known for the operator norm $\|C_\eta\|$. ■

We are now ready to prove the first of our main results:

Theorem A.3.10. *Let G be a second countable, locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. If $\eta \in \mathcal{E}_\Delta(G)$, then $\mathcal{G}(\eta;\Delta)$ is a Bessel family for $L^2(G)$. That is, there exists a $D > 0$ such that*

$$\int_\Delta |\langle \xi, \pi(z)\eta\rangle|^2 \,\mathrm{d}z \leq D\|\xi\|_2^2$$

for all $\xi \in L^2(G)$. Consequently, the analysis, synthesis and frame-like operators $C_\eta \colon L^2(G) \to L^2(\Delta)$, $D_\eta \colon L^2(\Delta) \to L^2(G)$, $S_{\eta,\gamma} \colon L^2(G) \to L^2(G)$ are all well-defined, bounded linear operators for $\eta, \gamma \in \mathcal{E}_\Delta(G)$.

Proof. Let $\eta \in \mathcal{E}_\Delta(G)$, and let $(\eta_n)_n$ be a sequence in $S_0(G)$ with

$$\lim_{n\to\infty} \|\eta - \eta_n\|_{\mathcal{E}_\Delta(G)} = 0.$$

Since $\eta_n \in S_0(G)$ for all n, $\mathcal{G}(\eta, \Delta)$ is a Bessel family for all n by Proposition A.2.10. Denote by D_n the optimal Bessel bound of $\mathcal{G}(\eta_n;\Delta)$ for each n, which by (A.3.10) in Lemma A.3.9 is equal to $\|\eta_n\|^2_{\mathcal{E}_\Delta(G)}$. Since $(\eta_n)_n$ is convergent in the Heisenberg module norm, it follows that $(\|\eta_n\|_{\mathcal{E}_\Delta(G)})_{n=1}^\infty$ is bounded, and so $(D_n)_{n=1}^\infty$ is bounded, by D say. We then have that

$$\int_\Delta |\left\langle \xi, \pi(z)\eta_n \right\rangle|^2 \,\mathrm{d}z \leq D_n\|\xi\|_2^2 \leq D\|\xi\|_2^2$$

for every $\xi \in L^2(G)$ and every $n \in \mathbb{N}$. Since $(\eta_n)_n \to \eta$ in $\mathcal{E}_\Delta(G)$, we have from Proposition A.3.7 that $(\eta_n)_n \to \eta$ in $L^2(G)$ as well. Hence, continuity of the inner product gives for each $z \in \Delta$ and each $\xi \in L^2(G)$ that

$$\lim_{n\to\infty} |\left\langle \xi, \pi(z)\eta_n \right\rangle|^2 = |\left\langle \xi, \pi(z)\eta \right\rangle|^2.$$

By Fatou's lemma, we obtain for every $\xi \in L^2(G)$ that

$$\int_\Delta |\left\langle \xi, \pi(z)\eta \right\rangle|^2 dz \leq \liminf_{n\to\infty} \int_\Delta |\left\langle \xi, \pi(z)\eta_n \right\rangle|^2 dz \leq D\|\xi\|^2.$$

This proves that $\mathcal{G}(\eta;\Delta)$ is a Bessel family. ∎

We are now able to extend the description of the Heisenberg module norm given in Lemma A.3.9 for functions in $S_0(G)$ to all of $\mathcal{E}_\Delta(G)$.

Proposition A.3.11. *Let $\eta \in \mathcal{E}_\Delta(G)$. Then the module norm of η can be expressed in the following ways:*

$$\begin{aligned} \|\eta\|_{\mathcal{E}_\Delta(G)} &= \|C_\eta\| && \text{(A.3.11)}\\ &= \|S_\eta\|^{1/2} && \text{(A.3.12)}\\ &= \sup_{\|\xi\|=1} \left(\int_\Delta |\left\langle \xi, \pi(z)\eta \right\rangle|^2 dz \right)^{1/2} && \text{(A.3.13)}\\ &= \inf\{D^{1/2} : D \text{ is a Bessel bound for } \mathcal{G}(\eta;\Delta)\}. && \text{(A.3.14)} \end{aligned}$$

Proof. Let $\eta \in \mathcal{E}_\Delta(G)$. We will show that $\|\eta\|_{\mathcal{E}_\Delta(G)} = \|C_\eta\|$. Once this is shown, the rest of the expressions for $\|\eta\|_{\mathcal{E}_\Delta(G)}$ follow just as in the proof of Lemma A.3.9.

Let $(\eta_n)_{n=1}^\infty$ be a sequence in $S_0(G)$ such that

$$\lim_{n\to\infty} \|\eta - \eta_n\|_{\mathcal{E}_\Delta(G)} = 0.$$

Then $(\eta_n)_n$ is a Cauchy sequence in the Heisenberg module norm, and so for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have that

$$\|\eta_m - \eta_n\|_{\mathcal{E}_\Delta(G)} < \epsilon.$$

Since $\eta_n \in S_0(G)$ for all $n \in \mathbb{N}$ and $S_0(G)$ is a subspace of $L^2(G)$, we have that $\eta_m - \eta_n \in S_0(G)$ for all $m, n \in \mathbb{N}$, and so by Lemma A.3.9, we can write

$$\|\eta_m - \eta_n\|_{\mathcal{E}_\Delta(G)} = \|C_{\eta_m - \eta_n}\| = \|C_{\eta_m} - C_{\eta_n}\|.$$

But then by the above, we obtain that the sequence of operators $(C_{\eta_n})_{n=1}^\infty$ is Cauchy in $\mathcal{L}(L^2(G), L^2(\Delta))$, and so by completeness, there exists $T \in \mathcal{L}(L^2(G), L^2(\Delta))$ such that

$$\lim_{n\to\infty} \|T - C_{\eta_n}\| = 0.$$

Now fix $\xi \in L^2(G)$. Then we have that

$$\lim_{n\to\infty} \|T\xi - C_{\eta_n}\xi\|_2 = 0.$$

It is well-known that this implies the existence of a subsequence $(C_{\eta_{n_k}}\xi)_{k=1}^\infty$ that converges pointwise almost everywhere to $T\xi$ (see for instance [100, Theorem 3.12]). However, since $(\eta_n)_n$ converges to η in the $L^2(G)$-norm by A.3.7, we have that

$$\lim_{n\to\infty} C_{\eta_n}\xi(z) = \lim_{n\to\infty} \langle \xi, \pi(z)\eta_n\rangle = \langle \xi, \pi(z)\eta\rangle = C_\eta\xi(z)$$

for every $z \in \Delta$. Hence $(C_{\eta_n}\xi)_n$ converges pointwise to $C_\eta\xi$, and it follows that $(C_{\eta_{n_k}}\xi)_k$ converges pointwise to $C_\eta\xi$ as well. This shows that $C_\eta\xi = T\xi$ almost everywhere, and so they represent the same element in $L^2(\Delta)$. Since ξ was arbitrary, it follows that $C_\eta = T$, and so we have that

$$\lim_{n\to\infty} \|C_\eta - C_{\eta_n}\| = 0.$$

This implies that

$$\|\eta\|_{\mathcal{E}_\Delta(G)} = \lim_{n\to\infty} \|\eta_n\|_{\mathcal{E}_\Delta(G)} = \lim_{n\to\infty} \|C_{\eta_n}\| = \|C_\eta\|.$$

■

Let $B_\Delta(G)$ denote the set of $\eta \in L^2(G)$ such that $\mathcal{G}(\eta; \Delta)$ is a Bessel family for $L^2(G)$. Then $B_\Delta(G)$ is a Banach space when equipped with the norm

$$\|\eta\|_{B_\Delta(G)} = \|C_\eta\| = \inf\{D^{1/2} : D \text{ is a Bessel bound for } \mathcal{G}(\eta; \Delta)\}. \tag{A.3.15}$$

By Proposition A.2.12, the Heisenberg module $\mathcal{E}_\Delta(G)$ is the completion of $S_0(G)$ with respect to the Heisenberg module norm. But by using our embedding of $\mathcal{E}_\Delta(G)$ into $L^2(G)$ in Proposition A.3.7 and the expression of the Heisenberg module norm provided in Proposition A.3.11, we obtain a concrete description of $\mathcal{E}_\Delta(G)$ as a subspace of $L^2(G)$. In the following proposition, we use the notation from (A.2.20).

Proposition A.3.12. *Let G be a second countable, locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. Then the Heisenberg module $\mathcal{E}_\Delta(G)$ is the completion of $S_0(G)$ in $B_\Delta(G)$. The bimodule structure can be described as follows: Let $a \in C^*(\Delta, c)$, $b \in C^*(\Delta^\circ, \overline{c})$ and $\xi \in \mathcal{E}_\Delta(G)$. Then*

$$a \cdot \xi = \pi_\Delta(a)\xi \tag{A.3.16}$$

$$\xi \cdot b = \pi^*_{\Delta^\circ}(b)\xi \tag{A.3.17}$$

Proof. By Proposition A.2.12, we know that $\mathcal{E}_\Delta(G)$ is the completion of $S_0(G)$ with respect to the Heisenberg module norm. By Proposition A.3.7, we know that $\mathcal{E}_\Delta(G)$ is continuously embedded into $L^2(G)$ in a way that respects the embedding of $S_0(G)$ into $L^2(G)$. By Proposition A.3.11, we have a description of the Heisenberg module norm as $\|\eta\|_{\mathcal{E}_\Delta(G)} = \|\eta\|_{B_\Delta(G)}$. It follows that $\mathcal{E}_\Delta(G)$ is the completion of $S_0(G)$ with respect to the norm of $B_\Delta(G)$.

To see that (A.3.16) holds, let $a \in C^*(\Delta, c)$ and $\xi \in \mathcal{E}_\Delta(G)$. Let $(a_n)_n$ be a sequence in $S_0(\Delta, c)$ such that $\lim_{n\to\infty} a_n = a$ in $C^*(\Delta, c)$. Let $(\xi_n)_n$ be a sequence in $S_0(G)$ such that $\lim_{n\to\infty} \xi_n = \xi$ in $\mathcal{E}_\Delta(G)$. Then by continuity of the left action of $C^*(\Delta, c)$ on $\mathcal{E}_\Delta(G)$, we have that

$$a \cdot \xi = \left(\lim_n a_n\right) \cdot \left(\lim_n \xi_n\right) = \lim_n \left(a_n \cdot \xi_n\right) = \lim_n \pi(a_n)\xi_n$$

in $\mathcal{E}_\Delta(G)$. The last equality follows from the description of $a \cdot \xi$ for $a \in S_0(\Delta, c)$ and $\xi \in S_0(G)$ as $\pi(a)\xi$ (see Proposition A.2.12). Since $\xi_n \to \xi$ in the Heisenberg module norm, we have that $\xi_n \to \xi$ in the $L^2(G)$-norm. Also, since $\pi(a_n) \to \pi(a)$ in the operator norm, we have that $\pi(a_n)\xi_n \to \pi(a)\xi$ in the $L^2(G)$-norm. Hence, interchanging the $\mathcal{E}_\Delta(G)$-limit in the equation above with an $L^2(G)$-limit, we obtain that $a \cdot \xi = \pi_\Delta(a)\xi$.

The argument for (A.3.17) is similar, as for $b \in S_0(\Delta^\circ, \overline{c})$ and $\xi \in S_0(G)$, the definition of $\xi \cdot b$ in Proposition A.2.12 is equal to $\pi^*_{\Delta^\circ}(b)\xi$. A similar approximation argument to the one above shows that $\xi \cdot b = \pi^*_{\Delta^\circ}(b)\xi$ also holds for $b \in C^*(\Delta^\circ, \overline{c})$ and $\xi \in \mathcal{E}_\Delta(G)$. ■

Example A.3.13. If one sets $\Delta = G \times \widehat{G}$, the Heisenberg module $\mathcal{E}_\Delta(G)$ is all of $L^2(G)$. To see this, note that $\Delta^\circ = \{0\}$. Thus, we have the identification $C^*(\Delta^\circ, \overline{c}) \cong \mathbb{C}$, where a sequence $a \in C^*(\Delta^\circ, \overline{c}) = \mathbb{C}\Delta^\circ$ is identified with its value $a(0)$ at 0. In this situation, the Heisenberg module $\mathcal{E}_\Delta(G)$ is a $C^*(\Delta, c)$-$\mathbb{C}$-imprimitivity bimodule. But then $\mathcal{E}_\Delta(G)$ is a right Hilbert C^*-module over $\mathbb{C}$, so it must be a Hilbert space (with linearity in the second argument of the inner product). The right action is given by

$$\xi \cdot b = \sum_{w \in \Delta^\circ} b(w)\pi(w)^*\xi = b(0)\xi,$$

which under the identification $C^*(\Delta^\circ, \overline{c}) \cong \mathbb{C}$ becomes $\xi \cdot \lambda = \xi\lambda$ for $\xi \in L^2(G)$ and $\lambda \in \mathbb{C}$, i.e. ordinary scalar multiplication. Furthermore, the inner product at the value 0 is given by $\langle \xi, \eta \rangle_{\bullet}(0) = \langle \pi(0)\eta, \xi \rangle = \langle \eta, \xi \rangle$, i.e. the right inner product is just the conjugate of the ordinary $L^2(G)$-inner product.

It follows immediately from Proposition A.2.4 that the Heisenberg module norm in this case is just the $L^2(G)$-norm, and so $\mathcal{E}_\Delta(G) = B_\Delta(G) = L^2(G)$. The statement $B_\Delta(G) = L^2(G)$ when Δ is the whole time-frequency plane is well known. Indeed, in this case the analysis operator C_η is the short-time Fourier transform. This is a bounded operator $L^2(G) \to L^2(G \times \widehat{G})$ for all $\eta \in L^2(G)$, and is invertible for any $\eta \neq 0$, see [52, Theorem 6.2.1].

Example A.3.14. Suppose G is a discrete group, and that Δ is a cocompact subgroup of $G \times \widehat{G}$ (which must then be a lattice). Then $S_0(G) = \ell^1(G)$ (Proposition A.2.10, 2), and so the Heisenberg module satisfies $\ell^1(G) \subseteq \mathcal{E}_\Delta(G) \subseteq \ell^2(G)$. In particular, if G is finite, then $\mathcal{E}_\Delta(G) = \ell^1(G) = \ell^2(G) = \mathbb{C}G \cong \mathbb{C}^{|G|}$.

The following theorem extends Lemma A.3.8 and is one of our main results.

Theorem A.3.15. *Let G be a second countable, locally compact abelian group, and let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$. Let $\eta, \gamma \in \mathcal{E}_\Delta(G)$. Then the module frame-like operator $\Theta_{\eta,\gamma} \colon \mathcal{E}_\Delta(G) \to \mathcal{E}_\Delta(G)$ extends via localization to the Gabor frame-like operator $S_{\eta,\gamma} \colon L^2(G) \to L^2(G)$.*

Proof. Let $(\eta_n)_{n=1}^\infty$ and $(\gamma_n)_{n=1}^\infty$ be sequences in $S_0(G)$ that converge towards η and γ respectively in the Heisenberg module norm. Let $\xi \in \mathcal{E}_\Delta(G)$. Then $(\Theta_{\eta_n,\gamma_n}\xi)_n$ converges towards $\Theta_{\eta,\gamma}\xi$ in the Heisenberg module norm. By Lemma A.3.8, we have that $\Theta_{\eta_n,\gamma_n}\xi = S_{\eta_n,\gamma_n}\xi$ for each n, and since convergence in the Heisenberg module norm implies convergence in the $L^2(G)$-norm, we have that

$$\lim_{n\to\infty} \|S_{\eta_n,\gamma_n}\xi_n - \Theta_{\eta,\gamma}\xi\|_2 = 0. \tag{A.3.18}$$

By Proposition A.3.11 and the identity $C_{\eta-\eta_n} = C_\eta - C_{\eta_n}$, the sequences of operators $(C_{\eta_n})_n$ and $(C^*_{\gamma_n})_n$ converge in the operator norm to C_η and C^*_γ respectively, and so $(S_{\eta_n,\gamma_n})_n$ converges in the operator norm towards $S_{\eta,\gamma}$. It follows that the sequence $(S_{\eta_n,\gamma_n}\xi)_n$ converges to $S_{\eta,\gamma}\xi$ in the L^2-norm. But then by (A.3.18), we have that $\Theta_{\eta,\gamma}\xi = S_{\eta,\gamma}\xi$. This shows that the restriction of $S_{\eta,\gamma}$ to $\mathcal{E}_\Delta(G)$ is equal to $\Theta_{\eta,\gamma}$, and so the unique extension of $\Theta_{\eta,\gamma}$ to a bounded linear operator on $L^2(G)$ must be $S_{\eta,\gamma}$. ∎

We now arrive at another one of our main results. The following result was previously only known for generators in $S_0(G)$ [65, 82]. It states that finite module frames for $\mathcal{E}_\Delta(G)$ are exactly the generators of multi-window Gabor frames for

$L^2(G)$, where the generators are allowed to come from $\mathcal{E}_\Delta(G)$. This gives a complete description of generators of Heisenberg modules in terms of multi-window Gabor frames.

Theorem A.3.16. *Let G be a second countable, locally compact abelian group, let Δ be a closed, cocompact subgroup of $G \times \widehat{G}$, and let $\eta_1, \ldots, \eta_k$ be elements of the Heisenberg module $\mathcal{E}_\Delta(G)$. Then the following are equivalent:*

1. *The set $\{\eta_1, \ldots, \eta_k\}$ generates $\mathcal{E}_\Delta(G)$ as a left $C^*(\Delta, c)$-module. That is, for all $\xi \in \mathcal{E}_\Delta(G)$ there exist $a_1, \ldots, a_k \in C^*(\Delta, c)$ such that*
$$\xi = \sum_{j=1}^{k} a_j \cdot \eta_j .$$
2. *The system*
$$\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta) = \{\pi(z)\eta_j : z \in \Delta, 1 \leq j \leq k\}$$
is a multi-window Gabor frame for $L^2(G)$.

Proof. By Proposition A.2.6, the set $\{\eta_1, \ldots, \eta_k\}$ is a generating set for $\mathcal{E}_\Delta(G)$ if and only if the sequence $(\eta_1, \ldots, \eta_k)$ is a module frame for $\mathcal{E}_\Delta(G)$. By Proposition A.2.5, this happens if and only if $\Theta = \Theta_{(\eta_j)_{j=1}^k} = \sum_{j=1}^k \Theta_{\eta_j}$ is invertible as an element of $\mathcal{L}_{C^*(\Delta,c)}(\mathcal{E}_\Delta(G))$. By Theorem A.3.15 and linearity of the localization map $\mathcal{L}_{C^*(\Delta,c)}(\mathcal{E}_\Delta(G)) \hookrightarrow \mathcal{L}(L^2(G))$, this operator extends via localization to the Gabor multi-window frame operator $S = S_{(\eta_j)_{j=1}^k} = \sum_{j=1}^k S_{\eta_j}$ on $L^2(G)$. Since the localization map $\mathcal{L}(\mathcal{E}_\Delta(G)) \hookrightarrow \mathcal{L}(L^2(G))$ is an inclusion of unital C^*-subalgebras, it follows by inverse closedness [86, Theorem 2.1.11] that Θ is invertible in $\mathcal{L}(\mathcal{E}_\Delta(G))$ if and only if S is invertible in $\mathcal{L}(L^2(G))$. But by Proposition A.2.9, the latter happens if and only if $\mathcal{G}(\eta_1, \ldots, \eta_k; \Delta)$ is a frame for $L^2(G)$. ∎

A.3.5 The fundamental identity of Gabor analysis

So far we have considered a closed subgroup Δ of $G \times \widehat{G}$, and from this we built the Heisenberg module $\mathcal{E}_\Delta(G)$, which is a $C^*(\Delta, c)$-$C^*(\Delta^\circ, \overline{c})$-imprimitivity bimodule. We focused specifically on the case when Δ is cocompact, since this implies Δ° is discrete and hence $C^*(\Delta^\circ, \overline{c})$ is unital. By [65, p. 5], Δ° is identical to the annihilator $\Delta^\perp$ (also defined in the same article) up to a measure-preserving change of coordinates, and it is also the case that $(\Delta^\perp)^\perp = \Delta$ by [33, Proposition 3.6.1]. Hence $(\Delta^\circ)^\circ = \Delta$. Imposing the restriction that Δ° also be cocompact, which implies that both Δ and Δ° are lattices, we could build $\mathcal{E}_{\Delta^\circ}(G)$ and ask how it relates to $\mathcal{E}_\Delta(G)$. The following proposition shows that the relationship is just about as good as we could hope for.

Proposition A.3.17. *Let Δ be a lattice in $G\times\widehat{G}$. Then $\mathcal{E}_\Delta(G) = \mathcal{E}_{\Delta^\circ}(G)$ as subspaces of $L^2(G)$, and $\|\eta\|_{\mathcal{E}_{\Delta^\circ}(G)} = s(\Delta)^{-1/2}\|\eta\|_{\mathcal{E}_\Delta(G)}$ for all $\eta \in \mathcal{E}_\Delta(G)$.*

Proof. Note first that since Δ is a lattice, so is Δ°. In particular, Δ° is a cocompact subgroup, so all the results in this section for Δ apply just as well for Δ°. Hence, by Proposition A.2.12 and Proposition A.3.12, one obtains the Heisenberg module $\mathcal{E}_{\Delta^\circ}(G)$ as a completion of $S_0(G)$ in $B_{\Delta^\circ}(G)$, which is a $C^*(\Delta^\circ, c)$-$C^*(\Delta, \overline{c})$-imprimitivity bimodule. Note that in the construction of $\mathcal{E}_{\Delta^\circ}(G)$ we put on Δ° the counting measure, and on Δ the counting measure scaled with $s(\Delta^\circ)^{-1}$ as per Convention A.3.6. Denote the left inner product on $\mathcal{E}_{\Delta^\circ}(G)$ by ${}_\bullet\langle\cdot,\cdot\rangle'$ and the right inner product by $\langle\cdot,\cdot\rangle_\bullet'$.

Let $\eta \in S_0(G)$. Denote by $\pi^*_{\Delta^\circ}$ the C^*-algebra representation as in the discussion following Proposition A.2.12. Denote by π_{Δ° the representation π_Δ in the same discussion, but with Δ replaced with Δ°. In other words, $\pi^*_{\Delta^\circ}$ is a representation of $C^*(\Delta^\circ, \overline{c})$ on $L^2(G)$, while π_{Δ° is a representation of $C^*(\Delta^\circ, c)$ on $L^2(G)$. Keeping in mind the right measures, we have that

$$\begin{aligned}\pi^*_{\Delta^\circ}(\langle\eta,\eta\rangle_\bullet) &= s(\Delta)^{-1}\sum_{w\in\Delta^\circ}\langle\pi(w)\eta,\eta\rangle\pi(w)^* \\ \pi_{\Delta^\circ}({}_\bullet\langle\eta,\eta\rangle') &= \sum_{w\in\Delta^\circ}\langle\eta,\pi(w)\eta\rangle\pi(w).\end{aligned}$$

From the above we see that

$$(\pi^*_{\Delta^\circ}(\langle\eta,\eta\rangle_\bullet))^* = s(\Delta)^{-1}\pi_{\Delta^\circ}({}_\bullet\langle\eta,\eta\rangle'). \tag{A.3.19}$$

Using the faithfulness of the integrated representations (Proposition A.2.11), we obtain for all $\eta \in S_0(G)$ that

$$\begin{aligned}\|\eta\|^2_{\mathcal{E}_{\Delta^\circ}(G)} &= \|{}_\bullet\langle\eta,\eta\rangle'\|_{C^*(\Delta^\circ,c)} \\ &= \|\pi_{\Delta^\circ}({}_\bullet\langle\eta,\eta\rangle')\| && \text{by faithfulness of } \pi_{\Delta^\circ} \\ &= \|\pi_{\Delta^\circ}({}_\bullet\langle\eta,\eta\rangle')^*\| \\ &= \|s(\Delta)^{-1}\pi^*_{\Delta^\circ}(\langle\eta,\eta\rangle_\bullet)\| && \text{by (A.3.19)} \\ &= s(\Delta)^{-1}\|\langle\eta,\eta\rangle_\bullet\|_{C^*(\Delta^\circ,\overline{c})} && \text{by faithfulness of } \pi^*_{\Delta^\circ} \\ &= s(\Delta)^{-1}\|{}_\bullet\langle\eta,\eta\rangle\|_{C^*(\Delta,c)} && \text{by Proposition A.2.4} \\ &= s(\Delta)^{-1}\|\eta\|^2_{\mathcal{E}_\Delta(G)}.\end{aligned}$$

By the above, we have that a sequence $(\eta_n)_n$ in $S_0(G)$ is Cauchy in the $\mathcal{E}_\Delta(G)$-norm if and only if it is Cauchy in the $\mathcal{E}_{\Delta^\circ}(G)$-norm. Thus, the sequence has a limit in $\mathcal{E}_\Delta(G)$-norm if and only if it has a limit in $\mathcal{E}_{\Delta^\circ}(G)$-norm. Since both of these norms dominate the $L^2(G)$-norm, it follows if $(\eta_n)_n$ is Cauchy in either of the norms, the limit in either of the norms give the same element in $L^2(G)$. It follows that $\mathcal{E}_{\Delta^\circ}(G) = \mathcal{E}_\Delta(G)$, with $\|\eta\|_{\mathcal{E}_{\Delta^\circ}(G)} = s(\Delta)^{1/2}\|\eta\|_{\mathcal{E}_\Delta(G)}$ for all $\eta \in \mathcal{E}_\Delta(G)$. ∎

Finally, we show that the fundamental identity of Gabor analysis (or FIGA) [43, Theorem 4.5] holds when all involved functions are in $\mathcal{E}_\Delta(G)$ when Δ is a lattice. In the following we let $S^\Delta_{\eta,\gamma}$ be the operator of (A.2.8), and let $S^{\Delta^\circ}_{\eta,\gamma}$ denote the operator of (A.2.8) but with Δ° instead of Δ. It is already known that FIGA holds for functions in $S_0(G)$ even when Δ is just a closed subgroup of the time-frequency plane $G\times\widehat{G}$ [65, Corollary 6.3]. However, in the proof of Proposition A.3.18 below we shall need to apply Theorem A.3.15 to frame operators with respect to Δ and with respect to Δ°. This then requires that both Δ and Δ° are cocompact, hence they are both lattices. With these restrictions, FIGA is the statement

$$\sum_{z\in\Delta} \langle \eta, \pi(z)\gamma\rangle \langle \pi(z)\xi, \psi\rangle = \frac{1}{s(\Delta)} \sum_{z^\circ\in\Delta^\circ} \langle \xi, \pi(z^\circ)\gamma\rangle \langle \pi(z^\circ)\eta, \psi\rangle \tag{A.3.20}$$

for $\eta, \gamma, \xi, \psi \in S_0(G)$. In short form it is just the statement

$$S^\Delta_{\gamma,\xi}\eta = s(\Delta)^{-1} S^{\Delta^\circ}_{\gamma,\eta}\xi \tag{A.3.21}$$

for $\eta, \gamma, \xi \in S_0(G)$. With the restriction that Δ is a lattice in $G\times\widehat{G}$, the following proposition extends the range for the FIGA (for the particular lattice Δ) to functions in $\mathcal{E}_\Delta(G)$.

Proposition A.3.18. *Let G be a second countable, locally compact abelian group, and let Δ be a lattice in $G\times\widehat{G}$. Then* (A.3.21) *holds for $\eta, \gamma, \xi \in \mathcal{E}_\Delta(G)$.*

Proof. Let $(\eta_n)_n$, $(\gamma_n)_n$ and $(\xi_n)_n$ be sequences in $S_0(G)$ that converge to η, γ and ξ, respectively, in the $\mathcal{E}_\Delta(G)$-norm. By A.3.17, the same is true in $\mathcal{E}_{\Delta^\circ}(G)$-norm. Then, since the fundamental identity of Gabor analysis applies for functions in $S_0(G)$ by [65, Corollary 6.3], we have

$$S^\Delta_{\gamma_n,\xi_n}\eta_n = s(\Delta)^{-1} S^{\Delta^\circ}_{\gamma_n,\eta_n}\xi_n$$

for all $n\in\mathbb{N}$. By Theorem A.3.15 we have

$$\lim_{n\to\infty} \|S^\Delta_{\gamma,\xi}\eta - S^\Delta_{\gamma_n,\xi_n}\eta_n\|_{\mathcal{E}_\Delta(G)} = 0$$
$$\lim_{n\to\infty} \|S^{\Delta^\circ}_{\gamma,\eta}\xi - S^{\Delta^\circ}_{\gamma_n,\eta_n}\xi_n\|_{\mathcal{E}_{\Delta^\circ}(G)} = 0.$$

Since convergence in the Heisenberg module norm implies convergence in the $L^2(G)$-norm, we conclude that the following equality holds in $L^2(G)$, were the limits are taken in $L^2(G)$:

$$S^\Delta_{\gamma,\xi}\eta = \lim_{n\to\infty} S^\Delta_{\gamma_n,\xi_n}\eta_n = \lim_{n\to\infty} s(\Delta)^{-1} S^{\Delta^\circ}_{\gamma_n,\eta_n}\xi_n = s(\Delta)^{-1} S^{\Delta^\circ}_{\gamma,\eta}\xi.$$

∎

Remark A.3.19. As already mentioned, the FIGA holds for functions in $S_0(G)$ even when Δ is only a closed subgroup for $G \times \widehat{G}$. The techniques in this paper are based on localization of A-B-imprimitivity bimodules, which requires that we have a finite trace on at least one of the algebras A and B. Therefore the assumption that Δ is a lattice in $G \times \widehat{G}$ is necessary for our approach to the FIGA. There might be another technique that allows for an extension of the FIGA to $\mathcal{E}_\Delta(G)$ for Δ only a closed subgroup of $G \times \widehat{G}$ that the authors are not aware of. We remark again that for the existence of Gabor frames over a closed subgroup Δ of $G \times \widehat{G}$, it is necessary that Δ is cocompact in $G \times \widehat{G}$, which is the setting for most of the results in the paper.

Paper B

Gabor duality theory for Morita equivalent C^*-algebras

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Paper B

Gabor duality theory for Morita equivalent C^*-algebras

Abstract

The duality principle for Gabor frames is one of the pillars of Gabor analysis. We establish a far-reaching generalization to Morita equivalence bimodules with some extra properties. For certain twisted group C^*-algebras the reformulation of the duality principle to the setting of Morita equivalence bimodules reduces to the well-known Gabor duality principle by localizing with respect to a trace. We may lift all results at the module level to matrix algebras and matrix modules, and in doing so, it is natural to introduce (n, d)-matrix Gabor frames, which generalize multi-window super Gabor frames. We are also able to establish density theorems for module frames on equivalence bimodules, and these localize to density theorems for (n, d)-matrix Gabor frames.

B.1 Introduction

Hilbert C^*-modules are well-studied objects in the theory of operator algebras and Rieffel made the crucial observation that they provide the correct framework for the extension of Morita equivalence of rings to C^*-algebras. In his seminal work [96] he noted that his equivalence bimodules between two C^*-algebras are bimodules where the left and right Hilbert C^*-module structures are compatible in a technical sense. We are interested in the features of these equivalence bimodules from the perspective of frame theory. In [47] the notion of standard module frame was introduced for countably generated Hilbert C^*-modules. Already in [97] Rieffel observed that finitely generated equivalence bimodules could be described in terms of finite standard module frames. He used this in his study of Heisenberg modules – a class of projective Hilbert C^*-modules over twisted group C^*-algebras. In [82] it was observed that these module frames are closely related to Gabor frames used in time-frequency analysis. Moreover, in [66] the properties of standard module frames for Heisenberg modules were studied from the perspective of the established duality theory of these Gabor frames.

The following central result of Gabor frames is due to the seminal work [31, 67, 99].

Theorem (Duality Theorem for Gabor systems)**.** For $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R})$ the Gabor system $\{e^{2\pi i \beta l(\,\cdot\,)} g(\,\cdot\, - \alpha k)\}_{k,l\in\mathbb{Z}}$ is a *frame* for $L^2(\mathbb{R})$ if and only if the Gabor system $\{e^{2\pi i l(\,\cdot\,)/\alpha} g(\,\cdot\, - k/\beta)\}_{k,l\in\mathbb{Z}}$ is a *Riesz sequence* for the closed span of $\{e^{2\pi i l(\,\cdot\,)/\alpha} g(\,\cdot\, - k/\beta)\}_{k,l\in\mathbb{Z}}$ in $L^2(\mathbb{R})$.

The possible extension of the duality principle from Gabor systems to other types of systems has been investigated in [1, 14, 15, 38] and [59] as well as in the form of the theory of R-duality [23, 26, 106, 107].

Motivated by the link between the duality theory of Gabor frames and the Morita equivalence of noncommutative tori [66, 82] we explore duality theory of module frames for equivalence bimodules between Morita equivalent C^*-algebras and show that this is a true generalization of the duality theory for Gabor frames.

Unlike the treatment of this topic in [66], here we do not rely on any results from time-frequency analysis. Indeed, we will consider a quite general situation, namely two Morita equivalent C^*-algebras A and B with Morita equivalence bimodule E, where E is finitely generated and projective as an A-module and B is equipped with a faithful finite trace. We show that module frames for E as an A-module in this situation admit a duality theorem and by localization with respect to a trace we are able to connect these module frame statements to results on frames in Hilbert spaces. Moreover, we show that some cornerstone results of Gabor analysis can be formulated as C^*-algebraic statements on Morita equivalence bimodules. Also, we establish density results for the existence of module frames, which seemingly have not been explored before.

The application of our duality results to Gabor systems on general locally compact abelian (LCA) groups with time-frequency shifts from closed cocompact subgroups of phase space yields exactly the known key results in duality theory of Gabor systems. Our general approach to duality principles has led us to the introduction of (n, d)-matrix Gabor frames that is a joint generalization of multi-window superframes and Riesz bases and we prove that their Gabor dual systems are (d, n)-matrix Riesz sequences.

Let us summarize the content of this paper. In Section B.2 we collect some facts about Hilbert C^*-modules which will be of use later, most importantly about localization of Hilbert C^*-modules. We then proceed in Section B.3 setting up for reformulating central results of Gabor analysis to the setting of Morita equivalence bimodules with some extra conditions. In this section we also obtain density theorems for existence of module frames. Lastly, in Section B.4 we localize with respect to a trace to recover the setting of Gabor analysis. Due to the setup of the previous section, we obtain easy proofs for some foundational results of Gabor analysis for a very general type of Gabor frame.

B.2 Preliminaries

We assume basic knowledge about Banach $*$-algebras, C^*-algebras, Banach modules, and Hilbert C^*-modules. In this section we collect definitions and basic facts of concepts crucial for this paper.

Suppose ϕ is a positive linear functional on a C^*-algebra B, and that E is a right Hilbert B-module. We define an inner product

$$\langle \cdot, \cdot \rangle_\phi : E \times E \to \mathbb{C}, \quad (f, g) \mapsto \phi(\langle g, f \rangle_B),$$

where $\langle \cdot, \cdot \rangle_B$ is the B-valued inner product. We may have to factor out the subspace $N_\phi := \{ f \in E \mid \langle f, f \rangle_B = 0\}$ and complete E/N_ϕ with respect to $\langle \cdot, \cdot \rangle_\phi$. This yields a Hilbert space which we will denote by H_E, and is known as the localization of E in ϕ. There is a natural map $\rho_\phi : E \to H_E$ which induces a map $\rho_\phi : \mathrm{End}_B(E) \to \mathbb{B}(H_E)$. We will focus entirely on the case in which ϕ is a faithful positive linear functional, that is, when $b \in B^+$ with $\phi(b) = 0$ implies $b = 0$. In that case $N_\phi = \{0\}$ and we have the following useful result from [74, p. 57-58].

Proposition B.2.1. *Let B be a C^*-algebra equipped with a faithful positive linear functional $\phi : B \to \mathbb{C}$, and let E be a Hilbert B-module. Then the map ρ_ϕ : $\mathrm{End}_B(E) \to \mathbb{B}(H_E)$ is an injective $*$-homomorphism.*

The Hilbert C^*-modules of interest will be A-B-equivalence bimodules for C^*-algebras A and B. We will denote the A-valued inner product by ${}_\bullet\langle \cdot, \cdot \rangle$, and the B-valued inner product by $\langle \cdot, \cdot \rangle_\bullet$.

Definition B.2.2. Let A and B be C^*-algebras. A *Morita equivalence bimodule between* A and B, or an *A-B-equivalence bimodule*, is a Hilbert C^*-module E satisfying the following conditions.

1. $\overline{{}_\bullet\langle E, E \rangle} = A$ and $\overline{\langle E, E \rangle_\bullet} = B$, where ${}_\bullet\langle E, E \rangle = \mathrm{span}_{\mathbb{C}}\{ {}_\bullet\langle f, g \rangle \mid f, g \in E\}$ and likewise for $\langle E, E \rangle_\bullet$.

2. For all $f, g \in E$, $a \in A$ and $b \in B$,

$$\langle af, g \rangle_\bullet = \langle f, a^* g \rangle_\bullet \text{ and } {}_\bullet\langle fb, g \rangle = {}_\bullet\langle f, gb^* \rangle.$$

3. For all $f, g, h \in E$,
$$_\bullet\langle f, g\rangle h = f\langle g, h\rangle_\bullet .$$

Now let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be dense Banach $*$-subalgebras such that the enveloping C^*-algebra of $\mathcal{A}$ is A and the enveloping C^*-algebra of $\mathcal{B}$ is B. Suppose further there is a dense $\mathcal{A}$-$\mathcal{B}$-inner product submodule $\mathcal{E} \subset E$ such that the conditions above hold with $\mathcal{A}, \mathcal{B}, \mathcal{E}$ instead of A, B, E. In that case we say $\mathcal{E}$ is an *$\mathcal{A}$-$\mathcal{B}$-pre-equivalence bimodule*.

It is a well-known result that if E is an A-B-equivalence bimodule, then $B \cong \mathbb{K}_A(E)$ through the identification $\Theta_{f,g} \mapsto \langle f, g\rangle_\bullet$. Here $\Theta_{f,g}$ is the compact module operator $\Theta_{f,g} : h \mapsto {}_\bullet\langle h, f\rangle g$. In particular, $\| {}_\bullet\langle f, f\rangle\| = \|\langle f, f\rangle_\bullet\|$ for all $f \in E$. We shall only have need for the case when E is a finitely generated Hilbert A-module. For future use we record the following result.

Proposition B.2.3. *Let E be an A-B-equivalence bimodule. Then E is a finitely generated projective A-module if and only if B is unital.*

The result is typically proved by finding sets $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ of elements of E for which

$$f = \sum_{I=1}^{n} {}_\bullet\langle f, g_i\rangle h_i = \sum_{i=1}^{n} f\langle g_i, h_i\rangle_\bullet$$

for all $f \in E$, as done in [98, Proposition 2.1] and later [97, Proposition 1.2]. Note that the systems $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are not necessarily A-linearly independent, but they still provide a reconstruction formula.

The following result concerns frame bounds for module frames consisting of a single element, though we do not formally introduce module frames until Definition B.3.6. It will turn out that it is enough to consider module frames consisting of only one element, see Remark B.3.8. The results will come into play when we relate module frames and Gabor frames in Section B.4.

Lemma B.2.4. *Let A be any C^*-algebra, and let E be a left Hilbert A-module. Let $T \in \mathrm{End}_A(E)$ be such that there exist $C, D > 0$ such that*

$$C\, {}_\bullet\langle f, f\rangle \leq {}_\bullet\langle Tf, f\rangle \leq D\, {}_\bullet\langle f, f\rangle, \tag{B.2.1}$$

for all $f \in E$. Then the smallest possible value of D is $\|T\|$, and the largest possible value for C is $\|T^{-1}\|^{-1}$.

Proof. Since T is positive we have $\|T\| = \sup_{\|f\|=1}\{\| {}_\bullet\langle Tf, f\rangle\|\}$. It follows that the smallest value for D is $\|T\|$. Furthermore, it is easy to see by (B.2.1) that ${}_\bullet\langle T^{-1}f, f\rangle \leq C^{-1}\,{}_\bullet\langle f, f\rangle$. Hence by the same argument applied to T^{-1} the smallest value of C^{-1} is $\|T^{-1}\|$, from which it follows that the largest value of C is $\|T^{-1}\|^{-1}$.

■

Since we aim to mimic the situation of Gabor analysis, the positive linear functional that we localize our Morita equivalence bimodule with respect to will have a particular form. In particular it will be a faithful trace. For unital Morita equivalent C^*-algebras A and B Rieffel showed in [98] that there is a bijection between non-normalized finite traces on A and non-normalized finite traces on B under which to a trace tr_B on B there is an associated trace tr_A on A satisfying

$$\mathrm{tr}_A({}_\bullet\langle f, g\rangle) = \mathrm{tr}_B(\langle g, f\rangle_\bullet) \tag{B.2.2}$$

for all $f, g \in E$. Here E is the Morita equivalence bimodule. We will in the sequel almost always consider A or B unital, and so instead we will suppose the existence of a finite faithful trace on one C^*-algebra (the unital one) and induce a possibly unbounded trace on the other C^*-algebra. The following was proved in [9, Proposition 3.1] and ensures that this procedure works. Note that if both C^*-algebras are unital then the induced trace is also a finite trace as in [98], the result can be deduced from [98, Proposition 2.2].

Proposition B.2.5. *Let E be an A-B-equivalence bimodule, and suppose tr_B is a faithful finite trace on B. Then the following hold:*

i) There is a unique lower semi-continuous trace on A, denoted tr_A, for which

$$\mathrm{tr}_A({}_\bullet\big\langle f, g\big\rangle) = \mathrm{tr}_B(\big\langle g, f\big\rangle_\bullet) \tag{B.2.3}$$

for all $f, g \in E$. Moreover, tr_A is faithful, and densely defined since it is finite on $\mathrm{span}\{{}_\bullet\big\langle f, g\big\rangle : f, g \in E\}$. Setting

$$\langle f, g\rangle_{\mathrm{tr}_A} = \mathrm{tr}_A({}_\bullet\big\langle f, g\big\rangle), \quad \langle f, g\rangle_{\mathrm{tr}_B} = \mathrm{tr}_B(\langle g, f\rangle_\bullet),$$

for $f, g \in E$ defines $\mathbb{C}$-valued inner products on E, with $\langle f, g\rangle_{\mathrm{tr}_A} = \langle f, g\rangle_{\mathrm{tr}_B}$ for all $f, g \in E$. Consequently, the Hilbert space obtained by completing E in the norm $\|f\|' = \mathrm{tr}_A({}_\bullet\big\langle f, f\big\rangle)^{1/2}$ is just the localization of E with respect to tr_B.

ii) If E and F are equivalence A-B-bimodules, then every adjointable A-linear operator $E \to F$ has a unique extension to a bounded linear operator $H_E \to H_F$. Furthermore, the map $\mathrm{End}_A(E, F) \to \mathrm{End}(H_E, H_F)$ given by sending T to its unique extension is a norm-decreasing linear map of Banach spaces. Finally, if $E = F$, the map $\mathrm{End}_A(E) \to \mathbb{B}(H_E)$ is an isometric $*$-homomorphism of C^*-algebras.

B.3 Duality for equivalence bimodules

B.3.1 The equivalence bimodule picture

In Gabor analysis one considers not just Gabor frames, but multi-window super Gabor frames. Indeed, we will in Section B.4 introduce matrix Gabor frames, which will turn out to generalize multi-window super Gabor frames. To aid in our approach to these types of frames, we shall want to lift an A-B-equivalence bimodule E to an equivalence module between matrix algebras over A and B. We will soon make this precise. For $k \in \mathbb{N}$ let $\mathbb{Z}_k$ denote the group $\mathbb{Z}/(k\mathbb{Z})$. To simplify formulas in the sequel, we will zero-index matrices, i.e. the top left corner will correspond to (0,0). For $n, d \in \mathbb{N}$ we then consider the space of $n \times d$-matrices with entries from E, denoted $M_{n,d}(E)$. $M_{n,d}(E)$ naturally becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule with actions and inner products defined below. Here $M_n(A)$ is the usual C^*-algebra consisting of $n \times n$-matrices over A, and likewise for $M_d(B)$. We will still let the A-valued inner product on E be denoted by ${}_\bullet\langle -, - \rangle$, and the B-valued inner product on E be denoted $\langle -, - \rangle_\bullet$. Define an $M_n(A)$-valued inner product on $M_{n,d}(E)$ by

$$
\begin{aligned}
{}_\bullet[-,-] &: M_{n,d}(E) \times M_{n,d}(E) \to M_n(A) \\
(f,g) &\mapsto \sum_{k \in \mathbb{Z}_d} \begin{pmatrix} {}_\bullet\langle f_{0,k}, g_{0,k} \rangle & {}_\bullet\langle f_{0,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{0,k}, g_{n-1,k} \rangle \\ {}_\bullet\langle f_{1,k}, g_{0,k} \rangle & {}_\bullet\langle f_{1,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{1,k}, g_{n-1,k} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ {}_\bullet\langle f_{n-1,k}, g_{0,k} \rangle & {}_\bullet\langle f_{n-1,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{n-1,k}, g_{n-1,k} \rangle \end{pmatrix}.
\end{aligned}
$$

The action of $M_n(A)$ on $M_{n,d}(E)$ is defined in the natural way, that is

$$
(af)_{i,j} = \sum_{k \in \mathbb{Z}_n} a_{i,k} f_{k,j},
$$

for $a \in M_n(A)$ and $f \in M_{n,d}(E)$. Likewise we define an $M_d(B)$-valued inner product on $M_{n,d}(E)$ in the following way

$$
\begin{aligned}
[-,-]_\bullet &: M_{n,d}(E) \times M_{n,d}(E) \to M_d(B) \\
(f,g) &\mapsto \sum_{k \in \mathbb{Z}_n} \begin{pmatrix} \langle f_{k,0}, g_{k,0} \rangle_\bullet & \langle f_{k,0}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,0}, g_{k,d-1} \rangle_\bullet \\ \langle f_{k,1}, g_{k,0} \rangle_\bullet & \langle f_{k,1}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,1}, g_{k,d-1} \rangle_\bullet \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{k,d-1}, g_{k,0} \rangle_\bullet & \langle f_{k,d-1}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,d-1}, g_{k,d-1} \rangle_\bullet \end{pmatrix}.
\end{aligned}
$$

The right action of $M_d(B)$ on $M_{n,d}(E)$ is defined by

$$
(fb)_{i,j} = \sum_{k \in \mathbb{Z}_d} f_{i,k} b_{k,j}
$$

for $f \in M_{n,d}(E)$ and $b \in M_d(B)$.

With this setup, $M_{n,d}(E)$ becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule. It is not hard to verify the three conditions of Definition B.2.2. Indeed, the setup above is just the one induced by the usual Morita equivalence of $\mathbb{C}$ with $M_k(\mathbb{C})$, $k \in \mathbb{N}$. In particular, we have for $f, g, h \in M_{n,d}(E)$ that

$$ {}_\bullet[f, g]h = f[g, h]_\bullet, $$

and also

$$ \begin{aligned} M_n(A) &= \mathbb{K}_{M_d(B)}(M_{n,d}(E)), \\ M_d(B) &= \mathbb{K}_{M_n(A)}(M_{n,d}(E)). \end{aligned} $$

Moreover, since the new inner products are defined using the inner products ${}_\bullet\langle -, -\rangle$ and $\langle -, -\rangle_\bullet$, we see that in case we have Banach $*$-subalgebras $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$, as well as an $\mathcal{A}$-$\mathcal{B}$-subbimodule $\mathcal{E} \subset E$ as above, we get

$$ {}_\bullet[M_{n,d}(\mathcal{E}), M_{n,d}(\mathcal{E})] \subset M_n(\mathcal{A}), \quad [M_{n,d}(\mathcal{E}), M_{n,d}(\mathcal{E})]_\bullet \subset M_d(\mathcal{B}), \tag{B.3.1} $$

as well as

$$ M_n(\mathcal{A})M_{n,d}(\mathcal{E}) \subset M_{n,d}(\mathcal{E}), \quad M_{n,d}(\mathcal{E})M_d(\mathcal{B}) \subset M_d(\mathcal{E}). \tag{B.3.2} $$

Remark B.3.1. While it is far from surprising that $M_{n,d}(E)$ becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule, the resulting actions and inner products above will in Section B.4 make natural the construction of a new type of Gabor frame which generalizes the n-multi-window d-super Gabor frames considered in [66], see Definition B.4.7 and Proposition B.4.29.

Definition B.3.2. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For $g \in M_{n,d}(E)$ we define the *analysis operator* by

$$ \begin{aligned} \Phi_g : M_{n,d}(E) &\to M_n(A) \\ f &\mapsto {}_\bullet[f, g], \end{aligned} $$

and the *synthesis operator*:

$$ \begin{aligned} \Psi_g : M_n(A) &\to M_{n,d}(E) \\ a &\mapsto a \cdot g. \end{aligned} $$

An elementary computation shows that $\Phi_g^* = \Psi_g$. We will not make n, and later d, explicit in the notation for the analysis and synthesis operators. It will be implicit from the atom g being in $M_{n,d}(E)$.

Remark B.3.3. As $M_{n,d}(E)$ is an $M_n(A)$-$M_d(B)$-bimodule, we could just as well have defined the analysis operator and the synthesis operator with respect to the $M_d(B)$-valued inner product. Indeed we will need this later, but it will then be indicated by writing Φ_g^B. Unless otherwise indicated the analysis operator and synthesis operator will be with respect to the left inner product module structure.

Definition B.3.4. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For $g, h \in M_{n,d}(E)$ we define the *frame-like operator* $\Theta_{g,h}$ to be

$$\begin{aligned}\Theta_{g,h} : E &\to E\\ f &\mapsto {}_\bullet[f, g] \cdot h.\end{aligned}$$

In other words, $\Theta_{g,h} = \Psi_h \Phi_g = \Phi_h^* \Phi_g$. The *frame operator of* g is the operator

$$\begin{aligned}\Theta_g := \Theta_{g,g} = \Phi_g^* \Phi_g : E &\to E\\ f &\mapsto {}_\bullet\langle f, g\rangle g.\end{aligned}$$

Remark B.3.5. The frame operator Θ_g is a positive operator since $\Theta_g = (\Phi_g)^* \Phi_g$.

Definition B.3.6. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. We say $g \in M_{n,d}(E)$ generates a *(single)* $M_n(A)$*-module frame for* $M_{n,d}(E)$ if Θ_g is an invertible operator $M_{n,d}(E) \to M_{n,d}(E)$. Equivalently, there exist constants $C, D > 0$ such that

$$C\, {}_\bullet[f, f] \leq {}_\bullet[f, g]\, {}_\bullet[g, f] \leq D\, {}_\bullet[f, f],$$

holds for all $f \in M_{n,d}(E)$.

Remark B.3.7. When g generates a module frame for E, Θ_g is a positive invertible operator on E.

Remark B.3.8. If we are willing to change the integer n in the above setup we can show that it is really always sufficient to consider a single generator. Indeed, suppose we have $g_1, \ldots, g_k \in M_{n,d}(E)$, $k \in \mathbb{N}$, such that $\sum_{i=1}^k \Theta_{g_i}$ is invertible $M_{n,d}(E) \to M_{n,d}(E)$. This is equivalent to existence of constants $C, D > 0$ such that

$$C\, {}_\bullet[f, f] \leq \sum_{i=1}^{k} {}_\bullet[f, g_i]\, {}_\bullet[g_i, f] \leq D\, {}_\bullet[f, f]$$

for all $f \in M_{n,d}(E)$. In other words, $(g_i)_{i=1}^k$ is what is typically known as an $M_n(A)$-module frame for $M_{n,d}(E)$. This is then equivalent to existence constants $C', D' > 0$ such that

$$C'\, {}_\bullet[f', f'] \leq {}_\bullet[f', g]\, {}_\bullet[g, f'] \leq D'\, {}_\bullet[f', f']$$

for all $f' \in M_{kn,d}(E)$ and where $g = (g_1, \ldots, g_k)^T \in M_{kn,d}(E)$. In the last equation the inner products are $M_{kn}(A)$-valued.

We will now begin to formulate the Morita equivalence bimodule versions of central results of Gabor analysis, and we will show in Section B.4 that the results localize to the well-known Gabor analysis results, but for the very general type of Gabor frame introduced in Definition B.4.7.

The following result, while quite obvious in this context, will localize to one of the cornerstones of Gabor analysis, namely the Wexler-Raz biorthogonality relations, see Proposition B.4.30.

Proposition B.3.9 (Wexler-Raz for equivalence bimodules)**.** *Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. Let $g, h \in M_{n,d}(E)$. Then $f = \Theta_{g,h} f = \Theta_{h,g} f$ for all $f \in M_{n,d}(E)$ if and only if $M_d(B)$ is unital and $\langle g, h \rangle_{\bullet} = \langle h, g \rangle_{\bullet} = 1_{M_d(B)}$.*

Proof. In the standard isomorphism $\mathbb{K}_{M_n(A)}(M_{n,d}(E)) \cong M_d(B)$ we send $\Theta_{g,h}$ to the element $[g, h]_{\bullet} \in M_d(B)$, from which the result follows immediately. ∎

We also record the following result for use in Section B.4.

Proposition B.3.10. *Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. Let $g, h \in M_{n,d}(E)$ be so that ${}_{\bullet}[f, h]g = f$ for all $f \in M_{n,d}(E)$. Then*

$$f = h[g, f]_{\bullet} \quad \text{for all } f \in \overline{h \cdot M_d(B)}.$$

Proof. By assumption $M_{n,d}(E)$ is finitely generated and projective as an $M_n(A)$-module, hence $M_d(B) \cong \mathbb{K}_{M_n(A)}(M_{n,d}(E)) = \mathrm{End}_{M_n(A)}(M_{n,d}(E))$ and $M_d(B)$ is unital. We may rewrite the equality to $f = f[h, g]_{\bullet}$ for all $f \in M_{n,d}(E)$, which implies $[h, g]_{\bullet} = 1_{M_d(B)}$ as $M_d(B)$ acts faithfully on $M_{n,d}(E)$. But then

$$[g, h]_{\bullet} = [h, g]_{\bullet}^{*} = 1_{M_d(B)}^{*} = 1_{M_d(B)}$$

as well. Then if we let $f \in h \cdot M_d(B)$ we may write $f = hb$ for some $b \in M_d(B)$, and we get

$$h[g, f]_{\bullet} = h[g, hb]_{\bullet} = h[g, h]_{\bullet} b = h 1_{M_d(B)} b = hb = f.$$

We extend the reconstruction formula to $\overline{h \cdot M_d(B)}$ by continuity. ∎

We shall have use for the following definition, which can be formulated on more general modules than equivalence bimodules, but we shall not need the more general setting.

Definition B.3.11. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that Θ_g is invertible on $M_{n,d}(E)$, then $h = (\Theta_g)^{-1} g$ is called the *canonical dual atom of* g.

Remark B.3.12. Note that if g is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then $M_n(A) \cdot g = M_{n,d}(E)$. To see this, let $f \in M_{n,d}(E)$. Then

$$f = \Theta_g(\Theta_g)^{-1} f = {}_\bullet[\Theta_g^{-1} f, g]g \in M_n(A) \cdot g.$$

There is a correspondence between projections in Morita equivalent C^*-algebras, see for example [97, Proposition 1.2]. We formulate the following variant.

Proposition B.3.13. *Let E be an A-B-equivalence bimodule between a C^*-algebra A and a unital C^*-algebra B, and let $n, d \in \mathbb{N}$. If $g, h \in M_{n,d}(E)$ are such that $[g, h]_\bullet = 1_{M_d(B)}$, then ${}_\bullet[g, h]$ is an idempotent in $M_n(A)$. If $h = \Theta_g^{-1} g$, then ${}_\bullet[g, h]$ yields a projection in $M_n(A)$.*

Proof. From $[g, h]_\bullet = 1_{M_d(B)} = 1^*_{M_d(B)} = [h, g]_\bullet$, we get

$${}_\bullet[g, h]\,{}_\bullet[g, h] = {}_\bullet[\,{}_\bullet[g, h]g, h] = {}_\bullet[g[h, g]_\bullet, h] = {}_\bullet[g \cdot 1_{M_d(B)}, h] = {}_\bullet[g, h],$$

so ${}_\bullet[g, h]$ is an idempotent in $M_n(A)$. If $h = (\Theta_g)^{-1} g$, we also have

$${}_\bullet[g, h] = {}_\bullet[g, \Theta_g^{-1} g] = {}_\bullet[\Theta_g^{-1} g, g] = {}_\bullet[h, g] = {}_\bullet[g, h]^*,$$

so ${}_\bullet[g, h]$ is a projection in $M_n(A)$. ∎

The duality principle is a cornerstone of the field of Gabor analysis, see for example [31, 67, 99]. One of the main intentions of this investigation is a reformulation of this duality principle in our module framework. Indeed, the duality principle in the Hilbert C^*-module picture turns out to be quite an elementary statement. To this end we introduce the following operator. As before, let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For an element $g \in M_{n,d}(E)$ we define the $M_d(B)$-coefficient operator by

$$\begin{aligned} \Phi_g^B : M_{n,d}(E) &\to M_d(B) \\ f &\mapsto [g, f]_\bullet\,. \end{aligned}$$

Note that this operator is $M_d(B)$-adjointable with adjoint

$$(\Phi_g^B)^* b \mapsto g \cdot b.$$

We are now in the position to state and prove the module version of the duality principle which will localize to the duality principle of Gabor analysis in Theorem B.4.31.

Proposition B.3.14 (Module Duality Principle)**.** *Let E be an A-B-equivalence bimodule, let $n, d \in \mathbb{N}$, and let $g \in M_{n,d}(E)$. The following are equivalent.*

1. $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ *is invertible.*

2. $\Phi_g^B(\Phi_g^B)^* : M_d(B) \to M_d(B)$ *is an isomorphism.*

Proof. We show that both statements are equivalent to $[g, g\,]_\bullet$ being invertible in $M_d(B)$. Suppose Θ_g is invertible. Then $M_{n,d}(E)$ is finitely generated and projective as an $M_n(A)$-module, so $M_d(B) \cong \mathbb{K}_{M_n(A)}(M_{n,d}(E))$ is unital. As

$$\Theta_g f = f[g, g\,]_\bullet,$$

statement (1) is equivalent to $[g, g\,]_\bullet$ being invertible in $M_d(B)$. On the other hand,

$$\Phi_g^B(\Phi_g^B)^* b = \Phi_g^B(g \cdot b) = [g, g \cdot b\,]_\bullet = [g, g\,]_\bullet\, b.$$

Since $\Phi_g^B(\Phi_g^B)^* \in \operatorname{End}_{M_d(B)}(M_d(B))$ and $M_d(B)$ is an ideal in $\operatorname{End}_{M_d(B)}(M_d(B))$, statement (2) implies that $M_d(B)$ is unital and the statement is equivalent to $[g, g\,]_\bullet$ being invertible in $M_d(B)$. ■

In Gabor analysis one is often concerned with the regularity of the atoms generating a Gabor frame, as these often have desirable properties. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ be as in the setup in (B.3.1) and (B.3.2). In case g is so that Θ_g is invertible on all of $M_{n,d}(E)$ with $g \in M_{n,d}(\mathcal{E})$, and $\mathcal{B} \subset B$ is a spectrally invariant Banach $*$-subalgebra with the same unit as B, the canonical dual atom has the following important property.

Proposition B.3.15. *Let E be an A-B-equivalence bimodule, with an $\mathcal{A}$-$\mathcal{B}$-pre-equivalence bimodule $\mathcal{E} \subset E$, and let $n, d \in \mathbb{N}$. Suppose $\mathcal{B} \subset B$ is spectrally invariant with the same unit. If $g \in M_{n,d}(\mathcal{E})$ is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then the canonical dual $(\Theta_g)^{-1}g \in M_{n,d}(\mathcal{E})$ as well.*

Proof. For $f \in M_{n,d}(E)$ we have

$$\Theta_g f = {}_\bullet[\,f, g]g = f[g, g\,]_\bullet\,.$$

We deduce that $[g, g\,]_\bullet$ is invertible in $M_d(B)$ and $(\Theta_g)^{-1}g = g[g, g\,]_\bullet^{-1}$. But as $g \in M_{n,d}(\mathcal{E})$ we have $[g, g\,]_\bullet \in M_d(\mathcal{B})$. By spectral invariance of $\mathcal{B}$ in B it follows that $[g, g\,]_\bullet^{-1} \in M_d(\mathcal{B})$, see [103, Theorem 2.1]. Then, since $M_{n,d}(\mathcal{E}) \cdot M_d(\mathcal{B}) \subset M_{n,d}(\mathcal{E})$, it follows that

$$(\Theta_g)^{-1}g = g[g, g\,]_\bullet^{-1} \in M_{n,d}(\mathcal{E}),$$

which is the desired assertion. ■

There are well-known theorems in Gabor analysis known as density theorems. Postponing the precise formulations and technicalities, they give restrictions on existence of certain spanning sets, e.g. Gabor frames, in terms of the volume of co-compact subgroups of phase space, see Proposition B.4.33 and Proposition B.4.34. We proceed to establish analogous statements for module frames on certain equivalence bimodules.

More precisely, let E be an A-B-equivalence bimodule, and let B be unital with faithful finite trace tr_B. We induce a trace tr_A on A by ways of Proposition B.2.5. Now let $n, d \in \mathbb{N}$, and consider $M_{n,d}(E)$ as an $M_n(A)$-$M_d(B)$ equivalence bimodule. Then there are traces on $M_n(A)$ and $M_d(B)$ satisfying

$$\mathrm{tr}_{M_n(A)}({}_\bullet[f, g]) = \mathrm{tr}_{M_d(B)}([g, f]_\bullet)$$

for all $f, g \in M_{n,d}(E)$. They can be written as

$$\mathrm{tr}_{M_n(A)}({}_\bullet[f, g]) = \frac{1}{n} \sum_{i \in \mathbb{Z}_n} \mathrm{tr}_A({}_\bullet[f, g]_{i,i}), \quad \mathrm{tr}_{M_d(B)}([f, g]_\bullet) = \frac{1}{n} \sum_{i \in \mathbb{Z}_d} \mathrm{tr}_B([f, g]_{\bullet i,i}). \tag{B.3.3}$$

The trace on $M_d(B)$ extends to a finite trace on the whole algebra, but the same might not be true for the densely defined trace on $M_n(A)$. It is however true if A, and hence also $M_n(A)$, is unital. It is easy to show that the lifting process on the traces preserves both finiteness and faithfulness. We may then present our density theorems for equivalence bimodules. These appear to be new in the literature, and we will in Section B.4 use them to deduce density theorems statements for matrix Gabor frames, which will also be introduced in the same section.

Theorem B.3.16. *Let E be an A-B-equivalence bimodule where both A and B are unital and equipped with faithful finite traces tr_A and tr_B related by (B.2.3), and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then*

$$d\,\mathrm{tr}_B(1_B) \leq n\,\mathrm{tr}_A(1_A). \tag{B.3.4}$$

Proof. The assumption that Θ_g is invertible implies $[g, g]_\bullet$ is invertible. Then

$$u = \Theta_g^{-1} g = g[g, g]_\bullet^{-1}$$

is the canonical dual frame for $M_{n,d}(E)$. We have $[g, u]_\bullet = [u, g]_\bullet = 1_{M_d(B)}$, and by Proposition B.3.13 ${}_\bullet[g, u]$ is a projection in $M_n(A)$. Observing that ${}_\bullet[g, u] \leq 1_{M_n(A)}$ in $M_n(A)$ and using (B.3.3), we get

$$\begin{aligned} d\,\mathrm{tr}_B(1_B) &= n \cdot \frac{1}{n} \sum_{i=1}^{d} \mathrm{tr}_B(1_B) = n\,\mathrm{tr}_{M_d(B)}(1_{M_d(B)}) = n\,\mathrm{tr}_{M_d(B)}([u, g]_\bullet) \\ &= n\,\mathrm{tr}_{M_n(A)}({}_\bullet[g, u]) \leq n\,\mathrm{tr}_{M_n(A)}(1_{M_n(A)}) = n \cdot \frac{1}{n} \sum_{i=1}^{n} \mathrm{tr}_A(1_A) = n\,\mathrm{tr}_A(1_A). \end{aligned}$$

∎

Theorem B.3.17. *Let E be an A-B-equivalence bimodule where both A and B are unital and equipped with faithful finite traces tr_A and tr_B related by Equation (B.2.3), and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that $\Phi_g \Phi_g^* : M_n(A) \to M_n(A)$ is an isomorphism, then*

$$d \,\mathrm{tr}_B(1_B) \geq n \,\mathrm{tr}_A(1_A). \tag{B.3.5}$$

Proof. The assumptions imply ${}_\bullet[g, g]^{-1} \in M_n(A)$, so it follows that

$$1_{M_n(A)} = {}_\bullet[g, g]^{-1}\, {}_\bullet[g, g] = {}_\bullet[\,{}_\bullet[g, g]^{-1} g, g],$$

and $[{}_\bullet[g, g]^{-1} g, g\,]_\bullet$ is a projection in $M_d(B)$ by Proposition B.3.13. Since B is unital, then by observing that $[{}_\bullet[g, g]^{-1} g, g\,]_\bullet \leq 1_{M_d(B)}$ in $M_d(B)$ and using (B.3.3), we get

$$\begin{aligned}
n \,\mathrm{tr}_A(1_A) &= n \cdot \frac{1}{n} \sum_{i=1}^{n} \mathrm{tr}_A(1_A) = n \,\mathrm{tr}_{M_n(A)}(1_{M_n(A)}) = n \,\mathrm{tr}_{M_n(A)}({}_\bullet[\,{}_\bullet[g, g]^{-1} g, g]) \\
&= n \,\mathrm{tr}_{M_d(B)}([g, {}_\bullet[g, g]^{-1} g\,]_\bullet) \leq n \,\mathrm{tr}_{M_d(B)}(1_{M_d(B)}) \\
&= n \cdot \frac{1}{n} \sum_{i=1}^{d} \mathrm{tr}_B(1_B) = d \,\mathrm{tr}_B(1_B).
\end{aligned}$$

∎

B.3.2 Passing to the localization

In [82] the existence of multi-window Gabor frames for $L^2(\mathbb{R}^d)$ with windows in Feichtinger's algebra was proved through considerations on a related Hilbert C^*-module. Furthermore, in [83] projections in noncommutative tori were constructed from Gabor frames with sufficiently regular windows. Thus being able to pass from an equivalence bimodule E to a localization H_E and back is quite important, and we dedicate this section to results on this procedure. We will interpret this in terms of Gabor analysis in Section B.4, and we will explain how $L^2(G)$, for G a second countable LCA group, relates to H_E for specific modules E which arise in the study of twisted group C^*-algebras.

In the following let E be an A-B-equivalence bimodule. We will make the presence of traces precise in the individual results. To ease notation we will not formulate the below results in the setting of $M_{n,d}(E)$ being an $M_n(A)$-$M_d(B)$-equivalence bimodule, $n, d \in \mathbb{N}$, as such a reformulation is easy but notationally tedious.

Proposition B.3.18. *Let E be an A-B-equivalence bimodule, where B is unital and equipped with a faithful finite trace tr_B. We induce a trace tr_A on A by (B.2.3) and denote by H_E the localization of E in tr_A, and by $(-,-)_E$ the inner product on the localization of E in tr_A, i.e. $(f_1, f_2)_E = \mathrm{tr}_A({}_\bullet\langle f_1, f_2\rangle)$ for all $f_1, f_2 \in E$. Now suppose $g \in E$. Then there exists an $h \in E$ such that we have ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$ if and only if there exist constants $C, D > 0$ such that*

$$C(f, f)_E \le (f\langle g, g\rangle_\bullet, f)_E \le D(f, f)_E \tag{B.3.6}$$

for all $f \in H_E$. In other words, g generates a module frame for E as an A-module if and only if the inequalities in (B.3.6) are satisfied for some $C, D > 0$.

Remark B.3.19. We should note that in the setting of Proposition B.3.18 it is possible to say that $\langle g, g\rangle_\bullet$ is invertible in B if and only if there is h such that $\langle g, h\rangle_\bullet = 1_B$. Indeed, one may obtain this by [47, Theorem 5.9] in the case A is unital, and by [9, Proposition 2.6] in the case that A is not unital. One could use this to deduce Proposition B.3.18. However, our proof gives frame bounds which are of independent interest, see Proposition B.4.36. Since we want to focus on the link between module frames and Hilbert space frames, we therefore offer a more direct argument.

Proof. Suppose first that there is an $h \in E$ such that ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$. By Morita equivalence this implies

$$f = {}_\bullet\langle f, g\rangle h = f\langle g, h\rangle_\bullet$$

for all $f \in E$. As before, this implies $1_B = \langle g, h\rangle_\bullet = \langle h, g\rangle_\bullet$. Since tr_B is a positive linear functional we obtain

$$\begin{aligned}
(f, f)_E &= \mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, h\rangle_\bullet\langle h, g\rangle_\bullet, f\rangle) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, h\langle h, g\rangle_\bullet\rangle_\bullet, f) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet, f\rangle) \\
&= \mathrm{tr}_B(\langle f, f\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet\rangle_\bullet) \\
&\le \mathrm{tr}_B(\langle f, f\langle g, g\rangle_\bullet \|{}_\bullet\langle h, h\rangle\|\rangle_\bullet) \\
&= \|{}_\bullet\langle h, h\rangle\|\, \mathrm{tr}_B(\langle f, f\langle g, g\rangle_\bullet\rangle_\bullet) \\
&= \|{}_\bullet\langle h, h\rangle\|\, \mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet, f\rangle) \\
&= \|{}_\bullet\langle h, h\rangle\|(f\langle g, g\rangle_\bullet, f)_E,
\end{aligned}$$

for all $f \in E$, where we have used

$$\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet \le \|{}_\bullet\langle h, h\rangle\|\langle g, g\rangle_\bullet,$$

see e.g. [93, Corollary 2.22]. We then get the lower frame bound with $C = \|{}_\bullet\langle h, h\rangle\|^{-1}$, that is

$$\frac{1}{\|{}_\bullet\langle h, h\rangle\|}(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E$$

for all $f \in E$. By Proposition B.2.1 all intermediate steps involve operators that extend to bounded operators on H_E, so we may extend by continuity. We get the upper frame bound by use of [93, Corollary 2.22] in the following manner

$$\begin{aligned}
(f\langle g, g\rangle_\bullet, f)_E &= \mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet, f\rangle) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet{}^{1/2}, f\langle g, g\rangle_\bullet{}^{1/2}\rangle) \\
&\leq \|\langle g, g\rangle_\bullet{}^{1/2}\|^2\, \mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\
&= \|\langle g, g\rangle_\bullet\|\, \mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\
&= \|{}_\bullet\langle g, g\rangle\|(f, f)_E,
\end{aligned}$$

for all $f \in E$. Once again all intermediate steps involve operators that extend to bounded operators on H_E by Proposition B.2.1, so we may extend the result to all of H_E. Thus we have shown that

$$\frac{1}{\|{}_\bullet\langle h, h\rangle\|}(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq \|{}_\bullet\langle g, g\rangle\|(f, f)_E$$

for all $f \in H_E$.

Conversely, suppose there are $C, D > 0$ such that

$$C(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq D(f, f)_E$$

for all $f \in H_E$. We wish to show that this implies there exist $h \in E$ such that ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$. The assumption implies that $f \mapsto f\langle g, g\rangle_\bullet$ is a positive, invertible operator on H_E. As C^*-algebras are inverse closed it follows that $\langle g, g\rangle_\bullet$ is invertible in B. Thus $f \mapsto f\langle g, g\rangle_\bullet$ is a positive, invertible operator on E as well. Hence the operator

$$\begin{aligned}
\Theta_g &: E \to E \\
f &\mapsto {}_\bullet\langle f, g\rangle g = f\langle g, g\rangle_\bullet
\end{aligned}$$

is invertible with inverse

$$\Theta_g^{-1} f = f\langle g, g\rangle_\bullet{}^{-1}.$$

Define $h := \Theta_g^{-1} g$, and let $f \in E$ be arbitrary. Then we have

$${}_\bullet\langle f, g\rangle h = {}_\bullet\langle f, g\rangle\Theta_g^{-1} g = \Theta_g^{-1}({}_\bullet\langle f, g\rangle g) = \Theta_g^{-1}\Theta_g f = f,$$

from which the result follows. ∎

We are interested in module frames and module Riesz sequences, and their relationship to frames and Riesz sequences in Gabor analysis for LCA groups. To get results on Riesz sequences in Section B.4 we need a module version of Riesz sequences which, when localized, yields the Riesz sequences we know from Gabor analysis. To make the transition to Gabor frames in Section B.4 easier, we will in the following result let A be unital with a faithful trace tr_A, and we will localize A as a Hilbert A-module in the trace tr_A, i.e. we let $(a_1, a_2)_A := \operatorname{tr}_A(a_1 a_2^*)$. The completion of A in this inner product will be denoted H_A, and the action of A on H_A is the continuous extension of the multiplication action of A on itself.

Proposition B.3.20. *Let E be an A-B-equivalence bimodule where A is unital and equipped with a faithful finite trace tr_A. We localize E as in the setting of Proposition B.3.18 and localize A as described above. Now suppose $g \in E$. Then $\Phi_g \Phi_g^* : A \to A$ is an isomorphism if and only if there exist $C, D > 0$ such that for all $a \in A$ it holds that*

$$C(a,a)_A \leq (ag, ag)_E \leq D(a,a)_A. \tag{B.3.7}$$

Proof. First suppose $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. Then, as ${}_\bullet\langle g, g\rangle \geq 0$ in A we have

$${}_\bullet\langle ag, ag\rangle = a\, {}_\bullet\langle g, g\rangle a^* \leq \| {}_\bullet\langle g, g\rangle \| aa^*,$$

and we may deduce

$$(ag, ag)_A = \operatorname{tr}_A({}_\bullet\langle ag, ag\rangle) \leq \| {}_\bullet\langle g, g\rangle \| \operatorname{tr}_A(aa^*) = \| {}_\bullet\langle g, g\rangle \| (a,a)_A.$$

Hence in (B.3.7) we may set $D = \| {}_\bullet\langle g, g\rangle \|$. Since $\Phi_g \Phi_g^* : A \to A$ is an isomorphism and $\Phi_g \Phi_g^* a = a\, {}_\bullet\langle g, g\rangle$, it follows that there is ${}_\bullet\langle g, g\rangle^{-1} \in A$. Then

$$\begin{aligned}
(a,a)_A &= \operatorname{tr}_A(aa^*) \\
&= \operatorname{tr}_A(a\, {}_\bullet\langle g, g\rangle^{1/2}\, {}_\bullet\langle g, g\rangle^{-1}\, {}_\bullet\langle g, g\rangle^{1/2} a^*) \\
&\leq \| {}_\bullet\langle g, g\rangle^{-1} \| \operatorname{tr}_A(a\, {}_\bullet\langle g, g\rangle a^*) \\
&= \| {}_\bullet\langle g, g\rangle^{-1} \| \operatorname{tr}_A({}_\bullet\langle ag, ag\rangle) \\
&= \| {}_\bullet\langle g, g\rangle^{-1} \| (ag, ag)_E,
\end{aligned}$$

which implies that we may set $C = \| {}_\bullet\langle g, g\rangle^{-1} \|^{-1}$ in (B.3.7). All intermediate steps extend to H_A by Proposition B.2.1.

Suppose now that (B.3.7) is satisfied. The lower inequality in (B.3.7) tells us that for all $a \in A$,

$$\begin{aligned}
(a({}_\bullet\langle g, g\rangle - C), a)_A &= \operatorname{tr}_A(a({}_\bullet\langle g, g\rangle - C)a^*) \\
&= \operatorname{tr}_A(a\, {}_\bullet\langle g, g\rangle a^*) - C \operatorname{tr}_A(aa^*) \\
&= \operatorname{tr}_A({}_\bullet\langle ag, ag\rangle) - C \operatorname{tr}_A(aa^*) \\
&= (ag, ag)_E - C(a,a)_A \geq 0.
\end{aligned}$$

Note that we need the upper inequality of (B.3.7) to extend all intermediate steps to H_A via Proposition B.2.1. It follows that ${}_\bullet\langle g, g\rangle$ is a positive invertible operator on $H_A \supset A$. As C^*-algebras are inverse closed it follows that ${}_\bullet\langle g, g\rangle$ is invertible in A. Then, since

$$\Phi_g \Phi_g^* a = a\,{}_\bullet\langle g, g\rangle,$$

it follows that $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. ■

Remark B.3.21. Note that in the proofs of the two preceding results the upper bounds in (B.3.6) and (B.3.7) were both satisfied with $D = \|{}_\bullet\langle g, g\rangle\|$. We will see in Section B.4 that in the Gabor analysis setting, this means that all atoms coming from the Hilbert C^*-module are Bessel vectors for the localized frame system.

For use in Section B.4, we introduce the following notion.

Definition B.3.22. Let E be an A-B-equivalence bimodule, let $n, d \in \mathbb{N}$, and let $g \in M_{n,d}(E)$. If $\Phi_g \Phi_g^* : M_n(A) \to M_n(A)$ is an isomorphism, we say g generates a *module Riesz sequence for $M_{n,d}(E)$ with respect to $M_n(A)$*.

B.4 The link to Gabor analysis

In this section we show how the above results reproduce some of the core results of Gabor analysis for LCA groups. We will find that some of the cornerstones of Gabor analysis on LCA groups are trivial consequences of the above framework.

To present the results we will need to explain how time-frequency analysis on LCA groups relates to Morita equivalence of twisted group C^*-algebras. In the interest of brevity, we refer the reader to [66] for a more in-depth treatment of time frequency analysis and its relation to twisted group C^*-algebras, and to [64] for a survey on the Feichtinger algebra. We can also not omit to mention [97], which is a major inspiration for a lot of work done in the intersection of Gabor analysis and operator algebras.

Throughout this section, we fix a second-countable LCA group G and let $\widehat{G}$ be its dual group. We fix a Haar measure μ_G on G and normalize the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that the Plancherel theorem holds. By Λ we denote a closed subgroup of the time-frequency plane $G \times \widehat{G}$. The induced topologies and group multiplications on Λ and $(G \times \widehat{G})/\Lambda$ turn them into LCA groups as well, and we may equip them with their respective Haar measures. Having fixed the Haar measures on G, $\widehat{G}$, and Λ, we will assume $(G \times \widehat{G})/\Lambda$ is equipped with the unique Haar measure such that Weil's formula holds, see e.g. [33, Theorem 1.5.3]. In this setting we can define the *size of* Λ by

$$s(\Lambda) := \int_{(G\times\widehat{G})/\Lambda} 1\, d\mu_{(G\times\widehat{G})/\Lambda}. \tag{B.4.1}$$

Note that $s(\Lambda)$ is finite if and only if Λ is cocompact in $G \times \widehat{G}$.

For any $\xi = (x, \omega) \in G \times \widehat{G}$ we may then define the time-frequency shift operator

$$\begin{aligned} \pi(\xi)\colon L^2(G) &\to L^2(G) \\ \pi(\xi)f(t) &= \omega(t)f(t-x) \end{aligned}$$

for $t \in G$ and $f \in L^2(G)$. We also define the 2-cocycle

$$\begin{aligned} c : (G \times \widehat{G}) \times (G \times \widehat{G}) &\to \mathbb{T} \\ (\xi_1, \xi_2) &\mapsto \overline{\omega_2(x_1)} \end{aligned}$$

for $\xi_1 = (x_1, \omega_1), \xi_2 = (x_2, \omega_2) \in G \times \widehat{G}$. Note then that

$$\pi(\xi_1)\pi(\xi_2) = c(\xi_1, \xi_2)\,\pi(\xi_1 + \xi_2).$$

For the reader's convenience we also note that

$$\pi(\xi)^* = c(\xi, \xi)\pi(-\xi)$$

for all $\xi \in G \times \widehat{G}$.

For the given closed subgroup $\Lambda \subseteq G \times \widehat{G}$ we define its *adjoint subgroup* Λ° by

$$\Lambda^\circ := \{\xi \in G \times \widehat{G} \mid \pi(\xi)\,\pi(\lambda) = \pi(\lambda)\,\pi(\xi) \text{ for all } \lambda \in \Lambda\,\}.$$

Note that $(\Lambda^\circ)^\circ = \Lambda$ and $\widehat{\Lambda^\circ} \cong (G \times \widehat{G})/\Lambda$, see for example [65]. Moreover, Λ is cocompact if and only if Λ° is discrete. With these identifications we put on Λ° the Haar measure such that the Plancherel theorem holds with respect to Λ° and $(G \times \widehat{G})/\Lambda$.

We want to reframe time-frequency analysis in terms of Morita equivalence bimodules for certain twisted group C^*-algebras. To do this we use the Feichtinger algebra. In order to introduce this, we first define the *short time Fourier transform* with respect to a function $g \in L^2(G)$ as the operator

$$V_g : L^2(G) \to L^2(G \times \widehat{G}),\ V_g f(\xi) = \langle f, \pi(\xi)g\rangle,$$

for $\xi \in G \times \widehat{G}$. The *Feichtinger algebra* $S_0(G)$ can be defined by

$$S_0(G) = \big\{f \in L^2(G) \mid V_f f \in L^1(G \times \widehat{G})\big\}.$$

A norm on $S_0(G)$ is given by

$$\|f\|_{S_0(G)} = \|V_g f\|_{L^1(G\times\widehat{G})} \text{ for some } g \in S_0(G) \setminus \{0\}.$$

It is a nontrivial fact that all elements of $S_0(G) \setminus \{0\}$ determine equivalent norms on $S_0(G)$. In case G is discrete one has $S_0(G) = \ell^1(G)$ with equivalent norms. Furthermore, $S_0(G)$ consists of continuous functions and is dense in both $L^1(G)$ and $L^2(G)$.

With the above norm, $S_0(\Lambda)$ becomes a Banach $*$-algebra when equipped with the twisted convolution and involution given by

$$F_1 \natural F_2(\lambda) := \int_\Lambda F_1(\lambda')F_2(\lambda - \lambda')c(\lambda', \lambda - \lambda')\, \mathrm{d}\lambda',$$
$$F_1^*(\lambda) := c(\lambda, \lambda)\overline{F_1(-\lambda)},$$

for $F_1, F_2 \in S_0(\Lambda)$ and $\lambda \in \Lambda$. We denote the resulting Banach $*$-algebra by $S_0(\Lambda, c)$.

It was shown in [66] that when Λ is a closed subgroup of $G \times \widehat{G}$ the map $\lambda \mapsto \pi(\lambda)$ is a faithful c-projective unitary representation of Λ, and the integrated representation becomes a nondegenerate $*$-representation of $S_0(\Lambda, c)$ as bounded operators on $L^2(G)$. In other words, given $a \in S_0(\Lambda, c)$, we have the representation given by

$$\pi(a)f = \int_\Lambda a(\lambda)\pi(\lambda)f\, \mathrm{d}\lambda,$$

for $f \in L^2(G)$, and where we interpret the integral weakly. It is well-known that this $*$-representation is faithful. Indeed it was shown in [97] for the case of the Schwartz-Bruhat space, and the arguments easily carry over to the Feichtinger algebra. By completing $S_0(\Lambda, c)$ in the C^*-algebra norm coming from the integrated representation we obtain a C^*-algebra which we denote by $C^*(\Lambda, c)$. It is well-known that this coincides with the enveloping C^*-algebra of $S_0(\Lambda, c)$. We do the same for $S_0(\Lambda^\circ, \overline{c})$, and denote its universal enveloping C^*-algebra by $C^*(\Lambda^\circ, \overline{c})$.

Now $S_0(G)$ becomes a pre-equivalence $S_0(\Lambda, c)$-$S_0(\Lambda^\circ, \overline{c})$-equivalence bimodule as in Definition B.2.2 when equipped with the actions

$$a \cdot f = \int_\Lambda a(\lambda)\pi(\lambda)f\, d\lambda, \quad f \cdot b = \int_{\Lambda^\circ} b(\lambda^\circ)\pi(\lambda^\circ)^* f\, d\lambda^\circ \tag{B.4.2}$$

for $a \in S_0(\Lambda, c)$, $b \in S_0(\Lambda^\circ, \overline{c})$, and $f, g \in S_0(G)$, and with algebra-valued inner products given by

$$_\bullet\langle f, g\rangle(\lambda) = \left\langle f, \pi(\lambda)g\right\rangle, \quad \langle f, g\rangle_\bullet(\lambda^\circ) = \left\langle g, \pi(\lambda^\circ)^* f\right\rangle \tag{B.4.3}$$

for $f, g \in S_0(G)$, $\lambda \in \Lambda$, $\lambda^\circ \in \Lambda^\circ$. The inner products on the right hand sides of the equality signs are those of $L^2(G)$. That these are well-defined was noted in Section 3 of [66]. As is typical we pass to the C^*-completions. The resulting completion of $S_0(G)$ will be denoted $E_\Lambda(G)$. As done in the Schwartz-Bruhat case in [97], we note that $E_\Lambda(G)$ is a $C^*(\Lambda, c)$-$C^*(\Lambda^\circ, \overline{c})$-equivalence bimodule.

Remark B.4.1. The fact that we get the same twisted group C^*-algebras by using $S_0(\Lambda, c)$ as we get when using the more traditional approach with $L^1(\Lambda, c)$ was noted in [9].

An important consequence of working with S_0 instead of L^1 is that we have well-defined traces on dense Banach $*$-subalgebras of $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$. Indeed, since S_0-functions are continuous, there are well-defined canonical faithful traces on $S_0(\Lambda, c)$ and $S_0(\Lambda^\circ, \overline{c})$ given by evaluation in 0. We will denote the trace on $S_0(\Lambda, c)$ by tr_Λ and the trace on $S_0(\Lambda^\circ, \overline{c})$ by $\mathrm{tr}_{\Lambda^\circ}$ in the sequel.

Remark B.4.2. Although the traces tr_Λ and $\mathrm{tr}_{\Lambda^\circ}$ do not in general extend to the C^*-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$, we can guarantee they extend in one case. Namely, tr_Λ extends to all of $C^*(\Lambda, c)$ if $C^*(\Lambda, c)$ is unital, which is equivalent to Λ being discrete. The same is of course true for $C^*(\Lambda^\circ, \overline{c})$ and $\mathrm{tr}_{\Lambda^\circ}$, with the discreteness condition on Λ°. This is due to the fact that the trace given by evaluation in the identity extends to twisted group C^*-algebras when the underlying group is discrete [19, p. 951].

The following result is straightforward to prove and explains why we in the sequel will focus mostly on the case where Λ is closed and cocompact.

Proposition B.4.3. *Let $\Lambda \subset G \times \widehat{G}$ be a closed subgroup. Then $E_\Lambda(G)$ is a finitely generated projective $C^*(\Lambda, c)$-module if and only if $\Lambda \subset G \times \widehat{G}$ is a cocompact subgroup.*

As a very last preparation before starting to connect our results of Section B.3 to Gabor analysis we note the following important result. It was shown in [58] in the case of lattices in $\mathbb{R}^{2d}$ and in the same paper it was claimed to hold for more general lattices in phase spaces of arbitrary LCA groups. It was shown for arbitrary discrete subgroups of phase spaces of LCA groups in [8].

Proposition B.4.4. *For a discrete subgroup Λ in $G \times \widehat{G}$ the involutive Banach algebra $S_0(\Lambda, c)$ is spectrally invariant in $C^*(\Lambda, c)$.*

To get results on Gabor frames for $L^2(G)$ with windows in E from the above setup, we will need to localize certain subsets of the C^*-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$, as well as the Morita equivalence bimodule $E_\Lambda(G)$, just as explained in Section B.2. For simplicity, let Λ be cocompact in $G \times \widehat{G}$ from now on, unless otherwise specified. Then Λ° is discrete and $\mathrm{tr}_{\Lambda^\circ}$ is defined on all of $C^*(\Lambda^\circ, \overline{c})$. The localization of $C^*(\Lambda^\circ, \overline{c})$ in $\mathrm{tr}_{\Lambda^\circ}$ is induced by the inner product $(-,-)_{\Lambda^\circ}$ given by

$$(b_1, b_2)_{\Lambda^\circ} := \mathrm{tr}_{\Lambda^\circ}(b_1^* b_2).$$

Since $S_0(\Lambda^\circ, \overline{c})$ is dense in $C^*(\Lambda^\circ, \overline{c})$ and $\mathrm{tr}_{\Lambda^\circ}$ is continuous with respect to the C^*-norm, it follows that their localizations in $\mathrm{tr}_{\Lambda^\circ}$ are the same. For $b_1, b_2 \in S_0(\Lambda^\circ, \overline{c})$ we then have

$$\begin{aligned}
(b_1, b_2)_{\Lambda^\circ} &= \mathrm{tr}_{\Lambda^\circ}(b_1^* b_2) \\
&= \mathrm{tr}_{\Lambda^\circ}\Big(\sum_{\lambda^\circ \in \Lambda^\circ} \overline{b_1(\lambda^\circ)} \pi(\lambda^\circ)^* \sum_{\xi \in \Lambda^\circ} b_2(\xi)\pi(\xi)\Big) \\
&= \mathrm{tr}_{\Lambda^\circ}\Big(\sum_{\lambda^\circ \in \Lambda^\circ} \sum_{\xi \in \Lambda^\circ} \overline{b_1(\lambda^\circ)} b_2(\xi) c(\lambda^\circ, \lambda^\circ) \pi(-\lambda^\circ)\pi(\xi)\Big) \\
&= \mathrm{tr}_{\Lambda^\circ}\Big(\sum_{\lambda^\circ \in \Lambda^\circ} \sum_{\xi \in \Lambda^\circ} \overline{b_1(\lambda^\circ)} b_2(\xi) c(\lambda^\circ, \lambda^\circ) c(-\lambda^\circ, \xi) \pi(-\lambda^\circ + \xi)\Big) \\
&= \mathrm{tr}_{\Lambda^\circ}\Big(\sum_{\lambda^\circ \in \Lambda^\circ} \sum_{\xi \in \Lambda^\circ} \overline{b_1(\lambda^\circ + \xi)} b_2(\xi) c(\lambda^\circ + \xi, \lambda^\circ + \xi) c(-\lambda^\circ - \xi, \xi) \pi(-\lambda^\circ)\Big) \\
&= \sum_{\xi \in \Lambda^\circ} \overline{b_1(\xi)} b_2(\xi) c(\xi, \xi) c(-\xi, \xi) \\
&= \sum_{\xi \in \Lambda^\circ} \overline{b_1(\xi)} b_2(\xi) \\
&= \langle b_1, b_2 \rangle_{\ell^2(\Lambda^\circ)}.
\end{aligned}$$

As $S_0(\Lambda^\circ, \overline{c}) = \ell^1(\Lambda^\circ, \overline{c})$ is dense in $\ell^2(\Lambda^\circ)$, we may identify the localization $H_{C^*(\Lambda^\circ, \overline{c})}$ of $C^*(\Lambda^\circ, \overline{c})$ with $\ell^2(\Lambda^\circ)$. By [9, Proposition 3.7] we also obtain that the localization of $E_\Lambda(G)$ in $\mathrm{tr}_{\Lambda^\circ}$ is $L^2(G)$. Note that this is the same as the localization of E in tr_Λ by construction, and that there is an action of $C^*(\Lambda, c)$ on $L^2(G)$ by extending the action of $C^*(\Lambda, c)$ on $E_\Lambda(G)$.

It is slightly more tricky to localize subsets of $C^*(\Lambda, c)$. Indeed, it is not in general possible as the trace might not be defined everywhere. However, even if $C^*(\Lambda, c)$ is not unital we may localize the algebraic ideal ${}_\bullet\langle E_\Lambda(G), E_\Lambda(G)\rangle \subset C^*(\Lambda, c)$ in the trace tr_Λ. Indeed, by [9, Theorem 3.10], elements of $E_\Lambda(G)$ are such that whenever $g \in E_\Lambda(G)$ and $f \in L^2(G)$, then $\{\langle f, \pi(\lambda)g\rangle\}_{\lambda \in \Lambda} \in L^2(\Lambda)$. This is the property of being a Bessel vector, which we will discuss in more detail below. Hence for any $f, g \in E_\Lambda(G)$, we may identify ${}_\bullet\langle f, g\rangle \in C^*(\Lambda, c)$ with $(\langle f, \pi(\lambda)g\rangle)_{\lambda \in \Lambda}$ in $L^2(\Lambda)$ by doing the analogous procedure with tr_Λ as for $\mathrm{tr}_{\Lambda^\circ}$ above.

We may do the same for the matrix algebras and matrix modules considered in Section B.3. Note that ${}_\bullet[M_{n,d}(E_\Lambda(G)), M_{n,d}(E_\Lambda(G))] = M_{n,d}({}_\bullet\langle E_\Lambda(G), E_\Lambda(G)\rangle)$ in the setup of Section B.3. Adapting the setting of twisted group C^*-algebras and Heisenberg modules above to the matrix algebra setting of Section B.3 we see that we obtain the following identifications

$$\begin{aligned} H_{M_d(C^*(\Lambda^\circ,\overline{c}))} &= \ell^2(\Lambda^\circ \times \mathbb{Z}_d \times \mathbb{Z}_d) \\ H_{M_n({}_\bullet\langle E_\Lambda(G),E_\Lambda(G)\rangle)} &= L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \\ H_{M_{n,d}(E_\Lambda(G))} &= L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d). \end{aligned} \tag{B.4.4}$$

Remark B.4.5. Should Λ° be cocompact and therefore Λ discrete, we do the obvious changes. Also if both Λ and Λ° are discrete, that is, they are both lattices, then we may localize all of $M_n(C^*(\Lambda, c))$ and all of $M_d(C^*(\Lambda^\circ, \overline{c}))$.

Remark B.4.6. Note that when we do the above lifting process to obtain the identifications of (B.4.4), we may still identify Λ as being in $G \times \widehat{G}$. That is, even though after the lifting process Λ is technically inside $G \times \mathbb{Z}_n \times \mathbb{Z}_d \times \widehat{G} \times \widehat{\mathbb{Z}_n} \times \widehat{\mathbb{Z}_d}$, Λ will be identified as embedded along the units of $\mathbb{Z}_n$, $\mathbb{Z}_d$ and their duals in this product space. This will be a standing assumption throughout the rest of the paper.

We are finally ready to present the material and constructions which constitute the main results and novelty of this paper in terms of time-frequency analysis. As a first step towards this, we will consider a novel type of Gabor frames. To ease notation we will for $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ write $f_{i,j}$ instead of $f(\cdot, i, j)$, and the same for elements of $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ and $L^2(\Lambda^\circ \times \mathbb{Z}_d \times \mathbb{Z}_d)$.

Definition B.4.7. Let Λ be a closed subgroup of $G \times \widehat{G}$. For $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ we define the coefficient operator C_g by

$$\begin{aligned} C_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) &\to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \\ C_g(f) &= \{ \sum_{m \in \mathbb{Z}_d} \langle f_{k,m}, \pi(\lambda) g_{l,m} \rangle \}_{\lambda \in \Lambda, k,l \in \mathbb{Z}_n} \end{aligned}$$

and the synthesis operator D_g by

$$\begin{aligned} D_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) &\to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \\ D_g a &= \{ \sum_{m \in \mathbb{Z}_n} \int_\Lambda a_{k,m}(\lambda) \pi(\lambda) g_{m,l} \, \mathrm{d}\lambda \}_{k \in \mathbb{Z}_n, l \in \mathbb{Z}_d}. \end{aligned}$$

Furthermore, we define the *frame-like operator* $S_{g,h} = D_h C_g$, and for brevity we write S_g for $D_g C_g$. We say S_g is the *frame operator associated to* g.

We say g generates an *(n,d)-matrix Gabor frame for $L^2(G)$* with respect to Λ if $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is an isomorphism. Equivalently, the collection of time-frequency shifts

$$\mathcal{G}(g; \Lambda) := \{\pi(\lambda) g_{i,j} \mid \lambda \in \Lambda\}_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d}$$

is a frame for $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$. We then say that $\mathcal{G}(g;\Lambda)$ is an *(n,d)-matrix Gabor frame for $L^2(G)$*. Equivalently, there exists $h\in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ such that for all $f\in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ we have

$$f_{r,s}=\sum_{k\in\mathbb{Z}_d}\sum_{l\in\mathbb{Z}_n}\int_\Lambda\langle f_{r,k},\pi(\lambda)g_{l,k}\rangle\pi(\lambda)h_{l,s}\,d\lambda, \tag{B.4.5}$$

for all $r\in\mathbb{Z}_n$ and $s\in\mathbb{Z}_d$. When g and h satisfy (B.4.5) we say $\mathcal{G}(g;\Lambda)$ and $\mathcal{G}(h;\Lambda)$ are a *dual pair of (n,d)-matrix Gabor frames*. If Λ is implicit, we may also say h is a *dual (n,d)-matrix Gabor atom for g*, or just a *dual atom of g*.

Remark B.4.8. The equivalence of the definitions of (n,d)-matrix Gabor frames given in Definition B.4.7 follows by [25, Lemma 6.3.2] and Proposition B.4.12 below.

Remark B.4.9. When $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, there is always a dual (n,d)-matrix Gabor atom for g, namely $h=S_g^{-1}g$. This is known as the *canonical dual of g*.

Remark B.4.10. One can verify that $C_g=D_g^*$. Thus S_g is always a positive operator between Hilbert spaces, just as for the module frame operator in Section B.3.

For general $g\in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ the operator C_g will not be bounded. Functions g such that C_g is bounded are of interest on their own.

Definition B.4.11. If $g\in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ is so that $C_g : L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)\to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ is a bounded operator we say g is an *(n,d)-matrix Gabor Bessel vector for $L^2(G)$ with respect to* Λ, or that $\mathcal{G}(g;\Lambda)$ is an *(n,d)-matrix Gabor Bessel system for $L^2(G)$*. Equivalently, there is $D>0$ such that for all $f\in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ we have

$$\langle f,f\rangle\le D\langle C_gf,C_gf\rangle, \tag{B.4.6}$$

which may also be written as

$$\sum_{i\in\mathbb{Z}_n}\sum_{j\in\mathbb{Z}_d}\int_G|f_{i,j}(\xi)|^2d\xi\le D\sum_{k,l\in\mathbb{Z}_n}\int_\Lambda|\sum_{m\in\mathbb{Z}_d}\langle f_{k,m},\pi(\lambda)g_{l,m}\rangle|^2d\lambda.$$

The smallest $D>0$ such that the condition of (B.4.6) holds is called the *optimal Bessel bound of $\mathcal{G}(g;\Lambda)$*.

The Gabor frames of Definition B.4.7 seemingly generalize the n-multi-window d-super Gabor frames of [66]. Indeed, we obtain n-multi-window d-super Gabor frames if we only require reconstruction of $f\in L^2(G\times\mathbb{Z}_d)$ and we identify $L^2(G\times\mathbb{Z}_d)\subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by embedding along a single element of $\mathbb{Z}_n$. Hence

(B.4.5) generalizes both multi-window Gabor frames and super Gabor frames as well, setting $d = 1$ or $n = 1$, respectively. However, we will in Proposition B.4.29 show that any n-multi-window d-super Gabor frame for $L^2(G)$ with respect to Λ is an (n, d)-matrix Gabor frame for $L^2(G)$ with respect to Λ. In spite of this we continue to call them by separate names, since, as mentioned above, they are used for reconstruction in different Hilbert spaces.

The following proposition was noted in the $(n, 1)$-matrix case in [9, Theorem 3.10], and its proof in the (n, d)-matrix Gabor case goes through the same except with more bookkeeping.

Proposition B.4.12. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E_\Lambda(G))$, $C_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ is a bounded operator. In other words, every $g \in M_{n,d}(E_\Lambda(G))$ is a Bessel vector.*

For ease of notation, the localization map in $M_n(C^*(\Lambda, c))$ will be denoted by ρ_Λ, though note that we might not be able to localize all of $M_n(C^*(\Lambda, c))$. With the above definitions, the following calculation is justified for $f, g \in M_{n,d}(E_\Lambda(G)) \subset L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by Proposition B.4.12

$$\begin{aligned}
\rho_\Lambda \Phi_g(f) &= \rho_\Lambda({}_\bullet[f, g]) \\
&= \rho_\Lambda(\{\sum_{m \in \mathbb{Z}_d} \int_\Lambda \langle f_{k,m}, \pi(\lambda) g_{l,m} \rangle \pi(\lambda)\}_{k,l \in \mathbb{Z}_n}) \\
&= \{\sum_{m \in \mathbb{Z}_d} \langle f_{k,m}, \pi(\lambda) g_{l,m} \rangle\}_{\lambda \in \Lambda, k,l \in \mathbb{Z}_n} = C_g \rho_{M_{n,d}(E_\Lambda(G))}(f).
\end{aligned}$$

Hence we obtain the following result.

Lemma B.4.13. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E_\Lambda(G))$, the module coefficient operator Φ_g localizes to give the coefficient operator C_g. Equivalently, the diagram*

$$\begin{tikzcd}
M_{n,d}(E_\Lambda(G)) \arrow[r, "\Phi_g"] \arrow[d, "\rho_{M_{n,d}(E_\Lambda(G))}"'] & M_n(C^*(\Lambda, c)) \arrow[d, "\rho_\Lambda"] \\
L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \arrow[r, "C_g"] & L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\end{tikzcd}$$

commutes for all $g \in M_{n,d}(E_\Lambda(G))$.

Likewise one may obtain $C_g^* \rho_\Lambda = \rho_{M_{n,d}(E_\Lambda(G))} \Phi_g^* : M_n({}_\bullet\langle E_\Lambda(G), E_\Lambda(G)\rangle) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ for all $g \in M_{n,d}(E_\Lambda(G))$. Note that the domain might be larger, but we cannot guarantee this unless $C^*(\Lambda, c)$ is unital, that is, when Λ is discrete.

Lemma B.4.14. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E_\Lambda(G))$, the module synthesis operator Φ_g^* localizes to the Gabor synthesis operator C_g^*. Equivalently, the diagram*

$$
\begin{tikzcd}
M_n({}_\bullet\langle E_\Lambda(G), E_\Lambda(G)\rangle) \arrow[r, "\Phi_g^*"] \arrow[d, "\rho_\Lambda"'] & M_{n,d}(E_\Lambda(G)) \arrow[d, "\rho_{M_{n,d}(E_\Lambda(G))}"] \\
L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \arrow[r, "C_g^*"] & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)
\end{tikzcd}
$$

commutes for every $g \in M_{n,d}(E_\Lambda(G))$.

Combining Lemma B.4.13 and Lemma B.4.14 we then obtain

Proposition B.4.15. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For all $g, h \in M_{n,d}(E_\Lambda(G))$, $S_{g,h}\rho_{M_{n,d}(E_\Lambda(G))} = \rho_{M_{n,d}(E_\Lambda(G))}\Theta_{g,h}$, meaning the module frame-like operator $\Theta_{g,h}$ localizes to the frame-like operator $S_{g,h}$. Equivalently, the diagram*

$$
\begin{tikzcd}
M_{n,d}(E_\Lambda(G)) \arrow[r, "\Theta_{g,h}"] \arrow[d, "\rho_{M_{n,d}(E_\Lambda(G))}"'] & M_{n,d}(E_\Lambda(G)) \arrow[d, "\rho_{M_{n,d}(E_\Lambda(G))}"] \\
L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \arrow[r, "S_{g,h}"] & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)
\end{tikzcd}
$$

commutes for all $g, h \in M_{n,d}(E_\Lambda(G))$.

As $\rho_{M_{n,d}(E_\Lambda(G))} : M_{n,d}(E_\Lambda(G)) \to \rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))$ is a linear bijection intertwining both the $C^*(\Lambda, c)$-actions and the $C^*(\Lambda^\circ, \overline{c})$-actions, we see by Proposition B.4.15 that for $g \in M_{n,d}(E_\Lambda(G))$, Θ_g is invertible if and only if $S_g|_{\rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))}$ is invertible. But we also have the following result.

Lemma B.4.16. *Let $\Lambda \subset G\times\widehat{G}$ be closed and cocompact, and let $g \in M_{n,d}(E_\Lambda(G))$. Then*

$$
\begin{aligned}
S_g|_{\rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))} : \rho_{M_{n,d}(E_\Lambda(G))}&(M_{n,d}(E_\Lambda(G))) \\
&\to \rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))
\end{aligned}
$$

is invertible if and only if

$$S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$$

is invertible.

Proof. Suppose first

$$S_g|_{\rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))} : \rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G))) \\ \to \rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))$$

is invertible. Since any $g \in M_{n,d}(E_\Lambda(G))$ is a Bessel vector by Proposition B.4.12, we may extend the operator by continuity to obtain that $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible as well.

Conversely, suppose $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible. Since S_g is the continuous extension of Θ_g, it then follows by Proposition B.2.1 and inverse closedness of C^*-algebras that Θ_g is invertible, which implies $S_g|_{\rho_{M_{n,d}(E_\Lambda(G))}(M_{n,d}(E_\Lambda(G)))}$ is invertible. ∎

Remark B.4.17. From now on we will identify $M_{n,d}(E)$ and its image in the localization, and we will do this without mention.

Combining Proposition B.4.15 and Lemma B.4.16 we obtain the following important result.

Proposition B.4.18. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For $g \in M_{n,d}(E_\Lambda(G))$ we have that Θ_g is invertible if and only if S_g is invertible. In other words, g generates a module frame for $M_{n,d}(E_\Lambda(G))$ as an $M_n(C^*(\Lambda, c))$-module if and only $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$.*

We also have the following important corollary.

Corollary B.4.19. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact, and let $g, h \in M_{n,d}(E_\Lambda(G))$. Then g and h generate dual (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ if and only if $[g, h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$.*

Proof. Suppose first $g, h \in M_{n,d}(E_\Lambda(G))$ generate dual (n, d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. Then we know that for all $f \in M_{n,d}(E_\Lambda(G))$ we have

$$f = {}_\bullet[f, g]h = f[g, h]_\bullet,$$

from which we as before deduce that $[g, h]_\bullet = 1_{M_d(C^*(\Lambda^\circ,\overline{c}))}$. This extends by continuity to the identity operator on all of $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$.

Conversely, if $[g, h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, then $[g, h]_\bullet$ acts as the identity on $M_{n,d}(E_\Lambda(G))$. For any $f \in M_{n,d}(E_\Lambda(G))$ we then have

$$f = f[g, h]_\bullet = {}_\bullet[f, g]h,$$

hence (B.4.5) holds for all $f \in M_{n,d}(E_\Lambda(G))$. But this extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by continuity, which implies that g and h generate dual (n, d)-matrix Gabor frames. ∎

Amongst other results, we wish to establish a duality principle for (n, d)-matrix Gabor frames. For this we also need to treat (n, d)-matrix Gabor Riesz sequences and relate them to Definition B.3.22.

Definition B.4.20. Let $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. We say g generates an *(n, d)-matrix Gabor Riesz sequence for $L^2(G)$ with respect to* Λ, or that $\mathcal{G}(g; \Lambda)$ is an *(n, d)-matrix Gabor Riesz sequence for $L^2(G)$*, if

$$C_g C_g^* : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$$

is an isomorphism. Equivalently, there exists $h \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ such that for all $a \in L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ we have

$$a_{r,s}(\mu) = \sum_{i \in \mathbb{Z}_d} \sum_{j \in \mathbb{Z}_n} \langle \int_\Lambda a_{r,j}(\lambda) \pi(\lambda) g_{j,i} \, d\lambda, \pi(\mu) h_{s,i} \rangle \tag{B.4.7}$$

for all $r, s \in \mathbb{Z}_n$ and all $\mu \in \Lambda$. If (B.4.7) is satisfied we will say h generates a *dual (n, d)-matrix Gabor Riesz sequence of* g.

Remark B.4.21. Note that the equivalence of the definitions of (n, d)-matrix Gabor Riesz sequences in Definition B.4.20 follows by [25, Theorem 3.6.6] and Proposition B.4.12.

Remark B.4.22. (B.4.7) is equivalent to $C_g C_h^* = C_h C_g^* = \mathrm{Id}_{L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)}$.

Before treating localization of module matrix Riesz sequences and how they relate to matrix Gabor Riesz sequences, we do a necessary but justified simplification. Recall that existence of finite module matrix Riesz sequences for $M_{n,d}(E_\Lambda(G))$ with respect to $M_n(C^*(\Lambda, c))$ requires $C^*(\Lambda, c)$ to be unital by Proposition B.3.14. In the following we therefore let Λ be discrete, but not necessarily cocompact. Hence $C^*(\Lambda, c)$ is unital with a faithful trace, but $C^*(\Lambda^\circ, \overline{c})$ might not have that property. By [65, p. 251] we know that $\mathcal{G}(g; \Lambda)$ is a Bessel system with Bessel bound D if and only if $\mathcal{G}(g; \Lambda^\circ)$ is a Bessel system with Bessel bound D. Applying Proposition B.4.12 we immediately get the following from Lemma B.4.13 and Lemma B.4.14.

Proposition B.4.23. *Let $\Lambda \subset G \times \widehat{G}$ be discrete. For all $g, h \in M_{n,d}(E_\Lambda(G))$ we have $(C_h C_g^*) \circ \rho_{M_n(C^*(\Lambda,c))} = \rho_\Lambda \circ (\Phi_h \Phi_g^*)$. Equivalently, the diagram*

$$\begin{tikzcd}
M_n(C^*(\Lambda, c)) \arrow[r, "\Phi_h \Phi_g^*"] \arrow[d, "\rho_\Lambda"'] & M_n(C^*(\Lambda, c)) \arrow[d, "\rho_\Lambda"] \\
L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \arrow[r, "C_h C_g^*"] & L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\end{tikzcd}$$

commutes.

As $\rho_\Lambda : M_n(C^*(\Lambda, c)) \to \rho_\Lambda(M_n(C^*(\Lambda, c))$ is a linear bijection respecting the actions of $C^*(\Lambda, c)$, we see by Proposition B.4.23 that for $g \in M_{n,d}(E_\Lambda(G))$, $\Phi_g\Phi_g^*$ is an isomorphism if and only if $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c))}$ is an isomorphism. In analogy with Lemma B.4.16 we have the following result.

Lemma B.4.24. *Let $\Lambda \subset G \times \widehat{G}$ be discrete. For $g \in M_{n,d}(E)$ we have that*

$$(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))} : \rho_\Lambda(M_n(C^*(\Lambda, c))) \to \rho_\Lambda(M_n(C^*(\Lambda, c)))$$

is invertible if and only if

$$C_gC_g^*\colon L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$$

is invertible.

Proof. Suppose $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))} : \rho_\Lambda(M_n(C^*(\Lambda, c))) \to \rho_\Lambda(M_n(C^*(\Lambda, c)))$ is invertible. Since any $g \in M_{n,d}(E_\Lambda(G))$ is a Bessel vector by Proposition B.4.12, we may extend the operator by continuity to obtain that $C_gC_g^*\colon L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ is invertible as well.

Conversely, suppose $C_gC_g^*\colon L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n) \to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ is invertible. Since $C_gC_g^*$ is the continuous extension of $\Phi_g\Phi_g^*$, it then follows by Proposition B.2.1 and inverse closedness of C^*-algebras that $\Phi_g\Phi_g^*$ is invertible as well, which implies $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))} : \rho_\Lambda(M_n(C^*(\Lambda, c))) \to \rho_\Lambda(M_n(C^*(\Lambda, c)))$ is invertible. ∎

Remark B.4.25. From now on we will identify $M_n({}_\bullet\langle E_\Lambda(G), E_\Lambda(G)\rangle)$ (and potentially a larger domain) and its localization. The same goes for $M_d(C^*(\Lambda^\circ, \overline{c}))$.

Now the following is an immediate consequence.

Proposition B.4.26. *Let $\Lambda \subset G\times\widehat{G}$ be discrete. For $g \in M_{n,d}(E_\Lambda(G))$ we have that $\Phi_g\Phi_g^*\colon M_n(C^*(\Lambda, c)) \to M_n(C^*(\Lambda, c))$ is invertible if and only if $C_gC_g^*\colon L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n)$ is invertible. In other words, g generates a module Riesz sequence for $M_{n,d}(E_\Lambda(G))$ as an $M_n(C^*(\Lambda, c))$-module if and only if $\mathcal{G}(g;\Lambda)$ is an (n, d)-matrix Gabor Riesz sequence for $L^2(G)$.*

By the proof of Lemma B.4.24 we then have the following statement.

Corollary B.4.27. *Let $\Lambda \subset G\times\widehat{G}$ be discrete. Suppose $g, h \in M_{n,d}(E_\Lambda(G))$. Then g and h generate dual (n, d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ if and only if ${}_\bullet[g, h]$ extends to the identity operator on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.*

Proof. Suppose first that g and h generate dual (n,d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. Then for all $a \in M_n(C^*(\Lambda, c))$ we have

$$(a_{r,s}) = \Big\{ \sum_{i\in\mathbb{Z}_d} \sum_{j\in\mathbb{Z}_n} \Big\langle \int_\Lambda a_{r,j}(\lambda)\pi(\lambda)g_{j,i}\,d\lambda, \pi(\mu)h_{s,i} \Big\rangle \Big\}_{\mu\in\Lambda, r,s\in\mathbb{Z}_n},$$

which is equivalent to $a = a\,{}_\bullet[g,h]$ for all $a \in M_n(C^*(\Lambda, c))$. But the first expression extends by continuity to $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$, so ${}_\bullet[g,h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.

Conversely, suppose ${}_\bullet[g,h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Once again, for all $a \in M_n(C^*(\Lambda, c))$ we then have

$$(a_{r,s}) = \Big\{ \sum_{i\in\mathbb{Z}_d} \sum_{j\in\mathbb{Z}_n} \Big\langle \int_\Lambda a_{r,j}(\lambda)\pi(\lambda)g_{j,i}\,d\lambda, \pi(\mu)h_{s,i} \Big\rangle \Big\}_{\mu\in\Lambda, r,s\in\mathbb{Z}_n},$$

which again extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Hence g and h are dual (n,d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. ∎

Note how the above results guarantee that when $\Lambda \subset G \times \widehat{G}$ is closed and cocompact and $g \in M_{n,d}(E_\Lambda(G))$ is such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, the canonical dual frame $S_g^{-1}g \in M_{n,d}(E_\Lambda(G))$. Indeed,

$$S_g^{-1}g = \Theta_g^{-1}g = g[g,g\,]_\bullet^{-1} \in M_{n,d}(E_\Lambda(G)).$$

Likewise, for Riesz sequences there is the notion of *canonical biorthogonal atom*, see for example [25, p. 160]. Restricting to Λ discrete, it is given by $(S_g^{\Lambda^\circ})^{-1}g$, where $S_g^{\Lambda^\circ}$ is the frame operator with respect to the right hand side, that is, with respect to Λ°. We see that for all $f \in M_{n,d}(E_\Lambda(G))$

$$S_g^{\Lambda^\circ} f = (\Phi_g^{C^*(\Lambda^\circ,\overline{c})})^* \Phi_g^{C^*(\Lambda^\circ,\overline{c})} f = (\Phi_g^{C^*(\Lambda^\circ,\overline{c})})^*([g,f\,]_\bullet) = g[g,f\,]_\bullet = {}_\bullet[g,g]f.$$

Thus it follows that

$$(S_g^{\Lambda^\circ})^{-1}g = (\Theta_g^{C^*(\Lambda^\circ,\overline{c})})^{-1}g = {}_\bullet[g,g]^{-1}g \in M_{n,d}(E).$$

Hence for both matrix Gabor frames and matrix Gabor Riesz sequences with generating atom in $M_{n,d}(E_\Lambda(G))$, the canonically associated dual atoms are also in $M_{n,d}(E_\Lambda(G))$. We have the following result which shows that in the cases we are interested in, if the generating atom is regular, the canonical dual atom has the same regularity.

Proposition B.4.28. *Let $g \in M_{n,d}(S_0(G))$.*

i) If $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$ and Λ is closed and cocompact in $G\times\widehat{G}$, then the canonical dual atom is in $M_{n,d}(S_0(G))$.

ii) If $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$ and Λ is discrete, then the canonical biorthogonal atom is also in $M_{n,d}(S_0(G))$.

Proof. For the proof of i), note that the assumption that Λ is cocompact implies that Λ° is discrete, so $M_d(C^*(\Lambda^\circ,\overline{c}))$ is unital. Also $M_d(C^*(\Lambda^\circ,\overline{c}))$ is a C^*-subalgebra of $\mathbb{B}(H_{M_{n,d}(E_\Lambda(G))})$ by Proposition B.2.1. That $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$ then means that (B.3.6) is satisfied for our current setting. We deduce, as in the proof of Proposition B.3.18, that $[g,g\,]_\bullet$ is invertible in $M_d(B)$. Since $[g,g\,]_\bullet\in M_d(S_0(\Lambda^\circ,\overline{c}))$ and $M_d(S_0(\Lambda^\circ,\overline{c}))$ is spectrally invariant in $M_d(C^*(\Lambda^\circ,\overline{c}))$ by Proposition B.4.4 and [103, Theorem 2.1] the canonical dual atom is $g[g,g\,]_\bullet^{-1}\in M_{n,d}(S_0(G))$.

For the proof of ii), note that the assumption that Λ is discrete implies $M_n(C^*(\Lambda,c))$ is unital. Also, $M_n(C^*(\Lambda,c))$ is a C^*-subalgebra of $\mathbb{B}(H_{M_n(C^*(\Lambda,c))})$ by Proposition B.2.1. That $\mathcal{G}(g;\Lambda)$ determines an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$ then means that (B.3.7) is satisfied for our current setting. The middle term of (B.3.7) can be written as $(a\,{}_\bullet[\,g,g],a)_{C^*(\Lambda,c)}$, so ${}_\bullet[\,g,g]$ extends to a positive, invertible operator on $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_n)$. We deduce as in the proof of Proposition B.3.20 that ${}_\bullet[\,g,g]$ is invertible in $M_n(C^*(\Lambda,c))$. Since $g\in M_{n,d}(S_0(G))$, we have ${}_\bullet[\,g,g]\in M_n(S_0(\Lambda,c))$, and again $M_n(S_0(\Lambda,c))$ is spectrally invariant in $M_n(C^*(\Lambda,c))$. It follows that the canonical dual atom $h:={}_\bullet[\,g,g]^{-1}g$ is in $M_{n,d}(S_0(G))$. ■

When applying the module setup of Section B.3 to Gabor analysis, we take as a pre-equivalence bimodule $\mathcal{E}=S_0(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$, which is a proper subspace of $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ unless G is a finite group. Even the Hilbert C^*-module completion $E_\Lambda(G)$ is properly contained in $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ for general Λ. As such, we cannot hope to treat general atoms in $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by applying just this method. But indeed the module reformulation is made exactly to guarantee some regularity of the atoms generating frames.

From Definition B.4.7 we see that (n,d)-matrix Gabor frames generalize n-multi-window d-super Gabor frames considered in [66]. However, we now make clear how they fit into the module framework. As mentioned earlier, we obtain n-multi-window d-super Gabor frames if we only require reconstruction of $f\in L^2(G\times\mathbb{Z}_d)$ and we identify $L^2(G\times\mathbb{Z}_d)\subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by embedding it along a single element in $\mathbb{Z}_n$. The module reformulation of this is that $g,h\in M_{n,d}(E_\Lambda(G))$ are dual n-multi-window d-super Gabor frames if for all $f\in M_{n,d}(E_\Lambda(G))$ supported only one row we have

$$f={}_\bullet[\,f,g]h=f[g,h\,]_\bullet\,.$$

Likewise, it is clear that the (n, d)-matrix Gabor Riesz sequences of Definition B.4.20 generalize the n-multi-window d-super Gabor Riesz sequences also considered in [66]. Indeed, we obtain n-multi-window d-super Gabor Riesz sequences if we only require reconstruction of $a \in L^2(\Lambda \times \mathbb{Z}_n)$ and we identify $L^2(\Lambda \times \mathbb{Z}_n) \subset L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ by embedding it along a single element in the middle copy of $\mathbb{Z}_n$. The module reformulation of this is that $g, h \in M_{n,d}(E_\Lambda(G))$ are dual n-multi-window d-super Gabor Riesz sequences if for all $a \in M_n(C^*(\Lambda, c))$ supported only one row we have

$$a = {}_\bullet[\, ag, h] = a\, {}_\bullet[\, g, h].$$

We proceed to prove that all n-multi-window d-super Gabor frames for $L^2(G)$ with respect to Λ are (n, d)-matrix Gabor frames for $L^2(G)$ with respect to Λ, as well as the analogous statement for Riesz sequences. The converse statements are true as well.

Proposition B.4.29. *Let g be in $M_{n,d}(E_\Lambda(G))$.*

i) If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor frame for $L^2(G)$ with a dual window $h \in M_{n,d}(E_\Lambda(G))$, then $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$ with dual window h.

ii) If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor Riesz sequence for $L^2(G)$ with a dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$ with $h \in M_{n,d}(E_\Lambda(G))$, then $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor Riesz sequence for $L^2(G)$ with dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$.

Proof. If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor frame for $L^2(G)$ with respect to Λ with a dual window $h \in M_{n,d}(E_\Lambda(G))$, we can, as noted above, reconstruct any $f \in M_{n,d}(E_\Lambda(G))$ supported on a single row, say the k'th row. In other words, $f = f[g, h\,]_\bullet$ for all $f \in M_{n,d}(E_\Lambda(G))$ supported on the k'th row. Writing out this expression we find that

$$f_{k,i} = \sum_{j \in \mathbb{Z}_d} f_{k,j} \cdot [g, h\,]_{\bullet\, i,j},$$

for all $i \in \mathbb{Z}_d$. Here $[g, h\,]_{\bullet\, i,j} \in C^*(\Lambda^\circ, \overline{c})$ denotes the entry (i, j) in $[g, h\,]_\bullet$. Since this holds for all f supported on the k'th row we deduce that $b_{i,j} = \delta_{i,j} 1_{C^*(\Lambda^\circ, \overline{c})}$, meaning $[g, h\,]_\bullet = 1_{M_d(C^*(\Lambda^\circ, \overline{c}))}$. The actions extend to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ and we deduce that $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$ with dual window h.

The proof of (ii) is completely analogous. ∎

At last we may present core results of time-frequency analysis for (n, d)-matrix Gabor frames. Due to all the work we have just put in to properly establishing the link between module frame theory on Morita equivalence bimodules and Gabor frame theory, we will see that the statements below more or less follow from the analogous statements in Section B.3.

Proposition B.4.30 (Wexler-Raz biorthogonality relations)**.** *Let $\Lambda \subset G \times \widehat{G}$ be a closed and cocompact subgroup, and let $g, h \in M_{n,d}(E_\Lambda(G))$. Then the following are equivalent:*

i) $\mathcal{G}(g;\Lambda)$ and $\mathcal{G}(h;\Lambda)$ are dual (n,d)-matrix Gabor frames for $L^2(G)$.

ii) For all $i, j \in \mathbb{Z}_d$ we have $\sum_{k\in\mathbb{Z}_n} \left\langle g_{k,i}, \pi(\lambda^\circ)h_{k,j}\right\rangle_{\ell^2(\Lambda^\circ)} = \delta_{0,\lambda^\circ}\delta_{i,j}s(\Lambda)$.

Proof. As Λ is cocompact we know Λ° is discrete, so $M_d(C^*(\Lambda^\circ,\overline{c}))$ is unital. Knowing this, we can see that both the above statements are equivalent to the statement $[g,h]_\bullet = [h,g]_\bullet = 1_{M_d(C^*(\Lambda^\circ,\overline{c}))}$. ∎

In the previous paragraphs we did quite a lot of work to establish a connection between module Riesz sequences and Riesz sequences in Gabor analysis, a connection we have yet to use for anything significant. However, as a result, we now obtain the following statement of the duality principle in Gabor analysis and a very short proof.

Theorem B.4.31 (Duality principle)**.** *Let $\Lambda \subset G \times \widehat{G}$ be a closed cocompact subgroup, and let $g \in M_{n,d}(E_\Lambda(G))$. Then the following are equivalent.*

i) $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$.

ii) $\mathcal{G}(g;\Lambda^\circ)$ is a (d,n)-matrix Gabor Riesz sequence for $L^2(G)$.

Proof. Statement i) can be seen to be equivalent to $[g,g]_\bullet$ being invertible in $M_d(C^*(\Lambda^\circ,\overline{c}))$ by Proposition B.4.18. But statement ii) is also equivalent to $[g,g]_\bullet$ being invertible in $M_d(C^*(\Lambda^\circ,\overline{c}))$ by Proposition B.4.26. This finishes the proof. ∎

For completeness we also include the following result related to the duality principle. This is a strengthening of the corresponding result in [66].

Proposition B.4.32. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact, and let $g, h \in M_{n,d}(E_\Lambda(G))$ be such that $[g,h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. Then ${}_\bullet[g,h]$ extends to an idempotent operator from $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ onto $\overline{\operatorname{span}}\{\bigoplus_{i\in\mathbb{Z}_n}\bigoplus_{j\in\mathbb{Z}_d}\pi(\lambda^\circ)g_{i,j}\}$.*

Proof. As $[g,h]_\bullet$ extends to the identity operator, we have $[g,h]_\bullet = [h,g]_\bullet = 1_{M_d(C^*(\Lambda^\circ,\overline{c}))}$. That ${}_\bullet[g,h]$ is an idempotent then follows by Proposition B.3.13.

By Proposition B.3.10 ${}_\bullet[g,h]$ is then an idempotent from $M_{n,d}(E_\Lambda(G))$ onto $\overline{gM_d(C^*(\Lambda^\circ,\overline{c}))}$. But this passes to the localization, and the localization of $\overline{gM_d(C^*(\Lambda^\circ,\overline{c}))}$ is

$$\overline{\operatorname{span}}\{\bigoplus_{i\in\mathbb{Z}_n}\bigoplus_{j\in\mathbb{Z}_d}\pi(\lambda^\circ)g_{i,j}\}\subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d).$$

■

Given a closed and cocompact subgroup Λ, we may ask if there are restrictions on $n,d\in\mathbb{N}$ for there to possibly exist (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. Conversely, if we fix n and d, we may ask if there are restrictions on the size of the subgroup Λ, see (B.4.1), for there to possibly exist (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. When Λ is a lattice, we have the following proposition.

Proposition B.4.33. *Let $\Lambda\subset G\times\widehat{G}$ be a lattice. If there is $g\in M_{n,d}(E_\Lambda(G))$ such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, then*

$$s(\Lambda)\leq\frac{n}{d},$$

where $s(\Lambda)$ is defined as in (B.4.1).

Proof. Since Λ is discrete and cocompact, both $C^*(\Lambda,c)$ and $C^*(\Lambda^\circ,\overline{c})$ are unital. We also know by Proposition B.4.28 that the canonical dual of g is in $M_{n,d}(E_\Lambda(G))$. Hence we are in the setting of Theorem B.3.16. Since module (n,d)-matrix frames localize to (n,d)-matrix Gabor frames for the localization, and we have $\operatorname{tr}_\Lambda(1_{C^*(\Lambda,c)})=1$, and $\operatorname{tr}_{\Lambda^\circ}(1_{C^*(\Lambda^\circ,\overline{c})})=s(\Lambda)$ (since the identity on $C^*(\Lambda^\circ,\overline{c})$ is $s(\Lambda)\delta_0$, where δ_0 is the indicator function in the group identity, see for example [97]), the result is immediate by Theorem B.3.16. ■

Likewise, given a lattice Λ, we may ask if there is a relationship between the size of Λ (B.4.1) and the integers n and d such that there can possibly exist (n,d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. This is the content of the following proposition.

Proposition B.4.34. *Let $\Lambda\subset G\times\widehat{G}$ be a lattice. If $g\in M_{n,d}(E_\Lambda(G))$ is such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$, then*

$$s(\Lambda)\geq\frac{n}{d},$$

where $s(\Lambda)$ is defined as in (B.4.1).

Proof. As before we know by the conditions on Λ that both $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$ are unital, and by Proposition B.4.28 the canonical dual of g is in $M_{n,d}(E_\Lambda(G))$. Thus we are in the setting of Theorem B.3.17. Since module (n, d)-matrix Riesz sequences localize to (n, d)-matrix Gabor Riesz sequences for the localization, and $\mathrm{tr}_\Lambda(1_{C^*(\Lambda,c)}) = 1$ and $\mathrm{tr}_{\Lambda^\circ}(1_{C^*(\Lambda^\circ,\overline{c})}) = s(\Lambda)$ (once again since the identity on B is $s(\Lambda)\delta_0$), the result is immediate by Theorem B.3.17. ∎

Remark B.4.35. The two preceding propositions contain statements known as density theorems in Gabor analysis. This is due to the fact that they give conditions on the density of a lattice for there to possibly exist Gabor frames and Riesz sequences.

Lastly in this paper, we prove that whenever Λ is cocompact, there is a close relationship between the module frame bounds and the Gabor frame bounds in the localization.

Proposition B.4.36. *Let $\Lambda \subset G \times \widehat{G}$ be a closed cocompact subgroup. Then $g \in M_{n,d}(E_\Lambda(G))$ generates a module (n, d)-matrix frame for $E_\Lambda(G)$ as a $C^*(\Lambda, c)$-module with lower frame bound C and upper frame bound D if and only if $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$ with lower frame bound C and upper frame bound D.*

Proof. By Lemma B.2.4 it suffices to prove that the optimal frame bounds are equal for both the module frame and the Gabor frame. We know that the localization of a module frame for $M_{n,d}(E_\Lambda(G))$ as an $M_n(C^*(\Lambda, c))$-module becomes an (n, d)-matrix Gabor frame for $L^2(G)$ with respect to Λ. Since Λ is cocompact, we also know that if $g \in M_{n,d}(E)$ is such that $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$, then the canonical dual $S_g^{-1} g \in M_{n,d}(E_\Lambda(G))$ also. By Proposition B.4.18 we have $\rho(\Theta_g) = S_g$. From standard Hilbert space frame theory we know that the optimal upper frame bound for S_g is $\|S_g\|$, and the optimal lower frame bound for S_g is $\|S_g^{-1}\|^{-1}$, see for example Section 5.1 of [53]. We know by Proposition B.2.1 that $\|\Theta_g\| = \|\rho(\Theta_g)\| = \|S_g\|$ and $\|\Theta_g^{-1}\| = \|\rho(\Theta_g^{-1})\| = \|S_g^{-1}\|$. The result then follows by Lemma B.2.4. ∎

Remark B.4.37. A straightforward calculation will show that $\|\Theta_g\| = \|{}_\bullet[g, g]\|$. Indeed, for an A-B-equivalence bimodule E this follows by the usual isomorphism $B \cong \mathbb{K}_A(E)$.

Corollary B.4.38. *Let $g \in M_{n,d}(E_\Lambda(G))$ and let $\Lambda \subset G \times \widehat{G}$ be a lattice. If D_Λ denotes the optimal Bessel bound for $\mathcal{G}(g; \Lambda)$ and D_{Λ° denotes the optimal Bessel bound for $\mathcal{G}(g; \Lambda^\circ)$, then $D_{\Lambda^\circ} = s(\Lambda)^{-1} D_\Lambda$.*

Proof. By Proposition B.4.36 and Remark B.4.37 it follows that $D_\Lambda = \|S_g\| = \|{}_\bullet[g,g]\|$. But the analogous argument can be made to work with Λ° instead of Λ, since the important part for the setup with localization as done in this paper is that Λ or Λ° is cocompact. Hence we may obtain D_{Λ° by similar considerations. We show how to do this. First we make $S_0(G)$ into a $S_0(\Lambda^\circ, c)$-$S_0(\Lambda, \overline{c})$-pre-equivalence bimodule similarly to what we did in (B.4.2) and (B.4.3), and then complete it to obtain a $C^*(\Lambda^\circ, c)$-$C^*(\Lambda, \overline{c})$-equivalence bimodule $E_{\Lambda^\circ}(G)$. By [9, Proposition 3.17] we have $E_\Lambda(G) = E_{\Lambda^\circ}(G)$ as function spaces (hence the same for the matrix cases). Denote by ${}_\bullet[\cdot,\cdot]'$ the $M_d(C^*(\Lambda^\circ, c))$-valued inner product on $M_{d,n}(E_{\Lambda^\circ}(G))$, and by $\|\cdot\|_{\Lambda^\circ}$ the resulting norm on $E_{\Lambda^\circ}(G)$. Then [9, Proposition 3.17] tells us that for any $f \in M_{d,n}(E_\Lambda(G))$ we have $\|{}_\bullet[f,f]'\|_{\Lambda^\circ} = s(\Lambda)^{-1}\|{}_\bullet[f,f]\|$. Following the first line of this proof for D_{Λ° instead, we then obtain $D_{\Lambda^\circ} = \|{}_\bullet[g,g]'\|_{\Lambda^\circ} = s(\Lambda)^{-1}\|{}_\bullet[g,g]\| = s(\Lambda)^{-1}D_\Lambda$, which is what we wanted to show. ■

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Paper C

Spectral invariance of $*$-representations of twisted convolution algebras with applications in Gabor analysis

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Paper C

Spectral invariance of $*$-representations of twisted convolution algebras with applications in Gabor analysis

Abstract

We show spectral invariance for faithful $*$-representations for a class of twisted convolution algebras. More precisely, if G is a locally compact group with a continuous 2-cocycle c for which the corresponding Mackey group G_c is C^*-unique and symmetric, then the twisted convolution algebra $L^1(G, c)$ is spectrally invariant in $\mathbb{B}(\mathcal{H})$ for any faithful $*$-representation of $L^1(G, c)$ as bounded operators on a Hilbert space $\mathcal{H}$. As an application of this result we give a proof of the statement that if Δ is a closed cocompact subgroup of the phase space of a locally compact abelian group G', and if g is some function in the Feichtinger algebra $S_0(G')$ that generates a Gabor frame for $L^2(G')$ over Δ, then both the canonical dual atom and the canonical tight atom associated to g are also in $S_0(G')$. We do this without the use of periodization techniques from Gabor analysis.

C.1 Introduction

The primary focus of this article is the concept of spectral invariance. In short, if $\mathcal{A}$ is a $*$-subalgebra of a Banach $*$-algebra $\mathcal{B}$, then $\mathcal{A}$ is said to be *spectrally invariant* in $\mathcal{B}$ if $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ for all $a \in \mathcal{A}$, where $\sigma_{\mathcal{A}}(a)$ denotes the spectrum of the element a in the algebra $\mathcal{A}$, and likewise for $\sigma_{\mathcal{B}}(a)$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are both unital with common unit, and if $a \in \mathcal{A}$ is invertible in $\mathcal{B}$, spectral invariance of $\mathcal{A}$ in $\mathcal{B}$ tells us that $a^{-1} \in \mathcal{A}$ as well. Spectral invariance of Banach $*$-algebras in C^*-algebras is a concept that has been extensively studied and is of importance in a number of different mathematical fields. Due to the seminal paper [49] the study of spectral invariance has been linked to Wiener's lemma, and variations of this result. As fields where spectral invariance is of importance we mention the theory of noncommutative tori [28, 58], Gabor analysis and window design in the theory of Gabor frames [58], convolution operators on locally compact groups [17, 44, 45], infinite-dimensional matrices [18, 50, 73, 105], and the theory of pseudodifferential operators [54, 55, 60, 105]. This list is by no means exhaustive. For an introduction to these variations on spectral invariance and Wiener's lemma we refer the reader to [56]. Moreover, we note that in recent years quite a bit of work has been done on spectral invariance of various algebras motivated by a plethora of different problems, see e.g. [20, 57, 84, 85].

The main motivations for this article are the uses of spectral invariance in noncommutative geometry [29] and in Gabor analysis [58] as spectral invariance of twisted convolution algebras appear frequently in both. Indeed, Gabor analysis has in recent years been used as a source of examples for concepts in noncommutative geometry, see e.g. [82, 83]. The original motivation for this article was to prove an extension of the main result of [58] in the case of closed cocompact subgroups of the phase space of a locally compact abelian group without using periodization techniques from Gabor analysis. We do this in Section C.4. Our focus will not be on general $*$-subalgebras of Banach $*$-algebras. Instead we will limit ourselves to a subclass of all twisted convolution algebras of locally compact groups where the twist is implemented by a continuous 2-cocycle, see Definition C.2.2. For such a locally compact group G and a continuous 2-cocycle c, the resulting twisted convolution algebra will be denoted $L^1(G, c)$. Given a faithful $*$-representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we wish to find conditions on G and π that guarantee that $\sigma_{L^1(G,c)}(f) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(f))$ for all $f \in L^1(G, c)$, i.e. that $L^1(G, c)$ is spectrally invariant in $\mathbb{B}(\mathcal{H})$. Key to our approach to this problem is the use of the Mackey group G_c associated to the locally compact group G and the continuous 2-cocycle c, and we define this group in Section C.2.1. Note that in general $L^1(G, c)$ and $L^1(G_c)$ are not isomorphic as Banach $*$-algebras. It will be of importance to us that the convolution algebra $L^1(G_c)$ is symmetric, which in short means that the positive elements of the Banach $*$-algebra $L^1(G_c)$ have positive spectra, see Definition C.2.6. We then apply Barnes' extension [17] of a result of Hulanicki [63], stated for the reader's convenience in Proposition C.2.9, to prove prove the main result of the article.

Due to the use of the result of Hulanicki, the argument for spectral invariance will depend on a norm condition on self-adjoint elements. This norm condition may be difficult to check in practice, so we describe a class of groups for which the condition is automatically satisfied. This leads us to C^*-unique groups, introduced by Boidol [22]. In short, a locally compact group G is C^*-unique if its convolution algebra $L^1(G)$ has a unique C^*-norm. A Banach $*$-algebra admitting a faithful $*$-representation is called C^*-unique if it has a unique C^*-completion. Examples of C^*-unique groups are semidirect products of abelian groups, connected metabelian groups, as well as groups where every compactly generated subgroup is of polynomial growth [22, p. 224]. We may now state the article's main theorem.

Theorem A (Theorem C.3.1)**.** Let G be a locally compact group with a continuous 2-cocycle c.

i) If $L^1(G_c)$ is C^*-unique, so is $L^1(G,c)$.

ii) If $L^1(G_c)$ is symmetric and C^*-unique and $\pi\colon L^1(G,c) \to \mathbb{B}(\mathcal{H})$ is a faithful $*$-representation, then $f \mapsto \|\pi(f)\|_{\mathbb{B}(\mathcal{H})}$, $f \in L^1(G,c)$, is the full C^*-norm on $L^1(G,c)$, and $\sigma_{L^1(G,c)}(f) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(f))$ for all $f \in L^1(G,c)$.

Though there are some known examples of C^*-unique groups, there are very few statements in the literature concerning the C^*-uniqueness of twisted convolution algebras. This is why we go via the convolution algebra of the Mackey group G_c, and why statement i) is of independent interest. Note also that for all unital Banach $*$-algebras, being symmetric is equivalent to being spectrally invariant in the enveloping C^*-algebra, see for example [79, p. 340].

Important to our proof of the main theorem is the observation that convolution in $L^1(G_c)$ can be expressed in terms of convolution in the algebras $L^1(G,c^n)$, $n \in \mathbb{Z}$, where c^n is the 2-cocycle c raised to the nth power, see Proposition C.3.6. As an immediate consequence, $L^1(G_c)$ can be decomposed in terms of the subalgebras $L^1(G,c^n)$ as in Corollary C.3.7, and this allows us to extend a faithful $*$-representation of $L^1(G,c)$ to a faithful $*$-representation of $L^1(G_c)$ in the proof of Theorem C.3.1. This is the crucial step in the proof.

Using our main theorem we are able to give a short proof on a problem concerning regularity of canonical dual atoms and canonical tight atoms in Gabor analysis. We will do this by restating the problem in operator algebraic terms and then use Theorem C.3.1. Exploring the interplay between Gabor analysis and operator algebras has gained much popularity in recent years [9, 10, 36, 66, 72, 82, 83]. The field of Gabor analysis has its origins in the seminal paper of Gabor [48], where he claimed that it is possible to obtain basis-like representations of functions in $L^2(\mathbb{R})$ in terms of the set $\{e^{2\pi i l x}\phi(x-k) : k,l \in \mathbb{Z}\}$, where ϕ denotes a Gaussian. A central problem of the field is still to find basis-like expansions of functions in terms of time-frequency shifts of the form (C.4.1). Although most research in this field is done on one or several real variables, it is possible, due to the nature of time-frequency shifts, to study Gabor analysis on phase spaces of locally compact abelian groups [52]. Let G be a locally compact abelian group. Then its phase space is the group $G \times \widehat{G}$, where $\widehat{G}$ is its Pontryagin dual. Let $\pi(z)$ be a time-frequency shift of the form (C.4.1) for some point $z = (x, \omega) \in G \times \widehat{G}$. Ignoring normalizations on the relevant Haar measures for the time being, one may then consider a closed cocompact subgroup $\Delta \subseteq G \times \widehat{G}$ and a function $g \in L^2(G)$ and ask when a set $\mathcal{G}(g; \Delta) := (\pi(z)g)_{z \in \Delta}$ is a frame for $L^2(G)$, i.e. when there exist constants $C, D > 0$ for which

$$C\|f\|_2^2 \leq \int_{\Delta} | \langle f, \pi(z)g \rangle |^2 \, \mathrm{d}z \leq D\|f\|_2^2$$

holds for all $f \in L^2(G)$, where $\mathrm{d}z$ is the chosen Haar measure on Δ. The reason for assuming that Δ is cocompact will be explained in Remark C.4.1. In time-frequency analysis it is often also of interest that the Gabor atom g has good time-frequency decay. One way of expressing good time-frequency decay is to say that g is in Feichtinger's algebra $S_0(G)$, see (C.4.5).

Equivalent to $\mathcal{G}(g; \Delta)$ being a Gabor frame for $L^2(G)$ is the invertibility of the frame operator $S \colon L^2(G) \to L^2(G)$ associated to $\mathcal{G}(g; \Delta)$. The form of S most suitable for our purposes is given in (C.4.6). Two functions of interest are then the canonical dual atom of g, which is $S^{-1}g$, and the canonical tight atom associated to g, which is $S^{-1/2}g$. They are of importance in Gabor analysis since they allow for perfect reconstution formulas for all functions in $L^2(G)$ in terms of $g, S^{-1}g$, and $S^{-1/2}g$, as illustrated by (C.4.3) and (C.4.4). If $g \in S_0(G)$ generates a frame $\mathcal{G}(g; \Delta)$ for $L^2(G)$, a natural question in Gabor analysis is then whether $S^{-1}g$ and $S^{-1/2}g$ are in $S_0(G)$ also. This leads us to our second main result.

Theorem B (Theorem C.4.2)**.** Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup, and suppose $g \in S_0(G)$ is such that $\mathcal{G}(g; \Delta)$ is a Gabor frame for $L^2(G)$. Then $S^{-1}g, S^{-1/2}g \in S_0(G)$ as well.

We note that the above result was proved in the case of separable lattices in $\mathbb{R}^{2d}$, and claimed to hold more generally for lattices in phase spaces of locally compact abelian groups, in the celebrated paper [58]. Though it is somewhat technical to prove Theorem C.3.1, our approach to Theorem C.4.2 presented below makes it simple to prove the extension of the main result of [58] for general closed cocompact subgroups rather than just lattices. It may be possible to adapt the proof from [58] to this setting as well, but we offer a proof which makes no use of periodization techniques available in the setting of Gabor analysis.

As mentioned, to prove Theorem C.4.2 we will restate the problem in operator algebraic language. For a Gabor frame $\mathcal{G}(g; \Delta)$ with $g \in S_0(G)$, the frame operator S can be rephrased in terms of a faithful (right) $*$-representation of the Banach $*$-algebra $\ell^1(\Delta^\circ, \overline{c})$, where Δ° is the adjoint lattice of Δ and c is the Heisenberg

2-cocycle, see (C.4.2) and (C.4.7). As we explain in the proof of Theorem C.4.2, any locally compact abelian group is C^*-unique and for any continuous 2-cocycle on it the associated Mackey group G_c is also C^*-unique. In addition, $L^1(G_c)$ will in this case be symmetric. Hence we may apply Theorem C.3.1 to obtain our second main result.

The strength in avoiding the periodization arguments of [58] and proving spectral invariance of a twisted L^1-algebra in terms of symmetry and C^*-uniqueness lies in the fact that the approach might be adaptable to other representations of groups where one has an analogous space to the Feichtinger algebra and an L^1-algebra acting on it, such as in the case of certain (projective) coorbit spaces [24, 40, 41].

The article is organized as follows. Section C.2 is dedicated to revising some results on how we obtain twisted convolution algebras and C^*-algebras through projective unitary representations of locally compact groups, as well as some results on symmetric convolution algebras and C^*-unique groups. Our first main result is Theorem C.3.1, and most of Section C.3 is dedicated to the proof of this theorem, though some results are of independent interest. In Section C.4 we rephrase a problem in Gabor analysis in terms of a faithful $*$-representation of a twisted convolution algebra, and apply Theorem C.3.1 to obtain a simple proof of the main result of this section, Theorem C.4.2.

C.2 Twisted convolution algebras

C.2.1 Projective unitary representations and twisted convolution algebras

We dedicate this section to explaining how we obtain twisted convolution algebras from projective unitary representations of locally compact groups.

Definition C.2.1. Let G be a locally compact group and let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on the Hilbert space $\mathcal{H}$ equipped with the strong topology. A *projective unitary representation* of G is a continuous group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$ satisfying

$$\pi(e) = \mathrm{Id}_{\mathcal{H}}, \quad \pi(x_1)\pi(x_2) = c(x_1, x_2)\pi(x_1 x_2)$$

where $x_1, x_2 \in G$, e is the unit of G, and $c : G \times G \to \mathbb{T}$ is some continuous map.

The map $c\colon G \times G \to \mathbb{T}$ associated to the projective group representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ in Definition C.2.1 has some important properties. Using associativity of π we realize that

$$c(x_1, x_2)c(x_1 x_2, x_3) = c(x_1, x_2 x_3)c(x_2, x_3), \quad x_1, x_2, x_3 \in G. \tag{C.2.1}$$

Moreover, $\pi(e) = \mathrm{Id}_{\mathcal{H}}$ forces

$$c(x, e) = c(e, x) = 1, \quad x \in G. \tag{C.2.2}$$

Definition C.2.2. Let G be a locally compact group. A continuous map $c\colon G \times G \to \mathbb{T}$ satisfying (C.2.1) and (C.2.2) is called a *continuous* 2*-cocycle for* G.

Continuous 2-cocycles are part of a cohomology theory for groups, though this is not something we will have much need for in the sequel. The following result lists some elementary results for 2-cocycles of groups.

Lemma C.2.3. *For a continuous* 2*-cocycle* c *for a locally compact group* G *we have*

i) For any $n \in \mathbb{Z}$*, the map* $c^n : G \times G \to \mathbb{T}$ *given by*

$$c^n(x_1, x_2) = (c(x_1, x_2))^n, \quad x_1, x_2 \in G,$$

is also a continuous 2*-cocycle.*

ii) For all $x \in G$ *we have*
$$c(x, x^{-1}) = c(x^{-1}, x).$$

iii) For all $x, y \in G$ *we have*

$$c(y, y^{-1})\overline{c(y^{-1}, x)} = c(y, y^{-1}x). \tag{C.2.3}$$

Proof. Statement i) is obvious. Statement ii) follows by setting $x_1 = x_3 = x$ and $x_2 = x^{-1}$ in (C.2.1) and then using (C.2.2). For iii) we may equivalently show that

$$c(y, y^{-1})c(yy^{-1}, x) = c(y, y^{-1}x)c(y^{-1}, x)$$

since $c(yy^{-1}, x) = 1$. Setting $x_1 = y$, $x_2 = y^{-1}$ and $x_3 = x$ in (C.2.1) and then using (C.2.2) we obtain the result. ∎

Given a locally compact group G and a continuous 2-cocycle c for G, there is always a distinguished c-projective unitary representation of G, namely the *c-twisted left regular representation*. It is the map $L^c : G \to \mathcal{U}(L^2(G))$ given by

$$L^c_y f(x) = c(y, y^{-1}x)f(y^{-1}x), \quad x, y \in G,\ f \in L^2(G).$$

If $c = 1$ we drop the c from the notation and just write L_y for $y \in G$.

Given a locally compact group G and a continuous 2-cocycle c, we can construct an associated group G_c known as the *Mackey group*. It has appeared in the literature numerous times before. As a topological space, G_c is just $G \times \mathbb{T}$ with the product topology. The binary operation is given by

$$(x_1, \tau_1)(x_2, \tau_2) = (x_1 x_2, \tau_1 \tau_2 \overline{c(x_1, x_2)}). \tag{C.2.4}$$

The identity is given by $(e, 1)$, and the inverse of an element $(x, \tau) \in G_c$ is given by $(x, \tau)^{-1} = (x^{-1}, \overline{\tau} c(x^{-1}, x))$. G_c is a locally compact group, and its left Haar measure is the product measure. Hence its modular function may be identified with the modular function of G. We normalize the measure of $\mathbb{T}$ to 1.

The usefulness of the Mackey group for us is in the fact that c-projective unitary representations of G induce unitary representations of G_c. Explicitly, given a c-projective unitary representation of G, say $\pi \colon G \to \mathcal{U}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we obtain a unitary representation $\pi_c \colon G_c \to \mathcal{U}(\mathcal{H})$ by setting

$$\pi_c(x, \tau) = \overline{\tau}\pi(x) \tag{C.2.5}$$

for $(x, \tau) \in G_c$.

We proceed to introduce twisted convolution algebras of these groups and show how we may complete them to C^*-algebras. For a locally compact group G with modular function m, we consider the space of measurable and integrable functions $L^1(G)$. For a continuous 2-cocycle c for G we define c-twisted convolution on $L^1(G)$ by

$$f_1 \natural_c f_2(x) = \int_G f_1(y) f_2(y^{-1}x) c(y, y^{-1}x)\, \mathrm{d}y,$$

for $f_1, f_2 \in L^1(G)$, where $\mathrm{d}y$ is the Haar measure on G. Should $f_2 \in L^p(G)$ and $p \in [1, \infty]$ we will use the same notation. We also define the c-twisted involution

$$f^{*_c}(x) = m(x^{-1})\overline{c(x, x^{-1}) f(x^{-1})}$$

for $f \in L^1(G)$. We denote the resulting $*$-algebra by $L^1(G, c)$. It becomes a Banach $*$-algebra when equipped with the usual L^1-norm.

Any c-projective unitary representation $\pi \colon G \to \mathcal{U}(\mathcal{H})$ now induces a nondegenerate $*$-representation $\pi \colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ by setting

$$\pi(f)\eta = \int_G f(x)\pi(x)\eta\, \mathrm{d}x, \qquad f \in L^1(G, c), \eta \in \mathcal{H},$$

where we interpret the integral weakly in $\mathcal{H}$. Note that $\|\pi(f)\| \leq \|f\|_{L^1(G)}$. If the integrated representation π is faithful this gives us a way of completing $L^1(G, c)$ to a C^*-algebra, namely for any $f \in L^1(G)$ we set $\|f\| := \|\pi(f)\|_{\mathbb{B}(\mathcal{H})}$. The integrated representation of the c-twisted left regular representation will be denoted by $f \mapsto L^c_f$. The following result, which will be important for us in the proof of Theorem C.3.1, is a special case of [77, Satz 6].

Proposition C.2.4. *Let G be an amenable locally compact group with a continuous 2-cocycle c. Then $f \mapsto \|L_f^c\|_{\mathbb{B}(L^2(G))}$ is the maximal C^*-norm on $L^1(G, c)$.*

Instead of twisting the convolution algebra of the locally compact group G by a continuous 2-cocycle c, we could first "twist" the group G by c to obtain the associated Mackey group G_c, and then consider the associated convolution algebra $L^1(G_c)$ with usual (untwisted) convolution and involution. We will have much use for this in the sequel. Any c-projective unitary representation of G induces a unitary representation π_c of G_c by (C.2.5), which in turn induces a nondegenerate $*$-representation π_c of $L^1(G_c)$. Note however that π_c is in general not a faithful $*$-representation of $L^1(G_c)$ even if π_c is a faithful unitary representation of G_c. Indeed, let $f \in L^1(G) \setminus \{0\}$ and define $F \in L^1(G_c)$ by $F(x, \tau) = \overline{\tau} f(x)$. Then

$$\begin{aligned}
\pi_c(F)\eta &= \int_G \int_{\mathbb{T}} F(x, \tau)\pi_c(x, \tau)\eta \,\mathrm{d}\tau \,\mathrm{d}x \\
&= \int_G \int_{\mathbb{T}} \overline{\tau} f(x)\overline{\tau}\pi(x)\eta \,\mathrm{d}\tau \,\mathrm{d}x = \int_G \int_{\mathbb{T}} \overline{\tau}^2 f(x)\pi(x)\eta \,\mathrm{d}\tau \,\mathrm{d}x = 0,
\end{aligned}$$

for all $\eta \in \mathcal{H}$, even though F is not the zero function.

Remark C.2.5. Note that if G is nondiscrete we may always extend a representation $\pi\colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ to its minimal unitization $L^1(G, c)^\sim$ by forcing the induced representation, also denoted π, to satisfy $\pi(1_{L^1(G,c)^\sim}) = \mathrm{Id}_{\mathcal{H}}$. If $L^1(G, c)$ is already unital it will always be implied that $\pi(1_{L^1(G,c)}) = \mathrm{Id}_{\mathcal{H}}$.

C.2.2 Symmetric group algebras and C^*-uniqueness

Two concepts that will be of great importance when proving our main result Theorem C.3.1 are that of symmetric convolution algebras and C^*-uniqueness.

In the sequel, if $\mathcal{A}$ is a $*$-algebra and $a \in \mathcal{A}$, we let $\sigma_{\mathcal{A}}(a)$ denote the spectrum of a in the algebra $\mathcal{A}$.

Definition C.2.6. A Banach $*$-algebra $\mathcal{A}$ is called *symmetric* if for all $a \in \mathcal{A}$ we have $\sigma_{\mathcal{A}}(a^*a) \subseteq [0, \infty)$. We will say that a locally compact group G is *symmetric* if $L^1(G)$ is a symmetric Banach $*$-algebra.

Remark C.2.7. By the famous Shirali-Ford theorem a Banach $*$-algebra $\mathcal{A}$ is symmetric if and only if it is *hermitian* (i.e. $a = a^* \in \mathcal{A}$ implies $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$).

Note that C^*-algebras are symmetric [86, Theorem 2.2.5]. Moreover, if $\mathcal{A}$ is a nonunital Banach $*$-algebra, $\mathcal{A}$ is symmetric if and only if its minimal unitization $\tilde{\mathcal{A}}$ is symmetric [94, Theorem (4.7.9)].

Locally compact groups G yielding symmetric (untwisted) convolution algebras $L^1(G)$ are of importance due to the following result shown in [58, Theorem 2.8] (though noted several times earlier). Note that we can omit the condition that G should be amenable, as it was recently shown that if $L^1(G)$ is symmetric, then G is amenable [101, Corollary 4.8].

Proposition C.2.8. *If G is a locally compact group the following statements are equivalent.*

i) $L^1(G)$ *is symmetric.*

ii) $\sigma_{L^1(G)}(f) = \sigma_{\mathbb{B}(L^2(G))}(L_f)$ *for all self-adjoint* $f \in L^1(G)$.

Note that for a locally compact group G and a continuous 2-cocycle c for G, the Mackey group G_c is amenable if and only if G is amenable [90, Proposition 1.13].

Like in [58], the proofs of some crucial steps will rely on the following result of Hulanicki, see [63], and the extension by Barnes, see [17]. For $a \in \mathcal{A}$, let $\rho_{\mathcal{A}}(a)$ denote the spectral radius of a in $\mathcal{A}$.

Proposition C.2.9. *Let $\mathcal{A}$ be a $*$-subalgebra of a Banach $*$-algebra $\mathcal{B}$, and suppose there is a faithful $*$-representation $\pi\colon \mathcal{B} \to \mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. If $\mathcal{B}$ is unital with unit $1_{\mathcal{B}}$ we require $\pi(1_{\mathcal{B}}) = \mathrm{Id}_{\mathcal{H}}$. If for all self-adjoint $a \in \mathcal{A}$ we have $\|\pi(a)\|_{\mathbb{B}(\mathcal{H})} = \rho_{\mathcal{A}}(a)$, then*

$$\sigma_{\mathcal{B}}(a') = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(a'))$$

for all $a' \in \mathcal{A}$.

Recall that for an element b in a Banach $*$-algebra $\mathcal{B}$, the spectral radius can be expressed as $\rho_{\mathcal{B}}(b) = \lim_{n\to\infty} \|b^n\|_{\mathcal{B}}^{1/n}$ [86, Theorem 1.2.7].

Locally compact groups yielding symmetric convolution algebras have been studied quite extensively. As examples we mention that all locally compact compactly generated groups of polynomial growth yield symmetric convolution algebras [80], as do all compact extensions of nilpotent groups [81, p. 191]. The latter fact will come into play in Section C.4. Note also that if a group G is locally compact and compactly generated of polynomial growth, so is its Mackey extension G_c for any continuous 2-cocycle c.

To deduce spectral invariance of $L^1(G, c)$ in Theorem C.3.1 the strategy in Section C.3 will be to use Proposition C.2.9. In order to do this, we will need a certain norm equality in order for the conditions of Proposition C.2.9 to be satisfied. We will restrict to a class of groups for which this is automatic.

Definition C.2.10. Let $\mathcal{B}$ be a Banach $*$-algebra admitting a faithful $*$-representation. We say $\mathcal{B}$ is C^**-unique* if the maximal C^*-norm $\|\cdot\|_*$ given by

$$\|b\|_* = \sup\{\|\pi(b)\|_{\mathbb{B}(\mathcal{H})} \mid \pi : \mathcal{B} \to \mathbb{B}(\mathcal{H}) \text{ is a } *\text{-representation of } \mathcal{B}\}$$

for $b \in \mathcal{B}$, is the unique C^*-norm on $\mathcal{B}$.

We say a locally compact group G is C^*-*unique* if $L^1(G)$ is C^*-unique as a Banach $*$-algebra.

A C^*-unique group G is amenable, since C^*-uniqueness in particular implies that the full and reduced group C^*-algebras of G coincide. The converse is not true [22, 92]. There are some known examples of C^*-unique groups. As examples we mention semidirect products of abelian groups, connected metabelian groups, as well as groups where every compactly generated subgroup is of polynomial growth [22, p. 224]. The latter will also come into play in Section C.4. Moreover, note that if G is a group where every compactly generated subgroup is of polynomial growth, so is its Mackey group G_c for any continuous 2-cocycle c.

C.3 Spectral invariance of twisted convolution algebras

All results below will be stated and proved in terms of left representations, i.e. left projective unitary representations of groups and left $*$-representations of the twisted convolution algebras we treated in Section C.2. This is only due to left representations being more common in the literature. We note that with proper restatements all results in this section also apply to the case of right representations. Indeed we shall need to consider right representations in Section C.4.

We start by presenting the main theorem of the article. The rest of the section will mostly be dedicated to its proof. Note that some of the results presented leading up to the proof of the main theorem were proved in [35] in a more abstract way. However, due to the specific setting of our results, considering $\mathbb{T}$-valued 2-cocycles directly (as opposed to the more general setting of [35]), we believe the clarity offered by the explicit calculations using the Fourier transform below makes the constructions much clearer. Indeed, applying the results of [35] to our setting would lead us to derive many of the same formulas as we do below.

Theorem C.3.1. *Let G be a locally compact group with a continuous 2-cocycle c.*

i) If $L^1(G_c)$ is C^*-unique, so is $L^1(G,c)$.

ii) If $L^1(G_c)$ is symmetric and C^*-unique and $\pi\colon L^1(G,c) \to \mathbb{B}(\mathcal{H})$ is a faithful $*$-representation, then $f \mapsto \|\pi(f)\|_{\mathbb{B}(\mathcal{H})}$, $f \in L^1(G,c)$, is the full C^*-norm on $L^1(G,c)$, and $\sigma_{L^1(G,c)}(f) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(f))$ for all $f \in L^1(G,c)$.

Remark C.3.2. Theorem C.3.1 also gives us sufficient conditions for $L^1(G,c)$ to be symmetric. Namely, from statement ii) in Theorem C.3.1 we see that if $L^1(G_c)$ is C^*-unique and symmetric, then $L^1(G,c)$ is spectrally invariant in its (unique) C^*-completion. Therefore it is spectrally invariant in its enveloping C^*-algebra, which we know happens if and only if $L^1(G,c)$ (and therefore also its minimal unitization if G is nondiscrete) is symmetric [79, p. 340].

Remark C.3.3. In general G_c is less tractable than the group G, so at first glance imposing requirements of symmetry and C^*-uniqueness on G_c in Theorem C.3.1 might not seem like an improvement. However, untwisted convolution algebras are more tractable than twisted ones, and have been studied to a much larger extent in the literature. In addition, as mentioned in Section C.2, some classes of symmetric groups and C^*-unique groups are closed under compact extensions, meaning, for groups G in those classes, we can impose symmetry and C^*-uniqueness on G itself rather than on G_c.

We begin by embedding $L^p(G)$ as a subspace of $L^p(G_c)$ for $1 \leq p \leq \infty$. Define the map $j\colon L^p(G) \to L^p(G_c)$ by

$$j(f)(x,\tau) = \tau f(x). \tag{C.3.1}$$

Lemma C.3.4. *Let G be a locally compact group and let c be a continuous 2-cocycle for G. Then j defined by (C.3.1) is an isometric $*$-homomorphism from $L^1(G,c)$ to $L^1(G_c)$, and an isometry from $L^p(G)$ to $L^p(G_c)$ for $1 < p \leq \infty$. Moreover, if $f \in L^1(G,c)$ and $g \in L^p(G)$, we have*

$$j(f \natural_c g) = j(f) * j(g) \tag{C.3.2}$$

for $p \in [1,\infty]$. Here $*$ denotes the usual (untwisted) convolution product.

Proof. We begin by verifying that j is an isometry for $1 \leq p < \infty$. Let $f \in L^p(G)$. Then

$$\begin{aligned}\|j(f)\|_{L^p(G_c)}^p &= \int_{G_c} |j(f)(x,\tau)|^p \,\mathrm{d}\tau\,\mathrm{d}x = \int_G \int_{\mathbb{T}} |\tau f(x)|^p \,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_G |f(x)|^p \,\mathrm{d}x = \|f\|_{L^p(G)}^p.\end{aligned}$$

Likewise, for $p = \infty$ and $f \in L^\infty(G)$ we get

$$\|j(f)\|_{L^\infty(G_c)} = \sup_{(x,\tau)\in G_c} |j(f)(x,\tau)| = \sup_{(x,\tau)\in G_c} |\tau f(x)| = \sup_{x\in G} |f(x)| = \|f\|_{L^\infty(G)}.$$

We now verify that j is a $*$-homomorphism when $p = 1$. Let $f_1, f_2 \in L^1(G,c)$.

Then for all $(x, \tau) \in G_c$ we have

$$\begin{aligned}
(j(f_1) * j(f_2))(x,\tau) &= \int_{G_c} j(f_1)(y,\xi) j(f_2)((y,\xi)^{-1}(x,\tau)) \,\mathrm{d}\xi \,\mathrm{d}y \\
&= \int_G \int_{\mathbb{T}} j(f_1)(y,\xi) j(f_2)(y^{-1}x, \overline{\xi} c(y,y^{-1}) \tau \overline{c(y^{-1},x)}) \,\mathrm{d}\xi \,\mathrm{d}y \\
&= \int_G \int_{\mathbb{T}} \xi f_1(y) \overline{\xi} \tau c(y,y^{-1}) \overline{c(y^{-1},x)} f_2(y^{-1}x) \,\mathrm{d}\xi \,\mathrm{d}y \\
&= \tau \int_G f_1(y) f_2(y^{-1}x) c(y,y^{-1}) \overline{c(y^{-1},x)} \,\mathrm{d}y \\
&= \tau \int_G f_1(y) f_2(y^{-1}x) c(y,y^{-1}x) \,\mathrm{d}y \\
&= j(f_1 \natural_c f_2)(x,\tau),
\end{aligned}$$

where we in the second to last line used (C.2.3). Doing the same calculation with $f_2 \in L^p(G)$ shows that (C.3.2) holds.

It then remains to show that j respects the involutions. For $f \in L^1(G, c)$ and all $(x, \tau) \in G_c$, we have

$$\begin{aligned}
j(f)^*(x,\tau) &= m(x^{-1}) \overline{j(f)((x,\tau)^{-1})} = m(x^{-1}) \overline{j(f)(x^{-1}, \overline{\tau} c(x,x^{-1}))} \\
&= m(x^{-1}) \overline{\overline{\tau} c(x,x^{-1}) f(x^{-1})} = m(x^{-1}) \tau \overline{c(x^{-1},x) f(x^{-1})} \\
&= \tau f^{*_c}(x) = j(f^{*_c})(x,\tau).
\end{aligned}$$

Hence $j(f)^* = j(f^{*_c})$ for all $f \in L^1(G, c)$. This finishes the proof. ■

Since j is an isometry and $L^p(G)$ is complete for all $p \in [1, \infty]$, we get that $j(L^p(G))$ is a closed subspace of $L^p(G_c)$. We may actually obtain a quite explicit description of this subspace. To do this, we expand functions in $L^p(G_c)$ as Fourier series with respect to their second argument, that is, in the $\mathbb{T}$-variable. Since the measure on G_c is the product measure coming from G and $\mathbb{T}$, we have that for any $F \in L^p(G_c)$, $1 \leq p \leq \infty$, and any $x \in G$, the function $\tau \mapsto F(x, \tau)$ is in $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$. Therefore the Fourier coefficients

$$F_k(x) = \int_{\mathbb{T}} F(x,\tau) \overline{\tau}^k \,\mathrm{d}\tau \tag{C.3.3}$$

are well defined, and the resulting Fourier series

$$F(x,\tau) = \sum_{k \in \mathbb{Z}} F_k(x) \tau^k$$

converges in $L^p(\mathbb{T})$ for $1 < p < \infty$. The following lemma then describes the range of j.

Lemma C.3.5. *Let G be a locally compact group and let c be a continuous 2-cocycle for G. For $1 \leq p \leq \infty$ we have $j(L^p(G)) = \{F \in L^p(G_c) \mid F_k = 0$ for $k \neq 1\}$.*

Proof. The inclusion $j(L^p(G)) \subseteq \{F \in L^p(G_c) \mid F_k = 0 \quad \text{for } k \neq 1\}$ is immediate by (C.3.1) and (C.3.3). For the converse containment note that if $F \in \{F \in L^p(G_c) \mid F_k = 0 \text{ for } k \neq 1\}$, then for all $(x, \tau) \in G_c$ we have $F(x, \tau) = \tau F_1(x)$. Since the measure on G_c is the product measure we must have that $x \mapsto F_1(x)$ is in $L^p(G)$. Hence $F = j(F_1)$, which proves the lemma. ■

To simplify notation in the sequel, denote by $L^1(G_c)_n$ the set

$$L^1(G_c)_n := \{F \in L^1(G_c) \mid F_k = 0 \text{ for } k \neq n\}.$$

It is then immediate that $L^1(G_c)_1 = j(L^1(G, c))$. We also have the following result.

Proposition C.3.6. *Let G be a locally compact group with a continuous 2-cocycle c, let $F \in L^1(G_c)$ and let $H \in L^p(G_c)$ for some $1 \leq p < \infty$. Then*

$$(F * H)(x, \tau) = \sum_{n \in \mathbb{Z}} (F_n \natural_{c^n} H_n)(x) \tau^n, \tag{C.3.4}$$

for all $(x, \tau) \in G_c$, where c^n is c to the nth power as in Lemma C.2.3. Moreover,

$$(F_n)^{*c^n} = (F^*)_n \tag{C.3.5}$$

for all $n \in \mathbb{Z}$.

Proof. Below we will make use of the Fourier expansions $F(y, \xi) = \sum_{m \in \mathbb{Z}} F_m(y) \xi^m$ and $H(y, \xi) = \sum_{m \in \mathbb{Z}} H_m(y) \xi^m$, where F_m and H_m are obtained through (C.3.3). We will assume both F and H have finite expansions of the form (C.3.3). This is sufficient since trigonometric polynomials are dense in $L^p(\mathbb{T})$, $1 \leq p < \infty$. The extension to the full statement follows by a standard density argument.

Since $\{\xi^m\}_{m \in \mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{T})$, we have for all $(x, \tau) \in G_c$

$$\begin{aligned}
(F * H)(x, \tau) &= \int_G \int_{\mathbb{T}} F(y, \xi) H((y, \xi)^{-1}(x, \tau)) \, \mathrm{d}\xi \, \mathrm{d}y \\
&= \int_G \int_{\mathbb{T}} F(y, \xi) H(y^{-1}x, \overline{\xi} c(y^{-1}, y) \tau \overline{c(y^{-1}, x)}) \, \mathrm{d}\xi \, \mathrm{d}y \\
&= \int_G \int_{\mathbb{T}} \sum_{m \in \mathbb{Z}} F_m(y) \xi^m \cdot \sum_{n \in \mathbb{Z}} H_n(y^{-1}x) \overline{\xi}^n \tau^n (c(y, y^{-1}x))^n \, \mathrm{d}\xi \, \mathrm{d}y \\
&= \int_G \sum_{n \in \mathbb{Z}} F_n(y) H_n(y^{-1}x) c^n(y, y^{-1}x) \tau^n \, \mathrm{d}y \\
&= \sum_{n \in \mathbb{Z}} \left(\int_G F_n(y) H_n(y^{-1}x) c^n(y, y^{-1}x) \, \mathrm{d}y \right) \tau^n \\
&= \sum_{n \in \mathbb{Z}} (F_n \natural_{c^n} H_n)(x) \tau^n
\end{aligned}$$

where, at the third equality, we have used (C.2.3). This establishes (C.3.4).

For any $F \in L^1(G_c)$ we also have

$$\begin{aligned}
(F^*)_n(x) &= \int_{\mathbb{T}} F^*(x,\tau)\overline{\tau}^n \, \mathrm{d}\tau \\
&= \int_{\mathbb{T}} m(x^{-1})\overline{F(x^{-1}, \overline{\tau} c(x^{-1},x))}\overline{\tau}^n \, \mathrm{d}\tau \\
&= m(x^{-1}) \int_{\mathbb{T}} \overline{F(x^{-1}, \tau c(x^{-1},x))\tau^n} \, \mathrm{d}\tau \\
&= m(x^{-1}) \int_{\mathbb{T}} \overline{F(x^{-1}, \tau)\tau^n}\,\overline{c(x^{-1},x)^n} \, \mathrm{d}\tau \\
&= m(x^{-1})\overline{c(x^{-1},x)^n}\,\overline{\int_{\mathbb{T}} F(x^{-1},\tau)\overline{\tau}^n \, \mathrm{d}\tau} \\
&= m(x^{-1})\overline{c^n(x^{-1},x)F_n(x^{-1})} \\
&= (F_n)^{*_{c^n}}(x),
\end{aligned}$$

for all $x \in G$, which establishes (C.3.5). ■

The following corollary is then immediate.

Corollary C.3.7. *Let G be a locally compact group and let c be a continuous 2-cocycle for G. Then $L^1(G_c)_n \cong L^1(G, c^n)$ as Banach $*$-algebras.*

As a final preparation before proving Theorem C.3.1, we need the following lemma.

Lemma C.3.8. *Let G be a locally compact group and let c be a continuous 2-cocycle for G. For $f \in L^1(G,c)$ we then have*

$$\rho_{L^1(G,c)}(f) = \rho_{L^1(G_c)}(j(f)).$$

If, in addition, f is self-adjoint we get

$$\rho_{\mathbb{B}(L^2(G))}(L_f^c) = \rho_{\mathbb{B}(L^2(G_c))}(L_{j(f)}). \tag{C.3.6}$$

Proof. Since $j \colon L^1(G,c) \to L^1(G_c)$ is an isometric $*$-homomorphism we have

$$\rho_{L^1(G,c)}(f) = \lim_{n\to\infty} \|f^n\|_{L^1(G,c)}^{1/n} = \lim_{n\to\infty} \|j(f)^n\|_{L^1(G_c)}^{1/n} = \rho_{L^1(G_c)}(j(f)),$$

which proves the first statement.

For the second statement, let $f \in L^1(G,c)$ be self-adjoint. Since f is self-adjoint and L_f^c and $L_{j(f)}$ realize f and $j(f)$ as bounded operators on Hilbert spaces, i.e. as elements of a C^*-algebra, we have

$$\rho_{\mathbb{B}(L^2(G))}(L_f^c) = \|L_f^c\|_{\mathbb{B}(L^2(G))} \quad \text{and} \quad \rho_{\mathbb{B}(L^2(G_c))}(L_{j(f)}) = \|L_{j(f)}\|_{\mathbb{B}(L^2(G_c))},$$

see [86, Theorem 2.1.1]. Thus it suffices to show $\|L_f^c\|_{\mathbb{B}(L^2(G))} = \|L_{j(f)}\|_{\mathbb{B}(L^2(G_c))}$. To do this, note first that by Lemma C.3.4

$$L_{j(f)}j(g) = j(f) * j(g) = j(f \natural_c g) = j(L_f^c g)$$

for any $g \in L^2(G)$. Moreover, by Proposition C.3.6 we see that $L_{j(f)}|_{j(L^2(G))^\perp} = 0$. Since $j : L^2(G) \to L^2(G_c)$ is an isometry it then follows that $\|L_f^c\|_{\mathbb{B}(L^2(G))} = \|L_{j(f)}\|_{\mathbb{B}(L^2(G_c))}$, which finishes the proof. ∎

We are finally ready to prove Theorem C.3.1.

Proof of Theorem C.3.1. We begin by proving i). Let $\pi \colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ be a faithful $*$-representation. As G_c is assumed to be C^*-unique, G_c is in particular amenable, so it follows that G is also amenable. Then Proposition C.2.4 gives that $f \mapsto \|L_f^c\|_{\mathbb{B}(L^2(G))}$, $f \in L^1(G, c)$, is the maximal C^*-norm on $L^1(G, c)$. Hence it suffices to prove that $\|\pi(f)\|_{\mathbb{B}(\mathcal{H})} = \|L_f^c\|_{\mathbb{B}(L^2(G))}$ for all $f \in L^1(G, c)$. To do this, we will first extend π to a faithful $*$-representation of $L^1(G_c)$. The obvious attempt at a $*$-representation of $L^1(G_c)$, namely the integrated representation of $\pi_c \colon G_c \to \mathcal{U}(\mathcal{H})$ as in (C.2.5), is in general not faithful as noted at the end of Section C.2.1. The construction of the desired faithful $*$-representation $\tilde{\pi}$ of $L^1(G_c)$ is therefore more involved.

For all $n \in \mathbb{Z}$ we know by Corollary C.3.7 that $L^1(G_c)_n \cong L^1(G, c^n)$ as Banach $*$-algebras, and in the sequel we make this identification to ease notation. For any $n \in \mathbb{Z} \setminus \{1\}$ we define

$$\pi^{(n)} := L^{c^n} : L^1(G, c^n) \to \mathbb{B}(L^2(G)),$$

and set

$$\pi^{(1)} := \pi \colon L^1(G, c) \to \mathbb{B}(\mathcal{H}).$$

Then $\pi^{(n)}$ is a faithful $*$-representation of $L^1(G, c^n)$ for all $n \in \mathbb{Z}$. Moreover, we set

$$\mathcal{H}^{(n)} = \begin{cases} L^2(G) & \text{if } n \in \mathbb{Z} \setminus \{1\} \\ \mathcal{H} & \text{if } n = 1. \end{cases}$$

Note that $\oplus_{k\in\mathbb{Z}}\mathbb{B}(\mathcal{H}^{(k)})$ becomes a C^*-algebra through $\oplus_{k\in\mathbb{Z}}\mathbb{B}(\mathcal{H}^{(k)}) \subseteq \mathbb{B}(\oplus_{k\in\mathbb{Z}}\mathcal{H}^{(k)})$, where $\oplus_{k\in\mathbb{Z}}\mathcal{H}^{(k)}$ is the Hilbert direct sum. We then consider the map $\tilde{\pi} : L^1(G_c) \to \oplus_{k\in\mathbb{Z}}\mathbb{B}(\mathcal{H}^{(k)})$ which for $F \in L^1(G_c)$ is given by

$$F \mapsto (F_k)_{k\in\mathbb{Z}} \mapsto \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}(F_k). \tag{C.3.7}$$

We must verify that this is a faithful $*$-homomorphism. Continuity will then follow since any $*$-homomorphism from a Banach $*$-algebra to a C^*-algebra is continuous [86, Theorem 2.1.7]

For $F, H \in L^1(G_c)$ it then follows from (C.3.4) that

$$\tilde{\pi}(F * H) = \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}(F_k \natural_{c^k} H_k) = \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}(F_k) \circ \pi^{(k)}(H_k) = \tilde{\pi}(F)\tilde{\pi}(H).$$

It also follows from (C.3.5) that

$$\tilde{\pi}(F^*) = \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}((F^*)_k) = \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}((F_k)^{*_{c^k}}) = \bigoplus_{k\in\mathbb{Z}} \pi^{(k)}(F_k)^* = \tilde{\pi}(F)^*.$$

We conclude that $\tilde{\pi}$ is a continuous $*$-homomorphism.

Now suppose $F \in L^1(G_c)$ is such that $\tilde{\pi}(F) = 0$. Then $\pi^{(k)}(F_k) = 0$ for all $k \in \mathbb{Z}$, and since $\pi^{(k)} \colon L^1(G, c^k) \to \mathbb{B}(\mathcal{H}^{(k)})$ are all faithful, we conclude that $F_k = 0$ for all $k \in \mathbb{Z}$. Since the Fourier transform is injective on L^1, this happens if and only if $F = 0$ almost everywhere, i.e. if $F = 0$ in $L^1(G_c)$. We deduce that $\tilde{\pi}$ is a faithful $*$-homomorphism.

Observe that since $\mathcal{H}^{(1)} = \mathcal{H}$, the two representations $\pi \colon L^1(G, c) \to \mathbb{B}(\mathcal{H})$ and $\tilde{\pi} \circ j \colon L^1(G, c) \to \mathbb{B}(\mathcal{H}^{(1)})$ can naturally be identified. Using the C^*-identity, C^*-uniqueness of G_c, and Lemma C.3.8 we then obtain

$$\begin{aligned} \|\pi(f)\|^2_{\mathbb{B}(\mathcal{H})} &= \|\pi(f^* \natural_c f)\|_{\mathbb{B}(\mathcal{H})} = \|\tilde{\pi}(j(f^* \natural_c f))\|_{\oplus_{k\in\mathbb{Z}}\mathbb{B}(\mathcal{H}^{(k)})} \\ &= \|L_{j(f^* \natural_c f)}\|_{\mathbb{B}(L^2(G_c))} = \|L^c_{f^* \natural_c f}\|_{\mathbb{B}(L^2(G))} = \|L^c_f\|^2_{\mathbb{B}(L^2(G))} \end{aligned}$$

for all $f \in L^1(G, c)$, which proves i).

To prove ii), let $f \in L^1(G, c)$ be self-adjoint. Using Lemma C.3.8, Proposition C.2.8 and i) of Theorem C.3.1, we have the following chain of equalities

$$\begin{aligned} \rho_{L^1(G,c)}(f) &= \rho_{L^1(G_c)}(j(f)) = \rho_{\mathbb{B}(L^2(G_c))}(L_{j(f)}) \\ &= \rho_{\mathbb{B}(L^2(G))}(L^c_f) = \|L^c_f\|_{\mathbb{B}(L^2(G))} = \|\pi(f)\|_{\mathbb{B}(\mathcal{H})}. \end{aligned}$$

By Proposition C.2.9 it then follows that $\sigma_{L^1(G,c)}(f) = \sigma_{\mathbb{B}(\mathcal{H})}(\pi(f))$ for all $f \in L^1(G, c)$. ■

Remark C.3.9. Looking at the proof of Theorem C.3.1 we might hope in light of results on symmetric (Banach) $*$-algebras in e.g. [20, 44, 45, 57, 84] that it is possible to obtain similar results for the algebras considered in these papers. However, considering the crucial role C^*-uniqueness plays in order to get spectral invariance for all $*$-representations for the $*$-algebra in Theorem C.3.1, it would look like a key ingredient in proofs of such results should be analogous C^*-uniqueness results for these algebras, and for the time being these remain elusive.

C.4 Applications to Gabor analysis

We begin by introducing the central concepts of Gabor analysis, before formulating the main result of this section. We then rephrase the setting of the problem in terms spectral invariance of a certain convolution algebra and use Theorem C.3.1 to prove the result.

Throughout this section G will be a locally compact abelian group and $\widehat{G}$ will be its Pontryagin dual. Note that we will write the group operation additively. Moreover, Δ will denote a closed cocompact subgroup of the time-frequency plane $G \times \widehat{G}$. The reason for restricting to cocompact subgroups will be made clear in Remark C.4.1. We fix a Haar measure on G and equip $\widehat{G}$ with the dual measure such that Plancherel's formula holds [33, Theorem 3.4.8]. We also fix a Haar measure on Δ (which in the sequel will be denoted $\mathrm{d}z$), and give $(G \times \widehat{G})/\Delta$ the unique measure such that Weil's formula holds [65, equation (2.4)]. The *size of* Δ is the quantity $s(\Delta) := \mu((G \times \widehat{G})/\Delta)$, where μ is the chosen Haar measure. As Δ is cocompact in $G \times \widehat{G}$, we have $s(\Delta) < \infty$.

We proceed to introduce the two unitary operators most relevant for Gabor analysis. Given $x \in G$ and $\omega \in \widehat{G}$, we define the *translation operator* T_x and *modulation operator* M_ω on $L^2(G)$ by

$$(T_x f)(t) = f(t - x), \quad (M_\omega f)(t) = \omega(t) f(t)$$

for $f \in L^2(G)$ and $t \in G$. Moreover, we define a *time-frequency shift* by

$$\pi(x, \omega) := M_\omega T_x \tag{C.4.1}$$

for $x \in G$ and $\omega \in \widehat{G}$.

Having introduced both translation and modulation we may define the subgroup of $G \times \widehat{G}$ which will be of greatest importance to us when proving Theorem C.4.2. This is due to the reformulations of the frame operator in (C.4.6) and (C.4.8) below. The *adjoint subgroup* of Δ, denoted Δ°, is the closed subgroup of $G \times \widehat{G}$ defined by

$$\Delta^\circ := \{ w \in G \times \widehat{G} \mid \pi(z)\pi(w) = \pi(w)\pi(z) \text{ for all } z \in \Delta \}. \tag{C.4.2}$$

Its importance for time-frequency analysis was first realized in [42]. We may identify Δ° with $((G \times \widehat{G})/\Delta)\widehat{\ }$ as in [65, p. 234] to pick the dual measure on Δ° corresponding to the measure on $(G \times \widehat{G})/\Delta$. Δ is cocompact in $G \times \widehat{G}$, so Δ° is discrete. The induced measure on Δ° is the counting measure scaled with the constant $s(\Delta)^{-1}$ [66, equation (13)].

Given $g \in L^2(G)$, the *Gabor system over* Δ *with generator* g is a family $\mathcal{G}(g; \Delta) := (\pi(z)g)_{z \in \Delta}$. It is called a *Gabor frame* if it is a (continuous) frame for $L^2(G)$ [4, 65, 69] in the sense that the following conditions are satisfied:

i) The family $\mathcal{G}(g, \Delta)$ is weakly measurable, i.e. for every $f \in L^2(G)$ the map $z \mapsto \langle f, \pi(z)g \rangle$ is measurable.

ii) There exist positive constants $C, D > 0$ such that for all $f \in L^2(G)$ we have that

$$C\|f\|_2^2 \leq \int_\Delta |\langle f, \pi(z)g \rangle|^2 \, \mathrm{d}z \leq D\|f\|_2^2.$$

Remark C.4.1. Gabor frames $\mathcal{G}(g; \Delta)$ for $L^2(G)$ with $g \in L^2(G)$ can only exist if Δ is cocompact [65, Theorem 5.1]. Indeed, this is also the case if we consider finitely many functions $g_1, \dots, g_k \in L^2(G)$ and a Gabor system $\mathcal{G}(g_1, \dots, g_k; \Delta)$ as in Remark C.4.5 below [66, Lemma 4.9]. The same is true if we consider matrix frames introduced in [10], see [10, Proposition 4.29].

If $\mathcal{G}(g; \Delta)$ is weakly measurable and $D < \infty$ for this family, we say $\mathcal{G}(g; \Delta)$ is a *Bessel system*. Associated to any Bessel system $\mathcal{G}(g; \Delta)$ is a linear bounded operator known as the *frame operator associated to* $\mathcal{G}(g; \Delta)$. It is the operator

$$\begin{aligned} S \colon L^2(G) &\to L^2(G) \\ f &\mapsto \int_\Delta \langle f, \pi(z)g \rangle \, \pi(z)g \, \mathrm{d}z, \end{aligned}$$

where we interpret the integral weakly in $L^2(G)$. It is well-known in frame theory that S commutes with all time-frequency shifts $\pi(z)$ when $z \in \Delta$, and that $\mathcal{G}(g; \Delta)$ is a Gabor frame for $L^2(G)$ if and only if S is invertible on $L^2(G)$. Moreover, it is not hard to see that S is a positive operator.

Now let $\mathcal{G}(g; \Delta)$ be a Gabor frame for $L^2(G)$. Using that the frame operator commutes with time-frequency shifts from Δ, we have

$$f = S^{-1}Sf = \int_\Delta \langle f, \pi(z)g \rangle \, \pi(z)S^{-1}g \, \mathrm{d}z \tag{C.4.3}$$

for all $f \in L^2(G)$. The function $S^{-1}g$ is known as *the canonical dual atom of* g. Moreover, we have

$$f = S^{-1/2}SS^{-1/2}f = \int_\Delta \left\langle f, \pi(z)S^{-1/2}g \right\rangle \pi(z)S^{-1/2}g \, \mathrm{d}z \tag{C.4.4}$$

for all $f \in L^2(G)$. The function $S^{-1/2}g$ is known as *the canonical tight atom associated to* g.

As a last preparation before presenting the main result of this section we must introduce a function space. Let $g \in L^2(G)$. We define the *short-time Fourier transform with respect to* g to be the operator $V_g : L^2(G) \to L^2(G \times \widehat{G})$ given by

$$V_g f(z) = \langle f, \pi(z)g \rangle$$

for $f \in L^2(G)$ and $z \in G \times \widehat{G}$. Using this, we define the *Feichtinger algebra* $S_0(G)$ by

$$S_0(G) := \{ f \in L^2(G) \mid V_f f \in L^1(G \times \widehat{G}) \}. \tag{C.4.5}$$

The Feichtinger algebra is known as a nice space of test functions for time-frequency analysis, and its elements have good decay in both time and frequency. We refer the reader to [64] for more information on the Feichtinger algebra. At last, we may state the main theorem of this section.

Theorem C.4.2. *Let $\Delta \subseteq G \times \widehat{G}$ be a closed cocompact subgroup, and suppose $g \in S_0(G)$ is such that $\mathcal{G}(g;\Delta)$ is a Gabor frame for $L^2(G)$. Then $S^{-1}g, S^{-1/2}g \in S_0(G)$ as well.*

In the case when Δ is a separable lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, Theorem C.4.2 was proved in [58], and it was claimed to hold for general lattices in phase spaces of locally compact abelian groups. It is possible that their techniques can be adapted to the setting of closed cocompact subgroups. However, it will turn out that the result is easier to deduce by using Theorem C.3.1, thereby circumventing any need to use the periodization techniques of [58]. In order to show Theorem C.4.2 we will reformulate the above setup to incorporate twisted convolution algebras. As a first step towards this we present the Fundamental Identity of Gabor Analysis. We refer the reader to [97, Proposition 2.11] for a proof. There the Schwartz-Bruhat space is used, but the proof can easily be adapted to the case of $S_0(G)$. For the case of modulation spaces, see for example [43] or [53].

Proposition C.4.3. *Let $f, g, h \in S_0(G)$. Then*

$$\int_\Delta \langle f, \pi(z)g \rangle \, \pi(z)h \, \mathrm{d}z = \frac{1}{s(\Delta)} \sum_{w \in \Delta^\circ} \langle \pi(w)h, g \rangle \, \pi(w)^* f$$

where we interpret the integral and the sum weakly in $L^2(G)$.

Proposition C.4.3 allows us to rewrite the frame operator S for $\mathcal{G}(g;\Delta)$ as

$$Sf = \int_\Delta \langle f, \pi(z)g \rangle \, \pi(z)g \, \mathrm{d}z = \frac{1}{s(\Delta)} \sum_{w \in \Delta^\circ} \langle \pi(w)g, g \rangle \, \pi(w)^* f. \tag{C.4.6}$$

This observation is key in rephrasing the problem. It is the right hand side which is of importance to us, and it will be most natural to restate the frame operator in terms of a right $*$-representation of a twisted convolution algebra, see Equation (C.4.8).

We will also need the continuous 2-cocycle on $G \times \widehat{G}$ known as the *Heisenberg 2-cocycle* [97, p. 263]. It is the map $c : (G \times \widehat{G}) \times (G \times \widehat{G}) \to \mathbb{T}$ given by

$$c((x,\omega),(y,\tau)) = \overline{\tau(x)} \tag{C.4.7}$$

for $(x,\omega),(y,\tau) \in G \times \widehat{G}$.

Remark C.4.4. Having introduced the Heisenberg 2-cocycle c we may also describe Δ° without using time-frequency shifts directly by ways of

$$\Delta^\circ = \{w \in G \times \widehat{G} \mid c(w,z)\overline{c}(z,w) = 1 \text{ for all } z \in \Delta\}$$

Restricting to Δ°, we construct the twisted convolution algebra $\ell^1(\Delta^\circ, \overline{c})$ as in Section C.2. Now the map

$$\begin{aligned} \pi^* \colon G \times \widehat{G} &\to \mathcal{U}(L^2(G)) \\ (x,\omega) &\mapsto \pi(x,\omega)^* \end{aligned}$$

defines a right $\overline{c}$-projective unitary representation. A right projective unitary representation of a group may also be viewed as a (left) projective unitary representation of its opposite group. We also get a right $\overline{c}$-projective unitary representation of Δ°, which we also denote by π^*. The integrated representation defines a right $*$-representation $\pi^* \colon \ell^1(\Delta^\circ, \overline{c}) \to \mathbb{B}(L^2(G))$. This representation leaves $S_0(G)$ invariant, i.e. $\pi^*(\ell^1(\Delta^\circ,\overline{c}))S_0(G) \subseteq S_0(G)$ [66, Theorem 3.4]. Given $a = (a_w)_{w\in\Delta^\circ} \in \ell^1(\Delta^\circ, \overline{c})$ and $f \in L^2(G)$ we have

$$\pi^*(a)f = \frac{1}{s(\Delta)} \sum_{w\in\Delta^\circ} a_w \pi(w)^* f.$$

Also, this $*$-representation is known to be faithful [97, Proposition 2.2]. Moreover, for $g \in S_0(G)$ we have $(\langle \pi(w)g, g\rangle)_{w\in\Delta^\circ} \in \ell^1(\Delta^\circ, \overline{c})$ [66, Theorem 3.4]. Using (C.4.6) for the Gabor system $\mathcal{G}(g;\Delta)$, $g \in S_0(G)$, we now see that

$$Sf = \pi^*((\langle \pi(w)g, g\rangle)_{w\in\Delta^\circ})f \tag{C.4.8}$$

for $f \in L^2(G)$. We are finally ready to prove Theorem C.4.2. The proof follows the same general outline as the proof of [58, Theorem 4.2], but without use of periodization techniques unique to the Gabor analysis setting.

Proof of Theorem C.4.2. If $g \in S_0(G)$ is such that $\mathcal{G}(g;\Delta)$ is a Gabor frame for $L^2(G)$, then the corresponding frame operator S is invertible. By (C.4.8) we may write $Sf = \pi^*((\langle \pi(w)g, g\rangle)_{w\in\Delta^\circ})f$ for any $f \in L^2(G)$. Since Δ° is abelian, every compactly generated subgroup of Δ° is of polynomial growth by the structure theorem for compactly generated locally compact abelian groups [33, Theorem 4.2.2]. Hence every compactly generated subgroup of $\Delta^\circ_{\overline{c}}$ is also of polynomial growth, since it is a compact extension of Δ°. It follows that $\Delta^\circ_{\overline{c}}$ is C^*-unique. Moreover, $\Delta^\circ_{\overline{c}}$ is nilpotent of class 1 as Δ° is abelian, so it follows that $\ell^1(\Delta^\circ_{\overline{c}})$ is symmetric. By Theorem C.3.1 we then have that $\ell^1(\Delta^\circ, \overline{c})$ is spectrally invariant in $\mathbb{B}(L^2(G))$. Hence there is $a = (a_w)_{w\in\Delta^\circ} \in \ell^1(\Delta^\circ, \overline{c})$ such that $a\natural_{\overline{c}}(\langle \pi(w)g, g\rangle)_{w\in\Delta^\circ} = 1_{\ell^1(\Delta^\circ,\overline{c})} = (\langle \pi(w)g, g\rangle)_{w\in\Delta^\circ}\natural_{\overline{c}}a$ and

$$S^{-1}f = \pi^*(a)f$$

for all $f \in L^2(G)$. Since $\pi^*(\ell^1(\Delta^\circ, \overline{c}))$ leaves $S_0(G)$ invariant, it follows that $S^{-1}g \in S_0(G)$.

Since S, hence also S^{-1}, is a positive operator, we may also take the square root of the image of a under π^* in $\mathbb{B}(L^2(G))$. By spectral invariance and the fact that Banach $*$-algebras are closed under holomorphic functional calculus [30, p. 212] it follows that there is $b = (b_w)_{w\in\Delta^\circ} \in \ell^1(\Delta^\circ, \overline{c})$ such that

$$S^{-1/2}f = \pi^*(b)f$$

for all $f \in L^2(G)$. Once again, since $\pi^*(\ell^1(\Delta^\circ, \overline{c}))$ leaves $S_0(G)$ invariant, it follows that $S^{-1/2}g \in S_0(G)$. This finishes the proof. ∎

Remark C.4.5. There are no issues extending this to multi-window Gabor frames, i.e. the case of $g_1, \ldots, g_k \in S_0(G)$ such that $\mathcal{G}(g_1, \ldots, g_k; \Delta) := \mathcal{G}(g_1; \Delta) \cup \cdots \cup \mathcal{G}(g_k; \Delta)$ is a frame for $L^2(G)$. Indeed, the only real difference is that we in (C.4.8) will need to consider $\pi^*((\sum_{i=1}^k \langle \pi(w)g_i, g_i \rangle)_{w\in\Delta^\circ})$. This is of no real consequence for the proofs. Hence we may conclude that for a multi-window Gabor frame $\mathcal{G}(g_1, \ldots, g_k; \Delta)$ for $L^2(G)$ with $g_1, \ldots, g_k \in S_0(G)$ and associated (multi-window) frame operator S we get $S^{-1}g_1, \ldots, S^{-1}g_k, S^{-1/2}g_1, \ldots, S^{-1/2}g_k \in S_0(G)$. Indeed one can go even further and do this for the matrix Gabor frames introduced in [10], which generalize multi-window super Gabor frames, using the setup from the same article. The key observation for doing this is that since $\ell^1(\Delta^\circ, \overline{c})$ is spectrally invariant in $\mathbb{B}(L^2(G))$ we also have that $M_n(\ell^1(\Delta^\circ, \overline{c}))$ is spectrally invariant in $M_n(\mathbb{B}(L^2(G)))$ for any $n \in \mathbb{N}$ [103].

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Paper D

C^*-uniqueness results for groupoids

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Paper D

C^*-uniqueness results for groupoids

Abstract

For a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with a continuous 2-cocycle σ we find conditions that guarantee that $\ell^1(\mathcal{G}, \sigma)$ has a unique C^*-norm.

D.1 Introduction

Given a reduced (Banach) $*$-algebra $\mathcal{A}$, the enveloping C^*-algebra $C^*(\mathcal{A})$ plays a fundamental role in the representation theory of $\mathcal{A}$. However, any faithful $*$-representation of $\mathcal{A}$ will yield a C^*-completion of $\mathcal{A}$, and one may ask if this completion is isomorphic to the enveloping C^*-algebra. In the particular case of a locally compact group G, we may for example consider the $*$-algebras $C_c(G)$ or $L^1(G)$. There are then two canonical C^*-norms, namely the one arising from the left regular representation and the maximal C^*-norm. It is well-known that G is an amenable group if and only if these two C^*-norms coincide. However, even for amenable groups we can not rule out that there are C^*-norms on $C_c(G)$ and $L^1(G)$ that are properly dominated by the norm induced by the left regular representation. Examples of this are given in [22, p. 230]. This invites the notion of *C^*-uniqueness*. A reduced $*$-algebra $\mathcal{A}$ is called C^*-unique if $C^*(\mathcal{A})$ is the unique C^*-completion of $\mathcal{A}$ up to isomorphism. This was extensively studied in [16] for $*$-algebras. Moreover, a more specialized study for convolution algebras of locally compact groups was conducted in [22], where C^*-uniqueness of $L^1(G)$ was studied by considering properties of the underlying group G. These two papers spawned investigations on C^*-uniqueness in the following decades, see for example [8, 32, 62, 78]. In later years, algebraic C^*-uniqueness of discrete groups has garnered some attention [3, 51, 102]. This is the study of C^*-uniqueness of the group ring $\mathbb{C}[\Gamma]$ for a discrete group Γ and is not equivalent to the study of C^*-uniqueness of $\ell^1(\Gamma)$, see Remark D.2.8.

We will in this paper study the C^*-uniqueness of certain Banach $*$-algebras associated to groupoids. To be more precise, given a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with a normalized continuous 2-cocycle σ, we will study the C^*-uniqueness of the I-norm completion of $C_c(\mathcal{G}, \sigma)$, which will be denoted by $\ell^1(\mathcal{G}, \sigma)$, see (D.2.3). Here $C_c(\mathcal{G}, \sigma)$ denotes the space $C_c(\mathcal{G})$ equipped with σ-twisted convolution and involution, see (D.2.1) and (D.2.2), and similarly for $\ell^1(\mathcal{G}, \sigma)$. Associated to $\ell^1(\mathcal{G}, \sigma)$ are two canonical C^*-norms, namely the one coming from the σ-twisted left regular representation, see (D.2.6), and the full C^*-norm. If these coincide we say $\mathcal{G}$ twisted by σ has the weak containment property. The technicalities will be postponed to Section D.2.3. Letting $\mathrm{Iso}(\mathcal{G})^\circ$ denote the interior of the isotropy subgroupoid of $\mathcal{G}$, we will first find that for $\ell^1(\mathcal{G}, \sigma)$ to be C^*-unique, it is sufficient that $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is C^*-unique. If we further let $\mathrm{Iso}(\mathcal{G})^\circ_x$ denote the fiber of $\mathrm{Iso}(\mathcal{G})^\circ$ in the point $x \in \mathcal{G}^{(0)}$, and let σ_x denote the restriction of σ to this fiber, we have the following main result.

Theorem D.1.1 (cf. Theorem D.3.1)**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid with a continuous 2-cocycle σ. Suppose that $\mathcal{G}$ twisted by σ has the weak containment property. Then $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique if all the twisted convolution algebras $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$, $x \in \mathcal{G}^{(0)}$, are C^*-unique.*

The theorem allows us to deduce C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ by considering C^*-uniqueness of the (twisted) convolution algebras of the discrete groups $\mathrm{Iso}(\mathcal{G})^\circ_x$, $x \in \mathcal{G}^{(0)}$. The latter has been studied earlier, the untwisted case in [22] and the twisted case in [8]. Using this we obtain several examples of groupoids $\mathcal{G}$ for which $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique in Section D.4. Additionally, we are able to deduce C^*-uniqueness of some wreath products using our groupoid approach, see Example D.4.4.

We will proceed in the following manner. In Section D.2 we will collect all results we will need regarding C^*-uniqueness of Banach $*$-algebras, C^*-algebra bundles, as well as cocycle-twisted convolution algebras associated to second-countable locally compact Hausdorff étale groupoids. In Section D.3 we first present our main theorem, Theorem D.3.1. The remainder of the section will be dedicated to its proof. Lastly, in Section D.4 we present examples of C^*-unique convolution algebras coming from groupoids, as well as deducing C^*-uniqueness of some wreath products.

D.2 Preliminaries

D.2.1 C^*-uniqueness for Banach $*$-algebras

A $*$-representation of a Banach $*$-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$, where $B(\mathcal{H})$ are the bounded linear operators on a Hilbert space $\mathcal{H}$. We say $\mathcal{A}$ is *reduced* if $\mathcal{A}_{\mathcal{R}} = \{a \in \mathcal{A} : \pi(a) = 0 \text{ for every } * \text{−representation } \pi \text{ of } \mathcal{A}\} = \{0\}$. All Banach $*$-algebras we consider in the sequel will be reduced. The *enveloping C^*-algebra* of a reduced Banach $*$-algebra $\mathcal{A}$ is the unique C^*-algebra $C^*(\mathcal{A})$ which admits the following universal property: there exists an injective $*$-homomorphism $\Phi : \mathcal{A} \to C^*(\mathcal{A})$ with dense range so that for every $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$, there exists a unique $*$-representation $\hat{\pi} : C^*(\mathcal{A}) \to B(\mathcal{H})$ so that $\pi = \hat{\pi} \circ \Phi$. In order to ease notation in the sequel we will identify $\mathcal{A}$ with the Banach $*$-subalgebra $\Phi(\mathcal{A})$ of $C^*(\mathcal{A})$ whenever it is natural to do so. The enveloping C^*-algebra of a Banach $*$-algebra always exists [89, Section 10.1].

Definition D.2.1. Let $\mathcal{A}$ be a reduced Banach $*$-algebra. We say that $\mathcal{A}$ is *C^*-unique* if the C^*-norm given by

$$\|a\| := \sup\{\|\pi(a)\| : \pi : \mathcal{A} \to B(\mathcal{H}) \text{ is a } *\text{-representation}\}$$

for every $a \in \mathcal{A}$, is the unique C^*-norm on $\mathcal{A}$. In other words, $\mathcal{A}$ is C^*-unique if $C^*(\mathcal{A})$ is the unique C^*-completion of $\mathcal{A}$ up to isomorphism.

We will make repeated use of the following result on C^*-uniqueness of Banach $*$-algebras, see [89, Proposition 10.5.19].

Proposition D.2.2. *Let $\mathcal{A}$ be a reduced Banach $*$-algebra with enveloping C^*-algebra $C^*(\mathcal{A})$. Then $\mathcal{A}$ is C^*-unique if and only if for every nonzero two-sided closed ideal $I \lhd C^*(\mathcal{A})$ we have $\mathcal{A} \cap I \neq \{0\}$.*

D.2.2 C^*-algebra bundles

The notion of a $C_0(X)$-algebra will be of importance in the proof of the main theorem. Hence we briefly revise some basic notions and results on $C_0(X)$-algebras and C^*-bundles.

Definition D.2.3. Let X be a locally compact Hausdorff space. A *$C_0(X)$-algebra* is a C^*-algebra A together with a non-degenerate injection $\iota : C_0(X) \to \mathcal{Z}(M(A))$, where the latter denotes the center of the multiplier algebra of A.

We shall also need to consider (upper semi-continuous) C^*-bundles.

Definition D.2.4. Let X be a locally compact Hausdorff space and let $\{B_x\}_{x\in X}$ be a family of C^*-algebras. A map f defined on X such that $f(x) \in B_x$ for all $x \in X$, is called a *section*. An *upper semi-continuous C^*-bundle* $\mathbf{B}$ *over* X is a triple $(X, \{B_x\}_{x\in X}, \Gamma_0(\mathbf{B}))$, where $\Gamma_0(\mathbf{B})$ is a family of sections, such that the following conditions are satisfied:

1. $\Gamma_0(\mathbf{B})$ is a C^*-algebra under pointwise operations and supremum norm,
2. for each $x \in X$, $B_x = \{f(x) : f \in \Gamma_0(\mathbf{B})\}$,
3. for each $f \in \Gamma_0(\mathbf{B})$ and each $\varepsilon > 0$, $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact,
4. $\Gamma_0(\mathbf{B})$ is closed under multiplication by $C_0(X)$, that is, for each $g \in C_0(X)$ and $f \in \Gamma_0(\mathbf{B})$, the section gf defined by $gf(x) = g(x)f(x)$ is in $\Gamma_0(\mathbf{B})$.

The two above concepts can be combined to obtain the main theorem of [87] which we present shortly for the reader's convenience. Suppose X is a locally compact Hausdorff space, and suppose A is a $C_0(X)$-algebra with map $\iota\colon C_0(X) \to \mathcal{Z}(M(A))$. For $x \in X$, denote by $J_x := C_0(X \setminus \{x\})$ and realize $J_x \subseteq C_0(X)$ in the natural way. Moreover, we define $I_x := \iota(J_x)A$, which is a closed two-sided ideal of A. We then have the following result which will play a major role in the proof of Theorem D.3.1.

Proposition D.2.5 ([87, Theorem 2.3])**.** *Let X be a locally compact Hausdorff space and let A be a $C_0(X)$-algebra. Then there exists a unique upper semi-continuous C^*-bundle* $\mathbf{B}$ *over X such that*

i) the fibers $B_x = A/I_x$, and

ii) there is an isomorphism $\phi\colon A \to \Gamma_0(\mathbf{B})$ satisfying $\phi(a)(x) = a + I_x$.

D.2.3 Groupoids, cocycle twists and associated algebras

Given a groupoid $\mathcal{G}$ we will denote by $\mathcal{G}^{(0)}$ its unit space and write $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ for the range and source maps, respectively. We will also denote by $\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} : s(\alpha) = r(\beta)\}$ the set of *composable elements*. In this paper, we will only consider groupoids $\mathcal{G}$ equipped with a second-countable locally compact Hausdorff topology making all the structure maps continuous. A groupoid $\mathcal{G}$ is called *étale* if the range map, and hence also the source map, is a local homeomorphism. A subset B of an étale groupoid $\mathcal{G}$ is called a *bisection* if there is an open set $U \subseteq \mathcal{G}$ containing B such that $r\colon U \to r(U)$ and $s\colon U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Second-countable locally compact Hausdorff étale groupoids have countable bases consisting of open bisections.

Given $x \in \mathcal{G}^{(0)}$ we define by $\mathcal{G}_x := \{\gamma \in \mathcal{G} : s(\gamma) = x\}$ and $\mathcal{G}^x := \{\gamma \in \mathcal{G} : r(\gamma) = x\}$. Observe that if $\mathcal{G}$ is étale the sets $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete for every $x \in \mathcal{G}^{(0)}$. The *isotropy group of* x is given by $\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$, and the *isotropy subgroupoid of* $\mathcal{G}$ is the subgroupoid $\mathrm{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x$ with the relative topology from $\mathcal{G}$. Let $\mathrm{Iso}(\mathcal{G})^\circ$ denote the interior of $\mathrm{Iso}(\mathcal{G})$. We then say that $\mathcal{G}$ is *topologically principal* if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$.

We will consider groupoid twists where the twist is implemented by a normalized continuous 2-cocycle. To be more precise, let $\mathcal{G}$ be a second countable locally compact étale groupoid. A *normalized continuous* 2*-cocycle* is then a continuous map $\sigma \colon \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying

$$\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$$

for all $\gamma \in \mathcal{G}$, and

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$$

whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$. The set of normalized continuous 2-cocycles on $\mathcal{G}$ will be denoted $Z^2(\mathcal{G}, \mathbb{T})$. Note that this is not the most general notion of a twist of a groupoid (see [104, Chapter 5]).

Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. We will define the σ-twisted convolution algebra $C_c(\mathcal{G}, \sigma)$ as follows: As a set it is just

$$C_c(\mathcal{G}, \sigma) = \{f : \mathcal{G} \to \mathbb{C} : f \text{ is continuous with compact support}\},$$

but equipped with σ-twisted convolution product

$$(f *_\sigma g)(\gamma) = \sum_{\mu \in \mathcal{G}_{s(\gamma)}} f(\gamma\mu^{-1})g(\mu)\sigma(\gamma\mu^{-1}, \mu), \quad f, g \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G}, \tag{D.2.1}$$

and σ-twisted involution

$$f^{*_\sigma}(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma)f(\gamma^{-1})}, \quad f \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G}. \tag{D.2.2}$$

We complete $C_c(\mathcal{G}, \sigma)$ in the ”fiberwise 1-norm”, also known as the I-norm, given by

$$\|f\|_I = \sup_{x \in \mathcal{G}^{(0)}} \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\} \tag{D.2.3}$$

for $f \in C_c(\mathcal{G}, \sigma)$. Denote by $\ell^1(\mathcal{G}, \sigma)$ the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the I-norm. This is a Banach $*$-algebra with the natural extensions of (D.2.1) and (D.2.2). For later use we record the following lemma.

Lemma D.2.6. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then for any $f \in \ell^1(\mathcal{G})$, the map defined by*

$$\mathcal{G}^{(0)} \ni x \mapsto \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\}, \tag{D.2.4}$$

is continuous.

Proof. By density it is enough to show this for $f \in C_c(\mathcal{G})$. It is well-known that $C_c(\mathcal{G}) = \operatorname{span}\{g \in C_c(\mathcal{G}) : g \text{ is supported on a bisection}\}$. Hence we may assume f is supported on a bisection U, i.e. $\operatorname{supp}(f) \subseteq U$. Furthermore, for f we denote the assignment of (D.2.4) by F. We thus wish to show that $F \in C(\mathcal{G}^{(0)})$.

To this end, fix $x \in \mathcal{G}^{(0)}$. As $f(x) = 0$ if $x \notin s(U)$, we assume $x \in s(U)$. Since $s(x) = x$ and $s\colon U \to s(U)$ is a homeomorphism, we therefore have $x \in U$. Moreover, let $(x_i)_i \subseteq \mathcal{G}^{(0)}$ be such that $x_i \to x$. Then eventually $x_i \in s(U)$ for all i large enough. For such i we have $F(x_i) = |f(\gamma_i)|$, where γ_i is the unique element of U with $s(\gamma_i) = x_i$. Now, as $s\colon U \to s(U)$ is a homeomorphism and $x_i \to x$, we have $\gamma_i \to \gamma \in U$, where γ is the unique element of U such that $s(\gamma) = x$. As $f \in C_c(\mathcal{G})$, it follows that $f(\gamma_i) \to f(\gamma)$, and hence $F(x_i) \to F(x)$. Hence $F \in C(\mathcal{G}^{(0)})$, and the result follows. ∎

We wish to understand when $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique, i.e. when it only permits one separating C^*-norm. To do this it will be of importance to use Proposition D.2.2.

The *(full) twisted groupoid C^*-algebra* $C^*(\mathcal{G}, \sigma)$ is the completion of $C_c(\mathcal{G}, \sigma)$ in the norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm bounded } *\text{-representation}\}, \tag{D.2.5}$$

for $f \in C_c(\mathcal{G}, \sigma)$. It was observed in [7, Lemma 3.3.19] that if $\mathcal{G}$ is étale, then every $*$-representation of $C_c(\mathcal{G}, \sigma)$ is bounded by the I-norm. Then, since we are completing with respect to a supremum over $*$-representations, $C^*(\mathcal{G}, \sigma)$ is just the C^*-envelope of $\ell^1(\mathcal{G}, \sigma)$.

Now we will construct a faithful representation of $\ell^1(\mathcal{G}, \sigma)$ called the *σ-twisted left regular representation*. In particular, we have that $\ell^1(\mathcal{G}, \sigma)$ is reduced. The completion of the image of $\ell^1(\mathcal{G}, \sigma)$ under the σ-twisted left regular representation is called the *σ-twisted reduced groupoid C^*-algebra of $\mathcal{G}$* and will be denoted $C^*_r(\mathcal{G}, \sigma)$. Let $x \in \mathcal{G}^{(0)}$. Then there is a representation $L^{\sigma,x}\colon C_c(\mathcal{G}, \sigma) \to B(\ell^2(\mathcal{G}_x))$ which is given by

$$L^{\sigma,x}(f)\delta_\gamma = \sum_{\mu \in \mathcal{G}^{r(\gamma)}} \sigma(\mu, \mu^{-1}\gamma) f(\mu)\delta_{\mu\gamma}, \quad \text{for } f \in C_c(\mathcal{G}, \sigma) \text{ and } \gamma \in \mathcal{G}_x. \tag{D.2.6}$$

We then obtain a faithful I-norm bounded $*$-representation of $C_c(\mathcal{G},\sigma)$ given by

$$\bigoplus_{x\in\mathcal{G}^{(0)}} L^{\sigma,x} : C_c(\mathcal{G},\sigma) \to \bigoplus_{x\in\mathcal{G}^{(0)}} B(\ell^2(\mathcal{G}_x)) \subset B(\bigoplus_{x\in\mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)). \tag{D.2.7}$$

$C_r^*(\mathcal{G},\sigma)$ is then the completion of the image of $C_c(\mathcal{G},\sigma)$ under the σ-twisted left regular representation. As the $*$-representation is I-norm bounded, $C_r^*(\mathcal{G},\sigma)$ is also the completion of $\ell^1(\mathcal{G},\sigma)$ in the same norm. Therefore, since $C^*(\mathcal{G},\sigma)$ is the C^*-envelope of $\ell^1(\mathcal{G},\sigma)$, by universality, there exists a natural (surjective) $*$-homomorphism $\lambda : C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$.

Definition D.2.7. Let $\mathcal{G}$ be a second-countable locally compact Hausdorff groupoid and let $\sigma \in Z^2(G,\mathbb{T})$. We say that $\mathcal{G}$ twisted by σ has the *weak containment property* when the natural map $\lambda\colon C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$ is an isomorphism.

If $\mathcal{G}$ is an amenable groupoid [5], we have that $C_r^*(\mathcal{G},\sigma) = C^*(\mathcal{G},\sigma)$ for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$ [5, Proposition 6.1.8], and hence $\mathcal{G}$ twisted by σ has the weak containment property for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. In [108] it was proved that amenability is not equivalent to having the weak containment property. On the other hand, it is not known to the authors whether the weak containment property is equivalent to the weak containment property with respect every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$.

Remark D.2.8. While both $\ell^1(\mathcal{G},\sigma)$ and $C_c(\mathcal{G},\sigma)$ complete to the same C^*-algebras $C^*(\mathcal{G},\sigma)$ and $C_r^*(\mathcal{G},\sigma)$ in the above setup, the question of C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ is not equivalent to C^*-uniqueness of the $*$-algebra $C_c(\mathcal{G},\sigma)$. To see this, let $\mathcal{G} = \mathbb{Z}$, the group of integers and consider the trivial twist $\sigma = 1$. Then $\ell^1(\mathbb{Z},1) = \ell^1(\mathbb{Z})$ is C^*-unique by [21], while $C_c(\mathbb{Z}) = \mathbb{C}[\mathbb{Z}]$ is not C^*-unique by [3, Proposition 2.4].

Denoting the restriction of σ to $\mathrm{Iso}(G)^\circ \subseteq \mathcal{G}$ also by σ, we define the Banach $*$-subalgebra $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ of $\ell^1(\mathcal{G},\sigma)$. We then have the following result.

Proposition D.2.9 ([7, Proposition 5.3.1])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. There is a $*$-homomorphism*

$$\iota\colon C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \to C^*(\mathcal{G},\sigma)$$

such that

$$\iota(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \mathrm{Iso}(\mathcal{G})^\circ, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. This homomorphism descends to an injective $*$-homomorphism

$$\iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \to C_r^*(\mathcal{G},\sigma).$$

We observe that the homomorphism ι is an isometry at the ℓ^1-level, i.e. that $\iota\colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \to \ell^1(\mathcal{G},\sigma)$ is an isometric $*$-homomorphism.

We then also have the following result from [7] which will be key to our approach to study C^*-uniqueness of twisted groupoid convolution algebras in Section D.3.

Proposition D.2.10 ([7, Theorem 5.3.13])**.** *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and let $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. Let $\iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \to C_r^*(\mathcal{G},\sigma)$ be the injective $*$-homomorphism of Proposition D.2.9. Suppose A is a C^*-algebra and that $\Psi\colon C_r^*(\mathcal{G},\sigma) \to A$ is a homomorphism. Then Ψ is injective if and only if $\Psi\circ\iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \to A$ is an injective homomorphism.*

D.3 C^*-uniqueness for cocycle-twisted groupoid convolution algebras

We begin this section by presenting our main theorem. The remainder of the section will be dedicated to proving it.

Given a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ and $\sigma \in Z^2(\mathcal{G},\mathbb{T})$, denote the restriction of σ to the fiber $\mathrm{Iso}(\mathcal{G})_x^\circ$ by σ_x. Note that σ_x is continuous as $\mathrm{Iso}(\mathcal{G})_x^\circ$ is discrete, i.e. $\sigma_x \in Z^2(\mathrm{Iso}(\mathcal{G})_x^\circ,\mathbb{T})$. The following then constitutes our main theorem.

Theorem D.3.1. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. Suppose that $\mathcal{G}$ twisted by σ has the weak containment property. Then $\ell^1(\mathcal{G},\sigma)$ is C^*-unique if all the twisted convolution algebras $\ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ,\sigma_x)$, $x \in \mathcal{G}^{(0)}$, are C^*-unique.*

As a first step towards proving Theorem D.3.1 we relate C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$.

Proposition D.3.2. *Suppose $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid with the weak containment property when twisted by $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. If $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is C^*-unique, then $\ell^1(\mathcal{G},\sigma)$ is C^*-unique.*

Proof. Suppose $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is C^*-unique. Then in particular $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) = C_r^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. Let $\{0\} \neq J \lhd C^*(\mathcal{G},\sigma) = C_r^*(\mathcal{G},\sigma)$ be a closed two-sided ideal. By Proposition D.2.2 it suffices to show that $J\cap\ell^1(\mathcal{G},\sigma) \neq \{0\}$. By Proposition D.2.10 we have $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)\cap J \neq \{0\}$ as the $*$-homomorphism $C^*(\mathcal{G},\sigma) \to C^*(\mathcal{G},\sigma)/J$ is not injective. Now define $I := J \cap C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. It is straightforward to verify that I is a two-sided ideal in $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$, and as both J and $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ are closed in $C^*(\mathcal{G},\sigma)$, I is also closed in $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. By C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ it then follows that $I \cap \ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \neq \{0\}$. From this we get

$$\{0\} \neq I \cap \ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) = J \cap \ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma) \subset J \cap \ell^1(\mathcal{G},\sigma),$$

from which we deduce by Proposition D.2.2 that $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique. ∎

Having related the question of C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ to a question regarding C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, we proceed to further relate this to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ for $x \in \mathcal{G}^{(0)}$. To do this we will show that for any $*$-representation $\pi\colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to B(\mathcal{H})$, the resulting C^*-algebra $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a $C_0(\mathcal{G}^{(0)})$-algebra. This is the content of Lemma D.3.3. However, we first do some preparatory work.

Observe that there exists a $*$-homomorphism $\phi : C_0(\mathcal{G}^{(0)}) \to \mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))$, the latter meaning the center of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Indeed, as $\mathcal{G}^{(0)}$ is open in $\mathrm{Iso}(\mathcal{G})^\circ$, we may take ϕ to be the inclusion where we extend functions in $C_0(\mathcal{G}^{(0)})$ by zero. The map ϕ is clearly isometric. As ϕ can be viewed as an inclusion, we omit writing it from now on to ease notation. Then given $g \in C_0(\mathcal{G}^{(0)})$ and $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ we have that

$$\begin{aligned}(g *_\sigma f)(\gamma) &= g(r(\gamma))f(\gamma)\sigma(r(\gamma), \gamma) = g(r(\gamma))f(\gamma)\\ &= f(\gamma)g(s(\gamma))\sigma(\gamma, s(\gamma)) = (f *_\sigma g)(\gamma),\end{aligned}$$

for every $\gamma \in \mathrm{Iso}(\mathcal{G})^\circ$. The resulting action of $C_0(\mathcal{G}^{(0)})$ on $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ can then be viewed as pointwise multiplication in the fibers of $\mathcal{G}^{(0)}$. By continuity we can extend ϕ to a continuous $*$-homomorphism from $C_0(\mathcal{G}^{(0)})$ to $\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))$. Let $\pi : \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to B(\mathcal{H})$ be a faithful $*$-representation and let $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ denote the completion in the operator norm of $B(\mathcal{H})$. Define the map $\iota := \pi \circ \phi : C_0(\mathcal{G}^{(0)}) \to \pi(\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)))$. We have that

$$\pi(\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))) = \mathcal{Z}(\pi(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))) \subseteq \mathcal{Z}(M(C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)))\,.$$

The following is then immediate.

Lemma D.3.3. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Let π be a $*$-representation of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Then $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a $C_0(\mathcal{G}^{(0)})$-algebra.*

Now fix $x \in \mathcal{G}^{(0)}$ and denote by $J_x = C_0(\mathcal{G}^{(0)} \setminus \{x\})$ the space of continuous functions of $\mathcal{G}^{(0)}$ vanishing at both infinity and x. As $C_0(\mathcal{G}^{(0)})$ is central in $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ and J_x is a closed two-sided ideal of $C_0(\mathcal{G}^{(0)})$, the space $I_x := J_x \cdot \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a closed two-sided ideal in $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Recall that we denote by σ_x the restriction of σ to the fiber $\mathrm{Iso}(\mathcal{G})^\circ_x$. We then have the following result.

Lemma D.3.4. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. For every $x \in \mathcal{G}^{(0)}$ the map $\psi_x\colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ given by restriction of functions is a continuous $*$-homomorphism inducing an isometric $*$-isomorphism between $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x$ and $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$.*

Proof. For $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ we have

$$\|\psi_x(f)\|_{\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x)} = \sum_{\gamma \in \mathrm{Iso}(\mathcal{G})^\circ_x} |f(\gamma)| \leq \sup_{y \in \mathcal{G}^{(0)}} \sum_{\mu \in \mathrm{Iso}(\mathcal{G})^\circ_y} |f(\mu)| = \|f\|_I$$

for all $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Thus ψ_x is a I-norm decreasing map, so it extends to a continuous $*$-homomorphism $\psi_x : \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$. It is surjective by Tietze's extension theorem.

Next we want to show that $\ker \psi_x = I_x$. First observe that given $g \in C_0(\mathcal{G}^{(0)})$ and $h \in C_c(\mathcal{G}, \sigma)$ we have that

$$\begin{aligned}\psi_x(g *_\sigma h)(\gamma) = (\psi_x(g) *_\sigma \psi_x(h))(\gamma) &= \sum_{\mu \in \mathrm{Iso}(\mathcal{G})^\circ_x} g(\mu)h(\mu^{-1}\gamma)\sigma(\mu, \mu^{-1}\gamma) \\ &= g(x)h(x\gamma)\sigma(x, \gamma) = g(x)h(\gamma),\end{aligned}$$

for every $\gamma \in \mathrm{Iso}(\mathcal{G})^\circ_x$.

Now let $f \in I_x$. We may then assume that f is the norm limit of elements f_n of the form $f_n = \sum_{i=1}^n g_i *_\sigma h_i$, where $g_i \in J_x$ and $h_i \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ for all $i \in \mathbb{N}$. It suffices to prove that $\psi_x(g_i *_\sigma h_i) = 0$ for all $i \in \mathbb{N}$. For any $\gamma \in \mathrm{Iso}(\mathcal{G})^\circ_x$ we then have $\psi_x(g_i *_\sigma h_i)(\gamma) = g_i(x)h_i(\gamma) = 0$ since $g_i(x) = 0$. Then it follows that $\psi_x(f_n) = 0$ for every $n \in \mathbb{N}$, and by continuity $\psi_x(f) = 0$. Thus, $I_x \subset \ker \psi_x$.

Conversely, suppose $f \in \ker \psi$. Then $f = \lim f_n$ for some $f_n \in C_c(\mathcal{G}, \sigma) \cap \ker \psi_x$, and hence $f_n(x) = 0$ for every $n \in \mathbb{N}$. Let $\{\rho_\lambda\}_{\lambda \in \Lambda} \subset C_0(\mathcal{G}^{(0)} \setminus \{x\})$ be a partition of the unit of $\mathcal{G}^{(0)} \setminus \{x\}$. Then given $n \in \mathbb{N}$ there exists a finite subset Λ_n of Λ, such that $g_n := \sum_{\lambda \in \Lambda_n} \rho_n \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) = J_x$ and $g_n(y) = 1$ for every $y \in r(\mathrm{supp}(f_n)) = s(\mathrm{supp}(f_n))$, and hence

$$f_n(\gamma) = g_n(r(\gamma))f_n(\gamma)\sigma(r(\gamma), \gamma) = (g_n *_\sigma f_n)(\gamma)$$

for every $\gamma \in \mathcal{G}$. Therefore we have that

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} (g_n *_\sigma f_n) \in \overline{J_x \cdot \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)} = I_x,$$

as we wanted. We would like to see that the isomorphism $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x \cong \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ is isometric. To do that, it is enough to check that

$$\inf\{\|f + h\| : h \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) \cdot C_c(\mathcal{G}, \sigma)\} = \|\psi_x(f)\|$$

for every $f \in C_c(\mathcal{G}, \sigma)$. Observe that by continuity of ψ_x we have $\|f + h\| \geq \|\psi_x(f)\|$ for every $h \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) \cdot C_c(\mathcal{G}, \sigma)$. As $\mathcal{G}$ is second-countable locally compact Hausdorff, so is $\mathcal{G}^{(0)} \setminus \{x\}$. Hence it is paracompact, and we can guarantee that there is a countable partition of unity $\{\rho_i\}_{i=1}^{\infty}$ for $\mathcal{G}^{(0)} \setminus \{x\}$. For $n \in \mathbb{N}$ let $U_n := \mathcal{G}^{(0)} \setminus \bigcup_{i=1}^{n} \operatorname{supp}(\rho_i)$. Then we have

$$\|f - (\sum_{i=0}^{n} \rho_i) f\| \leq \max_{y \in U_n} \|\psi_y(f)\| .$$

By Lemma D.2.6 the assignment $\mathcal{G}^{(0)} \ni x \mapsto \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\}$ is continuous. It follows that for every $\varepsilon > 0$ there exists n such that $|\|\psi_y(f)\| - \|\psi_x(f)\|| < \varepsilon$ for every $y \in U_n$. As $U_k \supset U_{k-1}$ for all k, it follows that $\|f - (\sum_{i=0}^{k} \rho_i) f\| \leq \|\psi_x(f)\| + \varepsilon$ for all $k \geq n$. As ε was arbitrary, this finishes the proof. ∎

We may finally prove Theorem D.3.1.

Proof of Theorem D.3.1. By Proposition D.3.2 it suffices to show that the condition implies that $\ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$ is C^*-unique. As above, denote by $J_x = C_0(\mathcal{G}^{(0)} \setminus \{x\})$ and by $I_x := \overline{J_x \cdot \ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)}$ the resulting closed two-sided ideal in $\ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$. Let $\pi\colon \ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma) \to B(\mathcal{H})$ be a faithful $*$-representation and denote by $C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$ the completion of $\pi(\ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma))$. Moreover, let I_x^π denote the closure of $\pi(I_x)$ in $C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$. By Proposition D.2.5 and Lemma D.3.3 there is an isomorphism $C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma) \cong \Gamma_0(\mathbf{B}^\pi)$, where the fibers B_x^π, $x \in \mathcal{G}^{(0)}$, are given by

$$B_x^\pi = C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)/I_x^\pi .$$

We will show that there is an injective $*$-homomorphism

$$\Psi_x\colon \ell^1(\operatorname{Iso}(\mathcal{G})_x^\circ, \sigma_x) \to B_x^\pi$$

for every $x \in \mathcal{G}^{(0)}$. To do this, fix $x \in \mathcal{G}^{(0)}$. First, we show that the composition

$$\ell^1(\operatorname{Iso}(\mathcal{G})_x^\circ, \sigma_x) \cong \ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)/I_x \to C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)/I_x^\pi \cong B_x^\pi$$

given by first applying the isomorphism of Lemma D.3.4 and then applying the map $f + I_x \mapsto f + I_x^\pi$ for $f \in \ell^1(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$ is a well-defined continuous $*$-homomorphism. This is our candidate for the map Ψ_x. Denote by I_x^π also the image of the ideal $I_x^\pi \lhd C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$ in $\Gamma_0(\mathbf{B}^\pi)$. It then suffices to show that if $F \in I_x^\pi$, then $F(x) = 0$.

To see this, note that we can let $C_0(\mathcal{G}^{(0)} \setminus \{x\})$ act on $C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma)$ by pointwise multiplication to obtain a have a continuous $*$-homomorphism

$$C_0(\mathcal{G}^{(0)} \setminus \{x\}) = J_x \to \mathcal{Z}(M(C^*_\pi(\operatorname{Iso}(\mathcal{G})^\circ, \sigma))),$$

which leaves I_x^π invariant, and as a result I_x^π becomes a Banach J_x-module. It is even non-degenerate as

$$\overline{J_x I_x^\pi} = \overline{J_x \overline{J_x C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)}} \supset \overline{J_x J_x C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)} = \overline{J_x C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)} = I_x^\pi,$$

since J_x, being a C^*-algebra, has an approximate identity. It then follows by Cohen-Hewitt factorization that if $F \in I_x^\pi$, then $F = f \cdot H$, where $f \in J_x$ and $H \in I_x^\pi$. Then $F(x) = f(x)H(x) = 0$, and the map Ψ_x is a well-defined $*$-homomorphism.

As $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is dense in its C^*-completion $C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, it follows that the image of Ψ_x is dense.

Lastly, if $\Psi_x(f) = 0$, then $\Psi_x(f) \in I_x^\pi$, and so $f|_{\mathrm{Iso}(\mathcal{G})_x^\circ} = 0$ by the above argument. Thus ψ_x is injective. Hence we have a continuous dense embedding

$$\Psi_x \colon \ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x) \hookrightarrow C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J_x^\pi.$$

Now $C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J_x^\pi$ becomes a C^*-completion of $\ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x)$. Since π is an arbitrary faithful $*$-representation of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, we deduce that this holds for all faithful $*$-representations. But as $\ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x)$ is assumed C^*-unique, we may then deduce

$$C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J_x^\pi \cong C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J_x^{\mathrm{full}}, \tag{D.3.1}$$

where $C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ and J_x^{full} denotes the completions in the maximal C^*-norm. As $x \in \mathcal{G}^{(0)}$ was arbitrary, we deduce that this holds for all $x \in \mathcal{G}^{(0)}$. Now let $B_x^{\mathrm{full}} = C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J_x^{\mathrm{full}}$. By Proposition D.2.5 and (D.3.1) we then have

$$C_\pi^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \cong \Gamma_0(\mathbf{B}^\pi) \cong \Gamma_0(\mathbf{B}^{\mathrm{full}}) \cong C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma).$$

From this we deduce that $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, and hence also $\ell^1(\mathcal{G}, \sigma)$, is C^*-unique. ∎

D.4 Examples

In this section we present some (classes of) examples of C^*-unique groupoids. Due to the nature of our main result, Theorem D.3.1, our examples will draw upon previously proved results on C^*-uniqueness of locally compact groups. We begin with a class of examples in the case of trivial cocycle twists.

Example D.4.1 (The untwisted case)**.** If we consider a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with the trivial 2-cocycle $\sigma = 1$, then C^*-uniqueness of $\ell^1(\mathcal{G}, 1) = \ell^1(\mathcal{G})$ can by Theorem D.3.1 be deduced by C^*-uniqueness of the Banach $*$-algebras $\ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x) = \ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ)$ for $x \in \mathcal{G}^{(0)}$. C^*-uniqueness of untwisted convolution algebras has been studied before, and it is known that for a locally compact group G, the Banach $*$-algebra $\ell^1(G)$ is C^*-unique if G is a semidirect product of abelian groups, or a group where every compactly generated subgroup is of polynomial growth [22, p. 224]. Hence if for every $x \in \mathcal{G}^{(0)}$ the discrete group $\mathrm{Iso}(\mathcal{G})^\circ_x$ is of one of these types, $\ell^1(\mathcal{G})$ will be C^*-unique.

In the case of locally compact groups it is well-known that amenability of the group is equivalent to the group having the weak containment property. Indeed, amenability is even equivalent to the weak containment property when twisted for all continuous 2-cocycles σ of the group. Moreover, it is easy to see that if a group is C^*-unique, then it is amenable. The converse is however not true [22, p. 230]. In stark contrast to the case of locally compact groups, the following example shows that groupoids can be C^*-unique without even being amenable.

Example D.4.2 (Non-amenable C^*-unique groupoid)**.** In [2, Theorem 2.7] the authors constructed a second-countable, locally compact, Hausdorff non-amenable étale groupoid $\mathcal{G}$ such that $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$ and $C^*_r(\mathcal{G}) = C^*(\mathcal{G})$. Then since $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ) = C_0(\mathcal{G}^{(0)}) \subseteq \ell^1(\mathcal{G})$, we have by Proposition D.2.10 that every nonzero two-sided ideal I of $C^*(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}^{(0)})$, and hence with $\ell^1(\mathcal{G})$. Therefore by Proposition D.2.2 we have that $\ell^1(\mathcal{G})$ is C^*-unique.

In this particular case we may also deduce C^*-uniqueness of $\ell^1(\mathcal{G})$ in another way. Namely, as $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, we have that $\mathrm{Iso}(\mathcal{G})^\circ_x$ is the trivial group for every $x \in \mathcal{G}^{(0)}$. Hence $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x)$ is C^*-unique by Example D.4.1. This argument of course carries over to any topologically principal groupoid. Indeed, this approach shows that whenever $\mathcal{G}$ is a second-countable, locally compact, Hausdorff topologically principal étale groupoid, then $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique for any $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$.

We also have classes of examples that includes more general cocycle twists.

Example D.4.3 (The twisted case)**.** Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid, and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. By Theorem D.3.1 C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ can be deduced by C^*-uniqueness of the Banach $*$-algebras $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$, for $x \in \mathcal{G}^{(0)}$, where σ_x as before denotes the restriction of σ to $\mathrm{Iso}(\mathcal{G})^\circ_x$. C^*-uniqueness of twisted convolution algebras of locally compact groups was studied in [8]. In [8, Theorem 3.1] it was found that if G is a locally compact group and $c \in Z^2(G, \mathbb{T})$, then $L^1(G, c)$ is C^*-unique if $L^1(G_c)$ is C^*-unique, where G_c denotes the Mackey group associated to G and c. As a topological space G_c is just $G \times \mathbb{T}$, but the binary operation is given by

$$(x, \tau) \cdot (y, \eta) = (xy, \tau\eta\overline{c(x, y)}).$$

We may relate C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x})$, where $\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x}$ denotes the Mackey group associated to $\mathrm{Iso}(\mathcal{G})^\circ_x$ and σ_x, and we deduce that $\ell^1(\mathcal{G},\sigma)$ is C^*-unique if $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x})$ is C^*-unique for every $x \in \mathcal{G}^{(0)}$. This happens if, for example, $\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x}$ is a group of one of the types discussed in Example D.4.1.

In the following example we are able to deduce C^*-uniqueness of a locally compact group not of the form discussed in Example D.4.1 by relating the question to C^*-uniqueness of a groupoid.

Example D.4.4 (The wreath product)**.** Let Γ denote the wreath product $H \wr G := \left(\bigoplus_G H\right) \rtimes G$ where H is a finite abelian group and where G is a countable discrete amenable group. We will show that $\ell^1(\Gamma)$ is C^*-unique.

To do this, let $\mathcal{G} = X \rtimes_\varphi G$ be the transformation groupoid where $X = \prod_G \hat{H}$, and φ is the shift homeomorphism of X by G. $\mathcal{G}$ is amenable since G is amenable. Then we have that

$$C^*(\Gamma) \cong C^*(\bigoplus_G H) \rtimes_\varphi G \cong C(X) \rtimes_\varphi G\,.$$

Now recall that by the Fourier transform $\ell^1(\bigoplus_G H) \cong A(X)$, where $A(X)$ is a dense subalgebra of $C(X)$. Indeed, it becomes a Banach $*$-subalgebra of $C(X)$ when equipped with the induced ℓ^1-norm through the Fourier transform, and then the isomorphism is also an isometry. It also follows that $C(X)$ is the completion of $\ell^1(\bigoplus_G H)$ with respect to some C^*-norm. We have that $\ell^1(\Gamma) \cong \ell^1(\ell^1\left(\bigoplus_G H\right), G) \cong \ell^1(A(X), G)$ (see for example [78, Remark and Notation 2.4]). Then there exists an isometric embedding $\iota : \ell^1(A(X), G) \hookrightarrow \ell^1(\mathcal{G})$ defined as follows. If $F \in \ell^1(A(X), G)$, we define $\iota(F)$ to be

$$\iota(F)(x, g) = \widehat{f_g}(x),$$

for $x \in X = \prod_G \hat{H}$ and $g \in G$, where f_g is the unique element of $\ell^1(\bigoplus_G H)$ with $\widehat{f_g} = F(g)$. Therefore by the isomorphisms $C^*(\ell^1(\Gamma)) \cong C^*(\ell^1(A(X), G)) \cong C^*(\ell^1(\mathcal{G}))$ it would be enough to check that any nonzero two-sided ideal I of $C^*(\mathcal{G})$ has a non-trivial intersection with the image of $\ell^1(A(X), G)$ by the inclusion ι. Observe that then $\ell^1(\bigoplus_G H) \subseteq \ell^1(\Gamma)$ can be identified with $\iota(A(X))$ in $C(X) \subseteq C^*(\mathcal{G})$. The groupoid $\mathcal{G}$ is clearly topologically principal, and hence $\ell^1(\mathcal{G})$ is C^*-unique. Moreover, for every closed two-sided ideal $\{0\} \neq I \trianglelefteq C^*(\mathcal{G})$ we have that $\{0\} \neq J := I \cap C(X)$ [71, Theorem 4.1]. But since $\bigoplus_G H$ is locally finite, then $\ell^1(\bigoplus_G H)$, and hence $A(X)$, are C^*-unique by [51]. Thus, $J \cap A(X) \neq \{0\}$, which further implies $J \cap \ell^1(A(X), G) \neq \{0\}$. It follows that $\ell^1(\Gamma)$ is C^*-unique.

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