Adaptive Control of a Scalar 1-D Linear Hyperbolic PDE with Uncertain Transport Speed Using Boundary Sensing

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***Abstract*— We solve an adaptive boundary control problem for an 1-D linear hyperbolic partial differential equation (PDE) with an uncertain in-domain source parameter and uncertain transport speed using boundary sensing only. Convergence of the parameters to their true values is achieved in finite-time. Since linear hyperbolic PDEs are finite-time convergent in the non-adaptive case, finite-time parameter convergence leads to the system state converging in finite-time. This is achieved by combining a recently derived transport speed estimation scheme using boundary sensing only, with the swapping scheme for hyperbolic PDEs and a least-squares identifier of an event-triggering type. The method is demonstrated in simulations.**

I. INTRODUCTION

A. Background

Systems of linear hyperbolic partial differential equations (PDEs) are used to model various flow and transport phenomena, ranging from electrical transmission lines and propagation of light in optical fibers, to flow in blood vessels and the propagation of epidemics (see e.g. [1] for an overview). These systems therefore naturally give rise to important estimation and control problems, with methods ranging from the use of control Lyapunov functions [2], frequency domain approaches [3] and active disturbance rejection control (ADRC) [4] to the backstepping method [5].

In the last few years, many results on adaptive estimation and control of systems of linear hyperbolic partial differential equations have been published. The first result was in [6], half a decade ago, where a 1-D system with a spatially varying in-domain source coefficient was adaptively stabilized using a filter based technique known as swapping design in conjunction with the backstepping method. Only boundary sensing was required. In [7], backstepping was used again, adaptively stabilizing the same type of systems using a Lyapunov approach, but requiring the use of full state measurements.

Before the result in [6] for hyperbolic PDEs, backstepping for parabolic PDEs had been extensively studied, particularly in the work by Smyshlyaev and Krstic. Important results include [8], [9] and [10] for the non-adaptive case, and [11], [12], [13] and [14] for the adaptive case.

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In [15] and [16], the method from [6] is extended to more complicated classes of hyperbolic PDEs, resulting in output-feedback solutions to systems of coupled hyperbolic PDE. Other results concerning uncertain boundary parameters include [17] and [18], and most of these backstepping-based adaptive results are summarized in [19].

However, common for the above mentioned results for hyperbolic PDEs, is that the system's transport speed is always assumed known. To the best of the authors' knowledge, no result exists on adaptive control of a hyperbolic PDE with an uncertain transport speed. The difficulty in having an uncertain transport speed comes from the fact that the transport speed appear in terms also including non-measured distributed spatial derivatives, which excludes commonly used techniques in adaptive control from being applied.

In [20], an event-triggered least-squares identifier was combined with a backstepping controller to achieve adaptive control of a constant-parameter parabolic PDE. The proposed methodology is an extension of the finite-time convergent event-triggered least-squares identification method and controller for nonlinear ordinary differential equations (ODEs) presented in [21]. However, the controller in [20] is a state-feedback one, requiring full-state measurements, and finite-time convergence of the state to zero can in general not be achieved for parabolic PDEs.

In this paper, we present the first result on backstepping-based adaptive boundary control of a linear hyperbolic PDE, where the transport speed is allowed to be uncertain. This is achieved by combining the boundary-sensing-based transport speed estimator from [22] with an event-triggered finite-time convergent estimator scheme which is motivated by the scheme presented in [21]. Since the transport speed is estimated in finite time, one can proceed as before with the swapping-based technique from [6] to estimate any of the other uncertain parameters considered in [6]. However, we slightly modify this technique to also estimate this parameter in finite time, for the scalar parameter case. Since linear hyperbolic PDEs of the type considered in [6] and this paper are finite-time convergent in the non-adaptive case, finite-time parameter convergence of the parameters leads to the system state converging in finite-time.

B. Notation

For two (possibly time-varying) signals $a(x), b(x)$ defined for $x \in [0,1]$, we define the operator $\equiv$ as

$$a \equiv b \Leftrightarrow \max_{x\in[0,1]} |a(x) - b(x)| = 0. \tag{1}$$

II. Problem statement

We consider the following simple type of scalar 1-D hyperbolic transport PDE with uncertain transport speed μ, and an uncertain in-domain source coefficient θ

$$u_t(x,t) - \mu u_x(x,t) = \theta u(0,t) \quad (2a)$$
$$u(1,t) = U(t) \quad (2b)$$
$$u(x,0) = u_0(x) \quad (2c)$$

where we assume

$$\mu \in \mathbb{R}, \quad \mu > 0, \quad \theta \in \mathbb{R}, \quad u_0 \in C^1([0,1]). \quad (3a)$$

The goal is to design a boundary control law U that adaptively stabilizes system (2), from boundary sensing only. We assume the following boundary measurements are available

$$y_0(t) = u(0,t), \qquad y_1(t) = u(1,t) \quad (4a)$$
$$\vartheta_0(t) = u_x(0,t), \qquad \vartheta_1(t) = u_x(1,t), \quad (4b)$$

where we note that $y_1 = U$.

Previous backstepping-based adaptive control results for hyperbolic PDEs employing boundary sensing have only used the Dirichlet-type boundary measurements on the form (4a) [19], however, none of these results considered an uncertain transport speed. All four methods proposed in [22] for estimation of the transport speeds in hyperbolic PDEs, required measurement of the spatial derivative, and only one of those four limited the measurements to be taken at the boundaries. The method considered in this paper is based on this method, and combines it with a finite-time convergent least-squares estimation scheme previously used in [21] and [20].

Note that we assume a continuously differentiable initial condition u_0. Moreover, we will assume that the boundary control law $U(t)$ is compatible so that the state u always will remain in $C^1([0,1])$. This is formally stated in the following assumption:

Assumption 1: We assume $u(t) \in C^1([0,1])$.

This is a property that is generally not the case for hyperbolic PDEs, but is required for the analysis that follows. We will achieve this by simply only consider solutions that are compatible with this assumption.

Regarding the uncertain parameters, we assume the following

Assumption 2: A lower bound on μ is known; specifically, we are in knowledge of a positive parameter $\underline{\mu}$ so that

$$\underline{\mu} \leq \mu. \quad (5)$$

III. Events and event triggers

We will use a time-triggered series of events. For simplicity, we will use a fixed interval between each event. This is justified by the fact that a linear hyperbolic PDE on the form (2) is finite-time observable [23], within a time d, defined as

$$d = \mu^{-1} \quad (6)$$

which is an unknown quantity. However, this also implies that system (2) is observable within a time T, defined as

$$T = \underline{\mu}^{-1} \quad (7)$$

which, by Assumption 2, satisfies $T \geq d$. We therefore choose to trigger a new event every T seconds; hence, the ith event happens at times τ_i, where

$$\tau_i = iT. \quad (8)$$

Of course, the performance, especially in terms of convergence times, can be improved by having non-periodic triggers, but we choose this periodic trigger due to its simplicity and such that the time length between triggers is sufficient to capture all necessary traits of the system.

IV. Estimation of the transport speed

A. Dynamics of u_x

We now derive a trigger-based finite-time convergent boundary estimator for the transport speed μ. Firstly, we derive the dynamics of the spatial derivative u_x, and define the new variable v as

$$v(x,t) = u_x(x,t). \quad (9)$$

By using the dynamics (2), we obtain

$$v_t(x,t) - \mu v_x(x,t) = 0 \quad (10a)$$
$$v(1,t) = d\dot{U}(t) - d\theta u(0,t) \quad (10b)$$
$$v(x,0) = v_0(x) \quad (10c)$$

where $v_0(x) = u_0'(x)$.

B. Parametric form

Define the measured, scalar signal

$$\begin{aligned}\eta(t) &= y_1(t) - y_0(t) \\ &= u(1,t) - u(0,t) = \int_0^1 u_x(x,t)dx \\ &= \int_0^1 v(x,t)dx. \end{aligned} \quad (11)$$

Differentiating (11) with respect to time, we obtain

$$\dot{\eta}(t) = \int_0^1 v_t(x,t)dx. \quad (12)$$

Inserting the dynamics (10a) and integrating gives

$$\begin{aligned}\dot{\eta}(t) &= \mu \int_0^1 v_x(x,t)dx = \mu(v(1,t) - v(0,t)) \\ &= \mu\omega(t) \end{aligned} \quad (13)$$

where

$$\omega(t) = u_x(1,t) - u_x(0,t) = \vartheta_1(t) - \vartheta_0(t) \quad (14)$$

is a measured signal. Integrating (13) from σ to t, we obtain

$$p(t,\sigma) = \mu q(t,\sigma) \quad (15)$$

where

$$p(t,\sigma) = \eta(t) - \eta(\sigma) \tag{16a}$$

$$q(t,\sigma) = \int_{\sigma}^{t} \omega(s)ds. \tag{16b}$$

From (15), it is evident that whenever $q(t,\sigma) \neq 0$, μ can be computed from $\mu = \frac{o(t,\sigma)}{q(t,\sigma)}$. However, we will use a technique similar to the one in [21], which is suited for estimation in the case of measurement noise and unmodeled disturbances.

Consider the cost function

$$J_i(\alpha) = \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{\tau_i} (p(t,\sigma) - \alpha q(t,\sigma))^2 d\sigma dt \tag{17}$$

which equally weights all measurements since the last event. It is clear that $\alpha = \mu$ minimizes (17). Evaluating $\frac{\partial J_i(\alpha)}{\partial \alpha} = 0$, we obtain

$$P_i = \mu Q_i \tag{18}$$

where

$$P_i = \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{\tau_i} p(t,\sigma) q(t,\sigma) d\sigma dt \tag{19a}$$

$$Q_i = \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{\tau_i} q^2(t,\sigma) d\sigma dt. \tag{19b}$$

Again, it is evident that if $Q_i \neq 0$, the transport speed μ can be computed as $\mu = \frac{P_i}{Q_i}$. In the next subsection, we investigate in what cases $Q_i = 0$, and an estimate will not be achievable.

C. Requirements for convergence of $\hat{\mu}$ to μ

In the case of an event, $t = \tau_i$, it is possible that $Q_i = 0$, and hence μ cannot be computed from (18). This is addressed in the following lemma.

Lemma 3: Let $U(t) = 0$ for all $t \leq \tau_i$. If either

- $\mu \neq \theta$ or
- $\mu = \theta$ and $u_0(x) \neq k(1-x)$ for any $k \in \mathbb{R}$,

then $Q_i = 0$ if and only if $u(t) \equiv 0$ for all $t \in [\tau_{i-1}, \tau_i]$, $i \geq 2$.

Proof: First of all, the case $k = 0$ is trivial, as $u_0 \equiv 0$ yields $u(t) \equiv 0$ for all $t \geq 0$. Hence, we only consider $k \neq 0$. From the definitions of Q_i and q in (19b) and (16b), respectively, and the assumption of having a continuous u, it is apparent that $Q_i = 0$ implies

$$q(t,\sigma) = 0, \ \forall t, \sigma \in [\tau_{i-1}, \tau_i], \tag{20}$$

which from (15), also implies

$$p(t,\sigma) = 0, \ \forall t, \sigma \in [\tau_{i-1}, \tau_i]. \tag{21}$$

From the definition of p and η in (16a) and (11), with the assumption $u(1,t) = U(t) = 0$ until μ is estimated, we then have

$$u(0,t) = u(0,\sigma), \ \forall t, \sigma \in [\tau_{i-1}, \tau_i], \tag{22}$$

and hence $u(0,t)$ is a constant, say

$$u(0,t) = c, \ \forall t \in [\tau_{i-1}, \tau_i]. \tag{23}$$

Using the method of characteristics, we find

$$\frac{d}{ds} u(x + \mu s, t - s) = -\theta u(0, t-s) \tag{24}$$

Integration from $s = 0$ to $s = d(1-x)$, with $U(t) = 0$, gives

$$u(x,t) = \theta \int_0^{d(1-x)} u(0, t-s) ds. \tag{25}$$

If $c = 0$, we then have $u(t) \equiv 0$, which is trivial. If $c \neq 0$, we insert $u(0,t) = c$ and evaluate (25) at $x = 0$ to obtain

$$c = \theta c d \tag{26}$$

and hence $Q_i = 0$ and $c \neq 0$ implies

$$\theta = \mu. \tag{27}$$

From (10), we have that $v(x,t) = v(1, t - d(1-x)) = -c$ for all $x \in [0,1]$, $t \leq \tau_i$, which by integration gives $u(x,t) = c(1-x)$ for all $t \leq \tau_i$. However, as we have assumed $u_0(x) \neq k(1-x)$ for any $k \in \mathbb{R}$, this cannot be the case. Hence, the conditions in the lemma and $Q_i = 0$ implies that $u(t) \equiv 0$. ■

Remark 4: For the special case $\theta = \mu$, all initial conditions on the form

$$u_0(x) = k(1-x) \tag{28}$$

give $Q_i = 0$ for all $i \geq 1$. This can be seen by inserting (28) into (2), showing that (2) has the unique solution $u(x,t) = u_0(x) = k(1-x)$ for all $t \geq 0$, which in the proof of Lemma 3 was shown to imply $Q_i = 0$. In this case, let $U(t) = U_{ex}(t)$ for $t = [0,d]$ for some $U_{ex}(t) \neq 0$ such that $u(x,d) \neq u_0(x)$ and the result of Lemma 3 holds for all initial conditions for all $i \geq 3$. We can therefore in the following assume without loss of generality that $u_0(x) \neq k(1-x)$, since the case $u_0(x) = k(1-x)$ can be handled by just applying $U = U_{ex}$ if $Q_1 = 0$. The signal U_{ex} is up to the designer; but a generic non-zero signal should suffice.

D. Estimation law

Based on the above derivations, we suggest Algorithm 1 for estimating μ, for which Theorem 5 holds.

Theorem 5: Consider system (2), with $U(t) = 0$ for all $t \leq \tau_1$. If $u(t)$ is not identically equal to zero for all $t \leq \tau_1$, then the estimation method of Algorithm 1 produces the correct estimate μ for $t \geq \tau_1$.

We will for the remainder of the paper denote the event for which μ is estimated as the i_μth event, so that τ_{i_μ} is the time for which $\hat{\mu}(t) = \mu$, $\forall t \geq \tau_{i_\mu}$.

V. Estimation of θ

A. Filter design and parametric form

With μ estimated using the method of Theorem 5, we can now proceed as before in estimation of the uncertain parameter θ, using the filter-based method presented in e.g. [6]. However, we will slightly modify this method so that

Algorithm 1 Estimation of μ

1) Let
$$\hat{\mu}(0) = \hat{\mu}_0. \tag{29}$$
for some $\hat{\mu}_0 \geq \underline{\mu}$.
2) At event i, set $\hat{\mu}$ as
$$\hat{\mu}(\tau_i) = \begin{cases} \dfrac{P_i}{Q_i} & \text{if } Q_i > 0 \\ \hat{\mu}(\tau_{i-1}) & \text{otherwise} \end{cases} \tag{30}$$
where P_i and Q_i are generated from (19).
3) For all times $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \dots$ between events, the estimate $\hat{\mu}$ is set to the most recent event-triggered estimate, that is
$$\hat{\mu}(t) = \hat{\mu}(\tau_i), \quad \forall t \in [\tau_i, \tau_{i+1}). \tag{31}$$

finite-time convergence is achieved. We introduce the filters

$$\phi_t(x,t) - \hat{\mu}(t)\phi_x(x,t) = y_0(t), \qquad \phi(1,t) = 0$$
$$\phi_0(x,0) = \phi_0(x) \tag{32a}$$
$$\psi_t(x,t) - \hat{\mu}(t)\psi_x(x,t) = 0, \qquad \psi(1,t) = U(t)$$
$$\psi_0(x,0) = \psi_0(x) \tag{32b}$$

for some initial conditions $\phi_0, \psi_0 \in \mathcal{C}^1([0,1])$ of choice. Using these filters, a non-adaptive estimate of the state u can be generated as

$$\bar{u}(x,t) = \psi(x,t) + \theta\phi(x,t). \tag{33}$$

The following lemma immediately follows

Lemma 6: Consider system (2), the state estimate (33) generated using the filters (32), with $\hat{\mu}$ generated using the method of Theorem 5. Then

$$\bar{u}(t) = u(t) \tag{34}$$

for all $t \geq \tau_{i_\mu} + d$.

Proof: We will prove that the signal

$$e(x,t) = u(x,t) - \bar{u}(x,t) \tag{35}$$

satisfies the dynamics

$$e_t(x,t) - \mu e_x(x,t) = 0 \tag{36a}$$
$$e(1,t) = 0 \tag{36b}$$
$$e(x,0) = e_0(x). \tag{36c}$$

We evaluate

$$\begin{aligned} e_t(x,t) &= u_t(x,t) - \psi_t(x,t) - \theta\phi_t(x,t) \\ &= \mu u_x(x,t) + \theta u(0,t) - \hat{\mu}(t)\psi_x(x,t) \\ &\quad - \theta\hat{\mu}(t)\phi_x(x,t) - \theta u(0,t) \end{aligned} \tag{37}$$

Utilizing that $\hat{\mu}(t) = \mu$ for $t \geq \tau_{i_\mu}$, we obtain

$$\begin{aligned} e_t(x,t) &= \mu(u_x(x,t) - \psi_x(x,t) - \theta\phi_x(x,t)) \\ &= \mu e_x(x,t). \end{aligned} \tag{38}$$

which proves (36a). Inserting $x = 1$ into (35) yields

$$\begin{aligned} e(1,t) &= u(1,t) - \psi(1,t) - \theta\phi(1,t) \\ &= U(t) - U(t) = 0 \end{aligned} \tag{39}$$

Lastly, the initial condition is given as

$$e_0(x) = u_0(x) - \psi_0(x) - \theta\phi_0(x). \tag{40}$$

From the simple dynamics (36) valid for $t \geq \tau_{i_\mu}$, it is evident that $e \equiv 0$ after an additional time $d = \mu^{-1}$, which produces the desired result. ■

We will for the remainder of the paper denote the event for which $\bar{u} = u$ as the $i_{\bar{u}}$th event, so that $\tau_{i_{\bar{u}}}$ is the time for which $\bar{u}(t) \equiv u(t)$, $\forall t \geq \tau_{i_{\bar{u}}}$.

B. Estimation

Following Lemma 6, we must have $\bar{u}(t) = u(t)$ for $t \geq \tau_{i_{\bar{u}}} = \tau_{i_\mu+1} = \tau_{i_\mu} + T \geq \tau_{i_\mu} + d$, which is the time of the event immediately after the i_μth event. Thus, for $t \geq \tau_{i_{\bar{u}}}$, we have

$$z(t) = y_0(t) - \psi(0,t) = \theta\phi(0,t) \tag{41}$$

Integrating from σ to t, and using the same techniques as for the transport speed, we obtain

$$A_i = \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{\tau_i} a(t,\sigma) b(t,\sigma) d\sigma dt \tag{42a}$$
$$B_i = \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^{\tau_i} b^2(t,\sigma) d\sigma dt \tag{42b}$$

where

$$a(t,\sigma) = \int_\sigma^t z(s) ds$$
$$b(t,\sigma) = \int_\sigma^t \phi(0,s) ds. \tag{43a}$$

As with the estimation of μ, it is evident that if $B_i \neq 0$, the parameter θ can be computed as $\theta = \frac{A_i}{B_i}$. In the next subsection, we investigate in what case $B_i = 0$.

C. Convergence of $\hat{\theta}$ to θ

Lemma 7: $B_i = 0$ if and only if $u(t) \equiv 0$ for all $t \in [\tau_{i-1}, \tau_i]$, for $i \geq i_{\bar{u}} + 1$.

Proof: From the definition of B_i, we conclude that $B_i = 0$ implies $b(t,\sigma) = 0$ for all $t, \sigma \in [\tau_{i-1}, \tau_i]$, which in turn implies $\phi(0,t) = 0$ for all $t \in [\tau_{i-1}, \tau_i]$. From the definition of the filter ϕ, we then conclude that $u(0,t)$ must be zero for $t \in [\tau_{i-1}, \tau_i]$, which, by the finite-time observability property of systems of the type (2) (see [23]), must imply $u \equiv 0$. ■

The control goal is $u(t) \equiv 0$ for some t. Hence, $B_i = 0$ if and only if $u(t) \equiv 0$, meaning that $u(t) \not\equiv 0$ will result in $B_i \neq 0$, and the possibility of estimating θ in finite time.

Algorithm 2 Estimation of θ

1) Let

$$\hat{\theta}(0) = \hat{\theta}_0. \tag{44}$$

for some $\hat{\theta}_0 \in \mathbb{R}$.

2) Consider the first event after the $i_{\bar{u}}$th event, that is the $(i_{\bar{u}}+1)$th event. Set $\hat{\theta}$ as

$$\hat{\theta}(\tau_{i_{\bar{u}}+1}) = \begin{cases} \frac{A_{i_{\bar{u}}+1}}{B_{i_{\bar{u}}+1}} & \text{if } B_{i_{\bar{u}}+1} > 0 \\ \hat{\theta}(\tau_{i_{\bar{u}}}) & \text{otherwise} \end{cases} \tag{45}$$

where $A_{i_{\bar{u}}+1}$ and $B_{i_{\bar{u}}+1}$ are generated from (42).

3) For all times $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \dots$ between events, the estimate $\hat{\theta}$ is set to the most recent event-triggered estimate, that is

$$\hat{\theta}(t) = \hat{\theta}(\tau_i), \quad \forall t \in [\tau_i, \tau_{i+1}). \tag{46}$$

D. *Estimation law*

Again, based on the above derivations, we suggest the following algorithm for estimating θ in finite time.

Theorem 8: Consider system (2). Assume μ has been estimated correctly using Algorithm 1. Then, provided $u(t)$ is not identically zero for all $\tau_{i_{\bar{u}}} \leq t \leq \tau_{i_{\bar{u}}}$, the estimation method of Algorithm 2 produces the correct estimate θ for $t \geq \tau_{i_{\bar{u}}+1}$.

For the remainder of the paper, we will denote the event for which θ is estimated as the i_θth event, so that $\hat{\theta}(t) = \theta$, $\forall t \geq \tau_{i_\theta}$.

VI. Adaptive control

Consider the control law

$$U(t) = -\frac{\hat{\theta}(t)}{\hat{\mu}(t)} \int_0^1 \exp\left(\frac{\hat{\theta}(t)}{\hat{\mu}(t)}(1-\xi)\right) \hat{u}(\xi, t) d\xi \tag{47}$$

where $\hat{u}$ is an adaptive state estimated generated as

$$\hat{u}(x,t) = \psi(x,t) + \hat{\theta}(t)\phi(x,t), \tag{48}$$

which is obtained by replacing θ by its estimate $\hat{\theta}$ in (33).

Theorem 9: Consider system (2), and the control law (47) with $\hat{\mu}$ and $\hat{\theta}$ generated using Theorems 5 and 8, respectively. Then

$$u(t) \equiv 0, \text{for } t \geq \tau_{i_\theta} + d \tag{49a}$$

$$\psi(t), \phi(t) \equiv 0, \text{for } t \geq \tau_{i_\theta} + 2d \tag{49b}$$

Proof: Since, by Theorems 5 and 8, we have $\hat{\mu}(t) = \mu$ and $\hat{\theta}(t) = \theta$ for $t \geq \tau_{i_\theta}$, the proof of (49a) follows directly from [19, Theorem 5.1]. From the structure of the filter in (32a), it is then evident that we must have $\phi(t) \equiv 0$ for $t \geq \tau_{i_\theta} + 2d$. From (33) and Lemma 6, $\psi(t) \equiv 0$ follows for $t \geq \tau_{i_\theta} + 2d$.

The explicit form of the controller gain in (47) is derived in e.g. [19, Example 3.1]. ■

Remark 10: In order to convey the main idea as clearly as possible, we have not payed attention to ensuring that solutions stay in $C^1([0,1])$, which is a premise for the analysis. To have solutions stay in $C^1([0,1])$, we would need modifications to ensure that $U(t) = \lim_{x\to 1} u(x,t)$ and $U \in C^1([0,\infty))$. Loosely speaking, we could restrict initial conditions to be compatible with $U(0) = 0$ and ensure smooth transitions between parameter updates in point 3) of the two algorithms.

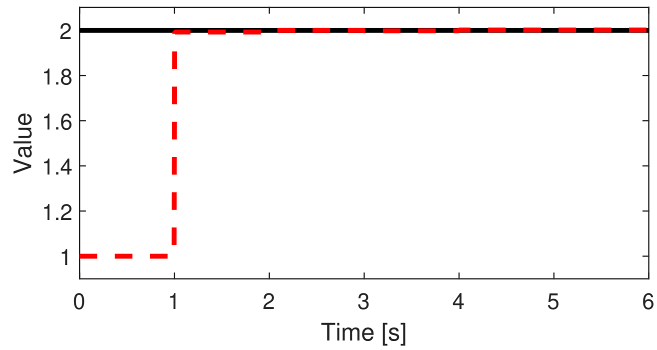

Fig. 1. Actual (solid black) and estimated (dashed red) μ for Case 1.

VII. Simulations

System (2) was implemented in MATLAB, along with the parameter observers of Algorithms (1) and (2), and the control law (47). Due to numerical issues when implementing on a computer, the requirements of Algorithms 1 and 2 of having $Q_i > 0$ and $B_i > 0$ respectively, are substituted by $Q_i > \epsilon_0$ and $B_i > \epsilon_0$, respectively, for a small number $\epsilon_0 > 0$. For the simulations in this section, we have chosen $\epsilon_0 = 10^{-3}$.

A. *Case 1*

The system parameters and initial condition were set to

$$\mu = 2 \qquad \theta = 3, \qquad u_0(x) = x \tag{50}$$

constituting an unstable system. The initial guesses and lower bound $\underline{\mu}$, were set to

$$\hat{\mu}_0 = 1, \qquad \hat{\theta}_0 = 0, \qquad \underline{\mu} = 1, \tag{51}$$

which corresponds to $T = 1$, and hence an event is triggered every second. The simulation results are found in Figures 1–4. From Figure 1, one can clearly se that the value of μ is correctly estimated at the first event, while θ is correctly estimated on the third event as predicted by theory. However, the estimated θ after the first event is very close to the actual value, resulting in the control managing to stabilize the system. Convergence to zero is not achieved before θ is estimated correctly and an additional time $d = \frac{1}{\mu} = 0.5$ seconds. Hence, the system norm converges to zero within a finite time $\tau_{i_\mu} + d = 3.5$ s, as predicted in Theorem 9.

B. *Case 2*

For this case, we choose

$$\mu = \theta = 2, \qquad u_0(x) = 2(1-x) \tag{52}$$

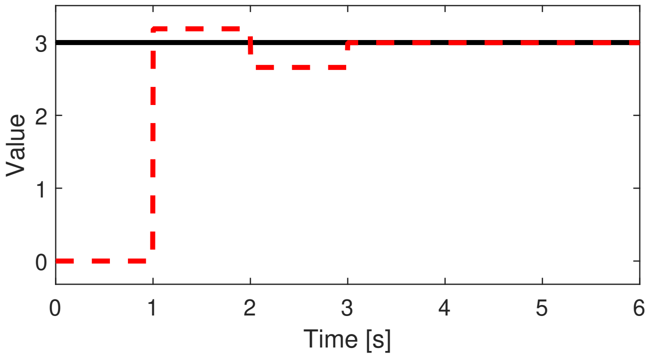

Fig. 2. Actual (solid black) and estimated (dashed red) θ for Case 1.

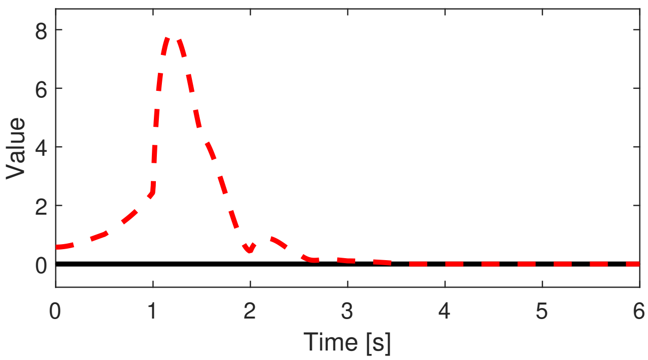

Fig. 3. System norm $||u||$ for Case 1.

which satisfy the conditions of Remark 4. The remaining parameters are the same as in Case VII-A. The exciting actuation signal U_{ex} which will be applied for $t \in [\tau_1, \tau_2]$ is set as

$$U_{ex}(t) = 1. \tag{53}$$

Note that the parameters and initial condition satisfying the requirements of Remark 4 occurs with probability zero. However, we include a simulation here for illustrative purposes.

The simulations results are found in Figures 5–8. It is observed that the system state now is stable in the uncontrolled case, as seen from the constant system norm shown in Figure 7 until $t = \tau_1 = 1$, when U is set to U_{ex} as seen in Figure 8. Due to this, the estimator for μ at the first event fails,

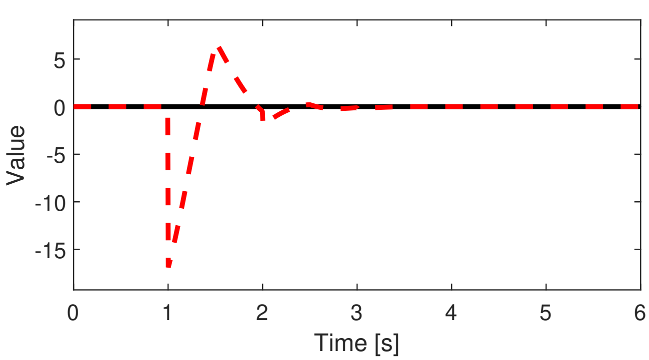

Fig. 4. Actuation signal U for Case 1.

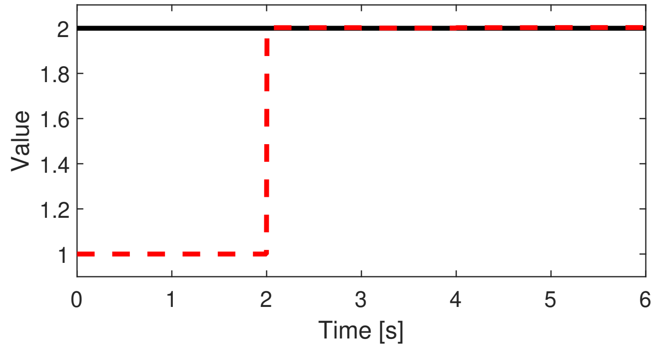

Fig. 5. Actual (solid black) and estimated (dashed red) μ for Case 2.

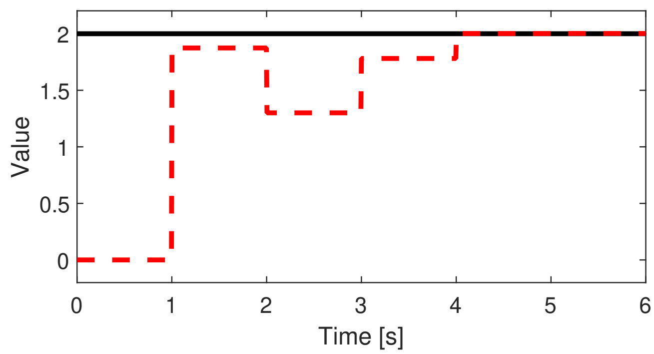

Fig. 6. Actual (solid black) and estimated (dashed red) θ for Case 2.

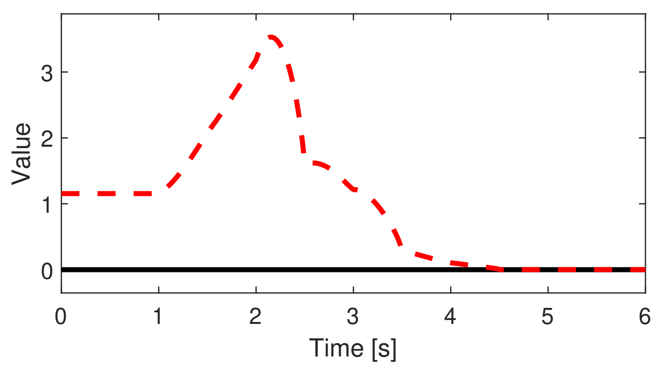

Fig. 7. System norm $||u||$ for Case 2.

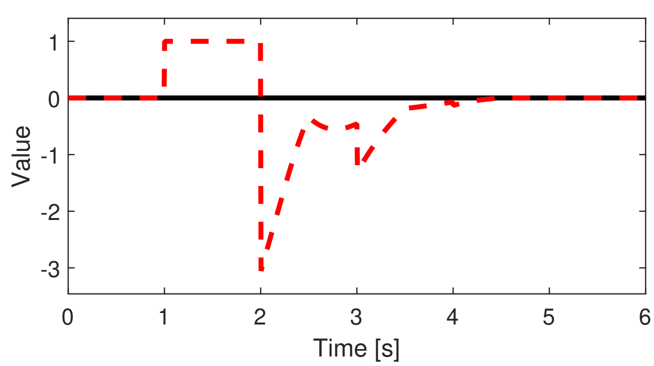

Fig. 8. Actuation signal U for Case 2.

as predicted by Theorem 5, to produce the correct estimate at $t = \tau_1$ as seen in Figure 5. However, it produces the correct estimate at the next event at $t = \tau_2 = 2$. The correct estimate for θ is then seen from Figure 6 to be computed at $t = \tau_4 = 4$ as predicted by theory, and the state norm converges to zero within $t = \tau_4 + d = 4.5\ s$.

VIII. Concluding remarks

We have combined a recently derived transport speed estimation scheme using boundary sensing only, with the swapping scheme for hyperbolic PDEs and a least-squares identifier of an event-triggering type, into an adaptive controller for a type of scalar 1-D liner hyperbolic PDE with an uncertain transport speed. Simulations showed finite-time convergence of the estimated parameters to their true values, and of the system state to zero.

The finite-time convergence of the state by using Algorithms 1, 2 and the control law (47) are done in the following four steps:

1) Step 1: Estimation of the transport speed: $\hat{\mu} = \mu$ for $t \geq \tau_i$
2) Step 2: The linear parametric form is valid: $\bar{u} = u$ for $t \geq \tau_{i+1}$
3) Step 3: Estimation of the in-domain source parameter: $\hat{\theta} = \theta$ for $t \geq \tau_{i+2}$
4) Step 4: Convergence of the state: $u \equiv 0$ for $t \geq \tau_{i+2}+d$

where τ_i denotes the ith event. If the system parameters and the initial condition are different from the special case considered in Remark 4, the above events will be triggered for $i = 1$, otherwise, $i = 2$, provided the excitation signal U_{ex} is chosen according to Algorithm 1.

Topics for further investigation include:

1) Non-periodic trigger times: The events are in the method presented in this paper triggered by time. However, it should be possible to achieve better convergence times if the events are allowed to be triggered by for instance having gathered measurements for a sufficiently long time series to produce a correct estimate.
2) Improved robustness should be possible to achieve by using measured data for a longer period backwards in time than just until the most recent trigger.
3) Systems with a spatially varying coefficient θ should be possible to handle. However, as this involves an infinite number of unknowns, finite-time convergence cannot be achieved.

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