Adaptive Control of a Linear, Scalar Hyperbolic PDE with Time-Varying Coefficients

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***Abstract*— We extend a previous result regarding adaptive control of a linear hyperbolic partial differential equation (PDE) with time-varying in-domain source coefficient in two ways. Firstly, we introduce a parametrization of the uncertain time-varying in-domain source coefficient that allows for a broader class of systems compared to previous result. Secondly, and more importantly, we introduce an uncertain scaling factor in the input boundary condition which is present in most applications, but wasn't handled in the previous result. All system parameters except the transport speed are uncertain and time-varying, although parametrizable as a linear combination of uncertain constants and certain time-variance. Closed-loop convergence of the state to the origin is proven, and performance is demonstrated for a numerical example.**

I. Introduction

A. Background

Systems of hyperbolic partial differential equations (PDEs) can be used to describe a vast range of different physical systems, ranging from predator-prey systems [1], to oil wells and road traffic [2]. Systems of this type have therefore been subject to extensive research, and several approaches have been used for design of estimators and controllers for such systems, including Lyapunov functions [3], Riemann invariants [4], and frequency domain approaches [5].

For the last two decades, the infinite-dimensional backstepping method has been used for controller and observer design of systems of PDEs. When using this method, an invertible Volterra integral transformation is introduced, and an accompanying control law or injection gains are designed that maps the system of interest into a target system designed to possess some desirable stability properties. Due to the invertibility of the transform, the stability properties of the two systems are the same. A huge advantage with this method, is that the designs are taken on the infinite-dimensional system directly, avoiding any uncertainties that results from discretization.

Infinite-dimensional backstepping was initially proposed for parabolic PDEs in [6], where it was used for non-adaptive stabilization of an unstable heat equation. The method was quickly expanded in numerous directions, including: Non-adaptive state-feedback control laws for a class of parabolic PDEs [7] and backstepping-based boundary observer design [8]. The method also found its applications to adaptive control of parabolic PDEs [9].

Application of the backstepping method to hyperbolic PDEs, on the other hand, was first done in [10], where a scalar, general 1–D linear hyperbolic PDE with time-invariant coefficients is stabilized using this method. The method has also been extended to systems of coupled hyperbolic PDEs [11], [12], [13], as well as adaptive solutions [14], [15], [16].

However, all aforementioned results only concern systems with time-invariant parameters, and very few results concern PDE systems with time-varying parameters. One such result is [17], where backstepping is used to construct an observer for a hyperbolic partial integro-differential equation. No control results were considered. The only result regarding infinite-dimensional backstepping-based controller design for time-varying linear hyperbolic PDEs is [18], where a PDE with a single parameter that is allowed to vary with both time and space, is stabilized using this technique. An adaptive control problem was also considered in [18], subject to the assumption of splitting the uncertain parameter into a known, scalar time-varying part, and uncertain scalar time-invariant part that was allowed to be spatially varying.

In this paper, we extend the result from [18], and solve an adaptive control problem for a more general type of systems by allowing the actuation to be scaled by an unknown, time-varying scalar, and allowing both uncertain parameters to be a superposition of an arbitrary number of known time-varying functions scaled by uncertain parameters. The paper is organized as follows. In Section II, we formally state the problem and necessary assumptions. The adaptive control problem is solved in Section III. The derived controller is demonstrated in a simulation in Section IV, while some concluding remarks are offered in Section V.

B. Notation

We define the domains:

$$\mathcal{D} = \{x \mid x \in [0,1]\}, \quad \mathcal{D}_1 = \{(x,t) \mid x \in \mathcal{D},\ t \geq 0\} \tag{1}$$

For a vectorized variable $u : \mathcal{D} \to \mathbb{R}^n$ (or $u : \mathcal{D}_1 \to \mathbb{R}^n$), we will use the following norm and associated vector space:

$$||u||_\infty = \sup_{x\in\mathcal{D}, i=1,2,\ldots,n} |u_i(x)| \tag{2a}$$

$$\mathcal{B}(\mathcal{D}) = \{u : \mathcal{D} \to \mathbb{R}^n \mid ||u||_\infty < \infty\}. \tag{2b}$$

Moreover, the operator $\equiv$ is defined as follows

$$u \equiv a \qquad \Leftrightarrow \qquad ||u-a||_\infty = 0. \tag{3}$$

II. Problem statement

We consider a 1–D linear hyperbolic partial differential equation with a constant transport speed, and time-varying

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source term and actuation scaling, in the form

$$u_t(x,t) - \mu u_x(x,t) = \theta(x,t)u(0,t) \tag{4a}$$
$$u(1,t) = \rho(t)U(t) \tag{4b}$$
$$u(x,0) = u_0(x) \tag{4c}$$
$$y(t) = u(0,t). \tag{4d}$$

defined for $(x,t) \in \mathcal{D}_1$. Any scalar, linear hyperbolic P(I)DE can be transformed to the form (4) [16, Lemma 2.1]. The goal is to design a control law U that adaptively stabilizes system (4) in the L_∞-sense despite the unknown θ and ρ, from using the boundary sensing y only.

Regarding the parameters ρ and θ, we assume the following, stated in Assumptions 1–3.

Assumption 1: There exist a parameter $\underline{\rho} > 0$ so that

$$\rho(t) \geq \underline{\rho} \tag{5}$$

for all $t \geq 0$.

The case $\rho(t) < 0$, $\forall t$ can be handled by simply defining the new control signal $\bar{U}(t) = -U(t)$.

Assumption 2: The parameters θ and ρ have the following form

$$\theta(x,t) = a^T(x)\nu(t) \tag{6a}$$
$$\rho(t) = b^T\vartheta(t) \tag{6b}$$

where

$$a(x) = \begin{bmatrix} a_1(x) & a_2(x) & \dots & a_n(x) \end{bmatrix}^T \tag{7a}$$
$$b = \begin{bmatrix} b_1 & b_2 & \dots & b_m \end{bmatrix}^T \tag{7b}$$

contain uncertain parameters, and

$$\nu(t) = \begin{bmatrix} \nu_1(t) & \nu_2(t) & \dots & \nu_n(t) \end{bmatrix}^T \tag{8a}$$
$$\vartheta(t) = \begin{bmatrix} \vartheta_1(t) & \vartheta_2(t) & \dots & \vartheta_m(t) \end{bmatrix}^T \tag{8b}$$

are known, bounded functions of time, and ν is for all t known a time d_1 into the future.

Assumption 3: We are in knowledge of non-negative constants $\bar{a}$ and $\bar{b}$ such that

$$||a||_\infty \leq \bar{a}, \qquad |b|_\infty \leq \bar{b}. \tag{9}$$

Additionally, the parameter b belongs to a known convex set Π, defined as

$$\Pi = \{\gamma \mid \mathcal{P}(\gamma) \leq 0\}, \tag{10}$$

for some function $\mathcal{P}(\gamma)$, so that

$$\gamma^T\vartheta(t) \geq \underline{\rho}, \ \forall t \geq 0, \ b \in \Pi. \tag{11}$$

The system considered in [18] is a subsystem of (4) subject to Assumptions 1–2, obtained by letting $n = 1$, and $b = 1$, $\vartheta(t) = 1$ for all $t \geq 0$.

Assumption 2 might seem quite restrictive. However, if for instance all system parameters are known to be periodic with a known period T, the parameters can be reasonably approximated using a truncated Fourier series. Such an approximation can be parameterized as (6).

III. Adaptive control

A. Filter design

Setting the stage for parameter estimation,, we introduce swapping filters that are used to express the system state as a linear combination of the filters and uncertain parameters. Define the filters

$$\psi_t(x,t) - \mu\psi_x(x,t) = 0, \qquad \psi(1,t) = \vartheta(t)U(t)$$
$$\psi(x,0) = \psi_0(x) \tag{12a}$$
$$\phi_t(x,t) - \mu\phi_x(x,t) = 0, \qquad \phi(1,t) = \nu(t)y(t)$$
$$\phi(x,0) = \phi_0(x) \tag{12b}$$

where ψ and ϕ are defined over $\mathcal{D}$ defined in (1), with initial conditions

$$\psi_0, \phi_0 \in \mathcal{B}(\mathcal{D}). \tag{13}$$

Note that the filters (12) can be generated using only known or measured quantities. Consider the non-adaptive state estimate of u generated as

$$\bar{u}(x,t) = b^T\psi(x,t) + d_1\mathcal{F}[a,\phi(t)](x). \tag{14}$$

where

$$\mathcal{F}[a,\phi(t)](x) = \int_x^1 a^T(\xi)\phi(1-(\xi-x),t)d\xi. \tag{15}$$

Lemma 4: Consider the system (4) and the non-adaptive state estimate generated from (14). Then

$$\bar{u} \equiv u \tag{16}$$

for $t \geq d_1$, where

$$d_1 = \mu^{-1}. \tag{17}$$

Proof: Define the non-adaptive state estimation error e as

$$e(x,t) = u(x,t) - \bar{u}(x,t). \tag{18}$$

We will show that e satisfies the dynamics

$$e_t(x,t) - \mu e_x(x,t) = 0 \tag{19a}$$
$$e(1,t) = 0 \tag{19b}$$
$$e(x,0) = e_0(x) \tag{19c}$$

for some $e_0 \in \mathcal{B}(\mathcal{D})$. From differentiating (18) with respect to time and space, and inserting the dynamics (4a) and (12), we obtain

$$\begin{aligned} e_t(x,t) &= u_t(x,t) - b^T\psi_t(x,t) - d_1\mathcal{F}[a,\phi_t(t)](x) \\ &= \mu u_x(x,t) + \theta(x,t)u(0,t) - b^T\mu\psi_x(x,t) \\ &\quad - \mathcal{F}[a,\phi_x(t)](x). \end{aligned} \tag{20}$$

and

$$\begin{aligned} e_x(x,t) &= u_x(x,t) - b^T\psi_x(x,t) - d_1\mathcal{F}[a,\phi_x(t)](x) \\ &\quad + d_1a^T(x)\nu(t)y(t), \end{aligned} \tag{21}$$

where we have inserting the boundary condition (12b). From (20) and (21), we find

$$e_t(x,t) - \mu e_x(x,t) = \theta(x,t)u(0,t) - a^T(x)\nu(t)y(t)$$

$$= 0 \quad (22)$$

where we have inserted the boundary measurement (4d) and used the parameterization (6a). Hence, the dynamics (19a) holds. Evaluating (18) at $x = 1$ and using the boundary conditions (4b) and (12a) gives (19b), since

$$\begin{aligned} e(1,t) &= u(1,t) - b^T\psi(1,t) \\ &= \rho U(t) - b^T\vartheta(t)U(t) = 0 \end{aligned} \quad (23)$$

where we have used (6b). Lastly, the the initial condition (19c) is given as

$$e_0(x) = u_0(x) - b^T\psi_0(0) - d_1\mathcal{F}[a,\phi_0](x). \quad (24)$$

Since $u_0, \psi_0, \phi_0 \in \mathcal{B}(\mathcal{D})$ it follows that $e_0 \in \mathcal{B}(\mathcal{D})$. From the simple transport dynamics (19) with zero as input, it is evident that $e \equiv 0$ and hence $\bar{u} \equiv u$ for $t \geq d_1$. ■

B. Adaptive laws

By Lemma 4, we now have the linear parametric model

$$\begin{aligned} y(t) = u(0,t) &= b^T\psi(0,t) + d_1\mathcal{F}[a,\phi(t)](0) + e(0,t) \\ &= b^T\psi(0,t) + d_1\int_0^1 a^T(\xi)\phi(1-\xi,t)d\xi + e(0,t) \end{aligned} \quad (25)$$

where $e(0,t) = 0$ for $t \geq d_1$.

We propose the adaptive laws

$$\hat{a}_t(x,t) = \text{proj}_{\bar{a}}\{\tau_1(x,t),\hat{a}(x,t)\}, \quad \hat{a}(x,0) = \hat{a}_0(x) \quad (26a)$$

$$\dot{\hat{b}}(t) = \text{proj}_{\Pi}\{\tau_2(t),\hat{b}(t)\}, \quad \hat{b}(0) = \hat{b}_0 \quad (26b)$$

where $\bar{a}$ and Π are defined in Assumption 3, and

$$\tau_1(x,t) = \gamma_1\frac{\hat{e}(0,t)\phi(1-x,t)}{1+f^2(t)} \quad (27a)$$

$$\tau_2(t) = \gamma_2\frac{\hat{e}(0,t)\psi(0,t)}{1+f^2(t)} \quad (27b)$$

for some positive gains γ_1, γ_2 of choice, with

$$f^2(t) = ||\phi(t)||^2 + |\psi(0,t)|^2. \quad (28)$$

and

$$\hat{e}(x,t) = u(x,t) - \hat{u}(x,t) \quad (29)$$

where

$$\hat{u}(x,t) = \hat{b}^T(t)\psi(x,t) + d_1\mathcal{F}[\hat{a}(t),\phi(t)](x) \quad (30)$$

is an adaptive estimate of the state u. The initial conditions $\hat{a}_0$ and $\hat{b}_0$ are assumed to satisfy

$$||\hat{a}_0||_\infty \leq \bar{a}, \qquad \hat{b}_0 \in \Pi. \quad (31)$$

The projection operators are standard, and defined as

$$\text{proj}_a(\tau,\omega) = \tau\begin{cases} 0 & \text{if } \omega = -a \text{ and } \tau \leq 0 \\ 0 & \text{if } \omega = a \text{ and } \tau \geq 0 \\ 1 & \text{otherwise} \end{cases} \quad (32)$$

and

$$\text{proj}_\Pi(\tau,\hat{b}) = \tau\begin{cases} 1 - \dfrac{\nabla_{\hat{b}}\mathcal{P}\nabla_{\hat{b}}\mathcal{P}^T}{\nabla_{\hat{b}}\mathcal{P}^T\nabla_{\hat{b}}\mathcal{P}} & \text{if } \hat{b} \in \partial\Pi \text{ and } \nabla_{\hat{b}}\mathcal{P}^T\tau \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (33)$$

Lemma 5: The adaptive laws (26) with initial conditions satisfying (31) provide the following signal properties

$$||\hat{a}(t)||_\infty \leq \bar{a}, \ \forall t \geq 0 \quad (34a)$$

$$\hat{b} \in \Pi, \ \forall t \geq 0 \quad (34b)$$

$$||\hat{a}_t||, \dot{\hat{b}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (34c)$$

$$\sigma \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (34d)$$

where

$$\sigma(t) = \frac{\hat{e}(0,t)}{\sqrt{1+f^2(t)}}. \quad (35)$$

Proof: Properties (34a)–(34b) follow from the projection operators and the initial conditions (31). Consider

$$\begin{aligned} V(t) = d_1\int_0^1 e^2(x,t)dx + \frac{d_1}{2\gamma_1}\int_0^1 \tilde{a}^T(x,t)\tilde{a}(x,t)dx \\ + \frac{1}{2\gamma_2}\tilde{b}^T(t)\tilde{b}(t) \end{aligned} \quad (36)$$

where

$$\tilde{a}(x,t) = a(x) - \hat{a}(x,t) \qquad \tilde{b}(t) = b - \hat{b}(t) \quad (37)$$

Differentiating with respect to time, inserting the adaptive law (34) and using the property $-\tilde{\theta}\text{proj}_{\bar{\theta}}(\tau,\hat{\theta}) \leq -\tilde{\theta}\tau$ ([19, Lemma E.1]), we find

$$\begin{aligned} \dot{V}(t) \leq -e^2(0,t) - d_1\int_0^1 \tilde{a}^T(x,t)\frac{\hat{e}(0,t)\phi(1-x,t)}{1+f^2(t)}dx \\ - \tilde{b}^T(t)\frac{\hat{e}(0,t)\psi(0,t)}{1+f^2(t)}. \end{aligned} \quad (38)$$

Inserting the relationship

$$\begin{aligned} \hat{e}(0,t) - e(0,t) = \tilde{b}^T(t)\psi(0,t) \\ + d_1\int_0^1 \tilde{a}^T(\xi,t)\phi(1-\xi,t)d\xi \end{aligned} \quad (39)$$

obtained from (14)–(15), (18) and (29)–(30), now gives

$$\dot{V}(t) \leq -e^2(0,t) - \frac{\hat{e}^2(0,t)}{1+f^2(t)} + \frac{\hat{e}(0,t)e(0,t)}{1+f^2(t)}. \quad (40)$$

Application of Young's inequality on the latter term yields

$$\dot{V}(t) \leq -\frac{1}{2}\sigma^2(t), \quad (41)$$

for σ defined in (35), which shows that V is non-increasing, and hence is bounded and has a limit as $t \to \infty$. Integrating (41) from $t = 0$ to ∞ gives

$$\sigma \in \mathcal{L}_2. \quad (42)$$

Moreover, we have

$$\begin{aligned} \sigma(t) &= \frac{\hat{e}(0,t)}{\sqrt{1+f^2(t)}} \\ &= \frac{1}{\sqrt{1+f^2(t)}}\big(e(0,t) + \tilde{b}^T(t)\psi(0,t) \end{aligned}$$

$$+ d_1\mathcal{F}[\tilde{a}(t),\phi(t)](0)\big)$$
$$\leq |\tilde{b}(t)| + ||\tilde{a}(t)|| + \frac{e(0,t)}{\sqrt{1+f^2(t)}}. \quad (43)$$

Due to the pure transport characteristic of e, and the fact that $e_0 \in \mathcal{B}(\mathcal{D})$, it follows that $e(0,\cdot) \in \mathcal{L}_\infty$, with $e(t) = 0$ for $t \geq d_1$, and hence $\sigma \in \mathcal{L}_\infty$.

From the adaptive law (26)–(27), we have

$$\begin{aligned} ||\tau_1(t)|| &= \left|\left|\gamma_1 \frac{\hat{e}(0,t)\phi(1-x,t)}{1+f^2(t)}\right|\right| \\ &\leq \gamma_1 \frac{|\hat{e}(0,t)|}{\sqrt{1+f^2(t)}} \frac{||\phi(t)||}{\sqrt{1+f^2(t)}} \\ &\leq \gamma_1|\sigma(t)| \end{aligned} \quad (44)$$

and hence

$$||\hat{a}_t|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (45)$$

follows. A similar line of reasoning gives $|\dot{\hat{b}}| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. ■

C. Main control law

Consider the control law

$$U(t) = \frac{1}{\hat{\rho}(t)} \int_0^1 \hat{k}(\xi,t)\hat{u}(\xi,t)d\xi \quad (46)$$

where $\hat{k}$ is the solution to the Volterra integral equation

$$\begin{aligned} &\mu\hat{k}(x,t) = -\hat{a}^T(1-x,t)\nu(t+d_1x) \\ &+ \int_x^1 \hat{k}(1+x-\xi,t)\hat{a}^T(1-\xi,t)d\xi\nu(t+d_1x). \end{aligned} \quad (47)$$

Theorem 6: Consider system (4) with the adaptive state estimate $\hat{u}$ generated from (30), and $\hat{k}$ as the solution to (47). If Assumptions 1–3 hold, then the control law (46) ensures

$$||u||_\infty, ||\psi||_\infty, ||\phi||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad (48a)$$
$$||u||_\infty, ||\psi||_\infty, ||\phi||_\infty \to 0. \quad (48b)$$

Proof of Theorem 6 is given in Section III-F.

D. Adaptive state estimate dynamics

It can straightforwardly be verified, using the filter dynamics (12), that the adaptive state estimate (30) has the dynamics

$$\begin{aligned} \hat{u}_t(x,t) - \mu\hat{u}_x(x,t) &= \hat{\theta}(x,t)u(0,t) + \dot{\hat{b}}^T(t)\psi(x,t) \\ &\quad + d_1\mathcal{F}[\hat{a}_t(t),\phi(t)](x) && (49a) \\ \hat{u}(1,t) &= \hat{\rho}(t)U(t) && (49b) \\ \hat{u}(x,0) &= \hat{u}_0(x) && (49c) \end{aligned}$$

where

$$\hat{\theta}(x,t) = \hat{a}^T(x,t)\nu(t), \qquad \hat{\rho}(t) = \hat{b}^T(t)\vartheta(t) \quad (50)$$

and for some initial condition

$$\hat{u}_0 \in \mathcal{B}(\mathcal{D}) \quad (51)$$

which can be obtain by evaluating (30) at $t = 0$.

E. Backstepping

We note that since $\hat{a}$ and ν are bounded, the former by projection and the latter by assumption, $\hat{k}$ as defined in (47) must be uniformly bounded, that is; there exists a constant $\bar{k}$ such that

$$||\hat{k}(t)||_\infty \leq \bar{k}, \quad \forall t \geq 0. \quad (52)$$

Consider the backstepping transformation T, given as

$$\begin{aligned} w(x,t) &= T[\hat{u}](x,t) \\ &= \hat{u}(x,t) - \int_0^x \hat{K}(x,\xi,t)\hat{u}(\xi,t)d\xi \end{aligned} \quad (53)$$

where $\hat{K}$ is given as

$$\hat{K}(x,\xi,t) = \hat{k}(1-(x-\xi), t-d_1(1-x)) \quad (54)$$

and $\hat{k}$ is the solution to (47).

Since $\hat{k}$ and hence also $\hat{K}$ are uniformly bounded, the backstepping transformation (53) is invertible, and by [16, Theorem 1.3], there exist constants $G_1, G_2 > 0$ so that

$$||w(t)|| \leq G_1||\hat{u}(t)||, \qquad ||\hat{u}(t)|| \leq G_2||w(t)|| \quad (55)$$

for all $t \geq 0$, where

$$\hat{u}(x,t) = T^{-1}[w](x,t). \quad (56)$$

Consider also the target system

$$\begin{aligned} w_t(x,t) - \mu w_x(x,t) &= T[\hat{a}^T\nu](x,t)\hat{e}(0,t) + T[\dot{\hat{b}}^T\psi](x,t) \\ &\quad + d_1T\left[\mathcal{F}[\hat{a}_t,\phi]\right](x,t) + \zeta(x,t)w(0,t) && (57a) \\ w(1,t) &= 0 && (57b) \\ w(x,0) &= w_0(x) && (57c) \end{aligned}$$

where

$$\zeta(x,t) = \int_{t-d_1(1-x)}^t \hat{a}_t^T(x,\tau)d\tau\nu(t) \quad (58)$$

for some $w_0 \in \mathcal{B}(\mathcal{D})$, where T is defined in (53).

Lemma 7: The backstepping transformation (53) and control law (46) map system (49) into the target system (57).

Proof: By differentiating (53) with respect to time and space, respectively, inserting the dynamics (49a), integrating by parts, and inserting the result into the dynamics (49a), we obtain

$$\begin{aligned} 0 &= \hat{u}_t(x,t) - \mu\hat{u}_x(x,t) - \hat{a}^T(x,t)\nu(t)u(0,t) \\ &\quad - \dot{\hat{b}}^T(t)\psi(x,t) - d_1\mathcal{F}[\hat{a}_t(t),\phi(t)](x) \\ &= w_t(x,t) - \mu w_x(x,t) + \int_0^x \big(\hat{K}_t(x,\xi,t) - \mu\hat{K}_x(x,\xi,t) \\ &\quad - \mu\hat{K}_\xi(x,\xi,t)\big)\hat{u}(\xi,t)d\xi - \hat{a}^T(x,t)\nu(t)\hat{e}(0,t) \\ &+ \int_0^x \hat{K}(x,\xi,t)\hat{a}^T(\xi,t)\nu(t)\hat{e}(0,t)d\xi - \Big(\hat{\theta}(x,t) \\ &\quad + \mu\hat{K}(x,0,t) - \int_0^x \hat{K}(x,\xi,t)\hat{a}^T(\xi,t)\nu(t)d\xi\Big)\hat{u}(0,t) \\ &- \dot{\hat{b}}^T(t)\psi(x,t) + \int_0^x \hat{K}(x,\xi,t)\dot{\hat{b}}^T(t)\psi(\xi,t)d\xi \end{aligned}$$

$$- d_1 \mathcal{F}[\hat{a}_t(t), \phi(t)](x)$$
$$+ d_1 \int_0^x \hat{K}(x,\xi,t)\mathcal{F}[\hat{a}_t(t), \phi(t)](\xi)d\xi. \tag{59}$$

Using the fact that $\hat{K}$ defined in (54) satisfies

$$0 = \hat{K}_t(x,\xi,t) - \mu\hat{K}_x(x,\xi,t) - \mu\hat{K}_\xi(x,\xi,t) \tag{60}$$

and defining

$$f(x,t) = \mu\hat{K}(x,0,t) + \hat{a}^T(x,t)\nu(t) - \int_0^x \hat{K}(x,\xi,t)\hat{a}^T(\xi,t)\nu(t)d\xi, \tag{61}$$

equation (59) can be written

$$w_t(x,t) - \mu w_x(x,t) = T[\hat{\theta}](x,t)\hat{e}(0,t) + f(x,t)\hat{u}(0,t) + T[\dot{\hat{b}}^T\psi](x,t) + d_1 T[\mathcal{F}[\hat{a}_t, \phi]](x,t). \tag{62}$$

Inserting the definition (54) for $\hat{K}$ into (61) gives

$$f(x,t) = \mu\hat{k}(1-x, t-d_1(1-x)) + \hat{a}^T(x,t)\nu(t) - \int_0^x \hat{k}(1-(x-\xi), t-d_1(1-x))\hat{a}^T(\xi,t)\nu(t)d\xi. \tag{63}$$

By equation (47) for $\hat{k}$, we have

$$\mu\hat{k}(1-x, t-d_1(1-x)) = -\hat{a}^T(x, t-d_1(1-x))\nu(t) + \int_0^x \hat{k}(1-x+\xi, t-d_1(1-x))\hat{a}^T(\xi,t)\nu(t)d\xi \tag{64}$$

and inserting this into (63) gives

$$f(x,t) = \left[\hat{a}^T(x,t) - \hat{a}^T(x, t-d_1(1-x))\right]\nu(t) = \int_{t-d_1(1-x)}^t \hat{a}_t^T(x,\tau)d\tau\nu(t) = \zeta(x,t) \tag{65}$$

and since $w(0,t) = \hat{u}(0,t)$, (57a) holds. Evaluating the backstepping transformation at $x = 1$ and inserting the boundary condition (4b) and the definition (54), we find

$$w(1,t) = \hat{\rho}(t)U(t) - \int_0^1 \hat{k}(\xi,t)\hat{u}(\xi,t)d\xi. \tag{66}$$

The control law (46) now gives the boundary condition (57b). Lastly, evaluating the backstepping transformation at $t = 0$ gives the initial condition (57c) from the initial condition (4c) as

$$w_0(x) = T[\hat{u}_0](x,0). \tag{67}$$

■

F. Proof of Theorem 6

Consider the non-negative functions

$$V_2(t) = \int_0^1 (1+x)w^2(x,t)dx \tag{68a}$$
$$V_3(t) = \int_0^1 (1+x)\psi^T(x,t)\psi(x,t)dx \tag{68b}$$
$$V_4(t) = \int_0^1 (1+x)\phi^T(x,t)\phi(x,t)dx. \tag{68c}$$

The following lemma states upper bounds on the derivatives of the functions (68).

Lemma 8: There exists positive constants h_1, h_2, h_3 and non-negative, integrable functions l_1, l_2, l_3 so that

$$\dot{V}_2(t) \le -\left(\mu - 32d_1||\zeta(t)||^2\right)w^2(0,t) + h_1\sigma^2(t)|\psi(0,t)|^2 - \frac{\mu}{4}V_2(t) + l_1(t)V_3(t) + l_2(t)V_4(t) + l_3(t) \tag{69a}$$
$$\dot{V}_3(t) \le -\mu|\psi(0,t)|^2 - \frac{\mu}{2}V_3(t) + h_2V_2(t) \tag{69b}$$
$$\dot{V}_4(t) \le h_3w^2(0,t) + h_3\sigma^2(t)V_3(t) + h_3\sigma^2(t)|\psi(0,t)|^2 + h_3\sigma^2(t) - \frac{\mu}{2}V_4(t). \tag{69c}$$

The proof of Lemma 8 is found in Appendix C.

Now forming the Lyapunov function candidate

$$V_5(t) = aV_2(t) + V_3(t) + V_4(t) \tag{70}$$

for some positive constant a, we find, using Lemma 8, the following upper bound on its derivative

$$\dot{V}_5(t) \le -\left(a(\mu - 32d_1||\zeta(t)||^2) - h_3\right)w^2(0,t) - (\mu - ah_1\sigma^2(t) - h_3\sigma^2(t))|\psi(0,t)|^2 - (a\frac{\mu}{4} - h_2)V_2(t) - \frac{\mu}{2}V_3(t) - \frac{\mu}{2}V_4(t) + (al_1(t) + h_3\sigma^2(t))V_3(t) + al_2(t)V_4(t) + al_3(t) + h_3\sigma^2(t). \tag{71}$$

It is evident that a sufficiently large value of a, will result in

$$\dot{V}_5(t) \le -\left(c_1 - c_2||\zeta(t)||^2\right)w^2(0,t) - (\mu - c_3\sigma^2(t))|\psi(0,t)|^2 - c_4V_5(t) + l_4(t)V_5(t) + l_5(t) \tag{72}$$

for some positive constants c_1, c_2, c_3, c_4, and non-negative, integrable functions l_4, l_5.

By the definition of ζ in (58), we have

$$||\zeta(t)||^2 = \int_0^1 \left(\int_{t-d(1-x)}^t \hat{a}_t^T(x,\tau)d\tau\nu(t)\right)^2 dx \le \int_0^1 \left(\int_{t-d}^t |\hat{a}_t^T(x,\tau)\nu(t)|d\tau\right)^2 dx \le n^2\bar{\nu}^2 \int_{t-d}^t ||\hat{a}_t(\tau)||d\tau \tag{73}$$

Now, since $||\hat{a}_t|| \in \mathcal{L}_2$, there must, for every $\epsilon_0 > 0$, exist a time $T_0 \ge 0$ so that

$$\int_{t-d}^t ||\hat{a}_t(\tau)||d\tau < \epsilon_0 \tag{74}$$

for all $t \ge T_0$. Choosing $\epsilon_0 = \frac{c_1}{c_2}$, we will for $t \ge T_0$ have

$$\dot{V}_5(t) \le -(\mu - h_3\sigma^2(t))|\psi(0,t)|^2 - c_3V_5(t) + l_5(t)V_5(t) + l_6(t). \tag{75}$$

Since $||u||_\infty$ for a system in the form (4), and $||\psi||_\infty$, $||\phi||_\infty$ of the filters generated from u, are all bounded in growth by an exponential function ([16, Theorem 1.1]), Lemma 9 in Appendix A gives

$$V_5 \in \mathcal{L}_1 \cap \mathcal{L}_\infty, \tag{76}$$

and hence

$$||w||, ||\psi||, ||\phi|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \tag{77}$$

From the definition of the filter ψ and the control law U, we then have

$$||\psi||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \tag{78}$$

and specifically $\psi^2(0, \cdot) \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, meaning that (75) can be written

$$\dot{V}_5(t) \leq -c_3 V_5(t) + l_5(t) V_5(t) + l_7(t) \tag{79}$$

where

$$l_7(t) = l_6(t) + |\psi(0, t)|^2 l_4(t) \tag{80}$$

Now since $\psi(0, \cdot) \in \mathcal{L}_\infty$ and $l_4 \in \mathcal{L}_1$, it follows that $l_7 \in \mathcal{L}_1$. Lemma 10 in Appendix B then gives $V_5 \to 0$, and thus

$$||w||, ||\psi||, ||\phi|| \to 0. \tag{81}$$

From the definition of the filter ψ and the control law U, as well as the invertibiliy of the transformation (53) (Lemma 7), we have

$$||\psi||_\infty \to 0 \tag{82}$$

and

$$||\hat{u}|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \qquad ||\hat{u}|| \to 0. \tag{83}$$

From (14) and Lemma 4, it follows that

$$||u||, ||u||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \qquad ||u||, ||u||_\infty \to 0, \tag{84}$$

and specifically $u(0, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $u(0, \cdot) \to 0$, which from the definition of the filter ϕ gives

$$||\phi||, ||\phi||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \qquad ||\phi||, ||\phi||_\infty \to 0. \tag{85}$$

IV. Simulation

System (4) and the adaptive controller of Theorem 6 were implemented in MATLAB, using the system parameters

$$\mu = 1 \tag{86a}$$
$$\theta(x, t) = \begin{bmatrix} e^{\frac{1}{2}x} & \frac{1}{5}(1 + x) \end{bmatrix} \nu(t) \tag{86b}$$
$$\rho(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \vartheta(t) \tag{86c}$$

where

$$\nu(t) = \begin{bmatrix} \frac{1}{2}(2 + \sin(\frac{\pi}{2}t)) & \frac{1}{2}\cos\left(\frac{\pi}{2\sqrt{2}}t\right) \end{bmatrix}^T \tag{87a}$$
$$\vartheta(t) = \begin{bmatrix} 1 + \dfrac{1}{2}\sin(\pi t) & \frac{1}{2}\cos\left(\frac{\pi}{2\sqrt{2}}t\right) \end{bmatrix}^T \tag{87b}$$

The system's initial condition was set to

$$u_0(x) = x, \tag{88}$$

and the bound $\bar{a}$ was set to

$$\bar{a} = 10^6, \tag{89}$$

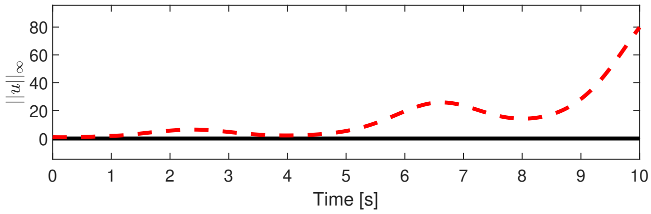

Fig. 1: State norm in uncontrolled case.

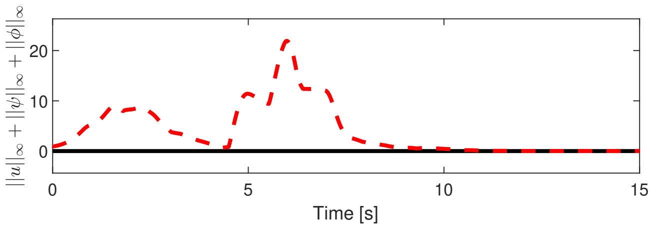

Fig. 2: State norms.

while the projection in (26b) ensuring that ρ stays positive, was implemented by forcing the components $\hat{b}_1$ and $\hat{b}_2$ of $\hat{b} = \begin{bmatrix} \hat{b}_1 & \hat{b}_2 \end{bmatrix}^T$ to satisfy

$$1 \leq \hat{b}_1(t), \qquad |\hat{b}_2(t)| \leq \hat{b}_1(t) \tag{90}$$

for all $t \geq 0$, which can be implemented as

$$\mathcal{P}(\hat{b}(t)) = \max\left\{ \left|1 - \hat{b}_1(t)\right|, \left||\hat{b}_2(t)| - \hat{b}_1(t)\right| \right\}. \tag{91}$$

The initial guesses were chosen as

$$\hat{a}_0 \equiv 0, \qquad \hat{b}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T. \tag{92}$$

System (4) with parameters (86a)–(87) is unstable in the uncontrolled case, as seen from Figure 1.

The simulation results from the closed loop case are shown in Figures 2–5. It is clearly seen from Figure 2 that the system state and filter states all converge to zero in the L_∞-sense, as guaranteed by Theorem 6. Also, the actuation signal seen in Figure 3 remains bounded. None of the components in a or b are anywhere near convergence, as seen in Figures 4–5, however, parameter convergence is not guaranteed in adaptive control.

V. Conclusions

We have extended the result from [18] for a time-varying PDE, and solved an adaptive control problem for a class of hyperbolic PDEs with an in-domain parameter and actuation

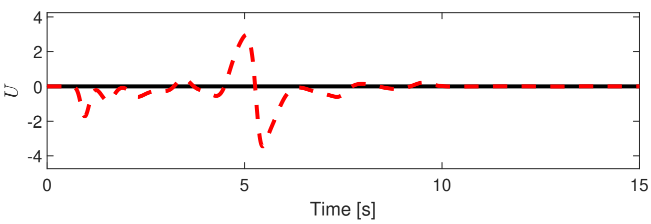

Fig. 3: Actuation.

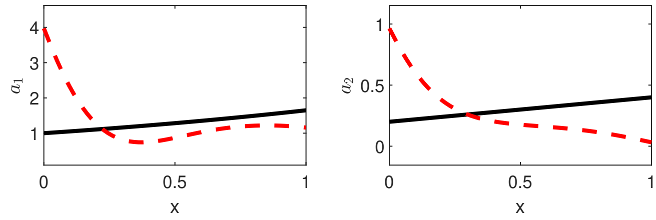

Fig. 4: Actual and final estimated $a(x)$.

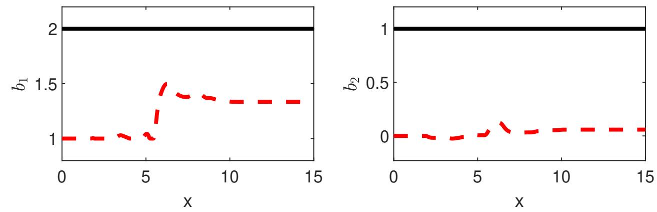

Fig. 5: Actual and estimated b.

scaling, that both are allowed to be uncertain and time-varying. The controller achieves asymptotic convergence of the system state's L_∞ norm to zero. Future work includes extensions to coupled PDEs.

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APPENDIX

A. Stability and convergence lemma

Lemma 9 (Lemma 12 from [20]): Let $v_1(t)$, $v_2(t)$, $\sigma(t)$, $l_1(t)$, $l_2(t)$ and $f(t)$, be real-valued, non-negative functions defined for $t \geq 0$. Suppose

$$l_1, l_2 \in \mathcal{L}_1 \tag{93a}$$

$$\int_0^t f(s)ds \leq Ae^{Bt} \tag{93b}$$

$$\sigma(t) \leq kv_1(t) \tag{93c}$$

$$\dot{v}_1(t) \leq -\sigma(t) \tag{93d}$$

$$\dot{v}_2(t) \leq -cv_2(t) + l_1(t)v_2(t) + l_2(t) - a(1 - b\sigma(t))f(t) \tag{93e}$$

for $t \geq 0$, where k, A, B, a, b and c are positive constants. Then $v_2 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$.

B. Convergence lemma

Lemma 10 (Lemma 3 from [21]): Let $v(t)$, $l_1(t)$, $l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{94a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{94b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{94c}$$

where c is a positive constant. Then

$$\lim_{t \to \infty} v(t) = 0. \tag{95}$$

C. Proof of Lemma 8

1) Bound on $\dot{V}_2$: Differentiating (68a), inserting the dynamics (57a) and integration by parts gives

$$\dot{V}_2(t) = 2\mu w^2(1,t) - \mu w^2(0,t) - \mu \int_0^1 w^2(x,t)dx + 2\int_0^1 (1+x)w(x,t)T[\hat{a}^T\nu](x,t)\hat{e}(0,t)dx$$

$$+ 2\int_0^1 (1+x)w(x,t)T[\dot{\hat{b}}^T\psi](x,t)dx$$
$$+ 2d\int_0^1 (1+x)w(x,t)T\left[\mathcal{F}[a_t,\phi](x)\right](x,t)dx$$
$$+ 2\int_0^1 (1+x)w(x,t)\zeta(x,t)u(0,t)dx \tag{96}$$

Inserting the boundary condition (57b) and definition (15), and using Young's inequality on the cross terms, gives

$$\dot{V}_2(t) \leq -\mu w^2(0,t) - \left(\frac{\mu}{2} - \sum_{i=1}^4 \rho_i\right)\int_0^1 (1+x)w^2(x,t)dx$$
$$+ \frac{1}{\rho_1}\int_0^1 (1+x)T^2[\hat{a}^T\nu](x,t)\hat{e}^2(0,t)dx$$
$$+ \frac{1}{\rho_2}\int_0^1 (1+x)T^2[\dot{\hat{b}}^T\psi](x,t) + \frac{1}{\rho_3}d_1^2\int_0^1 (1+x)$$
$$\times T\left[\int_x^1 \hat{a}_t^T(\xi,t)\phi(1-(\xi-x),t)d\xi\right]^2(x,t)dx$$
$$+ \frac{1}{\rho_4}\int_0^1 (1+x)\zeta^2(x,t)dx w^2(0,t). \tag{97}$$

for some arbitrary positive constants $\rho_i,\ i = 1,2,3,4$. Choosing

$$\rho_i = \frac{\mu}{16} \tag{98}$$

and using Cauchy-Schwarz' inequality gives

$$\dot{V}_2(t) \leq 2\mu w^2(1,t) - \left(\mu - 32d_1||\zeta(t)||^2\right)w^2(0,t) - \frac{\mu}{4}V_2(t)$$
$$+ 32dG_1^2||\hat{a}^T(t)\nu(t)||^2\hat{e}^2(0,t) + 32dG_1^2|\dot{\hat{b}}(t)|^2||\psi(t)||^2$$
$$+ 32d^3G_1^2||\hat{a}_t(t)||^2||\phi(t)||^2. \tag{99}$$

The term $\hat{e}^2(0,t)$ can be rewritten as

$$\begin{aligned}\hat{e}^2(0,t) &= \frac{\hat{e}^2(0,t)}{1+f^2(t)}(1+f^2(t)) \\ &= \sigma^2(t)(1+||\phi(t)||^2 + |\psi(0,t)|^2) \\ &\leq \sigma^2(t)V_3(t) + \sigma^2(t)|\psi(0,t)|^2 + \sigma^2(t)\end{aligned} \tag{100}$$

and inserting this, we obtain

$$\begin{aligned}\dot{V}_2(t) \leq &-\left(\mu - 32d_1||\zeta(t)||^2\right)w^2(0,t) + h_1\sigma^2(t)|\psi(0,t)|^2 \\ &- \frac{\mu}{4}V_2(t) + l_1(t)V_3(t) + l_2(t)V_4(t) + l_3(t)\end{aligned} \tag{101}$$

where

$$l_1(t) = 32d_1G_1^2|\dot{\hat{b}}(t)|^2 \tag{102a}$$
$$l_2(t) = 32d_1^3G_1^2||\hat{a}_t(t)||^2 + l_3(t) \tag{102b}$$
$$l_3(t) = h_1\sigma^2(t) \tag{102c}$$

are non-negative, integrable functions, with

$$h_1 = 32d_1G_1^2n^2\bar{a}^2\bar{\nu}^2 \tag{103}$$

as a positive constant, where $\bar{\nu}$ bounds $|\nu|_\infty$.

2) Bound on $\dot{V}_3$: Differentiating (68b), inserting the dynamics (12a) and integration by parts gives

$$\begin{aligned}\dot{V}_3(t) &= 2\mu|\psi(1,t)|^2 - \mu|\psi(0,t)|^2 \\ &\quad - \mu\int_0^1 \psi^T(x,t)\psi(x,t)dx \\ &\leq 2\mu m^2\bar{\vartheta}^2U^2(t) - \mu|\psi(0,t)|^2 - \frac{\mu}{2}V_3(t).\end{aligned} \tag{104}$$

where $\bar{\vartheta}$ is a positive constant bounding $|\vartheta|_\infty$. Inserting the boundary condition (12a) and control law (46) yields

$$\begin{aligned}\dot{V}_3(t) &\leq -\mu|\psi(0,t)|^2 - \frac{\mu}{2}V_3(t) \\ &\quad + 2\mu m^2\bar{\vartheta}^2\frac{1}{\hat{\rho}^2(t)}\left(\int_0^1 \hat{k}(\xi,t)\hat{u}(\xi,t)d\xi\right)^2 \\ &\leq -\mu|\psi(0,t)|^2 - \frac{\mu}{2}V_3(t) \\ &\quad + 2\mu m^2\bar{\vartheta}^2M_\rho^2||\hat{k}(t)||||\hat{u}(t)|| \\ &\leq -\mu|\psi(0,t)|^2 - \frac{\mu}{2}V_3(t) + h_2||w(t)||\end{aligned} \tag{105}$$

where

$$h_2 = 2\mu m^2\bar{\vartheta}^2M_\rho^2G_2^2\bar{k}^2. \tag{106}$$

with

$$M_\rho = \left(\inf_{b\in\Pi, t\geq 0}|b^T\vartheta(t)|\right)^{-1} \tag{107}$$

3) Bound on $\dot{V}_4$: Differentiating (68c), inserting the dynamics (12b) and integration by parts gives

$$\begin{aligned}\dot{V}_4(t) &= 2\mu|\phi(1,t)|^2 - \mu|\phi(0,t)|^2 \\ &\quad - \mu\int_0^1 \phi^T(x,t)\phi(x,t)dx \\ \dot{V}_4(t) &\leq 2\mu\nu^T(t)\nu(t))u^2(0,t) - \frac{\mu}{2}V_4(t) \\ &\leq 2\mu n^2\bar{\nu}^2(\hat{u}(0,t) + \hat{e}(0,t))^2 - \frac{\mu}{2}V_4(t) \\ &\leq h_3w^2(0,t) + h_3\hat{e}^2(0,t) - \frac{\mu}{2}V_4(t)\end{aligned} \tag{108}$$

where

$$h_3 = 4\mu n^2\bar{\nu}^2 \tag{109}$$

is a positive constant. Inserting (100) yields

$$\begin{aligned}\dot{V}_4(t) \leq\ & h_3w^2(0,t) + h_3\sigma^2(t)V_3(t) + h_3\sigma^2(t)|\psi(0,t)|^2 \\ &+ h_3\sigma^2(t) - \frac{\mu}{2}V_4(t).\end{aligned} \tag{110}$$