Stabilization and Tracking Control of a Time-Variant Linear Hyperbolic PIDE Using Backstepping

Henrik Anfinsen [a], Ole Morten Aamo [a],

[a] *Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), Trondheim N-7491, Norway.*

Abstract

We extend previous results regarding infinite-dimensional backstepping-based controller design for linear hyperbolic partial (integro-)differential equations (P(I)DEs), and derive a state-feedback controller for a PIDE system with time-varying system parameters. The system state converges to zero in the ∞-norm, in a finite time corresponding to the propagation time between the boundaries. Secondly, the controller is slightly modified to solve an output tracking problem. The derived controllers are demonstrated in simulations. The derived state-feedback controllers can also be combined with state observers into output-feedback controllers.

Key words: Distributed parameter systems. Hyperbolic systems. Linear systems. Boundary control.

1 Introduction

Background: Infinite-dimensional backstepping is a systematic method for design of controllers and observers for systems of partial (integro-)differential equations (P(I)DEs). It is based on the introduction of invertible Volterra integral transforms and accompanying control laws that map the system of interest into a carefully designed target system possessing some desirable stability properties. Due to the invertibility of the transform, the stability properties of the two systems are the same. The method was originally proposed for parabolic PDEs in [9], and was thereafter quickly expanded in numerous directions with key references including: Non-adaptive state-feedback control laws for a class of parabolic PDEs [11] and Backstepping-based boundary observer design [12]. The method also found its way into adaptive solutions [13].

Extension of the backstepping method to hyperbolic PDEs is done in [8], where a scalar, general 1–D linear hyperbolic PDE with time-invariant coefficients is stabilized using this method. The method has also been extended to systems of coupled hyperbolic PDEs [14], [5], [6], as well as adaptive solutions [3].

All above mentioned solutions consider systems with time-invariant parameters. Very few results exist regarding control or estimation of time-varying systems of linear hyperbolic PDEs using infinite-dimensional backstepping. An observer based on backstepping is derived in [4] for a hyperbolic partial integro-differential equation, which is a subclass of the type of systems considered in this paper. The resulting observer achieves exponential convergence of the estimated state to its true value in the L_2-norm. The only result regarding infinite-dimensional backstepping-based controller design for time-varying linear hyperbolic PDEs is [2], where a PDE with a single parameter that is allowed to vary with both time and space, is stabilized using this technique.

In this paper, we will derive a control law for the more general type of systems investigated in [8]. We allow all system parameters, except the transport speed, to be time-varying, and derive a backstepping-based control law that achieves convergence to zero in a finite time in the ∞-norm. We also slightly modify the controller for solving an output tracking problem.

Notation: We define the domains: $\mathcal{D} = \{x \mid x \in [0,1]\}, \mathcal{D}_1 = \{(x,t) \mid x \in \mathcal{D},\ t \in [0,T_1]\}, \mathcal{T} = \{(x,\xi) \mid 0 \leq \xi \leq x \leq 1\}, \mathcal{T}_1 = \{(x,\xi,t) \mid (x,\xi) \in \mathcal{T},\ t \in [0,T_1]\}$, for some constant $T_1 > 0$. For two variables $u, v : \mathcal{D} \to \mathbb{R}$ (or $u, v : \mathcal{D}_1 \to \mathbb{R}$), we will use the following norm and associated vector space: $||u||_\infty = \sup_{x\in\mathcal{D}} |u(x)|, \mathcal{B}(\mathcal{D}) =$

⋆ Corresponding author: H. Anfinsen.

Email addresses: henrik.anfinsen@ntnu.no (Henrik Anfinsen), aamo@ntnu.no (Ole Morten Aamo).

$\{u : \mathcal{D} \to \mathbb{R} \mid ||u||_\infty < \infty\}$, and the operator $\equiv$ is defined as $u \equiv v \Leftrightarrow ||u - v||_\infty = 0$, and $u \equiv 0 \Leftrightarrow ||u||_\infty = 0$.

2 Problem statement

We consider a 1–D linear hyperbolic partial integro-differential equation in the system state $u : \mathcal{D}_1 \to \mathbb{R}$

$$u_t(x,t) - \mu(x)u_x(x,t) = f(x,t)u(x,t) + g(x,t)u(0,t) + \int_0^x h(x,\xi,t)u(\xi,t)d\xi \quad (1a)$$

$$u(1,t) = U(t) \quad (1b)$$

$$u(x,0) = u_0(x) \quad (1c)$$

$$y(t) = u(0,t). \quad (1d)$$

The system parameters are assumed to satisfy $\mu : \mathcal{D} \to \mathbb{R}$, $\mu \in C^1([0,1])$, $\mu(x) > 0$, $\forall x \in \mathcal{D}$, $f, g : \mathcal{D}_1 \to \mathbb{R}$, $h : \mathcal{T}_1 \to \mathbb{R}$, while the initial condition u_0 is assumed to satisfy $u_0 \in \mathcal{B}(\mathcal{D})$.

The systems considered in [4] is the subclass of (1), obtained by letting $\mu \equiv 1$, and, moreover, [4] deals with the state observation problem, only. The stabilization problem for the case $f \equiv 0, h \equiv 0$ and $\mu \equiv$ const was considered in [2]. Thus, the present paper is the first to provide stabilization and tracking controllers for the general system (1). The PIDE (1) is related to the Korteweg-de Vries PDE, which models shallow-water waves and ion acoustic waves in plasma, and is also an approximation of the linearized Boussinesq PDE that models complex water waves such as tidal bores [7, Section 14.3].

Firstly, we design a state-feedback control law $U(t)$ so that system (1) is stabilized in $\mathcal{B}(\mathcal{D})$. Specifically, the state converges to zero in $\mathcal{B}(\mathcal{D})$ (i.e. $u(t) \equiv 0$) in a finite time d_1 given as

$$d_1 = \int_0^1 \frac{d\gamma}{\mu(\gamma)}. \quad (2)$$

The quantity d_1 corresponds to the propagation time from $x = 1$ to $x = 0$ in system (1). Secondly, we consider a tracking problem, where the goal is to design a control law so that the output y tracks an arbitrary reference signal r, and simultaneously stabilizes the system in $\mathcal{B}(\mathcal{D})$.

Both controllers will be derived subject to Assumptions 1 and 2, while the tracking controller also employs Assumption 3.

Assumption 1 *There exist constants $\bar{f}, \bar{g}, \bar{h}$ so that $|f(x,t)| \le \bar{f}$, $|g(x,t)| \le \bar{g}$, $\forall (x,t) \in \mathcal{D}_1$ and $|h(x,\xi,t)| \le \bar{h}$, $\forall (x,\xi,t) \in \mathcal{T}_1$.*

Assumption 2 *The parameter $\mu(x)$ is known for all $x \in \mathcal{D}$, $f(x,t)$ and $g(x,t)$ are known for all $(x,t) \in \mathcal{D}_1$ and $h(x,\xi,t)$ is known for all $(x,\xi,t) \in \mathcal{T}_1$.*

Assumption 3 *At any time t, the signal $r(t)$ is known and predictable a time d_1 into the future. Moreover, there exists a constant $\bar{r}$ so that $|r(t)| \le \bar{r}$, $\forall t \ge 0$.*

3 Finite-time convergent state-feedback controller

3.1 Controller

Consider a function $K(x,\xi,t)$ defined on $\mathcal{T}_1$ and satisfying the PIDE

$$K_t(x,\xi,t) - \mu(x)K_x(x,\xi,t) - \mu(\xi)K_\xi(x,\xi,t) = (\mu'(\xi) - f(\xi,t))K(x,\xi,t) - \int_\xi^x K(x,s,t)h(s,\xi,t)ds + \alpha(x,t)h(x,\xi,t) \quad (3)$$

with boundary condition

$$\mu(0)K(x,0,t) + \alpha(x,t)g(x,t) - \int_0^x K(x,\xi,t)g(\xi,t)d\xi = 0 \quad (4)$$

where

$$\alpha(x,t) = \exp\left(-\int_x^1 \frac{f(s,t+\phi(x)-\phi(s))}{\mu(s)}ds\right), \quad (5)$$

is a bounded function (under Assumption 1), with

$$\phi(x) = \int_0^x \frac{d\gamma}{\mu(\gamma)}. \quad (6)$$

Theorem 4 *Consider system* (1) *in closed loop with*

$$U(t) = \int_0^1 K(1,\xi,t)u(\xi,t)d\xi \quad (7)$$

where $K : \mathcal{T}_1 \to \mathbb{R}$ is a solution to (3)–(4)*. If Assumption 1 holds, then $u(t) \equiv 0$ for $t \in [d_1, T_1]$, where d_1 is defined in* (2)*.*

The existence and computation of the kernel K is addressed in Section 3.2.

PROOF. Consider the same type of backstepping transformation used in [4]

$$w(x,t) = \alpha(x,t)u(x,t) - \int_0^x K(x,\xi,t)u(\xi,t)d\xi, \quad (8)$$

from the variable u into a new variable $w : \mathcal{D}_1 \to \mathbb{R}$, where α is given by (5), and K satisfies (3)–(4). The transformation (8) is invertible for all $\alpha \neq 0$, which, by the definition of α in (5) is the case here.

We will show that the transformation (8) and the control law (7) map system (1) into the following target system

$$w_t(x,t) - \mu(x)w_x(x,t) = 0 \tag{9a}$$
$$w(1,t) = 0 \tag{9b}$$
$$w(x,0) = w_0(x) \tag{9c}$$

for some $w_0 \in \mathcal{B}(\mathcal{D})$. By differentiating (8) with respect to time, inserting the dynamics (1a) and integrating by parts, we obtain

$$\begin{aligned}\alpha(x,t)u_t(x,t) &= -\alpha_t(x,t)u(x,t) + w_t(x,t)\\ &+ K(x,x,t)\mu(x)u(x,t) - K(x,0,t)\mu(0)u(0,t)\\ &- \int_0^x K_\xi(x,\xi,t)\mu(\xi)u(\xi,t)d\xi\\ &- \int_0^x K(x,\xi,t)(\mu'(\xi) - f(\xi,t))u(\xi,t)d\xi\\ &+ \int_0^x K(x,\xi,t)g(\xi,t)d\xi u(0,t)\\ &+ \int_0^x \int_\xi^x K(x,s,t)h(s,\xi,t)dsu(\xi,t)d\xi\\ &+ \int_0^x K_t(x,\xi,t)u(\xi,t)d\xi \end{aligned} \tag{10}$$

where we have changed the order of integration in the double integral involving the term in h. Similarly, differentiating (8) with respect to space results in

$$\begin{aligned}\alpha(x,t)u_x(x,t) &= -\alpha_x(x,t)u(x,t) + w_x(x,t)\\ &+ K(x,x,t)u(x,t) + \int_0^x K_x(x,\xi,t)u(\xi,t)d\xi.\end{aligned} \tag{11}$$

Multiplying (1a) by $\alpha(x,t)$ and inserting (10) and (11), and using the dynamics (3), the fact that $\alpha_t(x,t) - \mu(x)\alpha_x(x,t) = -\alpha(x,t)f(x,t)$ which is easily verified from (5) by direct substitution, and using the boundary condition (4), we obtain (9a). Evaluating (8) at $x = 1$ and inserting the boundary condition (1b) gives

$$w(1,t) = \alpha(1,t)U(t) - \int_0^1 K(1,\xi,t)u(\xi,t)d\xi. \tag{12}$$

The control law (7) results in (9b) when noting from (5) that $\alpha(1,t) = 1$. The initial condition w_0 is given from u_0, K and α as

$$w_0(x) = \alpha(x,0)u_0(x) - \int_0^x K(x,\xi,0)u_0(\xi)d\xi, \tag{13}$$

found by evaluating (8) at $t = 0$. The target system (9) is a pure transport equation with zero as input, and hence $w \equiv 0$ for $t \in [d_1, T_1]$. The kernel governed by (3)–(4) is uniformly bounded since all the coefficients are uniformly bounded in light of Assumption 1 (see Theorem 5). The backstepping transformation (8) is therefore invertible, and the result follows. □

3.2 Computing the controller gain

We now prove that there in fact exists a controller gain $K(1,\xi,t)$ so that (3)–(4) holds. We use a proof similar to what was done in [4].

Theorem 5 *Suppose Assumption 1 holds. Then, the PIDE (3)–(4) has a solution in $\mathcal{T}_1$ with a bound depending on $\bar{f}$, $\bar{g}$, $\bar{h}$, but not on the state u.*

PROOF. The PIDE (3) with (4) imposed corresponds to a PIDE propagating backwards in time. Instead, the boundary condition must be set as $K(1,\xi,t)$, and the problem then boils down to deciding what function $K(1,\xi,t)$, defined over $\mathcal{D}_1$, when propagated through the dynamics (3) to the boundary $\xi = 0$ ensures that (4) holds.

We will prove that such a function K exists by reversing time. Consider the new kernel L also defined over $\mathcal{T}_1$, in the variables (x,ξ,τ), with parameter T_1, defined as

$$L(x,\xi,\tau) = K(x,\xi,T_1 - \tau). \tag{14}$$

Using the equations (3)–(4), we can straightforwardly find the following PIDE for L

$$\begin{aligned}L_\tau(x,\xi,\tau) + \mu(x)L_x(x,\xi,\tau) + \mu(\xi)L_\xi(x,\xi,\tau)\\ = -(\mu'(\xi) - f(\xi,T_1-\tau))L(x,\xi,\tau)\\ + \int_\xi^x L(x,s,\tau)h(s,\xi,T_1-\tau)ds\\ - \alpha(x,T_1-\tau)h(x,\xi,T_1-\tau)\end{aligned} \tag{15a}$$
$$\begin{aligned}\mu(0)L(x,0,\tau) = \int_0^x L(x,\xi,\tau)g(\xi,T_1-\tau)d\xi\\ - \alpha(x,T_1-\tau)g(x,T_1-\tau)\end{aligned} \tag{15b}$$
$$L(x,\xi,0) = L_0(x,\xi) \tag{15c}$$

for the initial condition

$$L_0(x,\xi) = K(x,\xi,T_1). \tag{16}$$

The PIDE (15) is well-posed, and $\sup_{(x,\xi,t)\in\mathcal{T}_1}|L(x,\xi,t)|$ has an upper bound that depends on $\bar{f}$, $\bar{g}$ and $\bar{h}$. This follows from slightly modifying the proof of well-posedness in [4]. Moreover, the gain $K(1,\xi,t)$ can be obtained from L as

$$K(1,\xi,t) = L(1,\xi,T_1 - t). \tag{17}$$

The PIDE (15) is independent of the state u, so therefore, its solution is independent of u. □

The kernel L can be computed from (15) numerically in software off-line prior to implementation of system (1), in view of Assumption 2.

Note that the initial condition L_0 given as (16) is generally unknown. To cope with this, the computation of L can be done over the extended domain

$$\tau \in [-t_0, T_1] \tag{18}$$

for some $t_0 \geq d_1 > 0$, acting as a "startup time", and using an arbitrary initial condition

$$L(x, \xi, -t_0) = L_{-t_0}(x, \xi). \tag{19}$$

We also note that if the system parameters f, g, h are periodic in t, with a common period of say T, the kernel K will also be periodic with period T. It will in this case suffice to compute K over *any* interval of length T.

4 Finite-time convergent output tracking controller

We now seek to design a control law U so that the tracking objective

$$y(t) = r(t) \tag{20}$$

is achieved for $t \geq d_1$. This is achieved by adding a term to the control law (7) as follows

$$\begin{aligned} U(t) = &\int_0^1 K(1, \xi, t) u(\xi, t) d\xi \\ &+ \alpha(0, t + d_1) r(t + d_1) \end{aligned} \tag{21}$$

where α is given from (5).

Theorem 6 *Consider system* (1) *in closed loop with* (21)*, where* $K : \mathcal{T}_1 \to \mathbb{R}$ *is a solution to* (3)–(4) *and* α *is given by* (5)*. If Assumptions 1–3 hold, then* $y(t) = r(t)$ *for* $t \in [d_1, T_1]$*. Moreover, there exists a constant* $\nu > 0$ *so that* $||u(t)||_\infty \leq \nu \bar{r}$ *for all* $t \in [d_1, T_1]$*.*

PROOF. With the control law (21), the backstepping transformation (8) used in the proof of Theorem 4, maps system (1) into the target system

$$w_t(x, t) - \mu(x) w_x(x, t) = 0 \tag{22a}$$
$$w(1, t) = \alpha(0, t + d_1) r(t + d_1) \tag{22b}$$
$$w(x, 0) = w_0(x). \tag{22c}$$

System (22) can be solved explicitly to yield

$$\begin{aligned} w(x, t) &= w(1, t - \phi(1) + \phi(x)) \\ &= \alpha(0, t + \phi(x)) r(t + \phi(x)) \end{aligned} \tag{23}$$

for $t \geq \phi(1) - \phi(x)$, where ϕ is defined in (6), and we have used that $d_1 = \phi(1)$. Inserting this into (1d) gives $y(t) = u(0, t) = \frac{1}{\alpha(0,t)} w(0, t) = \frac{1}{\alpha(0,t)} w(1, t - d_1) = \frac{1}{\alpha(0,t)} \alpha(0, t) r(t) = r(t)$.

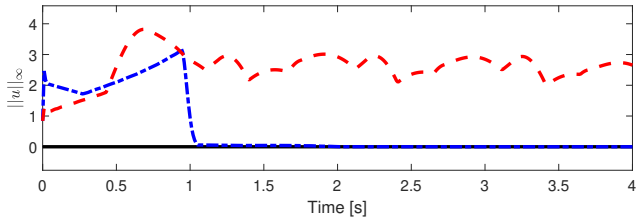

Fig. 1. State norms for the stabilization (dashed-dotted blue) and tracking (dashed red) cases.

We note from (23) that $||w||_\infty \leq \bar{\alpha} \bar{r}$ for $t \geq d_1$, where $\bar{\alpha}$ bounds α. Such a bound $\bar{\alpha}$ exists since Assumption 1 holds. From the invertibility of the transform (8) with a bounded kernel K and bounded α, the result follows. □

5 Simulations

The controllers of Theorems 4 and 6 are implemented in MATLAB, using the system parameters

$$\mu \equiv 1, \qquad f \equiv 0 \tag{24a}$$
$$g(x, t) = 1 + \frac{1}{2}(x + \sin(0.2\pi t)) \tag{24b}$$
$$h(x, \xi, t) = 1 + (x + \xi) \cos(0.5\pi t) \tag{24c}$$

and initial condition $u_0(x) = \sin(x)$. We note that $d_1 = 1$ in this case. System (1) with parameters (24a) is open loop unstable.

5.1 *Smooth reference signal*

The reference signal is set to

$$r(t) = 1 + 2\sin(2\pi t). \tag{25}$$

The controller gain is computed off-line using the method described in Section 3.2, with $t_0 = 2d_1$, and using a second order upwind variant of the directional discretization method proposed in [1] for approximating the spatial derivatives. The system states is implemented using the method of lines [10] with a second order upwind scheme.

In all closed loop cases, it is observed from Figure 1 that the stabilizing controller of Theorem 4 achieves convergence of the system state to zero for $t \geq d_1$. The actuation signals are also bounded in both cases, as seen in Figure 2, with the control signal converging to zero for the stabilizing controller. The tracking goal is seen in Figure 3 to be achieved for (approximately) $t \geq d_1$, as stated in Theorem 6.

5.2 *Discontinuous reference signal*

We now set the reference signal to a square wave with period of 1 second. Even though the method proposed in

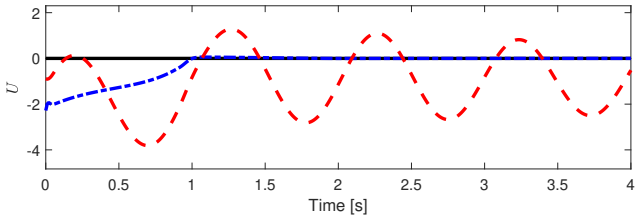

Fig. 2. Actuation signals for the stabilization (dashed-dotted blue) and tracking (dashed red) cases.

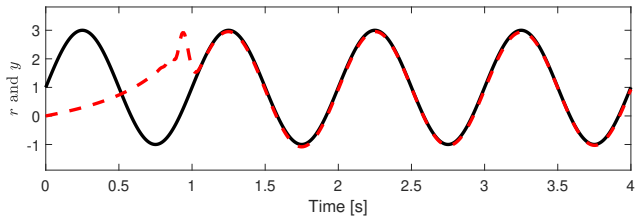

Fig. 3. Reference signal r (solid black) and measured output y (dashed red) during output tracking.

this paper indeed handles a discontinuous reference signal, the second order upwind scheme used for simulating the system in Section 5.1 requires spatial derivatives to exist. To deal with discontinuous references, we therefore compute the closed loop dynamics by exploiting the explicit solution (23) of target system (22) and apply the inverse backstepping transformation, that is the inverse of (8), to obtain u. In other words, we solve an integral equation, thereby avoiding the need to compute spatial derivatives. The results are shown in Figures 4–6.

The system state is bounded, as shown in Figure 4, and the sharp edges induced by the square reference are clearly visible throughout the domain. It is observed from Figure 5 that the measured boundary $y(t) = u(0, t)$ successfully tracks the square wave after one second of simulation. The control input that achieves this is shown in Figure 2, and is bounded as predicted by theory.

From Figures 4 and 6, it is observed that the system state and actuation are bounded.

6 Discussions and further work

We have derived a controller for a 1–D linear hyperbolic PIDE with spatially and time-varying source parameters. The controller achieved convergence in the ∞-norm in finite time d_1, corresponding to the propagation time between the boundaries. Secondly, the state-feedback controller was modified to solve an output tracking problem, where the PIDE's unactuated boundary successfully tracked a bounded reference signal of choice. The controller was also shown to stabilize the system in the ∞-norm. The tracking goal was achieved in finite time d_1. The derived controllers were implemented in MATLAB, illustrating the theory. The derived controllers can

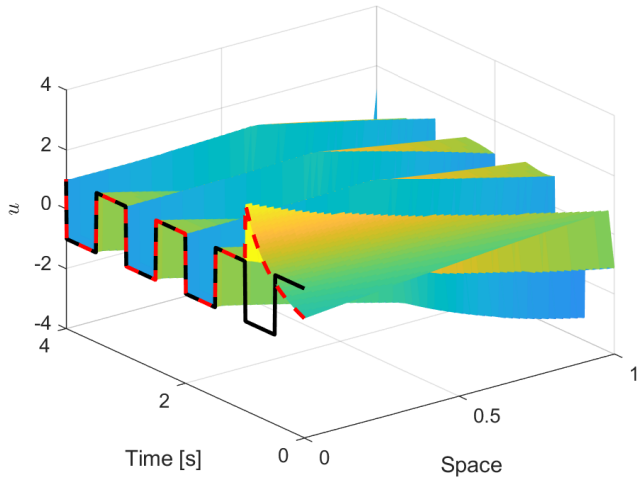

Fig. 4. System state with a square wave reference signal (solid black). The measured boundary $y(t) = u(0, t)$ is highlighted (dashed red).

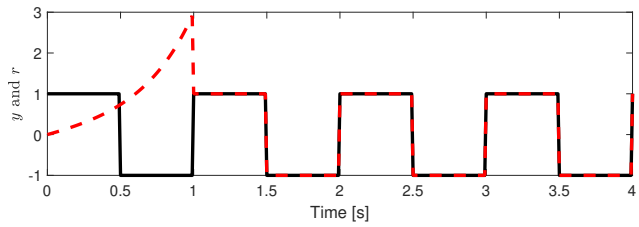

Fig. 5. Measured output (dashed red) and the square wave reference signal (solid black).

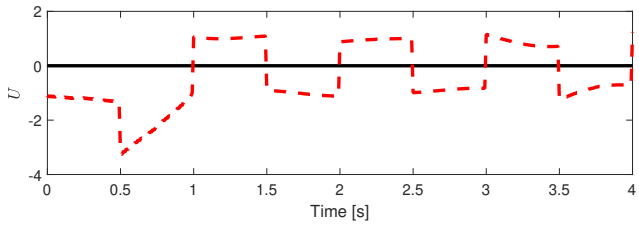

Fig. 6. Actuation (dashed red) with a square wave reference signal.

also be combined with an observer generating a full state estimate that converge to its true value in d_1 time, in the sense of the ∞-norm. One such observer was derived in [4] for a slightly less general type of systems. However, the extension to systems on the form (1) is straightforward.

The proposed theory is valid for a finite time interval $t \in [0, T_1]$. However, the case of having T_1 at an arbitrary (possible infinite) length can be handled by splitting the horizon into periods of, say T seconds. The controller gain for the $(i-1)$th period can be computed during the ith period, and hence, the horizon length T_1 is not needed to be known beforehand.

Although optimality is not considered here, the fact

that the controller gains are obtained by solving a PIDE backwards in time from a boundary condition at the terminal time is reminiscent of conditions for well-posedness in optimal control. Optimal controllers follow from Pontryagin's maximum principle, working backwards in time from a terminal cost. The most famous example is, of course, the differential Riccati equation which provides the solution of the linear quadratic control problem.

A natural next step is to solve the above problem for more complicated systems of P(I)DEs with time-varying coefficients, for instance 2×2 systems which is stabilized using infinite-dimensional backstepping for the time-invariant case in [14].

A subclass of (1), where

$$\mu \equiv \text{const}, \qquad f \equiv 0, \qquad h \equiv 0 \tag{26}$$

is investigated in [2]. It is shown that subject to (26), the kernel equation (3) is significantly simplified. A trivial observer for (1) with (26) is also derived in [2]. An adaptive problem is also solved in [2], under the assumption (26) and the additional assumption that $g(x,t)$ can be split into a time-varying part and a spatially varying part, i.e. $g(x,t) = g_1(x)g_2(t)$.

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