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Audun Reigstad

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Audun Reigstad
Trondheim, September 9, 2020

INTRODUCTION

This thesis is concerned with the nonlinear variational wave (NVW) equation

$$u_{tt} - c(u)(c(u)u_x)_x = 0, \tag{1}$$

where the function $u = u(t, x)$ is such that $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, and $c : \mathbb{R} \to \mathbb{R}$ is a given function depending on u. The equation was first introduced by Saxton in [29], where it is derived as the Euler–Lagrange equation for the variational integral

$$\int_0^\infty \int_{-\infty}^\infty (u_t^2 - c^2(u)u_x^2)\, dx\, dt.$$

The equation appears in the study of liquid crystals, where it describes the director field of a nematic liquid crystal, and where the function c is given by

$$c^2(u) = \alpha \sin^2(u) + \beta \cos^2(u), \tag{2}$$

where α and β are positive physical constants. We refer to [24] and [29] for information about liquid crystals, and the derivation of the equation. From a mathematical point of view it is possible to study (1) with other choices of the function c. Commonly it is assumed that c is continuous, strictly positive and bounded. In addition, one often requires some smoothness on the derivatives of c.

The study of the Cauchy problem, i.e., solving (1) with initial data

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1$$

has been of interest ever since the derivation of the equation. A key property of (1) is that solutions can loose regularity in finite time, even for smooth initial data. The loss of regularity is due to the formation of singularities in the derivatives of u. A singularity means that either u_x or u_t becomes unbounded pointwise while u remains continuous. Therefore, one has to consider weak solutions of (1).

For smooth solutions of the NVW equation, the energy

$$\frac{1}{2} \int_{\mathbb{R}} (u_t^2 + c^2(u)u_x^2)\, dx$$

is independent of time. The singularities in the derivatives are characterized by the fact that $u_x(t, \cdot)$ and $u_t(t, \cdot)$ remain in $L^2(\mathbb{R})$ after they become pointwise unbounded. In other words, we have concentration of energy at points where the derivative blows up. Thus, it is reasonable to look for weak solutions with bounded energy. This naturally leads to the two following notions of solutions. For conservative solutions the energy is constant in time, while for dissipative solutions the energy is decreasing in time. The difference between these solutions is in the continuation after the formation of a singularity. For dissipative solutions the energy decreases at the blow-up time, while for conservative solutions the energy remains unchanged.

The fact that singularities may appear complicates the study of existence, uniqueness and stability of solutions to the NVW equation. Moreover, there are no known explicit solutions of (1) that exhibit the singular behavior of the derivative. When c is equal to a constant we have the classical wave equation whose solutions are known, but does not have singularities.

This is in contrast to for example the Camassa–Holm equation, which has several known solutions with singularities that have served as illuminating examples. The lack of explicit solutions for the NVW equation means that one has to draw inspiration from other equations whose solutions exhibit similar phenomena as (1). Therefore, the study of (1) has been closely related to the Camassa–Holm equation and the Hunter–Saxton equation.

Background

An asymptotic equation for (1) has served as a rich source of inspiration in the study of the NVW equation. The asymptotic equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2 \tag{3}$$

was first derived by Hunter and Saxton in [24], and is known as the Hunter–Saxton (HS) equation. The equation describes small-amplitude and high-frequency perturbations of a constant state of (1). In [24] it is shown that smooth solutions of (3) break down in finite time, meaning that at some finite point in time, the derivative u_x becomes unbounded pointwise. Therefore, one has to consider weak solutions. Next, the authors construct weak solutions which remain continuous after the spatial derivative blows up, which comes from the fact that u_x is square-integrable. Their construction of weak solutions reveals that they in general are not unique.

The non-uniqueness of weak solutions suggests that one should introduce admissibility criteria for selecting weak solutions. Motivated by the fact that the energy

$$\int_{\mathbb{R}} u_x^2 \, dx$$

is bounded for weak solutions, Hunter and Zheng introduce the concept of conservative and dissipative solutions for (3) in [25, 26]. For conservative solutions the energy is constant, even after the solution looses regularity. For dissipative solutions, the energy is nonincreasing, and decreases when singularities appear. The authors establish global existence of weak solutions of both types, for initial data where $u_{0,x}$ has compact support and is of bounded variation. They prove the interesting property, which was observed for the constructed solutions in [24], that both types of weak solutions remain continuous after the derivative blows up.

An important contribution to the study of the NVW equation is [17], where Glassey, Hunter and Zheng prove the corresponding singularity formation for (1). They show that the first order derivatives u_t and u_x can become unbounded pointwise in finite time, even when starting from smooth initial data. This corresponds to concentration of energy in a single point. Hence, global smooth solutions of (1) does not exist.

Moreover, a bounded traveling wave solution is constructed, corresponding to the function $c(u)$ in (2). The constructed wave is a weak solution, which is continuous and piecewise smooth. In particular, the smooth parts are monotone and at their endpoints cusp singularities might turn up, i.e., the derivative is unbounded while the solution itself is bounded.

The authors also point out the difficulty of concentration of energy at points where $c' = 0$.

In a series of papers [30, 31, 32], Zhang and Zheng strengthen the well-posedness results obtained for the HS equation. They establish global existence and uniqueness of conservative and dissipative weak solutions with initial data $u_{0,x}$ belonging to $L^2(\mathbb{R})$ and with compact support. The authors apply methods from the theory of Young measures.

Using similar methods, the same authors study the NVW equation in [33, 34, 36, 37], where they obtain global existence of weak dissipative solutions u of the NVW equation with the following assumptions on the initial data: $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$ and $c'(u_0) > 0$. It is also assumed that $c'(u) \geq 0$ for all u. The functions $u(t, \cdot)$, $u_t(t, \cdot)$ and $u_x(t, \cdot)$ belong to $L^2(\mathbb{R})$ for all $t \geq 0$.

As observed for the constructed weak solution in [17, p. 70], singularities at points where $c' = 0$ are particularly challenging. Because of the expression (2) for the function $c(u)$ appearing in the context of liquid crystals, it is of interest to study (1) with a sign changing c'. It turns out that the first order asymptotic equation (3) is not appropriate for studying these type of singularities.

In the derivation of the HS equation in [24] it is assumed that $c' \neq 0$ at the constant state which is perturbed. The NVW equation allows for a second order asymptotic equation

$$(u_t + u^2 u_x)_x = u u_x^2, \tag{4}$$

which was also introduced in [24]. Here, one requires that $c' = 0$ and $c'' \neq 0$ at the constant state which is perturbed.

In [35], Zhang and Zheng studied the second order asymptotic equation. The authors show that the derivative of the solution blows up in finite time, starting from smooth initial data. From initial data such that $u_{0,x}$ has bounded variation and compact support, they obtain existence of weak dissipative solutions, where the solution and the first order derivatives belong to $L^2_{\text{loc}}(\mathbb{R})$ for all times.

To further study the challenging singularities of (1), Bressan, Zhang and Zheng studied in [13] the more general equation

$$(u_t + f(u)_x)_x = \frac{1}{2} f''(u) u_x^2, \tag{5}$$

where f is a function belonging to $C^2(\mathbb{R})$. With $f(u) = \frac{u^2}{2}$ we get the HS equation, and with $f(u) = \frac{u^3}{3}$ we end up with the second order asymptotic equation (4). The authors construct a semigroup of both conservative and dissipative solutions. A fundamental problem in such a construction is the fact that the derivative of solutions to (5) can become unbounded pointwise in finite time. This corresponds to energy concentrating in a single point. To overcome this problem, the authors consider, in addition to the solution itself, a nonnegative Radon measure μ whose absolutely continuous part corresponds to the classical energy. The singular part of the measure contains information about energy concentration. With this framework, one can prescribe singular initial data. Under certain assumptions on the function f, it is shown that there exists a semigroup of global, weak, conservative solutions. The solution $u(t, \cdot)$ is locally Hölder continuous, and depends continuously on the initial data. The corresponding result holds for dissipative solutions provided that the function f is convex. For both solutions, uniqueness is shown under certain conditions. The conservative solutions are constructed by a change of coordinates based on the characteristics, and the solution is obtained by a contraction argument.

A semigroup of global, dissipative solution of the HS equation is constructed in [5].

With a similar approach, in [6, 7] Bressan and Constantin construct a semigroup of global, conservative and dissipative solutions of the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - u_{xxx} = 0, \tag{6}$$

where κ is a constant. In [6] a set of independent and dependent variables based on the characteristics are introduced, which transforms the equation into a semilinear system of equations. In the new variables, the time variable is still present. Existence and uniqueness of solutions of the semilinear system is obtained by a contraction argument. These solutions exist globally, even after the formation of singularities. By returning to the original variables, the authors obtain a semigroup of global conservative weak solutions.

The idea of rewriting the equation into a system of equations based on the characteristics, is used for the NVW equation in [14] by Bressan and Zheng. A fundamental difference from the HS and CH equation, is that the NVW equation, like the classical wave equation, has two families of characteristics: forward and backward characteristics, while the HS and CH equation has one family of characteristics. Loosely speaking, singularity formation may occur in both families, and one must take this into account in the new coordinates. A consequence of this is that the time variable is not present in the new coordinates. By introducing dependent and independent variables based on the characteristics, (1) transforms into a semilinear system of equations. Existence and uniqueness of solutions to this system follows by a contraction argument. Returning to the original variables, the authors obtain a global semigroup of conservative solutions of (1).

In [22], Holden and Raynaud construct a semigroup of weak, global, conservative solutions of the NVW equation. The approach is related to [14]. As in their work on the CH equation in [21], the equation is rewritten into Lagrangian variables. In the next section, we will describe the method developed in [22].

Dissipative solutions of (1) have been studied by Bressan and Huang in [9]. The corresponding semilinear system of differential equations in the new variables now have discontinuous right-hand side. Existence of solutions to this system follows by a compactness argument. By mapping the solution back to the original variables, the authors show that it provides a dissipative solution of the NVW equation, assuming that $c'(u) > 0$ for all u.

Uniqueness of weak solutions to the NVW equation is a delicate subject, as the characteristics in general are not unique. The uniqueness of conservative solutions is studied in [1, 4], where uniqueness is established for the solutions constructed in [14] given that certain conditions hold, which yield unique characteristics.

A result on the regularity of conservative solutions to (1) has been established by Bressan and Chen in [2]. For initial data satisfying certain smoothness conditions, it is shown that the solution u is piecewise smooth and that the derivative u_x can become pointwise unbounded at finitely many characteristics. An asymptotic description of these solutions in a neighborhood of the singularities is shown in [10] by

Bressan, Huang and Yu. Moreover, in this setting a Lipschitz metric for conservative solutions has been constructed by Bressan and Chen in [3].

Outline of the used method

The main article of this thesis uses the framework from [22]. We therefore give a short description of the method.

We assume that c belongs to $C^2(\mathbb{R})$ and satisfies

$$\frac{1}{\kappa} \leq c(u) \leq \kappa \tag{7}$$

for some $\kappa \geq 1$. In addition, we assume that

$$\max_{u\in\mathbb{R}} |c'(u)| \leq k_1 \quad \text{and} \quad \max_{u\in\mathbb{R}} |c''(u)| \leq k_2 \tag{8}$$

for positive constants k_1 and k_2.

In the study of (1), the functions $R = u_t + c(u)u_x$ and $S = u_t - c(u)u_x$ are often introduced. Then, for smooth solutions the energy rewrites as

$$\frac{1}{4}\int_{\mathbb{R}} (R^2 + S^2)\, dx.$$

The functions R^2 and S^2 are the left and right traveling part of the energy density, respectively. In contrast to the classical wave equation, where c is constant, the right and left part of the energy can interact with each other. That is, energy can swap back and forth between the two parts, while the total energy remains unchanged, in the case of conservative solutions.

To take into account energy concentration in both directions, two positive Radon measures μ and ν are added to the solution and the initial data. The absolutely continuous part of the measures are equal to the left and right traveling part of the energy in the smooth case, i.e., $\mu_{\text{ac}} = \frac{1}{4}R^2\, dx$ and $\nu_{\text{ac}} = \frac{1}{4}S^2\, dx$.

By considering the tuple (u, R, S, μ, ν), one has a complete description of the solution u and possible energy concentration at any time. Thus, one considers these five elements as a solution to (1). The set of all solutions is denoted by $\mathcal{D}$, in which the functions u, R and S belong to $L^2(\mathbb{R})$.

As for the classical wave equation, the NVW equation has two families of characteristics: forward and backward characteristics. The backward characteristics transport the energy described by the measure μ, while the forward characteristics transport the energy described by the measure ν. We interpret the characteristics as particles. At points where the measures are nonsingular there is a finite amount of energy, and there is exactly one forward and one backward characteristic starting from that point. This particle is mapped to one point in the new coordinates (X, Y).

At a point where one of the measures is singular and the other is not, there is an infinite amount of energy. There are infinitely many characteristics corresponding to the singular measure starting from that point, while the nonsingular measure yields one characteristic. This single point is mapped to a horizontal or vertical line in the new coordinates, depending on which measure is singular.

The situation where both measures are singular at a point, means that there is an infinite amount of both backward and forward energy at that point. Infinitely many characteristics of both types start out from that point, and all these particles correspond to a box in the (X, Y)-plane.

The derivation of the system of equations corresponding to (1) in the new coordinates is illustrated by assuming that u is smooth, and μ and ν are absolutely continuous. Then, the method of characteristics yields solutions X and Y of the equations

$$X_t - c(u)X_x = 0 \quad \text{and} \quad Y_t + c(u)Y_x = 0. \tag{9}$$

The operators acting on X and Y are the two factors $\frac{\partial}{\partial t} - c(u)\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t} + c(u)\frac{\partial}{\partial x}$ corresponding to the highest order derivatives in (1). This defines new coordinates (X, Y). The characteristics for the equations in (9) are given by

$$x_t(t) = -c(u(t, x(t))) \quad \text{and} \quad x_t(t) = +c(u(t, x(t))), \tag{10}$$

respectively, for some starting value $x(0) = x_0$. Note that X and Y are constant along characteristics, meaning that particle paths are mapped to straight lines.

Considering the original variables t and x as functions of X and Y, we define $U(X, Y) = u(t(X, Y), x(X, Y))$. We introduce functions J and K, where J corresponds to the energy distribution in the new coordinates. Denoting $Z = (t, x, U, J, K)$, we end up with the identities

$$x_X = c(U)t_X, \qquad x_Y = -c(U)t_Y, \tag{11a}$$

$$J_X = c(U)K_X, \qquad J_Y = -c(U)K_Y, \tag{11b}$$

$$2J_X x_X = (c(U)U_X)^2, \qquad 2J_Y x_Y = (c(U)U_Y)^2, \tag{11c}$$

and a semilinear system of equations

$$Z_{XY} = F(Z)(Z_X, Z_Y), \tag{12}$$

where $F(Z)$ is a bilinear and symmetric tensor from $\mathbb{R}^5 \times \mathbb{R}^5$ to $\mathbb{R}^5$. From (11) it is clear that the vector Z consists of dependent and independent elements. A fixed point argument is used to prove existence of solutions to the system. This requires a curve $(\mathcal{X}(s), \mathcal{Y}(s))$ parametrized by $s \in \mathbb{R}$ in the (X, Y)-plane that corresponds to the initial time $t = 0$. In the smooth case, the set of points $(X, Y) \in \mathbb{R}^2$ such that $t(X, Y) = 0$ defines this curve, which is monotone. For general initial data this set is the union of strictly monotone curves, horizontal and vertical lines, and boxes. In the case of a box there are in principle infinitely many possible ways of choosing the curve. One has to take this into account when defining initial data in the Lagrangian coordinates.

The initial data in $\mathcal{D}$ is mapped to the Lagrangian variables in $\mathcal{G}_0$ in two steps. First we define a map $\mathbf{L}$ from $\mathcal{D}$ to the set $\mathcal{F}$, consisting of elements $\psi = (\psi_1, \psi_2)$ where $\psi_1(X)$ and $\psi_2(Y)$ are five dimensional vectors.

Loosely speaking, the map $\mathbf{L}$ yields the value of Z and its derivatives in each characteristic direction, i.e., in the X and the Y direction. Linking the values of ψ_1 and ψ_2 yields the set of points in the (X, Y)-plane where time equals zero. For instance, in the case of initial data where both measures are singular at the same point, this set is a box.

The next map picks one curve $(\mathcal{X}, \mathcal{Y})$ from the set where time equals zero, and sets the value of Z and its derivatives on the curve. This map is denoted by $\mathbf{C}$ and maps $\mathcal{F}$ to the set $\mathcal{G}_0$, which is the set of elements $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ corresponding to time equals zero. An element Θ consists of the initial curve $(\mathcal{X}, \mathcal{Y})$ parametrized by $s \in \mathbb{R}$, and yields the value of Z, Z_X and Z_Y on the curve. This means that $\mathcal{Z}(s) =$

$Z(\mathcal{X}(s),\mathcal{Y}(s))$, $\mathcal{V}(X)=Z_X(X,\mathcal{Y}(\mathcal{X}^{-1}(X)))$ and $\mathcal{W}(Y)=Z_Y(\mathcal{X}(\mathcal{Y}^{-1}(Y)),Y)$, and we denote this by $\Theta=Z\bullet(\mathcal{X},\mathcal{Y})$. In the case of a box, the map $\mathbf{C}$ picks the curve consisting of the left vertical side and the upper horizontal side of the box. We consider curves $(\mathcal{X},\mathcal{Y})$ of a certain type. The functions $\mathcal{X}$ and $\mathcal{Y}$ are continuous, nondecreasing, and have finite distance to the identity. Moreover, the functions satisfy $\mathcal{X}+\mathcal{Y}=\mathrm{Id}$. The set of such curves is denoted by $\mathcal{C}$. The functions $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{W}$ belong, with some modifications, to $L^\infty(\mathbb{R})$.

Existence and uniqueness of solutions to (12) with initial data Θ follows from a fixed point argument. The solution is first constructed on small rectangular domains Ω in the (X,Y)-plane, where the initial curve $(\mathcal{X},\mathcal{Y})$ connects the lower left corner with the upper right corner of the rectangle. Here, a solution basically means that Z, Z_X and Z_Y are pointwise bounded in the box, and that (12) is satisfied almost everywhere in Ω. We consider solutions satisfying some additional properties, i.e., they satisfy the identities in (11) and some monotonicity conditions. The set of such solutions is denoted by $\mathcal{H}(\Omega)$. For the initial data $\Theta\in\mathcal{G}_0$ we have $\mathcal{V}_2+\mathcal{V}_4>0$ and $\mathcal{W}_2+\mathcal{W}_4>0$ almost everywhere. This property is preserved in the solution and is important in proving that the solution operator from $\mathcal{D}$ to $\mathcal{D}$ is a semigroup.

A pointwise uniform bound on the functions Z, Z_X and Z_Y in strip like domains containing small rectangles allows us to prove, by an induction argument, existence and uniqueness of solutions in $\mathcal{H}(\Omega)$ on arbitrarily large rectangular domains Ω.

If a function Z in $\mathcal{H}(\Omega)$ is a solution on any rectangular domain Ω, and there exists a curve $(\mathcal{X},\mathcal{Y})\in\mathcal{C}$ such that $Z\bullet(\mathcal{X},\mathcal{Y})\in\mathcal{G}$, we say that Z is a global solution to (12). Here, $\mathcal{G}$ is the analogue of $\mathcal{G}_0$, corresponding to time t different from zero. The set of global solutions is denoted by $\mathcal{H}$. The functions Z, Z_X and Z_Y are, with some modifications, pointwise bounded globally. In particular, the Lagrangian counterpart to the energy is bounded.

A global solution can be constructed by using local solutions in boxes. The procedure is as follows. First we construct solutions on rectangles with diagonal points that lie on the initial curve $(\mathcal{X},\mathcal{Y})\in\mathcal{C}$. These solutions are then used to construct initial data for adjacent rectangles, and we obtain solutions there as well. Continuing like this one obtains a global solution. We denote the solution map that to any initial data $\Theta\in\mathcal{G}$ yields a unique solution $Z\in\mathcal{H}$ by $\mathbf{S}$.

Having constructed a global solution $Z\in\mathcal{H}$, the goal is to map it back to Eulerian coordinates $\mathcal{D}$ for any time $T>0$. As addressed before, the points $(X,Y)\in\mathbb{R}^2$ such that $t(X,Y)=T$ may contain boxes. In order to use the sets previously defined for time equal to zero, we shift time to zero, i.e., for $Z\in\mathcal{H}$ we define $\bar{Z}\in\mathcal{H}$ where $\bar{t}(X,Y)=t(X,Y)-T$. The other elements of $\bar{Z}$ are identical to Z. We call this map $\mathbf{t}_T$.

In the case of a box, the curve $(\mathcal{X},\mathcal{Y})$ corresponding to time T is defined as the left vertical side and the upper horizontal side of the box. The element $\Theta\in\mathcal{G}_0$ is then defined as $\Theta=Z\bullet(\mathcal{X},\mathcal{Y})$, and we denote the map by $\mathbf{E}:\mathcal{H}\to\mathcal{G}_0$. Because of the monotonicity of the function t, the curve corresponding to time T is below the initial curve. For any $\Theta\in\mathcal{G}_0$ we define a map $\mathbf{D}$ that associates an element $\psi\in\mathcal{F}$. The operator

$$S_T=\mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T\circ\mathbf{S}\circ\mathbf{C}$$

that for any initial data in $\mathcal{F}$ yields a solution in $\mathcal{F}$ corresponding to time $T>0$, is a semigroup. The remaining step to Eulerian coordinates is the map $\mathbf{M}:\mathcal{F}\to\mathcal{D}$, and yields an element $(u, R, S, \mu, \nu)(T) \in \mathcal{D}$ at time $T > 0$. Thus, the map

$$\bar{S}_T = \mathbf{M} \circ S_T \circ \mathbf{L}$$

yields an element in $\mathcal{D}$ given any initial data in $\mathcal{D}$. If $\mathbf{L} \circ \mathbf{M} = \mathrm{Id}$, it follows from the semigroup property of S_T that also $\bar{S}_T$ is a semigroup. However, this identity does not hold in general. This is because for any element $\psi \in \mathcal{F}$ we have $x_i + J_i \in G$, where the group G is given by all invertible functions f such that $f - \mathrm{Id}$, $f^{-1} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ and $(f - \mathrm{Id})' \in L^2(\mathbb{R})$, while the element $\bar{\psi} \in \mathcal{F}$ given by the map $\mathbf{L}$ satisfies $\bar{x}_i + \bar{J}_i = \mathrm{Id}$. Therefore, in general one has $\psi \neq \bar{\psi}$.

To overcome this problem, one considers the following approach. Assume that $x_1 + J_1 = f$ and $x_2 + J_2 = g$ where $f, g \in G$, and consider $\tilde{\psi}$ defined by $\tilde{x}_1 = x_1 \circ f^{-1}$, $\tilde{J}_1 = J_1 \circ f^{-1}$, $\tilde{x}_2 = x_2 \circ g^{-1}$ and $\tilde{J}_2 = J_2 \circ g^{-1}$. It follows that $\tilde{x}_i + \tilde{J}_i = \mathrm{Id}$, $i = 1, 2$. We denote $\phi = (f^{-1}, g^{-1})$ and $\tilde{\psi} = \psi \cdot \phi$. This defines an action of G^2 on the set $\mathcal{F}$. Moreover, the transformation of ψ to $\tilde{\psi}$ defines a projection Π from $\mathcal{F}$ on the set

$$\mathcal{F}_0 = \{\psi = (\psi_1, \psi_2) \in \mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \text{ and } x_2 + J_2 = \mathrm{Id}\}.$$

Thus, we have $\tilde{\psi} = \Pi(\psi)$. It turns out that the map $S_T : \mathcal{F} \to \mathcal{F}$ is invariant under the group acting on $\mathcal{F}$, i.e.,

$$S_T(\psi \cdot \phi) = S_T(\psi) \cdot \phi,$$

where $\phi \in G^2$. This is a consequence of the fact that the maps which S_T is composed of, are invariant under the group action. The action of G^2 on the set of curves $\mathcal{C}$ and $\mathcal{G}$ naturally follows from the definition of the action on $\mathcal{F}$. On the set of curves $\mathcal{C}$, the action corresponds to stretching of the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ in the X and Y direction. For the set $\mathcal{H}$, the action is defined such that it commutes with the $\bullet$ operation.

A key result is that the map $\mathbf{M}$ satisfies $\mathbf{M} = \mathbf{M} \circ \Pi$, and that $\mathcal{F}_0$ contains exactly one element of each equivalence class of $\mathcal{F}$ with respect to G^2. This implies that to each element in $\mathcal{D}$ there correspond infinitely many elements in $\mathcal{F}$, all belonging to the same equivalence class. The mapping $\mathbf{L} : \mathcal{D} \to \mathcal{F}_0$ on the other hand picks one member of each equivalence class, but we could also pick a different one. Applying the solution operator to all elements belonging to the same equivalence class yields infinitely many solutions in $\mathcal{F}$, which form an equivalence class. Using the mapping $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ on all of these solutions yields the same element in $\mathcal{D}$. Since we get the same solution in the end, we can think of each member of the equivalence class as a different "parametrization" of the initial data in $\mathcal{F}$, which are connected through relabeling. Hence, the map $\bar{S}_T$ is a semigroup. Moreover, the solution produced by the map is a global weak solution of (1). It is conservative in the sense that for all $T \geq 0$,

$$\mu(T)(\mathbb{R}) + \nu(T)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}),$$

where $\mu(T)$ and $\nu(T)$ are the measures at time T, and μ_0 and ν_0 are the initial measures. This is a consequence of the fact that the energy function J in Lagrangian coordinates is such that the limit

$$\lim_{s \to \pm\infty} J(\mathcal{X}(s), \mathcal{Y}(s))$$

is independent of the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. Thus, the same limiting values of J are obtained for curves corresponding to different times.

Paper I: A Regularized System for the Nonlinear Variational Wave Equation

The main part of this thesis is the study of a regularizing system for the NVW equation. This is the content of Paper I, which is an extended version of an article that will be submitted for publication, see [19]. The system reads

$$u_{tt} - c(u)(c(u)u_x)_x = -\frac{c'(u)}{4}(\rho^2 + \sigma^2), \tag{13a}$$

$$\rho_t - (c(u)\rho)_x = 0, \tag{13b}$$

$$\sigma_t + (c(u)\sigma)_x = 0. \tag{13c}$$

We study (13) by adapting the method used in [22] for the NVW equation, which is described in the previous section.

As in [22], we consider initial data with measures to allow for energy concentration at time equals zero. In the smooth case, the energy associated to (13) is

$$\frac{1}{4}\int_{\mathbb{R}} \left(R^2 + c(u)\rho^2 + S^2 + c(u)\sigma^2\right) dx.$$

Here, $R^2 + c(u)\rho^2$ and $S^2 + c(u)\sigma^2$ are the left and right traveling part of the energy density, respectively. The functions ρ and σ are added to the set $\mathcal{D}$. Now $\mathcal{D}$ consists of elements $(u, R, S, \rho, \sigma, \mu, \nu)$, where u, R, S, ρ and σ belong to $L^2(\mathbb{R})$, and the measures μ and ν satisfy $\mu_{\text{ac}} = \frac{1}{4}(R^2 + c(u)\rho^2)\,dx$ and $\nu_{\text{ac}} = \frac{1}{4}(S^2 + c(u)\sigma^2)\,dx$.

In the new coordinates (X, Y) we introduce $P(X, Y) = \rho(t(X, Y), x(X, Y))$, $p = Px_X$, $Q(X, Y) = \sigma(t(X, Y), x(X, Y))$, and $q = Qx_Y$. We denote $Z = (t, x, U, J, K)$ and obtain the same identities as in (11a) and (11b), while the third identity (11c) now takes the form

$$2J_X x_X = (c(U)U_X)^2 + c(U)p^2 \quad \text{and} \quad 2J_Y x_Y = (c(U)U_Y)^2 + c(U)q^2. \tag{14}$$

Moreover, we get the same system of differential equations (12), and two additional equations

$$p_Y = 0 \quad \text{and} \quad q_X = 0, \tag{15}$$

which correspond to (13b) and (13c). Note the difference between (11c) and (14), which shows that the solutions of (12) corresponding to (1) and (13) are not identical. In particular, from (14) we see that the solutions t, x, U, J, K of (12) are not independent of the solutions p, q of (15).

The construction of a semigroup of weak, global, conservative solutions of (13) follows to a large extent the procedure for the NVW equation. We add two functions, describing ρ and σ, to the sets that correspond to the solution in Lagrangian variables. The mappings between the sets are modified accordingly.

Our main results are the following. We consider smooth initial data u_0, R_0, S_0, ρ_0 and σ_0, and absolutely continuous measures μ_0 and ν_0 on a finite interval $[x_l, x_r]$. If ρ_0 and σ_0 are strictly positive on this interval, then for every time $t \in \left[0, \frac{1}{2\kappa}(x_r - x_l)\right]$, the solutions $\rho(t, x)$ and $\sigma(t, x)$ will also be strictly positive for all $x \in [x_l + \kappa t, x_r - \kappa t]$. The strict positivity of ρ_0 and σ_0 is preserved by (15). This has a regularizing effect on the solution at time t in the sense that $u(t, x)$, $R(t, x)$, $S(t, x)$, $\rho(t, x)$ and $\sigma(t, x)$ are smooth, and the measures $\mu(t)$ and $\nu(t)$ are absolutely continuous.

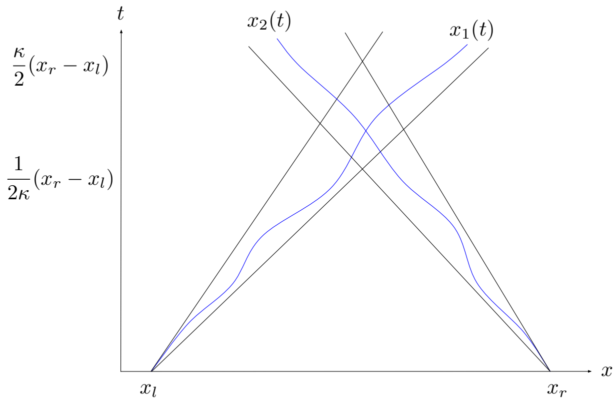

FIGURE 1. Characteristics of the NVW equation. The forward characteristic $x_1(t)$ starting from x_l is given by $x_{1,t}(t) = c(u(t, x_1(t)))$, $x_1(0) = x_l$, and the backward characteristic $x_2(t)$ starting from x_r is given by $x_{2,t}(t) = -c(u(t, x_2(t)))$, $x_2(0) = x_r$. Because of (7), they intersect at a time t such that $\frac{1}{2\kappa}(x_r - x_l) \le t \le \frac{\kappa}{2}(x_r - x_l)$.

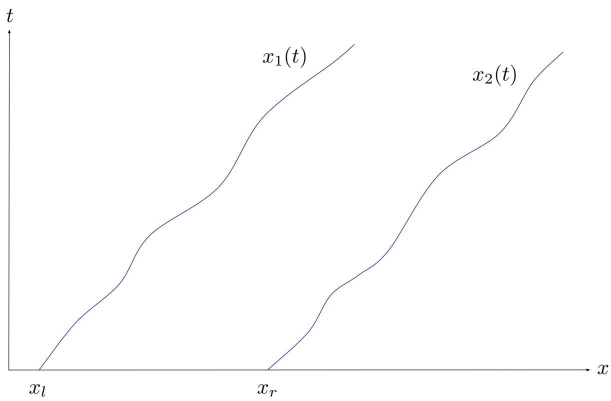

FIGURE 2. Characteristics of the CH equation. The characteristic $x_1(t)$ starting from x_l is given by $x_{1,t}(t) = u(t, x_1(t))$, $x_1(0) = x_l$, and the characteristic $x_2(t)$ starting from x_r is given by $x_{2,t}(t) = u(t, x_2(t))$, $x_2(0) = x_r$.

The region where regularity holds comes from the characteristics in (10), see Figure 1.

Next, we consider a sequence of smooth solutions $(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)$ with initial data satisfying $u_0^n \to u_0$ in $L^\infty([x_l, x_r])$, $R_0^n \to R_0$, $S_0^n \to S_0$, $\rho_0^n \to 0$, $\sigma_0^n \to 0$ in $L^2([x_l, x_r])$, where u_0, R_0 and S_0 are smooth, and the associated measures μ_0 and ν_0 are absolutely continuous on $[x_l, x_r]$. Then, $u^n(t, \cdot) \to u(t, \cdot)$ in $L^\infty([x_l + \kappa t, x_r - \kappa t])$ for all $t \in \left[0, \frac{1}{2\kappa}(x_r - x_l)\right]$, where u is a solution of the NVW equation with initial data $(u_0, R_0, S_0, \mu_0, \nu_0)$. A central ingredient in the proof is a Gronwall inequality in two variables, see [15].

We point out that these are local results. The main reason for this is that we require the initial data ρ_0 and σ_0 corresponding to the equations (13b) and (13c) to be bounded from below by a strictly positive constant and to belong to L^2, which is not possible globally.

We hope that further studies of the smooth approximations will be helpful in the understanding of singularities to (1).

The motivation for studying (13) comes from the two-component Camassa–Holm system

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x = 0, \tag{16a}$$

$$\rho_t + (u\rho)_x = 0, \tag{16b}$$

where $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$ are given numbers. In [18], global, weak, conservative solutions of (16) are constructed. It is shown that the solution of (16) is regular if initially ρ_0 is strictly positive. Moreover, a sequence of regular solutions, with $\rho_0^n \to 0$, converge in $L^\infty(\mathbb{R})$ to the global conservative weak solution of the CH equation.

Loosely speaking, since the CH equation has one family of characteristics, see Figure 2, an extra variable ρ is needed to preserve the positivity of ρ_0 in the characteristic direction. Since the NVW equation has both forward and backward characteristics, we need two extra variables ρ and σ to preserve the positivity of ρ_0 and σ_0 in each characteristic direction.

We mention that a regularizing system has been studied for the HS equation in [28].

Paper II: Traveling Waves for the Nonlinear Variational Wave Equation

The second part of this thesis deals with traveling wave solutions of the NVW equation. Paper II is an extended version of an article that will be submitted for publication, see [20].

We consider traveling wave solutions of (1) with wave speed $s \in \mathbb{R}$, i.e., solutions $u(t, x) = w(x - st)$, where w denotes some continuous and bounded function. Classical traveling wave solutions of (1) satisfy the equation

$$\left[s^2 - c^2(w)\right] w_{\xi\xi} - c(w)c'(w)w_\xi^2 = 0.$$

We assume that the function c belongs to $C^2(\mathbb{R})$ and that there exists $0 < \alpha < \beta < \infty$, such that

$$\alpha = \min_{u \in \mathbb{R}} c(u) \quad \text{and} \quad \beta = \max_{u \in \mathbb{R}} c(u).$$

Moreover, we assume that (8) holds.

We study whether we can glue together local, classical traveling wave solutions with wave speed $s \in \mathbb{R}$ to obtain globally, bounded, continuous traveling waves. Our main result is that we can only glue at points $\xi \in \mathbb{R}$ where $|s| = c(w(\xi))$ and $c'(w(\xi)) \neq 0$, where w denotes the traveling wave composed of the local solutions. At the point ξ, w has a singularity, meaning that the derivative w_ξ is pointwise unbounded and w is bounded. Near singularities, the traveling wave is a monotone, classical solution. Moreover, denoting $u(t,x) = w(x-st)$, we prove that u is a weak solution of (1).

There are three possible ways of gluing at the singular point ξ. The derivative of w can have the same sign on both sides of the singularity, in which case there is an inflection point at ξ. Another possibility is that the derivative has opposite sign on each side of the singularity. Then the traveling wave is either convex or concave on both sides of ξ, and the singularity is a cusp. The third possibility is that w is constant on one side of the singularity and monotone on the other side.

If $|s|$ does not belong to the interval $[\alpha, \beta]$, then w is a monotone, classical solution, which is globally unbounded.

The approach we use is similar to the derivation of the Rankine–Hugoniot condition for hyperbolic conservation laws, see e.g. [23]. Applying the method of proof to the CH equation, we recover a well-known result by Lenells on traveling wave solutions, see [27]. Classical traveling wave solutions of (6), where $\kappa = 0$, satisfy

$$w^2(w-s) - w_\xi^2(w-s) = 2aw + b \tag{17}$$

for some constants a and b. Gluing two local, classical traveling wave solutions with speed s together at a point ξ to obtain a bounded, continuous wave w can only be done if $s = w(\xi)$, and the constant a corresponding to the two solutions are identical.

Paper III: Competition Models for Plant Stems

The final article [8] of this thesis deals with models for plants competing for sunlight. It was written during a research stay at Penn State University the academic year 2018/2019, where Professor Alberto Bressan was visited.

We consider a large number of similar plants, uniformly distributed in the plane. Moreover, we assume that each plant consists of a single stem, which is described by a curve $\gamma(s) = (x(s), y(s))$ parametrized by arc length. We consider the situation where sunlight comes from the direction of the unit vector $\mathbf{n} = (n_1, n_2)$, where $n_2 < 0 < n_1$. We denote by $\theta_0 \in (0, \frac{\pi}{2})$ the angle such that $(-n_2, n_1) = (\cos(\theta_0), \sin(\theta_0))$.

A functional describing the amount of sunlight captured by each stem is to be maximized, subject to certain conditions. The functional depends on the intensity of light, which we assume is a given nondecreasing function $I(y)$ depending on the height above ground. The derivation of the sunlight functional follows the procedure described in [12].

We analyze two models. In the first one we assume that all stems have the same given length $l > 0$ and thickness $\kappa > 0$. Then, the optimization problem for a single stem consists of finding the height $h > 0$ of the stem and angle $\theta(y)$ between the stem and the x-axis that maximizes the gathered sunlight

$$\int_0^h I(y)\left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta(y) - \theta_0)}\right\}\right)\cos(\theta(y) - \theta_0)\, dy.$$

We prove that under certain conditions on the light intensity function, there exists a unique optimal solution (h^*, θ^*). The solution satisfies $\theta^*(h^*) = \theta_0$, i.e., the tip of the stem is orthogonal to the light rays. Next, we assume that we have a continuous distribution of identical stems given by the optimal solution. We use this to compute the new intensity of light at height y. Given this intensity function, we can continue the procedure of finding a new optimal stem. We prove that there exists a unique competitive equilibrium provided that the density of vegetation is sufficiently small.

In the second model we give no constraint on the length $l > 0$ of the stem, and we allow the density $u(s) > 0$ to be variable along the stem. Now the optimization problem is to maximize the sunlight gathered by the stem,

$$\int_0^\infty I(y(s))\left(1 - \exp\left\{\frac{-u(s)}{\cos(\theta(s) - \theta_0)}\right\}\right)\cos(\theta(s) - \theta_0)\,ds,$$

among all admissible (θ, u), subject to a cost of transporting water and nutrients from the root to the leaves, given by

$$\int_0^\infty \left(\int_s^\infty u(t)\,dt\right)^\alpha ds$$

for some $0 < \alpha < 1$. We prove that an optimal solution (θ^*, u^*) exists, which corresponds to a stem of finite length. The optimization problem can be formulated as an optimal control problem with both initial and terminal constraints. The Pontryagin maximum principle, see [11, Section 6.5] and [16, Chapter II, §5] leads to a two-point boundary value problem for a system of ordinary equations for the adjoint variables. By analyzing this problem, uniqueness of the optimal solution is established provided that the density of external vegetation is small. Moreover, the tip of the stem is orthogonal to the vector $\mathbf{n}$. For the second model we also prove that there exists a unique competitive equilibrium, provided that the density of stems is sufficiently small.

We would like to mention that there is an ongoing project with Bressan on the optimal design of a marine reserve.

References

[1] A. Bressan: Uniqueness of conservative solutions for nonlinear wave equations via characteristics. Bull. Braz. Math. Soc. (N.S.) **47**, 157–169 (2016).

[2] A. Bressan and G. Chen: Generic regularity of conservative solutions to a nonlinear wave equation. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 335–354 (2017).

[3] A. Bressan and G. Chen: Lipschitz metrics for a class of nonlinear wave equations. Arch. Ration. Mech. Anal. **226**, 1303–1343 (2017).

[4] A. Bressan, G. Chen, and Q. Zhang: Unique conservative solutions to a variational wave equation. Arch. Ration. Mech. Anal. **217**, 1069–1101 (2015).

[5] A. Bressan and A. Constantin: Global solutions of the Hunter–Saxton equation. SIAM J. Math. Anal. **37**, 996–1026 (2005).

[6] A. Bressan and A. Constantin: Global conservative solutions of the Camassa–Holm equation. Arch. Ration. Mech. Anal. **183**, 215–239 (2007).

[7] A. Bressan and A. Constantin: Global dissipative solutions of the Camassa–Holm equation. Anal. Appl. (Singap.) **5**, 1–27 (2007).

[8] A. Bressan, S. T. Galtung, A. Reigstad, and J. Ridder: Competition models for plant stems. J. Differential Equations **269**, 1571–1611 (2020).

[9] A. Bressan and T. Huang: Representation of dissipative solutions to a nonlinear variational wave equation. Commun. Math. Sci. **14**, 31–53 (2016).

[10] A. Bressan, T. Huang, and F. Yu: Structurally stable singularities for a nonlinear wave equation. Bull. Inst. Math. Acad. Sin. (N.S.) **10**, 449–478 (2015).

[11] A. Bressan and B. Piccoli: Introduction to the mathematical theory of control. AIMS Series on Applied Mathematics, 2. American Institute of Mathematical Sciences, Springfield, MO, 2007.

[12] A. Bressan and Q. Sun: On the optimal shape of tree roots and branches. Math. Models Methods Appl. Sci. **28**, 2763–2801 (2018).

[13] A. Bressan, P. Zhang, and Y. Zheng: Asymptotic variational wave equations. Arch. Ration. Mech. Anal. **183**, 163–185 (2007).
[14] A. Bressan and Y. Zheng: Conservative solutions to a nonlinear variational wave equation. Comm. Math. Phys. **266**, 471–497 (2006).
[15] A. M. Fink, D. S. Mitrinović, and J. E. Pečarić: Inequalities involving functions and their integrals and derivatives. Mathematics and its Applications (East European Series), vol. 53., Kluwer Academic Publishers Group, Dordrecht (1991).
[16] W. H. Fleming and R. W. Rishel: Deterministic and stochastic optimal control. Applications of Mathematics, no. 1, Springer-Verlag, Berlin-New York (1975).
[17] R. T. Glassey, J. K. Hunter, and Y. Zheng: Singularities of a variational wave equation. J. Differential Equations **129**, 49–78 (1996).
[18] K. Grunert, H. Holden, and X. Raynaud: Global solutions for the two-component Camassa–Holm system. Comm. Partial Differential Equations **37**, 2245–2271 (2012).
[19] K. Grunert and A. Reigstad: A regularized system for the nonlinear variational wave equation. `http://arxiv.org/abs/2008.13003` (2020).
[20] K. Grunert and A. Reigstad: Traveling waves for the nonlinear variational wave equation. `http://arxiv.org/abs/2009.03178` (2020).
[21] H. Holden and X. Raynaud: Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. Comm. Partial Differential Equations **32**, 1511–1549 (2007).
[22] H. Holden and X. Raynaud: Global semigroup of conservative solutions of the nonlinear variational wave equation. Arch. Ration. Mech. Anal. **201**, 871–964 (2011).
[23] H. Holden and N. H. Risebro: Front tracking for hyperbolic conservation laws. Appl. Math. Sci., vol. 152, Springer-Verlag, New York (2002).
[24] J. K. Hunter and R. Saxton: Dynamics of director fields. SIAM J. Appl. Math. **51**, 1498–1521 (1991).
[25] J. K. Hunter and Y. Zheng: On a nonlinear hyperbolic variational equation: I. Global existence of weak solutions. Arch. Ration. Mech. Anal. **129**, 305–353 (1995).
[26] J. K. Hunter and Y. Zheng: On a nonlinear hyperbolic variational equation: II. The zero-viscosity and dispersion limits. Arch. Ration. Mech. Anal. **129**, 355–383 (1995).
[27] J. Lenells: Traveling wave solutions of the Camassa–Holm equation. J. Differential Equations **217**, 393–430 (2005).
[28] A. Nordli: A Lipschitz metric for conservative solutions of the two-component Hunter–Saxton system. Methods Appl. Anal. **23**, 215–232 (2016).
[29] R. A. Saxton: Dynamic instability of the liquid crystal director. In: W. B. Lindquist (ed.) Current Progress in Hyperbolic Systems: Riemann Problems and Computations (Brunswick, ME, 1988), 325–330. Contemp. Math., Vol. 100, Amer. Math. Soc., Providence, RI (1989).
[30] P. Zhang and Y. Zheng: On oscillations of an asymptotic equation of a nonlinear variational wave equation. Asymptot. Anal. **18**, 307–327 (1998).
[31] P. Zhang and Y. Zheng: On the existence and uniqueness of solutions to an asymptotic equation of a variational wave equation. Acta Math. Sin. (Engl. Ser.) **15**, 115–130 (1999).
[32] P. Zhang and Y. Zheng: Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data. Arch. Ration. Mech. Anal. **155**, 49–83 (2000).
[33] P. Zhang and Y. Zheng: Rarefactive solutions to a nonlinear variational wave equation of liquid crystals. Comm. Partial Differential Equations **26**, 381–419 (2001).
[34] P. Zhang and Y. Zheng: Singular and rarefactive solutions to a nonlinear variational wave equation. Chin. Ann. Math. **22B**, 159–170 (2001).
[35] P. Zhang and Y. Zheng: On the second-order asymptotic equation of a variational wave equation. Proc. Roy. Soc. Edinburgh Sect. A **132**, 483–509 (2002).
[36] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation. Arch. Rat. Mech. Anal. **166**, 303–319 (2003).
[37] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation with general data. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 207–226 (2005).

[38] P. Zhang and Y. Zheng: On the global weak solutions to a variational wave equation. In: C. M. Dafermos and E. Feireisl (eds.) Handbook of Differential Equations. Evolutionary Equations. Volume 2, 561–648. Elsevier, Amsterdam (2005).

Paper I

A Regularized System for the Nonlinear Variational Wave Equation

K. Grunert and A. Reigstad

A REGULARIZED SYSTEM FOR THE NONLINEAR VARIATIONAL WAVE EQUATION

KATRIN GRUNERT AND AUDUN REIGSTAD

Abstract. We present a new generalization of the nonlinear variational wave equation. We prove existence of local, smooth solutions for this system. As a limiting case, we recover the nonlinear variational wave equation.

Contents

1. Introduction

The nonlinear variational wave equation (NVW) is given by

$$u_{tt} - c(u)(c(u)u_x)_x = 0, \tag{1.1}$$

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where $u = u(t,x)$, $t \geq 0$ and $x \in \mathbb{R}$, with initial data

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1. \tag{1.2}$$

It was introduced by Saxton in [11], where it is derived by applying the variational principle to the functional

$$\int_0^\infty \int_{-\infty}^\infty (u_t^2 - c^2(u)u_x^2)\, dx\, dt.$$

It is well known that derivatives of solutions of this equation can develop singularities in finite time even for smooth initial data, see e.g. [8]. The continuation past singularities is highly nontrivial, and allows for various distinct solutions. The most common way of continuing the solution is to require that the energy is non-increasing, which naturally leads to the two following notions of solutions: Dissipative solutions for which the energy is decreasing in time, see [3, 12, 13, 14], and conservative solutions for which the energy is constant in time. In the latter case a semigroup of solutions has been constructed in [4, 10].

In this paper we modify (1.1) by adding two transport equations and coupling terms. The resulting system is given by

$$u_{tt} - c(u)(c(u)u_x)_x = -\frac{c'(u)}{4}(\rho^2 + \sigma^2), \tag{1.3a}$$

$$\rho_t - (c(u)\rho)_x = 0, \tag{1.3b}$$

$$\sigma_t + (c(u)\sigma)_x = 0, \tag{1.3c}$$

with initial data

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \rho|_{t=0} = \rho_0, \quad \sigma|_{t=0} = \sigma_0. \tag{1.4}$$

It is clear that when $\rho = \sigma = 0$ we recover (1.1). We assume that $c \in C^2(\mathbb{R})$ and satisfies

$$\frac{1}{\kappa} \leq c(u) \leq \kappa \tag{1.5}$$

for some $\kappa \geq 1$. In addition, we assume that

$$\max_{u\in\mathbb{R}} |c'(u)| \leq k_1 \quad \text{and} \quad \max_{u\in\mathbb{R}} |c''(u)| \leq k_2 \tag{1.6}$$

for positive constants k_1 and k_2.

We are interested in studying conservative solutions of the initial value problem (1.3)-(1.4) for initial data u_0, $u_{0,x}$, u_1, ρ_0, $\sigma_0 \in L^2(\mathbb{R})$. For smooth and bounded solutions such that u, u_t, u_x, ρ and σ vanish at $\pm\infty$ the energy is given by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \Big(u_t^2 + c^2(u)u_x^2 + \frac{1}{2}c(u)\rho^2 + \frac{1}{2}c(u)\sigma^2\Big)\, dx, \tag{1.7}$$

and independent of time. One way to see this is to consider the quantity

$$K(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2\, dx,$$

which we can think of as the "kinetic energy". We compute $K'(t)$ and find by using (1.3a),

$$K'(t) = \int_{\mathbb{R}} \Big(c(u)u_t(c(u)u_x)_x - \frac{1}{4}c'(u)u_t(\rho^2 + \sigma^2)\Big)\, dx.$$

For the first term we get, by integration by parts,

$$\int_{\mathbb{R}} c(u)u_t(c(u)u_x)_x\,dx = -\int_{\mathbb{R}} \big(c^2(u)u_x u_{xt} + c(u)c'(u)u_t u_x^2\big)\,dx = -\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}c^2(u)u_x^2\,dx,$$

and for the second term we obtain from (1.3b) and (1.3c),

$$\begin{aligned}\int_{\mathbb{R}} c'(u)u_t(\rho^2+\sigma^2)\,dx &= \frac{d}{dt}\int_{\mathbb{R}} c(u)(\rho^2+\sigma^2)\,dx - 2\int_{\mathbb{R}} c(u)(\rho\rho_t+\sigma\sigma_t)\,dx\\ &= \frac{d}{dt}\int_{\mathbb{R}} c(u)(\rho^2+\sigma^2)\,dx - \int_{\mathbb{R}}\Big((c^2(u)\rho^2)_x - (c^2(u)\sigma^2)_x\Big)\,dx\\ &= \frac{d}{dt}\int_{\mathbb{R}} c(u)(\rho^2+\sigma^2)\,dx.\end{aligned}$$

Therefore we get

$$K'(t) = -\frac{d}{dt}\bigg(\int_{\mathbb{R}}\frac{1}{2}c^2(u)u_x^2\,dx + \frac{1}{4}\int_{\mathbb{R}} c(u)(\rho^2+\sigma^2)\,dx\bigg),$$

which implies that $E(t)$ is constant. In particular, we have

$$E(t) = \frac{1}{2}\int_{\mathbb{R}}\Big(u_1^2 + c^2(u_0)u_{0,x}^2 + \frac{1}{2}c(u_0)\rho_0^2 + \frac{1}{2}c(u_0)\sigma_0^2\Big)\,dx.$$

Next, we introduce the functions R and S defined as

$$\begin{cases} R = u_t + c(u)u_x,\\ S = u_t - c(u)u_x.\end{cases} \tag{1.8}$$

Note that R and S are smooth by assumption. Using (1.8) we can express the energy in (1.7) as

$$E(t) = \frac{1}{4}\int_{\mathbb{R}}\big(R^2 + c(u)\rho^2 + S^2 + c(u)\sigma^2\big)\,dx. \tag{1.9}$$

As we shall see, we can think of $R^2 + c(u)\rho^2$ and $S^2 + c(u)\sigma^2$ as the left and right traveling part of the energy density, respectively. Indeed, from (1.3a) we have

$$\begin{cases} R_t - c(u)R_x = \dfrac{c'(u)}{4c(u)}(R^2-S^2) - \dfrac{c'(u)}{4}(\rho^2+\sigma^2),\\[2ex] S_t + c(u)S_x = -\dfrac{c'(u)}{4c(u)}(R^2-S^2) - \dfrac{c'(u)}{4}(\rho^2+\sigma^2).\end{cases} \tag{1.10}$$

Multiplying the first equation in (1.10) by R and the second by S, using

$$(c(u)R^2)_x = \frac{c'(u)}{2c(u)}R^2(R-S) + c(u)(R^2)_x,$$

and

$$(c(u)S^2)_x = \frac{c'(u)}{2c(u)}S^2(R-S) + c(u)(S^2)_x,$$

yields

$$\begin{cases} (R^2)_t - (c(u)R^2)_x = \dfrac{c'(u)}{2c(u)}(R^2S - RS^2) - \dfrac{c'(u)}{2}R(\rho^2+\sigma^2),\\[2ex] (S^2)_t + (c(u)S^2)_x = -\dfrac{c'(u)}{2c(u)}(R^2S - RS^2) - \dfrac{c'(u)}{2}S(\rho^2+\sigma^2).\end{cases}$$

Moreover, using (1.3b) and (1.3c) we get

$$\begin{cases} (\rho^2)_t - (c(u)\rho^2)_x = \dfrac{c'(u)}{2c(u)}(R-S)\rho^2, \\ (\sigma^2)_t + (c(u)\sigma^2)_x = -\dfrac{c'(u)}{2c(u)}(R-S)\sigma^2, \end{cases}$$

which implies

$$\begin{cases} (c(u)\rho^2)_t - (c^2(u)\rho^2)_x = \dfrac{c'(u)}{2}(R+S)\rho^2, \\ (c(u)\sigma^2)_t + (c^2(u)\sigma^2)_x = \dfrac{c'(u)}{2}(R+S)\sigma^2. \end{cases}$$

This leads to

$$\begin{cases} (R^2 + c(u)\rho^2)_t - \Big(c(u)(R^2 + c(u)\rho^2)\Big)_x \\ \quad = \dfrac{c'(u)}{2c(u)}(R^2S - RS^2) + \dfrac{c'(u)}{2}(\rho^2 S - \sigma^2 R), \\ (S^2 + c(u)\sigma^2)_t + \Big(c(u)(S^2 + c(u)\sigma^2)\Big)_x \\ \quad = -\dfrac{c'(u)}{2c(u)}(R^2S - RS^2) - \dfrac{c'(u)}{2}(\rho^2 S - \sigma^2 R). \end{cases} \tag{1.11}$$

From (1.11) we get

$$\begin{cases} \left(\dfrac{1}{c(u)}\big(R^2 + c(u)\rho^2\big)\right)_t - (R^2 + c(u)\rho^2)_x \\ \quad = -\dfrac{c'(u)}{2c^2(u)}(R^2S + RS^2) - \dfrac{c'(u)}{2c(u)}(\rho^2 S + \sigma^2 R), \\ \left(\dfrac{1}{c(u)}\big(S^2 + c(u)\sigma^2\big)\right)_t + (S^2 + c(u)\sigma^2)_x \\ \quad = -\dfrac{c'(u)}{2c^2(u)}(R^2S + RS^2) - \dfrac{c'(u)}{2c(u)}(\rho^2 S + \sigma^2 R). \end{cases} \tag{1.12}$$

Combining (1.11) and (1.12), we finally obtain

$$\begin{cases} \Big(R^2 + c(u)\rho^2 + S^2 + c(u)\sigma^2\Big)_t \\ \quad - \Big(c(u)\big(R^2 + c(u)\rho^2 - S^2 - c(u)\sigma^2\big)\Big)_x = 0, \\ \left(\dfrac{1}{c(u)}\big(R^2 + c(u)\rho^2 - S^2 - c(u)\sigma^2\big)\right)_t \\ \quad - \Big(R^2 + c(u)\rho^2 + S^2 + c(u)\sigma^2\Big)_x = 0. \end{cases} \tag{1.13}$$

Let

$$v = R^2 + c(u)\rho^2 + S^2 + c(u)\sigma^2 \quad \text{and} \quad w = \frac{1}{c(u)}\big(R^2 + c(u)\rho^2 - S^2 - c(u)\sigma^2\big),$$

then (1.13) rewrites as

$$\begin{pmatrix} v \\ w \end{pmatrix}_t - \begin{pmatrix} c^2(u)w \\ v \end{pmatrix}_x = 0, \tag{1.14}$$

which is a system of conservation laws, see [4, (2.4)-(2.6)] for the NVW equation.

Conservation of v and w here means that

$$\frac{d}{dt}\int_{x_1}^{x_2} v(t,x)\,dx = (c^2(u)w)(t,x_2) - (c^2(u)w)(t,x_1) \tag{1.15}$$

and

$$\frac{d}{dt}\int_{x_1}^{x_2} w(t,x)\,dx = v(t,x_2) - v(t,x_1).$$

Note that we have, by (1.9)

$$E(t) = \frac{1}{4}\int_{\mathbb{R}} v(t,x)\,dx.$$

So far we assumed that u, u_t, u_x, ρ and σ are smooth and bounded functions that vanish at $\pm\infty$. Under these assumptions we get by letting $x_1 \to -\infty$ and $x_2 \to +\infty$ in (1.15),

$$\frac{d}{dt}\int_{-\infty}^{\infty} v(t,x)\,dx = 0$$

and we recover the condition $E'(t) = 0$.

In the view of (1.11), we interpret $R^2 + c(u)\rho^2$ and $S^2 + c(u)\sigma^2$ as the left and right traveling part of the energy density, respectively. Moreover, the right-hand sides of the two equations in (1.11) are equal with opposite sign, which means that the right and the left part can interact with each other. That is, energy can swap back and forth between the two parts, while the total energy remains unchanged because of (1.13).

In contrast to the linear wave equation, solutions to (1.1), and hence also to (1.3), can develop singularities in finite time, even for smooth initial data, see e.g. [8]. Here, a singularity means that either u_x or u_t becomes unbounded pointwise while u remains continuous, and $u(t,\cdot)$,$u_x(t,\cdot)$, $u_t(t,\cdot) \in L^2(\mathbb{R})$ for all $t \geq 0$. This means that the energy densities $\frac{1}{4}(R^2 + c(u)\rho^2)$ and $\frac{1}{4}(S^2 + c(u)\sigma^2)$ may become unbounded pointwise. In other words, the energy density measures $\frac{1}{4}(R^2+c(u)\rho^2)\,dx$ and $\frac{1}{4}(S^2+c(u)\sigma^2)\,dx$ can have singular parts, meaning that energy concentrates on sets of measure zero. Thus if we want to obtain a semigroup of solutions of (1.3) we must be able to deal with both singular initial data and singularities turning up at later times. Assume that we have a singularity at time $t = t_0$. A central question is: if we want to solve the equation for $t \geq t_0$, how do we prescribe initial data at $t = t_0$? By computing $\frac{1}{4}(R^2 + c(u)\rho^2)(t_0,\cdot)$ and $\frac{1}{4}(S^2 + c(u)\sigma^2)(t_0,\cdot)$ in $L^2(\mathbb{R})$, we cannot conclude whether or not energy has concentrated. On the other hand, the presence of singularities in the initial data greatly affects the analysis of the equation and this is information we need to have available at the initial time. The solution to this problem is to add to the initial data two positive Radon measures μ_0 and ν_0, such that the absolutely continuous parts equal the classical energy densities, i.e., $(\mu_0)_{\text{ac}} = \frac{1}{4}(R_0^2+c(u_0)\rho_0^2)\,dx$ and $(\nu_0)_{\text{ac}} = \frac{1}{4}(S_0^2+c(u_0)\sigma_0^2)\,dx$. The singular parts of the measures on the other hand contain information about the concentration of energy.

Next, we illustrate the formation of a singularity with the following example. We consider a function $f(t,x)$, where f^2 should be thought of as either $\frac{1}{4}(R^2+c(u)\rho^2)$ or $\frac{1}{4}(S^2+c(u)\sigma^2)$. The function $f(t,\cdot)$ belongs to $L^2(\mathbb{R})$ for all $t\geq 0$. At $t=0$, f is smooth and bounded for all x. At a later time $t=t_0>0$, f becomes unbounded at the origin and $f^2(t,x)\,dx$ converges weak-star in the sense of measures to the Dirac delta at zero as $t\to t_0$.

Let $t_0>0$ and consider the function

$$f(t,x)=\Big(\frac{2}{\pi}\Big)^{\frac{1}{4}}\frac{1}{\sqrt{|t-t_0|}}e^{-(\frac{x}{t-t_0})^2},$$

where $t\geq 0$. We have $f(t,\cdot)\in L^2(\mathbb{R})$ for all $t\geq 0$ since

$$\int_{\mathbb{R}} f^2(t,x)\,dx=1. \tag{1.16}$$

Note that $f(t,x)\to 0$ for $x\neq 0$ and $f(t,0)\to +\infty$ as $t\to t_0$. Moreover, direct calculations yield

$$\lim_{t\to t_0}\int_{\mathbb{R}}\phi(x)f(t,x)\,dx=0 \quad \text{and} \quad \lim_{t\to t_0}\int_{\mathbb{R}}\phi(x)f^2(t,x)\,dx=\phi(0)$$

for all $\phi\in C_c^\infty(\mathbb{R})$. In other words, $f(t,\cdot)\overset{*}{\rightharpoonup} 0$ and $f^2(t,x)\,dx\overset{*}{\rightharpoonup}\delta_0$, where δ_0 is the Dirac delta at zero. Also note from (1.16) that $f(t,\cdot)$ does not converge to zero in $L^2(\mathbb{R})$, and since $f(t,x)\to 0$ almost everywhere it means that $f(t,\cdot)$ does not converge in $L^2(\mathbb{R})$. In fact we have

$$\lim_{t\to t_0}\int_{\mathbb{R}} f^p(t,x)\,dx=\begin{cases}0, & 1\leq p<2,\\ 1, & p=2,\\ \infty, & 2<p<\infty,\end{cases}$$

and since $f\geq 0$ this implies that $f(t,\cdot)\to 0$ in $L^p(\mathbb{R})$ for $1\leq p<2$.

2. Equivalent System

In this section we introduce a change of coordinates based on the method of characteristics. As a motivation for the approach we use for (1.1) and (1.3) we start out with the linear wave equation.

2.1. **The Linear Wave Equation.**

Consider the linear wave equation

$$u_{tt}-c^2u_{xx}=0, \tag{2.1}$$

where c is constant. We factorize the wave operator and can write the equation either as

$$\Big[\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big]\Big[\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big]u=0 \tag{2.2}$$

or

$$\Big[\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big]\Big[\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big]u=0. \tag{2.3}$$

In both cases we find that the characteristics are given by $x\pm ct=$ constant, and that every solution is of the form

$$u(t,x)=F(x+ct)+G(x-ct), \tag{2.4}$$

for some functions F and G. In other words, the solution consists of a left and right traveling part.

Let $X(t,x) = x + ct$ and $Y(t,x) = x - ct$. Then the functions X and Y satisfy

$$X_t - cX_x = 0 \quad \text{and} \quad Y_t + cY_x = 0. \tag{2.5}$$

Note that the operators acting on X and Y are the two factors of the wave operator. We consider the mapping from (t,x)-plane to (X,Y)-plane defined by the above equations. To make sure that the transformation is non-degenerate we compute

$$\det\left(\begin{bmatrix} X_t & X_x \\ Y_t & Y_x \end{bmatrix}\right) = 2c,$$

which implies that we must have $c \neq 0$. We observe that the characteristics $x + ct =$ constant and $x - ct =$ constant are mapped to horizontal and vertical lines in the (X,Y)-plane, respectively. Let $U(X,Y) = u(t(X,Y), x(X,Y))$. We compute the derivatives of $u(t,x) = U(X(t,x), Y(t,x))$ and get

$$u_t = U_X X_t + U_Y Y_t, \tag{2.6a}$$

$$u_{tt} = U_{XX} X_t^2 + U_X X_{tt} + 2U_{XY} X_t Y_t + U_{YY} Y_t^2 + U_Y Y_{tt}, \tag{2.6b}$$

$$u_x = U_X X_x + U_Y Y_x, \tag{2.6c}$$

$$u_{xx} = U_{XX} X_x^2 + U_X X_{xx} + 2U_{XY} X_x Y_x + U_{YY} Y_x^2 + U_Y Y_{xx}. \tag{2.6d}$$

Inserting (2.6) in (2.1) yields

$$\begin{aligned} 0 &= u_{tt} - c^2 u_{xx} \\ &= U_{XX}(X_t^2 - c^2 X_x^2) + U_X(X_{tt} - c^2 X_{xx}) + 2U_{XY}(X_t Y_t - c^2 X_x Y_x) \\ &\quad + U_{YY}(Y_t^2 - c^2 Y_x^2) + U_Y(Y_{tt} - c^2 Y_{xx}) \\ &= -4c^2 X_x Y_x U_{XY}, \end{aligned}$$

where we used (2.5). The functions X_x and Y_x are nonzero and finite, as $X_x = 1$ and $Y_x = 1$. Furthermore, since we assume $c \neq 0$, we get

$$U_{XY} = 0. \tag{2.7}$$

In particular, we have $U(X,Y) = F(X) + G(Y)$ and once again we obtain (2.4).

2.2. **The Nonlinear Variational Wave Equation.** Now we turn to the nonlinear variational wave equation

$$u_{tt} - c(u)(c(u)u_x)_x = u_{tt} - c(u)c'(u)u_x^2 - c^2(u)u_{xx} = 0.$$

We first note that a factorization of the operator like we did in (2.2) and (2.3) is not possible because of the function $c(u)$. Instead we look for a factorization of the terms containing the highest order derivatives. We compute

$$\left[\frac{\partial}{\partial t} - c(u)\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} + c(u)\frac{\partial}{\partial x}\right]u = u_{tt} - c(u)(c(u)u_x)_x + c'(u)u_t u_x = +c'(u)u_t u_x \tag{2.8}$$

and

$$\left[\frac{\partial}{\partial t} + c(u)\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} - c(u)\frac{\partial}{\partial x}\right]u = u_{tt} - c(u)(c(u)u_x)_x - c'(u)u_t u_x = -c'(u)u_t u_x. \tag{2.9}$$

Both these factorizations take care of the higher order derivatives, and we end up with a lower order term on the right-hand side. Note the difference in sign of this term depending on which operator is used first, showing that the operators do not commute. Therefore it is natural to consider both equations corresponding to the two factorizations. Note that (2.8) and (2.9) are the equations for $R = u_t + c(u)u_x$ and $S = u_t - c(u)u_x$ and we see once more that it is convenient to work with these functions. Note that R and S are the directional derivatives of u in the directions $(1, c(u))$ and $(1, -c(u))$, respectively. From (2.8) and (2.9) we see that the directional derivative of R in the direction $(1, -c(u))$ and the directional derivative of S in the direction $(1, c(u))$ are equal with opposite sign.

In the following we assume that u is sufficiently smooth and bounded.

We consider the characteristics corresponding to the highest order derivatives, i.e., the characteristics corresponding to the two factors $\frac{\partial}{\partial t} - c(u)\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t} + c(u)\frac{\partial}{\partial x}$ of the nonlinear variational wave operator. More specifically, we consider the functions $X(t,x)$ and $Y(t,x)$ satisfying

$$X_t - c(u)X_x = 0 \quad \text{and} \quad Y_t + c(u)Y_x = 0. \tag{2.10}$$

First, we want to solve the equation for $X(t,x)$ with the method of characteristics. Let t and x be functions of parameters s and ξ. We compute

$$\frac{d}{ds}X(t(s,\xi), x(s,\xi)) = X_t t_s + X_x x_s \tag{2.11}$$

which is equal to zero if

$$t_s(s,\xi) = 1 \quad \text{and} \quad x_s(s,\xi) = -c(u(t(s,\xi), x(s,\xi))). \tag{2.12}$$

We assume that $t(0,\xi) = 0$ and $x(0,\xi) = \xi$ for all $\xi \in \mathbb{R}$. Then we get

$$t(s,\xi) = s \quad \text{and} \quad x_s(s,\xi) = -c(u(s, x(s,\xi))). \tag{2.13}$$

We integrate the last equation in (2.13) and get

$$x(s,\xi) = \xi - \int_0^s c(u(r, x(r,\xi)))\, dr. \tag{2.14}$$

Recalling assumption (1.5), we get $-\kappa \leq x_s(s,\xi) \leq -\frac{1}{\kappa}$ and

$$\xi - \kappa s \leq x(s,\xi) \leq \xi - \frac{1}{\kappa}s$$

for all $s \geq 0$ and $\xi \in \mathbb{R}$.

For fixed ξ we consider the differential equation in (2.13). Assuming that $u_x(t,\cdot) \in L^\infty(\mathbb{R})$, the right-hand side is Lipschitz continuous with respect to the x-argument, i.e.,

$$|c(u(s,x_2)) - c(u(s,x_1))| \leq k_1 \sup_{t\geq 0} ||u_x(t,\cdot)||_{L^\infty(\mathbb{R})}|x_2 - x_1|$$

for all x_1, $x_2 \in \mathbb{R}$. Thus, there exists a unique local solution $x(\cdot,\xi)$ with initial data $x(0,\xi) = \xi$. This means that only one characteristic starts from the point given by $t = 0$ and $x = \xi$.

Now we can have three scenarios:

If $0 < x_\xi(s,\xi) < \infty$ for all $s < s_0$, then the solution $X(s, x(s,\xi))$ is well-defined and there is at least a chance of continuing $X(s, x(s,\xi))$ for $s \geq s_0$.

If $x_\xi(s_0,\xi) = 0$ at some point (s_0,ξ) with $s_0 > 0$, characteristics starting from different values of ξ may intersect at $s = s_0$ and it is not clear that $X(s, x(s,\xi))$ is well-defined for $s > s_0$.

If $x_\xi(s,\xi)\to\infty$ as s tends to some point $s_0>0$ and some ξ, then the solution $x(s,\xi)$ is not defined for $s\geq s_0$.

Differentiating the last equation in (2.13) with respect to ξ gives us

$$x_{s\xi}(s,\xi)=-c'(u(s,x(s,\xi)))u_x(s,x(s,\xi))x_\xi(s,\xi),$$

which we integrate to get

$$x_\xi(s,\xi)=\exp\Big\{-\int_0^s c'(u(r,x(r,\xi)))u_x(r,x(r,\xi))\,dr\Big\}. \tag{2.15}$$

Since u_x is bounded we have $0<x_\xi(s,\xi)<\infty$ for all $0\leq s<\infty$ and all ξ. This is because

$$\exp\Big\{-k_1 s\sup_{t\in[0,s]}||u_x(t,\cdot)||_{L^\infty(\mathbb{R})}\Big\}\leq x_\xi(s,\xi)\leq\exp\Big\{k_1 s\sup_{t\in[0,s]}||u_x(t,\cdot)||_{L^\infty(\mathbb{R})}\Big\}.$$

Thus, in the smooth case we do not end up with the two challenging scenarios described above.

We compute the determinant of the Jacobian corresponding to the map $(s,\xi)\to(t,x)$ and get

$$\det\left(\begin{bmatrix} t_s & t_\xi\\ x_s & x_\xi\end{bmatrix}\right)=t_s x_\xi-t_\xi x_s=x_\xi.$$

Since $0<x_\xi(s,\xi)<\infty$ we have from the inverse function theorem that the Jacobian corresponding to the map $(t,x)\to(s,\xi)$ satisfies

$$\begin{bmatrix} s_t & s_x\\ \xi_t & \xi_x\end{bmatrix}=\frac{1}{x_\xi}\begin{bmatrix} x_\xi & -t_\xi\\ -x_s & t_s\end{bmatrix}.$$

From (2.13) we get

$$s_t=1,\quad s_x=0,\quad \xi_t=-\frac{x_s}{x_\xi},\quad \xi_x=\frac{1}{x_\xi}, \tag{2.16}$$

so that

$$s(t,x)=t$$

and

$$\xi_t(t,x)=-x_s(t,\xi(t,x))\xi_x(t,x)=c(u(t,\xi(t,x)))\xi_x(t,x).$$

Furthermore, (2.10)–(2.12) imply that

$$X(t(s,\xi),x(s,\xi))=X(0,\xi)=g(\xi),$$

for some strictly increasing function $g\in C^1(\mathbb{R})$. Differentiation, combined with (2.12) and (2.16) yields

$$X_t=g'(\xi)\xi_t=-g'(\xi)\frac{x_s}{x_\xi}\quad\text{and}\quad X_x=g'(\xi)\xi_x=g'(\xi)\frac{1}{x_\xi}, \tag{2.17}$$

which implies $0<X_t<\infty$ and $0<X_x<\infty$.

Next, we study $Y(t,x)$ with the method of characteristics. We obtain

$$\frac{d}{ds}Y(t(s,\xi),x(s,\xi))=0$$

with the characteristics given by

$$t_s(s,\xi)=1\quad\text{and}\quad x_s(s,\xi)=c(u(t(s,\xi),x(s,\xi))). \tag{2.18}$$

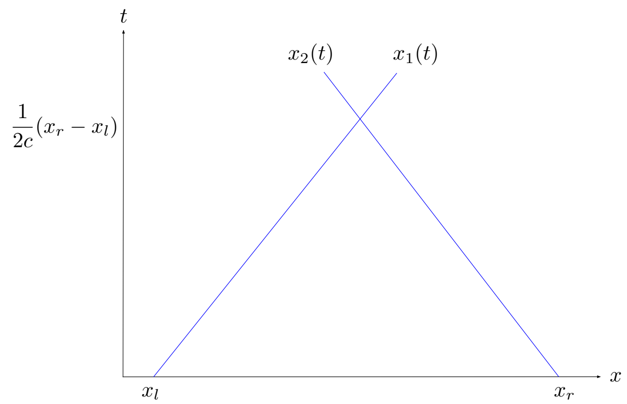

FIGURE 1. Characteristics of the linear wave equation, i.e., c is constant. The forward characteristic $x_1(t) = x_l + ct$ starting from x_l, and the backward characteristic $x_2(t) = x_r - ct$ starting from x_r, intersect at $t = \frac{1}{2c}(x_r - x_l)$.

Assuming that $t(0,\xi) = 0$ and $x(0,\xi) = \xi$ for all $\xi \in \mathbb{R}$ we get

$$t(s,\xi) = s \quad \text{and} \quad x_s(s,\xi) = c(u(s,x(s,\xi))). \tag{2.19}$$

If $Y(0,\xi) = h(\xi)$ for some strictly increasing function $h \in C^1(\mathbb{R})$, then

$$Y(s,x(s,\xi)) = h(\xi).$$

As in the computations above we find

$$x_\xi(s,\xi) = \exp\left\{ \int_0^s c'(u(r,x(r,\xi)))u_x(r,x(r,\xi))\,dr \right\}, \tag{2.20}$$

and since u_x is bounded, $0 < x_\xi(s,\xi) < \infty$ for all $0 \le s < \infty$ and all ξ. We also find that (2.16) holds with x_ξ as defined in (2.20), and

$$Y_t = h'(\xi)\xi_t = -h'(\xi)\frac{x_s}{x_\xi} \quad \text{and} \quad Y_x = h'(\xi)\xi_x = h'(\xi)\frac{1}{x_\xi}, \tag{2.21}$$

so that $-\infty < Y_t < 0$ and $0 < Y_x < \infty$.

Figure 1 and 2 show the characteristics for the linear wave equation and the NVW equation, respectively.

Now we consider the mapping from the (t,x)-plane to the (X,Y)-plane. The determinant of the Jacobian of this map reads

$$d = \det\left(\begin{bmatrix} X_t & X_x \\ Y_t & Y_x \end{bmatrix} \right) = X_tY_x - X_xY_t = 2c(u)X_xY_x = -\frac{2X_tY_t}{c(u)} \tag{2.22}$$

and once again we see that we must assume that $c(u)$ is strictly positive and finite. Since $0 < X_x < \infty$ and $0 < Y_x < \infty$, we have $0 < d < \infty$. The inverse function theorem then implies that the Jacobian corresponding to the map $(X,Y) \to (t,x)$ satisfies

$$\begin{bmatrix} t_X & t_Y \\ x_X & x_Y \end{bmatrix} = \frac{1}{d}\begin{bmatrix} Y_x & -X_x \\ -Y_t & X_t \end{bmatrix}.$$

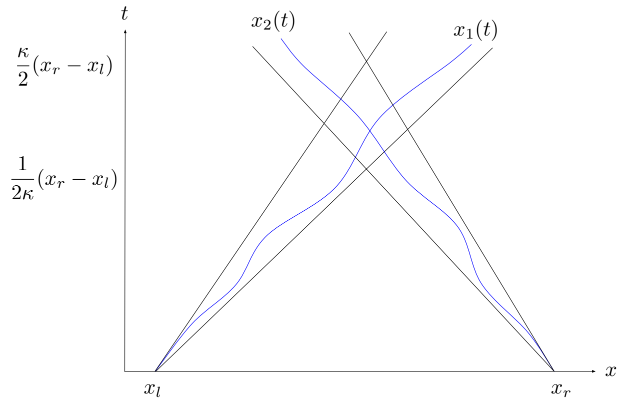

FIGURE 2. Characteristics of the NVW equation. The forward characteristic $x_1(t)$ starting from x_l is given by $x_{1,t}(t) = c(u(t, x_1(t)))$, $x_1(0) = x_l$, and the backward characteristic $x_2(t)$ starting from x_r is given by $x_{2,t}(t) = -c(u(t, x_2(t)))$, $x_2(0) = x_r$. Because of (1.5), they intersect at a time t such that $\frac{1}{2\kappa}(x_r - x_l) \leq t \leq \frac{\kappa}{2}(x_r - x_l)$.

From the above equality many identities can be read off, and we only mention some of them. By using (2.10) and (2.22), we obtain

$$2c(u)t_X X_x = 1, \quad -2c(u)t_Y Y_x = 1, \quad 2x_X X_x = 1, \quad 2x_Y Y_x = 1, \tag{2.23}$$

which imply

$$x_X = c(u)t_X \quad \text{and} \quad x_Y = -c(u)t_Y. \tag{2.24}$$

We observe from (2.23) that t_X, t_Y, x_X and x_Y are nonzero and finite.

Let $U(X,Y) = u(t(X,Y), x(X,Y))$. We insert the derivatives of $u(t,x) = U(X(t,x), Y(t,x))$ from (2.6) into (1.1) and get

$$\begin{aligned} 0 &= u_{tt} - c(u)(c(u)u_x)_x \\ &= U_{XX}(X_t^2 - c^2(u)X_x^2) + U_X(X_{tt} - c^2(u)X_{xx}) \\ &\quad + 2U_{XY}(X_tY_t - c^2(u)X_xY_x) \\ &\quad + U_{YY}(Y_t^2 - c^2(u)Y_x^2) + U_Y(Y_{tt} - c^2(u)Y_{xx}) \\ &\quad - c(u)c'(u)(U_X^2X_x^2 + 2U_XU_YX_xY_x + U_Y^2Y_x^2). \end{aligned} \tag{2.25}$$

Due to (2.10) all second order derivatives of U drop out except for the term containing the mixed derivative U_{XY}. We compute the remaining terms. From (2.6) and (2.10) we have

$$R = u_t + c(u)u_x = U_X(X_t + c(u)X_x) + U_Y(Y_t + c(u)Y_x) = 2c(u)U_XX_x$$

and

$$S = u_t - c(u)u_x = U_X(X_t - c(u)X_x) + U_Y(Y_t - c(u)Y_x) = -2c(u)U_YY_x,$$

and after using (2.23) we get

$$R = c(u)\frac{U_X}{x_X} \tag{2.26}$$

and

$$S = -c(u)\frac{U_Y}{x_Y}. \tag{2.27}$$

By differentiating (2.10) and using (2.23) we obtain

$$X_{tt} - c^2(u)X_{xx} = c'(u)X_xR = \frac{c'(u)R}{2x_X} \quad \text{and} \quad Y_{tt} - c^2(u)Y_{xx} = -c'(u)Y_xS = -\frac{c'(u)S}{2x_Y}.$$

From (2.10) and (2.23) we have

$$X_tY_t - c^2(u)X_xY_x = -2c^2(u)X_xY_x = -\frac{c^2(u)}{2x_Xx_Y}.$$

Thus (2.25) is equivalent to

$$U_{XY} = \frac{c'(u)}{4c^3(u)}(R^2x_Xx_Y + S^2x_Yx_X) - \frac{c'(u)}{2c(u)}U_XU_Y. \tag{2.28}$$

Let

$$J_X = \frac{1}{2}R^2x_X \quad \text{and} \quad J_Y = \frac{1}{2}S^2x_Y, \tag{2.29}$$

which we think of as the left and right traveling part of the energy density in the new variables, respectively. Now (2.28) yields

$$U_{XY} = \frac{c'(u)}{2c^3(U)}(J_Xx_Y + J_Yx_X) - \frac{c'(u)}{2c(u)}U_XU_Y.$$

Using (2.26), (2.27) and (2.29) we get

$$2x_XJ_X = c^2(U)U_X^2 \quad \text{and} \quad 2x_YJ_Y = c^2(U)U_Y^2. \tag{2.30}$$

We find it convenient to introduce the function K defined by

$$K_X = \frac{1}{2c(u)}R^2x_X \quad \text{and} \quad K_Y = -\frac{1}{2c(u)}S^2x_Y, \tag{2.31}$$

which satisfies

$$J_X = c(U)K_X \quad \text{and} \quad J_Y = -c(U)K_Y. \tag{2.32}$$

In view of (1.12) and (1.13) (with $\rho = \sigma = 0$) we can think of K_X and K_Y as the left and right traveling part of the second conserved quantity $\frac{1}{c(u)}(R^2 - S^2)$ in the new coordinates, respectively.

Next, let us derive the equations for t_{XY}, x_{XY}, J_{XY} and K_{XY}. We have $x_{XY} = x_{YX}$, which by using (2.24) is the same as $(c(U)t_X)_Y = (-c(U)t_Y)_X$. This leads to

$$t_{XY} = -\frac{c'(U)}{2c(U)}(U_Yt_X + U_Xt_Y).$$

We find the equation for x_{XY} by using (2.24) in $t_{XY} = t_{YX}$, which yields $(\frac{x_X}{c(U)})_Y = (-\frac{x_Y}{c(U)})_X$ and finally

$$x_{XY} = \frac{c'(U)}{2c(U)}(U_Yx_X + U_Xx_Y).$$

Using $J_{XY}=J_{YX}$, $K_{XY}=K_{YX}$ and (2.32) we get

$$J_{XY}=\frac{c'(U)}{2c(U)}(U_YJ_X+U_XJ_Y)$$

and

$$K_{XY}=-\frac{c'(U)}{2c(U)}(U_YK_X+U_XK_Y).$$

Finally we end up with the following system of differential equations

$$t_{XY}=-\frac{c'(U)}{2c(U)}(U_Yt_X+U_Xt_Y), \tag{2.33a}$$

$$x_{XY}=\frac{c'(U)}{2c(U)}(U_Yx_X+U_Xx_Y), \tag{2.33b}$$

$$U_{XY}=\frac{c'(u)}{2c^3(U)}(J_Xx_Y+J_Yx_X)-\frac{c'(u)}{2c(u)}U_XU_Y, \tag{2.33c}$$

$$J_{XY}=\frac{c'(U)}{2c(U)}(U_YJ_X+U_XJ_Y), \tag{2.33d}$$

$$K_{XY}=-\frac{c'(U)}{2c(U)}(U_YK_X+U_XK_Y). \tag{2.33e}$$

2.3. **The Regularized System.** We derive a set of equations corresponding to (1.3) in the new variables. We consider characteristics $X(t,x)$ and $Y(t,x)$ given by (2.10). We assume that u, R, S, ρ, and σ are smooth and bounded. As above we get that t_X, t_Y, x_X, and x_Y are nonzero and finite.

Denote $u(t,x)=U(X(t,x),Y(t,x))$. By calculations like those that led to (2.28) we find

$$U_{XY}=\frac{c'(u)}{4c^3(u)}\Big((R^2+c(u)\rho^2)x_Xx_Y+(S^2+c(u)\sigma^2)x_Yx_X\Big)-\frac{c'(u)}{2c(u)}U_XU_Y.$$

We introduce

$$J_X=\frac{1}{2}(R^2+c(u)\rho^2)x_X \quad\text{and}\quad J_Y=\frac{1}{2}(S^2+c(u)\sigma^2)x_Y. \tag{2.34}$$

Using (2.26) and (2.27) we obtain

$$U_{XY}=\frac{c'(u)}{2c^3(U)}(J_Xx_Y+J_Yx_X)-\frac{c'(u)}{2c(u)}U_XU_Y.$$

Accordingly, we define

$$K_X=\frac{1}{2c(u)}(R^2+c(u)\rho^2)x_X \quad\text{and}\quad K_Y=-\frac{1}{2c(u)}(S^2+c(u)\sigma^2)x_Y. \tag{2.35}$$

The derivation of the system of equations is similar to the one in Section 2.2 for the NVW equation (1.1). In fact, we obtain the same equations as in (2.33). In addition we get two equations corresponding to (1.3b) and (1.3c). Let $\rho(t,x)=P(X(t,x),Y(t,x))$. By (1.3b) we get

$$\begin{aligned}0&=\rho_t-(c(u)\rho)_x\\&=P_X(X_t-c(u)X_x)+P_Y(Y_t-c(u)Y_x)-c'(u)P(U_XX_x+U_YY_x).\end{aligned}$$

From (2.10) and (2.23) we have

$$c(U)P_Y x_X + \frac{c'(U)}{2}P(U_X x_Y + U_Y x_X) = 0$$

and from (2.33) we see that this is the same as

$$P_Y x_X + P x_{XY} = (P x_X)_Y = 0.$$

We define $p = P x_X$, so that

$$p_Y = 0.$$

Let $\sigma(t,x) = Q(X(t,x), Y(t,x))$. From (1.3c) we have

$$\begin{aligned} 0 &= \sigma_t + (c(u)\sigma)_x \\ &= Q_X(X_t + c(u)X_x) + Q_Y(Y_t + c(u)Y_x) + c'(u)Q(U_X X_x + U_Y Y_x). \end{aligned}$$

Using (2.10) and (2.23) we get

$$c(U)Q_X x_Y + \frac{c'(U)}{2}Q(U_X x_Y + U_Y x_X) = 0$$

and by (2.33) we find

$$Q_X x_Y + Q x_{XY} = (Q x_Y)_X = 0.$$

We define $q = Q x_Y$, so that

$$q_X = 0.$$

By (2.26), (2.27) and (2.34) we get

$$2x_X J_X = c^2(U)U_X^2 + c(U)p^2 \quad \text{and} \quad 2x_Y J_Y = c^2(U)U_Y^2 + c(U)q^2. \tag{2.36}$$

Furthermore, we note that the relations

$$x_X = c(U)t_X, \qquad x_Y = -c(U)t_Y, \tag{2.37a}$$

$$J_X = c(U)K_X, \qquad J_Y = -c(U)K_Y \tag{2.37b}$$

hold. To summarize, we obtain the following system of equations

$$t_{XY} = -\frac{c'(U)}{2c(U)}(U_Y t_X + U_X t_Y), \tag{2.38a}$$

$$x_{XY} = \frac{c'(U)}{2c(U)}(U_Y x_X + U_X x_Y), \tag{2.38b}$$

$$U_{XY} = \frac{c'(U)}{2c^3(U)}(x_Y J_X + x_X J_Y) - \frac{c'(U)}{2c(U)}U_X U_Y, \tag{2.38c}$$

$$J_{XY} = \frac{c'(U)}{2c(U)}(U_Y J_X + U_X J_Y), \tag{2.38d}$$

$$K_{XY} = -\frac{c'(U)}{2c(U)}(U_Y K_X + U_X K_Y), \tag{2.38e}$$

$$p_Y = 0, \tag{2.38f}$$

$$q_X = 0. \tag{2.38g}$$

We introduce the vector $Z = (t, x, U, J, K)$. The system (2.38a)-(2.38e) then rewrites as

$$Z_{XY} = F(Z)(Z_X, Z_Y), \tag{2.39}$$

where $F(Z)$ is a bilinear and symmetric tensor from $\mathbb{R}^5 \times \mathbb{R}^5$ to $\mathbb{R}^5$. Due to the relations (2.37a), either one of the equations in (2.38a) and (2.38b) is redundant: one could remove one of them, and the system would remain well-posed, and one retrieves t or x by using (2.37a). Similarly, either one of the equations (2.38d) and (2.38e) becomes redundant by (2.37b). However, we find it convenient to work with the complete set of variables, i.e., $Z = (t, x, U, J, K)$.

We observe that the equations (2.38a)-(2.38e) are not coupled with the last two (2.38f)-(2.38g). Furthermore they are identical to the set of equations (2.33) found for the NVW equation. However, we see that (2.30) is different from (2.36), so the solutions will not be identical. Also, from (2.36) we see that the solutions t, x, U, J, K of the five first equations are not independent of the solutions p, q of the last two equations.

To prove the existence of solutions of (2.38) we use a fixed point argument which is similar to the one found in [10]. In order to do so we need a curve $(\mathcal{X}(s), \mathcal{Y}(s))$ parametrized by $s \in \mathbb{R}$ in the (X, Y)-plane that corresponds to the initial time, i.e., it consists of all points $(X, Y) \in \mathbb{R}^2$ such that $t(X, Y) = 0$. We will admit curves of the following type.

Definition 2.1. *We denote by $\mathcal{C}$ the set of curves in the plane $\mathbb{R}^2$ parametrized by $(\mathcal{X}(s), \mathcal{Y}(s))$ with $s \in \mathbb{R}$, such that*

$$\mathcal{X} - \mathrm{Id},\ \mathcal{Y} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R}), \tag{2.40a}$$

$$\dot{\mathcal{X}} \geq 0,\ \dot{\mathcal{Y}} \geq 0 \tag{2.40b}$$

with the normalization

$$\frac{1}{2}(\mathcal{X}(s) + \mathcal{Y}(s)) = s \quad \text{for all } s \in \mathbb{R}. \tag{2.40c}$$

We set

$$\|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}} = \|\mathcal{X} - \mathrm{Id}\|_{L^\infty(\mathbb{R})} + \|\mathcal{Y} - \mathrm{Id}\|_{L^\infty(\mathbb{R})}\,. \tag{2.40d}$$

In the above derivation where we assumed that the solutions are smooth and bounded we found that $0 < t_X < \infty$ and $-\infty < t_Y < 0$. This implies that both $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are strictly increasing functions. Indeed, by differentiating $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$ and using (2.40c) we get

$$\dot{\mathcal{X}} = -\frac{2t_Y}{t_X - t_Y} \quad \text{and} \quad \dot{\mathcal{Y}} = \frac{2t_X}{t_X - t_Y},$$

which implies $\dot{\mathcal{X}} > 0$ and $\dot{\mathcal{Y}} > 0$. Thus, in this case $(\mathcal{X}(s), \mathcal{Y}(s))$ is a strictly monotone curve.

For general initial data the set

$$\Gamma_0 = \{(X, Y) \in \mathbb{R}^2 \mid t(X, Y) = 0\}$$

will be the union of strictly monotone curves, horizontal and vertical lines, and boxes. We define this set implicitly in Definition 3.4 and 3.7, and the examples following the definitions show how the set Γ_0 depends on the initial data $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$.

The idea is the following. The backward characteristics transports the energy described by the measure μ_0, while the forward characteristics transports the energy described by the measure ν_0.

At points $(0, x_0)$ where the initial data is smooth and bounded and the measures are absolutely continuous, there is a finite amount of energy, and there is exactly one forward and one backward characteristic starting from $(0, x_0)$. This point is mapped to one point in Lagrangian coordinates. An interval of such points yields a strictly monotone curve in Lagrangian coordinates, like we showed above.

At points $(0, x_0)$ where only one of the measures is singular, say μ_0, there is a finite amount of forward energy and an infinite amount of backward energy. Thus there are infinitely many backward characteristics, but only one forward characteristic starting from $(0, x_0)$. If we think of characteristics as particles, then the infinite amount of backward energy at $(0, x_0)$ is distributed over infinitely many particles. To label these particles, we map this point to a horizontal line in the (X, Y)-plane.

If only ν_0 is singular at $(0, x_0)$, the point is mapped to a vertical line in the (X, Y)-plane.

At points $(0, x_0)$ where both measures are singular, there is an infinite amount of both forward and backward energy. Thus there are infinitely many forward and backward characteristics starting from $(0, x_0)$. If we think of characteristics as particles, then the infinite amount of both forward and backward energy at $(0, x_0)$ is distributed over infinitely many particles and we need a rectangular box in the (X, Y)-plane to label all these particles.

From the set Γ_0 we have to choose a unique curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. In the case of a box there are in principle infinitely many possible ways of doing this. We define the curve in Definition 3.7, where we in the case of a box roughly speaking define it to be the union of the left vertical side and the upper horizontal side of the box.

Having defined the curve $(\mathcal{X}, \mathcal{Y})$, we have to assign the values of Z, Z_X, Z_Y, p and q on it in order to solve (2.38). In addition we require that several properties derived in this section for the smooth case hold on the curve, see Definition 4.7. Later we will prove that solutions of (2.38) satisfy the same properties. In Section 3 we explain how to define the functions on the curve for general initial data $u_0, R_0, S_0, \rho_0, \sigma_0 \in L^2(\mathbb{R})$ and measures μ_0 and ν_0. Here we present how to proceed for initial data such that the functions $u_0, R_0, S_0, \rho_0, \sigma_0$ are smooth and bounded, and such that the measures μ_0 and ν_0 are absolutely continuous. We have to specify the values of 19 functions. Let us see how many equations we have available to determine these values. Let

$$t(\mathcal{X}(s), \mathcal{Y}(s)) = 0, \tag{2.41}$$

and

$$U(\mathcal{X}(s), \mathcal{Y}(s)) = u_0(x(\mathcal{X}(s), \mathcal{Y}(s))). \tag{2.42}$$

From (2.26), (2.27) and (2.34) we have

$$U_X(\mathcal{X}, \mathcal{Y}) = x_X(\mathcal{X}, \mathcal{Y}) \frac{R_0(x(\mathcal{X}, \mathcal{Y}))}{c(u_0(x(\mathcal{X}, \mathcal{Y})))}, \tag{2.43}$$

$$U_Y(\mathcal{X}, \mathcal{Y}) = -x_Y(\mathcal{X}, \mathcal{Y}) \frac{S_0(x(\mathcal{X}, \mathcal{Y}))}{c(u_0(x(\mathcal{X}, \mathcal{Y})))}, \tag{2.44}$$

$$J_X(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} x_X(\mathcal{X}, \mathcal{Y})(R_0^2 + c(u_0)\rho_0^2)(x(\mathcal{X}, \mathcal{Y})), \tag{2.45}$$

$$J_Y(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} x_Y(\mathcal{X}, \mathcal{Y})(S_0^2 + c(u_0)\sigma_0^2)(x(\mathcal{X}, \mathcal{Y})), \tag{2.46}$$

$$p(\mathcal{X},\mathcal{Y}) = x_X(\mathcal{X},\mathcal{Y})\rho_0(x(\mathcal{X},\mathcal{Y})), \tag{2.47}$$

$$q(\mathcal{X},\mathcal{Y}) = x_Y(\mathcal{X},\mathcal{Y})\sigma_0(x(\mathcal{X},\mathcal{Y})) \tag{2.48}$$

and from (2.37a) and (2.37b) we have the relations

$$x_X(\mathcal{X},\mathcal{Y}) = c(u_0(x(\mathcal{X},\mathcal{Y})))t_X(\mathcal{X},\mathcal{Y}), \tag{2.49}$$

$$x_Y(\mathcal{X},\mathcal{Y}) = -c(u_0(x(\mathcal{X},\mathcal{Y})))t_Y(\mathcal{X},\mathcal{Y}), \tag{2.50}$$

$$J_X(\mathcal{X},\mathcal{Y}) = c(u_0(x(\mathcal{X},\mathcal{Y})))K_X(\mathcal{X},\mathcal{Y}), \tag{2.51}$$

$$J_Y(\mathcal{X},\mathcal{Y}) = -c(u_0(x(\mathcal{X},\mathcal{Y})))K_Y(\mathcal{X},\mathcal{Y}). \tag{2.52}$$

We also want to use the fact that Z_X and Z_Y are derivatives of Z to assign values of t, x, U, J, K, and we can write this condition as

$$\dot{Z}(\mathcal{X}(s),\mathcal{Y}(s)) = Z_X(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{X}}(s) + Z_Y(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{Y}}(s), \tag{2.53}$$

where the notation means $\dot{Z}(\mathcal{X}(s),\mathcal{Y}(s)) = \frac{d}{ds}Z(\mathcal{X}(s),\mathcal{Y}(s))$. We have 19 unknowns $(\mathcal{X},\mathcal{Y},p,q,Z,Z_X,Z_Y)$ and 17 equations, given by (2.40c) and (2.41)-(2.53). We use the two remaining degrees of freedom to obtain Z_X, p, Z_Y and q bounded. We set

$$2x_X(\mathcal{X}(s),\mathcal{Y}(s)) + J_X(\mathcal{X}(s),\mathcal{Y}(s)) = 1, \tag{2.54}$$

$$2x_Y(\mathcal{X}(s),\mathcal{Y}(s)) + J_Y(\mathcal{X}(s),\mathcal{Y}(s)) = 1. \tag{2.55}$$

In view of (2.36), x_X and J_X have the same sign, so that (2.54) implies that they are non-negative and bounded. Similarly we find that $x_Y \geq 0$, $J_Y \geq 0$, and that they are bounded from above. From (2.49) and (2.50) it then follows that t_X, K_X, t_Y and K_Y are bounded and $t_X \geq 0$, $K_X \geq 0$, $t_Y \leq 0$ and $K_Y \leq 0$. The relation (2.36) also implies that U_X, p, U_Y and q are bounded.

Using (2.45) and (2.46) in (2.54) and (2.55) yields

$$x_X(\mathcal{X},\mathcal{Y}) = \left(\frac{2}{4+R_0^2+c(u_0)\rho_0^2}\right)(x(\mathcal{X},\mathcal{Y})), \tag{2.56}$$

$$x_Y(\mathcal{X},\mathcal{Y}) = \left(\frac{2}{4+S_0^2+c(u_0)\sigma_0^2}\right)(x(\mathcal{X},\mathcal{Y})), \tag{2.57}$$

which implies by (2.49) and (2.50) that

$$t_X(\mathcal{X},\mathcal{Y}) = \left(\frac{2}{c(u_0)(4+R_0^2+c(u_0)\rho_0^2)}\right)(x(\mathcal{X},\mathcal{Y})),$$

$$t_Y(\mathcal{X},\mathcal{Y}) = -\left(\frac{2}{c(u_0)(4+S_0^2+c(u_0)\sigma_0^2)}\right)(x(\mathcal{X},\mathcal{Y})).$$

Now (2.43)-(2.48) take the form

$$U_X(\mathcal{X},\mathcal{Y}) = \left(\frac{2R_0}{c(u_0)(4+R_0^2+c(u_0)\rho_0^2)}\right)(x(\mathcal{X},\mathcal{Y})),$$

$$U_Y(\mathcal{X},\mathcal{Y}) = -\left(\frac{2S_0}{c(u_0)(4+S_0^2+c(u_0)\sigma_0^2)}\right)(x(\mathcal{X},\mathcal{Y})),$$

$$J_X(\mathcal{X},\mathcal{Y}) = \left(\frac{R_0^2+c(u_0)\rho_0^2}{4+R_0^2+c(u_0)\rho_0^2}\right)(x(\mathcal{X},\mathcal{Y})),$$

$$J_Y(\mathcal{X},\mathcal{Y}) = \left(\frac{S_0^2+c(u_0)\sigma_0^2}{4+S_0^2+c(u_0)\sigma_0^2}\right)(x(\mathcal{X},\mathcal{Y})),$$

$$p(\mathcal{X},\mathcal{Y}) = \left(\frac{2\rho_0}{4+R_0^2+c(u_0)\rho_0^2}\right)(x(\mathcal{X},\mathcal{Y})),$$
$$q(\mathcal{X},\mathcal{Y}) = \left(\frac{2\sigma_0}{4+S_0^2+c(u_0)\sigma_0^2}\right)(x(\mathcal{X},\mathcal{Y}))$$

and from (2.51) and (2.52) we get

$$K_X(\mathcal{X},\mathcal{Y}) = \left(\frac{R_0^2+c(u_0)\rho_0^2}{c(u_0)(4+R_0^2+c(u_0)\rho_0^2)}\right)(x(\mathcal{X},\mathcal{Y})),$$
$$K_Y(\mathcal{X},\mathcal{Y}) = -\left(\frac{S_0^2+c(u_0)\sigma_0^2}{c(u_0)(4+S_0^2+c(u_0)\sigma_0^2)}\right)(x(\mathcal{X},\mathcal{Y})).$$

It remains to determine $(\mathcal{X},\mathcal{Y})$ and the value of x, J and K on the curve.

Differentiating (2.41) with respect to s yields

$$t_X(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{X}}(s) + t_Y(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$$

and after using (2.49) and (2.50) we get

$$x_X(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{X}}(s) = x_Y(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{Y}}(s).$$

Using this in (2.53) implies

$$\dot{x}(\mathcal{X}(s),\mathcal{Y}(s)) = 2x_X(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{X}}(s) = 2x_Y(\mathcal{X}(s),\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{2.58}$$

We use (2.58) then (2.56) and (2.57) in (2.40c), and get

$$\begin{aligned}
2 &= \dot{\mathcal{X}}(s) + \dot{\mathcal{Y}}(s)\\
&= \frac{1}{2}\left(\frac{1}{x_X(\mathcal{X}(s),\mathcal{Y}(s))} + \frac{1}{x_Y(\mathcal{X}(s),\mathcal{Y}(s))}\right)\dot{x}(\mathcal{X}(s),\mathcal{Y}(s))\\
&= \left(2 + \frac{1}{4}(R_0^2+c(u_0)\rho_0^2+S_0^2+c(u_0)\sigma_0^2)\right)(x(\mathcal{X}(s),\mathcal{Y}(s)))\dot{x}(\mathcal{X}(s),\mathcal{Y}(s)).
\end{aligned}$$

We define $x(\mathcal{X}(s),\mathcal{Y}(s))$ implicitly as

$$2x(\mathcal{X}(s),\mathcal{Y}(s)) + \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))}(R_0^2+c(u_0)\rho_0^2+S_0^2+c(u_0)\sigma_0^2)(z)\,dz = 2s. \tag{2.59}$$

Note that the left-hand side is a strictly increasing function with respect to x, so that (2.59) uniquely defines $x(\mathcal{X}(s),\mathcal{Y}(s))$.

From (2.58) and (2.56) it follows that

$$\dot{\mathcal{X}}(s) = \left(1+\frac{1}{4}(R_0^2+c(u_0)\rho_0^2)\right)(x(\mathcal{X}(s),\mathcal{Y}(s)))\dot{x}(\mathcal{X}(s),\mathcal{Y}(s)) \tag{2.60}$$

and we define

$$\mathcal{X}(s) = x(\mathcal{X}(s),\mathcal{Y}(s)) + \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))}(R_0^2+c(u_0)\rho_0^2)(z)\,dz.$$

Similarly, by (2.58) and (2.57), we get

$$\dot{\mathcal{Y}}(s) = \left(1+\frac{1}{4}(S_0^2+c(u_0)\sigma_0^2)\right)(x(\mathcal{X}(s),\mathcal{Y}(s)))\dot{x}(\mathcal{X}(s),\mathcal{Y}(s)) \tag{2.61}$$

and we set

$$\mathcal{Y}(s) = x(\mathcal{X}(s), \mathcal{Y}(s)) + \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} (S_0^2 + c(u_0)\sigma_0^2)(z)\,dz.$$

From (2.54), (2.55) and (2.40c) we have

$$\begin{aligned}\dot{J}(\mathcal{X}, \mathcal{Y}) &= J_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} + J_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}} \\ &= (1 - 2x_X(\mathcal{X}, \mathcal{Y}))\dot{\mathcal{X}} + (1 - 2x_Y(\mathcal{X}, \mathcal{Y}))\dot{\mathcal{Y}} \\ &= 2 - 2\dot{x}(\mathcal{X}, \mathcal{Y}),\end{aligned} \tag{2.62}$$

so that

$$2x(\mathcal{X}(s), \mathcal{Y}(s)) + J(\mathcal{X}(s), \mathcal{Y}(s)) = 2s,$$

which combined with (2.59) yields

$$J(\mathcal{X}(s), \mathcal{Y}(s)) = \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} (R_0^2 + c(u_0)\rho_0^2 + S_0^2 + c(u_0)\sigma_0^2)(z)\,dz.$$

From (2.51) and (2.52) we have

$$\dot{K}(\mathcal{X}, \mathcal{Y}) = K_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} + K_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}} = \frac{J_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} - J_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}}}{c(u_0(x(\mathcal{X}, \mathcal{Y})))}.$$

Multiplying (2.54) by $\dot{\mathcal{X}}$ and (2.55) by $\dot{\mathcal{Y}}$, yields

$$J_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} = \dot{\mathcal{X}} - 2x_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} \quad \text{and} \quad J_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}} = \dot{\mathcal{Y}} - 2x_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}}.$$

Using (2.58), we get

$$J_X(\mathcal{X}, \mathcal{Y})\dot{\mathcal{X}} - J_Y(\mathcal{X}, \mathcal{Y})\dot{\mathcal{Y}} = \dot{\mathcal{X}} - \dot{\mathcal{Y}},$$

which implies by (2.60) and (2.61) that

$$\dot{K}(\mathcal{X}, \mathcal{Y}) = \frac{(R_0^2 + c(u_0)\rho_0^2 - S_0^2 - c(u_0)\sigma_0^2)(x(\mathcal{X}, \mathcal{Y}))}{4c(u_0(x(\mathcal{X}, \mathcal{Y})))}\dot{x}(\mathcal{X}, \mathcal{Y}).$$

We define

$$K(\mathcal{X}(s), \mathcal{Y}(s)) = \int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} \frac{1}{4c(u_0)}(R_0^2 + c(u_0)\rho_0^2 - S_0^2 - c(u_0)\sigma_0^2)(z)\,dz.$$

In Section 6 we prove the existence of global, weak, conservative solutions of (1.3). Our approach follows closely [10]. The solutions we construct will be conservative in the sense that for all $t \geq 0$,

$$\mu(t)(\mathbb{R}) + \nu(t)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}),$$

where we denote the solution at time t by $(u, R, S, \rho, \sigma, \mu, \nu)(t)$, see Theorem 6.2. This is a consequence of the fact that the energy function J in Lagrangian coordinates satisfies that the limit

$$\lim_{s\to\pm\infty} J(\mathcal{X}(s), \mathcal{Y}(s))$$

is independent of the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. Thus, the same limiting values of J are obtained for curves corresponding to different times, see Lemma 4.14. We do not address uniqueness of conservative solutions.

The main results of this paper is contained in Section 7, where we first prove that under certain conditions we have local smooth solutions of (1.3). More specifically, on a finite interval $[x_l, x_r]$ we assume that the initial data satisfies the following: $u_0, R_0, S_0, \rho_0, \sigma_0$ are smooth and bounded, μ_0 and ν_0 are absolutely continuous, and the functions ρ_0 and σ_0 are strictly positive. Then we prove that for every time $t \in \big[0, \frac{1}{2\kappa}(x_r - x_l)\big]$, the functions $\rho(t,x)$ and $\sigma(t,x)$ are strictly positive for all $x \in [x_l + \kappa t, x_r - \kappa t]$. This has a regularizing effect on the solution in the sense that the solution at time t will then satisfy the same regularity conditions as the initial data does on the interval $[x_l + \kappa t, x_r - \kappa t]$, see Corollary 7.2. Roughly speaking, the variables p and q contain information about ρ_0 and σ_0, respectively. In particular, since $p_Y = 0$ and $q_X = 0$, the strict positivity of p and q is preserved in the characteristic directions. The identities in (2.36) then imply the strict positivity of x_X and x_Y.

In Theorem 7.3 we prove that we can locally approximate weak solutions of (1.1) by smooth solutions of (1.3) in L^∞, provided that certain regularity and convergence conditions hold, see Section 7.

3. From Eulerian to Lagrangian Coordinates

We first define the set $\mathcal{D}$ which consists of possible initial data corresponding to (1.3) in Eulerian coordinates.

Definition 3.1. *The set $\mathcal{D}$ consists of the elements $(u, R, S, \rho, \sigma, \mu, \nu)$ such that*

$$u, R, S, \rho, \sigma \in L^2(\mathbb{R}),$$

$$u_x = \frac{1}{2c(u)}(R - S), \tag{3.1}$$

and μ and ν are finite positive Radon measures with

$$\mu_{\mathrm{ac}} = \frac{1}{4}(R^2 + c(u)\rho^2)\,dx \quad \text{and} \quad \nu_{\mathrm{ac}} = \frac{1}{4}(S^2 + c(u)\sigma^2)\,dx. \tag{3.2}$$

Note that this definition allows for initial data with concentrated energy. The next step is to map elements from $\mathcal{D}$ to a set $\mathcal{F}$ which is defined as follows.

Definition 3.2. *The group G is given by all invertible functions f such that*

$$f - \mathrm{Id} \ \text{ and } f^{-1} - \mathrm{Id} \ \text{ both belong to } W^{1,\infty}(\mathbb{R}), \tag{3.3}$$

and

$$(f - \mathrm{Id})' \in L^2(\mathbb{R}). \tag{3.4}$$

Note that if f, $g \in G$, then also f^{-1}, g^{-1} and $f \circ g$ belong to G.

Definition 3.3. *The set $\mathcal{F}$ consists of all functions $\psi = (\psi_1, \psi_2)$ such that*

$$\psi_1(X) = (x_1(X), U_1(X), J_1(X), K_1(X), V_1(X), H_1(X))$$

and

$$\psi_2(Y) = (x_2(Y), U_2(Y), J_2(Y), K_2(Y), V_2(Y), H_2(Y))$$

satisfy the following regularity and decay conditions

$$x_1 - \mathrm{Id},\ x_2 - \mathrm{Id},\ J_1,\ J_2,\ K_1,\ K_2 \in W^{1,\infty}(\mathbb{R}), \tag{3.5a}$$

$$x_1' - 1,\ x_2' - 1,\ J_1',\ J_2',\ K_1',\ K_2',\ H_1,\ H_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.5b}$$

$$U_1,\ U_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.5c}$$

$$V_1,\ V_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.5d}$$

and the additional conditions

$$x_1', x_2', J_1', J_2' \geq 0, \tag{3.6}$$

$$J_1' = c(U_1)K_1', \quad J_2' = -c(U_2)K_2', \tag{3.7}$$

$$x_1'J_1' = (c(U_1)V_1)^2 + c(U_1)H_1^2, \quad x_2'J_2' = (c(U_2)V_2)^2 + c(U_2)H_2^2, \tag{3.8}$$

$$x_1 + J_1,\ x_2 + J_2 \in G, \tag{3.9}$$

$$\lim_{X\to-\infty} J_1(X) = \lim_{Y\to-\infty} J_2(Y) = 0. \tag{3.10}$$

Moreover, for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ *such that*

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \text{ for all } s \in \mathbb{R},$$

we have

$$U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)) \tag{3.11a}$$

for all $s \in \mathbb{R}$ *and*

$$\frac{d}{ds}U_1(\mathcal{X}(s)) = \frac{d}{ds}U_2(\mathcal{Y}(s)) = V_1(\mathcal{X}(s))\dot{\mathcal{X}}(s) + V_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \tag{3.11b}$$

for almost all $s \in \mathbb{R}$.

Condition (3.9) will be important in Section 5.4 where we prove that the solution operator from $\mathcal{D}$ to $\mathcal{D}$ is a semigroup. In the proof we use $x_i + J_i$ for $i = 1, 2$ as relabeling functions.

For any strictly monotone curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ we introduce

$$\mathcal{X}(Y) = \mathcal{X}(\mathcal{Y}^{-1}(Y)) \quad \text{and} \quad \mathcal{Y}(X) = \mathcal{Y}(\mathcal{X}^{-1}(X)).$$

In the context of the previous section where we derived the system (2.38) for smooth solutions (Z, p, q), the elements (ψ_1, ψ_2) should be thought of

$$\begin{aligned}
x_1(X) &= x(X, \mathcal{Y}(X)), & x_2(Y) &= x(\mathcal{X}(Y), Y),\\
U_1(X) &= U(X, \mathcal{Y}(X)), & U_2(Y) &= U(\mathcal{X}(Y), Y),\\
J_1(X) &= \int_{-\infty}^{X} J_X(Z, \mathcal{Y}(Z))\, dZ, & J_2(Y) &= \int_{-\infty}^{Y} J_Y(\mathcal{X}(Z), Z)\, dZ,\\
K_1(X) &= \int_{-\infty}^{X} K_X(Z, \mathcal{Y}(Z))\, dZ, & K_2(Y) &= \int_{-\infty}^{Y} K_Y(\mathcal{X}(Z), Z)\, dZ,\\
V_1(X) &= U_X(X, \mathcal{Y}(X)), & V_2(Y) &= U_Y(\mathcal{X}(Y), Y),\\
H_1(X) &= p(X, \mathcal{Y}(X)), & H_2(Y) &= q(\mathcal{X}(Y), Y)
\end{aligned}$$

and

$$\begin{aligned}
x_1'(X) &= 2x_X(X, \mathcal{Y}(X)), & x_2'(Y) &= 2x_Y(\mathcal{X}(Y), Y),\\
J_1'(X) &= J_X(X, \mathcal{Y}(X)), & J_2'(Y) &= J_Y(\mathcal{X}(Y), Y),\\
K_1'(X) &= K_X(X, \mathcal{Y}(X)), & K_2'(Y) &= K_Y(\mathcal{X}(Y), Y).
\end{aligned}$$

We define the map from $\mathcal{D}$ to $\mathcal{F}$.

Definition 3.4. *Given $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$, we define $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$ and $\psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$ as*

$$x_1(X) = \sup\{x \in \mathbb{R} \mid x' + \mu((-\infty, x')) < X \text{ for all } x' < x\}, \tag{3.12a}$$

$$x_2(Y) = \sup\{x \in \mathbb{R} \mid x' + \nu((-\infty, x')) < Y \text{ for all } x' < x\} \tag{3.12b}$$

and

$$J_1(X) = X - x_1(X), \qquad J_2(Y) = Y - x_2(Y), \tag{3.12c}$$

$$U_1(X) = u(x_1(X)), \qquad U_2(Y) = u(x_2(Y)), \tag{3.12d}$$

$$V_1(X) = x_1'(X)\frac{R(x_1(X))}{2c(U_1(X))}, \qquad V_2(Y) = -x_2'(Y)\frac{S(x_2(Y))}{2c(U_2(Y))}, \tag{3.12e}$$

$$K_1(X) = \int_{-\infty}^{X} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X}, \qquad K_2(Y) = -\int_{-\infty}^{Y} \frac{J_2'(\bar{Y})}{c(U_2(\bar{Y}))}\, d\bar{Y}, \tag{3.12f}$$

$$H_1(X) = \frac{1}{2}\rho(x_1(X))x_1'(X), \qquad H_2(Y) = \frac{1}{2}\sigma(x_2(Y))x_2'(Y). \tag{3.12g}$$

We let $\mathbf{L} : \mathcal{D} \to \mathcal{F}$ denote the mapping which to any $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$ associates the element $\psi = (\psi_1, \psi_2) \in \mathcal{F}$ as defined above.

As mentioned before solutions can develop singularities in finite time and energy can concentrate on sets of measure zero. If this is the case one has to put some extra effort into understanding (3.12e) and (3.12g) since they might be of the form $0 \cdot \infty$, when $x_1'(X) = 0$. One has, in the smooth case for $X_1 < X_2$ that

$$\int_{x_1(X_1)}^{x_1(X_2)} \frac{R}{2c(u)}(x)\, dx = \int_{X_1}^{X_2} \frac{R(x_1(\tilde{X}))}{2c(u(x_1(\tilde{X})))} x_1'(\tilde{X})\, d\tilde{X} = \int_{X_1}^{X_2} V_1(\tilde{X})\, d\tilde{X}$$

and

$$\begin{aligned} \left| \int_{x_1(X_1)}^{x_1(X_2)} \frac{R}{2c(u)}(x)\, dx \right| &\leq \kappa \sqrt{x_1(X_2) - x_1(X_1)} \sqrt{\mu_{\mathrm{ac}}((x_1(X_1), x_1(X_2))} \\ &\leq \kappa \sqrt{x_1(X_2) - x_1(X_1)} \sqrt{J_1(X_2) - J_1(X_1)}. \end{aligned} \tag{3.13}$$

If we now consider the nonsmooth case, (3.13) still holds and the above calculations imply that $V_1(X)$ exists and is bounded. Furthermore, if $x_1'(X) = 0$, we must have that $V_1(X) = 0$.

In the case of initial data $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$ such that μ_0 and ν_0 are absolutely continuous with respect to the Lebesgue measure, we check that we end up with the same expressions as at the end of Section 2. By (3.2) we have

$$\mu_0((-\infty, x)) = \frac{1}{4} \int_{-\infty}^{x} (R_0^2 + c(u_0)\rho_0^2)(z)\, dz$$

and

$$\nu_0((-\infty, x)) = \frac{1}{4} \int_{-\infty}^{x} (S_0^2 + c(u_0)\sigma_0^2)(z)\, dz.$$

Since the functions $x+\mu_0((-\infty,x))$ and $x+\nu_0((-\infty,x))$ are continuous and strictly increasing, we get from (3.12a) and (3.12b),

$$x_1(X)+\frac{1}{4}\int_{-\infty}^{x_1(X)}(R_0^2+c(u_0)\rho_0^2)(z)\,dz=X$$

and

$$x_2(Y)+\frac{1}{4}\int_{-\infty}^{x_2(Y)}(S_0^2+c(u_0)\sigma_0^2)(z)\,dz=Y.$$

We add these equalities and since $x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))=x(\mathcal{X}(s),\mathcal{Y}(s))$ and $\mathcal{X}(s)+\mathcal{Y}(s)=2s$ we get

$$2x(\mathcal{X}(s),\mathcal{Y}(s))+\frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))}(R_0^2+c(u_0)\rho_0^2+S_0^2+c(u_0)\sigma_0^2)(z)\,dz=2s$$

and we recover (2.59). In a similar way we can show by using Definition 3.4 that we get the other expressions that we derived in Section 2.

We illustrate the mappings in this section with a series of examples. We study three possible situations where the initial measures are absolutely continuous and discrete. We want to illustrate how the region where $x_1(X)=x_2(Y)$ and $Y=2s-X$ in the (X,Y)-plane looks like in each situation. This is important in (3.28), the definition of the initial curve $(\mathcal{X},\mathcal{Y})$.

Example 1. We first consider the case where both μ_0 and ν_0 are absolutely continuous. More specifically, let

$$\mu_0((-\infty,x])=\nu_0((-\infty,x])=\arctan(x)+\frac{\pi}{2}.$$

The measures are absolutely continuous. Let $f(x)=\arctan(x)+x+\frac{\pi}{2}$, which is strictly increasing and continuous. We have $x_1(X)=f^{-1}(X)$ and $x_2(Y)=f^{-1}(Y)$. By differentiating the identity $f(f^{-1}(X))=X$ we obtain

$$x_1'(X)=\frac{1}{1+\frac{1}{1+x_1(X)^2}}\geq\frac{1}{2},$$

and similarly we get $x_2'(Y)\geq\frac{1}{2}$, so that both x_1 and x_2 are strictly increasing functions.

Example 2. We consider the case where one of the measures is absolutely continuous and the other is not. Let

$$\mu_0((-\infty,x))=\begin{cases}0, & x\leq 0,\\ 1, & x>0\end{cases}\quad\text{and}\quad\nu_0((-\infty,x))=\arctan(x)+\frac{\pi}{2}.$$

The measure μ_0 is not absolutely continuous with respect to the Lebesgue measure, as

$$\mu_0(\{0\})=\mu_0\Big(\bigcap_{n=1}^{\infty}\Big[0,\frac{1}{n}\Big)\Big)=\lim_{n\to\infty}\mu_0\Big(\Big[0,\frac{1}{n}\Big)\Big)=1.$$

Using Definition 3.4, we find that

$$x_1(X)=\begin{cases}X & \text{if } X\leq 0,\\ 0 & \text{if } 0\leq X\leq 1,\\ X-1, & \text{if } 1\leq X,\end{cases}$$

and as in Example 1 we have $x_2(Y) = f^{-1}(Y)$.

Example 3. We consider the case where both measures are singular at the same point. Let $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$ be such that

$$\mu_0((-\infty, x)) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0 \end{cases} \quad \text{and} \quad \nu_0((-\infty, x)) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

From Definition 3.4 we get

$$x_1(X) = \begin{cases} X & \text{if } X \leq 0, \\ 0 & \text{if } 0 \leq X \leq 1, \\ X - 1, & \text{if } 1 \leq X \end{cases} \quad \text{and} \quad x_2(Y) = \begin{cases} Y & \text{if } Y \leq 0, \\ 0 & \text{if } 0 \leq Y \leq 1, \\ Y - 1, & \text{if } 1 \leq Y. \end{cases}$$

Proof of the well-posedness of Definition 3.4. We only show that ψ_1 as defined above satisfies the conditions in the definition of $\mathcal{F}$. The corresponding proof for ψ_2 is similar. First we prove that the derivatives of x_1 and J_1 are well-defined. Let us show that x_1 is Lipschitz continuous. Consider $X, X' \in \mathbb{R}$ such that $X < X'$ and $x_1(X) < x_1(X')$. The definition of x_1 implies that there exists an increasing sequence, z_i', and a decreasing one, z_i, such that $\lim_{i\to\infty} z_i' = x_1(X')$ and $\lim_{i\to\infty} z_i = x_1(X)$ with $z_i' + \mu((-\infty, z_i')) < X'$ and $z_i + \mu((-\infty, z_i)) \geq X$. Combining these two inequalities gives

$$\mu((-\infty, z_i')) - \mu((-\infty, z_i)) + z_i' - z_i < X' - X.$$

For sufficiently large i, we have $z_i' > z_i$, so that $\mu((-\infty, z_i')) - \mu((-\infty, z_i)) = \mu([z_i, z_i')) \geq 0$. Hence, $z_i' - z_i < X' - X$. Letting i tend to infinity, we obtain $x_1(X') - x_1(X) \leq X' - X$ and x_1 is Lipschitz continuous with Lipschitz constant at most one. Thus, x_1 is differentiable almost everywhere. Then, by (3.12c) it follows that J_1 is Lipschitz continuous with Lipschitz constant at most two, so that J_1 is differentiable almost everywhere.

Next, we show (3.5a)-(3.5d) and that K_1 is well-defined and differentiable almost everywhere. It is clear from (3.12a) that x_1 yields a nondecreasing function. For any $z > x_1(X)$, we have $z + \mu((-\infty, z)) \geq X$. Hence, $X - z \leq \mu(\mathbb{R})$ and, since we can choose z arbitrarily close to $x_1(X)$, we obtain $X - x_1(X) \leq \mu(\mathbb{R})$. Since $x_1(X) \leq X$, we have

$$|X - x_1(X)| \leq \mu(\mathbb{R}) \tag{3.14}$$

and $x_1 - \mathrm{Id} \in L^\infty(\mathbb{R})$. Since x_1 is nondecreasing and has Lipschitz constant at most one, we have $0 \leq x_1' \leq 1$ almost everywhere, so that $x_1' - 1 \in L^\infty(\mathbb{R})$. From (3.14), we obtain $|J_1(X)| \leq \mu(\mathbb{R})$ and $J_1 \in L^\infty(\mathbb{R})$. We have $J_1' = 1 - x_1'$ a.e. and therefore $0 \leq J_1' \leq 1$ a.e., which implies that $J_1' \in L^\infty(\mathbb{R})$. Thus, $J_1' \in L^1(\mathbb{R})$ as $\int_{-\infty}^X J_1'(\bar{X})\, d\bar{X} \leq ||J_1||_{L^\infty(\mathbb{R})}$. By Hölder's inequality, we obtain

$$\|J_1'\|_{L^2(\mathbb{R})}^2 \leq \|J_1'\|_{L^\infty(\mathbb{R})} \|J_1'\|_{L^1(\mathbb{R})} \leq \|J_1\|_{L^\infty(\mathbb{R})} \leq \mu(\mathbb{R}).$$

Hence, $J_1' \in L^2(\mathbb{R})$ and since $J_1' = 1 - x_1'$ a.e., we have that $x_1' - 1 \in L^2(\mathbb{R})$. Note that, by the above, the inequalities for x_1' and J_1' in (3.6) are satisfied. The fact that J_1' is integrable also implies that K_1 is well-defined and differentiable almost everywhere. By differentiating (3.12f), we obtain $K_1' = \frac{J_1'}{c(U_1)}$, so that $K_1 \in W^{1,\infty}(\mathbb{R})$

and $K_1' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By a change of variables and, using the fact that $x_1' \leq 1$ a.e., we get

$$\int_{\mathbb{R}} H_1^2(X)\,dX \leq \frac{1}{4}\int_{\mathbb{R}} \rho^2(x)\,dx < \infty$$

and

$$\int_{\mathbb{R}} V_1^2(X)\,dX \leq \frac{\kappa^2}{4}\int_{\mathbb{R}} R^2(x)\,dx < \infty,$$

so that H_1 and V_1 belong to $L^2(\mathbb{R})$.

Next, we prove that U_1 is in $L^2(\mathbb{R})$. Let $B_3 = \{X \in \mathbb{R} \mid x_1'(X) < \frac{1}{2}\}$. Since $J_1' = 1 - x_1'$, $B_3 = \{X \in \mathbb{R} \mid J_1'(X) > \frac{1}{2}\}$, and $J_1' \in L^2(\mathbb{R})$, we obtain $\operatorname{meas}(B_3) < \infty$ after using Chebyshev's inequality. We have, since $x_1' \geq \frac{1}{2}$ in B_3^c,

$$\begin{aligned}\int_{\mathbb{R}} U_1^2(X)\,dX &= \int_{B_3} U_1^2(X)\,dX + \int_{B_3^c} U_1^2(X)\,dX\\ &\leq \operatorname{meas}(B_3)\,\|u\|^2_{L^\infty(\mathbb{R})} + 2\int_{B_3^c} u^2(x_1(X))x_1'(X)\,dX\\ &\leq \operatorname{meas}(B_3)\,\|u\|^2_{L^\infty(\mathbb{R})} + 2\,\|u\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Since $u \in H^1(\mathbb{R})$, we have that $u \in L^\infty(\mathbb{R})$ and we conclude that $U_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. It remains to show that H_1 and V_1 belong to $L^\infty(\mathbb{R})$. In order to prove this and (3.8), we have to compute the derivative of x_1. Following [7], we decompose μ into its absolutely continuous, singular continuous and discrete part, denoted by μ_{ac}, μ_{sc} and μ_{d}, respectively. The support of μ_{d} consists of a countable set of points. The function $G(x) = \mu((-\infty, x))$ is lower semi-continuous and its points of discontinuity coincide exactly with the support of μ_{d}. Let A denote the complement of $x_1^{-1}(\operatorname{supp}(\mu_{\mathrm{d}}))$, that is, $A = \{X \in \mathbb{R} \mid x_1(X) \in \operatorname{supp}(\mu_{\mathrm{d}})^c\}$. We claim that for any $X \in A$, we have

$$\mu((-\infty, x_1(X))) + x_1(X) = X. \tag{3.15}$$

By (3.12a) there exists an increasing sequence z_i which converges to $x_1(X)$ such that $G(z_i) + z_i < X$. Since G is lower semi-continuous, $\lim_{i\to\infty} G(z_i) = G(x_1(X))$ and therefore

$$G(x_1(X)) + x_1(X) \leq X.$$

Assume that $G(x_1(X)) + x_1(X) < X$. Since $x_1(X)$ is a point of continuity of G, we can find an x such that $x > x_1(X)$ and $G(x) + x < X$. This contradicts the definition of $x_1(X)$ and proves our claim (3.15). Let

$$B_1 = \Big\{x \in \mathbb{R} \mid \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\mu((x-\varepsilon, x+\varepsilon)) = \frac{1}{4}(R^2(x) + c(u(x))\rho^2(x))\Big\}.$$

Since $\frac{1}{4}(R^2 + c\rho^2)\,dx$ is the absolutely continuous part of μ, we have from Besicovitch's derivation theorem that $\operatorname{meas}(B_1^c) = 0$. The proof can be found in [1]. Given $X \in x_1^{-1}(B_1)$, we denote $x = x_1(X)$. We claim that for all $i \in \mathbb{N}$, there exists $0 < \varepsilon < \frac{1}{i}$ such that $x - \varepsilon$ and $x + \varepsilon$ both belong to $\operatorname{supp}(\mu_{\mathrm{d}})^c$. Let us assume the opposite. Then, there exists $i \in \mathbb{N}$ such that for all $0 < \varepsilon < \frac{1}{i}$, $x - \varepsilon$ and $x + \varepsilon$ both belong to $\operatorname{supp}(\mu_{\mathrm{d}})$. Since the set $(0, \frac{1}{i})$ is uncountable, this implies that uncountably many points belong to $\operatorname{supp}(\mu_{\mathrm{d}})$. This is a contradiction, and our claim is proved. Hence, we can find two sequences X_i and X_i' in A such that

$\frac{1}{2}(x_1(X_i)+x_1(X_i'))=x_1(X)$ and $0<X_i'-X_i<\frac{1}{i}$. We have by (3.15), since X_i and X_i' belong to A,

$$\mu([x_1(X_i),x_1(X_i')))+x_1(X_i')-x_1(X_i)=X_i'-X_i. \tag{3.16}$$

Since $x_1(X_i)\notin \mathrm{supp}(\mu_{\mathrm{d}})$, we infer that $\mu(\{x_1(X_i)\})=0$ and $\mu([x_1(X_i),x_1(X_i')))=\mu((x_1(X_i),x_1(X_i')))$. Dividing (3.16) by $X_i'-X_i$, we get

$$\frac{x_1(X_i')-x_1(X_i)}{X_i'-X_i}\frac{\mu((x_1(X_i),x_1(X_i')))}{x_1(X_i')-x_1(X_i)}+\frac{x_1(X_i')-x_1(X_i)}{X_i'-X_i}=1$$

and letting i tend to infinity, we obtain

$$x_1'(X)\frac{1}{4}(R^2+c(u)\rho^2)(x_1(X))+x_1'(X)=1 \tag{3.17}$$

for almost every $X\in x_1^{-1}(B_1)$. Since $\mathbb{R}=x_1^{-1}(B_1)\cup x_1^{-1}(B_1^c)$, it remains to study the behavior of x_1' in $x_1^{-1}(B_1^c)$. We proved above that $\mathrm{meas}(B_1^c)=0$, which does not imply in general that $\mathrm{meas}(x_1^{-1}(B_1^c))=0$.[1] Therefore, we need the following result.

Lemma 3.5 ([9, Lemma 3.9])**.** *Given an increasing Lipschitz continuous function $f:\mathbb{R}\to\mathbb{R}$, for any set B of measure zero, we have $f'=0$ almost everywhere in $f^{-1}(B)$.*

We apply Lemma 3.5 and get, since $\mathrm{meas}(B_1^c)=0$, that $x_1'=0$ almost everywhere in $x_1^{-1}(B_1^c)$. From (3.17), we get

$$\begin{aligned}x_1'(X)J_1'(X)&=x_1'(X)^2\frac{1}{4}(R^2(x_1(X))+c(U_1(X))\rho^2(x_1(X)))\\&=(c(U_1(X))V_1(X))^2+c(U_1(X))H_1^2(X)\end{aligned}$$

and (3.8) follows. The relation in (3.7) follows by differentiating (3.12f). Now we can prove that H_1 and V_1 belong to $L^\infty(\mathbb{R})$. By (3.8), we have

$$\begin{aligned}0&\le\left(\frac{1}{\kappa}|V_1|+\frac{1}{\sqrt{\kappa}}|H_1|\right)^2\\&\le\big(c(U_1)|V_1|+\sqrt{c(U_1)}|H_1|\big)^2\\&\le 2\big(c^2(U_1)V_1^2+c(U_1)H_1^2\big)\\&=2x_1'J_1'\le 2\end{aligned}$$

since $x_1',J_1'\in[0,1]$. This implies that $H_1,V_1\in L^\infty(\mathbb{R})$.

Since $x_1+J_1=\mathrm{Id}$, all conditions in Definition 3.2 are satisfied and hence also (3.9).

The function J_1 vanishes at $-\infty$ since J_1 is non-decreasing and non-negative and $x_1(X)\le X$. Hence, (3.10) is satisfied for J_1. Let us verify that (3.11a) and (3.11b) hold. Consider a curve $(\mathcal{X},\mathcal{Y})\in\mathcal{C}$ such that $x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))$. By (3.12d), we have

$$U_1(\mathcal{X}(s))=u(x_1(\mathcal{X}(s)))=u(x_2(\mathcal{Y}(s)))=U_2(\mathcal{Y}(s)).$$

We obtain

$$U_1(\mathcal{X}(\bar{s}))-U_1(\mathcal{X}(s))=\int_{x_1(\mathcal{X}(s))}^{x_1(\mathcal{X}(\bar{s}))}u_x(x)\,dx$$

[1]If $\mu=\delta_0$, then $B_1^c=\{0\}$, but $x_1^{-1}(\{0\})=[0,1]$.

$$
\begin{aligned}
&= \int_{x_1(\mathcal{X}(s))}^{x_1(\mathcal{X}(\bar{s}))} \frac{(R-S)}{2c(u)}(x)\,dx \\
&= \int_{x_1(\mathcal{X}(s))}^{x_1(\mathcal{X}(\bar{s}))} \frac{R}{2c(u)}(x)dx - \int_{x_2(\mathcal{Y}(s))}^{x_2(\mathcal{Y}(\bar{s}))} \frac{S}{2c(u)}(x)\,dx \\
&= \int_s^{\bar{s}} (V_1(\mathcal{X})\dot{\mathcal{X}} + V_2(\mathcal{Y})\dot{\mathcal{Y}})(r)\,dr
\end{aligned}
$$

where we used that $u_x = \frac{1}{2c(u)}(R-S)$ and $x_1(\mathcal{X}) = x_2(\mathcal{Y})$. Dividing both sides by $\bar{s} - s$ and letting $\bar{s} \to s$ yields (3.11b). □

Given an element in $\mathcal{F}$ we want to define a curve $(\mathcal{X}, \mathcal{Y})$ and the values of ψ on that curve. We define the set $\mathcal{G}$ which consists of curves $(\mathcal{X}, \mathcal{Y})$ and five functions $\mathcal{Z}$, $\mathcal{V}$, $\mathcal{W}$, $\mathfrak{p}$ and $\mathfrak{q}$, next. Recalling Section 2 the idea is that these functions in the smooth case are given through

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s))$$

and

$$
\begin{aligned}
\mathcal{V}(\mathcal{X}(s)) &= Z_X(\mathcal{X}(s), \mathcal{Y}(s)), & \mathcal{W}(\mathcal{Y}(s)) &= Z_Y(\mathcal{X}(s), \mathcal{Y}(s)), \\
\mathfrak{p}(\mathcal{X}(s)) &= p(\mathcal{X}(s), \mathcal{Y}(s)), & \mathfrak{q}(\mathcal{Y}(s)) &= q(\mathcal{X}(s), \mathcal{Y}(s)),
\end{aligned}
$$

and hence motivate some of the regularity conditions that are imposed in the definition of the set $\mathcal{G}$. For example, from the derivation in the previous section we know that the function $x(X, Y)$ is increasing with respect to both its arguments and is therefore unbounded. However, from (2.59) we get

$$|x(\mathcal{X}(s), \mathcal{Y}(s)) - s| \leq \frac{1}{2}(\mu_0(\mathbb{R}) + \nu_0(\mathbb{R}))$$

which belongs to $L^\infty(\mathbb{R})$. Therefore, we require that $\mathcal{Z}_2 - \mathrm{Id}$ belongs to $L^\infty(\mathbb{R})$.

It is convenient to introduce the following notation: to any triplet $(\mathcal{Z}, \mathcal{V}, \mathcal{W})$ of five dimensional vector functions we associate a triplet $(\mathcal{Z}^a, \mathcal{V}^a, \mathcal{W}^a)$ given by

$$\mathcal{Z}_1^a = \mathcal{Z}_1 - \frac{1}{c(0)}(\mathcal{X} - \mathrm{Id}), \quad \mathcal{V}_1^a = \mathcal{V}_1 - \frac{1}{2c(0)}, \quad \mathcal{W}_1^a = \mathcal{W}_1 + \frac{1}{2c(0)}, \tag{3.18a}$$

$$\mathcal{Z}_2^a = \mathcal{Z}_2 - \mathrm{Id}, \quad \mathcal{V}_2^a = \mathcal{V}_2 - \frac{1}{2}, \quad \mathcal{W}_2^a = \mathcal{W}_2 - \frac{1}{2}, \tag{3.18b}$$

$$\mathcal{Z}_i^a = \mathcal{Z}_i, \quad \mathcal{V}_i^a = \mathcal{V}_i, \quad \mathcal{W}_i^a = \mathcal{W}_i \tag{3.18c}$$

for $i \in \{3, 4, 5\}$.

Definition 3.6. *The set $\mathcal{G}$ is the set of all elements $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ which consist of a curve $(\mathcal{X}(s), \mathcal{Y}(s)) \in \mathcal{C}$, three vector-valued functions*

$$
\begin{aligned}
\mathcal{Z}(s) &= (\mathcal{Z}_1(s), \mathcal{Z}_2(s), \mathcal{Z}_3(s), \mathcal{Z}_4(s), \mathcal{Z}_5(s)), \\
\mathcal{V}(X) &= (\mathcal{V}_1(X), \mathcal{V}_2(X), \mathcal{V}_3(X), \mathcal{V}_4(X), \mathcal{V}_5(X)), \\
\mathcal{W}(Y) &= (\mathcal{W}_1(Y), \mathcal{W}_2(Y), \mathcal{W}_3(Y), \mathcal{W}_4(Y), \mathcal{W}_5(Y)),
\end{aligned}
$$

and two functions $\mathfrak{p}(X)$ and $\mathfrak{q}(Y)$. We set

$$\|\Theta\|_{\mathcal{G}}^2 = \|\mathcal{Z}_3\|_{L^2(\mathbb{R})}^2 + \|\mathcal{V}^a\|_{L^2(\mathbb{R})}^2 + \|\mathcal{W}^a\|_{L^2(\mathbb{R})}^2 + \|\mathfrak{p}\|_{L^2(\mathbb{R})}^2 + \|\mathfrak{q}\|_{L^2(\mathbb{R})}^2 \tag{3.19}$$

and

$$|||\Theta|||_{\mathcal{G}} = \|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}} + \left\|\frac{1}{\mathcal{V}_2+\mathcal{V}_4}\right\|_{L^\infty(\mathbb{R})} + \left\|\frac{1}{\mathcal{W}_2+\mathcal{W}_4}\right\|_{L^\infty(\mathbb{R})} + \|\mathcal{Z}^a\|_{L^\infty(\mathbb{R})} + \|\mathcal{V}^a\|_{L^\infty(\mathbb{R})} + \|\mathcal{W}^a\|_{L^\infty(\mathbb{R})} + \|\mathfrak{p}\|_{L^\infty(\mathbb{R})} + \|\mathfrak{q}\|_{L^\infty(\mathbb{R})}. \tag{3.20}$$

The element Θ *belongs to* $\mathcal{G}$ *if*

(i)

$$\|\Theta\|_{\mathcal{G}} < \infty \quad \textit{and} \quad |||\Theta|||_{\mathcal{G}} < \infty; \tag{3.21}$$

(ii)

$$\mathcal{V}_2, \mathcal{W}_2, \mathcal{V}_4, \mathcal{W}_4 \geq 0; \tag{3.22}$$

(iii) for almost every $s \in \mathbb{R}$*, we have*

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \tag{3.23}$$

(iv)

$$\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_1(\mathcal{X}(s)), \qquad \mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_1(\mathcal{Y}(s)), \tag{3.24a}$$

$$\mathcal{V}_4(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_5(\mathcal{X}(s)), \qquad \mathcal{W}_4(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_5(\mathcal{Y}(s)) \tag{3.24b}$$

and

$$2\mathcal{V}_4(\mathcal{X}(s))\mathcal{V}_2(\mathcal{X}(s)) = (c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{p}^2(\mathcal{X}(s)), \tag{3.24c}$$

$$2\mathcal{W}_4(\mathcal{Y}(s))\mathcal{W}_2(\mathcal{Y}(s)) = (c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{q}^2(\mathcal{Y}(s)); \tag{3.24d}$$

(v)

$$\lim_{s\to-\infty} \mathcal{Z}_4(s) = 0. \tag{3.25}$$

We denote by $\mathcal{G}_0$ *the subset of* $\mathcal{G}$ *which parametrize the data at time* $t = 0$*, that is,*

$$\mathcal{G}_0 = \{\Theta \in \mathcal{G} \mid \mathcal{Z}_1 = 0\}.$$

For $\Theta \in \mathcal{G}_0$, we get by using (3.23) and (3.24a), that

$$\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{3.26}$$

This implies that

$$\dot{\mathcal{Z}}_2(s) = 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = 2\mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{3.27}$$

Note that for an element $\Theta \in \mathcal{G}$ we have $\mathcal{V}_2 + \mathcal{V}_4 > 0$ and $\mathcal{W}_2 + \mathcal{W}_4 > 0$ almost everywhere. As we shall see, this property is preserved in the solution and is important in proving that the solution operator from $\mathcal{D}$ to $\mathcal{D}$ is a semigroup.

Definition 3.7. *For any* $\psi = (\psi_1, \psi_2) \in \mathcal{F}$*, we define* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ *as*

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid x_1(X') < x_2(2s - X') \textit{ for all } X' < X\} \tag{3.28}$$

and set $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$*. We have*

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)). \tag{3.29}$$

We define

$$\mathcal{Z}_1(s) = 0, \tag{3.30a}$$

$$\mathcal{Z}_2(s) = x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)), \tag{3.30b}$$

$$\mathcal{Z}_3(s) = U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)), \tag{3.30c}$$
$$\mathcal{Z}_4(s) = J_1(\mathcal{X}(s)) + J_2(\mathcal{Y}(s)), \tag{3.30d}$$
$$\mathcal{Z}_5(s) = K_1(\mathcal{X}(s)) + K_2(\mathcal{Y}(s)) \tag{3.30e}$$

and

$$\mathcal{V}_1(X) = \frac{1}{2c(U_1(X))}x_1'(X), \qquad \mathcal{W}_1(Y) = -\frac{1}{2c(U_2(Y))}x_2'(Y), \tag{3.31a}$$
$$\mathcal{V}_2(X) = \frac{1}{2}x_1'(X), \qquad \mathcal{W}_2(Y) = \frac{1}{2}x_2'(Y), \tag{3.31b}$$
$$\mathcal{V}_3(X) = V_1(X), \qquad \mathcal{W}_3(Y) = V_2(Y), \tag{3.31c}$$
$$\mathcal{V}_4(X) = J_1'(X), \qquad \mathcal{W}_4(Y) = J_2'(Y), \tag{3.31d}$$
$$\mathcal{V}_5(X) = K_1'(X), \qquad \mathcal{W}_5(Y) = K_2'(Y), \tag{3.31e}$$
$$\mathfrak{p}(X) = H_1(X), \qquad \mathfrak{q}(Y) = H_2(Y). \tag{3.31f}$$

Let $\mathbf{C} : \mathcal{F} \to \mathcal{G}_0$ *denote the mapping which to any* $\psi \in \mathcal{F}$ *associates the element* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ *as defined above.*

Example 1 continued. The function $\mathcal{X}(s)$ is given as the unique point of intersection X between $x_1(X)$ and $x_2(2s - X)$, i.e.,
$$x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s)),$$
which implies that
$$\mathcal{X}(s) = s \quad \text{and} \quad \mathcal{Y}(s) = s.$$
Hence, $(\mathcal{X}, \mathcal{Y})$ is a strictly monotone curve.

Example 2 continued. For $s \leq \frac{\pi}{4}$, $\mathcal{X}(s)$ is given implicitly as the solution of the equation $X = x_2(2s - X)$, or
$$\arctan(\mathcal{X}(s)) + 2\mathcal{X}(s) + \frac{\pi}{2} = 2s.$$
By differentiating, we get
$$\dot{\mathcal{X}}(s) = \frac{1}{1 + \frac{1}{2+2\mathcal{X}(s)^2}},$$
so that $\frac{2}{3} \leq \dot{\mathcal{X}}(s) \leq 1$ which implies that $\dot{\mathcal{Y}}(s) \geq 1$. Hence, for $s \leq \frac{\pi}{4}$ $(\mathcal{X}, \mathcal{Y})$ is a strictly monotone curve.

For $\frac{\pi}{4} \leq s \leq \frac{\pi}{4} + \frac{1}{2}$, we have
$$\mathcal{X}(s) = 2s - \frac{\pi}{2}$$
and $\mathcal{Y}(s) = \frac{\pi}{2}$, that is, $\mathcal{Y}(s)$ is constant.

For $s > \frac{\pi}{4} + \frac{1}{2}$, $\mathcal{X}(s)$ is given as the solution of the equation $X - 1 = x_2(2s - X)$, that is
$$\arctan(\mathcal{X}(s) - 1) + 2\mathcal{X}(s) - 1 + \frac{\pi}{2} = 2s.$$
We differentiate and get
$$\dot{\mathcal{X}}(s) = \frac{1}{1 + \frac{1}{2+2(\mathcal{X}(s)-1)^2}},$$
so that $\frac{2}{3} \leq \dot{\mathcal{X}}(s) \leq 1$ and $\dot{\mathcal{Y}}(s) \geq 1$. Hence, for $s > \frac{\pi}{4} + \frac{1}{2}$ the curve $(\mathcal{X}, \mathcal{Y})$ is the graph of a strictly increasing function.

We conclude that the curve $(\mathcal{X},\mathcal{Y})$ consists of two strictly increasing parts (when $s \leq \frac{\pi}{4}$ and $s > \frac{\pi}{4}+\frac{1}{2}$) which are joined by a horizontal line segment (when $\frac{\pi}{4} \leq s \leq \frac{\pi}{4}+\frac{1}{2}$).

Example 3 continued. In order to compute $\mathcal{X}$ as defined in (3.28) we study the region (X,Y) such that $Y = 2s - X$ and $x_1(X) = x_2(Y)$. We have

$$x_2(2s-X) = \begin{cases} 2s - X - 1 & \text{if } X \leq 2s-1, \\ 0 & \text{if } 2s-1 \leq X \leq 2s, \\ 2s - X, & \text{if } 2s \leq X. \end{cases}$$

If $2s \leq 0$, the two functions intersect at only one point $X = s$. The same holds for $2s - 1 \geq 1$, where they intersect at $X = s$.

If $0 \leq 2s \leq 1$ then $x_1(X) = x_2(2s-X)$ for all $0 \leq X \leq 2s$. Since $Y = 2s-X$ this corresponds to straight line segments in the (X,Y)-plane with endpoints in $(0,2s)$ and $(2s,0)$. When we consider all $0 \leq 2s \leq 1$ we therefore get a triangle in the (X,Y)-plane with corner points at $(0,0)$, $(1,0)$ and $(0,1)$.

If $0 \leq 2s-1 \leq 1$ then $x_1(X) = x_2(2s-X)$ for all $2s-1 \leq X \leq 1$. Since $Y = 2s-X$ this corresponds to straight line segments in the (X,Y)-plane with endpoints in $(2s-1,1)$ and $(1,2s-1)$. When we consider all $0 \leq 2s-1 \leq 1$ we therefore get a triangle in the (X,Y)-plane with corner points at $(1,0)$, $(1,1)$ and $(0,1)$.

Therefore, for $0 \leq s \leq 1$ the region in the (X,Y)-plane where $x_1(X) = x_2(Y)$ for $Y = 2s - X$ is a box with corners at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

In principle we could pick any curve $(\mathcal{X}(s),\mathcal{Y}(s))$ in the box which satisfies Definition 2.1. This is because for any such curve in the box we have $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$. From (3.30a) we would then have $\mathcal{Z}_1(s) = 0$ so that $t(\mathcal{X}(s),\mathcal{Y}(s)) = 0$ for any such curve. Hence, $t(X,Y) = 0$ for all $(X,Y) = [0,1]\times[0,1]$. In other words, time is equal to zero in the box. In order to proceed we must pick one of these curves, and from (3.28) this curve consists of the straight line between $(0,0)$ and $(0,1)$, and the straight line between $(0,1)$ and $(1,1)$, that is,

$$\mathcal{X}(s) = \begin{cases} s & \text{if } s \leq 0, \\ 0 & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2s-1 & \text{if } \frac{1}{2} \leq s \leq 1, \\ s & \text{if } s \geq 1 \end{cases} \quad \text{and} \quad \mathcal{Y}(s) = \begin{cases} s & \text{if } s \leq 0, \\ 2s & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq s \leq 1, \\ s & \text{if } s \geq 1, \end{cases}$$

see Figure 3 and 4.

We mention that if the measures are not discrete at the same point, we do not get boxes. Instead, the region in the (X,Y)-plane where $x_1(X) = x_2(Y)$ for $Y = 2s-X$ is a curve consisting of the graph of strictly increasing functions and horizontal and vertical line segments.

Proof of the well-posedness of Definition 3.7. Let us verify that $(\mathcal{X},\mathcal{Y})$ belongs to $\mathcal{C}$. We first prove that $\mathcal{X}$ is nondecreasing. Let $s < \bar{s}$ and consider a sequence X_i such that $\lim_{i\to\infty} X_i = \mathcal{X}(s)$ with $X_i < \mathcal{X}(s)$. By (3.28) and since x_2 is nondecreasing, we have

$$x_1(X_i) < x_2(2s - X_i) \leq x_2(2\bar{s} - X_i).$$

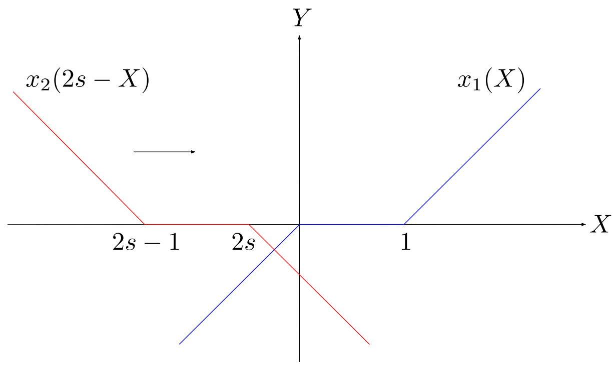

FIGURE 3. The functions $x_1(X)$ and $x_2(2s - X)$, from Example 3, for some $s < 0$. The functions intersect at $X = \mathcal{X}(s)$. As s increases, the graph of $x_2(2s - X)$ moves to the right. Eventually, the functions will intersect on intervals, which correspond to the box in Figure 4.

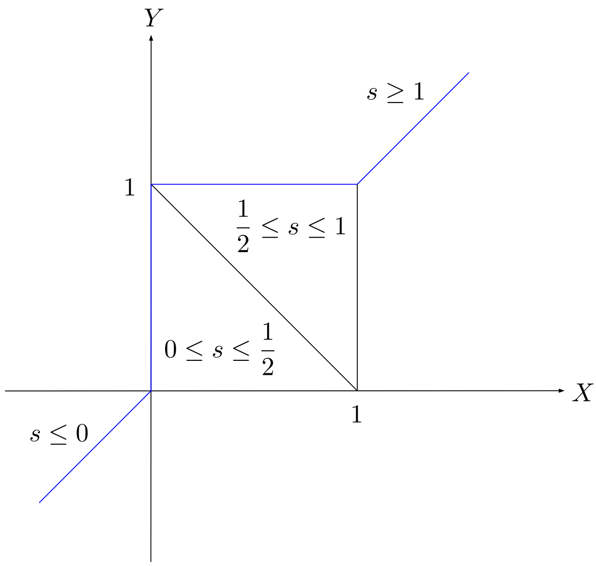

FIGURE 4. The set of all X such that $x_1(X) = x_2(2s - X)$, from Example 3, for different values of s. The curve $(\mathcal{X}(s), \mathcal{Y}(s))$ is marked in blue.

Hence, $X_i < \mathcal{X}(\bar{s})$. By letting i tend to infinity, we conclude that $\mathcal{X}(s) \leq \mathcal{X}(\bar{s})$. By the continuity of x_1 and x_2, we obtain (3.29). We show that $\mathcal{X}$ is differentiable almost everywhere. We claim that $\mathcal{X}$ is Lipschitz continuous with Lipschitz constant bounded by two, that is,

$$|\mathcal{X}(\bar{s}) - \mathcal{X}(s)| \leq 2|\bar{s} - s|. \tag{3.32}$$

We may assume without loss of generality that $s < \bar{s}$. Assume that (3.32) does not hold, so that

$$\mathcal{X}(\bar{s}) - \mathcal{X}(s) > 2(\bar{s} - s) \tag{3.33}$$

for some $\bar{s} > s \in \mathbb{R}$. Thus, $\mathcal{Y}(s) > \mathcal{Y}(\bar{s})$. Then, since x_2 is nondecreasing,

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \geq x_2(\mathcal{Y}(\bar{s})) = x_1(\mathcal{X}(\bar{s})).$$

This implies that $x_1(\mathcal{X}(s)) = x_1(\mathcal{X}(\bar{s}))$ because $x_1' \geq 0$ and $\mathcal{X}(s) < \mathcal{X}(\bar{s})$. Hence, x_1 is constant on $[\mathcal{X}(s), \mathcal{X}(\bar{s})]$. One proves similarly that x_2 is constant on $[\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$. Consider the point (X, Y) given by $Y = \mathcal{Y}(s)$ and $X = 2\bar{s} - \mathcal{Y}(s)$. We have

$$\mathcal{X}(s) = 2s - \mathcal{Y}(s) < X < 2\bar{s} - \mathcal{Y}(\bar{s}) = \mathcal{X}(\bar{s}),$$

so that $(X, Y) \in [\mathcal{X}(s), \mathcal{X}(\bar{s})] \times [\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$. It follows that $x_1(X) = x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) = x_2(2\bar{s} - X)$ and $X < \mathcal{X}(\bar{s})$, which contradicts the definition of $\mathcal{X}$. Therefore, (3.33) cannot hold and we have proved (3.32). Then, by Rademacher's theorem, $\mathcal{X}$ is differentiable almost everywhere. Let us prove that $\mathcal{X} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. This follows since

$$\mathcal{X}(s) - s = \frac{1}{2}(\mathcal{X}(s) - \mathcal{Y}(s)) = \frac{1}{2}(\mathcal{X}(s) - x_1(\mathcal{X}(s)) + x_2(\mathcal{Y}(s)) - \mathcal{Y}(s))$$

and $x_1 - \mathrm{Id}, x_2 - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. Since $\dot{\mathcal{X}} \leq 2$, it follows that $\dot{\mathcal{Y}} = 2 - \dot{\mathcal{X}} \geq 0$. As above, one can show that $\mathcal{Y} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. Hence, $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. We prove that $||\Theta||_{\mathcal{G}}$ and $|||\Theta|||_{\mathcal{G}}$ are finite. In order to prove that $\mathcal{Z}_3 \in L^2(\mathbb{R})$ we define the set

$$B = \{s \in \mathbb{R} \mid \dot{\mathcal{X}}(s) \geq 1\}.$$

Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$, we have $\dot{\mathcal{Y}} > 1$ on B^c. Thus,

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{Z}_3^2(s)\, ds &= \int_B U_1^2(\mathcal{X}(s))\, ds + \int_{B^c} U_2^2(\mathcal{Y}(s))\, ds \\
&\leq \int_B U_1^2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds + \int_{B^c} U_2^2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\, ds \\
&\leq \|U_1\|_{L^2(\mathbb{R})}^2 + \|U_2\|_{L^2(\mathbb{R})}^2
\end{aligned}$$

and $\mathcal{Z}_3 \in L^2(\mathbb{R})$. The fact that $\mathcal{Z}_3^a \in L^\infty(\mathbb{R})$ follows from $U_1, U_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Next we show that the components of $\mathcal{V}^a$ belong to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By (3.18) and (3.31a), we have

$$\begin{aligned}
|\mathcal{V}_1^a(X)| &= \left| \frac{1}{2c(U_1(X))} x_1'(X) - \frac{1}{2c(0)} \right| \\
&= \left| \frac{1}{2c(U_1(X))}(x_1'(X) - 1) + \frac{c(0) - c(U_1(X))}{2c(U_1(X))c(0)} \right| \\
&\leq \frac{\kappa}{2}|x_1'(X) - 1| + \frac{\kappa^2}{2}|c(0) - c(U_1(X))| \\
&= \frac{\kappa}{2}|x_1'(X) - 1| + \frac{\kappa^2}{2}\left| \int_{U_1(X)}^0 c'(\tilde{U})\, d\tilde{U} \right| \\
&\leq \frac{\kappa}{2}|x_1'(X) - 1| + \frac{\kappa^2 k_1}{2}|U_1(X)|,
\end{aligned}$$

which implies that $\mathcal{V}_1^a$ belongs to $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, as $x_1'-1, U_1\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. We have

$$\mathcal{V}_2^a=\mathcal{V}_2-\frac{1}{2}=\frac{1}{2}(x_1'-1),$$

so that $\mathcal{V}_2^a\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Since $\mathcal{V}_3^a=V_1$, $\mathcal{V}_4^a=J_1'$ and $\mathcal{V}_5^a=K_1'$, we conclude that all the components of $\mathcal{V}^a$ belong to $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Similarly, one shows that the components of $\mathcal{W}^a$ belong to $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Since $\mathfrak{p}=H_1$ and $\mathfrak{q}=H_2$, we have $\mathfrak{p},\mathfrak{q}\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. We have that $\mathcal{Z}_2^a\in L^\infty(\mathbb{R})$ because

$$\mathcal{Z}_2^a(s)=\mathcal{Z}_2(s)-s=x_1(\mathcal{X}(s))-s=x_1(\mathcal{X}(s))-\mathcal{X}(s)+\mathcal{X}(s)-s$$

and $x_1-\mathrm{Id},\mathcal{X}-\mathrm{Id}\in L^\infty(\mathbb{R})$. Since $\mathcal{Z}_1=0$, we have

$$\mathcal{Z}_1^a(s)=-\frac{1}{c(0)}(\mathcal{X}(s)-s),$$

so that $\mathcal{Z}_1^a\in L^\infty(\mathbb{R})$. From the relations $\mathcal{Z}_4^a=J_1(\mathcal{X})+J_2(\mathcal{Y})$ and $\mathcal{Z}_5^a=K_1(\mathcal{X})+K_2(\mathcal{Y})$, it follows from (3.5a) that they belong to $L^\infty(\mathbb{R})$. To check that $\frac{1}{\mathcal{V}_2+\mathcal{V}_4}$ and $\frac{1}{\mathcal{W}_2+\mathcal{W}_4}$ are bounded, we need the following result.

Lemma 3.8 ([9, Lemma 3.2]). *If $f\in G$ satisfies $||f-\mathrm{Id}||_{W^{1,\infty}(\mathbb{R})}+||f^{-1}-\mathrm{Id}||_{W^{1,\infty}(\mathbb{R})}\leq\alpha$ for some $\alpha\geq 0$, then $\frac{1}{1+\alpha}\leq f'\leq 1+\alpha$ almost everywhere. Conversely, if f is absolutely continuous, $f-\mathrm{Id}\in L^\infty(\mathbb{R})$, $f'-1\in L^2(\mathbb{R})$ and there exists $c\geq 1$ such that $\frac{1}{c}\leq f'\leq c$ almost everywhere, then f belongs to G and satisfies $||f-\mathrm{Id}||_{W^{1,\infty}(\mathbb{R})}+||f^{-1}-\mathrm{Id}||_{W^{1,\infty}(\mathbb{R})}\leq\alpha$, where $\alpha\geq 0$ only depends on c and $||f-\mathrm{Id}||_{L^\infty(\mathbb{R})}$.*

Since $x_1+J_1\in G$, Lemma 3.8 implies that for some $\alpha\geq 0$, $1/(1+\alpha)\leq x_1'+J_1'\leq 1+\alpha$ almost everywhere and it follows that $\frac{1}{\mathcal{V}_2+\mathcal{V}_4}\in L^\infty(\mathbb{R})$. Similarly, one can show that $\frac{1}{\mathcal{W}_2+\mathcal{W}_4}\in L^\infty(\mathbb{R})$. Hence, (3.21) holds. From (3.31b), (3.31d) and (3.6), we can check that (3.22) is satisfied. We verify that (3.23) holds. By differentiating (3.29), we obtain $x_1'(\mathcal{X})\dot{\mathcal{X}}=x_2'(\mathcal{Y})\dot{\mathcal{Y}}$, which after using (3.31b) yields (3.26). It follows that

$$\dot{\mathcal{Z}}_2=\frac{1}{2}x_1'(\mathcal{X})\dot{\mathcal{X}}+\frac{1}{2}x_2'(\mathcal{Y})\dot{\mathcal{Y}}=\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}}+\mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}}.$$

By (3.30a), we have $\dot{\mathcal{Z}}_1=0$, and by (3.31a), (3.30b) and (3.30c), we obtain

$$\mathcal{V}_1(\mathcal{X})\dot{\mathcal{X}}+\mathcal{W}_1(\mathcal{Y})\dot{\mathcal{Y}}=\frac{1}{2c(U_1(\mathcal{X}))}x_1'(\mathcal{X})\dot{\mathcal{X}}-\frac{1}{2c(U_2(\mathcal{Y}))}x_2'(\mathcal{Y})\dot{\mathcal{Y}}=0.$$

Using (3.30c), (3.11b) and (3.31c), we find

$$\dot{\mathcal{Z}}_3=\frac{1}{2}U_1'(\mathcal{X})\dot{\mathcal{X}}+\frac{1}{2}U_2'(\mathcal{Y})\dot{\mathcal{Y}}=V_1(\mathcal{X})\dot{\mathcal{X}}+V_2(\mathcal{Y})\dot{\mathcal{Y}}=\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}}+\mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}}.$$

The relations for $\dot{\mathcal{Z}}_4$ and $\dot{\mathcal{Z}}_5$ in (3.23) follow by differentiating (3.30d) and (3.30e), respectively. The two identities in (3.24a) follow from (3.31a) and (3.31b). Using (3.7), we can verify that (3.24b) holds. The last two identities (3.24c) and (3.24d) follows from (3.8). It remains to prove (3.25). Since $\mathcal{Z}_4(s)=J_1(\mathcal{X}(s))+J_2(\mathcal{Y}(s))$, $\lim_{s\to-\infty}\mathcal{X}(s)=-\infty$ and $\lim_{s\to-\infty}\mathcal{Y}(s)=-\infty$, it follows from (3.10) that $\lim_{s\to-\infty}\mathcal{Z}_4(s)=0$, so that (3.25) is satisfied. □

4. Existence of Solutions for the Equivalent System

4.1. **Existence of Short-Range Solutions.** In the following we denote rectangular domains by

$$\Omega = [X_l, X_r] \times [Y_l, Y_r]$$

and we set $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. We define curves in rectangular domains as follows.

Definition 4.1. *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we denote by $\mathcal{C}(\Omega)$ the set of curves in Ω parametrized by $(\mathcal{X}(s), \mathcal{Y}(s))$ with $s \in [s_l, s_r]$ such that $(\mathcal{X}(s_l), \mathcal{Y}(s_l)) = (X_l, Y_l)$, $(\mathcal{X}(s_r), \mathcal{Y}(s_r)) = (X_r, Y_r)$ and*

$$\mathcal{X} - \mathrm{Id}, \quad \mathcal{Y} - \mathrm{Id} \in W^{1,\infty}([s_l, s_r]), \tag{4.1a}$$

$$\dot{\mathcal{X}} \geq 0, \quad \dot{\mathcal{Y}} \geq 0, \tag{4.1b}$$

$$\frac{1}{2}(\mathcal{X}(s) + \mathcal{Y}(s)) = s \quad \text{for all } s \in [s_l, s_r]. \tag{4.1c}$$

We set

$$\|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}(\Omega)} = \|\mathcal{X} - \mathrm{Id}\|_{L^\infty([s_l,s_r])} + \|\mathcal{Y} - \mathrm{Id}\|_{L^\infty([s_l,s_r])}.$$

We introduce the counterpart of $\mathcal{G}$ on bounded domains, which we denote by $\mathcal{G}(\Omega)$.

Definition 4.2. *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we denote by $\mathcal{G}(\Omega)$ the set of all elements which consist of a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$, three vector-valued functions $\mathcal{Z}(s)$, $\mathcal{V}(X)$ and $\mathcal{W}(Y)$, and two functions $\mathfrak{p}(X)$ and $\mathfrak{q}(Y)$. We denote $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ and set*

$$\|\Theta\|^2_{\mathcal{G}(\Omega)} = \|\mathcal{Z}_3\|^2_{L^2([s_l,s_r])} + \|\mathcal{V}^a\|^2_{L^2([X_l,X_r])} + \|\mathcal{W}^a\|^2_{L^2([Y_l,Y_r])} + \|\mathfrak{p}\|^2_{L^2([X_l,X_r])} + \|\mathfrak{q}\|^2_{L^2([Y_l,Y_r])}$$

and

$$\begin{aligned}|||\Theta|||_{\mathcal{G}(\Omega)} = \|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}(\Omega)} &+ \left\|\frac{1}{\mathcal{V}_2 + \mathcal{V}_4}\right\|_{L^\infty([X_l,X_r])} + \left\|\frac{1}{\mathcal{W}_2 + \mathcal{W}_4}\right\|_{L^\infty([Y_l,Y_r])} \\ &+ \|\mathcal{Z}^a\|_{L^\infty([s_l,s_r])} + \|\mathcal{V}^a\|_{L^\infty([X_l,X_r])} + \|\mathcal{W}^a\|_{L^\infty([Y_l,Y_r])} \\ &+ \|\mathfrak{p}\|_{L^\infty([X_l,X_r])} + \|\mathfrak{q}\|_{L^\infty([Y_l,Y_r])}.\end{aligned}$$

The element Θ belongs[2] *to $\mathcal{G}(\Omega)$, if*

(i)

$$|||\Theta|||_{\mathcal{G}(\Omega)} < \infty,$$

(ii)

$$\mathcal{V}_2, \mathcal{W}_2, \mathcal{Z}_4, \mathcal{V}_4, \mathcal{W}_4 \geq 0,$$

(iii) for almost every $s \in \mathbb{R}$, we have

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s), \tag{4.2}$$

[2]Note that condition (i) implies $\|\Theta\|_{\mathcal{G}(\Omega)} < \infty$ because Ω is bounded.

(iv)

$$\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_1(\mathcal{X}(s)), \qquad \mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_1(\mathcal{Y}(s)), \tag{4.3a}$$

$$\mathcal{V}_4(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_5(\mathcal{X}(s)), \qquad \mathcal{W}_4(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_5(\mathcal{Y}(s)) \tag{4.3b}$$

and

$$2\mathcal{V}_4(\mathcal{X}(s))\mathcal{V}_2(\mathcal{X}(s)) = (c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{p}^2(\mathcal{X}(s)), \tag{4.3c}$$

$$2\mathcal{W}_4(\mathcal{Y}(s))\mathcal{W}_2(\mathcal{Y}(s)) = (c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{q}^2(\mathcal{Y}(s)). \tag{4.3d}$$

By definition we have for any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$ that the functions $\mathcal{X}$ and $\mathcal{Y}$ are nondecreasing. To any nondecreasing function one can associate its generalized inverse, a concept which is presented in, e.g., [2].

Definition 4.3. *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$, we define the generalized inverse of $\mathcal{X}$ and $\mathcal{Y}$ as*

$$\alpha(X) = \sup\{s \in [s_l, s_r] \mid \mathcal{X}(s) < X\} \quad \text{for } X \in (X_l, X_r],$$
$$\beta(Y) = \sup\{s \in [s_l, s_r] \mid \mathcal{Y}(s) < Y\} \quad \text{for } Y \in (Y_l, Y_r],$$

respectively. We denote $\mathcal{X}^{-1} = \alpha$ and $\mathcal{Y}^{-1} = \beta$.

The generalized inverse functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ satisfy the following properties.

Lemma 4.4. *The functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ are lower semicontinuous and nondecreasing. We have*

$$\mathcal{X} \circ \mathcal{X}^{-1} = \mathrm{Id} \quad \text{and} \quad \mathcal{Y} \circ \mathcal{Y}^{-1} = \mathrm{Id}, \tag{4.4a}$$

$$\mathcal{X}^{-1} \circ \mathcal{X}(s) = s \text{ for any } s \text{ such that } \dot{\mathcal{X}}(s) > 0 \tag{4.4b}$$

and

$$\mathcal{Y}^{-1} \circ \mathcal{Y}(s) = s \text{ for any } s \text{ such that } \dot{\mathcal{Y}}(s) > 0. \tag{4.4c}$$

We refer to [10, Lemma 3] for a proof.

Now we define solutions of (2.38) on rectangular domains. Consider the Banach spaces

$$L^\infty_X(\Omega) = L^\infty([Y_l, Y_r], C([X_l, X_r])), \qquad L^\infty_Y(\Omega) = L^\infty([X_l, X_r], C([Y_l, Y_r])),$$
$$W^{1,\infty}_X(\Omega) = L^\infty([Y_l, Y_r], W^{1,\infty}([X_l, X_r])), \quad W^{1,\infty}_Y(\Omega) = L^\infty([X_l, X_r], W^{1,\infty}([Y_l, Y_r])).$$

The corresponding norms for $f : \Omega \mapsto \mathbb{R}$ are defined as

$$||f||_{L^\infty_X(\Omega)} = \operatorname{ess\,sup}_{Y \in [Y_l, Y_r]} ||f(\cdot, Y)||_{L^\infty([X_l, X_r])},$$
$$||f||_{L^\infty_Y(\Omega)} = \operatorname{ess\,sup}_{X \in [X_l, X_r]} ||f(X, \cdot)||_{L^\infty([Y_l, Y_r])},$$
$$||f||_{W^{1,\infty}_X(\Omega)} = \operatorname{ess\,sup}_{Y \in [Y_l, Y_r]} ||f(\cdot, Y)||_{W^{1,\infty}([X_l, X_r])},$$
$$||f||_{W^{1,\infty}_Y(\Omega)} = \operatorname{ess\,sup}_{X \in [X_l, X_r]} ||f(X, \cdot)||_{W^{1,\infty}([Y_l, Y_r])}.$$

We introduce the function Z^a, defined as

$$Z^a_1(X, Y) = Z_1(X, Y) - \frac{1}{2c(0)}(X - Y), \tag{4.5a}$$

$$Z^a_2(X, Y) = Z_2(X, Y) - \frac{1}{2}(X + Y), \tag{4.5b}$$

$$Z_i^a(X,Y) = Z_i(X,Y) \quad \text{for } i \in \{3,4,5\} \tag{4.5c}$$

in order to conveniently express the decay of Z at infinity in the diagonal direction. Although we are not yet concerned with the behavior at infinity, the notation will be useful when introducing global solutions.

Definition 4.5. *We say that (Z,p,q) is a solution of (2.38) in $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ if*

(i)

$$Z^a \in [W^{1,\infty}(\Omega)]^5, \quad Z_X^a \in [W_Y^{1,\infty}(\Omega)]^5, \quad Z_Y^a \in [W_X^{1,\infty}(\Omega)]^5, \tag{4.6}$$
$$p \in W_Y^{1,\infty}(\Omega), \quad q \in W_X^{1,\infty}(\Omega),$$

(ii) for almost every $X \in [X_l, X_r]$,

$$(Z_X(X,Y))_Y = F(Z)(Z_X, Z_Y)(X,Y), \tag{4.7}$$

(iii) for almost every $Y \in [Y_l, Y_r]$,

$$(Z_Y(X,Y))_X = F(Z)(Z_X, Z_Y)(X,Y), \tag{4.8}$$

(iv) for almost every $X \in [X_l, X_r]$,

$$p_Y(X,Y) = 0, \tag{4.9}$$

(v) for almost every $Y \in [Y_l, Y_r]$,

$$q_X(X,Y) = 0. \tag{4.10}$$

We say that (Z,p,q) is a global solution of (2.38), if these conditions hold for any rectangular domain Ω.

The following lemma, whose proof follows the same lines as the one of [10, Lemma 4], shows that the imposed regularity in Definition 4.5 is necessary to extract relevant data from a curve. Slightly abusing the notation, we denote

$$\mathcal{X}(Y) = \mathcal{X} \circ \mathcal{Y}^{-1}(Y) \quad \text{and} \quad \mathcal{Y}(X) = \mathcal{Y} \circ \mathcal{X}^{-1}(X). \tag{4.11}$$

Lemma 4.6. *Let Ω be a rectangular domain in $\mathbb{R}^2$ and assume that*

$$Z^a \in [W^{1,\infty}(\Omega)]^5, \quad Z_X^a \in [W_Y^{1,\infty}(\Omega)]^5, \quad Z_Y^a \in [W_X^{1,\infty}(\Omega)]^5,$$
$$p \in W_Y^{1,\infty}(\Omega), \quad q \in W_X^{1,\infty}(\Omega).$$

Given a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$, let $(\mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ be defined as

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for all } s \in \mathbb{R}$$

and

$$\mathcal{V}(X) = Z_X(X, \mathcal{Y}(X)) \text{ for a.e. } X \in \mathbb{R},$$
$$\mathcal{W}(Y) = Z_Y(\mathcal{X}(Y), Y) \text{ for a.e. } Y \in \mathbb{R},$$
$$\mathfrak{p}(X) = p(X, \mathcal{Y}(X)) \text{ for a.e. } X \in \mathbb{R},$$
$$\mathfrak{q}(Y) = q(\mathcal{X}(Y), Y) \text{ for a.e. } Y \in \mathbb{R}$$

or equivalently

$$\mathcal{V}(\mathcal{X}(s)) = Z_X(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{X}}(s) > 0,$$

$$\mathcal{W}(\mathcal{Y}(s)) = Z_Y(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{Y}}(s) > 0,$$
$$\mathfrak{p}(\mathcal{X}(s)) = p(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{X}}(s) > 0,$$
$$\mathfrak{q}(\mathcal{Y}(s)) = q(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{Y}}(s) > 0.$$

Then $\mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q} \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ *and we denote* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ *by*

$$(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}).$$

We now introduce the set $\mathcal{H}(\Omega)$ of all solutions of (2.38) on rectangular domains, which satisfy (2.36), (2.37), and some additional constraints.

Definition 4.7. *Given* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, *let* $\mathcal{H}(\Omega)$ *be the set of all solutions* (Z, p, q) *to (2.38) in the sense of Definition 4.5 which satisfy the following properties*

$$x_X = c(U)t_X, \qquad x_Y = -c(U)t_Y, \tag{4.12a}$$
$$J_X = c(U)K_X, \qquad J_Y = -c(U)K_Y, \tag{4.12b}$$
$$2J_Xx_X = (c(U)U_X)^2 + c(U)p^2, \qquad 2J_Yx_Y = (c(U)U_Y)^2 + c(U)q^2, \tag{4.12c}$$
$$x_X \geq 0, \qquad x_Y \geq 0, \tag{4.12d}$$
$$J_X \geq 0, \qquad J_Y \geq 0, \tag{4.12e}$$
$$x_X + J_X > 0, \qquad x_Y + J_Y > 0. \tag{4.12f}$$

We have the following short-range existence theorem.

Theorem 4.8. *Given* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, *then for any* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$, *there exists a unique solution* $(Z, p, q) \in \mathcal{H}(\Omega)$ *such that*

$$\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}), \tag{4.13}$$

if $s_r - s_l \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega)})$. *Here* C *denotes an increasing function dependent on* Ω, κ, k_1, *and* k_2.

Proof. We aim to use the Banach fixed-point theorem. Define $\mathcal{B}$ as the set of all elements (Z_h, Z_v, V, W) such that

$$Z_h \in [L^\infty_X(\Omega)]^5, \quad Z_v \in [L^\infty_Y(\Omega)]^5, \quad V \in [L^\infty_Y(\Omega)]^5, \quad W \in [L^\infty_X(\Omega)]^5$$

and

$$\sum_{i=1}^{5} (||Z^a_{h,i}||_{L^\infty_X(\Omega)} + ||Z^a_{v,i}||_{L^\infty_Y(\Omega)} + ||V^a_i||_{L^\infty_Y(\Omega)} + ||W^a_i||_{L^\infty_X(\Omega)}) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)}, \tag{4.14}$$

where we used the same notation for Z_h and Z_v as in (4.5) for Z. For V and W we used the same notation as in (3.18) for $\mathcal{V}$ and $\mathcal{W}$, that is,

$$V^a_1 = V_1 - \frac{1}{2c(0)}, \qquad W^a_1 = W_1 + \frac{1}{2c(0)}, \tag{4.15a}$$
$$V^a_2 = V_2 - \frac{1}{2}, \qquad W^a_2 = W_2 - \frac{1}{2}, \tag{4.15b}$$
$$V^a_i = V_i, \qquad W^a_i = W_i \qquad \text{for } i \in \{3, 4, 5\}. \tag{4.15c}$$

As we shall see, for the fixed point, the functions Z_h and Z_v coincide and are equal to the solution Z, but we find it convenient to define both quantities in order to keep the symmetry of the problem with respect to the X and Y variables. For any $(Z_h, Z_v, V, W) \in \mathcal{B}$, we introduce the mapping $\mathcal{T}$ given by

$$(\bar{Z}_h, \bar{Z}_v, \bar{V}, \bar{W}) = \mathcal{T}(Z_h, Z_v, V, W),$$

where

$$\bar{Z}_h(X,Y) = \mathcal{Z}(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V(\tilde{X}, Y)\, d\tilde{X} \tag{4.16}$$

for a.e. $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$,

$$\bar{Z}_v(X,Y) = \mathcal{Z}(\mathcal{X}^{-1}(X)) + \int_{\mathcal{Y}(X)}^{Y} W(X, \tilde{Y})\, d\tilde{Y} \tag{4.17}$$

for a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$,

$$\bar{V}(X,Y) = \mathcal{V}(X) + \int_{\mathcal{Y}(X)}^{Y} F(Z_h)(V, W)(X, \tilde{Y})\, d\tilde{Y} \tag{4.18}$$

for a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$,

$$\bar{W}(X,Y) = \mathcal{W}(Y) + \int_{\mathcal{X}(Y)}^{X} F(Z_h)(V, W)(\tilde{X}, Y)\, d\tilde{X} \tag{4.19}$$

for a.e. $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$.

Let us compare this mapping with a solution Z of (2.38a)-(2.38e) (in the sense of Definition 4.5) which satisfies (4.13). For any $(X, Y) \in \Omega$, we have

$$Z(X, \mathcal{Y}(s)) = \mathcal{Z}(s) + \int_{\mathcal{X}(s)}^{X} Z_X(\tilde{X}, \mathcal{Y}(s))\, d\tilde{X},$$

which after setting $s = \mathcal{Y}^{-1}(Y)$ yields

$$Z(X,Y) = \mathcal{Z}(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} Z_X(\tilde{X}, Y)\, d\tilde{X}.$$

Similarly, we obtain, by setting $s = \mathcal{X}^{-1}(X)$,

$$Z(X,Y) = \mathcal{Z}(\mathcal{X}^{-1}(X)) + \int_{\mathcal{Y}(X)}^{Y} Z_Y(X, \tilde{Y})\, d\tilde{Y}.$$

For a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$, we have

$$Z_X(X,Y) = Z_X(X, \mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^{Y} F(Z)(Z_X, Z_Y)(X, \tilde{Y})\, d\tilde{Y}$$

which by (4.13) rewrites as

$$Z_X(X,Y) = \mathcal{V}(X) + \int_{\mathcal{Y}(X)}^{Y} F(Z)(Z_X, Z_Y)(X, \tilde{Y})\, d\tilde{Y}.$$

By a similar argument, we get

$$Z_Y(X,Y) = \mathcal{W}(Y) + \int_{\mathcal{X}(Y)}^{X} F(Z)(Z_X, Z_Y)(\tilde{X}, Y)\, d\tilde{X}$$

for a.e. $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$. Thus, if Z is a solution of (2.38a)-(2.38e) which satisfies (4.13), then (Z, Z, Z_X, Z_Y) is a fixed point of $\mathcal{T}$.

Let us show that the mapping $\mathcal{T}$ maps $\mathcal{B}$ into $\mathcal{B}$. To begin with we derive some estimates. We set $\delta = s_r - s_l$. By (4.4a) and since $0 \leq \dot{\mathcal{X}} \leq 2$, we have

$$|X - \mathcal{X}(Y)| = |\mathcal{X}(\alpha(X)) - \mathcal{X}(\beta(Y))| \leq \mathcal{X}(s_r) - \mathcal{X}(s_l) \leq 2(s_r - s_l) = 2\delta \tag{4.20}$$

and similarly, we get

$$|Y - \mathcal{Y}(X)| \leq 2(s_r - s_l) = 2\delta. \tag{4.21}$$

For the first component of $\bar{Z}_h^a$, we have

$$\begin{aligned}
\bar{Z}_{h,1}^a(X,Y) &= \bar{Z}_{h,1}(X,Y) - \frac{1}{2c(0)}(X-Y) \quad \text{by (4.5)} \\
&= \mathcal{Z}_1(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V_1(\tilde{X},Y)\,d\tilde{X} - \frac{1}{2c(0)}(X-Y) \quad \text{by (4.16)} \\
&= \mathcal{Z}_1^a(\mathcal{Y}^{-1}(Y)) + \frac{1}{c(0)}(\mathcal{X}(Y) - \mathcal{Y}^{-1}(Y)) \quad \text{by (3.18) and (4.15)} \\
&\quad + \int_{\mathcal{X}(Y)}^{X} \left(V_1^a(\tilde{X},Y) + \frac{1}{2c(0)}\right) d\tilde{X} - \frac{1}{2c(0)}(X-Y) \\
&= \mathcal{Z}_1^a(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V_1^a(\tilde{X},Y)\,d\tilde{X} \\
&\quad + \frac{1}{c(0)}\left(\frac{1}{2}\mathcal{X}(Y) + \frac{1}{2}Y - \mathcal{Y}^{-1}(Y)\right).
\end{aligned}$$

From (4.1c), we obtain

$$\frac{1}{2}\mathcal{X}(Y) + \frac{1}{2}Y - \mathcal{Y}^{-1}(Y) = \frac{1}{2}\mathcal{X}(Y) + \frac{1}{2}Y - \frac{1}{2}(\mathcal{X}(Y) + Y) = 0.$$

Hence, by (4.20),

$$||\bar{Z}_{h,1}^a||_{L_X^\infty(\Omega)} \leq ||\mathcal{Z}_1^a||_{L^\infty([s_l,s_r])} + 2\delta ||V_1^a||_{L_Y^\infty(\Omega)}.$$

We proceed similarly for the second component of $\bar{Z}_h^a$ and get

$$\begin{aligned}
\bar{Z}_{h,2}^a(X,Y) &= \bar{Z}_{h,2}(X,Y) - \frac{1}{2}(X+Y) \\
&= \mathcal{Z}_2(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V_2(\tilde{X},Y)\,d\tilde{X} - \frac{1}{2}(X+Y) \\
&= \mathcal{Z}_2^a(\mathcal{Y}^{-1}(Y)) + \mathcal{Y}^{-1}(Y) + \int_{\mathcal{X}(Y)}^{X} \left(V_2^a(\tilde{X},Y) + \frac{1}{2}\right) d\tilde{X} - \frac{1}{2}(X+Y) \\
&= \mathcal{Z}_2^a(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V_2^a(\tilde{X},Y)\,d\tilde{X} + \mathcal{Y}^{-1}(Y) - \frac{1}{2}\mathcal{X}(Y) - \frac{1}{2}Y.
\end{aligned}$$

By (4.1c), we have

$$\mathcal{Y}^{-1}(Y) - \frac{1}{2}\mathcal{X}(Y) - \frac{1}{2}Y = \frac{1}{2}(\mathcal{X}(Y) + Y) - \frac{1}{2}\mathcal{X}(Y) - \frac{1}{2}Y = 0,$$

which leads to the estimate

$$||\bar{Z}_{h,2}^a||_{L_X^\infty(\Omega)} \leq ||\mathcal{Z}_2^a||_{L^\infty([s_l,s_r])} + 2\delta ||V_2^a||_{L_Y^\infty(\Omega)}.$$

For $i \in \{3,4,5\}$, we have

$$||\bar{Z}^a_{h,i}||_{L^\infty_X(\Omega)} \leq ||\mathcal{Z}^a_i||_{L^\infty([s_l,s_r])} + 2\delta||V^a_i||_{L^\infty_Y(\Omega)}.$$

Similar bounds hold for the components of $\bar{Z}^a_v$. Let us consider $\bar{V}^a_1$. By using the governing equations (2.38), we obtain

$$\begin{aligned}\bar{V}^a_1(X,Y) &= \bar{V}_1(X,Y) - \frac{1}{2c(0)}\\ &= \mathcal{V}_1(X) - \frac{1}{2c(0)} + \int_{\mathcal{Y}(X)}^Y F_1(Z_h)(V,W)(X,\tilde{Y})\,d\tilde{Y}\\ &= \mathcal{V}^a_1(X) - \int_{\mathcal{Y}(X)}^Y \left(\frac{c'(Z_{h,3})}{2c(Z_{h,3})}(V_3W_1+W_3V_1)\right)(X,\tilde{Y})\,d\tilde{Y}.\end{aligned}$$

We have

$$\begin{aligned}|V_3W_1+W_3V_1| &= \left|V^a_3W^a_1 + W^a_3V^a_1 - \frac{1}{2c(0)}(W^a_3 - V^a_3)\right|\\ &\leq ||V^a_3||_{L^\infty_Y(\Omega)}||W^a_1||_{L^\infty_X(\Omega)} + ||W^a_3||_{L^\infty_X(\Omega)}||V^a_1||_{L^\infty_Y(\Omega)}\\ &\quad + \frac{\kappa}{2}\big(||W^a_3||_{L^\infty_X(\Omega)} + ||V^a_3||_{L^\infty_Y(\Omega)}\big)\\ &\leq 4|||\Theta|||^2_{\mathcal{G}(\Omega)} + \kappa|||\Theta|||_{\mathcal{G}(\Omega)} \quad \text{by (4.14)}.\end{aligned}$$

Hence,

$$||\bar{V}^a_1||_{L^\infty_Y(\Omega)} \leq ||\mathcal{V}^a_1||_{L^\infty([X_l,X_r])} + \delta\kappa k_1\big(4|||\Theta|||^2_{\mathcal{G}(\Omega)} + \kappa|||\Theta|||_{\mathcal{G}(\Omega)}\big).$$

After doing the same for the other components of $\bar{V}$ and $\bar{W}$, we get

$$\sum_{i=1}^5(||\bar{Z}^a_{h,i}||_{L^\infty_X(\Omega)} + ||\bar{Z}^a_{v,i}||_{L^\infty_Y(\Omega)} + ||\bar{V}^a_i||_{L^\infty_Y(\Omega)} + ||\bar{W}^a_i||_{L^\infty_X(\Omega)}) \leq |||\Theta|||_{\mathcal{G}(\Omega)} + \delta C_1(|||\Theta|||_{\mathcal{G}(\Omega)})$$

for some increasing function $C_1(|||\Theta|||_{\mathcal{G}(\Omega)})$. Hence, by setting δ small enough, the mapping $\mathcal{T}$ maps $\mathcal{B}$ into $\mathcal{B}$.

It remains to show that $\mathcal{T}$ is a contraction. Let (Z_h,Z_v,V,W) and (Z'_h,Z'_v,V',W') belong to $\mathcal{B}$, and fix $(X,Y) \in \Omega$. For the first component of F, we have

$$\begin{aligned}&|\bar{V}^a_1(X,Y) - (\bar{V}'_1)^a(X,Y)|\\ &= |\bar{V}_1(X,Y) - \bar{V}'_1(X,Y)|\\ &\leq \int_{\mathcal{Y}(X)}^Y |F_1(Z_h)(V,W) - F_1(Z'_h)(V',W')|(X,\tilde{Y})\,d\tilde{Y}\end{aligned} \tag{4.22}$$

and by (2.38), we get

$$\begin{aligned}&F_1(Z_h)(V,W) - F_1(Z'_h)(V',W')\\ &= -\frac{c'(Z_{h,3})}{2c(Z_{h,3})}(V_3W_1+W_3V_1) + \frac{c'(Z'_{h,3})}{2c(Z'_{h,3})}(V'_3W'_1+W'_3V'_1)\\ &= -\frac{c'(Z^a_{h,3})}{2c(Z^a_{h,3})}\left(V^a_3\left(W^a_1 - \frac{1}{2c(0)}\right) + W^a_3\left(V^a_1 + \frac{1}{2c(0)}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{c'((Z'_{h,3})^a)}{2c((Z'_{h,3})^a)}\Big((V'_3)^a\Big((W'_1)^a-\frac{1}{2c(0)}\Big)+(W'_3)^a\Big((V'_1)^a+\frac{1}{2c(0)}\Big)\Big)\\
&=\frac{c'((Z'_{h,3})^a)}{2c((Z'_{h,3})^a)}\Big((V'_3)^a((W'_1)^a-W_1^a)+W_1^a((V'_3)^a-V_3^a)\\
&\qquad+(W'_3)^a((V'_1)^a-V_1^a)+V_1^a((W'_3)^a-W_3^a)\\
&\qquad-\frac{1}{2c(0)}((V'_3)^a-V_3^a)+\frac{1}{2c(0)}((W'_3)^a-W_3^a)\Big)\\
&+\frac{1}{2}\Big(V_3^aW_1^a+W_3^aV_1^a-\frac{1}{2c(0)}V_3^a+\frac{1}{2c(0)}W_3^a\Big)\Big(\frac{c'((Z'_{h,3})^a)}{c((Z'_{h,3})^a)}-\frac{c'(Z_{h,3}^a)}{c(Z_{h,3}^a)}\Big).
\end{aligned}
$$

Since

$$
\frac{c'((Z'_{h,3})^a)}{c((Z'_{h,3})^a)}-\frac{c'(Z_{h,3}^a)}{c(Z_{h,3}^a)}=\int_{Z_{h,3}^a}^{(Z'_{h,3})^a}\Big(\frac{c''c-(c')^2}{c^2}\Big)(Z)\,dZ,
$$

this leads to the estimate

$$
\begin{aligned}
&|F_1(Z_h)(V,W)-F_1(Z'_h)(V',W')|\\
&\le\frac{\kappa k_1}{2}\Big(2|||\Theta|||_{\mathcal{G}(\Omega)}\Big(||(W'_1)^a-W_1^a||_{L^\infty_X(\Omega)}+||(V'_3)^a-V_3^a||_{L^\infty_Y(\Omega)}\\
&\qquad+||(V'_1)^a-V_1^a||_{L^\infty_Y(\Omega)}+||(W'_3)^a-W_3^a||_{L^\infty_X(\Omega)}\Big)\\
&\qquad+\frac{\kappa}{2}\Big(||(V'_3)^a-V_3^a||_{L^\infty_Y(\Omega)}+||(W'_3)^a-W_3^a||_{L^\infty_X(\Omega)}\Big)\Big)\\
&\quad+\frac{1}{2}\Big(4|||\Theta|||^2_{\mathcal{G}(\Omega)}+\kappa|||\Theta|||_{\mathcal{G}(\Omega)}\Big)||(Z'_{h,3})^a-Z_{h,3}^a||_{L^\infty_X(\Omega)}\\
&\quad\times(\kappa k_2+(\kappa k_1)^2),
\end{aligned}
$$

where we used (4.14). We insert this into (4.22) and obtain

$$
\begin{aligned}
&|\bar V_1^a(X,Y)-(\bar V'_1)^a(X,Y)|\\
&\le\delta K\big(||W^a-(W')^a||_{L^\infty_X(\Omega)}+||V^a-(V')^a||_{L^\infty_Y(\Omega)}+||Z_h^a-(Z'_h)^a||_{L^\infty_X(\Omega)}\big),
\end{aligned}
$$

where K depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$, κ, k_1 and k_2 because of (1.5) and (1.6). Following the same lines, we obtain

$$
\begin{aligned}
&||\bar Z_h^a-(\bar Z'_h)^a||_{L^\infty_X(\Omega)}+||\bar Z_v^a-(\bar Z'_v)^a||_{L^\infty_Y(\Omega)}+||\bar V^a-(\bar V')^a||_{L^\infty_Y(\Omega)}+||\bar W^a-(\bar W')^a||_{L^\infty_X(\Omega)}\\
&\le\delta C_2\big(||Z_h^a-(Z'_h)^a||_{L^\infty_X(\Omega)}+||Z_v^a-(Z'_v)^a||_{L^\infty_Y(\Omega)}\\
&\qquad+||V^a-(V')^a||_{L^\infty_Y(\Omega)}+||W^a-(W')^a||_{L^\infty_X(\Omega)}\big)
\end{aligned}
$$

for some increasing function C_2 depending on $|||\Theta|||_{\mathcal{G}(\Omega)}$, Ω, κ, k_1 and k_2. By setting $\delta>0$ so small that $\delta C_2<1$, we conclude that $\mathcal{T}$ is a contraction. Hence, $\mathcal{T}$ admits a unique fixed point that we denote (Z_h,Z_v,V,W).

Let us prove that $Z_h=Z_v$, $V=Z_X$ and $W=Z_Y$. We denote by $\mathcal{N}_X$ the set of points $X\in[X_l,X_r]$ for which (4.17) and (4.18) hold. Similarly, we denote by $\mathcal{N}_Y$ the set of points $Y\in[Y_l,Y_r]$ for which (4.16) and (4.19) hold. We have $\mathrm{meas}([X_l,X_r]\setminus\mathcal{N}_X)=\mathrm{meas}([Y_l,Y_r]\setminus\mathcal{N}_Y)=0$ such that $\mathrm{meas}(\Omega\setminus\mathcal{N}_X\times\mathcal{N}_Y)=0$.

For any $(X,Y)\in\mathcal{N}_X\times\mathcal{N}_Y$, we have

$$Z_h(X,Y)-Z_v(X,Y)=\mathcal{Z}(\mathcal{Y}^{-1}(Y))+\int_{\mathcal{X}(Y)}^{X}V(\tilde X,Y)\,d\tilde X - \mathcal{Z}(\mathcal{X}^{-1}(X))-\int_{\mathcal{Y}(X)}^{Y}W(X,\tilde Y)\,d\tilde Y. \tag{4.23}$$

From (4.18) and (4.19), we obtain

$$\begin{aligned}
&\int_{\mathcal{X}(Y)}^{X}V(\tilde X,Y)\,d\tilde X-\int_{\mathcal{Y}(X)}^{Y}W(X,\tilde Y)\,d\tilde Y\\
&=\int_{\mathcal{X}(Y)}^{X}\Big(\mathcal{V}(\tilde X)+\int_{\mathcal{Y}(\tilde X)}^{Y}F(Z_h)(V,W)(\tilde X,\tilde Y)\,d\tilde Y\Big)\,d\tilde X\\
&\quad-\int_{\mathcal{Y}(X)}^{Y}\Big(\mathcal{W}(\tilde Y)+\int_{\mathcal{X}(\tilde Y)}^{X}F(Z_h)(V,W)(\tilde X,\tilde Y)\,d\tilde X\Big)\,d\tilde Y\\
&=\int_{\mathcal{X}(Y)}^{X}\mathcal{V}(\tilde X)\,d\tilde X-\int_{\mathcal{Y}(X)}^{Y}\mathcal{W}(\tilde Y)\,d\tilde Y\\
&=\int_{\mathcal{X}\circ\mathcal{Y}^{-1}(Y)}^{\mathcal{X}\circ\mathcal{X}^{-1}(X)}\mathcal{V}(\tilde X)\,d\tilde X-\int_{\mathcal{Y}\circ\mathcal{X}^{-1}(X)}^{\mathcal{Y}\circ\mathcal{Y}^{-1}(Y)}\mathcal{W}(\tilde Y)\,d\tilde Y\quad\text{by (4.4}a\text{) and (4.11)}\\
&=-\int_{\mathcal{X}^{-1}(X)}^{\mathcal{Y}^{-1}(Y)}\big(\mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s)+\mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\big)\,ds\quad\text{by a change of variables}\\
&=-\int_{\mathcal{X}^{-1}(X)}^{\mathcal{Y}^{-1}(Y)}\dot{\mathcal{Z}}(s)\,ds\quad\text{by (4.2)}\\
&=\mathcal{Z}(\mathcal{X}^{-1}(X))-\mathcal{Z}(\mathcal{Y}^{-1}(Y)).
\end{aligned}$$

By inserting this into (4.23), we conclude that $Z_h(X,Y)=Z_v(X,Y)$ for all $(X,Y)\in\mathcal{N}_X\times\mathcal{N}_Y$, that is, almost everywhere in Ω.

We denote $Z=Z_h=Z_v$. The function Z is only defined in $\mathcal{N}_X\times\mathcal{N}_Y$. We define $Z(X,Y)$ for all $(X,Y)\in\Omega$ by setting

$$Z(X,Y)=\lim_{n\to\infty}Z(X_n,Y_n), \tag{4.24}$$

where (X_n,Y_n) is a squence in $\mathcal{N}_X\times\mathcal{N}_Y$ such that $(X_n,Y_n)\to(X,Y)$ as $n\to\infty$. Let us prove that this is well-defined. First we show that Z is Lipschitz continuous in $\mathcal{N}_X\times\mathcal{N}_Y$. Let $(X_1,Y_1),(X_2,Y_2)\in\mathcal{N}_X\times\mathcal{N}_Y$. By (4.16), we have

$$\begin{aligned}
Z(X_2,Y_2)-Z(X_1,Y_2)&=\int_{\mathcal{X}(Y_2)}^{X_2}V(\tilde X,Y_2)\,d\tilde X-\int_{\mathcal{X}(Y_2)}^{X_1}V(\tilde X,Y_2)\,d\tilde X\\
&=\int_{X_1}^{X_2}V(\tilde X,Y_2)\,d\tilde X
\end{aligned}$$

and, from (4.17), we get

$$Z(X_1,Y_2)-Z(X_1,Y_1)=\int_{\mathcal{Y}(X_1)}^{Y_2}W(X_1,\tilde Y)\,d\tilde Y-\int_{\mathcal{Y}(X_1)}^{Y_1}W(X_1,\tilde Y)\,d\tilde Y$$

$$= \int_{Y_1}^{Y_2} W(X_1, \tilde{Y})\, d\tilde{Y}.$$

This implies that

$$\begin{aligned}(4.25)\qquad &|Z(X_2,Y_2) - Z(X_1,Y_1)| \\ &\leq |Z(X_2,Y_2) - Z(X_1,Y_2)| + |Z(X_1,Y_2) - Z(X_1,Y_1)| \\ &\leq (||V||_{L^\infty_Y(\Omega)} + ||W||_{L^\infty_X(\Omega)})(|X_2 - X_1| + |Y_2 - Y_1|) \\ &\leq C(|X_2 - X_1| + |Y_2 - Y_1|),\end{aligned}$$

where C only depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$, and we conclude that Z is Lipschitz continuous in $\mathcal{N}_X \times \mathcal{N}_Y$.

For any $(X, Y) \in \Omega$, there exists a sequence $(X_n, Y_n) \in \mathcal{N}_X \times \mathcal{N}_Y$ such that $(X_n, Y_n) \to (X, Y)$ as $n \to \infty$, since $\text{meas}(\Omega \setminus \mathcal{N}_X \times \mathcal{N}_Y) = 0$. From (4.25), we have

$$|Z(X_n,Y_n) - Z(X_m,Y_m)| \leq C(|X_n - X_m| + |Y_n - Y_m|),$$

so that the limit $\lim_{n\to\infty} Z(X_n, Y_n)$ exists. We claim that the limit is independent of the particular choice of the sequence in $\mathcal{N}_X \times \mathcal{N}_Y$ converging to (X, Y). Let $(\bar{X}_m, \bar{Y}_m)$ be another sequence in $\mathcal{N}_X \times \mathcal{N}_Y$ such that $(\bar{X}_m, \bar{Y}_m) \to (X, Y)$ as $n \to \infty$. By (4.25), we have

$$\begin{aligned}|Z(X_n,Y_n) - Z(\bar{X}_m,\bar{Y}_m)| &\leq C(|X_n - \bar{X}_m| + |Y_n - \bar{Y}_m|) \\ &\leq C(|X_n - X| + |X - \bar{X}_m| + |Y_n - Y| + |Y - \bar{Y}_m|)\end{aligned}$$

which implies that $\lim_{n\to\infty} Z(X_n, Y_n) = \lim_{m\to\infty} Z(\bar{X}_m, \bar{Y}_m)$. This proves the claim and (4.24) is well-defined. It now follows from (4.25) that Z is Lipschitz continuous in Ω. Indeed, for any $(X, Y), (\bar{X}, \bar{Y}) \in \Omega$, there exist sequences $(X_n, Y_n), (\bar{X}_n, \bar{Y}_n) \in \mathcal{N}_X \times \mathcal{N}_Y$, such that $(X_n, Y_n) \to (X, Y)$ and $(\bar{X}_n, \bar{Y}_n) \to (\bar{X}, \bar{Y})$, which yields

$$\begin{aligned}|Z(X,Y) - Z(\bar{X},\bar{Y})| &= \lim_{n\to\infty} |Z(X_n,Y_n) - Z(\bar{X}_n,\bar{Y}_n)| \\ &\leq C \lim_{n\to\infty} (|X_n - \bar{X}_n| + |Y_n - \bar{Y}_n|) \\ &= C(|X - \bar{X}| + |Y - \bar{Y}|).\end{aligned}$$

Thus, Z is Lipschitz continuous and therefore differentiable almost everywhere in Ω. It follows from (4.16) and (4.17), that

$$Z_X(X,Y) = V(X,Y) \quad \text{and} \quad Z_Y(X,Y) = W(X,Y)$$

for almost every $X \in [X_l, X_r]$ and $Y \in [Y_l, Y_r]$. Now we define the solutions p and q of (2.38f) and (2.38g), respectively. For almost every $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$, we set

$$p(X,Y) = \mathfrak{p}(X) \tag{4.26}$$

and for almost every $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$, we set

$$q(X,Y) = \mathfrak{q}(Y). \tag{4.27}$$

Let us check that (4.6) is satisfied. Since Z is Lipschitz continuous, it follows that all the components of Z belong to $W^{1,\infty}(\Omega)$. As in the argument above where we showed that Z is Lipschitz continuous, one can show that for almost every X, Z_X is Lipschitz continuous with respect to Y, and that for almost every Y, Z_Y is Lipschitz continuous with respect to X. It implies that $Z_{XY}(X,\cdot) \in L^\infty([Y_l,Y_r])$ for almost every $X \in [X_l,X_r]$ and $Z_{YX}(\cdot,Y) \in L^\infty([X_l,X_r])$ for almost every $Y \in [X_l,X_r]$, which in turn implies that

$$\operatorname{ess\,sup}_{X\in[X_l,X_r]} ||Z_X(X,\cdot)||_{W^{1,\infty}([Y_l,Y_r])}$$

and

$$\operatorname{ess\,sup}_{Y\in[Y_l,Y_r]} ||Z_Y(\cdot,Y)||_{W^{1,\infty}([X_l,X_r])}$$

are bounded by a constant. Hence, $Z_X \in W_Y^{1,\infty}(\Omega)$ and $Z_Y \in W_X^{1,\infty}(\Omega)$. The fact that $p \in W_Y^{1,\infty}(\Omega)$ and $q \in W_X^{1,\infty}(\Omega)$ follows by (4.26) and (4.27), since $\mathfrak{p} \in L^\infty([X_l,X_r])$ and $\mathfrak{q} \in L^\infty([Y_l,Y_r])$. Thus, (4.6) holds.

Next, we verify that the relations (4.7)-(4.10) are satisfied. Since $(Z,\mathcal{Z},\mathcal{Z}_X,\mathcal{Z}_Y)$ is a fixed point of $\mathcal{T}$, we have, by differentiating (4.18) and (4.19), that (4.7) and (4.8) hold. The relations (4.9) and (4.10) follow by differentiating (4.26) and (4.27), respectively.

We prove that (Z,p,q) satisfies (4.13). Since $(Z,\mathcal{Z},\mathcal{Z}_X,\mathcal{Z}_Y)$ is a fixed point of $\mathcal{T}$, we get, by (4.18) and (4.19), that

$$Z_X(X,\mathcal{Y}(X)) = \mathcal{V}(X) \quad \text{and} \quad Z_Y(\mathcal{X}(Y),Y) = \mathcal{W}(Y).$$

By (4.17), we have

$$Z(\mathcal{X}(s),\mathcal{Y}(s)) = \mathcal{Z}(\mathcal{X}^{-1}(\mathcal{X}(s)))$$

and, by (4.4b), this implies (4.13) for all $s \in [s_l,s_r]$ such that $\dot{\mathcal{X}}(s) > 0$. Similarly, by (4.16), we have

$$Z(\mathcal{X}(s),\mathcal{Y}(s)) = \mathcal{Z}(\mathcal{Y}^{-1}(\mathcal{Y}(s)))$$

and, by (4.4c), this implies (4.13) for all $s \in [s_l,s_r]$ such that $\dot{\mathcal{Y}}(s) > 0$. Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$, the set of all $s \in [s_l,s_r]$ such that both $\dot{\mathcal{X}}(s) = 0$ and $\dot{\mathcal{Y}}(s) = 0$ has zero measure. Hence, for almost every $s \in [s_l,s_r]$, (4.13) holds, and since Z is continuous, we get that $Z(\mathcal{X}(s),\mathcal{Y}(s)) = \mathcal{Z}(s)$ for all $s \in [s_l,s_r]$. By (4.26) and (4.27), it follows that

$$p(X,\mathcal{Y}(X)) = \mathfrak{p}(X)$$

for almost every $X \in [X_l,X_r]$ and

$$q(\mathcal{X}(Y),Y) = \mathfrak{q}(Y)$$

for almost every $Y \in [Y_l,Y_r]$, respectively, and we conclude that (4.13) holds. Hence, we have shown that Z is a solution of (2.38a)-(2.38e) which satisfies (4.13) if and only if it is a fixed point for $\mathcal{T}$. Since the fixed point exists and is unique, we have proved the existence and uniqueness of the solution Z to (2.38a)-(2.38e). Furthermore, since the functions p and q, as defined in (4.26) and (4.27), respectively, satisfy Definition 4.5 and (4.13), we have proved the existence of a unique solution (Z,p,q) of (2.38). Next we prove that the solution (Z,p,q) belongs to $\mathcal{H}(\Omega)$. We define the functions $v \in W_Y^{1,\infty}(\Omega)$ and $w \in W_X^{1,\infty}(\Omega)$ as

$$v = x_X - c(U)t_X \quad \text{and} \quad w = x_Y + c(U)t_Y.$$

We want to prove that v and w are both zero. By using the governing equations (2.38), we obtain

$$v_Y = x_{XY} - c'(U)U_Y t_X - c(U)t_{XY} = \frac{c'(U)}{2c(U)}(U_Y v + U_X w)$$

and

$$w_X = x_{XY} + c'(U)U_X t_Y + c(U)t_{XY} = \frac{c'(U)}{2c(U)}(U_Y v + U_X w).$$

By (4.13) and (4.3a), we have $v(X, \mathcal{Y}(X)) = 0$ and $w(\mathcal{X}(Y), Y) = 0$. It follows that

$$v(X,Y) = \int_{\mathcal{Y}(X)}^{Y} \Big(\frac{c'(U)}{2c(U)}(U_Y v + U_X w)\Big)(X, \tilde{Y})\, d\tilde{Y},$$

$$w(X,Y) = \int_{\mathcal{X}(Y)}^{X} \Big(\frac{c'(U)}{2c(U)}(U_Y v + U_X w)\Big)(\tilde{X}, Y)\, d\tilde{X},$$

which implies, by using (4.20) and (4.21), that

$$||v||_{L^\infty_Y(\Omega)} \leq \delta\kappa k_1\big(||U_Y||_{L^\infty_X(\Omega)} + ||U_X||_{L^\infty_Y(\Omega)}\big)\big(||v||_{L^\infty_Y(\Omega)} + ||w||_{L^\infty_X(\Omega)}\big),$$

$$||w||_{L^\infty_X(\Omega)} \leq \delta\kappa k_1\big(||U_Y||_{L^\infty_X(\Omega)} + ||U_X||_{L^\infty_Y(\Omega)}\big)\big(||v||_{L^\infty_Y(\Omega)} + ||w||_{L^\infty_X(\Omega)}\big).$$

Hence, by setting $\delta > 0$ smaller if necessary, we get $||v||_{L^\infty_Y(\Omega)} = ||w||_{L^\infty_X(\Omega)} = 0$. One proceeds similarly in order to prove (4.12b). We show (4.12c). Define $z \in W^{1,\infty}_Y(\Omega)$ as

$$z = 2J_X x_X - (c(U)U_X)^2 - c(U)p^2.$$

We have

$$\begin{aligned} z_Y &= 2J_{XY}x_X + 2J_X x_{XY} - 2c(U)^2 U_X U_{XY} - 2c(U)c'(U)U_Y U_X^2 \\ &\quad - c'(U)U_Y p^2 - 2c(U)pp_Y \\ &= \frac{c'(U)}{c(U)} U_Y z \end{aligned}$$

and by (4.3c), $z(X, \mathcal{Y}(X)) = 0$ for $X \in [X_l, X_r]$, since $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) = \Theta \in \mathcal{G}(\Omega)$. After integrating, we obtain, since $|U_Y| \leq 2|||\Theta|||_{\mathcal{G}(\Omega)}$ and $|Y - \mathcal{Y}(X)| \leq 2\delta$, that

$$|z(X,Y)| \leq |z(X, \mathcal{Y}(X))| e^{4\delta\kappa k_1 |||\Theta|||_{\mathcal{G}(\Omega)}}.$$

Hence, $z = 0$. Similarly, one proves that $2J_Y x_Y = (c(U)U_Y)^2 + c(U)q^2$. Now we prove (4.12d)-(4.12f). Since

$$\frac{1}{x_X + J_X}(X, \mathcal{Y}(X)) = \frac{1}{\mathcal{V}_2 + \mathcal{V}_4}(X)$$

for almost every $X \in [X_l, X_r]$, and since $\mathcal{V}_2$ and $\mathcal{V}_4$ belong to $\mathcal{G}(\Omega)$, we have

$$\Big\|\frac{1}{x_X + J_X}(\cdot, \mathcal{Y}(\cdot))\Big\|_{L^\infty([X_l, X_r])} \leq |||\Theta|||_{\mathcal{G}(\Omega)}.$$

Let $X \in [X_l, X_r]$ such that

$$\frac{1}{x_X + J_X}(X, \mathcal{Y}(X)) \leq |||\Theta|||_{\mathcal{G}(\Omega)},$$

and we define

$$Y_* = \inf\{Y \in [Y_l, Y_r] \mid Y \leq \mathcal{Y}(X) \text{ and } (x_X + J_X)(X, Y') > 0 \text{ for all } Y' > Y\}$$

and

$$Y^* = \sup\{Y \in [Y_l, Y_r] \mid Y \geq \mathcal{Y}(X) \text{ and } (x_X + J_X)(X, Y') > 0 \text{ for all } Y' < Y\}.$$

For $Y \in (Y_*, Y^*)$, we have $(x_X + J_X)(X,Y) > 0$ and we define

$$\eta(Y) = \frac{1}{(x_X + J_X)(X,Y)}.$$

Let us assume that $Y^* < Y_r$. Then

$$(x_X + J_X)(X, Y^*) = 0. \tag{4.28}$$

Since $\eta(Y) \geq 0$ for $Y \in (Y_*, Y^*)$, and $J_X x_X \geq 0$ by (4.12c), we have that

$$x_X(X,Y) \geq 0 \quad \text{and} \quad J_X(X,Y) \geq 0$$

for $Y \in (Y_*, Y^*)$. By (2.38), we have

$$\eta_Y = -\frac{x_{XY} + J_{XY}}{(x_X + J_X)^2} = -\frac{c'(U)}{2c(U)} \frac{U_Y(x_X + J_X) + U_X(x_Y + J_Y)}{(x_X + J_X)^2}$$

and from (4.12c), we obtain

$$|U_X| \leq \frac{1}{c(U)} \sqrt{2J_X x_X} \leq \frac{1}{\sqrt{2}c(U)}(J_X + x_X).$$

Hence,

$$\eta_Y \leq \frac{|c'(U)|}{2c(U)} \Big(|U_Y| + \frac{1}{\sqrt{2}c(U)}(x_Y + J_Y) \Big) \eta \leq C\eta$$

for some constant C which only depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$. From Gronwall's inequality it follows that

$$\frac{1}{x_X + J_X}(X,Y) \leq \frac{1}{\mathcal{V}_2 + \mathcal{V}_4}(X) e^{C|Y - \mathcal{Y}(X)|}, \tag{4.29}$$

which contradicts (4.28), so that we must have $Y^* = Y_r$. In the same way one proves that $Y_* = Y_l$. Hence,

$$x_X(X,Y) \geq 0, \quad J_X(X,Y) \geq 0 \quad \text{and} \quad (x_X + J_X)(X,Y) > 0$$

for almost every $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$. This concludes the proof of the first identities in (4.12d)–(4.12f). The second identities in (4.12d) – (4.12f) can be proved in a similar way. ☐

4.2. **Existence of Local Solutions.** We begin with some a priori estimates.

Given a positive constant L, we call domains of the type

$$\{(X,Y) \in \mathbb{R}^2 \mid |Y - X| \leq 2L\}$$

strip domains, which correspond to domains where time is bounded. We have the following a priori estimates for the solution of (2.38).

Lemma 4.9. *Given* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ *and* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$, *let* $(Z, p, q) \in \mathcal{H}(\Omega)$ *be a solution of (2.38) such that* $\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})$. *Let* $\mathcal{E}_0 = ||\mathcal{Z}_4||_{L^\infty([s_l, s_r])} + ||\mathcal{Z}_5||_{L^\infty([s_l, s_r])}$. *Then the following statements hold:*

(i) Boundedness of the energy, that is,

$$0 \leq J(X,Y) \leq \mathcal{E}_0 \quad \textit{for all } (X,Y) \in \Omega \tag{4.30a}$$

and

$$||K||_{L^\infty(\Omega)} \leq (1+\kappa)\mathcal{E}_0. \tag{4.30b}$$

(ii) The functions Z, Z_X, Z_Y, p and q remain uniformly bounded in strip domains which contain Ω, that is, there exists a nondecreasing function $C_1 = C_1(|||\Theta|||_{\mathcal{G}(\Omega)}, L)$ such that for any $L > 0$ and any $(X,Y) \in \Omega$ such that $|X - Y| \leq 2L$, we have

$$|Z^a(X,Y)| \leq C_1, \quad |Z_X(X,Y)| \leq C_1, \quad |Z_Y(X,Y)| \leq C_1, \tag{4.31a}$$

$$|p(X,Y)| \leq C_1, \quad |q(X,Y)| \leq C_1 \tag{4.31b}$$

and

$$\frac{1}{x_X + J_X}(X,Y) \leq C_1, \quad \frac{1}{x_Y + J_Y}(X,Y) \leq C_1. \tag{4.31c}$$

Condition (ii) *is equivalent to the following:*

(iii) For any curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, we have

$$|||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}(\Omega)} \leq C_1, \tag{4.32}$$

where $C_1 = C_1(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||\Theta|||_{\mathcal{G}(\Omega)})$ is an increasing function with respect to both its arguments.

Proof. Given $P = (X,Y) \in \Omega$, let $s_0 = \mathcal{Y}^{-1}(Y)$ and $s_1 = \mathcal{X}^{-1}(X)$, where we assume that $s_0 \leq s_1$ (the proof for the other case is similar). We denote $P_0 = (\mathcal{X}(s_0), \mathcal{Y}(s_0))$ and $P_1 = (\mathcal{X}(s_1), \mathcal{Y}(s_1))$. Since $\mathcal{X}$ and $\mathcal{Y}$ are increasing functions, we have that $X = \mathcal{X}(s_1) \geq \mathcal{X}(s_0)$ and $Y = \mathcal{Y}(s_0) \leq \mathcal{Y}(s_1)$. Then, because $\mathcal{Z}_4 \geq 0$, $J_X \geq 0$ and $J_Y \geq 0$, we have

$$0 \leq \mathcal{Z}_4(s_0) = J(P_0) \leq J(P) \leq J(P_1) = \mathcal{Z}_4(s_1) \leq \mathcal{E}_0, \tag{4.33}$$

which proves (4.30a). By (4.12b), we have

$$K(P) - K(P_0) = \int_{\mathcal{X}(s_0)}^{X} \left(\frac{J_X}{c(U)}\right)(\tilde{X}, Y)\, d\tilde{X}.$$

Hence,

$$|K(P)| \leq |K(P_0)| + \kappa(J(P) - J(P_0)) \leq (1+\kappa)\mathcal{E}_0$$

by (4.30a). This proves (4.30b). Next we show (4.31a)-(4.31b). Since $x_X \geq 0$, we have

$$x(P) \geq x(P_0) = \mathcal{Z}_2(s_0) = \mathcal{Z}_2^a(s_0) + s_0 \geq -|||\Theta|||_{\mathcal{G}(\Omega)} + s_0$$

and since $\frac{1}{2}(X+Y) = Y + \frac{1}{2}(X-Y) \leq \mathcal{Y}(s_0) + L$, it follows that

$$x(P) - \frac{1}{2}(X+Y) \geq -|||\Theta|||_{\mathcal{G}(\Omega)} + (s_0 - \mathcal{Y}(s_0)) - L \geq -2|||\Theta|||_{\mathcal{G}(\Omega)} - L.$$

Similarly, we find $x(P) \leq |||\Theta|||_{\mathcal{G}(\Omega)} + s_1$ and $\frac{1}{2}(X+Y) \geq \mathcal{X}(s_1) - L$, which implies that

$$x(P) - \frac{1}{2}(X+Y) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + L.$$

Hence,

$$|Z_2^a(P)| = \left|x(P) - \frac{1}{2}(X+Y)\right| \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + L.$$

By (4.12a), we have

$$|t(P)| \leq |t(P_1)| + \int_{Y}^{\mathcal{Y}(s_1)} |t_Y(X, \tilde{Y})|\, d\tilde{Y}$$

$$= |t(P_1)| + \int_Y^{\mathcal{Y}(s_1)} \Big(\frac{x_Y}{c(U)}\Big)(X,\tilde{Y})\,d\tilde{Y}$$
$$\leq |t(P_1)| + \kappa(x(P_1) - x(P))$$

and since

$$x(P_1)-x(P) \leq x(P_1)-x(P_0) = \mathcal{Z}_2^a(s_1)+s_1-\mathcal{Z}_2^a(s_0)-s_0 \leq 2|||\Theta|||_{\mathcal{G}(\Omega)}+s_1-s_0 \tag{4.34}$$

and

$$s_1 - s_0 = (s_1 - \mathcal{X}(s_1)) + (\mathcal{Y}(s_0) - s_0) + (X - Y) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + 2L, \tag{4.35}$$

it follows that

$$|t(P)| = |Z_1(P)| \leq |||\Theta|||_{\mathcal{G}(\Omega)} + \kappa(4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L).$$

Hence,

$$|Z_1^a(P)| = \Big|t(P) - \frac{1}{2c(0)}(X-Y)\Big| \leq |||\Theta|||_{\mathcal{G}(\Omega)} + \kappa(4|||\Theta|||_{\mathcal{G}(\Omega)} + 3L).$$

We have

$$|U(P)| \leq |U(P_1)| + \int_Y^{\mathcal{Y}(s_1)} |U_Y(X,\tilde{Y})|\,d\tilde{Y}.$$

By (4.12c), we have $2J_Yx_Y \geq (c(U)U_Y)^2$, which implies that

$$|U_Y| \leq \frac{\kappa}{\sqrt{2}}(J_Y + x_Y). \tag{4.36}$$

Hence, by (4.33), (4.34) and (4.35), we obtain

$$\begin{aligned} |U(P)| &\leq |U(P_1)| + \frac{\kappa}{\sqrt{2}}(J(P_1) + x(P_1) - J(P) - x(P)) \\ &\leq |||\Theta|||_{\mathcal{G}(\Omega)} + \frac{\kappa}{\sqrt{2}}(\mathcal{E}_0 + 4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L). \end{aligned} \tag{4.37}$$

We prove that Z_X and Z_Y remain bounded. As above, we assume that $Y \leq \mathcal{Y}(X)$. For almost every $X \in [X_l, X_r]$, we have

$$|Z_X(X,Y)| \leq |Z_X(X,\mathcal{Y}(X))| + \int_Y^{\mathcal{Y}(X)} (|t_{XY}|+|x_{XY}|+|U_{XY}|+|J_{XY}|+|K_{XY}|)(X,\tilde{Y})\,d\tilde{Y}.$$

From the governing equations (2.38), we find that

$$|t_{XY}| + |x_{XY}| + |U_{XY}| + |J_{XY}| + |K_{XY}| \leq C|Z_X||Z_Y|$$

for some constant C dependent on Ω, κ, and k_1. By Gronwall's lemma we obtain

$$\begin{aligned} |Z_X(X,Y)| &\leq |Z_X(X,\mathcal{Y}(X))| \exp\Big(\int_Y^{\mathcal{Y}(X)} C|Z_Y(X,\tilde{Y})|\,d\tilde{Y}\Big) \\ &= |\mathcal{V}(X)| \exp\Big(C\int_Y^{\mathcal{Y}(X)} |Z_Y(X,\tilde{Y})|\,d\tilde{Y}\Big). \end{aligned} \tag{4.38}$$

From (3.18), we have

$$|\mathcal{V}(X)| \leq |\mathcal{V}^a(X)| + C \leq |||\Theta|||_{\mathcal{G}(\Omega)} + C$$

for some constant C dependent on Ω, κ, and k_1. Furthermore, using (4.12a), (4.12b) and (4.36), we obtain

$$|Z_Y| = \frac{1}{c(U)}(x_Y + J_Y) + x_Y + J_Y + |U_Y| \leq C(x_Y + J_Y).$$

Hence,

$$\begin{aligned}\int_Y^{\mathcal{Y}(X)} |Z_Y(X,\tilde{Y})|\,d\tilde{Y} &\leq C\int_Y^{\mathcal{Y}(X)} (x_Y + J_Y)(X,\tilde{Y})\,d\tilde{Y}\\ &= C(x(X,\mathcal{Y}(X)) + J(X,\mathcal{Y}(X)) - x(X,Y) - J(X,Y))\\ &\leq C(\mathcal{E}_0 + 4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L),\end{aligned}$$

where we used the same estimate as in (4.37). Combined with (4.38), this yields

$$|Z_X(X,Y)| \leq C_1$$

for some constant C_1 which only depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$ and L. Similarly, one proves the bound on Z_Y. The estimates in (4.31b) follows from the fact that $p_Y = 0$ and $q_X = 0$. Indeed, we have

$$|p(X,Y)| = |p(X,\mathcal{Y}(X))| = |\mathfrak{p}(X)| \leq |||\Theta|||_{\mathcal{G}(\Omega)}$$

and similarly for q. Let us prove (4.31c). In (4.29), we found that

$$\frac{1}{x_X + J_X}(X,Y) \leq \frac{1}{\mathcal{V}_2 + \mathcal{V}_4}(X)e^{C|Y-\mathcal{Y}(X)|} \tag{4.39}$$

for a constant C which only depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$. We have

$$\begin{aligned}|Y - \mathcal{Y}(X)| &= |Y - X + \mathcal{X}(\mathcal{X}^{-1}(X)) - \mathcal{X}^{-1}(X) + \mathcal{X}^{-1}(X) - \mathcal{Y}(\mathcal{X}^{-1}(X))|\\ &\leq L + \|\mathcal{X} - \mathrm{Id}\|_{L^\infty([s_l,s_r])} + \|\mathcal{Y} - \mathrm{Id}\|_{L^\infty([s_l,s_r])}\\ &\leq L + |||\Theta|||_{\mathcal{G}(\Omega)},\end{aligned}$$

which combined with (4.39) yields the first inequality in (4.31c). The other inequality in (4.31c) can be proved in a similar way. We show that the conditions (ii) and (iii) are equivalent. Given a curve $(\bar{\mathcal{X}},\bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, we have, since $\bar{\mathcal{X}} + \bar{\mathcal{Y}} = 2s$, that

$$||\bar{\mathcal{X}} - \bar{\mathcal{Y}}||_{L^\infty} = 2||\bar{\mathcal{X}} - \mathrm{Id}\,||_{L^\infty} = ||\bar{\mathcal{X}} - \mathrm{Id}\,||_{L^\infty} + ||\bar{\mathcal{Y}} - \mathrm{Id}\,||_{L^\infty} = ||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$$

and (4.32) follows. □

We have the following existence and uniqueness result.

Lemma 4.10 (Existence and uniqueness on arbitrarily large rectangles)**.** *Given a rectangular domain* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ *and* $\Theta = (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) \in \mathcal{G}(\Omega)$, *there exists a unique solution* $(Z,p,q) \in \mathcal{H}(\Omega)$ *such that*

$$\Theta = (Z,p,q) \bullet (\mathcal{X},\mathcal{Y}).$$

Proof. Let $\delta = \frac{s_r - s_l}{N}$, where N is an integer that we will specifiy later. For $i = 0,\dots,N$, let $s_i = i\delta + s_l$ and $P_i = (X_i,Y_i) = (\mathcal{X}(s_i),\mathcal{Y}(s_i))$. For $i,j = 0,\dots,N$, we construct a grid which consists of the points $P_{i,j} = (X_{i,j},Y_{i,j})$, where $X_{i,j} = X_i$ and $Y_{i,j} = Y_j$. We denote by $\Omega_{i,j}$ the rectangle with diagonal points $P_{i,j}$ and $P_{i+1,j+1}$, and by Ω_n the rectangle with diagonal points (X_0,Y_0) and (X_n,Y_n). We prove by induction that there exists a unique $(Z,p,q) \in \mathcal{H}(\Omega_n)$ such that $\Theta = (Z,p,q) \bullet (\mathcal{X},\mathcal{Y}) \in \mathcal{G}(\Omega_n)$. There is an increasing function $C = C(|||\Theta|||_{\mathcal{G}(\Omega)})$ such that

$$s_1 - s_0 = \delta \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega)}) \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega_1)}),$$

provided that N is large enough. Hence, by Theorem 4.8, there exists a unique solution $(Z,p,q) \in \mathcal{H}(\Omega_1)$ such that $\Theta = (Z,p,q) \bullet (\mathcal{X},\mathcal{Y}) \in \mathcal{G}(\Omega_1)$. We assume that there exists a unique solution (Z,p,q) on Ω_n and prove that there exists a solution on Ω_{n+1}. By Theorem 4.8, there exists a unique solution on $\Omega_{n,n}$ since

$$s_{n+1} - s_n = \delta \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega)}) \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega_{n,n})}).$$

For $j = n-1, \dots, 0$, we iteratively construct the unique solution in $\Omega_{n,j}$ and $\Omega_{j,n}$ as follows. We only treat the case of $\Omega_{n,j}$. We assume that the solution is known on $\Omega_{n,j+1}$, then we define $\tilde{\Theta} = (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}, \tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}) \in \mathcal{G}(\Omega_{n,j})$ as follows: the curve $(\tilde{\mathcal{X}}(s), \tilde{\mathcal{Y}}(s))$ is given by

$$\tilde{\mathcal{X}}(s) = X_n, \quad \tilde{\mathcal{Y}}(s) = 2s - X_n$$

for $\frac{1}{2}(X_n + Y_j) \leq s \leq \frac{1}{2}(X_n + Y_{j+1})$ and

$$\tilde{\mathcal{X}}(s) = 2s - Y_{j+1}, \quad \tilde{\mathcal{Y}}(s) = Y_{j+1}$$

for $\frac{1}{2}(X_n + Y_{j+1}) \leq s \leq \frac{1}{2}(X_{n+1} + Y_{j+1})$. We set

$$\begin{aligned}
\tilde{Z}(s) &= Z(\tilde{\mathcal{X}}(s), \tilde{\mathcal{Y}}(s)) \quad \text{for } s \in \Big[\frac{1}{2}(X_n + Y_j), \frac{1}{2}(X_{n+1} + Y_{j+1})\Big],\\
\tilde{\mathcal{V}}(X) &= Z_X(X, Y_{j+1}) \quad \text{for a.e. } X \in [X_n, X_{n+1}],\\
\tilde{\mathcal{W}}(Y) &= Z_Y(X_n, Y) \quad \text{for a.e. } Y \in [Y_j, Y_{j+1}],\\
\tilde{\mathfrak{p}}(X) &= p(X, Y_{j+1}) \quad \text{for a.e. } X \in [X_n, X_{n+1}],\\
\tilde{\mathfrak{q}}(Y) &= q(X_n, Y) \quad \text{for a.e. } Y \in [Y_j, Y_{j+1}],
\end{aligned}$$

where (Z,p,q) is the solution on $\Omega_n \cup (\cup_{i=j+1}^n \Omega_{n,i})$. By Lemma 4.9, we have that $|||\tilde{\Theta}|||_{\mathcal{G}(\Omega_{n,j})} \leq C_1(|||\Theta|||_{\mathcal{G}(\Omega)}, L)$. We have

$$\frac{1}{2}(X_{n+1} + Y_{j+1}) - \frac{1}{2}(X_n + Y_j) = \frac{1}{2}(\mathcal{X}(s_{n+1}) - \mathcal{X}(s_n) + \mathcal{Y}(s_{j+1}) - \mathcal{Y}(s_j)) \leq 2\delta,$$

because $\mathcal{X}$ and $\mathcal{Y}$ are Lipschitz continuous with Lipschitz constant smaller than 2. By setting N so large that $2\delta \leq 1/C(C_1)$ it follows that $2\delta \leq 1/C(|||\tilde{\Theta}|||_{\mathcal{G}(\Omega_{n,j})})$, so that we can apply Theorem 4.8 to $\Omega_{n,j}$ and obtain the existence of a unique solution in $\mathcal{H}(\Omega_{n,j})$. Similarly we obtain the existence of a unique solution in $\mathcal{H}(\Omega_{j,n})$. Since

$$\Omega_{n+1} = \Omega_n \cup (\cup_{j=0}^n \Omega_{n,j}) \cup (\cup_{j=0}^{n-1} \Omega_{j,n}),$$

we have proved the existence of a unique solution in Ω_{n+1}. □

Lemma 4.11 (A Gronwall lemma for curves)**.** *Let* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ *and assume that* $(Z,p,q) \in \mathcal{H}(\Omega)$ *and* $(\mathcal{X},\mathcal{Y}) \in \mathcal{C}(\Omega)$*. Then, for any* $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$*, we have*

$$||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega)} \leq C||(Z,p,q) \bullet (\mathcal{X},\mathcal{Y})||_{\mathcal{G}(\Omega)},$$

where $C = C(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||(Z,p,q) \bullet (\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\Omega)})$ *is an increasing function with respect to both its arguments.*

Proof. Note that for any function $f \in W_Y^{1,\infty}(\Omega)$, f_Y is well-defined, but not f_X. This means that the form $f(X,Y)\,dX$ is well-defined, while the form $f(X,Y)\,dY$ is not. Similarly, for any function $g \in W_X^{1,\infty}(\Omega)$, the form $g(X,Y)\,dY$ is well-defined and $g(X,Y)\,dX$ is not. Thus, given $(Z,p,q) \in \mathcal{H}(\Omega)$, we can consider the forms $U^2\,dX$, $U^2\,dY$, $|Z_X^a|^2\,dX$, $|Z_Y^a|^2\,dY$, $p^2\,dX$ and $q^2\,dY$. For any curve $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$ and $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}})$ such that $\bar{\Theta} = (Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$, we have

$$\int_{\bar{\Gamma}} U^2(X,Y)\,dX + U^2(X,Y)\,dY = 2\int_{s_l}^{s_r} \bar{\mathcal{Z}}_3^2(s)\,ds$$

and, since $Z_X^a(X,Y) = Z_X^a(X, \bar{\mathcal{Y}}(X))$ and $Z_Y^a(X,Y) = Z_Y^a(\bar{\mathcal{X}}(Y), Y)$ on $\bar{\Gamma}$, we have

$$\int_{\bar{\Gamma}} |Z_X^a(X,Y)|^2\,dX = \int_{X_l}^{X_r} |\bar{\mathcal{V}}^a(X)|^2\,dX, \quad \int_{\bar{\Gamma}} |Z_Y^a(X,Y)|^2\,dY = \int_{Y_l}^{Y_r} |\bar{\mathcal{W}}^a(Y)|^2\,dY.$$

Similarly, since $p(X,Y) = p(X, \bar{\mathcal{Y}}(X))$ and $q(X,Y) = q(\bar{\mathcal{X}}(Y), Y)$ on $\bar{\Gamma}$, we obtain

$$\int_{\bar{\Gamma}} p(X,Y)^2\,dX = \int_{X_l}^{X_r} \bar{\mathfrak{p}}(X)^2\,dX, \quad \int_{\bar{\Gamma}} q(X,Y)^2\,dY = \int_{Y_l}^{Y_r} \bar{\mathfrak{q}}(Y)^2\,dY.$$

Now we can rewrite

$$\begin{aligned} &||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega)}^2 \\ &\quad = \int_{\bar{\Gamma}} \left(\frac{1}{2}U^2\,(dX + dY) + |Z_X^a|^2\,dX + |Z_Y^a|^2\,dY + p^2\,dX + q^2\,dY\right). \end{aligned} \tag{4.40}$$

Thus, we want to prove that

$$\begin{aligned} &\int_{\bar{\Gamma}} \left(\frac{1}{2}U^2\,(dX + dY) + |Z_X^a|^2\,dX + |Z_Y^a|^2\,dY + p^2\,dX + q^2\,dY\right) \\ &\quad \leq C\int_{\Gamma} \left(\frac{1}{2}U^2\,(dX + dY) + |Z_X^a|^2\,dX + |Z_Y^a|^2\,dY + p^2\,dX + q^2\,dY\right). \end{aligned} \tag{4.41}$$

We decompose the proof into three steps.

Step 1. We first prove that (4.41) holds for small domains. We claim that there exist constants δ and C, which depend on $||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$ and $|||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$, such that for any rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ with $s_r - s_l \leq \delta$, (4.41) holds. By Lemma 4.9, we have

$$||U||_{L^\infty(\Omega)} + ||Z_X^a||_{L^\infty(\Omega)} + ||Z_Y^a||_{L^\infty(\Omega)} + ||p||_{L^\infty(\Omega)} + ||q||_{L^\infty(\Omega)} \leq C,$$

where $C = C(\Omega, |||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ which is increasing with respect to its second argument. Let

$$A = \sup_{\bar{\Gamma}} \int_{\bar{\Gamma}} \left(\frac{1}{2}U^2\,(dX + dY) + |Z_X^a|^2\,dX + |Z_Y^a|^2\,dY + p^2\,dX + q^2\,dY\right),$$

where the supremum is taken over all $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$. We have

$$\begin{aligned} \left(x_X - \frac{1}{2}\right)^2(X,Y) &= \left(x_X - \frac{1}{2}\right)^2(X, \mathcal{Y}(X)) \\ &\quad + \int_{\mathcal{Y}(X)}^{Y} 2\left(\left(x_X - \frac{1}{2}\right)x_{XY}\right)(X, \tilde{Y})\,d\tilde{Y}. \end{aligned} \tag{4.42}$$

Since (Z,p,q) is a solution of (2.38), we have for almost every $X\in[X_l,X_r]$ and all $Y\in[Y_l,Y_r]$, that

$$\begin{aligned}2\Big(x_X-\frac{1}{2}\Big)x_{XY}&=\Big(x_X-\frac{1}{2}\Big)\frac{c'(U)}{c(U)}(U_Yx_X+U_Xx_Y)\\&=\frac{c'(U)}{c(U)}\Big(U_Y\Big(x_X-\frac{1}{2}\Big)^2+\frac{1}{2}U_Y\Big(x_X-\frac{1}{2}\Big)\\&\qquad+U_X\Big(x_Y-\frac{1}{2}\Big)\Big(x_X-\frac{1}{2}\Big)+\frac{1}{2}U_X\Big(x_X-\frac{1}{2}\Big)\Big).\end{aligned}$$

Using (1.5), (1.6) and Young's inequality, we get

$$\begin{aligned}&\Big|2\Big(x_X-\frac{1}{2}\Big)x_{XY}\Big|\\&\le\kappa k_1\Big(||Z^a_Y||_{L^\infty(\Omega)}\Big(x_X-\frac{1}{2}\Big)^2+\frac{1}{2}|U_Y|\Big|x_X-\frac{1}{2}\Big|\\&\qquad+||Z^a_X||_{L^\infty(\Omega)}\Big|x_Y-\frac{1}{2}\Big|\Big|x_X-\frac{1}{2}\Big|+\frac{1}{2}|U_X|\Big|x_X-\frac{1}{2}\Big|\Big)\\&\le\kappa k_1\Big(||Z^a_Y||_{L^\infty(\Omega)}\Big(x_X-\frac{1}{2}\Big)^2+\frac{1}{4}U_Y^2+\frac{1}{4}\Big(x_X-\frac{1}{2}\Big)^2\\&\qquad+\frac{1}{2}||Z^a_X||_{L^\infty(\Omega)}\Big(x_Y-\frac{1}{2}\Big)^2+\frac{1}{2}||Z^a_X||_{L^\infty(\Omega)}\Big(x_X-\frac{1}{2}\Big)^2\\&\qquad+\frac{1}{4}U_X^2+\frac{1}{4}\Big(x_X-\frac{1}{2}\Big)^2\Big)\\&\le\kappa k_1\Big(C\Big(\Big(x_X-\frac{1}{2}\Big)^2+\Big(x_Y-\frac{1}{2}\Big)^2\Big)\\&\qquad+\frac{1}{2}\Big(\Big(x_X-\frac{1}{2}\Big)^2+U_X^2+U_Y^2\Big)\Big)\\&\le\kappa k_1\Big(C+\frac{1}{2}\Big)\big(|Z^a_X|^2+|Z^a_Y|^2\big).\end{aligned}\tag{4.43}$$

Inserting this into (4.42) and integrating over $[X_l,X_r]$ gives

$$\begin{aligned}\int_{\bar\Gamma}\Big(x_X-\frac{1}{2}\Big)^2dX&\le\int_\Gamma\Big(x_X-\frac{1}{2}\Big)^2dX\\&\qquad+\kappa k_1\Big(C+\frac{1}{2}\Big)\int_{X_l}^{X_r}\int_{Y_l}^{Y_r}\big(|Z^a_X|^2+|Z^a_Y|^2\big)\,dY\,dX.\end{aligned}\tag{4.44}$$

For any $Y\in[Y_l,Y_r]$, the integral $\int_{X_l}^{X_r}|Z^a_X|^2(X,Y)\,dX$ can be seen as part of the integral of the form $|Z^a_X|^2\,dX$ on the piecewise linear path going through the points (X_l,Y_l), (X_l,Y), (X_r,Y) and (X_r,Y_r), which implies that $\int_{X_l}^{X_r}|Z^a_X|^2(X,Y)\,dX\le A$. Similarly, for any $X\in[X_l,X_r]$, $\int_{Y_l}^{Y_r}|Z^a_Y|^2(X,Y)\,dY\le A$. Hence, by (4.44),

$$\int_{\bar\Gamma}\big(Z^a_{2,X}\big)^2\,dX\le\int_\Gamma\big(Z^a_{2,X}\big)^2\,dX+\kappa k_1\big(C+\frac{1}{2}\big)A(Y_r-Y_l+X_r-X_l)$$

$$\leq \int_{\Gamma} \left(Z^a_{2,X}\right)^2 dX + 2\delta\kappa k_1 \big(C + \frac{1}{2}\big) A$$

since $Y_r - Y_l + X_r - X_l = 2(s_r - s_l)$. By treating the other components of Z^a_X and Z^a_Y similarly, we get

$$\int_{\bar\Gamma} |Z^a_X|^2 \, dX \leq \int_{\Gamma} |Z^a_X|^2 \, dX + \delta \bar{C} A, \tag{4.45}$$

$$\int_{\bar\Gamma} |Z^a_Y|^2 \, dY \leq \int_{\Gamma} |Z^a_Y|^2 \, dY + \delta \bar{C} A. \tag{4.46}$$

where $\bar{C}$ depends on $||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$, $|||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$, κ and k_1. For almost every $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$, we have $p_Y(X,Y) = 0$, so that $p(X,Y) = p(X, \mathcal{Y}(X))$. By squaring this expression and integrating over $[X_l, X_r]$, we obtain

$$\int_{\bar\Gamma} p^2 \, dX = \int_{\Gamma} p^2 \, dX. \tag{4.47}$$

Since $q_X(X,Y) = 0$ for almost every $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$, we get $q(X,Y) = q(\mathcal{X}(Y), Y)$ which implies, after squaring and integrating over $[Y_l, Y_r]$, that

$$\int_{\bar\Gamma} q^2 \, dY = \int_{\Gamma} q^2 \, dY. \tag{4.48}$$

For U, we have

$$\begin{aligned} U^2(X,Y) &= U^2(X, \mathcal{Y}(X)) + 2 \int_{\mathcal{Y}(X)}^{Y} (U U_Y)(X, \tilde{Y}) \, d\tilde{Y} \\ &\leq U^2(X, \mathcal{Y}(X)) + \left| \int_{\mathcal{Y}(X)}^{Y} U^2(X, \tilde{Y}) \, d\tilde{Y} \right| + \left| \int_{\mathcal{Y}(X)}^{Y} U_Y^2(X, \tilde{Y}) \, d\tilde{Y} \right|. \end{aligned}$$

As above, it follows that

$$\int_{\bar\Gamma} U^2 \, dX \leq \int_{\Gamma} U^2 \, dX + 2\delta A. \tag{4.49}$$

Similarly, we obtain

$$\int_{\bar\Gamma} U^2 \, dY \leq \int_{\Gamma} U^2 \, dY + 2\delta A. \tag{4.50}$$

By adding (4.45) – (4.50), we obtain

$$\begin{aligned} &\int_{\bar\Gamma} \left(\frac{1}{2} U^2 \, (dX + dY) + |Z^a_X|^2 \, dX + |Z^a_Y|^2 \, dY + p^2 \, dX + q^2 \, dY \right) \\ &\leq \int_{\Gamma} \left(\frac{1}{2} U^2 \, (dX + dY) + |Z^a_X|^2 \, dX + |Z^a_Y|^2 \, dY + p^2 \, dX + q^2 \, dY \right) + 2\delta\bar{C}A + 2\delta A, \end{aligned}$$

which yields, after taking the supremum over all curves $\bar\Gamma$,

$$(1 - 2\delta\bar{C} - 2\delta) A \leq \int_{\Gamma} \left(\frac{1}{2} U^2 \, (dX + dY) + |Z^a_X|^2 \, dX + |Z^a_Y|^2 \, dY + p^2 \, dX + q^2 \, dY \right)$$

and (4.41) follows.

Step 2. For an arbitrarily large rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we now prove that (4.41) holds for curves $\bar\Gamma = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$ such that

$$\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) > \mathcal{Y}(s) - \mathcal{X}(s) \quad \text{for all } s \in (s_l, s_r). \tag{4.51}$$

We claim that (4.51) implies that the curve $\bar\Gamma$ lies above Γ and intersects Γ only at the end points. From (4.51) and (4.1c), we find that $\bar{\mathcal{Y}}(s) > \mathcal{Y}(s)$ and $\mathcal{X}(s) > \bar{\mathcal{X}}(s)$. If $\bar{\mathcal{X}}(\bar s) = \mathcal{X}(s)$ for some $\bar s \in (s_l, s_r)$, then $\bar{\mathcal{X}}(\bar s) = \mathcal{X}(s) > \bar{\mathcal{X}}(s)$, so that $\bar s \geq s$ because $\bar{\mathcal{X}}$ is nondecreasing. This implies, since also $\bar{\mathcal{Y}}$ is nondecreasing, that $\mathcal{Y}(s) < \bar{\mathcal{Y}}(s) \leq \bar{\mathcal{Y}}(\bar s)$. Hence, $\bar\Gamma$ is above Γ except at the end points. The proof in the case when $\bar\Gamma$ is below Γ is similar. For some constant $K > 0$ that will be determined later, we have for almost every $X \in [X_l, X_r]$, that

$$\begin{aligned}
&e^{-K(\bar{\mathcal{Y}}(X)-X)}\Big(x_X - \frac{1}{2}\Big)^2(X, \bar{\mathcal{Y}}(X)) - e^{-K(\mathcal{Y}(X)-X)}\Big(x_X - \frac{1}{2}\Big)^2(X, \mathcal{Y}(X)) \\
&= -K\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}\Big(x_X - \frac{1}{2}\Big)^2(X,Y)\, dY \\
&\quad + \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}\Big(2\Big(x_X - \frac{1}{2}\Big)x_{XY}\Big)(X,Y)\, dY.
\end{aligned}$$

By integrating over $[X_l, X_r]$ and using (4.43), we get that

$$\begin{aligned}
&\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}\Big(x_X - \frac{1}{2}\Big)^2(X, \bar{\mathcal{Y}}(X))\, dX - \int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}\Big(x_X - \frac{1}{2}\Big)^2(X, \mathcal{Y}(X))\, dX \\
&\leq -K\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}\Big(x_X - \frac{1}{2}\Big)^2(X,Y)\, dY\, dX \\
&\quad + \kappa k_1\Big(C + \frac{1}{2}\Big)\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}\big(|Z_X^a|^2 + |Z_Y^a|^2\big)(X,Y)\, dY\, dX.
\end{aligned}$$

We treat the other components of Z_X^a in the same way and obtain

$$\begin{aligned}
&\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}|Z_X^a(X, \bar{\mathcal{Y}}(X))|^2\, dX - \int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}|Z_X^a(X, \mathcal{Y}(X))|^2\, dX \\
&\quad\leq -K\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}|Z_X^a(X,Y)|^2\, dY\, dX \\
&\qquad + M\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}\big(|Z_X^a|^2 + |Z_Y^a|^2\big)(X,Y)\, dY\, dX,
\end{aligned} \tag{4.52}$$

where M depends on $||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$, $|||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$, κ and k_1. Similarly, for Z_Y^a, we obtain

$$\begin{aligned}
&\int_{\bar\Gamma} e^{-K(Y-\bar{\mathcal{X}}(Y))}|Z_Y^a(\bar{\mathcal{X}}(Y), Y)|^2\, dY - \int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))}|Z_Y^a(\mathcal{X}(Y), Y)|^2\, dY \\
&\quad\leq -K\int_{Y_l}^{Y_r}\int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)} e^{-K(Y-X)}|Z_Y^a(X,Y)|^2\, dX\, dY \\
&\qquad + M\int_{Y_l}^{Y_r}\int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)} e^{-K(Y-X)}\big(|Z_X^a|^2 + |Z_Y^a|^2\big)(X,Y)\, dX\, dY.
\end{aligned} \tag{4.53}$$

We claim that the sets

$$\mathcal{N}_1 = \{(X,Y) \mid X_l < X < X_r,\ \mathcal{Y}(X) < Y < \bar{\mathcal{Y}}(X)\}$$

and

$$\mathcal{N}_2 = \{(X,Y) \mid Y_l < Y < Y_r,\ \bar{\mathcal{X}}(Y) < X < \mathcal{X}(Y)\}$$

are equal up to a set of zero measure. Let $(X,Y) \in \mathcal{N}_1$ and set $s_1 = \mathcal{X}^{-1}(X)$, $s_2 = \mathcal{Y}^{-1}(Y)$, $s_3 = \bar{\mathcal{Y}}^{-1}(Y)$ and $s_4 = \bar{\mathcal{X}}^{-1}(X)$. We have

$$\mathcal{Y}(s_1) = \mathcal{Y}(X) < Y = \mathcal{Y}(s_2) = \bar{\mathcal{Y}}(s_3) < \bar{\mathcal{Y}}(X) = \bar{\mathcal{Y}}(s_4)$$

so that $s_1 < s_2$ and $s_3 < s_4$. It follows that

$$\bar{\mathcal{X}}(Y) = \bar{\mathcal{X}}(s_3) \leq \bar{\mathcal{X}}(s_4) = X = \mathcal{X}(s_1) \leq \mathcal{X}(s_2) = \mathcal{X}(Y).$$

Hence, $\mathcal{N}_1 \subset \mathcal{N}_2$ up to a set of measure zero. Similarly, one proves the reverse inclusion. Now (4.52) and (4.53) take the form

$$\begin{aligned}(4.54)\quad &\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}|Z_X^a(X,\bar{\mathcal{Y}}(X))|^2\,dX - \int_\Gamma e^{-K(\mathcal{Y}(X)-X)}|Z_X^a(X,\mathcal{Y}(X))|^2\,dX\\ &\leq -K\iint_{\mathcal{N}_1} e^{-K(Y-X)}|Z_X^a(X,Y)|^2\,dX\,dY\\ &\quad + M\iint_{\mathcal{N}_1} e^{-K(Y-X)}\big(|Z_X^a|^2+|Z_Y^a|^2\big)(X,Y)\,dX\,dY\end{aligned}$$

and

$$\begin{aligned}(4.55)\quad &\int_{\bar\Gamma} e^{-K(Y-\bar{\mathcal{X}}(Y))}|Z_Y^a(\bar{\mathcal{X}}(Y),Y)|^2\,dY - \int_\Gamma e^{-K(Y-\mathcal{X}(Y))}|Z_Y^a(\mathcal{X}(Y),Y)|^2\,dY\\ &\leq -K\iint_{\mathcal{N}_1} e^{-K(Y-X)}|Z_Y^a(X,Y)|^2\,dX\,dY\\ &\quad + M\iint_{\mathcal{N}_1} e^{-K(Y-X)}\big(|Z_X^a|^2+|Z_Y^a|^2\big)(X,Y)\,dX\,dY.\end{aligned}$$

A similar computation as above yields

$$\begin{aligned}(4.56)\quad &\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}U^2(X,\bar{\mathcal{Y}}(X))\,dX - \int_\Gamma e^{-K(\mathcal{Y}(X)-X)}U^2(X,\mathcal{Y}(X))\,dX\\ &= \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-KU^2+2UU_Y)(X,Y)\,dX\,dY\\ &\leq \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-KU^2+U^2+U_Y^2)(X,Y)\,dX\,dY\end{aligned}$$

and

$$\begin{aligned}(4.57)\quad &\int_{\bar\Gamma} e^{-K(Y-\bar{\mathcal{X}}(Y))}U^2(\bar{\mathcal{X}}(Y),Y)\,dY - \int_\Gamma e^{-K(Y-\mathcal{X}(Y))}U^2(\mathcal{X}(Y),Y)\,dY\\ &\leq \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-KU^2+U^2+U_X^2)(X,Y)\,dX\,dY.\end{aligned}$$

Furthermore, we have

$$(4.58)\quad \int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}p^2(X,\bar{\mathcal{Y}}(X))\,dX - \int_\Gamma e^{-K(\mathcal{Y}(X)-X)}p^2(X,\mathcal{Y}(X))\,dX$$

$$= -K \iint_{\mathcal{N}_1} e^{-K(Y-X)} p^2(X,Y)\, dX\, dY$$

and

$$\int_{\bar{\Gamma}} e^{-K(Y-\bar{\mathcal{X}}(Y))} q^2(\bar{\mathcal{X}}(Y),Y)\, dY - \int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))} q^2(\mathcal{X}(Y),Y)\, dY \tag{4.59}$$
$$= -K \iint_{\mathcal{N}_1} e^{-K(Y-X)} q^2(X,Y)\, dX\, dY.$$

By combining (4.54)-(4.59), we obtain

$$\begin{aligned}
&\int_{\bar{\Gamma}} e^{-K(\bar{\mathcal{Y}}(X)-X)} \left(\frac{1}{2}U^2 + |Z_X^a|^2 + p^2\right)(X, \bar{\mathcal{Y}}(X))\, dX \\
&\quad + \int_{\bar{\Gamma}} e^{-K(Y-\bar{\mathcal{X}}(Y))} \left(\frac{1}{2}U^2 + |Z_Y^a|^2 + q^2\right)(\bar{\mathcal{X}}(Y), Y)\, dY \\
&\quad - \int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)} \left(\frac{1}{2}U^2 + |Z_X^a|^2 + p^2\right)(X, \mathcal{Y}(X))\, dX \\
&\quad - \int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))} \left(\frac{1}{2}U^2 + |Z_Y^a|^2 + q^2\right)(\mathcal{X}(Y), Y)\, dY \\
&\leq \iint_{\mathcal{N}_1} e^{-K(Y-X)} \Big(-KU^2 + U^2 + \frac{1}{2}U_Y^2 + \frac{1}{2}U_X^2 - K|Z_X^a|^2 - K|Z_Y^a|^2 \\
&\qquad\qquad + 2M|Z_X^a|^2 + 2M|Z_Y^a|^2 - Kp^2 - Kq^2\Big)(X,Y)\, dX\, dY \\
&\leq (2M + 1 - K) \iint_{\mathcal{N}_1} e^{-K(Y-X)} (U^2 + |Z_X^a|^2 + |Z_Y^a|^2 + p^2 + q^2)(X,Y)\, dX\, dY.
\end{aligned}$$

By choosing K so large that the right-hand side is negative and get that

$$\begin{aligned}
&e^{-K||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}} \int_{\bar{\Gamma}} \left(\frac{1}{2}U^2\,(dX+dY) + |Z_X^a|^2\, dX + |Z_Y^a|^2\, dY + p^2\, dX + q^2\, dY\right) \\
&\leq e^{K||(\mathcal{X},\mathcal{Y})||_{\mathcal{C}(\Omega)}} \int_{\Gamma} \left(\frac{1}{2}U^2\,(dX+dY) + |Z_X^a|^2\, dX + |Z_Y^a|^2\, dY + p^2\, dX + q^2\, dY\right)
\end{aligned}$$

and (4.41) follows.

Step 3. Given any rectangle $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we consider a sequence of rectangular domains $\Omega_i = [X_i, X_{i+1}] \times [Y_i, Y_{i+1}]$ for $i = 0, \dots, N-1$ such that X_i and Y_i are increasing, $(X_0, Y_0) = (X_l, Y_l)$, $(X_N, Y_N) = (X_r, Y_r)$, and $(\mathcal{X}, \mathcal{Y}), (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega_i)$ for $s \in [s_i, s_{i+1}]$. We construct the sequence of rectangles such that either $s_{i+1} - s_i \leq \delta$ (and Step 1 applies) or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) > \mathcal{Y}(s) - \mathcal{X}(s)$ or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) < \mathcal{Y}(s) - \mathcal{X}(s)$ for $s \in (s_i, s_{i+1})$ (and Step 2 applies). Then

$$\begin{aligned}
||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||^2_{\mathcal{G}(\Omega)} &= \sum_{i=0}^{N-1} ||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||^2_{\mathcal{G}(\Omega_i)} \\
&\leq \sum_{i=0}^{N-1} C||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})||^2_{\mathcal{G}(\Omega_i)} \\
&= C||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})||^2_{\mathcal{G}(\Omega)}.
\end{aligned}$$

□

Lemma 4.12 (Stability in L^2). *Let $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and assume that (Z, p, q), $(\tilde{Z}, \tilde{p}, \tilde{q}) \in \mathcal{H}(\Omega)$ and $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$. Then, for any $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, we have*

$$||(Z - \tilde{Z}, p - \tilde{p}, q - \tilde{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega)} \leq D||(Z - \tilde{Z}, p - \tilde{p}, q - \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})||_{\mathcal{G}(\Omega)},$$

where $D = D(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}, |||(\tilde{Z}, \tilde{p}, \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ is an increasing function with respect to all its arguments.

Proof. As in the proof of Lemma 4.11, we can consider the forms $(U - \tilde{U})^2\, dX$, $(U - \tilde{U})^2\, dY$, $|Z_X^a - \tilde{Z}_X^a|^2\, dX$, $|Z_Y^a - \tilde{Z}_Y^a|^2\, dY$, $(p - \tilde{p})^2\, dX$ and $(q - \tilde{q})^2\, dY$. For any curve $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, we find

$$\begin{aligned}
&||(Z - \tilde{Z}, p - \tilde{p}, q - \tilde{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||^2_{\mathcal{G}(\Omega)} \\
&= \int_{\bar{\Gamma}} \bigg(\frac{1}{2}(U - \tilde{U})^2\, (dX + dY) + |Z_X^a - \tilde{Z}_X^a|^2\, dX \\
&\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\, dY + (p - \tilde{p})^2\, dX + (q - \tilde{q})^2\, dY \bigg).
\end{aligned}$$

Thus, we want to prove that

$$\begin{aligned}
&\int_{\bar{\Gamma}} \bigg(\frac{1}{2}(U - \tilde{U})^2\, (dX + dY) + |Z_X^a - \tilde{Z}_X^a|^2\, dX \\
&\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\, dY + (p - \tilde{p})^2\, dX + (q - \tilde{q})^2\, dY \bigg) \\
&\leq D \int_{\Gamma} \bigg(\frac{1}{2}(U - \tilde{U})^2\, (dX + dY) + |Z_X^a - \tilde{Z}_X^a|^2\, dX \\
&\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\, dY + (p - \tilde{p})^2\, dX + (q - \tilde{q})^2\, dY \bigg).
\end{aligned} \tag{4.60}$$

We decompose the proof into three steps.

Step 1. We prove that (4.60) holds for small domains. We claim that there exist constants δ and D, which depend on $||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$, $|||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$ and $|||(\tilde{Z}, \tilde{p}, \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$, such that for any rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ with $s_r - s_l \leq \delta$, (4.60) holds. By Lemma 4.9, we have

$$\begin{aligned}
&||U||_{L^\infty(\Omega)} + ||Z_X^a||_{L^\infty(\Omega)} + ||Z_Y^a||_{L^\infty(\Omega)} + ||p||_{L^\infty(\Omega)} + ||q||_{L^\infty(\Omega)} \leq C, \\
&||\tilde{U}||_{L^\infty(\Omega)} + ||\tilde{Z}_X^a||_{L^\infty(\Omega)} + ||\tilde{Z}_Y^a||_{L^\infty(\Omega)} + ||\tilde{p}||_{L^\infty(\Omega)} + ||\tilde{q}||_{L^\infty(\Omega)} \leq \tilde{C}
\end{aligned} \tag{4.61}$$

where $C = C(\Omega, |||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ and $\tilde{C} = \tilde{C}(\Omega, |||(\tilde{Z}, \tilde{p}, \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ are increasing with respect to the second argument. Let

$$\begin{aligned}
A = \sup_{\bar{\Gamma}} &\int_{\bar{\Gamma}} \bigg(\frac{1}{2}(U - \tilde{U})^2\, (dX + dY) + |Z_X^a - \tilde{Z}_X^a|^2\, dX \\
&\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\, dY + (p - \tilde{p})^2\, dX + (q - \tilde{q})^2\, dY \bigg),
\end{aligned}$$

where the supremum is taken over all $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$. We have

$$(x_X - \tilde{x}_X)^2(X, Y) = (x_X - \tilde{x}_X)^2(X, \mathcal{Y}(X)) \tag{4.62}$$

$$+\int_{\mathcal{Y}(X)}^{Y} 2(x_X-\tilde{x}_X)(x_{XY}-\tilde{x}_{XY})(X,\tilde{Y})\,d\tilde{Y}.$$

Since (Z,p,q) is a solution of (2.38), we have for almost every $X\in[X_l,X_r]$ and all $Y\in[Y_l,Y_r]$, that

$$\begin{aligned}
x_{XY}-\tilde{x}_{XY} &= \frac{c'(U)}{2c(U)}(U_Yx_X+U_Xx_Y)-\frac{c'(\tilde{U})}{2c(\tilde{U})}(\tilde{U}_Y\tilde{x}_X+\tilde{U}_X\tilde{x}_Y)\\
&= \frac{c'(U)}{2c(U)}(U_Yx_X+U_Xx_Y-\tilde{U}_Y\tilde{x}_X-\tilde{U}_X\tilde{x}_Y)\\
&\quad+\left(\frac{c'(U)}{2c(U)}-\frac{c'(\tilde{U})}{2c(\tilde{U})}\right)(\tilde{U}_Y\tilde{x}_X+\tilde{U}_X\tilde{x}_Y)\\
&= \frac{c'(U)}{2c(U)}\bigg(U_Y(x_X-\tilde{x}_X)+\left(\left(\tilde{x}_X-\frac{1}{2}\right)+\frac{1}{2}\right)(U_Y-\tilde{U}_Y)\\
&\qquad\qquad+U_X(x_Y-\tilde{x}_Y)+\left(\left(\tilde{x}_Y-\frac{1}{2}\right)+\frac{1}{2}\right)(U_X-\tilde{U}_X)\bigg)\\
&\quad+\frac{1}{2}\left(\tilde{U}_Y\left(\left(\tilde{x}_X-\frac{1}{2}\right)+\frac{1}{2}\right)+\tilde{U}_X\left(\left(\tilde{x}_Y-\frac{1}{2}\right)+\frac{1}{2}\right)\right)\\
&\qquad\times\int_{\tilde{U}}^{U}\left(\frac{c''(u)c(u)-c'(u)^2}{c(u)^2}\right)du.
\end{aligned}$$

Using (1.5), (1.6) and (4.61), this implies that

$$\begin{aligned}
&\big|2(x_X-\tilde{x}_X)(x_{XY}-\tilde{x}_{XY})\big|\\
&\le \kappa k_1\bigg(||Z_Y^a||_{L^\infty(\Omega)}(x_X-\tilde{x}_X)^2+\left(||\tilde{Z}_X^a||_{L^\infty(\Omega)}+\frac{1}{2}\right)|U_Y-\tilde{U}_Y||x_X-\tilde{x}_X|\\
&\qquad+||Z_X^a||_{L^\infty(\Omega)}|x_Y-\tilde{x}_Y||x_X-\tilde{x}_X|+\left(||\tilde{Z}_Y^a||_{L^\infty(\Omega)}+\frac{1}{2}\right)|U_X-\tilde{U}_X||x_X-\tilde{x}_X|\bigg)\\
&\quad+\left(||\tilde{Z}_Y^a||_{L^\infty(\Omega)}\left(||\tilde{Z}_X^a||_{L^\infty(\Omega)}+\frac{1}{2}\right)+||\tilde{Z}_X^a||_{L^\infty(\Omega)}\left(||\tilde{Z}_Y^a||_{L^\infty(\Omega)}+\frac{1}{2}\right)\right)\\
&\qquad\times(\kappa k_2+\kappa^2k_1^2)|U-\tilde{U}||x_X-\tilde{x}_X|\\
&\le \kappa k_1\left(C+\tilde{C}+\frac{1}{2}\right)\bigg((x_X-\tilde{x}_X)^2+|U_Y-\tilde{U}_Y||x_X-\tilde{x}_X|\\
&\qquad+|x_Y-\tilde{x}_Y||x_X-\tilde{x}_X|+|U_X-\tilde{U}_X||x_X-\tilde{x}_X|\bigg)\\
&\quad+2\left(\tilde{C}+\frac{1}{2}\right)^2(\kappa k_2+\kappa^2k_1^2)|U-\tilde{U}||x_X-\tilde{x}_X|\\
&\le \kappa k_1\left(C+\tilde{C}+\frac{1}{2}\right)\bigg((x_X-\tilde{x}_X)^2+\frac{1}{2}(U_Y-\tilde{U}_Y)^2+\frac{1}{2}(x_X-\tilde{x}_X)^2\\
&\qquad+\frac{1}{2}(x_Y-\tilde{x}_Y)^2+\frac{1}{2}(x_X-\tilde{x}_X)^2+\frac{1}{2}(U_X-\tilde{U}_X)^2+\frac{1}{2}(x_X-\tilde{x}_X)^2\bigg)
\end{aligned}$$

$$+ 2\Big(\tilde{C} + \frac{1}{2}\Big)^2 (\kappa k_2 + \kappa^2 k_1^2)\Big(\frac{1}{2}(U - \tilde{U})^2 + \frac{1}{2}(x_X - \tilde{x}_X)^2\Big)$$
$$\leq \frac{5}{2}\kappa k_1\Big(C + \tilde{C} + \frac{1}{2}\Big)\Big((x_X - \tilde{x}_X)^2 + (U_Y - \tilde{U}_Y)^2 + (x_Y - \tilde{x}_Y)^2 + (U_X - \tilde{U}_X)^2\Big)$$
$$+ \Big(\tilde{C} + \frac{1}{2}\Big)^2 (\kappa k_2 + \kappa^2 k_1^2)\Big((U - \tilde{U})^2 + (x_X - \tilde{x}_X)^2\Big).$$

We set

$$m = \max\Big\{\frac{5}{2}\kappa k_1\Big(C + \tilde{C} + \frac{1}{2}\Big), \Big(\tilde{C} + \frac{1}{2}\Big)^2 (\kappa k_2 + \kappa^2 k_1^2)\Big\}$$

and get

$$\begin{aligned}
&\big|2(x_X - \tilde{x}_X)(x_{XY} - \tilde{x}_{XY})\big| \\
&\leq m\big((U_Y - \tilde{U}_Y)^2 + (x_Y - \tilde{x}_Y)^2 + (U_X - \tilde{U}_X)^2 + (U - \tilde{U})^2\big) \\
&\quad + 2m(x_X - \tilde{x}_X)^2 \\
&\leq 2m\big((U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2 + |Z_Y^a - \tilde{Z}_Y^a|^2\big).
\end{aligned} \tag{4.63}$$

We insert this into (4.62) and get

$$(x_X - \tilde{x}_X)^2(X,Y) \leq (x_X - \tilde{x}_X)^2(X, \mathcal{Y}(X)) + 2m\int_{Y_l}^{Y_r} \big((U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2 + |Z_Y^a - \tilde{Z}_Y^a|^2\big)(X, \tilde{Y})\, d\tilde{Y}$$

which, after integrating over $[X_l, X_r]$, yields

$$\int_{\bar{\Gamma}} (x_X - \tilde{x}_X)^2\, dX \leq \int_{\Gamma} (x_X - \tilde{x}_X)^2\, dX + 2m\int_{X_l}^{X_r}\int_{Y_l}^{Y_r} \big((U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2 + |Z_Y^a - \tilde{Z}_Y^a|^2\big)\, dY\, dX.$$

For any $Y \in [Y_l, Y_r]$, $\int_{X_l}^{X_r} \big(\frac{1}{2}(U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2\big)(X,Y)\, dX$ can be seen as part of the integral of the form $\big(\frac{1}{2}(U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2\big)\, dX$ on the piecewise linear path going through the points (X_l, Y_l), (X_l, Y), (X_r, Y) and (X_r, Y_r). This implies that $\int_{X_l}^{X_r} \big(\frac{1}{2}(U - \tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2\big)(X,Y)\, dX \leq A$. Similarly, for any $X \in [X_l, X_r]$, we get $\int_{Y_l}^{Y_r} \big(\frac{1}{2}(U - \tilde{U})^2 + |Z_Y^a - \tilde{Z}_Y^a|^2\big)(X,Y)\, dY \leq A$. Hence,

$$\begin{aligned}
\int_{\bar{\Gamma}} \big(Z_{2,X}^a - \tilde{Z}_{2,X}^a\big)^2\, dX &\leq \int_{\Gamma} \big(Z_{2,X}^a - \tilde{Z}_{2,X}^a\big)^2\, dX + 2mA(Y_r - Y_l + X_r - X_l) \\
&\leq \int_{\Gamma} \big(Z_{2,X}^a - \tilde{Z}_{2,X}^a\big)^2\, dX + 4\delta m A
\end{aligned}$$

because $Y_r - Y_l + X_r - X_l = 2(s_r - s_l)$. By treating the other components of $Z_X^a - \tilde{Z}_X^a$ and $Z_Y^a - \tilde{Z}_Y^a$ similarly, we get

$$\int_{\bar{\Gamma}} |Z_X^a - \tilde{Z}_X^a|^2\, dX \leq \int_{\Gamma} |Z_X^a - \tilde{Z}_X^a|^2\, dX + MA\delta, \tag{4.64}$$

$$\int_{\bar{\Gamma}} |Z_Y^a - \tilde{Z}_Y^a|^2\, dY \leq \int_{\Gamma} |Z_Y^a - \tilde{Z}_Y^a|^2\, dY + MA\delta, \tag{4.65}$$

where M depends on $||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$, $|||(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\Omega)}$, $|||(\tilde{Z},\tilde{p},\tilde{q})\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\Omega)}$, κ, k_1 and k_2. For almost every $X\in[X_l,X_r]$ and all $Y\in[Y_l,Y_r]$, we have $p_Y(X,Y)=0$ and $\tilde{p}_Y(X,Y)=0$, so that $p(X,Y)-\tilde{p}(X,Y)=p(X,\mathcal{Y}(X))-\tilde{p}(X,\mathcal{Y}(X))$. By squaring this expression and integrating over $[X_l,X_r]$, we obtain

$$\int_{\bar{\Gamma}}(p-\tilde{p})^2\,dX=\int_{\Gamma}(p-\tilde{p})^2\,dX. \tag{4.66}$$

Since $q_X(X,Y)=0$ and $\tilde{q}_X(X,Y)=0$ for almost every $Y\in[Y_l,Y_r]$ and all $X\in[X_l,X_r]$, we get $q(X,Y)-\tilde{q}(X,Y)=q(\mathcal{X}(Y),Y)-\tilde{q}(\mathcal{X}(Y),Y)$ which implies, after squaring and integrating over $[Y_l,Y_r]$, that

$$\int_{\bar{\Gamma}}(q-\tilde{q})^2\,dY=\int_{\Gamma}(q-\tilde{q})^2\,dY. \tag{4.67}$$

We have

$$\begin{aligned}
&(U-\tilde{U})^2(X,Y)\\
&=(U-\tilde{U})^2(X,\mathcal{Y}(X))+2\int_{\mathcal{Y}(X)}^{Y}(U-\tilde{U})(U_Y-\tilde{U}_Y)(X,\tilde{Y})\,d\tilde{Y}\\
&\leq(U-\tilde{U})^2(X,\mathcal{Y}(X))+\int_{\mathcal{Y}(X)}^{Y}(U-\tilde{U})^2(X,\tilde{Y})\,d\tilde{Y}+\int_{\mathcal{Y}(X)}^{Y}(U_Y-\tilde{U}_Y)^2(X,\tilde{Y})\,d\tilde{Y}.
\end{aligned}$$

As above, it follows that

$$\int_{\bar{\Gamma}}(U-\tilde{U})^2\,dX\leq\int_{\Gamma}(U-\tilde{U})^2\,dX+2A\delta. \tag{4.68}$$

Similarly, we obtain

$$\int_{\bar{\Gamma}}(U-\tilde{U})^2\,dY\leq\int_{\Gamma}(U-\tilde{U})^2\,dY+2A\delta. \tag{4.69}$$

By adding (4.64)-(4.69), we obtain

$$\begin{aligned}
&\int_{\bar{\Gamma}}\bigg(\frac{1}{2}(U-\tilde{U})^2\,(dX+dY)+|Z_X^a-\tilde{Z}_X^a|^2\,dX\\
&\qquad+|Z_Y^a-\tilde{Z}_Y^a|^2\,dY+(p-\tilde{p})^2\,dX+(q-\tilde{q})^2\,dY\bigg)\\
&\leq\int_{\Gamma}\bigg(\frac{1}{2}(U-\tilde{U})^2\,(dX+dY)+|Z_X^a-\tilde{Z}_X^a|^2\,dX\\
&\qquad+|Z_Y^a-\tilde{Z}_Y^a|^2\,dY+(p-\tilde{p})^2\,dX+(q-\tilde{q})^2\,dY\bigg)+2\delta MA+2\delta A
\end{aligned}$$

which yields, after taking the supremum over all curves $\bar{\Gamma}$,

$$\begin{aligned}
(1-2\delta M-2\delta)A\leq\int_{\Gamma}\bigg(&\frac{1}{2}(U-\tilde{U})^2\,(dX+dY)+|Z_X^a-\tilde{Z}_X^a|^2\,dX\\
&+|Z_Y^a-\tilde{Z}_Y^a|^2\,dY+(p-\tilde{p})^2\,dX+(q-\tilde{q})^2\,dY\bigg)
\end{aligned}$$

and (4.60) follows.

Step 2. For an arbitrarily large rectangular domain $\Omega=[X_l,X_r]\times[Y_l,Y_r]$, we now prove that (4.60) holds for curves $\bar\Gamma=(\bar{\mathcal{X}},\bar{\mathcal{Y}})\in\mathcal{C}(\Omega)$ such that

$$\bar{\mathcal{Y}}(s)-\bar{\mathcal{X}}(s)>\mathcal{Y}(s)-\mathcal{X}(s)\quad\text{for all } s\in(s_l,s_r),$$

that is, the curve $\bar\Gamma$ lies above Γ and intersects Γ only at the end points, as in the proof of Lemma 4.11. The proof in the case when $\bar\Gamma$ is below Γ is similar. For a constant $K>0$ that will be determined later, we have for almost every $X\in[X_l,X_r]$, that

$$\begin{aligned}
&e^{-K(\bar{\mathcal{Y}}(X)-X)}(x_X-\tilde{x}_X)^2(X,\bar{\mathcal{Y}}(X))-e^{-K(\mathcal{Y}(X)-X)}(x_X-\tilde{x}_X)^2(X,\mathcal{Y}(X))\\
&=-K\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}(x_X-\tilde{x}_X)^2(X,Y)\,dY\\
&\quad+\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}(2(x_X-\tilde{x}_X)(x_{XY}-\tilde{x}_{XY}))(X,Y)\,dY.
\end{aligned}$$

We integrate over $[X_l,X_r]$ and get, by using (4.63), that

$$\begin{aligned}
&\int_{\bar\Gamma}e^{-K(\bar{\mathcal{Y}}(X)-X)}(x_X-\tilde{x}_X)^2(X,\bar{\mathcal{Y}}(X))\,dX-\int_{\Gamma}e^{-K(\mathcal{Y}(X)-X)}(x_X-\tilde{x}_X)^2(X,\mathcal{Y}(X))\,dX\\
&\leq -K\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}(x_X-\tilde{x}_X)^2(X,Y)\,dY\,dX\\
&\quad+2m\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}\big((U-\tilde{U})^2+|Z_X^a-\tilde{Z}_X^a|^2+|Z_Y^a-\tilde{Z}_Y^a|^2\big)(X,Y)\,dY\,dX.
\end{aligned}$$

By treating the other components of $Z_X^a-\tilde{Z}_X^a$ in the same way, we obtain

$$\begin{aligned}
&\int_{\bar\Gamma}e^{-K(\bar{\mathcal{Y}}(X)-X)}|Z_X^a-\tilde{Z}_X^a|^2(X,\bar{\mathcal{Y}}(X))\,dX-\int_{\Gamma}e^{-K(\mathcal{Y}(X)-X)}|Z_X^a-\tilde{Z}_X^a|^2(X,\mathcal{Y}(X))\,dX\\
&\leq -K\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}|Z_X^a-\tilde{Z}_X^a|^2(X,Y)\,dY\,dX\\
&\quad+M\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)}e^{-K(Y-X)}\big((U-\tilde{U})^2+|Z_X^a-\tilde{Z}_X^a|^2+|Z_Y^a-\tilde{Z}_Y^a|^2\big)(X,Y)\,dY\,dX
\end{aligned}$$

where M depends on $||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}$, $|||(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\Omega)}$, $|||(\tilde{Z},\tilde{p},\tilde{q})\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\Omega)}$, κ, k_1 and k_2. Similarly, for $Z_Y^a-\tilde{Z}_Y^a$, we get

$$\begin{aligned}
&\int_{\bar\Gamma}e^{-K(Y-\bar{\mathcal{X}}(Y))}|Z_Y^a-\tilde{Z}_Y^a|^2(\bar{\mathcal{X}}(Y),Y)\,dY-\int_{\Gamma}e^{-K(Y-\mathcal{X}(Y))}|Z_Y^a-\tilde{Z}_Y^a|^2(\mathcal{X}(Y),Y)\,dY\\
&\leq -K\int_{Y_l}^{Y_r}\int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)}e^{-K(Y-X)}|Z_Y^a-\tilde{Z}_Y^a|^2(X,Y)\,dX\,dY\\
&\quad+M\int_{Y_l}^{Y_r}\int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)}e^{-K(Y-X)}\big((U-\tilde{U})^2+|Z_X^a-\tilde{Z}_X^a|^2+|Z_Y^a-\tilde{Z}_Y^a|^2\big)(X,Y)\,dX\,dY.
\end{aligned}$$

In the proof of Lemma 4.11, we showed that the sets

$$\mathcal{N}_1=\{(X,Y)\mid X_l<X<X_r,\ \mathcal{Y}(X)<Y<\bar{\mathcal{Y}}(X)\}$$

and

$$\mathcal{N}_2 = \{(X,Y) \mid Y_l < Y < Y_r,\ \bar{\mathcal{X}}(Y) < X < \mathcal{X}(Y)\}$$

are equal up to a set of zero measure. Hence, we get

$$\begin{aligned}
&\int_{\bar{\Gamma}} e^{-K(\bar{\mathcal{Y}}(X)-X)}|Z_X^a - \tilde{Z}_X^a|^2(X,\bar{\mathcal{Y}}(X))\,dX \\
&- \int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}|Z_X^a - \tilde{Z}_X^a|^2(X,\mathcal{Y}(X))\,dX \\
&\leq -K \iint_{\mathcal{N}_1} e^{-K(Y-X)}|Z_X^a - \tilde{Z}_X^a|^2(X,Y)\,dX\,dY \\
&\quad + M \iint_{\mathcal{N}_1} e^{-K(Y-X)}\big((U-\tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2 \\
&\qquad\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\big)(X,Y)\,dX\,dY
\end{aligned} \tag{4.70}$$

and

$$\begin{aligned}
&\int_{\bar{\Gamma}} e^{-K(Y-\bar{\mathcal{X}}(Y))}|Z_Y^a - \tilde{Z}_Y^a|^2(\bar{\mathcal{X}}(Y),Y)\,dY \\
&- \int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))}|Z_Y^a - \tilde{Z}_Y^a|^2(\mathcal{X}(Y),Y)\,dY \\
&\leq -K \iint_{\mathcal{N}_1} e^{-K(Y-X)}|Z_Y^a - \tilde{Z}_Y^a|^2(X,Y)\,dX\,dY \\
&\quad + M \iint_{\mathcal{N}_1} e^{-K(Y-X)}\big((U-\tilde{U})^2 + |Z_X^a - \tilde{Z}_X^a|^2 \\
&\qquad\qquad + |Z_Y^a - \tilde{Z}_Y^a|^2\big)(X,Y)\,dX\,dY.
\end{aligned} \tag{4.71}$$

A similar computation as above yields

$$\begin{aligned}
&\int_{\bar{\Gamma}} e^{-K(\bar{\mathcal{Y}}(X)-X)}(U-\tilde{U})^2(X,\bar{\mathcal{Y}}(X))\,dX \\
&- \int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}(U-\tilde{U})^2(X,\mathcal{Y}(X))\,dX \\
&= \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-K(U-\tilde{U})^2 + 2(U-\tilde{U})(U_Y-\tilde{U}_Y))(X,Y)\,dX\,dY \\
&\leq \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-K(U-\tilde{U})^2 + (U-\tilde{U})^2 + (U_Y-\tilde{U}_Y)^2)(X,Y)\,dX\,dY
\end{aligned} \tag{4.72}$$

and

$$\begin{aligned}
&\int_{\bar{\Gamma}} e^{-K(Y-\bar{\mathcal{X}}(Y))}(U-\tilde{U})^2(\bar{\mathcal{X}}(Y),Y)\,dY \\
&- \int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))}(U-\tilde{U})^2(\mathcal{X}(Y),Y)\,dY \\
&\leq \iint_{\mathcal{N}_1} e^{-K(Y-X)}(-K(U-\tilde{U})^2 + (U-\tilde{U})^2 + (U_X-\tilde{U}_X)^2)(X,Y)\,dX\,dY.
\end{aligned} \tag{4.73}$$

Furthermore, we have

$$\begin{aligned}(4.74)\qquad &\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}(p-\tilde p)^2(X,\bar{\mathcal{Y}}(X))\,dX \\ &-\int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}(p-\tilde p)^2(X,\mathcal{Y}(X))\,dX \\ &= -K\iint_{\mathcal{N}_1} e^{-K(Y-X)}(p-\tilde p)^2(X,Y)\,dX\,dY\end{aligned}$$

and

$$\begin{aligned}(4.75)\qquad &\int_{\bar\Gamma} e^{-K(Y-\bar{\mathcal{X}}(Y))}(q-\tilde q)^2(\bar{\mathcal{X}}(Y),Y)\,dY \\ &-\int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))}(q-\tilde q)^2(\mathcal{X}(Y),Y)\,dY \\ &= -K\iint_{\mathcal{N}_1} e^{-K(Y-X)}(q-\tilde q)^2(X,Y)\,dX\,dY.\end{aligned}$$

Combining (4.70)-(4.75), we obtain

$$\begin{aligned}&\int_{\bar\Gamma} e^{-K(\bar{\mathcal{Y}}(X)-X)}\Big(\frac12(U-\tilde U)^2+|Z_X^a-\tilde Z_X^a|^2+(p-\tilde p)^2\Big)(X,\bar{\mathcal{Y}}(X))\,dX \\ &\quad+\int_{\bar\Gamma} e^{-K(Y-\bar{\mathcal{X}}(Y))}\Big(\frac12(U-\tilde U)^2+|Z_Y^a-\tilde Z_Y^a|^2+(q-\tilde q)^2\Big)(\bar{\mathcal{X}}(Y),Y)\,dY \\ &\quad-\int_{\Gamma} e^{-K(\mathcal{Y}(X)-X)}\Big(\frac12(U-\tilde U)^2+|Z_X^a-\tilde Z_X^a|^2+(p-\tilde p)^2\Big)(X,\mathcal{Y}(X))\,dX \\ &\quad-\int_{\Gamma} e^{-K(Y-\mathcal{X}(Y))}\Big(\frac12(U-\tilde U)^2+|Z_Y^a-\tilde Z_Y^a|^2+(q-\tilde q)^2\Big)(\mathcal{X}(Y),Y)\,dY \\ &\le \iint_{\mathcal{N}_1} e^{-K(Y-X)}\Big(-K(U-\tilde U)^2+(U-\tilde U)^2+\frac12(U_Y-\tilde U_Y)^2 \\ &\qquad\qquad+\frac12(U_X-\tilde U_X)^2-K|Z_X^a-\tilde Z_X^a|^2-K|Z_Y^a-\tilde Z_Y^a|^2 \\ &\qquad\qquad+2M(U-\tilde U)^2+2M|Z_X^a-\tilde Z_X^a|^2+2M|Z_Y^a-\tilde Z_Y^a|^2 \\ &\qquad\qquad-K(p-\tilde p)^2-K(q-\tilde q)^2\Big)(X,Y)\,dX\,dY \\ &\le (2M+1-K)\iint_{\mathcal{N}_1} e^{-K(Y-X)}\Big((U-\tilde U)^2+|Z_X^a-\tilde Z_X^a|^2+|Z_Y^a-\tilde Z_Y^a|^2 \\ &\qquad\qquad+(p-\tilde p)^2+(q-\tilde q)^2\Big)(X,Y)\,dX\,dY.\end{aligned}$$

We choose K so large that the right-hand side becomes negative. This implies that

$$\begin{aligned}e^{-K\|(\bar{\mathcal{X}},\bar{\mathcal{Y}})\|_{\mathcal{C}(\Omega)}}\int_{\bar\Gamma}\Big(&\frac12(U-\tilde U)^2\,(dX+dY)+|Z_X^a-\tilde Z_X^a|^2\,dX \\ &+|Z_Y^a-\tilde Z_Y^a|^2\,dY+(p-\tilde p)^2\,dX+(q-\tilde q)^2\,dY\Big)\end{aligned}$$

$$\leq e^{K||(\mathcal{X},\mathcal{Y})||_{\mathcal{C}(\Omega)}} \int_\Gamma \bigg(\frac{1}{2}(U-\tilde{U})^2\,(dX+dY) + |Z^a_X - \tilde{Z}^a_X|^2\,dX + |Z^a_Y - \tilde{Z}^a_Y|^2\,dY + (p-\tilde{p})^2\,dX + (q-\tilde{q})^2\,dY\bigg)$$

and (4.60) follows.

Step 3. Given any rectangle $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we consider a sequence of rectangular domains $\Omega_i = [X_i, X_{i+1}] \times [Y_i, Y_{i+1}]$ for $i = 0, \dots, N-1$ such that X_i and Y_i are increasing, $(X_0, Y_0) = (X_l, Y_l)$, $(X_N, Y_N) = (X_r, Y_r)$, and $(\mathcal{X},\mathcal{Y}), (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega_i)$ for $s \in [s_i, s_{i+1}]$. We construct the sequence of rectangles such that either $s_{i+1} - s_i \leq \delta$ (and Step 1 applies) or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) > \mathcal{Y}(s) - \mathcal{X}(s)$ or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) < \mathcal{Y}(s) - \mathcal{X}(s)$ for $s \in (s_i, s_{i+1})$ (and Step 2 applies). Then

$$\begin{aligned} ||(Z-\tilde{Z}, p-\tilde{p}, q-\tilde{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||^2_{\mathcal{G}(\Omega)} &= \sum_{i=0}^{N-1} ||(Z-\tilde{Z}, p-\tilde{p}, q-\tilde{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||^2_{\mathcal{G}(\Omega_i)} \\ &\leq \sum_{i=0}^{N-1} D||(Z-\tilde{Z}, p-\tilde{p}, q-\tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})||^2_{\mathcal{G}(\Omega_i)} \\ &= D||(Z-\tilde{Z}, p-\tilde{p}, q-\tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})||^2_{\mathcal{G}(\Omega)}. \end{aligned}$$

□

4.3. **Existence of Global Solutions in $\mathcal{H}$.**

Definition 4.13 (Global solutions)**.** *Let $\mathcal{H}$ be the set of all functions (Z, p, q) such that*

(i) $(Z, p, q) \in \mathcal{H}(\Omega)$ for all rectangular domains Ω;
(ii) there exists a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$.

The following lemma shows that condition (ii) does not depend on the particular curve for which it holds. In particular, we can replace this condition by the requirement that $(Z, p, q) \bullet (\mathcal{X}_d, \mathcal{Y}_d) \in \mathcal{G}$ for the diagonal, which is given by $\mathcal{X}_d(s) = \mathcal{Y}_d(s) = s$.

Lemma 4.14. *Given $(Z, p, q) \in \mathcal{H}$, we have $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$ for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. Moreover, the limit $\lim_{s\to\infty} J(\mathcal{X}(s), \mathcal{Y}(s))$ is independent of the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$.*

Proof. Since $(Z, p, q) \in \mathcal{H}$, we know that there exists a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$. We have to check that the conditions (i)-(v) of Definition 3.6 are satisfied for $\bar{\Theta} = (Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. For any curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$, we have to prove that

$$|||(Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}} < \infty \quad \text{and} \quad ||(Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}} < \infty. \tag{4.76}$$

For any positive number $\bar{s}$, we denote $\Omega_{\bar{s}} = [\bar{\mathcal{X}}(-\bar{s}), \bar{\mathcal{X}}(\bar{s})] \times [\bar{\mathcal{Y}}(-\bar{s}), \bar{\mathcal{Y}}(\bar{s})]$. Let

$$s_{\max} = \begin{cases} \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(\bar{s})) & \text{if } \mathcal{Y}(\mathcal{X}^{-1}(\bar{\mathcal{X}}(\bar{s}))) \leq \bar{\mathcal{Y}}(\bar{s}) \\ \mathcal{X}^{-1}(\bar{\mathcal{X}}(\bar{s})) & \text{otherwise} \end{cases}$$

and

$$s_{\min} = \begin{cases} \mathcal{X}^{-1}(\bar{\mathcal{X}}(-\bar{s})) & \text{if } \bar{\mathcal{X}}(-\bar{s}) \leq \mathcal{X}(\mathcal{Y}^{-1}(\bar{\mathcal{Y}}(-\bar{s}))) \\ \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(-\bar{s})) & \text{otherwise.} \end{cases}$$

We claim that $s_{\min} \leq -\bar{s} \leq \bar{s} \leq s_{\max}$.

If $\mathcal{Y}(\mathcal{X}^{-1}(\bar{\mathcal{X}}(\bar{s}))) \leq \bar{\mathcal{Y}}(\bar{s})$, then $\mathcal{X}^{-1}(\bar{\mathcal{X}}(\bar{s})) \leq \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(\bar{s})) = s_{\max}$, so that $\bar{\mathcal{X}}(\bar{s}) \leq \mathcal{X}(s_{\max})$. This implies, since $\mathcal{Y}(s_{\max}) = \bar{\mathcal{Y}}(\bar{s})$, that

$$2\bar{s} - \bar{\mathcal{Y}}(\bar{s}) \leq 2s_{\max} - \mathcal{Y}(s_{\max}) = 2s_{\max} - \bar{\mathcal{Y}}(\bar{s})$$

and we conclude that $\bar{s} \leq s_{\max}$.

If $\bar{\mathcal{X}}(-\bar{s}) \leq \mathcal{X}(\mathcal{Y}^{-1}(\bar{\mathcal{Y}}(-\bar{s})))$, then $s_{\min} = \mathcal{X}^{-1}(\bar{\mathcal{X}}(-\bar{s})) \leq \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(-\bar{s}))$ and we get $\mathcal{Y}(s_{\min}) \leq \bar{\mathcal{Y}}(-\bar{s})$, which implies, since $\mathcal{X}(s_{\min}) = \bar{\mathcal{X}}(-\bar{s})$, that

$$2s_{\min} - \mathcal{X}(s_{\min}) \leq -2\bar{s} - \bar{\mathcal{X}}(-\bar{s}) = -2\bar{s} - \mathcal{X}(s_{\min}).$$

Hence, $s_{\min} \leq -\bar{s}$. The other cases can be treated in a similar way.

We denote $\tilde{\Omega}_{\bar{s}} = [\mathcal{X}(s_{\min}), \mathcal{X}(s_{\max})] \times [\mathcal{Y}(s_{\min}), \mathcal{Y}(s_{\max})]$ and, since $s_{\min} \leq -\bar{s} \leq \bar{s} \leq s_{\max}$, we have $\Omega_{\bar{s}} \subset \tilde{\Omega}_{\bar{s}}$. We define the curve

$$(\tilde{\mathcal{X}}(s), \tilde{\mathcal{Y}}(s))$$

$$= \begin{cases} (\mathcal{X}(s), \mathcal{Y}(s)) & \text{if } s < s_{\min}, \\ \text{straight line joining } (\mathcal{X}(s_{\min}), \mathcal{Y}(s_{\min})) \text{ and } (\bar{\mathcal{X}}(-\bar{s}), \mathcal{Y}(-\bar{s})) & \text{if } s_{\min} \leq s < -\bar{s}, \\ (\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) & \text{if } -\bar{s} \leq s \leq \bar{s}, \\ \text{straight line joining } (\bar{\mathcal{X}}(\bar{s}), \mathcal{Y}(\bar{s})) \text{ and } (\mathcal{X}(s_{\max}), \mathcal{Y}(s_{\max})) & \text{if } \bar{s} < s \leq s_{\max}, \\ (\mathcal{X}(s), \mathcal{Y}(s)) & \text{if } s_{\max} < s. \end{cases}$$

We have that $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ and $(\mathcal{X}, \mathcal{Y})$ belong to $\mathcal{C}(\tilde{\Omega}_{\bar{s}})$. By Lemma 4.9, we get

$$\begin{aligned} |||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}(\Omega_{\bar{s}})} &\leq |||(Z,p,q) \bullet (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})} \\ &\leq C_1\Big(||(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})||_{\mathcal{C}(\tilde{\Omega}_{\bar{s}})}, |||\Theta|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}\Big) \\ &\leq C_1\big(||(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})||_{\mathcal{C}}, |||\Theta|||_{\mathcal{G}}\big), \end{aligned}$$

and by letting $\bar{s}$ tend to infinity, we obtain $|||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}} < \infty$. By Lemma 4.11, we obtain

$$\begin{aligned} ||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega_{\bar{s}})} &\leq ||(Z,p,q) \bullet (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})} \\ &\leq C||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})} \\ &\leq C||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})||_{\mathcal{G}}, \end{aligned}$$

where

$$\begin{aligned} C &= C\Big(||(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})||_{\mathcal{C}(\Omega_{\bar{s}})}, |||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}\Big) \\ &\leq C\big(||(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})||_{\mathcal{C}}, |||(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}}\big). \end{aligned}$$

By letting $\bar{s}$ tend to infinity, we get $||(Z,p,q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}} < \infty$ and we have proved (4.76). Hence, condition (i) of Definition 3.6 is satisfied. The conditions (ii)-(iv) follow directly since $(Z,p,q) \in \mathcal{H}$ and $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. We prove that

$$\lim_{s \to \pm\infty} J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = \lim_{s \to \pm\infty} J(\mathcal{X}(s), \mathcal{Y}(s)). \tag{4.77}$$

For any $s \in \mathbb{R}$, let $s_1 = \mathcal{X}^{-1}(\bar{\mathcal{X}}(s))$ and $s_2 = \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(s))$. If $s_1 \leq s_2$, then $\bar{\mathcal{X}}(s) = \mathcal{X}(s_1) \leq \mathcal{X}(s_2)$ and $\bar{\mathcal{Y}}(s) = \mathcal{Y}(s_2) \geq \mathcal{Y}(s_1)$. Since $J_X, J_Y \geq 0$, we get

$$J(\mathcal{X}(s_1), \mathcal{Y}(s_1)) \leq J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) \leq J(\mathcal{X}(s_2), \mathcal{Y}(s_2)).$$

Similarly, if $s_2 \leq s_1$, we obtain

$$J(\mathcal{X}(s_2), \mathcal{Y}(s_2)) \leq J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) \leq J(\mathcal{X}(s_1), \mathcal{Y}(s_1)),$$

so that

$$\min\{J(\mathcal{X}(s_1), \mathcal{Y}(s_1)), J(\mathcal{X}(s_2), \mathcal{Y}(s_2))\} \leq J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) \tag{4.78}$$

and

$$J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) \leq \max\{J(\mathcal{X}(s_1), \mathcal{Y}(s_1)), J(\mathcal{X}(s_2), \mathcal{Y}(s_2))\}. \tag{4.79}$$

Since

$$|s_1 - s| \leq |\mathcal{X}(s_1) - s_1| + |\bar{\mathcal{X}}(s) - s| \leq ||(\mathcal{X}, \mathcal{Y})||_{\mathcal{C}} + ||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}},$$

we have that $\lim_{s\to\pm\infty} s_1 = \lim_{s\to\pm\infty} \mathcal{X}^{-1}(\bar{\mathcal{X}}(s)) = \pm\infty$. Similarly, we find that $\lim_{s\to\pm\infty} s_2 = \lim_{s\to\pm\infty} \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(s)) = \pm\infty$. Hence, (4.78) and (4.79) yield (4.77), since J is bounded and monotone. In particular, we have that these limits are independent of which curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ is chosen. Furthermore, by (3.25),

$$\lim_{s\to-\infty} J(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = \lim_{s\to-\infty} J(\mathcal{X}(s), \mathcal{Y}(s)) = 0,$$

which shows that the last condition (v) in Definition 3.6 is satisfied for $\bar{\Theta} = (Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. Hence, $\bar{\Theta} \in \mathcal{G}$. □

We have the following global existence theorem.

Theorem 4.15 (Existence and uniqueness of global solutions)**.** *For any initial data $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}$, there exists a unique solution $(Z, p, q) \in \mathcal{H}$ such that $\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})$. We denote this solution mapping by*

$$\mathbf{S} : \mathcal{G} \to \mathcal{H}. \tag{4.80}$$

Proof. First we show how to construct the solution on rectangles with diagonal points which lie on the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$.

We consider two points, $(\bar{X}_l, \bar{Y}_l)$ and $(\bar{X}_r, \bar{Y}_r)$, on the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $\bar{X}_l < \bar{X}_r$ and $\bar{Y}_l < \bar{Y}_r$. Set $\bar{s}_l = \frac{1}{2}(\bar{X}_l + \bar{Y}_l)$ and $\bar{s}_r = \frac{1}{2}(\bar{X}_r + \bar{Y}_r)$. Let the restriction of Θ to $\bar{\Omega} = [\bar{X}_l, \bar{X}_r] \times [\bar{Y}_l, \bar{Y}_r]$, which we denote by $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}})$, be such that $\bar{\mathcal{X}}(s) = \mathcal{X}(s)$, $\bar{\mathcal{Y}}(s) = \mathcal{Y}(s)$, $\bar{\mathcal{Z}}(s) = \mathcal{Z}(s)$, $\bar{\mathcal{V}}(\bar{\mathcal{X}}(s)) = \mathcal{V}(\mathcal{X}(s))$, $\bar{\mathcal{W}}(\bar{\mathcal{Y}}(s)) = \mathcal{W}(\mathcal{Y}(s))$, $\bar{\mathfrak{p}}(\bar{\mathcal{X}}(s)) = \mathfrak{p}(\mathcal{X}(s))$ and $\bar{\mathfrak{q}}(\bar{\mathcal{Y}}(s)) = \mathfrak{q}(\mathcal{Y}(s))$ for $s \in [\bar{s}_l, \bar{s}_r]$. Then, $\bar{\Theta} \in \mathcal{G}(\bar{\Omega})$ and, by Lemma 4.10, there exists a unique solution $(\bar{Z}, \bar{p}, \bar{q}) \in \mathcal{H}(\bar{\Omega})$ such that $\bar{\Theta} = (\bar{Z}, \bar{p}, \bar{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$.

We can consider a new rectangle $\tilde{\Omega} = [\tilde{X}_l, \tilde{X}_r] \times [\tilde{Y}_l, \tilde{Y}_r]$ such that the upper right diagonal point of $\bar{\Omega}$ is the lower left diagonal point of $\tilde{\Omega}$, that is, $\tilde{X}_l = \bar{X}_r$ and $\tilde{Y}_l = \bar{Y}_r$. We set the upper right diagonal point $(\tilde{X}_r, \tilde{Y}_r)$ to be on the curve $(\mathcal{X}, \mathcal{Y})$. By the above argument, we obtain a unique solution $(\tilde{Z}, \tilde{p}, \tilde{q}) \in \mathcal{H}(\tilde{\Omega})$ such that $\tilde{\Theta} = (\tilde{Z}, \tilde{p}, \tilde{q}) \bullet (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$, where $\tilde{\Theta} = (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}, \tilde{\mathfrak{p}}, \tilde{\mathfrak{q}})$ is the restriction of Θ to $\tilde{\Omega}$. By repeating this argument on rectangles above $\tilde{\Omega}$ and below $\bar{\Omega}$, we obtain unique solutions in rectangles which cover the curve $(\mathcal{X}, \mathcal{Y})$.

Now we show how to construct solutions on rectangles with diagonal points which does not lie on the given curve $(\mathcal{X},\mathcal{Y})\in\mathcal{C}$. We consider the above setting.

Let $\hat{\Omega}=[\hat{X}_l,\hat{X}_r]\times[\hat{Y}_l,\hat{Y}_r]$ be the rectangle which lies above $\bar{\Omega}$ and to the left of $\tilde{\Omega}$, that is, $\hat{X}_l=\bar{X}_l$, $\hat{X}_r=\bar{X}_r=\tilde{X}_l$, $\hat{Y}_l=\tilde{Y}_l=\bar{Y}_r$ and $\hat{Y}_r=\tilde{Y}_r$. Set

$$(\hat{\mathcal{X}}(s),\hat{\mathcal{Y}}(s))=\begin{cases}(2s-\hat{Y}_l,\hat{Y}_l) & \text{if } \hat{s}_l\leq s\leq\tilde{s}_l,\\ (\hat{X}_r,2s-\hat{X}_r) & \text{if } \tilde{s}_l<s\leq\hat{s}_r,\end{cases}$$

where $\hat{s}_l$, $\tilde{s}_l$ and $\hat{s}_r$ are defined similarly to $\bar{s}_l$ and $\bar{s}_r$ above. We have $(\hat{\mathcal{X}},\hat{\mathcal{Y}})\in\mathcal{C}(\hat{\Omega})$, and we denote $\hat{\Theta}=(\hat{\mathcal{X}},\hat{\mathcal{Y}},\hat{\mathcal{Z}},\hat{\mathcal{V}},\hat{\mathcal{W}},\hat{\mathfrak{p}},\hat{\mathfrak{q}})$, where

$$\hat{\mathcal{Z}}(s)=\bar{\mathcal{Z}}(s),\quad \hat{\mathcal{V}}(\hat{\mathcal{X}}(s))=\bar{\mathcal{V}}(\hat{\mathcal{X}}(s)),\quad \hat{\mathcal{W}}(\hat{\mathcal{Y}}(s))=\bar{\mathcal{W}}(\hat{\mathcal{Y}}(s)),$$
$$\hat{\mathfrak{p}}(\hat{\mathcal{X}}(s))=\bar{\mathfrak{p}}(\hat{\mathcal{X}}(s)),\quad \hat{\mathfrak{q}}(\hat{\mathcal{Y}}(s))=\bar{\mathfrak{q}}(\hat{\mathcal{Y}}(s))$$

for $\hat{s}_l\leq s\leq\tilde{s}_l$ and

$$\hat{\mathcal{Z}}(s)=\tilde{\mathcal{Z}}(s),\quad \hat{\mathcal{V}}(\hat{\mathcal{X}}(s))=\tilde{\mathcal{V}}(\hat{\mathcal{X}}(s)),\quad \hat{\mathcal{W}}(\hat{\mathcal{Y}}(s))=\tilde{\mathcal{W}}(\hat{\mathcal{Y}}(s)),$$
$$\hat{\mathfrak{p}}(\hat{\mathcal{X}}(s))=\tilde{\mathfrak{p}}(\hat{\mathcal{X}}(s)),\quad \hat{\mathfrak{q}}(\hat{\mathcal{Y}}(s))=\tilde{\mathfrak{q}}(\hat{\mathcal{Y}}(s))$$

for $\tilde{s}_l<s\leq\hat{s}_r$. We have $\hat{\Theta}\in\mathcal{G}(\hat{\Omega})$. By Lemma 4.10, there exists a unique solution $(\hat{Z},\hat{p},\hat{q})\in\mathcal{H}(\hat{\Omega})$ such that $\hat{\Theta}=(\hat{Z},\hat{p},\hat{q})\bullet(\hat{\mathcal{X}},\hat{\mathcal{Y}})$. By repeatedly applying this argument to rectangles that are adjacent to rectangles where we have a solution, we obtain unique solutions in any rectangular domain. Hence, condition (i) of Definition 4.13 is satisfied.

We define (Z,p,q) to be the unique solution in each rectangle. Then, we have $(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})=\Theta\in\mathcal{G}$ and condition (ii) of Definition 4.13 is satisfied, so that $(Z,p,q)\in\mathcal{H}$. □

5. From Lagrangian to Eulerian Coordinates

5.1. **Mapping from $\mathcal{H}$ to $\mathcal{F}$.** Given an element (Z,p,q) in $\mathcal{H}$ we now want to map it to an element in the set $\mathcal{G}$ and then further to one in $\mathcal{F}$. For a solution in $\mathcal{H}$ corresponding to time $T>0$, i.e., $t(X,Y)=T$, we find it convenient to first shift the time to zero so that we can map the solution to an element in $\mathcal{G}_0$ in the next step.

Definition 5.1. *Given $T\geq 0$ and $(Z,p,q)\in\mathcal{H}$, we define*

$$\bar{t}(X,Y)=t(X,Y)-T \tag{5.1a}$$

and

$$\bar{x}(X,Y)=x(X,Y),\qquad \bar{U}(X,Y)=U(X,Y), \tag{5.1b}$$
$$\bar{J}(X,Y)=J(X,Y),\qquad \bar{K}(X,Y)=K(X,Y), \tag{5.1c}$$
$$\bar{p}(X,Y)=p(X,Y),\qquad \bar{q}(X,Y)=q(X,Y). \tag{5.1d}$$

We denote by $\mathbf{t}_T:\mathcal{H}\to\mathcal{H}$ the mapping which associates to any $(Z,p,q)\in\mathcal{H}$ the element $(\bar{Z},\bar{p},\bar{q})\in\mathcal{H}$. We have

$$\mathbf{t}_{T+T'}=\mathbf{t}_T\circ\mathbf{t}_{T'}. \tag{5.2}$$

Definition 5.2. *Given* $(Z,p,q) \in \mathcal{H}$*, we define*

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid t(X', 2s - X') < 0 \text{ for all } X' < X\} \tag{5.3}$$

and $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$*. Then, we have* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ *and* $(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$*. We denote by* $\mathbf{E} : \mathcal{H} \to \mathcal{G}_\mathbf{0}$ *the mapping which associates to any* $(Z,p,q) \in \mathcal{H}$ *the element* $(Z,p,q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$*.*

Proof of the well-posedness of Definition 5.2. We prove that $(\mathcal{X}, \mathcal{Y})$ belongs to $\mathcal{C}$. Let us verify that $\mathcal{X}$ is nondecreasing. Let $s < \bar{s}$ and consider a sequence X_i such that $X_i < \mathcal{X}(s)$ and $\lim_{i\to\infty} X_i = \mathcal{X}(s)$. We have $t(X_i, 2s - X_i) < 0$ and, since $t_Y \leq 0$, $t(X_i, 2\bar{s} - X_i) < 0$, which implies that $X_i < \mathcal{X}(\bar{s})$. By letting i tend to infinity, we obtain $\mathcal{X}(s) \leq \mathcal{X}(\bar{s})$, so that $\mathcal{X}$ is nondecreasing.

Next we show that $\mathcal{X}$ is differentiable almost everywhere. We claim that $\mathcal{X}$ is Lipschitz continuous with Lipschitz constant at most two. Let us assume the opposite, that is, there exists $\bar{s} > s$ such that

$$\mathcal{X}(\bar{s}) - \mathcal{X}(s) > 2(\bar{s} - s).$$

This implies that $\mathcal{Y}(s) > \mathcal{Y}(\bar{s})$ and we denote $\Omega = [\mathcal{X}(s), \mathcal{X}(\bar{s})] \times [\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$. Since $t_X \geq 0$ and $t_Y \leq 0$, we have, for any $(X,Y) \in \Omega$, that

$$0 = t(\mathcal{X}(s), \mathcal{Y}(s)) \leq t(X, \mathcal{Y}(s)) \leq t(X,Y) \leq t(X, \mathcal{Y}(\bar{s})) \leq t(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})) = 0,$$

so that $t(X,Y) = 0$ for all $(X,Y) \in \Omega$. Consider the point (X,Y) given by $Y = \mathcal{Y}(s)$ and $X = 2\bar{s} - \mathcal{Y}(s)$. It belongs to Ω since $X = 2\bar{s} - \mathcal{Y}(s) < 2\bar{s} - \mathcal{Y}(\bar{s}) = \mathcal{X}(\bar{s})$. We have $t(X,Y) = 0$, $X + Y = 2\bar{s}$ and $X < \mathcal{X}(\bar{s})$, which contradicts the definition of $\mathcal{X}(\bar{s})$. Hence, we have proved that $\mathcal{X}$ is Lipschitz continuous with Lipschitz constant at most two and therefore differentiable almost everywhere. Furthermore, it follows that $\mathcal{Y}$ is nondecreasing and differentiable almost everywhere. Since $\dot{\mathcal{X}}, \dot{\mathcal{Y}} \in [0,2]$, we find that $\dot{\mathcal{X}} - 1, \dot{\mathcal{Y}} - 1 \in L^\infty(\mathbb{R})$. It remains to prove that $\mathcal{X} - \mathrm{Id}$ and $\mathcal{Y} - \mathrm{Id}$ belong to $L^\infty(\mathbb{R})$. We will prove that

$$\limsup_{s\to\pm\infty} |\mathcal{X}(s) - s| \leq \frac{L}{2} \tag{5.4}$$

for a constant L that will be set later, and which depends on κ and $|||(Z,p,q) \bullet (\mathcal{X}_d, \mathcal{Y}_d)|||_\mathcal{G}$. First we prove that

$$\limsup_{s\to\infty} (\mathcal{X}(s) - s) \leq \frac{L}{2}. \tag{5.5}$$

Assume the opposite. Introducing $f(s) = \sup_{r\geq s}(\mathcal{X}(r) - r)$, which is a nonincreasing function, we then have

$$\inf_{s\geq 0} f(s) > \frac{L}{2}.$$

This implies that

$$f(s) > \frac{L}{2}$$

for all $s \geq 0$. The function $\mathcal{X} - \mathrm{Id}$ can only be unbounded at infinity since it is continuous with bounded derivative. If $\mathcal{X} - \mathrm{Id}$ is bounded at infinity, (5.5) is immediately satisfied. Thus, we assume that $\mathcal{X}(s) - s$ tends to infinity as $s \to \infty$.

This implies that $f(s) \to \infty$ as $s \to \infty$. Then there is an increasing sequence $s_n \to \infty$ as $n \to \infty$, such that

$$\mathcal{X}(s_n) - s_n > \frac{L}{2}$$

for all n, and $\mathcal{X}(s_n) - s_n \to \infty$ as $n \to \infty$. We have $\mathcal{Y}(s_n) = 2s_n - \mathcal{X}(s_n) \le s_n - \frac{L}{2}$. Since $t_X \ge 0$ and $t_Y \le 0$ we have

$$0 = t(\mathcal{X}(s_n), \mathcal{Y}(s_n)) \ge t\Big(s_n + \frac{L}{2}, s_n - \frac{L}{2}\Big).$$

Next we prove that

$$\liminf_{n\to\infty} t\Big(s_n + \frac{L}{2}, s_n - \frac{L}{2}\Big) \ge 1, \tag{5.6}$$

which will lead to the contradiction

$$0 \ge \liminf_{n\to\infty} t\Big(s_n + \frac{L}{2}, s_n - \frac{L}{2}\Big) \ge 1,$$

and (5.5) follows.

By (4.12a) and (4.12d), we have

$$\begin{aligned}
(5.7)\quad & t\Big(s_n + \frac{L}{2}, s_n - \frac{L}{2}\Big) \\
&= t\Big(s_n - \frac{L}{2}, s_n - \frac{L}{2}\Big) + \int_{s_n-\frac{L}{2}}^{s_n+\frac{L}{2}} \Big(\frac{x_X}{c(U)}\Big)\Big(\tilde{X}, s_n - \frac{L}{2}\Big)\, d\tilde{X} \\
&\ge -|||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}} + \frac{L}{2\kappa} + \frac{1}{\kappa}\int_{s_n-\frac{L}{2}}^{s_n+\frac{L}{2}} \Big(x_X - \frac{1}{2}\Big)\Big(\tilde{X}, s_n - \frac{L}{2}\Big)\, d\tilde{X}.
\end{aligned}$$

Let $\Omega_{n,L} = [s_n - \frac{L}{2}, s_n + \frac{L}{2}] \times [s_n - \frac{L}{2}, s_n + \frac{L}{2}]$ and consider the curve

$$(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = \begin{cases} (\mathcal{X}_d(s), \mathcal{Y}_d(s)) & \text{for } s < s_n - \frac{L}{2}, \\ (2s - (s_n - \frac{L}{2}), s_n - \frac{L}{2}) & \text{for } s_n - \frac{L}{2} \le s \le s_n, \\ (s_n + \frac{L}{2}, 2s - (s_n + \frac{L}{2})) & \text{for } s_n \le s \le s_n + \frac{L}{2}, \\ (\mathcal{X}_d(s), \mathcal{Y}_d(s)) & \text{for } s > s_n + \frac{L}{2}. \end{cases}$$

Both $(\mathcal{X}_d, \mathcal{Y}_d)$ and $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ belong to $\mathcal{C}(\Omega_{n,L})$. By the Cauchy–Schwarz inequality and Lemma 4.11, we find

$$\begin{aligned}
(5.8)\quad & \Big|\int_{s_n-\frac{L}{2}}^{s_n+\frac{L}{2}} \Big(x_X - \frac{1}{2}\Big)\Big(\tilde{X}, s_n - \frac{L}{2}\Big)\, d\tilde{X}\Big| \\
&\le \sqrt{L}\Big(\int_{s_n-\frac{L}{2}}^{s_n+\frac{L}{2}} \Big(x_X - \frac{1}{2}\Big)^2\Big(\tilde{X}, s_n - \frac{L}{2}\Big)\, d\tilde{X}\Big)^{\frac{1}{2}} \\
&\le \sqrt{L}\,||(Z,p,q)\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{G}(\Omega_{n,L})} \\
&\le \sqrt{L}C||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)||_{\mathcal{G}(\Omega_{n,L})}.
\end{aligned}$$

Here, C is an increasing function with respect to both its arguments and we have

$$\begin{aligned}
C &= C\big(||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega_{n,L})}, |||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}(\Omega_{n,L})}\big) \\
&\le C\big(L, |||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}}\big),
\end{aligned}$$

where we used that $||(\bar{\mathcal{X}},\bar{\mathcal{Y}})||_{\mathcal{C}(\Omega_{n,L})}=L$. From (4.40), we have

$$\begin{aligned}&\lim_{n\to\infty}||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)||^2_{\mathcal{G}(\Omega_{n,L})}\\&=\lim_{n\to\infty}\int_{s_n-\frac{L}{2}}^{s_n+\frac{L}{2}}\big(U^2+|Z_X^a|^2+|Z_Y^a|^2+p^2+q^2\big)(\tilde{X},\tilde{X})\,d\tilde{X}=0,\end{aligned}$$

which combined with (5.8) and (5.7) yields

$$\liminf_{n\to\infty}t\Big(s_n+\frac{L}{2},s_n-\frac{L}{2}\Big)\geq -|||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}}+\frac{L}{2\kappa}.$$

Setting $L\geq 2\kappa(|||(Z,p,q)\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}}+1)$ implies (5.6). Thus, we have proved (5.5).

It remains to prove that $\liminf_{s\to\infty}(\mathcal{X}(s)-s)\geq -\frac{L}{2}$ in order to conclude that

$$\limsup_{s\to\infty}|\mathcal{X}(s)-s|\leq\frac{L}{2}.$$

The proof is similar to the one above. Now one has to show that

$$\limsup_{n\to\infty}t\Big(s_n-\frac{L}{2},s_n+\frac{L}{2}\Big)\leq -1, \tag{5.9}$$

for an increasing sequence $s_n\to\infty$ as $n\to\infty$, in order to get a contradiction.

The proof of

$$\limsup_{s\to-\infty}|\mathcal{X}(s)-s|\leq\frac{L}{2}$$

is similar to the argument above. To show

$$\limsup_{s\to-\infty}(\mathcal{X}(s)-s)\leq\frac{L}{2}\quad\text{and}\quad\liminf_{s\to-\infty}(\mathcal{X}(s)-s)\geq-\frac{L}{2},$$

one proves

$$\liminf_{n\to\infty}t\Big(s_n+\frac{L}{2},s_n-\frac{L}{2}\Big)\geq 1\quad\text{and}\quad\limsup_{n\to\infty}t\Big(s_n-\frac{L}{2},s_n+\frac{L}{2}\Big)\leq -1,$$

respectively, for a carefully chosen decreasing sequence $s_n\to-\infty$ as $n\to\infty$.

This concludes the proof of (5.4), and we have showed that $\mathcal{X}-\mathrm{Id}$ and $\mathcal{Y}-\mathrm{Id}$ belong to $L^\infty(\mathbb{R})$, so that $(\mathcal{X},\mathcal{Y})\in\mathcal{C}$. Then, by Lemma 4.14, we have

$$(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})=(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})\in\mathcal{G}$$

and by construction $\mathcal{Z}_1(s)=t(\mathcal{X}(s),\mathcal{Y}(s))=0$ so that $(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})\in\mathcal{G}_0$. □

Definition 5.3. *Given* $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})\in\mathcal{G}_0$*, let* $\psi_1=(x_1,U_1,J_1,K_1,V_1,H_1)$ *and* $\psi_2=(x_2,U_2,J_2,K_2,V_2,H_2)$ *be defined as*

$$x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))=\mathcal{Z}_2(s), \tag{5.10a}$$

$$U_1(\mathcal{X}(s))=U_2(\mathcal{Y}(s))=\mathcal{Z}_3(s), \tag{5.10b}$$

$$J_1(\mathcal{X}(s))=\int_{-\infty}^{s}\mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau,\quad J_2(\mathcal{Y}(s))=\int_{-\infty}^{s}\mathcal{W}_4(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau)\,d\tau, \tag{5.10c}$$

$$K_1(\mathcal{X}(s))=\int_{-\infty}^{s}\mathcal{V}_5(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau,\quad K_2(\mathcal{Y}(s))=\int_{-\infty}^{s}\mathcal{W}_5(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau)\,d\tau, \tag{5.10d}$$

and

$$V_1 = \mathcal{V}_3, \qquad V_2 = \mathcal{W}_3, \tag{5.10e}$$

$$H_1 = \mathfrak{p}, \qquad H_2 = \mathfrak{q}. \tag{5.10f}$$

We denote by $\mathbf{D} : \mathcal{G}_0 \to \mathcal{F}$ *the mapping which to any* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ *associates the element* $\psi \in \mathcal{F}$ *as defined above.*

Proof of the well-posedness of Definition 5.3. We check the well-posedness of (5.10a) and (5.10b). Consider $s < \bar{s}$ such that $\mathcal{X}(s) = \mathcal{X}(\bar{s})$. Since $\mathcal{X}$ is nondecreasing and continuous, we have $\dot{\mathcal{X}}(\tilde{s}) = 0$ and $\dot{\mathcal{Y}}(\tilde{s}) = 2$ for all $\tilde{s} \in [s, \bar{s}]$. From (3.26), it follows that $\mathcal{W}_2(\mathcal{Y}(\tilde{s})) = 0$ for all $\tilde{s} \in [s, \bar{s}]$. Hence,

$$\dot{\mathcal{Z}}_2(\tilde{s}) = \mathcal{V}_2(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s}) + \mathcal{W}_2(\mathcal{Y}(\tilde{s}))\dot{\mathcal{Y}}(\tilde{s}) = 0$$

for all $\tilde{s} \in [s, \bar{s}]$, so that $\mathcal{Z}_2(\tilde{s}) = \mathcal{Z}_2(\bar{s})$ and (5.10a) is well-posed.

From (3.24d), we obtain

$$0 = 2\mathcal{W}_4(\mathcal{Y}(\tilde{s}))\mathcal{W}_2(\mathcal{Y}(\tilde{s})) = (c(\mathcal{Z}_3(\tilde{s}))\mathcal{W}_3(\mathcal{Y}(\tilde{s})))^2 + c(\mathcal{Z}_3(\tilde{s}))\mathfrak{q}^2(\mathcal{Y}(\tilde{s})),$$

which implies that $\mathcal{W}_3(\mathcal{Y}(\tilde{s})) = 0$ and $\mathfrak{q}(\mathcal{Y}(\tilde{s})) = 0$ for all $\tilde{s} \in [s, \bar{s}]$. Thus,

$$\dot{\mathcal{Z}}_3(\tilde{s}) = \mathcal{V}_3(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s}) + \mathcal{W}_3(\mathcal{Y}(\tilde{s}))\dot{\mathcal{Y}}(\tilde{s}) = 0$$

so that $\mathcal{Z}_3(\tilde{s}) = \mathcal{Z}_3(\bar{s})$ and (5.10b) is well-posed.

Let us prove that J_1 and K_1 given by (5.10c) and (5.10d) are well-posed. The proof is similar for J_2 and K_2. Since $\mathcal{V}_4, \mathcal{W}_4, \dot{\mathcal{X}}, \dot{\mathcal{Y}} \geq 0$, we have $J_1 \geq 0$ and

$$\begin{aligned} J_1(\mathcal{X}(s)) &= \int_{-\infty}^{s} \mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau \\ &\leq \int_{-\infty}^{s} (\mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) + \mathcal{W}_4(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau))\,d\tau \\ &= \int_{-\infty}^{s} \dot{\mathcal{Z}}_4(\tau)\,d\tau \leq ||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})}, \end{aligned} \tag{5.11}$$

so that the function $\mathcal{V}_4$ belongs to $L^1(\mathbb{R})$ and J_1 is bounded. If $s, \bar{s} \in \mathbb{R}$ are such that $s < \bar{s}$ and $\mathcal{X}(s) = \mathcal{X}(\bar{s})$, we have $\dot{\mathcal{X}}(\tilde{s}) = 0$ for all $\tilde{s} \in [s, \bar{s}]$ since $\mathcal{X}$ is nondecreasing. Hence,

$$\int_{-\infty}^{\tilde{s}} \mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau = \int_{-\infty}^{s} \mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau, \tag{5.12}$$

and J_1 is well-posed. For K_1, we have by (3.24b), that

$$K_1(\mathcal{X}(s)) = \int_{-\infty}^{s} \mathcal{V}_5(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\,d\tau = \int_{-\infty}^{s} \frac{\mathcal{V}_4(\mathcal{X}(\tau))}{c(\mathcal{Z}_3(\tau))}\dot{\mathcal{X}}(\tau)\,d\tau \leq \kappa J_1(\mathcal{X}(s)), \tag{5.13}$$

which by (5.11) and since $\mathcal{V}_4 \geq 0$ and $c > 0$, implies that $0 \leq K_1(X) \leq \kappa||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})}$. By an argument as in (5.12) applied to K_1, we conclude that also K_1 is well-posed.

Next we show that $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$ as defined in (5.10a)-(5.10f) satisfies the conditions in the definition of the set $\mathcal{F}$. The proof for ψ_2 is similar.

Let us show that x_1 is Lipschitz continuous and therefore differentiable almost everywhere. Consider $s, \bar{s} \in \mathbb{R}$ and set $X = \mathcal{X}(s)$ and $\bar{X} = \mathcal{X}(\bar{s})$. We have

$$|x_1(\bar{X}) - x_1(X)| = |\mathcal{Z}_2(\bar{s}) - \mathcal{Z}_2(s)|$$

$$
\begin{aligned}
&= \left| \int_s^{\bar{s}} \dot{\mathcal{Z}}_2(\tilde{s})\, d\tilde{s} \right| \\
&= \left| \int_s^{\bar{s}} (\mathcal{V}_2(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s}) + \mathcal{W}_2(\mathcal{Y}(\tilde{s}))\dot{\mathcal{Y}}(\tilde{s}))\, d\tilde{s} \right| \\
&= \left| 2\int_s^{\bar{s}} \mathcal{V}_2(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s})\, d\tilde{s} \right| \quad \text{by (3.26)} \\
&= \left| 2\int_s^{\bar{s}} \mathcal{V}_2^a(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s})\, d\tilde{s} + \int_s^{\bar{s}} \dot{\mathcal{X}}(\tilde{s})\, d\tilde{s} \right| \quad \text{by (3.18)} \\
&\leq (2\left\| \mathcal{V}_2^a \right\|_{L^\infty(\mathbb{R})} + 1)|\bar{X} - X|.
\end{aligned}
$$

From (5.11), we have that J_1 is increasing and hence differentiable almost everywhere. Similarly, one shows that K_1 is differentiable almost everywhere.

Next we show (3.5a)-(3.5d). We have

$$
x_1(\mathcal{X}(s)) - \mathcal{X}(s) = \mathcal{Z}_2(s) - s + s - \mathcal{X}(s) = \mathcal{Z}_2^a + s - \mathcal{X}(s), \tag{5.14}
$$

so that $x_1 - \mathrm{Id} \in L^\infty(\mathbb{R})$ since $\mathcal{Z}_2^a$ and $\mathcal{X} - \mathrm{Id}$ belong to $L^\infty(\mathbb{R})$. Differentiating (5.10a) and using (3.26), we obtain $x_1'(\mathcal{X})\dot{\mathcal{X}} = \dot{\mathcal{Z}}_2 = 2\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}}$. Hence, $x_1' = 2\mathcal{V}_2$ and we get that

$$
x_1' - 1 = 2\mathcal{V}_2 - 1 = 2\mathcal{V}_2^a,
$$

which shows that $x_1' - 1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By (5.11), we have $J_1 \in L^\infty(\mathbb{R})$. We differentiate (5.10c) and obtain $J_1' = \mathcal{V}_4 = \mathcal{V}_4^a$, which implies that $J_1' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, from (5.13) it follows that $K_1 \in L^\infty(\mathbb{R})$ and $K_1' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Since $\mathfrak{p}, \mathfrak{q} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have by (5.10f) that H_1 and H_2 belong to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The function U_1 belongs to $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$, as $U_1(\mathcal{X}) = \mathcal{Z}_3 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We have $V_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by (5.10e) and since $\mathcal{V}_3 = \mathcal{V}_3^a \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Hence, we have proved (3.5a)-(3.5d). Let us verify (3.6). We showed above that $x_1' = 2\mathcal{V}_2$ and $J_1' = \mathcal{V}_4$. Thus, $x_1', J_1' \geq 0$ because $\mathcal{V}_2, \mathcal{V}_4 \geq 0$. The identity (3.7) follows from (3.24b) since $J_1' = \mathcal{V}_4$ and $K_1' = \mathcal{V}_5$. We can check that the relation (3.8) holds by using (3.24c), (5.10e) and (5.10f). Let us prove (3.9) by using Lemma 3.8. We found above that J_1 is absolutely continuous. Since x_1 is Lipschitz continuous, it follows that $x_1 + J_1$ is absolutely continuous. By (5.11), we have

$$
|x_1 + J_1 - \mathrm{Id}\,| \leq |x_1 - \mathrm{Id}\,| + ||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})}
$$

which, by (5.14), implies that $x_1 + J_1 - \mathrm{Id} \in L^\infty(\mathbb{R})$.

We proved above that $x_1' - 1, J_1' \in L^2(\mathbb{R})$, which implies that $x_1' + J_1' - 1 \in L^2(\mathbb{R})$.

The fact that $\frac{1}{\mathcal{V}_2 + \mathcal{V}_4} \in L^\infty(\mathbb{R})$ implies that there exists a number $k > 0$ such that $\mathcal{V}_2(X) + \mathcal{V}_4(X) \geq k$ for almost every $X \in \mathbb{R}$. Then, since $x_1' + J_1' = 2\mathcal{V}_2 + \mathcal{V}_4$, we obtain

$$
k \leq \mathcal{V}_2 + \mathcal{V}_4 \leq x_1' + J_1' \leq 2(\mathcal{V}_2 + \mathcal{V}_4) = 2(\mathcal{V}_2^a + \mathcal{V}_4^a) + 1 \leq 2\big(||\mathcal{V}_2^a||_{L^\infty(\mathbb{R})} + ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})}\big) + 1,
$$

so that the remaining condition in Lemma 3.8 holds. Hence, $x_1 + J_1 \in G$ and we have proved (3.9).

By (5.10c) and (3.25), we have

$$
0 \leq J_1(\mathcal{X}(s)) + J_2(\mathcal{Y}(s)) = \int_{-\infty}^s \dot{\mathcal{Z}}_4(\tau)\, d\tau = \mathcal{Z}_4(s),
$$

and since $\lim_{s\to-\infty} \mathcal{X}(s) = -\infty$ and $\lim_{s\to-\infty} \mathcal{Y}(s) = -\infty$, (3.25) implies (3.10). The relation (3.11a) follows directly from (5.10b). Using (5.10b), (3.23), and (5.10e), we obtain

$$\frac{d}{ds}U_1(\mathcal{X}) = \frac{d}{ds}U_2(\mathcal{Y}) = \dot{\mathcal{Z}}_3 = \mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}} = V_1(\mathcal{X})\dot{\mathcal{X}} + V_2(\mathcal{Y})\dot{\mathcal{Y}},$$

so that (3.11b) holds.

□

5.2. **Semigroup of Solutions in $\mathcal{F}$.** We define the solution operator on the set $\mathcal{F}$.

Definition 5.4. *For any $T \geq 0$, we define the mapping $S_T : \mathcal{F} \to \mathcal{F}$ by*

$$S_T = \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C}.$$

In order to show that S_T is a semigroup we need the following result.

Lemma 5.5. *We have*

$$\mathbf{C} \circ \mathbf{D} \circ \mathbf{E} = \mathbf{E}, \quad \mathbf{D} \circ \mathbf{C} = \mathrm{Id} \tag{5.15}$$

and

$$\mathbf{E} \circ \mathbf{S} \circ \mathbf{C} = \mathbf{C}, \quad \mathbf{S} \circ \mathbf{E} = \mathrm{Id}\,. \tag{5.16}$$

It follows that $\mathbf{S} \circ \mathbf{C} = (\mathbf{D} \circ \mathbf{E})^{-1}$ and the sets $\mathcal{F}$ and $\mathcal{H}$ are in bijection.

Proof. We first prove (5.15). Given $(Z, p, q) \in \mathcal{H}$, let

$$\begin{aligned}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) &= \mathbf{E}(Z, p, q),\\ (\psi_1, \psi_2) &= \mathbf{D}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}),\\ (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}) &= \mathbf{C}(\psi_1, \psi_2).\end{aligned}$$

We want to prove that $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}) = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$. Let us show that $\bar{\mathcal{X}} = \mathcal{X}$. We claim that for any $s \in \mathbb{R}$ and (X, Y) such that $X < \mathcal{X}(s)$ and $X + Y = 2s$, we have either

$$x_1(X) < x_1(\mathcal{X}(s)) \quad \text{or} \quad x_2(Y) > x_2(\mathcal{Y}(s)). \tag{5.17}$$

Let us assume the opposite, that is, there exist $\bar{s}$ and $(\bar{X}, \bar{Y})$ such that $\bar{X} < \mathcal{X}(\bar{s})$, $\bar{X} + \bar{Y} = 2\bar{s}$ and

$$x_1(\bar{X}) = x_1(\mathcal{X}(\bar{s})) = \mathcal{Z}_2(\bar{s}) = x_2(\mathcal{Y}(\bar{s})) = x_2(\bar{Y}),$$

where we used (5.10a). Let $s_0 = \mathcal{X}^{-1}(\bar{X})$ and $s_1 = \mathcal{Y}^{-1}(\bar{Y})$. Since $\bar{X} < \mathcal{X}(\bar{s})$, $\mathcal{Y}(\bar{s}) < \bar{Y}$ and $\dot{\mathcal{X}}, \dot{\mathcal{Y}} \geq 0$, we have $s_0 < \bar{s} < s_1$. Consider the rectangular domain $\Omega = [\mathcal{X}(s_0), \mathcal{X}(s_1)] \times [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. We want to construct a solution $(\tilde{Z}, \tilde{p}, \tilde{q})$ of (2.38) in Ω.

Since

$$\begin{aligned}\mathcal{Z}_2(s_0) &= x_1(\mathcal{X}(s_0)) = x_1(\bar{X}) = \mathcal{Z}_2(\bar{s}),\\ \mathcal{Z}_2(s_1) &= x_2(\mathcal{Y}(s_1)) = x_2(\bar{Y}) = \mathcal{Z}_2(\bar{s})\end{aligned}$$

and $\mathcal{Z}_2$ is nondecreasing, we have $\mathcal{Z}_2(s) = \mathcal{Z}_2(s_0) = \mathcal{Z}_2(s_1)$ for all $s \in [s_0, s_1]$. We have $\dot{\mathcal{Z}}_2(s) = \mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$ for all $s \in [s_0, s_1]$, which implies that $\mathcal{V}_2(X) = 0$ for almost every $X \in [\mathcal{X}(s_0), \mathcal{X}(s_1)]$ and $\mathcal{W}_2(Y) = 0$ for almost every $Y \in [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. Then, by (3.24a), (3.24c) and (3.24d), we have $\mathcal{V}_1(X) = \mathcal{V}_3(X) = \mathfrak{p}(X) = 0$ for almost every $X \in [\mathcal{X}(s_0), \mathcal{X}(s_1)]$ and $\mathcal{W}_1(Y) = \mathcal{W}_3(Y) = \mathfrak{q}(Y) = 0$ for almost every $Y \in [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. Hence, $\mathcal{Z}_1(s)$ is constant for all $s \in [s_0, s_1]$ and we define[3] $\tilde{t}(X, Y) = 0$ in Ω. Let

$$\begin{aligned} \tilde{x}(X,Y) &= \mathcal{Z}_2(s), & \tilde{U}(X,Y) &= \mathcal{Z}_3(s), \\ \tilde{J}(X,Y) &= J_1(X) + J_2(Y), & \tilde{K}(X,Y) &= K_1(X) + K_2(Y), \\ \tilde{p}(X,Y) &= \mathfrak{p}(X), & \tilde{q}(X,Y) &= \mathfrak{q}(Y). \end{aligned}$$

Then, $(\tilde{Z}, \tilde{p}, \tilde{q})$ is a solution of (2.38) in Ω. By the uniqueness of the solution, we get $(\tilde{Z}, \tilde{p}, \tilde{q}) = (Z, p, q)$. In particular, we have $t(\bar{X}, \bar{Y}) = 0$ such that $\bar{X} < \mathcal{X}(\bar{s})$ and $\bar{X} + \bar{Y} = 2\bar{s}$, which contradicts the definition of $\mathcal{X}$ given by (5.3). Hence, we conclude that (5.17) holds. By (5.10a), we have $x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s))$. Thus, (3.28) implies that $\bar{\mathcal{X}}(s) \leq \mathcal{X}(s)$ and it follows that $\bar{\mathcal{Y}}(s) \geq \mathcal{Y}(s)$. From (3.29), we have

$$x_1(\bar{\mathcal{X}}(s)) = x_2(\bar{\mathcal{Y}}(s)). \tag{5.18}$$

Let us assume that $\bar{\mathcal{X}}(s) < \mathcal{X}(s)$. Then, by (5.17), we have either $x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s))$ or $x_2(\bar{\mathcal{Y}}(s)) > x_2(\mathcal{Y}(s))$. If $x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s))$, then

$$x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \leq x_2(\bar{\mathcal{Y}}(s)),$$

which contradicts (5.18). Similarly, if $x_2(\bar{\mathcal{Y}}(s)) > x_2(\mathcal{Y}(s))$, we obtain the contradiction

$$x_2(\bar{\mathcal{Y}}(s)) > x_2(\mathcal{Y}(s)) = x_1(\mathcal{X}(s)) \geq x_1(\bar{\mathcal{X}}(s)).$$

Hence, $\bar{\mathcal{X}} = \mathcal{X}$ and therefore $\bar{\mathcal{Y}} = \mathcal{Y}$. Then, by (3.30b) and (5.10a), we have $\bar{\mathcal{Z}}_2(s) = x_1(\bar{\mathcal{X}}(s)) = x_1(\mathcal{X}(s)) = \mathcal{Z}_2(s)$. Similarly, one finds that $\bar{\mathcal{Z}}_3 = \mathcal{Z}_3$. By (3.30a) and since $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$, we have $\bar{\mathcal{Z}}_1 = \mathcal{Z}_1 = 0$. We have

$$\begin{aligned} \bar{\mathcal{Z}}_4(s) &= J_1(\bar{\mathcal{X}}(s)) + J_2(\bar{\mathcal{Y}}(s)) \quad \text{by (3.30d)} \\ &= J_1(\mathcal{X}(s)) + J_2(\mathcal{Y}(s)) \\ &= \int_{-\infty}^{s} (\mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) + \mathcal{W}_4(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau))\, d\tau \quad \text{by (5.10c)} \\ &= \int_{-\infty}^{s} \dot{\mathcal{Z}}_4(\tau)\, d\tau = \mathcal{Z}_4(s) \quad \text{by (3.23)} \end{aligned}$$

and by a similar calculation, we obtain $\bar{\mathcal{Z}}_5 = \mathcal{Z}_5$. Let us verify that $\bar{\mathcal{V}} = \mathcal{V}$ (one shows that $\bar{\mathcal{W}} = \mathcal{W}$ in a similar way). By differentiating $x_1(\mathcal{X}(s)) = \mathcal{Z}_2(s)$ and using (3.26), we obtain $x_1' = 2\mathcal{V}_2$. This yields

$$\bar{\mathcal{V}}_1(\bar{\mathcal{X}}) = \bar{\mathcal{V}}_1(\mathcal{X}) = \frac{1}{2c(U_1(\mathcal{X}))} x_1'(\mathcal{X}) = \frac{1}{c(\mathcal{Z}_3)} \mathcal{V}_2(\mathcal{X}) = \mathcal{V}_1(\mathcal{X})$$

by (3.31a) and (3.24a), and

$$\bar{\mathcal{V}}_2(\bar{\mathcal{X}}) = \bar{\mathcal{V}}_2(\mathcal{X}) = \frac{1}{2} x_1'(\mathcal{X}) = \mathcal{V}_2(\mathcal{X})$$

[3] In a rectangular domain where $t = 0$, $(\mathcal{X}, \mathcal{Y})$ as defined by (5.3) has to consist of the vertical straight line connecting the lower left diagonal point with the upper left corner, and the horizontal straight line connecting the upper left corner with the upper right diagonal point, since $t_X \geq 0$ and $t_Y \leq 0$

by (3.31b). From (3.31c)-(3.31f) and (5.10c)-(5.10f), we obtain

$$\begin{aligned}
\bar{\mathcal{V}}_3(\bar{\mathcal{X}}) &= \bar{\mathcal{V}}_3(\mathcal{X}) = V_1(\mathcal{X}) = \mathcal{V}_3(\mathcal{X}),\\
\bar{\mathcal{V}}_4(\bar{\mathcal{X}}) &= \bar{\mathcal{V}}_4(\mathcal{X}) = J_1'(\mathcal{X}) = \mathcal{V}_4(\mathcal{X}),\\
\bar{\mathcal{V}}_5(\bar{\mathcal{X}}) &= \bar{\mathcal{V}}_5(\mathcal{X}) = K_1'(\mathcal{X}) = \mathcal{V}_5(\mathcal{X}),\\
\bar{\mathfrak{p}}(\bar{\mathcal{X}}) &= \bar{\mathfrak{p}}(\mathcal{X}) = H_1(\mathcal{X}) = \mathfrak{p}(\mathcal{X}),\\
\bar{\mathfrak{q}}(\bar{\mathcal{Y}}) &= \bar{\mathfrak{q}}(\mathcal{Y}) = H_2(\mathcal{Y}) = \mathfrak{q}(\mathcal{X}).
\end{aligned}$$

Hence, we have proved that $\mathbf{C}\circ\mathbf{D}\circ\mathbf{E}=\mathbf{E}$. By a straightforward calculation, using Definition 3.7 and Definition 5.3, one proves that $\mathbf{D}\circ\mathbf{C}=\mathrm{Id}$. This concludes the proof of (5.15).

Next we prove (5.16). Given $(\psi_1,\psi_2)\in\mathcal{F}$, let

$$\begin{aligned}
(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) &= \mathbf{C}(\psi_1,\psi_2),\\
(Z,p,q) &= \mathbf{S}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}),\\
(\bar{\mathcal{X}},\bar{\mathcal{Y}},\bar{\mathcal{Z}},\bar{\mathcal{V}},\bar{\mathcal{W}},\bar{\mathfrak{p}},\bar{\mathfrak{q}}) &= \mathbf{E}(Z,p,q).
\end{aligned}$$

As before, we first show that $\bar{\mathcal{X}}=\mathcal{X}$. Since $(Z,p,q)\in\mathcal{H}$ is a solution with $(Z,p,q)\bullet(\mathcal{X},\mathcal{Y})=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})\in\mathcal{G}_0$, we have that $t(\mathcal{X}(s),\mathcal{Y}(s))=0$. Hence, by (5.3), we get $\bar{\mathcal{X}}(s)\le\mathcal{X}(s)$. Assume that there exists $s\in\mathbb{R}$ such that $\bar{\mathcal{X}}(s)<\mathcal{X}(s)$. Let $s_0=\mathcal{X}^{-1}(\bar{\mathcal{X}}(s))$ and $s_1=\mathcal{Y}^{-1}(\bar{\mathcal{Y}}(s))$. Since $\mathcal{X}(s_0)=\bar{\mathcal{X}}(s)<\mathcal{X}(s)$ and $\mathcal{Y}(s_1)=\bar{\mathcal{Y}}(s)>\mathcal{Y}(s)$, we have $s_0<s<s_1$. By (4.12a), (4.12d) and since $t(\mathcal{X}(s_0),\mathcal{Y}(s_0))=t(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s))=t(\mathcal{X}(s_1),\mathcal{Y}(s_1))=0$, we get $t_Y(\mathcal{X}(s_0),Y)=0$ for $Y\in[\mathcal{Y}(s_0),\mathcal{Y}(s_1)]$ and $t_X(X,\mathcal{Y}(s_1))=0$ for $X\in[\mathcal{X}(s_0),\mathcal{X}(s_1)]$. This implies, by (4.12a), that $x_Y(\mathcal{X}(s_0),Y)=0$ for $Y\in[\mathcal{Y}(s_0),\mathcal{Y}(s_1)]$ and $x_X(X,\mathcal{Y}(s_1))=0$ for $X\in[\mathcal{X}(s_0),\mathcal{X}(s_1)]$. Then,

$$\begin{aligned}
x_1(\bar{\mathcal{X}}(s)) &= x_1(\mathcal{X}(s_0))\\
&= \mathcal{Z}_2(s_0) \quad \text{by (3.30b)}\\
&= x(\mathcal{X}(s_0),\mathcal{Y}(s_0)) \quad \text{since } (Z,p,q)\bullet(\mathcal{X},\mathcal{Y})=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})\\
&= x(\mathcal{X}(s_0),\mathcal{Y}(s_1))\\
&= x(\mathcal{X}(s_1),\mathcal{Y}(s_1)) = \mathcal{Z}_2(s_1) = x_2(\mathcal{Y}(s_1)) = x_2(\bar{\mathcal{Y}}(s)).
\end{aligned}$$

However, the fact that $x_1(\bar{\mathcal{X}}(s))=x_2(\bar{\mathcal{Y}}(s))$ and $\bar{\mathcal{X}}(s)<\mathcal{X}(s)$ contradicts the definition of $\mathcal{X}$ in (3.28). Hence, we must have $\bar{\mathcal{X}}=\mathcal{X}$, which yields $\bar{\mathcal{Y}}=\mathcal{Y}$ and

$$\begin{aligned}
\bar{\mathcal{Z}}(s) &= Z(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s)) = Z(\mathcal{X}(s),\mathcal{Y}(s)) = \mathcal{Z}(s),\\
\bar{\mathcal{V}}(\bar{\mathcal{X}}) &= \bar{\mathcal{V}}(\mathcal{X}) = Z_X(\mathcal{X},\mathcal{Y}) = \mathcal{V}(\mathcal{X}),\\
\bar{\mathcal{W}}(\bar{\mathcal{Y}}) &= \bar{\mathcal{W}}(\mathcal{Y}) = Z_Y(\mathcal{X},\mathcal{Y}) = \mathcal{W}(\mathcal{Y}),\\
\bar{\mathfrak{p}}(\bar{\mathcal{X}}) &= \bar{\mathfrak{p}}(\mathcal{X}) = p(\mathcal{X},\mathcal{Y}) = \mathfrak{p}(\mathcal{X}),\\
\bar{\mathfrak{q}}(\bar{\mathcal{Y}}) &= \bar{\mathfrak{q}}(\mathcal{Y}) = q(\mathcal{X},\mathcal{Y}) = \mathfrak{q}(\mathcal{Y}).
\end{aligned}$$

Hence, we have proved that $\mathbf{E}\circ\mathbf{S}\circ\mathbf{C}=\mathbf{C}$. By the uniqueness of the solution for given data $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})\in\mathcal{G}_0$, we have that $\mathbf{S}\circ\mathbf{E}=\mathrm{Id}$. This concludes the proof of (5.16). □

Theorem 5.6. *The mapping S_T is a semigroup.*

Proof. We have

$$\begin{aligned} S_T \circ S_{T'} &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C} \circ \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} \quad \text{by Lemma 5.5} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T+T'} \circ \mathbf{S} \circ \mathbf{C} \quad \text{by (5.2)} \\ &= S_{T+T'}. \end{aligned}$$

□

5.3. Mapping from $\mathcal{F}$ to $\mathcal{D}$.

Definition 5.7. *Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we define $(u, R, S, \rho, \sigma, \mu, \nu)$ as*

$$u(x) = U_1(X) \quad \text{if } x_1(X) = x \tag{5.19a}$$

or, equivalently,

$$u(x) = U_2(Y) \quad \text{if } x_2(Y) = x, \tag{5.19b}$$

$$R(x)\,dx = (x_1)_\#(2c(U_1(X))V_1(X)\,dX), \tag{5.19c}$$

$$S(x)\,dx = (x_2)_\#(-2c(U_2(Y))V_2(Y)\,dY), \tag{5.19d}$$

$$\rho(x)\,dx = (x_1)_\#(2H_1(X)\,dX), \tag{5.19e}$$

$$\sigma(x)\,dx = (x_2)_\#(2H_2(Y)\,dY), \tag{5.19f}$$

$$\mu = (x_1)_\#(J_1'(X)\,dX), \tag{5.19g}$$

$$\nu = (x_2)_\#(J_2'(Y)\,dY). \tag{5.19h}$$

The relations (5.19c)-(5.19f) *are equivalent to*

$$R(x_1(X))x_1'(X) = 2c(U_1(X))V_1(X), \tag{5.20a}$$

$$S(x_2(Y))x_2'(Y) = -2c(U_2(Y))V_2(Y), \tag{5.20b}$$

$$\rho(x_1(X))x_1'(X) = 2H_1(X), \tag{5.20c}$$

$$\sigma(x_2(Y))x_2'(Y) = 2H_2(Y), \tag{5.20d}$$

respectively, for almost every X and Y. We denote by $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ the mapping which to any $\psi \in \mathcal{F}$ associates the element $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$ as defined above.

The push-forward of a measure λ by a function f is the measure $f_\#\lambda$ defined by $f_\#\lambda(B) = \lambda(f^{-1}(B))$ for Borel sets B.

The well-posedness of Definition 5.7 is part of the proof of the following lemma.

Lemma 5.8. *Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, let $(u, R, S, \rho, \sigma, \mu, \nu) = \mathbf{M}(\psi_1, \psi_2)$. Then, for any $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ such that $(\psi_1, \psi_2) = \mathbf{D}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$, we have*

$$u(x) = \mathcal{Z}_3(s) \quad \text{if } x = \mathcal{Z}_2(s), \tag{5.21a}$$

$$R(x)\,dx = (\mathcal{Z}_2)_\#(2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), \tag{5.21b}$$

$$S(x)\,dx = (\mathcal{Z}_2)_\#(-2c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds), \tag{5.21c}$$

$$\rho(x)\,dx = (\mathcal{Z}_2)_\#(2\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), \tag{5.21d}$$

$$\sigma(x)\,dx = (\mathcal{Z}_2)_\#(2\mathfrak{q}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds), \tag{5.21e}$$

$$\mu = (\mathcal{Z}_2)_\#(\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), \tag{5.21f}$$

$$\nu = (\mathcal{Z}_2)_\#(\mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds). \tag{5.21g}$$

The relations (5.21b) *and* (5.21d) *are equivalent to*

$$R(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)), \tag{5.22a}$$

$$\rho(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s)) = \mathfrak{p}(\mathcal{X}(s)) \tag{5.22b}$$

for any s such that $\dot{\mathcal{X}}(s) > 0$, respectively. The relations (5.21c) *and* (5.21e) *are equivalent to*

$$S(\mathcal{Z}_2(s))\mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)), \tag{5.23a}$$

$$\sigma(\mathcal{Z}_2(s))\mathcal{W}_2(\mathcal{Y}(s)) = \mathfrak{q}(\mathcal{Y}(s)) \tag{5.23b}$$

for any s such that $\dot{\mathcal{Y}}(s) > 0$, respectively.

Proof. We decompose the proof into five steps.

Step 1. We first prove that (5.19) implies (5.21). If $x = x_1(X)$, let $s = \mathcal{X}^{-1}(X)$. Then, by (5.10a), $x = x_1(X) = x_1(\mathcal{X}(s)) = \mathcal{Z}_2(s)$, and, by (5.10b), $u(x) = U_1(X) = U_1(\mathcal{X}(s)) = \mathcal{Z}_3(s)$. Similarly, if $x = x_2(Y)$, we let $s = \mathcal{Y}^{-1}(Y)$ and obtain $x = x_2(Y) = x_2(\mathcal{Y}(s)) = \mathcal{Z}_2(s)$ and $u(x) = U_2(Y) = U_2(\mathcal{Y}(s)) = \mathcal{Z}_3(s)$. Hence, both (5.19a) and (5.19b) imply (5.21a). The identity (5.21b) follows from (5.19c) since, for any Borel set A, we have

$$\begin{aligned}\int_A R(x)\,dx &= \int_{x_1^{-1}(A)} 2c(U_1(X))V_1(X)\,dX\\ &= \int_{(x_1\circ\mathcal{X})^{-1}(A)} 2c(U_1(\mathcal{X}(s)))V_1(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \quad \text{by a change of variables}\\ &= \int_{\mathcal{Z}_2^{-1}(A)} 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \quad \text{by (5.10a), (5.10b) and (5.10e).}\end{aligned}$$

In the same way, one proves that (5.19d) implies (5.21c). By a similar calculation as above, we obtain

$$\begin{aligned}\int_A \rho(x)\,dx &= \int_{x_1^{-1}(A)} 2H_1(X)\,dX\\ &= \int_{(x_1\circ\mathcal{X})^{-1}(A)} 2H_1(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\\ &= \int_{\mathcal{Z}_2^{-1}(A)} 2\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\end{aligned}$$

and

$$\begin{aligned}\int_A \sigma(x)\,dx &= \int_{x_2^{-1}(A)} 2H_2(Y)\,dY\\ &= \int_{(x_2\circ\mathcal{Y})^{-1}(A)} 2H_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds\\ &= \int_{\mathcal{Z}_2^{-1}(A)} 2\mathfrak{q}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds,\end{aligned}$$

which shows that (5.19e) and (5.19f) imply (5.21d) and (5.21e), respectively. From (5.19g), we find

$$\mu(A) = \int_{x_1^{-1}(A)} J_1'(X)\, dX = \int_{(x_1\circ\mathcal{X})^{-1}(A)} J_1'(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds = \int_{\mathcal{Z}_2^{-1}(A)} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds,$$

so that (5.19g) leads to (5.21f). By a similar calculation, one shows that (5.19h) yields (5.21g).

Step 2. We prove that u is a well-defined function that belongs to $L^2(\mathbb{R})$. Consider $s_0, s_1 \in \mathbb{R}$ such that $s_0 < s_1$ and $x = \mathcal{Z}_2(s_0) = \mathcal{Z}_2(s_1)$. Since $\mathcal{Z}_2$ is continuous and nondecreasing, we have $\dot{\mathcal{Z}}_2(s) = \mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$ for $s \in [s_0, s_1]$, which implies that $\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$. Then, by multiplying (3.24c) with $\dot{\mathcal{X}}(s)^2$ and (3.24d) with $\dot{\mathcal{Y}}(s)^2$, we obtain $\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$ and therefore $\dot{\mathcal{Z}}_3(s) = \mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = 0$ for $s \in [s_0, s_1]$. Hence, $\mathcal{Z}_3(s_0) = \mathcal{Z}_3(s_1)$ and (5.21a) is well-defined. We have

$$\begin{aligned}
\int_{\mathbb{R}} u^2(x)\, dx &= \int_{\mathbb{R}} u^2(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\, ds \quad \text{by a change of variables}\\
&= \int_{\mathbb{R}} \mathcal{Z}_3^2(s)\dot{\mathcal{Z}}_2(s)\, ds \quad \text{by (5.21a)}\\
&= 2\int_{\mathbb{R}} \mathcal{Z}_3^2(s)\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds \quad \text{by (3.26)}\\
&\leq 4\int_{\mathbb{R}} \mathcal{Z}_3^2(s)\mathcal{V}_2(\mathcal{X}(s))\, ds \quad \text{since } 0 \leq \dot{\mathcal{X}} \leq 2 \text{ and } \mathcal{V}_2 \geq 0\\
&= 4\int_{\mathbb{R}} \mathcal{Z}_3^2(s)\Big(\mathcal{V}_2^a(\mathcal{X}(s)) + \frac{1}{2}\Big)\, ds \quad \text{by (3.18)}\\
&\leq 4\Big(||\mathcal{V}_2^a||_{L^\infty(\mathbb{R})} + \frac{1}{2}\Big)||\mathcal{Z}_3||^2_{L^2(\mathbb{R})}
\end{aligned}$$

and $u \in L^2(\mathbb{R})$.

Step 3. We show that the definitions (5.21b)-(5.21e) are well-defined, and that the relations (5.22a)-(5.23b) hold. First we prove that the measures

$$\begin{aligned}
&(\mathcal{Z}_2)_\#(2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds),\\
&(\mathcal{Z}_2)_\#(-2c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\, ds),\\
&(\mathcal{Z}_2)_\#(2\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds),\\
&(\mathcal{Z}_2)_\#(2\mathfrak{q}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\, ds)
\end{aligned}$$

are absolutely continuous with respect to Lebesgue measure. We claim that the function

$$F(x) = \int_{\mathcal{Z}_2^{-1}((-\infty,x])} 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds$$

is absolutely continuous. Let $(x_i, \bar{x}_i)$, $i = 1, \dots, N$, be non-intersecting intervals. We have

$$\sum_{i=1}^N |F(\bar{x}_i) - F(x_i)| = \sum_{i=1}^N \Big| \int_{\mathcal{Z}_2^{-1}((x_i,\bar{x}_i])} 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds\Big|.$$

The set $\mathcal{Z}_2^{-1}((x_i,\bar{x}_i])$ is an interval, since the function $\mathcal{Z}_2$ is nondecreasing and continuous. Denote $s_i = \sup\{s\in\mathbb{R} \mid \mathcal{Z}_2(s)\le x_i\}$ and $\bar{s}_i = \sup\{s\in\mathbb{R} \mid \mathcal{Z}_2(s)\le \bar{x}_i\}$. We have $\mathcal{Z}_2(s_i)=x_i$, $\mathcal{Z}_2(\bar{s}_i)=\bar{x}_i$ and $\mathcal{Z}_2^{-1}((x_i,\bar{x}_i]) = (s_i,\bar{s}_i]$. Then

$$\sum_{i=1}^N |F(\bar{x}_i)-F(x_i)| \le 2\kappa \int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} |\mathcal{V}_3(\mathcal{X}(s))|\dot{\mathcal{X}}(s)\,ds.$$

From (3.24c), we obtain $|\mathcal{V}_3(\mathcal{X})|\le \kappa(2\mathcal{V}_4(\mathcal{X})\mathcal{V}_2(\mathcal{X}))^{\frac12}$. This implies, by the Cauchy–Schwarz inequality, that

$$\begin{aligned}(5.24)\quad &\sum_{i=1}^N |F(\bar{x}_i)-F(x_i)|\\ &\le 2\kappa^2 \int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} (\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s))^{\frac12}(2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s))^{\frac12}\,ds\\ &\le 2\kappa^2 \Big(\int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\Big)^{\frac12}\Big(\int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\Big)^{\frac12}.\end{aligned}$$

Inserting the estimates

$$\begin{aligned}\int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds &\le \int_{\mathbb{R}} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\\ &\le \int_{\mathbb{R}} (\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)+\mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s))\,ds\\ &= \int_{\mathbb{R}} \dot{\mathcal{Z}}_4(s)\,ds\\ &\le ||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})} \quad \text{by (3.25)}\end{aligned}$$

and

$$\int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds = \int_{\bigcup_{i=1}^N (s_i,\bar{s}_i]} \dot{\mathcal{Z}}_2(s)\,ds = \sum_{i=1}^N |\bar{x}_i - x_i|$$

into (5.24), we get

$$\sum_{i=1}^N |F(\bar{x}_i)-F(x_i)| \le C\Big(\sum_{i=1}^N |\bar{x}_i-x_i|\Big)^{\frac12}$$

for a constant C which only depends on $|||\Theta|||_{\mathcal{G}}$ and κ. This implies that F is absolutely continuous. Then, the measure $(\mathcal{Z}_2)_\#(2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds)$ is absolutely continuous. In the same way, one proves that measures $(\mathcal{Z}_2)_\#(-2c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds)$, $(\mathcal{Z}_2)_\#(2\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds)$ and $(\mathcal{Z}_2)_\#(2\mathfrak{q}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds)$ are absolutely continuous, so that the functions R, S, ρ and σ as given by (5.21b)-(5.21e) are well-defined.

Let us prove (5.22a). We have

$$\int_{\mathcal{Z}_2^{-1}(A)} R(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\,ds = \int_{\mathcal{Z}_2^{-1}(A)} 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \tag{5.25}$$

for any Borel set A, and we want to show that for any measurable set B,

$$\int_B R(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\,ds = \int_B 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds. \tag{5.26}$$

For any measurable set B, we have the decomposition $\mathcal{Z}_2^{-1}(\mathcal{Z}_2(B)) = B \cup (B^c \cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(B)))$. Let us prove that $\dot{\mathcal{Z}}_2 = 0$ on $B^c \cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(B))$. Consider a point $\bar{s} \in B^c \cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(B))$. There exists $\tilde{s} \in B$ such that $\mathcal{Z}_2(\bar{s}) = \mathcal{Z}_2(\tilde{s})$, which implies, since $\mathcal{Z}_2$ is nondecreasing, that $\dot{\mathcal{Z}}_2 = 0$ on the interval joining the points $\bar{s}$ and $\tilde{s}$. Since $\bar{s}$ was arbitrary, we conclude that $\dot{\mathcal{Z}}_2(s) = 0$ for all $s \in B^c \cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(B))$. Then, by an estimate as above, we get

$$\begin{aligned}
&\left| \int_{B^c\cap\mathcal{Z}_2^{-1}(\mathcal{Z}_2(B))} 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \right| \\
&\leq 2\kappa^2 ||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \left(\int_{B^c\cap\mathcal{Z}_2^{-1}(\mathcal{Z}_2(B))} \dot{\mathcal{Z}}_2(s)\,ds \right)^{\frac{1}{2}} = 0.
\end{aligned}$$

Hence, by taking $A = \mathcal{Z}_2(B)$ in (5.25), we obtain (5.26). Thus,

$$R(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s) = 2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)$$

which yields, because $\dot{\mathcal{Z}}_2(s) = 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)$,

$$R(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))$$

for any s such that $\dot{\mathcal{X}}(s) > 0$. Similarly, one proves (5.22b), (5.23a) and (5.23b).

Step 4. We show that R, S, ρ and σ belong to $L^2(\mathbb{R})$, and $u_x = \frac{R-S}{2c(u)}$. Since

$$\begin{aligned}
\int_{\mathbb{R}} R^2(x)\,dx &= \int_{\mathbb{R}} R^2(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\,ds \quad \text{by a change of variables} \\
&= 2\int_{\mathbb{R}} R^2(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \quad \text{by (3.26)} \\
&= 2\int_{\{s\in\mathbb{R}\,|\,\mathcal{V}_2(\mathcal{X}(s))>0\}} \frac{\big(R(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s))\big)^2}{\mathcal{V}_2(\mathcal{X}(s))}\dot{\mathcal{X}}(s)\,ds \\
&= 2\int_{\{s\in\mathbb{R}\,|\,\mathcal{V}_2(\mathcal{X}(s))>0\}} \frac{\big(c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\big)^2}{\mathcal{V}_2(\mathcal{X}(s))}\dot{\mathcal{X}}(s)\,ds \quad \text{by (5.22a)} \\
&\leq 4\int_{\mathbb{R}} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \quad \text{by (3.24c)} \\
&\leq 4\int_{\mathbb{R}} \big(\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\big)\,ds \quad \text{since } \mathcal{W}_4 \geq 0 \\
&= 4\int_{\mathbb{R}} \dot{\mathcal{Z}}_4(s)\,ds \\
&\leq 4||\mathcal{Z}_4^a||_{L^\infty(\mathbb{R})} \quad \text{by (3.25)},
\end{aligned}$$

R belongs to $L^2(\mathbb{R})$. Similarly, using (5.22b), (5.23a) and (5.23b), one proves that ρ, S and σ belong to $L^2(\mathbb{R})$, respectively.

Let ϕ be a smooth test function with compact support. We have

$$
\begin{aligned}
&\int_{\mathbb{R}} u(x)\phi_x(x)\,dx \\
&= \int_{\mathbb{R}} u(\mathcal{Z}_2(s))\phi_x(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\,ds \quad \text{by a change of variables} \\
&= \int_{\mathbb{R}} \mathcal{Z}_3(s)(\phi(\mathcal{Z}_2(s)))_s\,ds \quad \text{by (5.21a)} \\
&= -\int_{\mathbb{R}} \dot{\mathcal{Z}}_3(s)\phi(\mathcal{Z}_2(s))\,ds \quad \text{by integrating by parts} \\
&= -\int_{\mathbb{R}} \big(\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\big)\phi(\mathcal{Z}_2(s))\,ds \\
&= -\int_{\mathbb{R}} \frac{1}{c(\mathcal{Z}_3(s))}\big(R(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) \\
&\qquad\qquad - S(\mathcal{Z}_2(s))\mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\big)\phi(\mathcal{Z}_2(s))\,ds \quad \text{by (5.22)} \\
&= -\int_{\mathbb{R}} \frac{1}{2c(\mathcal{Z}_3(s))}\big(R(\mathcal{Z}_2(s)) - S(\mathcal{Z}_2(s))\big)\phi(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\,ds \quad \text{by (3.26)} \\
&= -\int_{\mathbb{R}} \frac{1}{2c(u(x))}\big(R(x) - S(x)\big)\phi(x)\,ds \quad \text{by a change of variables.}
\end{aligned}
$$

Hence, $u_x = \frac{R-S}{2c(u)}$ in the sense of distributions.

Step 5. We prove that $\mu_{\text{ac}} = \frac{1}{4}(R^2 + c(u)\rho^2)\,dx$ and $\nu_{\text{ac}} = \frac{1}{4}(S^2 + c(u)\sigma^2)\,dx$. Let

$$A = \{s \in \mathbb{R} \mid \mathcal{V}_2(\mathcal{X}(s)) > 0\} \quad \text{and} \quad B = (\mathcal{Z}_2(A^c))^c. \tag{5.27}$$

Since $\dot{\mathcal{Z}}_2 = 2\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}}$, we have $\dot{\mathcal{Z}}_2 = 0$ on A^c, so that

$$\operatorname{meas}(B^c) = \int_{A^c} \dot{\mathcal{Z}}_2(s)\,ds = 0.$$

Since $\mathcal{Z}_2^{-1}(B^c) = \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c)) \supset A^c$, we have $\mathcal{Z}_2^{-1}(B) \subset A$[4]. Let M be any Borel set. We have $\mu(M) = \mu(M \cap B) + \mu(M \cap B^c)$, and we claim that $\mu(M \cap B)$ is the absolutely continuous part of μ. Since $M \cap B \subset B$, $\mathcal{Z}_2^{-1}(M \cap B) \subset A$. Hence,

$$
\begin{aligned}
&\mu(M \cap B) \\
&= \int_{\mathcal{Z}_2^{-1}(M\cap B)} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \\
&= \int_{\mathcal{Z}_2^{-1}(M\cap B)} \frac{\mathcal{V}_4(\mathcal{X}(s))\mathcal{V}_2(\mathcal{X}(s))}{\mathcal{V}_2(\mathcal{X}(s))}\dot{\mathcal{X}}(s)\,ds \\
&= \int_{\mathcal{Z}_2^{-1}(M\cap B)} \frac{(c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{p}^2(\mathcal{X}(s))}{2\mathcal{V}_2(\mathcal{X}(s))}\dot{\mathcal{X}}(s)\,ds \quad \text{by (3.24c)}
\end{aligned}
$$

[4]The following example is useful to have in mind. Suppose that $\mathcal{Z}_2$ is strictly increasing outside an interval I on which $\dot{\mathcal{Z}}_2 = 0$. Assume further that A^c is a subinterval of I. We have $\mathcal{V}_2(\mathcal{X}) = 0$ on A^c, and $\dot{\mathcal{X}} = 0$ on $I \setminus A^c$. In this case it is not hard to check that $\mathcal{Z}_2^{-1}(B) \subset A$

$$= \frac{1}{4}\int_{\mathcal{Z}_2^{-1}(M\cap B)} (R^2(\mathcal{Z}_2(s)) + c(\mathcal{Z}_3(s))\rho^2(\mathcal{Z}_2(s)))2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \quad \text{by (5.22)}$$
$$= \frac{1}{4}\int_{\mathcal{Z}_2^{-1}(M\cap B)} (R^2(\mathcal{Z}_2(s)) + c(\mathcal{Z}_3(s))\rho^2(\mathcal{Z}_2(s)))\dot{\mathcal{Z}}_2(s)\,ds$$
$$= \frac{1}{4}\int_{M\cap B} (R^2(x) + c(u(x))\rho^2(x))\,dx \quad \text{by a change of variables.}$$

It follows that for any Borel set M with measure zero, $\mu(M\cap B) = 0$, so that $\mu_{\mathrm{ac}} = \frac{1}{4}(R^2(x) + c(u(x))\rho^2(x))\,dx$. Similarly, one proves that $\nu_{\mathrm{ac}} = \frac{1}{4}(S^2(x) + c(u(x))\sigma^2(x))\,dx$.

For further reference, let us prove that the singular part of μ, $\mu_{\mathrm{sing}}(M) = \mu(M\cap B^c) = \mu(M\cap \mathcal{Z}_2(A^c))$, can be written as

$$\mu_{\mathrm{sing}}(M) = \int_{\mathcal{Z}_2^{-1}(M)\cap A^c} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds. \tag{5.28}$$

We have

$$\mu_{\mathrm{sing}}(M) = \int_{\mathcal{Z}_2^{-1}(M\cap \mathcal{Z}_2(A^c))} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \tag{5.29}$$

and

$$\begin{aligned}\mathcal{Z}_2^{-1}(M\cap \mathcal{Z}_2(A^c)) &= \mathcal{Z}_2^{-1}(M)\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c))\\ &= \mathcal{Z}_2^{-1}(M)\cap (A^c\cup (A\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c))))\\ &= (\mathcal{Z}_2^{-1}(M)\cap A^c)\cup (\mathcal{Z}_2^{-1}(M)\cap (A\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c)))).\end{aligned}$$

Either the set $A\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c))$ is empty, or $\mathcal{Z}_2$ is constant on $A\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c))$, and that $\dot{\mathcal{Z}}_2 = 2\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} = 0$ and, since $\mathcal{V}_2(\mathcal{X}) > 0$ on A, we must have that $\dot{\mathcal{X}} = 0$ on $A\cap \mathcal{Z}_2^{-1}(\mathcal{Z}_2(A^c))$. Then (5.28) follows from (5.29). In a similar way, one shows that

$$\nu_{\mathrm{sing}}(M) = \int_{\mathcal{Z}_2^{-1}(M)\cap A^c} \mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds.$$

□

By using the semigroup S_T we can, together with the mappings from $\mathcal{D}$ to $\mathcal{F}$ and vica versa, study the solution in the original set of variables, for given initial data in $\mathcal{D}$.

Lemma 5.9. *Given* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, *let*

$$(u, R, S, \rho, \sigma, \mu, \nu)(T) = \mathbf{M}\circ S_T\circ \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$$

and

$$(Z, p, q) = \mathbf{S}\circ \mathbf{C}\circ \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0).$$

Then, we have

$$u(t(X,Y), x(X,Y)) = U(X,Y) \tag{5.30}$$

for all $(X,Y)\in\mathbb{R}^2$,

$$R(t(X,Y), x(X,Y))x_X(X,Y) = c(U(X,Y))U_X(X,Y), \tag{5.31a}$$

$$\rho(t(X,Y), x(X,Y))x_X(X,Y) = p(X,Y) \tag{5.31b}$$

for almost every $(X,Y)\in\mathbb{R}^2$ *such that* $x_X(X,Y)>0$, *and*

$$S(t(X,Y),x(X,Y))x_Y(X,Y) = -c(U(X,Y))U_Y(X,Y), \tag{5.32a}$$

$$\sigma(t(X,Y),x(X,Y))x_Y(X,Y) = q(X,Y) \tag{5.32b}$$

for almost every $(X,Y)\in\mathbb{R}^2$ *such that* $x_Y(X,Y)>0$. *Furthermore, we have*

$$u_t = \frac{1}{2}(R+S) \quad \textit{and} \quad u_x = \frac{1}{2c(u)}(R-S) \tag{5.33}$$

in the sense of distributions.

Proof. Given $(X,Y)\in\mathbb{R}^2$, we denote $\bar t = t(X,Y)$ and $\bar x = x(X,Y)$. Let

$$(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) = \mathbf{E}\circ\mathbf{t}_{\bar t}(Z,p,q).$$

We have $t(\mathcal{X}(s),\mathcal{Y}(s)) = \bar t$, $\mathcal{Z}_2(s) = x(\mathcal{X}(s),\mathcal{Y}(s))$ and $\mathcal{Z}_3(s) = U(\mathcal{X}(s),\mathcal{Y}(s))$. Notice that we can write

$$\begin{aligned}(u,R,S,\rho,\sigma,\mu,\nu)(\bar t) &= \mathbf{M}\circ S_{\bar t}\circ\mathbf{L}(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0)\\ &= \mathbf{M}\circ\mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_{\bar t}\circ\mathbf{S}\circ\mathbf{C}\circ\mathbf{L}(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0)\\ &= \mathbf{M}\circ\mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_{\bar t}(Z,p,q)\\ &= \mathbf{M}\circ\mathbf{D}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}),\end{aligned}$$

so that we can apply Lemma 5.8, from which we have that $u(\bar t,\bar x) = \mathcal{Z}_3(s)$ for any s such that $\bar x = \mathcal{Z}_2(s)$. This implies that, for any $\bar s$ such that

$$t(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) = \bar t = t(X,Y) \quad \text{and} \quad x(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) = \bar x = x(X,Y), \tag{5.34}$$

we have

$$u(\bar t,\bar x) = U(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)).$$

Then, (5.30) will be proved once we have proved that

$$U(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) = U(X,Y). \tag{5.35}$$

We show that when (5.34) holds, then either $(X,Y) = (\mathcal{X}(\bar s),\mathcal{Y}(\bar s))$ or

$$x_X = x_Y = U_X = U_Y = p = q = 0 \tag{5.36}$$

in the rectangle with corners at (X,Y) and $(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))$, so that (5.35) holds in both cases. We first consider the rectangle where $\mathcal{X}(\bar s)\le X$ and $\mathcal{Y}(\bar s)\le Y$. Since $x_X\ge 0$ and $x_Y\ge 0$, (5.34) implies that $x_X=0$ and $x_Y=0$ in $[\mathcal{X}(\bar s),X]\times[\mathcal{Y}(\bar s),Y]$. By (4.12c), we have $U_X=U_Y=p=q=0$ in $[\mathcal{X}(\bar s),X]\times[\mathcal{Y}(\bar s),Y]$, so that U is constant and we have proved (5.35). In the case where $\mathcal{X}(\bar s)\le X$ and $\mathcal{Y}(\bar s)\ge Y$, we find, since $t_X\ge 0$ and $t_Y\le 0$, that $t_X=0$ and $t_Y=0$ in $[\mathcal{X}(\bar s),X]\times[Y,\mathcal{Y}(\bar s)]$. By (4.12a), it follows that $x_X=x_Y=0$ and we prove (5.35) as before. The other cases can be treated in the same way. Thus, (5.35) holds and we have proved (5.30). We prove (5.31a) and (5.31b). By (5.22a), (5.22b) and the definition of $\mathbf{E}$, we have

$$\begin{aligned}&R(\bar t,x(\mathcal{X}(s),\mathcal{Y}(s)))x_X(\mathcal{X}(s),\mathcal{Y}(s)) = c(U(\mathcal{X}(s),\mathcal{Y}(s)))U_X(\mathcal{X}(s),\mathcal{Y}(s)),\\ &\rho(\bar t,x(\mathcal{X}(s),\mathcal{Y}(s)))x_X(\mathcal{X}(s),\mathcal{Y}(s)) = p(\mathcal{X}(s),\mathcal{Y}(s)),\end{aligned}$$

so that

$$\begin{aligned}&R(t(X,Y),x(X,Y))x_X(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) = c(U(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)))U_X(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)),\\ &\rho(t(X,Y),x(X,Y))x_X(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) = p(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))\end{aligned}$$

for any $\bar{s}$ such that (5.34) holds. This implies (5.31a) and (5.31b) because when (5.34) is satisfied, then either $(X,Y)=(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s}))$ or (5.36) holds. Similarly, from (5.23a) and (5.23b), we obtain

$$\begin{aligned}&S(t(X,Y),x(X,Y))x_Y(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s}))=-c(U(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s})))U_Y(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s})),\\&\sigma(t(X,Y),x(X,Y))x_Y(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s}))=q(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s}))\end{aligned}$$

for any $\bar{s}$ such that (5.34) holds, so that (5.32a) and (5.32b) follows. Now we prove (5.33). Let $\phi(t,x)$ be a smooth test function with compact support. We have

$$\begin{aligned}&\iint_{\mathbb{R}^2}(u\phi_t)(t,x)\,dt\,dx\\
&=\iint_{\mathbb{R}^2}\big((u\phi_t)\circ(t,x)(t_Xx_Y-t_Yx_X)\big)(X,Y)\,dX\,dY\quad\text{by a change of variables}\\
&=\iint_{\mathbb{R}^2}\big(U\phi_t\circ(t,x)(t_Xx_Y-t_Yx_X)\big)(X,Y)\,dX\,dY\quad\text{by (5.30)}\\
&=\iint_{\mathbb{R}^2}\big(U(\phi_X\circ(t,x)x_Y-\phi_Y\circ(t,x)x_X)\big)(X,Y)\,dX\,dY\quad\text{by calculating }\phi_X\text{ and }\phi_Y\\
&=-\iint_{\mathbb{R}^2}\big(((Ux_Y)_X-(Ux_X)_Y)\phi\circ(t,x)\big)(X,Y)\,dX\,dY\quad\text{by integrating by parts}\\
&=-\iint_{\mathbb{R}^2}\big((U_Xx_Y-U_Yx_X)\phi\circ(t,x)\big)(X,Y)\,dX\,dY\\
&=-\iint_{\mathbb{R}^2}\left(\left(\frac{R+S}{c(u)}\phi\right)\circ(t,x)x_Xx_Y\right)(X,Y)\,dX\,dY\quad\text{by (5.31a) and (5.32a)}\\
&=-\iint_{\mathbb{R}^2}\left(\left(\frac{1}{2}(R+S)\phi\right)\circ(t,x)(t_Xx_Y-t_Yx_X)\right)(X,Y)\,dX\,dY\quad\text{by (4.12a)}\\
&=-\iint_{\mathbb{R}^2}\left(\frac{1}{2}(R+S)\phi\right)(t,x)\,dt\,dx,\end{aligned}$$

which proves the first identity in (5.33). The second one is proven in the same way. □

5.4. **Semigroup of Solutions in $\mathcal{D}$.** Now we can define a mapping on the original set of variables, $\mathcal{D}$.

Definition 5.10. *For any $T>0$, let $\bar{S}_T:\mathcal{D}\to\mathcal{D}$ be defined as*

$$\bar{S}_T=\mathbf{M}\circ S_T\circ\mathbf{L}.$$

Since

$$\bar{S}_T\circ\bar{S}_{T'}=\mathbf{M}\circ S_T\circ\mathbf{L}\circ\mathbf{M}\circ S_{T'}\circ\mathbf{L}$$

it would immediately follow from the semigroup property of S_T that $\bar{S}_T$ is also a semigroup if we had $\mathbf{L}\circ\mathbf{M}=\mathrm{Id}$, but this identity does not hold in general. To see this consider an element ψ in $\mathcal{F}$. By Definition 3.3 we have $x_1+J_1\in G$, which in particular means that $x_1+J_1-\mathrm{Id}\in L^\infty(\mathbb{R})$. Let $\xi=\mathbf{M}(\psi)$ and $\bar{\psi}=\mathbf{L}(\xi)$. From Definition 3.4 we have $\bar{x}_1+\bar{J}_1=\mathrm{Id}$, and it is clear that in general we have $\psi\neq\bar{\psi}$.

It is the aim of this section to prove that $\bar{S}_T$ is a semigroup. The idea is loosely speaking the following. Assume that $x_1(X)+J_1(X)=f(X)$ where $f\in G$. We associate to x_1 and J_1 the functions $\bar{x}_1$ and $\bar{J}_1$ such that $\bar{x}_1+\bar{J}_1=\mathrm{Id}$. We observe that the transformation $\bar{x}_1(X) = x_1(f^{-1}(X))$ and $\bar{J}_1(X) = J_1(f^{-1}(X))$ is such a mapping. The identities (3.7) and (3.8) allow us to define the remaining elements of $\bar{\psi}$, see Definition 5.11 below. The transformation of ψ to $\bar{\psi}$ defines an action of G^2 on the set $\mathcal{F}$. It defines a projection Π from $\mathcal{F}$ on the set

$$\mathcal{F}_0 = \{\psi = (\psi_1, \psi_2) \in \mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \text{ and } x_2 + J_2 = \mathrm{Id}\}.$$

Thus, we have $\bar{\psi} = \Pi(\psi)$.

We prove that the map S_T is invariant under the group acting on $\mathcal{F}$. Thus, we have to define the action of G^2 on the sets $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{H}$. The definition of the action on $\mathcal{F}$ will naturally lead to the definition of the action on the set of curves $\mathcal{C}$ and the set $\mathcal{G}$. We define the action of G^2 on the set of solutions $\mathcal{H}$ so that it commutes with the $\bullet$ operation, see Lemma 5.15 below.

Then we prove that the map $\mathbf{M}$ satisfies $\mathbf{M} = \mathbf{M} \circ \Pi$, and that $\mathcal{F}_0$ contains exactly one element of each equivalence class of $\mathcal{F}$ with respect to G^2.

Definition 5.11. *For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$ and $f, g \in G$, we define $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ as*

$$\bar{x}_1(X) = x_1(f(X)), \qquad \bar{x}_2(Y) = x_2(g(Y)), \tag{5.37a}$$

$$\bar{U}_1(X) = U_1(f(X)), \qquad \bar{U}_2(Y) = U_2(g(Y)), \tag{5.37b}$$

$$\bar{J}_1(X) = J_1(f(X)), \qquad \bar{J}_2(Y) = J_2(g(Y)), \tag{5.37c}$$

$$\bar{K}_1(X) = K_1(f(X)), \qquad \bar{K}_2(Y) = K_2(g(Y)), \tag{5.37d}$$

$$\bar{V}_1(X) = f'(X)V_1(f(X)), \qquad \bar{V}_2(Y) = g'(Y)V_2(g(Y)), \tag{5.37e}$$

$$\bar{H}_1(X) = f'(X)H_1(f(X)), \qquad \bar{H}_2(Y) = g'(Y)H_2(g(Y)). \tag{5.37f}$$

The mapping $\mathcal{F} \times G^2 \to \mathcal{F}$ given by $\psi \times (f, g) \mapsto \bar{\psi}$ defines an action of the group G^2 on $\mathcal{F}$ and we denote $\bar{\psi} = \psi \cdot (f, g)$.

Proof of the well-posedness of Definition 5.11. We prove that $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ belongs to $\mathcal{F}$. We only show that $\bar{\psi}_1$ satisfies the conditions in Definition 3.3. The proof is similar for $\bar{\psi}_2$. First we show that $\bar{\psi}_1$ satisfies the regularity conditions in (3.5). We will use the following result throughout the proof. By Lemma 3.8, there exists $\alpha \geq 0$ such that $\frac{1}{1+\alpha} \leq f'(X) \leq 1 + \alpha$ for almost every $X \in \mathbb{R}$ and $\frac{1}{1+\alpha} \leq g'(Y) \leq 1 + \alpha$ for almost every $Y \in \mathbb{R}$.

Since $x_1 - \mathrm{Id}, f - \mathrm{Id} \in L^\infty(\mathbb{R})$,

$$\bar{x}_1(X) - X = (x_1(f(X)) - f(X)) + (f(X) - X)$$

and $\bar{x}_1 - \mathrm{Id}$ belongs to $L^\infty(\mathbb{R})$. We differentiate and obtain

$$\bar{x}_1'(X) - 1 = (x_1'(f(X)) - 1)f'(X) + (f'(X) - 1).$$

Since $x_1' - 1, f' - 1 \in L^\infty(\mathbb{R})$ and $\frac{1}{1+\alpha} \leq f' \leq 1 + \alpha$, $\bar{x}_1' - 1$ belongs to $L^\infty(\mathbb{R})$. Moreover, by a straightforward calculation, we get

$$\begin{aligned}
\int_{\mathbb{R}} (\bar{x}_1'(X) - 1)^2 \, dX &\leq 2 \int_{\mathbb{R}} (x_1'(f(X)) - 1)^2 f'(X)^2 \, dX + 2 \int_{\mathbb{R}} (f'(X) - 1)^2 \, dX \\
&\leq 2(1+\alpha) \int_{\mathbb{R}} (x_1'(f(X)) - 1)^2 f'(X) \, dX + 2 \int_{\mathbb{R}} (f'(X) - 1)^2 \, dX \\
&= 2(1+\alpha) \int_{\mathbb{R}} (x_1'(X) - 1)^2 \, dX + 2 \int_{\mathbb{R}} (f'(X) - 1)^2 \, dX,
\end{aligned}$$

where we used a change of variables in the last equality. This shows that $\bar{x}_1' - 1 \in L^2(\mathbb{R})$ because $x_1' - 1, f' - 1 \in L^2(\mathbb{R})$. Since $J_1 \in L^\infty(\mathbb{R})$ it follows immediately from the definition of $\bar{J}_1$ that it also belongs to $L^\infty(\mathbb{R})$. We have $\bar{J}_1'(X) = J_1'(f(X))f'(X)$, so that $\bar{J}_1' \in L^\infty(\mathbb{R})$ because $J_1' \in L^\infty(\mathbb{R})$ and $\frac{1}{1+\alpha} \leq f' \leq 1 + \alpha$. The function $\bar{J}_1'$ also belongs to $L^2(\mathbb{R})$ since

$$\begin{aligned} \int_{\mathbb{R}} \bar{J}_1'(X)^2\, dX &= \int_{\mathbb{R}} J_1'(f(X))^2 f'(X)^2\, dX \\ &\leq (1+\alpha) \int_{\mathbb{R}} J_1'(f(X))^2 f'(X)\, dX \\ &= (1+\alpha) \int_{\mathbb{R}} J_1'(X)^2\, dX \quad \text{by a change of variables} \end{aligned} \tag{5.38}$$

and $J_1' \in L^2(\mathbb{R})$. In a similar way, one shows that $\bar{K}_1 \in L^\infty(\mathbb{R})$ and $\bar{K}_1', \bar{H}_1, \bar{V}_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We have

$$\begin{aligned} \int_{\mathbb{R}} \bar{U}_1(X)^2\, dX &= \int_{\mathbb{R}} U_1(f(X))^2 \frac{f'(X)}{f'(X)}\, dX \\ &\leq (1+\alpha) \int_{\mathbb{R}} U_1(f(X))^2 f'(X)\, dX \\ &= (1+\alpha) \int_{\mathbb{R}} U_1(X)^2\, dX \quad \text{by a change of variables,} \end{aligned}$$

so that $\bar{U}_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as $U_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Hence, we have proved (3.5). By differentiating $\bar{x}_1$ and $\bar{J}_1$, we find that the inequalities in (3.6) are satisfied since $x_1', J_1' \geq 0$ and $f' \geq \frac{1}{1+\alpha} > 0$.

The relations in (3.7)-(3.8) follow by direct calculation, for example, we have

$$\begin{aligned} \bar{x}_1'(X)\bar{J}_1'(X) &= f'(X)^2 x_1'(f(X)) J_1'(f(X)) \\ &= f'(X)^2 \big[(c(U_1(f(X)))V_1(f(X)))^2 + c(U_1(f(X)))H_1^2(f(X)) \big] \\ &= (c(\bar{U}_1(X))\bar{V}_1(X))^2 + c(\bar{U}_1(X))\bar{H}_1^2(X). \end{aligned}$$

Let us prove that $\bar{x}_1 + \bar{J}_1 \in G$. We proved above that $\bar{x}_1 - \mathrm{Id}, \bar{J}_1 \in L^\infty(\mathbb{R})$ and $\bar{x}_1' - 1, \bar{J}_1' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, which implies that $\bar{x}_1 + \bar{J}_1 - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ and $\bar{x}_1' + \bar{J}_1' - 1 \in L^2(\mathbb{R})$. Since $x_1 + J_1 \in G$ we get by Lemma 3.8 that there exists $\alpha_1 \geq 0$ such that $\frac{1}{1+\alpha_1} \leq x_1' + J_1' \leq 1 + \alpha_1$ and $\bar{x}_1 + \bar{J}_1$ is invertible because

$$\bar{x}_1'(X) + \bar{J}_1'(X) = f'(X)(x_1'(f(X)) + J_1'(f(X))) \geq \frac{1}{(1+\alpha)(1+\alpha_1)} > 0.$$

We prove that $(\bar{x}_1 + \bar{J}_1)^{-1} - \mathrm{Id}$ belongs to $W^{1,\infty}(\mathbb{R})$. Since

$$\begin{aligned} (\bar{x}_1 + \bar{J}_1)^{-1} - \mathrm{Id} &= f^{-1} \circ (x_1 + J_1)^{-1} - \mathrm{Id} \\ &= f^{-1} \circ (x_1 + J_1)^{-1} - (x_1 + J_1)^{-1} + (x_1 + J_1)^{-1} - \mathrm{Id} \end{aligned}$$

and $f^{-1} - \mathrm{Id}, (x_1 + J_1)^{-1} - \mathrm{Id} \in L^\infty(\mathbb{R})$, $(\bar{x}_1 + \bar{J}_1)^{-1} - \mathrm{Id}$ belongs to $L^\infty(\mathbb{R})$. Let $v = \bar{x}_1 + \bar{J}_1$. We have

$$(v^{-1})' = \frac{1}{v'(v^{-1})} = \frac{1}{f'(v^{-1})(x_1'(f(v^{-1})) + J_1'(f(v^{-1})))},$$

so that

$$\frac{1}{(1+\alpha)(1+\alpha_1)} - 1 \leq (v^{-1})' - 1 \leq (1+\alpha)(1+\alpha_1) - 1$$

and $((\bar{x}_1 + \bar{J}_1)^{-1})' - 1 \in L^\infty(\mathbb{R})$. Hence, we have proved (3.9).

We prove (3.10). Since $f - \mathrm{Id} \in L^\infty(\mathbb{R})$, we have $\lim_{X\to-\infty} f(X) = -\infty$. This implies, by (3.10), that

$$\lim_{X\to-\infty} \bar{J}_1(X) = \lim_{X\to-\infty} J_1(f(X)) = 0.$$

In order to prove the identities (3.11a) and (3.11b) we have to define the action of G^2 on $\mathcal{C}$.

Definition 5.12. *For any $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ and $f, g \in G$, we define $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ as*

$$\bar{\mathcal{X}} = f^{-1} \circ \mathcal{X} \circ h \quad \text{and} \quad \bar{\mathcal{Y}} = g^{-1} \circ \mathcal{Y} \circ h, \tag{5.39}$$

where $h \in G$ is the re-normalizing function which yields $\bar{\mathcal{X}} + \bar{\mathcal{Y}} = 2\,\mathrm{Id}$, that is,

$$(f^{-1} \circ \mathcal{X} + g^{-1} \circ \mathcal{Y}) \circ h = 2\,\mathrm{Id}\,. \tag{5.40}$$

The mapping $\mathcal{C} \times G^2 \to \mathcal{C}$ given by $(\mathcal{X}, \mathcal{Y}) \times (f, g) \mapsto (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ defines an action of the group G^2 on $\mathcal{C}$ and we denote $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (\mathcal{X}, \mathcal{Y}) \cdot (f, g)$.

The action corresponds to a stretching of the curve in the X and Y directions.

Proof of the well-posedness of Definition 5.12. Let $v = \frac{1}{2}(f^{-1} \circ \mathcal{X} + g^{-1} \circ \mathcal{Y})$. We want to prove that v belongs to G by using Lemma 3.8. We have

$$\begin{aligned}
v - \mathrm{Id} &= \frac{1}{2}(f^{-1} \circ \mathcal{X} - \mathcal{X} + \mathcal{X} + g^{-1} \circ \mathcal{Y}) - \mathrm{Id} \\
&= \frac{1}{2}(f^{-1} \circ \mathcal{X} - \mathcal{X} + 2\,\mathrm{Id} - \mathcal{Y} + g^{-1} \circ \mathcal{Y}) - \mathrm{Id} \\
&= \frac{1}{2}(f^{-1} \circ \mathcal{X} - \mathcal{X} + g^{-1} \circ \mathcal{Y} - \mathcal{Y})
\end{aligned}$$

which belongs to $L^\infty(\mathbb{R})$ because $f - \mathrm{Id}, g - \mathrm{Id} \in L^\infty(\mathbb{R})$. Since f and g belong to G, we have, by Lemma 3.8, that there exists $\alpha \geq 0$ such that $\frac{1}{1+\alpha} \leq (f^{-1})' \leq 1 + \alpha$ and $\frac{1}{1+\alpha} \leq (g^{-1})' \leq 1 + \alpha$ almost everywhere. Then,

$$v' = \frac{1}{2}(((f^{-1})' \circ \mathcal{X})\dot{\mathcal{X}} + ((g^{-1})' \circ \mathcal{Y})\dot{\mathcal{Y}}) \leq \frac{1}{2}(1+\alpha)(\dot{\mathcal{X}} + \dot{\mathcal{Y}}) = 1 + \alpha$$

and, similarly, we obtain that

$$v' \geq \frac{1}{1+\alpha}. \tag{5.41}$$

We show that v is absolutely continuous. Let $(s_i, \bar{s}_i)$, $i = 1, \ldots, N$, be non-intersecting intervals. We have

$$\begin{aligned}
\sum_{i=1}^{N} |v(\bar{s}_i) - v(s_i)| &= \frac{1}{2} \sum_{i=1}^{N} |f^{-1} \circ \mathcal{X}(\bar{s}_i) + g^{-1} \circ \mathcal{Y}(\bar{s}_i) - f^{-1} \circ \mathcal{X}(s_i) - g^{-1} \circ \mathcal{Y}(s_i)| \\
&\leq \frac{1}{2} \sum_{i=1}^{N} \left(\left| \int_{\mathcal{X}(s_i)}^{\mathcal{X}(\bar{s}_i)} (f^{-1})'(X)\, dX \right| + \left| \int_{\mathcal{Y}(s_i)}^{\mathcal{Y}(\bar{s}_i)} (g^{-1})'(Y)\, dY \right| \right)
\end{aligned}$$

$$\leq \frac{1}{2}(1+\alpha)\sum_{i=1}^{N}(|\mathcal{X}(\bar{s}_i)-\mathcal{X}(s_i)|+|\mathcal{Y}(\bar{s}_i)-\mathcal{Y}(s_i)|)$$
$$\leq 2(1+\alpha)\sum_{i=1}^{N}|\bar{s}_i-s_i|,$$

where we used that $\dot{\mathcal{X}}$ and $\dot{\mathcal{Y}}$ are bounded by 2. Hence, v is absolutely continuous.

Let us prove that if $w\in G$, then $(w^{-1})'-1$ belongs to $L^2(\mathbb{R})$. Since $w\circ w^{-1}=\mathrm{Id}$, we have $(w'\circ w^{-1})(w^{-1})'=1$, so that, by Lemma 3.8, $(w^{-1})'\leq 1+\beta$ for some $\beta>0$. This implies that

$$\begin{aligned}
(5.42)\qquad &\int_{\mathbb{R}}((w^{-1})'(x)-1)^2\,dx\\
&=\int_{\mathbb{R}}((w^{-1})'(x)-(w'\circ(w^{-1})(x))(w^{-1})'(x))^2\,dx\\
&=\int_{\mathbb{R}}(1-w'\circ(w^{-1})(x))^2(w^{-1})'(x)^2\,dx\\
&\leq(1+\beta)\int_{\mathbb{R}}(1-w'\circ(w^{-1})(x))^2(w^{-1})'(x)\,dx\\
&\leq(1+\beta)\int_{\mathbb{R}}(1-w'(x))^2\,dx\quad\text{by a change of variables,}
\end{aligned}$$

which is bounded because $w'-1\in L^2(\mathbb{R})$, and we conclude that $(w^{-1})'-1\in L^2(\mathbb{R})$.

Since $\dot{\mathcal{X}}+\dot{\mathcal{Y}}=2$, we have

$$\begin{aligned}
v'-1&=\frac{1}{2}(((f^{-1})'\circ\mathcal{X})\dot{\mathcal{X}}+((g^{-1})'\circ\mathcal{Y})\dot{\mathcal{Y}}-2)\\
&=\frac{1}{2}(((f^{-1})'\circ\mathcal{X})\dot{\mathcal{X}}-\dot{\mathcal{X}}+((g^{-1})'\circ\mathcal{Y})\dot{\mathcal{Y}}-\dot{\mathcal{Y}}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}}(v'(s)-1)^2\,ds\\
&=\frac{1}{4}\int_{\mathbb{R}}(((f^{-1})'\circ\mathcal{X}(s))\dot{\mathcal{X}}(s)-\dot{\mathcal{X}}(s)+((g^{-1})'\circ\mathcal{Y}(s))\dot{\mathcal{Y}}(s)-\dot{\mathcal{Y}}(s))^2\,ds\\
&\leq\frac{1}{2}\int_{\mathbb{R}}(((f^{-1})'\circ\mathcal{X}(s))\dot{\mathcal{X}}(s)-\dot{\mathcal{X}}(s))^2\,ds\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}(((g^{-1})'\circ\mathcal{Y}(s))\dot{\mathcal{Y}}(s)-\dot{\mathcal{Y}}(s))^2\,ds\\
&\leq\int_{\mathbb{R}}((f^{-1})'\circ\mathcal{X}(s)-1)^2\dot{\mathcal{X}}(s)\,ds\\
&\quad+\int_{\mathbb{R}}((g^{-1})'\circ\mathcal{Y}(s)-1)^2\dot{\mathcal{Y}}(s)\,ds\quad\text{since }\dot{\mathcal{X}}\leq 2\text{ and }\dot{\mathcal{Y}}\leq 2\\
&=\int_{\mathbb{R}}((f^{-1})'(X)-1)^2\,dX+\int_{\mathbb{R}}((g^{-1})'(Y)-1)^2\,dY\quad\text{by a change of variables}
\end{aligned}$$

and by (5.42), we conclude that $v'-1\in L^2(\mathbb{R})$. Then, by Lemma 3.8, $v\in G$.

We define $h = v^{-1}$. Since $v \circ h = \mathrm{Id}$, (5.40) holds. We claim that $h \in G$. Since v belongs to G, we have that $h - \mathrm{Id}$ and $h^{-1} - \mathrm{Id}$ belong to $W^{1,\infty}(\mathbb{R})$. From (5.42), we get that $h' - 1 \in L^2(\mathbb{R})$. Therefore, $h \in G$.

Now we prove that $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. We have $\bar{\mathcal{X}} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ since

$$\bar{\mathcal{X}} - \mathrm{Id} = f^{-1} \circ \mathcal{X} \circ h - \mathcal{X} \circ h + \mathcal{X} \circ h - h + h - \mathrm{Id}$$

and $f^{-1} - \mathrm{Id}, \mathcal{X} - \mathrm{Id}, h - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. Since $\dot{\mathcal{X}} \geq 0$, we get

$$\dot{\bar{\mathcal{X}}} = ((f^{-1})' \circ \mathcal{X} \circ h)(\dot{\mathcal{X}} \circ h)h' \geq \frac{1}{(1+\alpha)^2} \dot{\mathcal{X}} \circ h \geq 0.$$

Similarly, one shows that $\bar{\mathcal{Y}} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ and $\dot{\bar{\mathcal{Y}}} \geq 0$. The identity (2.40c) is satisfied since $v \circ h = \mathrm{Id}$, which we proved above. Hence, $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. □

End of proof of the well-posedness of Definition 5.11.

It remains to prove (3.11a) and (3.11b). Let $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. Then $(\mathcal{X}, \mathcal{Y}) = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \cdot (f^{-1}, g^{-1})$ belongs to $\mathcal{C}$. In particular, $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$ for all $s \in \mathbb{R}$ and the identities (3.11a) and (3.11b) hold for the elements corresponding to (ψ_1, ψ_2). Furthermore, $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (\mathcal{X}, \mathcal{Y}) \cdot (f, g)$, which implies, using the same notation as in Definition 5.12, that

$$\bar{x}_1(\bar{\mathcal{X}}(s)) = x_1(\mathcal{X}(h(s))) = x_2(\mathcal{Y}(h(s))) = \bar{x}_2(\bar{\mathcal{Y}}(s)).$$

We find

$$\bar{U}_1(\bar{\mathcal{X}}(s)) = U_1(\mathcal{X}(h(s))) = U_2(\mathcal{Y}(h(s))) = \bar{U}_2(\bar{\mathcal{Y}}(s)),$$

which proves (3.11a). Since

$$\dot{\bar{\mathcal{X}}}(s) = \frac{\dot{\mathcal{X}}(h(s))h'(s)}{f'(\bar{\mathcal{X}}(s))} \quad \text{and} \quad \dot{\bar{\mathcal{Y}}}(s) = \frac{\dot{\mathcal{Y}}(h(s))h'(s)}{g'(\bar{\mathcal{Y}}(s))}$$

we have

$$\bar{V}_1(\bar{\mathcal{X}}(s))\dot{\bar{\mathcal{X}}}(s) + \bar{V}_2(\bar{\mathcal{Y}}(s))\dot{\bar{\mathcal{Y}}}(s) = h'(s)\big[V_1(\mathcal{X}(h(s)))\dot{\mathcal{X}}(h(s)) + V_2(\mathcal{Y}(h(s)))\dot{\mathcal{Y}}(h(s))\big].$$

and we obtain

$$\begin{aligned}\bar{V}_1(\bar{\mathcal{X}}(s))\dot{\bar{\mathcal{X}}}(s) + \bar{V}_2(\bar{\mathcal{Y}}(s))\dot{\bar{\mathcal{Y}}}(s) &= \frac{d}{ds}U_1(\mathcal{X}(h(s))) = \frac{d}{ds}\bar{U}_1(\bar{\mathcal{X}}(s)) \\ &= \frac{d}{ds}U_2(\mathcal{Y}(h(s))) = \frac{d}{ds}\bar{U}_2(\bar{\mathcal{Y}}(s)).\end{aligned}$$

This proves (3.11b). □

Definition 5.13. *For any* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}$ *and* $f, g \in G$, *we define* $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}})$ *as*

$$(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (\mathcal{X}, \mathcal{Y}) \cdot (f, g), \tag{5.43a}$$

$$\bar{\mathcal{Z}} = \mathcal{Z} \circ h, \tag{5.43b}$$

where h *is given by* (5.40), *and*

$$\bar{\mathcal{V}}(X) = f'(X)\mathcal{V}(f(X)), \qquad \bar{\mathcal{W}}(Y) = g'(Y)\mathcal{W}(g(Y)), \tag{5.43c}$$

$$\bar{\mathfrak{p}}(X) = f'(X)\mathfrak{p}(f(X)), \qquad \bar{\mathfrak{q}}(Y) = g'(Y)\mathfrak{q}(g(Y)). \tag{5.43d}$$

The mapping $\mathcal{G} \times G^2 \to \mathcal{G}$ *given by* $\Theta \times (f, g) \mapsto \bar{\Theta}$ *defines an action of the group* G^2 *on* $\mathcal{G}$ *and we denote* $\bar{\Theta} = \Theta \cdot (f, g)$.

Proof of the well-posedness of Definition 5.13. We have to prove that $\bar{\Theta}$ belongs to $\mathcal{G}$. In the proof of the well-posedness of Definition 5.12, we showed that $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. We check that $||\bar{\Theta}||_{\mathcal{G}}$ and $|||\bar{\Theta}|||_{\mathcal{G}}$ are finite. Since $f, h \in G$, we have from (3.3) that $f - \mathrm{Id}, f^{-1} - \mathrm{Id}, h - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ and therefore, by (3.18), we conclude that the quantities

$$\begin{aligned}
\bar{\mathcal{Z}}_1^a &= \bar{\mathcal{Z}}_1 - \frac{1}{c(0)}(\bar{\mathcal{X}} - \mathrm{Id}) \\
&= \mathcal{Z}_1 \circ h - \frac{1}{c(0)}(f^{-1} \circ \mathcal{X} \circ h - \mathrm{Id}) \\
&= \mathcal{Z}_1 \circ h - \frac{1}{c(0)}(f^{-1} \circ \mathcal{X} \circ h - \mathcal{X} \circ h + \mathcal{X} \circ h - h + h - \mathrm{Id}) \\
&= \mathcal{Z}_1^a \circ h - \frac{1}{c(0)}(f^{-1} \circ \mathcal{X} \circ h - \mathcal{X} \circ h + h - \mathrm{Id}),
\end{aligned}$$

$$\bar{\mathcal{Z}}_2^a = \bar{\mathcal{Z}}_2 - \mathrm{Id} = \mathcal{Z}_2 \circ h - \mathrm{Id} = \mathcal{Z}_2 \circ h - h + h - \mathrm{Id} = \mathcal{Z}_2^a \circ h + h - \mathrm{Id}$$

and

$$\bar{\mathcal{Z}}_i^a = \bar{\mathcal{Z}}_i = \mathcal{Z}_i \circ h = \mathcal{Z}_i^a \circ h \quad \text{for } i \in \{3, 4, 5\}$$

belong to $L^\infty(\mathbb{R})$.

By Lemma 3.8, there exists $\delta > 0$ such that

$$f'(X) \geq \delta, \quad g'(X) \geq \delta, \quad \text{and} \quad h'(X) \geq \delta, \tag{5.44}$$

for almost every $X \in \mathbb{R}$. This yields

$$\int_{\mathbb{R}} \bar{\mathcal{Z}}_3(s)^2 \, ds = \int_{\mathbb{R}} (\mathcal{Z}_3 \circ h(s))^2 \, ds \leq \frac{1}{\delta} \int_{\mathbb{R}} (\mathcal{Z}_3 \circ h(s))^2 h'(s) \, ds = \frac{1}{\delta} \int_{\mathbb{R}} \mathcal{Z}_3(s)^2 \, ds$$

by a change of variables, and we conclude that $\bar{\mathcal{Z}}_3 \in L^2(\mathbb{R})$. Furthermore, we have

$$\begin{aligned}
\bar{\mathcal{V}}_1^a &= \bar{\mathcal{V}}_1 - \frac{1}{2c(0)} \\
&= f'\mathcal{V}_1 \circ f - \frac{1}{2c(0)} \\
&= f'\left(\mathcal{V}_1 \circ f - \frac{1}{2c(0)}\right) + \frac{1}{2c(0)}(f' - 1) \\
&= f'\mathcal{V}_1^a \circ f + \frac{1}{2c(0)}(f' - 1),
\end{aligned}$$

$$\bar{\mathcal{V}}_2^a = \bar{\mathcal{V}}_2 - \frac{1}{2} = f'\mathcal{V}_2 \circ f - \frac{1}{2} = f'\left(\mathcal{V}_2 \circ f - \frac{1}{2}\right) + \frac{1}{2}(f' - 1) = f'\mathcal{V}_2^a \circ f + \frac{1}{2}(f' - 1),$$

$$\bar{\mathcal{V}}_i^a = \bar{\mathcal{V}}_i = f'\mathcal{V}_i \circ f = f'\mathcal{V}_i^a \circ f \quad \text{for } i \in \{3, 4, 5\},$$

$$\bar{\mathfrak{p}} = f'\mathfrak{p} \circ f$$

and

$$\bar{\mathfrak{q}} = g'\mathfrak{p} \circ g$$

which implies, by (3.3), (3.4) and (5.44), that all the components of $\bar{\mathcal{V}}^a$ and $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{q}}$ belong to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Similarly, one shows that the components of $\bar{\mathcal{W}}^a$ belong to the same set. From (5.43c) and (5.44), we see that the inequalities in (3.22) are satisfied for $\bar{\Theta}$. Since

$$\frac{1}{\bar{\mathcal{V}}_2+\bar{\mathcal{V}}_4}=\frac{1}{f'(\mathcal{V}_2\circ f+\mathcal{V}_4\circ f)}\leq\frac{1}{\delta(\mathcal{V}_2\circ f+\mathcal{V}_4\circ f)}$$

and

$$\frac{1}{\bar{\mathcal{W}}_2+\bar{\mathcal{W}}_4}=\frac{1}{g'(\mathcal{W}_2\circ g+\mathcal{W}_4\circ g)}\leq\frac{1}{\delta(\mathcal{W}_2\circ g+\mathcal{W}_4\circ g)},$$

$\frac{1}{\bar{\mathcal{V}}_2+\bar{\mathcal{V}}_4}$ and $\frac{1}{\bar{\mathcal{W}}_2+\bar{\mathcal{W}}_4}$ belong to $L^\infty(\mathbb{R})$. Hence, $||\bar{\Theta}||_{\mathcal{G}}$ and $|||\bar{\Theta}|||_{\mathcal{G}}$ are finite. By (5.39), we obtain

$$\begin{aligned}\dot{\bar{\mathcal{Z}}}(s)&=\dot{\mathcal{Z}}(h(s))\dot{h}(s)\\&=\big[\mathcal{V}(\mathcal{X}(h(s)))\dot{\mathcal{X}}(h(s))+\mathcal{W}(\mathcal{Y}(h(s)))\dot{\mathcal{Y}}(h(s))\big]\dot{h}(s)\\&=\mathcal{V}(f(\bar{\mathcal{X}}(s)))f'(\bar{\mathcal{X}}(s))\dot{\bar{\mathcal{X}}}(s)+\mathcal{W}(g(\bar{\mathcal{Y}}(s)))g'(\bar{\mathcal{Y}}(s))\dot{\bar{\mathcal{Y}}}(s)\\&=\bar{\mathcal{V}}(\bar{\mathcal{X}}(s))\dot{\bar{\mathcal{X}}}(s)+\bar{\mathcal{W}}(\bar{\mathcal{Y}}(s))\dot{\bar{\mathcal{Y}}}(s)\end{aligned}$$

and we have proved that Θ satisfies (3.23). The relations (3.24a)-(3.24d) follows by direct computation, for instance, we have

$$\begin{aligned}2\bar{\mathcal{V}}_4(\bar{\mathcal{X}})\bar{\mathcal{V}}_2(\bar{\mathcal{X}})&=2f'(\bar{\mathcal{X}})^2\mathcal{V}_4(f(\bar{\mathcal{X}}))\mathcal{V}_2(f(\bar{\mathcal{X}}))\\&=2f'(\bar{\mathcal{X}})^2\mathcal{V}_4(\mathcal{X}(h))\mathcal{V}_2(\mathcal{X}(h))\\&=f'(\bar{\mathcal{X}})^2\big[(c(\mathcal{Z}_3(h))\mathcal{V}_3(\mathcal{X}(h)))^2+c(\mathcal{Z}_3(h))\mathfrak{p}(\mathcal{X}(h))^2\big]\\&=f'(\bar{\mathcal{X}})^2\big[(c(\bar{\mathcal{Z}}_3)\mathcal{V}_3(f(\bar{\mathcal{X}})))^2+c(\bar{\mathcal{Z}}_3)\mathfrak{p}(f(\bar{\mathcal{X}}))^2\big]\\&=(c(\bar{\mathcal{Z}}_3)\bar{\mathcal{V}}_3(\bar{\mathcal{X}}))^2+c(\bar{\mathcal{Z}}_3)\bar{\mathfrak{p}}(\bar{\mathcal{X}})^2.\end{aligned}$$

It remains to prove that (3.25) holds for $\bar{\Theta}$. Since $h-\mathrm{Id}\in L^\infty(\mathbb{R})$, $\lim_{s\to-\infty}h(s)=-\infty$, so that

$$\lim_{s\to-\infty}\bar{\mathcal{Z}}_4(s)=\lim_{s\to-\infty}\mathcal{Z}_4(h(s))=0$$

and we conclude that $\bar{\Theta}\in\mathcal{G}$. □

Definition 5.14. *For any $(Z,p,q)\in\mathcal{H}$ and $f,g\in G$, we define*

$$\bar{Z}(X,Y)=Z(f(X),g(Y)),\tag{5.45a}$$

$$\bar{p}(X,Y)=f'(X)p(f(X),g(Y)),\tag{5.45b}$$

$$\bar{q}(X,Y)=g'(Y)q(f(X),g(Y)).\tag{5.45c}$$

The mapping $\mathcal{H}\times G^2\to\mathcal{H}$ given by $(Z,p,q)\times(f,g)\mapsto(\bar{Z},\bar{p},\bar{q})$ defines an action of the group G^2 on $\mathcal{H}$ and we denote $(\bar{Z},\bar{p},\bar{q})=(Z,p,q)\cdot(f,g)$.

Proof of the well-posedness of Definition 5.14. Condition (i) of Definition 4.13 implies that for any $(Z,p,q)\in\mathcal{H}$, we have

$$Z^a\in[W^{1,\infty}(\mathbb{R}^2)]^5,\quad Z^a_X\in[W^{1,\infty}_Y(\mathbb{R}^2)]^5,\quad Z^a_Y\in[W^{1,\infty}_X(\mathbb{R}^2)]^5,\tag{5.46}$$
$$p\in W^{1,\infty}_Y(\mathbb{R}^2),\quad q\in W^{1,\infty}_X(\mathbb{R}^2),$$

$$(Z_X(X,Y))_Y=F(Z)(Z_X,Z_Y)(X,Y)\tag{5.47}$$

for almost every $X \in \mathbb{R}$,

$$(Z_Y(X,Y))_X = F(Z)(Z_X, Z_Y)(X,Y) \tag{5.48}$$

for almost every $Y \in \mathbb{R}$,

$$p_Y(X,Y) = 0 \tag{5.49}$$

for almost every $X \in \mathbb{R}$,

$$q_X(X,Y) = 0 \tag{5.50}$$

for almost every $Y \in \mathbb{R}$. We first prove that the same regularity conditions hold for $(\bar{Z}, \bar{p}, \bar{q})$. Let us consider the first component of $\bar{Z}$. We have, by (4.5),

$$\begin{aligned}
&\bar{Z}_1^a(X,Y)\\
&= Z_1(f(X), g(Y)) - \frac{1}{2c(0)}(X-Y)\\
&= Z_1(f(X), g(Y)) - \frac{1}{2c(0)}(f(X)-g(Y)) + \frac{1}{2c(0)}(f(X)-g(Y)) - \frac{1}{2c(0)}(X-Y)\\
&= Z_1^a(f(X), g(Y)) + \frac{1}{2c(0)}(f(X) - X + Y - g(Y)).
\end{aligned}$$

Since $f - \mathrm{Id}, g - \mathrm{Id} \in L^\infty(\mathbb{R})$ and $c \geq \frac{1}{\kappa}$, we get, by (5.46), that $\bar{Z}_1^a \in L^\infty(\mathbb{R}^2)$. We differentiate and obtain

$$\bar{Z}_{1,X}^a(X,Y) = f'(X) Z_{1,X}^a(f(X), g(Y)) + \frac{1}{2c(0)}(f'(X) - 1)$$

and

$$\bar{Z}_{1,Y}^a(X,Y) = g'(Y) Z_{1,Y}^a(f(X), g(Y)) - \frac{1}{2c(0)}(g'(Y) - 1).$$

Since f and g belong to G, we have, by Lemma 3.8, that there exists $\alpha \geq 0$ such that $\frac{1}{1+\alpha} \leq f'(X) \leq 1+\alpha$ for almost every $X \in \mathbb{R}$ and $\frac{1}{1+\alpha} \leq g'(Y) \leq 1+\alpha$ for almost every $Y \in \mathbb{R}$. This implies, by (5.46), that $\bar{Z}_{1,X}^a, \bar{Z}_{1,Y}^a \in L^\infty(\mathbb{R}^2)$ as $f' - 1, g' - 1 \in L^\infty(\mathbb{R})$ and $c \geq \frac{1}{\kappa}$. We have

$$\sup_{X\in\mathbb{R}} ||\bar{Z}_{1,X}^a(X,\cdot)||_{L^\infty(\mathbb{R})} \leq (1+\alpha) \sup_{X\in\mathbb{R}} ||Z_{1,X}^a(X,\cdot)||_{L^\infty(\mathbb{R})} + \frac{\kappa}{2}(\alpha+2)$$

and

$$\sup_{Y\in\mathbb{R}} ||\bar{Z}_{1,Y}^a(\cdot,Y)||_{L^\infty(\mathbb{R})} \leq (1+\alpha) \sup_{Y\in\mathbb{R}} ||Z_{1,Y}^a(\cdot,Y)||_{L^\infty(\mathbb{R})} + \frac{\kappa}{2}(\alpha+2).$$

Differentiating $\bar{Z}_{1,X}^a$ and $\bar{Z}_{1,Y}^a$ yields

$$\sup_{X\in\mathbb{R}} ||\bar{Z}_{1,XY}^a(X,\cdot)||_{L^\infty(\mathbb{R})} \leq (1+\alpha)^2 \sup_{X\in\mathbb{R}} ||Z_{1,XY}^a(X,\cdot)||_{L^\infty(\mathbb{R})}$$

and

$$\sup_{Y\in\mathbb{R}} ||\bar{Z}_{1,YX}^a(\cdot,Y)||_{L^\infty(\mathbb{R})} \leq (1+\alpha)^2 \sup_{Y\in\mathbb{R}} ||Z_{1,YX}^a(\cdot,Y)||_{L^\infty(\mathbb{R})}.$$

From (5.46), we conclude that $\bar{Z}_{1,X}^a \in W_Y^{1,\infty}(\mathbb{R}^2)$ and $\bar{Z}_{1,Y}^a \in W_X^{1,\infty}(\mathbb{R}^2)$. One proves the same inclusions for the other components of $\bar{Z}$ in a similar way. We have

$$\sup_{X\in\mathbb{R}} ||\bar{p}(X,\cdot)||_{L^\infty(\mathbb{R})} \leq (1+\alpha) \sup_{X\in\mathbb{R}} ||p(X,\cdot)||_{L^\infty(\mathbb{R})}$$

and

$$\sup_{Y\in\mathbb{R}} ||\bar{q}(\cdot,Y)||_{L^\infty(\mathbb{R})} \leq (1+\alpha)\sup_{Y\in\mathbb{R}} ||q(\cdot,Y)||_{L^\infty(\mathbb{R})}$$

which are bounded by (5.46). From (5.49) and (5.50), we have

$$\bar{p}_Y(X,Y) = f'(X)g'(Y)p_Y(f(X),g(Y)) = 0$$

and

$$\bar{q}_X(X,Y) = f'(X)g'(Y)q_X(f(X),g(Y)) = 0,$$

so that condition (iv) and (v) of Definition 4.5 are satisfied. Moreover, $\bar{p} \in W_Y^{1,\infty}(\mathbb{R}^2)$ and $\bar{q} \in W_X^{1,\infty}(\mathbb{R}^2)$, so that conditions (i) of Definition 4.5 holds. By using the linearity of the mapping $F(Z)$, we obtain

$$\begin{aligned}
\bar{Z}_{XY} &= f'g'Z_{XY}(f,g)\\
&= f'g'F(Z(f,g))(Z_X(f,g),Z_Y(f,g))\\
&= F(Z(f,g))(f'Z_X(f,g),g'Z_Y(f,g))\\
&= F(\bar{Z})(\bar{Z}_X,\bar{Z}_Y),
\end{aligned}$$

so that conditions (ii) and (iii) of Definition 4.5 are satisfied. It remains to show the relations in (4.12). The identities (4.12a)-(4.12c) follows by direct computation. For instance, we have

$$\begin{aligned}
2\bar{J}_X\bar{x}_X &= 2(f')^2J_X(f,g)x_X(f,g)\\
&= (f')^2[(c(U(f,g))U_X(f,g))^2 + c(U(f,g))p^2(f,g)]\\
&= (c(U(f,g))f'U_X(f,g))^2 + c(U(f,g))(f'p(f,g))^2\\
&= (c(\bar{U})\bar{U}_X)^2 + c(\bar{U})\bar{p}^2.
\end{aligned}$$

Since $x_X + J_X > 0$ and $f' \geq \frac{1}{1+\alpha} > 0$, we get

$$\bar{x}_X + \bar{J}_X = f'(x_X(f,g) + J_X(f,g)) \geq \frac{1}{1+\alpha}(x_X(f,g)+J_X(f,g)) > 0.$$

Similarly, one shows the other inequalities in (4.12d)-(4.12f). Hence, we have proved that $(\bar{Z},\bar{p},\bar{q})$ satisfies condition (i) of Definition 4.13. To prove that the second condition is satisfied, we need the following lemma.

Lemma 5.15. *For any $(Z,p,q)\in\mathcal{H}$, $\Gamma = (\mathcal{X},\mathcal{Y})\in\mathcal{C}$ and $\phi = (f,g)\in G^2$, we have*

$$((Z,p,q)\bullet\Gamma)\cdot\phi = ((Z,p,q)\cdot\phi)\bullet(\Gamma\cdot\phi).$$

Proof. Let

$$\begin{aligned}
\Theta &= (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) = (Z,p,q)\bullet\Gamma,\\
\bar{\Theta} &= (\bar{\mathcal{X}},\bar{\mathcal{Y}},\bar{\mathcal{Z}},\bar{\mathcal{V}},\bar{\mathcal{W}},\bar{\mathfrak{p}},\bar{\mathfrak{q}}) = \Theta\cdot\phi,\\
\bar{\Gamma} &= (\bar{\mathcal{X}},\bar{\mathcal{Y}}) = \Gamma\cdot\phi,\\
(\bar{Z},\bar{p},\bar{q}) &= (Z,p,q)\cdot\phi,\\
\tilde{\Theta} &= (\bar{\mathcal{X}},\bar{\mathcal{Y}},\tilde{\mathcal{Z}},\tilde{\mathcal{V}},\tilde{\mathcal{W}},\tilde{\mathfrak{p}},\tilde{\mathfrak{q}}) = (\bar{Z},\bar{p},\bar{q})\bullet\bar{\Gamma}.
\end{aligned}$$

We want to prove that $\bar{\Theta} = \tilde{\Theta}$. By (5.45a), (5.39) and (5.43b), we get

$$\tilde{\mathcal{Z}} = \bar{Z}(\bar{\mathcal{X}},\bar{\mathcal{Y}}) = Z(f\circ\bar{\mathcal{X}}, g\circ\bar{\mathcal{Y}}) = Z(\mathcal{X}\circ h,\mathcal{Y}\circ h) = \mathcal{Z}\circ h = \bar{\mathcal{Z}}.$$

We have

$$\begin{aligned}\tilde{\mathcal{V}}(\bar{\mathcal{X}}) &= \bar{Z}_X(\bar{\mathcal{X}},\bar{\mathcal{Y}})\\ &= f'(\bar{\mathcal{X}})Z_X(f\circ\bar{\mathcal{X}},g\circ\bar{\mathcal{Y}}) \quad \text{by (5.45a)}\\ &= f'(\bar{\mathcal{X}})Z_X(\mathcal{X}\circ h,\mathcal{Y}\circ h) \quad \text{by (5.39)}\\ &= f'(\bar{\mathcal{X}})\mathcal{V}(\mathcal{X}\circ h)\\ &= f'(\bar{\mathcal{X}})\mathcal{V}(f\circ\bar{\mathcal{X}}) \quad \text{by (5.39)}\\ &= \bar{\mathcal{V}}(\bar{\mathcal{X}}) \quad \text{by (5.43c)}.\end{aligned}$$

In a similar way, one shows that $\tilde{\mathcal{W}} = \bar{\mathcal{W}}$. Moreover,

$$\begin{aligned}\tilde{\mathfrak{p}}(\bar{\mathcal{X}}) &= \bar{p}(\bar{\mathcal{X}},\bar{\mathcal{Y}})\\ &= f'(\bar{\mathcal{X}})p(f\circ\bar{\mathcal{X}},g\circ\bar{\mathcal{Y}}) \quad \text{by (5.45b)}\\ &= f'(\bar{\mathcal{X}})p(\mathcal{X}\circ h,\mathcal{Y}\circ h) \quad \text{by (5.39)}\\ &= f'(\bar{\mathcal{X}})\mathfrak{p}(\mathcal{X}\circ h)\\ &= f'(\bar{\mathcal{X}})\mathfrak{p}(f\circ\bar{\mathcal{X}}) \quad \text{by (5.39)}\\ &= \bar{\mathfrak{p}}(\bar{\mathcal{X}}) \quad \text{by (5.43d)}.\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathfrak{q}}(\bar{\mathcal{Y}}) &= \bar{q}(\bar{\mathcal{X}},\bar{\mathcal{Y}})\\ &= g'(\bar{\mathcal{Y}})q(f\circ\bar{\mathcal{X}},g\circ\bar{\mathcal{Y}}) \quad \text{by (5.45c)}\\ &= g'(\bar{\mathcal{Y}})q(\mathcal{X}\circ h,\mathcal{Y}\circ h) \quad \text{by (5.39)}\\ &= g'(\bar{\mathcal{Y}})\mathfrak{q}(\mathcal{Y}\circ h)\\ &= g'(\bar{\mathcal{Y}})\mathfrak{q}(g\circ\bar{\mathcal{Y}}) \quad \text{by (5.39)}\\ &= \bar{\mathfrak{q}}(\bar{\mathcal{Y}}) \quad \text{by (5.43d)}\end{aligned}$$

and we have proved that $\bar{\Theta} = \tilde{\Theta}$. □

End of proof of the well-posedness of Definition 5.14. Now we prove that for any $\phi = (f,g) \in G^2$ and $(Z,p,q) \in \mathcal{H}$, $(\bar{Z},\bar{p},\bar{q})$, as defined in (5.45), satisfies condition (ii) in Definition 4.13. Since $(Z,p,q) \in \mathcal{H}$, there exists a curve $\Gamma \in \mathcal{C}$ such that $(Z,p,q) \bullet \Gamma \in \mathcal{G}$. Consider the curve $\bar{\Gamma} = \Gamma \cdot \phi$ which, by Definition 5.12, belongs to $\mathcal{C}$. By Definition 5.13, $((Z,p,q)\bullet\Gamma)\cdot\phi$ belongs to $\mathcal{G}$. This implies that $(\bar{Z},\bar{p},\bar{q})\bullet\bar{\Gamma} \in \mathcal{G}$, as

$$\begin{aligned}(\bar{Z},\bar{p},\bar{q}) \bullet \bar{\Gamma} &= ((Z,p,q)\cdot\phi) \bullet (\Gamma\cdot\phi) && \text{by (5.45)}\\ &= ((Z,p,q)\bullet\Gamma)\cdot\phi && \text{by Lemma 5.15}\end{aligned}$$

and we have proved the last condition. Hence, we conclude that $(\bar{Z},\bar{p},\bar{q}) \in \mathcal{H}$. □

Lemma 5.16. *The mappings* $\mathbf{E}$, $\mathbf{t}_T$, $\mathbf{S}$, $\mathbf{D}$ *and* $\mathbf{C}$ *are* G^2*-equivariant, that is, for all* $\phi = (f,g) \in G^2$*, we have*

$$\mathbf{E}((Z,p,q)\cdot\phi) = \mathbf{E}(Z,p,q)\cdot\phi, \tag{5.51a}$$

$$\mathbf{t}_T((Z,p,q)\cdot\phi) = \mathbf{t}_T(Z,p,q)\cdot\phi \tag{5.51b}$$

for all $(Z,p,q)\in\mathcal{H}$,

$$\mathbf{S}(\Theta\cdot\phi)=\mathbf{S}(\Theta)\cdot\phi \tag{5.51c}$$

for all $\Theta\in\mathcal{G}$,

$$\mathbf{D}(\Theta\cdot\phi)=\mathbf{D}(\Theta)\cdot\phi \tag{5.51d}$$

for all $\Theta\in\mathcal{G}_0$, *and*

$$\mathbf{C}(\psi\cdot\phi)=\mathbf{C}(\psi)\cdot\phi \tag{5.51e}$$

for all $\psi\in\mathcal{F}$. *Therefore* S_T *is* G^2*-equivariant, that is,*

$$S_T(\psi\cdot\phi)=S_T(\psi)\cdot\phi \tag{5.51f}$$

for all $\psi\in\mathcal{F}$.

Proof. We decompose the proof into six steps.

Step 1. We prove (5.51a). Let

$$\begin{aligned}
\Theta&=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})=\mathbf{E}(Z,p,q),\\
\tilde\Theta&=(\tilde{\mathcal{X}},\tilde{\mathcal{Y}},\tilde{\mathcal{Z}},\tilde{\mathcal{V}},\tilde{\mathcal{W}},\tilde{\mathfrak{p}},\tilde{\mathfrak{q}})=\Theta\cdot\phi,\\
(\bar Z,\bar p,\bar q)&=(Z,p,q)\cdot\phi,\\
\bar\Theta&=(\bar{\mathcal{X}},\bar{\mathcal{Y}},\bar{\mathcal{Z}},\bar{\mathcal{V}},\bar{\mathcal{W}},\bar{\mathfrak{p}},\bar{\mathfrak{q}})=\mathbf{E}(\bar Z,\bar p,\bar q).
\end{aligned}$$

We want to prove that $\tilde\Theta=\bar\Theta$. First we show that $(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})=(\bar{\mathcal{X}},\bar{\mathcal{Y}})$. By (5.3) and (5.45), we have

$$\bar{\mathcal{X}}(s)=\sup\{X\in\mathbb{R}\mid t(f(X'),g(2s-X'))<0\text{ for all }X'<X\}. \tag{5.52}$$

From (5.3), we have $t(\mathcal{X}(s),\mathcal{Y}(s))=0$, so that $t(\mathcal{X}\circ h(s),\mathcal{Y}\circ h(s))=0$ which implies, by (5.39), that $t(f\circ\tilde{\mathcal{X}}(s),g\circ\tilde{\mathcal{Y}}(s))=0$. Hence, by (5.52), $\bar{\mathcal{X}}\leq\tilde{\mathcal{X}}$. Let us assume that $\bar{\mathcal{X}}(s)<\tilde{\mathcal{X}}(s)$ for some $s\in\mathbb{R}$. Since f and g are strictly increasing functions, we have $f(\bar{\mathcal{X}}(s))<f(\tilde{\mathcal{X}}(s))$ and $g(\tilde{\mathcal{Y}}(s))<g(\bar{\mathcal{Y}}(s))$ which implies, by (4.12a) and (4.12d), that

$$t(f\circ\bar{\mathcal{X}}(s),g\circ\bar{\mathcal{Y}}(s))\leq t(f\circ\tilde{\mathcal{X}}(s),g\circ\bar{\mathcal{Y}}(s))\leq t(f\circ\tilde{\mathcal{X}}(s),g\circ\tilde{\mathcal{Y}}(s)).$$

By (5.52), we have $t(f\circ\bar{\mathcal{X}}(s),g\circ\bar{\mathcal{Y}}(s))=0$ and since $t(f\circ\tilde{\mathcal{X}}(s),g\circ\tilde{\mathcal{Y}}(s))=t(\mathcal{X}\circ h(s),\mathcal{Y}\circ h(s))=0$, the monotonicity of t implies that $t(X,Y)=0$ for all $(X,Y)\in[f\circ\bar{\mathcal{X}}(s),f\circ\tilde{\mathcal{X}}(s)]\times[g\circ\tilde{\mathcal{Y}}(s),g\circ\bar{\mathcal{Y}}(s)]$. If $f\circ\bar{\mathcal{X}}(s)\leq 2h(s)-g\circ\bar{\mathcal{Y}}(s)$, set $X'=2h(s)-g\circ\bar{\mathcal{Y}}(s)$ and $Y'=g\circ\bar{\mathcal{Y}}(s)$. We get

$$f\circ\bar{\mathcal{X}}(s)\leq X'<2h(s)-g\circ\tilde{\mathcal{Y}}(s)=2h(s)-\mathcal{Y}\circ h(s)=\mathcal{X}\circ h(s)=f\circ\tilde{\mathcal{X}}(s),$$

that is, $X'\in[f\circ\bar{\mathcal{X}}(s),f\circ\tilde{\mathcal{X}}(s)]$, so that $t(X',Y')=0$. Thus, we have $t(X',Y')=0$, $X'<\mathcal{X}\circ h(s)$ and $X'+Y'=2h(s)$, which contradicts the definition (5.3) of $(\mathcal{X},\mathcal{Y})$ at $h(s)$. If $f\circ\bar{\mathcal{X}}(s)>2h(s)-g\circ\bar{\mathcal{Y}}(s)$, let $X'=f\circ\bar{\mathcal{X}}(s)$ and $Y'=2h(s)-f\circ\bar{\mathcal{X}}(s)$, which implies that $X'<f\circ\tilde{\mathcal{X}}(s)=\mathcal{X}\circ h(s)$ and

$$\begin{aligned}
g\circ\tilde{\mathcal{Y}}(s)&=\mathcal{Y}\circ h(s)=2h(s)-\mathcal{X}\circ h(s)=2h(s)-f\circ\tilde{\mathcal{X}}(s)\\
&<2h(s)-f\circ\bar{\mathcal{X}}(s)=Y'<g\circ\bar{\mathcal{Y}}(s),
\end{aligned}$$

so that $Y' \in [g\circ\tilde{\mathcal{Y}}(s), g\circ\bar{\mathcal{Y}}(s)]$. Thus, $t(X',Y')=0$ which is a contradiction, because $X' < \mathcal{X}\circ h(s)$ and $X'+Y' = 2h(s)$. Hence, we must have $\bar{\mathcal{X}} = \tilde{\mathcal{X}}$, which implies that $\bar{\mathcal{Y}} = \tilde{\mathcal{Y}}$. Then, we get

$$\begin{aligned}
\mathbf{E}((Z,p,q)\cdot\phi) &= ((Z,p,q)\cdot\phi)\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})\\
&= ((Z,p,q)\cdot\phi)\bullet(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})\\
&= ((Z,p,q)\cdot\phi)\bullet((\mathcal{X},\mathcal{Y})\cdot\phi)\\
&= ((Z,p,q)\bullet(\mathcal{X},\mathcal{Y}))\cdot\phi \quad \text{by Lemma 5.15}\\
&= \mathbf{E}(Z,p,q)\cdot\phi
\end{aligned}$$

and we have proved (5.51a).

Step 2. Let us prove (5.51b). We denote

$$\begin{aligned}
(\hat{Z},\hat{p},\hat{q}) &= (Z,p,q)\cdot\phi,\\
(\tilde{Z},\tilde{p},\tilde{q}) &= \mathbf{t}_T(\hat{Z},\hat{p},\hat{q}),\\
(\check{Z},\check{p},\check{q}) &= \mathbf{t}_T(Z,p,q),\\
(\bar{Z},\bar{p},\bar{q}) &= (\check{Z},\check{p},\check{q})\cdot\phi
\end{aligned}$$

and we want to show that $(\tilde{Z},\tilde{p},\tilde{q}) = (\bar{Z},\bar{p},\bar{q})$. By (5.45) and (5.1), we obtain

$$\begin{aligned}
\tilde{t} &= \hat{t} - T = t(f,g) - T = \check{t}(f,g) = \bar{t},\\
\tilde{x} &= \hat{x} = x(f,g) = \check{x}(f,g) = \bar{x},\\
\tilde{p} &= \hat{p} = f'p(f,g) = f'\check{p}(f,g) = \bar{p},\\
\tilde{q} &= \hat{q} = g'q(f,g) = g'\check{q}(f,g) = \bar{q}.
\end{aligned}$$

By a calculation similar to the one where it is shown that $\tilde{x} = \bar{x}$, one obtains for the remaining components that $\tilde{Z} = \bar{Z}$.

Step 3. We prove (5.51c). For any $\Theta = (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})$, we denote

$$\begin{aligned}
(\tilde{Z},\tilde{p},\tilde{q}) &= \mathbf{S}(\Theta\cdot\phi),\\
(Z,p,q) &= \mathbf{S}(\Theta),\\
(\bar{Z},\bar{p},\bar{q}) &= (Z,p,q)\cdot\phi.
\end{aligned}$$

We want to prove that $(\tilde{Z},\tilde{p},\tilde{q}) = (\bar{Z},\bar{p},\bar{q})$. From the definition of the solution operator $\mathbf{S}$ in (4.80), we have

$$(\tilde{Z},\tilde{p},\tilde{q})\bullet((\mathcal{X},\mathcal{Y})\cdot\phi) = \Theta\cdot\phi \quad\text{and}\quad (Z,p,q)\bullet(\mathcal{X},\mathcal{Y}) = \Theta$$

which implies, along with Lemma 5.15, that

$$\begin{aligned}
(\bar{Z},\bar{p},\bar{q})\bullet((\mathcal{X},\mathcal{Y})\cdot\phi) &= ((Z,p,q)\cdot\phi)\bullet((\mathcal{X},\mathcal{Y})\cdot\phi)\\
&= ((Z,p,q)\bullet(\mathcal{X},\mathcal{Y}))\cdot\phi = \Theta\cdot\phi.
\end{aligned}$$

Hence, $(\tilde{Z},\tilde{p},\tilde{q})$ and $(\bar{Z},\bar{p},\bar{q})$ are two solutions to the same data $\Theta\cdot\phi$. By Theorem 4.15, the solution is unique, so that $(\tilde{Z},\tilde{p},\tilde{q}) = (\bar{Z},\bar{p},\bar{q})$.

Step 4. Now we prove (5.51d). For any $\Theta = (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) \in \mathcal{G}_0$, let

$$\begin{aligned}
\tilde{\Theta} &= (\tilde{\mathcal{X}},\tilde{\mathcal{Y}},\tilde{\mathcal{Z}},\tilde{\mathcal{V}},\tilde{\mathcal{W}},\tilde{\mathfrak{p}},\tilde{\mathfrak{q}}) = \Theta\cdot\phi,\\
\tilde{\psi} &= (\tilde{\psi}_1,\tilde{\psi}_2) = \mathbf{D}(\tilde{\Theta}),
\end{aligned}$$

$$\psi = (\psi_1, \psi_2) = \mathbf{D}(\Theta),$$
$$\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) = \psi \cdot \phi,$$

where we denote $\tilde{\psi}_1 = (\tilde{x}_1, \tilde{U}_1, \tilde{J}_1, \tilde{K}_1, \tilde{V}_1, \tilde{H}_1)$ and $\tilde{\psi}_2 = (\tilde{x}_2, \tilde{U}_2, \tilde{J}_2, \tilde{K}_2, \tilde{V}_2, \tilde{H}_2)$, and similar for ψ and $\bar{\psi}$. We want to show that $\tilde{\psi} = \bar{\psi}$. The proofs of $\tilde{\psi}_1 = \bar{\psi}_1$ and $\tilde{\psi}_2 = \bar{\psi}_2$ are similar and we only show that $\tilde{\psi}_1 = \bar{\psi}_1$. We have

$$\begin{aligned}
\tilde{x}_1(\tilde{\mathcal{X}}(s)) &= \tilde{\mathcal{Z}}_2(s) \quad \text{by (5.10a)}\\
&= \mathcal{Z}_2(h(s)) \quad \text{by (5.43b)}\\
&= x_1(\mathcal{X} \circ h(s)) \quad \text{by (5.10a)}\\
&= x_1(f \circ \tilde{\mathcal{X}}(s)) \quad \text{by (5.39)}\\
&= \bar{x}_1(\tilde{\mathcal{X}}(s)) \quad \text{by (5.37a).}
\end{aligned}$$

Similarly, one proves that $\tilde{U}_1 = \bar{U}_1$.

$$\begin{aligned}
\tilde{J}_1(\tilde{\mathcal{X}}(s)) &= \int_{-\infty}^{\tilde{\mathcal{X}}(s)} \tilde{\mathcal{V}}_4(X)\, dX \quad \text{by (5.10c)}\\
&= \int_{-\infty}^{f^{-1}\circ\mathcal{X}\circ h(s)} f'(X)\mathcal{V}_4(f(X))\, dX \quad \text{by (5.39) and (5.43c)}\\
&= \int_{-\infty}^{\mathcal{X}\circ h(s)} \mathcal{V}_4(X)\, dX \quad \text{by a change of variables}\\
&= J_1(\mathcal{X} \circ h(s)) \quad \text{by (5.10c)}\\
&= J_1(f \circ \tilde{\mathcal{X}}(s)) \quad \text{by (5.39)}\\
&= \bar{J}_1(\tilde{\mathcal{X}}(s)) \quad \text{by (5.37c).}
\end{aligned}$$

By a similar calculation, one shows that $\tilde{K}_1 = \bar{K}_1$. From (5.10f), (5.43d) and (5.37f), we obtain

$$\tilde{H}_1(\tilde{\mathcal{X}}(s)) = \tilde{\mathfrak{p}}(\tilde{\mathcal{X}}(s)) = f'(\tilde{\mathcal{X}}(s))\mathfrak{p}(f \circ \tilde{\mathcal{X}}(s)) = f'(\tilde{\mathcal{X}}(s))H_1(f \circ \tilde{\mathcal{X}}(s)) = \bar{H}_1(\tilde{\mathcal{X}}(s)),$$
$$\tilde{H}_2(\tilde{\mathcal{Y}}(s)) = \tilde{\mathfrak{q}}(\tilde{\mathcal{Y}}(s)) = g'(\tilde{\mathcal{Y}}(s))\mathfrak{q}(g \circ \tilde{\mathcal{Y}}(s)) = g'(\tilde{\mathcal{Y}}(s))H_2(g \circ \tilde{\mathcal{Y}}(s)) = \bar{H}_2(\tilde{\mathcal{Y}}(s)).$$

Similarly, one proves that $\tilde{V}_1 = \bar{V}_1$.

Step 5. We prove (5.51e). Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we denote

$$\begin{aligned}
\bar{\psi} &= (\bar{\psi}_1, \bar{\psi}_2) = \psi \cdot \phi,\\
\bar{\Theta} &= (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}) = \mathbf{C}(\bar{\psi}),\\
\Theta &= (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) = \mathbf{C}(\psi),\\
\tilde{\Theta} &= (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}, \tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}) = \Theta \cdot \phi.
\end{aligned}$$

We want to prove that $\tilde{\Theta} = \bar{\Theta}$. We first show that $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}) = (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. By (3.28) and (5.37a), we have

$$\bar{\mathcal{X}}(s) = \sup\{X \in \mathbb{R} \mid x_1 \circ f(X') < x_2 \circ g(2s - X') \text{ for all } X' < X\}. \tag{5.53}$$

From (3.29), we obtain $x_1 \circ \mathcal{X} \circ h = x_2 \circ \mathcal{Y} \circ h$ which implies, by (5.39), that $x_1 \circ f \circ \tilde{\mathcal{X}} = x_2 \circ g \circ \tilde{\mathcal{Y}}$. Hence, by (5.53), $\bar{\mathcal{X}} \leq \tilde{\mathcal{X}}$. Assume that $\bar{\mathcal{X}}(s) < \tilde{\mathcal{X}}(s)$ for some

$s \in \mathbb{R}$. Since f and g are strictly increasing functions, we have $f(\bar{\mathcal{X}}(s)) < f(\tilde{\mathcal{X}}(s))$ and $g(\tilde{\mathcal{Y}}(s)) < g(\bar{\mathcal{Y}}(s))$ which implies, by (3.6), that

$$x_1 \circ f \circ \bar{\mathcal{X}}(s) \leq x_1 \circ f \circ \tilde{\mathcal{X}}(s) = x_2 \circ g \circ \tilde{\mathcal{Y}}(s) \leq x_2 \circ g \circ \bar{\mathcal{Y}}(s).$$

By (5.53), we have $x_1 \circ f \circ \bar{\mathcal{X}}(s) = x_2 \circ g \circ \bar{\mathcal{Y}}(s)$, so we must have $x_1 \circ f \circ \bar{\mathcal{X}}(s) = x_1 \circ f \circ \tilde{\mathcal{X}}(s)$ and $x_2 \circ g \circ \tilde{\mathcal{Y}}(s) = x_2 \circ g \circ \bar{\mathcal{Y}}(s)$. This implies, since x_1 and x_2 are nondecreasing, that x_1 and x_2 are constant on $[f \circ \bar{\mathcal{X}}(s), f \circ \tilde{\mathcal{X}}(s)]$ and $[g \circ \tilde{\mathcal{Y}}(s), g \circ \bar{\mathcal{Y}}(s)]$, respectively. If $f \circ \bar{\mathcal{X}}(s) \leq 2h(s) - g \circ \bar{\mathcal{Y}}(s)$, set $X' = 2h(s) - g \circ \bar{\mathcal{Y}}(s)$ and $Y' = g \circ \bar{\mathcal{Y}}(s)$. We get

$$f \circ \bar{\mathcal{X}}(s) \leq X' < 2h(s) - g \circ \tilde{\mathcal{Y}}(s) = 2h(s) - \mathcal{Y} \circ h(s) = \mathcal{X} \circ h(s) = f \circ \tilde{\mathcal{X}}(s),$$

that is, $X' \in [f \circ \bar{\mathcal{X}}(s), f \circ \tilde{\mathcal{X}}(s)]$, so that $x_1(X') = x_1(f \circ \tilde{\mathcal{X}}(s))$. Similarly, we get that $x_2(Y') = x_2(g \circ \tilde{\mathcal{Y}}(s))$. Then, by (5.39) and (3.29), we obtain

$$x_1(X') = x_1(f \circ \tilde{\mathcal{X}}(s)) = x_1(\mathcal{X} \circ h(s)) = x_2(\mathcal{Y} \circ h(s)) = x_2(g \circ \tilde{\mathcal{Y}}(s)) = x_2(Y').$$

Thus, we have $x_1(X') = x_2(Y')$, $X' < \mathcal{X} \circ h(s)$ and $X' + Y' = 2h(s)$, which contradicts the definition (3.28) of $(\mathcal{X}, \mathcal{Y})$ at $h(s)$. If $f \circ \bar{\mathcal{X}}(s) > 2h(s) - g \circ \bar{\mathcal{Y}}(s)$, let $X' = f \circ \bar{\mathcal{X}}(s)$ and $Y' = 2h(s) - f \circ \bar{\mathcal{X}}(s)$, which implies that $X' < f \circ \tilde{\mathcal{X}}(s) = \mathcal{X} \circ h(s)$ and $x_1(X') = x_1(f \circ \tilde{\mathcal{X}}(s))$. We get

$$\begin{aligned} g \circ \tilde{\mathcal{Y}}(s) = \mathcal{Y} \circ h(s) = 2h(s) - \mathcal{X} \circ h(s) &= 2h(s) - f \circ \tilde{\mathcal{X}}(s) \\ &< 2h(s) - f \circ \bar{\mathcal{X}}(s) = Y' < g \circ \bar{\mathcal{Y}}(s), \end{aligned}$$

so that $Y' \in [g \circ \tilde{\mathcal{Y}}(s), g \circ \bar{\mathcal{Y}}(s)]$ and $x_2(Y') = x_2(g \circ \tilde{\mathcal{Y}}(s))$. Then, as before, we obtain $x_1(X') = x_2(Y')$ which is a contradiction, because $X' < \mathcal{X} \circ h(s)$ and $X' + Y' = 2h(s)$. Hence, we must have $\bar{\mathcal{X}} = \tilde{\mathcal{X}}$, which implies that $\bar{\mathcal{Y}} = \tilde{\mathcal{Y}}$. Then, by a straightforward calculation, one proves that $\bar{\mathcal{Z}} = \tilde{\mathcal{Z}}$, $\bar{\mathcal{V}} = \tilde{\mathcal{V}}$, $\bar{\mathcal{W}} = \tilde{\mathcal{W}}$, $\bar{\mathfrak{p}} = \tilde{\mathfrak{p}}$ and $\bar{\mathfrak{q}} = \tilde{\mathfrak{q}}$. For example, we have

$$\begin{aligned} \bar{\mathcal{Z}}_3(s) &= \bar{U}_1(\bar{\mathcal{X}}(s)) \quad \text{by (3.30c)} \\ &= \bar{U}_1(\tilde{\mathcal{X}}(s)) \\ &= U_1(f \circ \tilde{\mathcal{X}}(s)) \quad \text{by (5.37b)} \\ &= U_1(\mathcal{X} \circ h(s)) \quad \text{by (5.39)} \\ &= \mathcal{Z}_3(h(s)) \quad \text{by (3.30c)} \\ &= \tilde{\mathcal{Z}}_3(s) \quad \text{by (5.43b)}, \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{V}}_1(\bar{\mathcal{X}}(s)) &= \frac{1}{2c(\bar{U}_1 \circ \tilde{\mathcal{X}}(s))} \bar{x}_1'(\tilde{\mathcal{X}}(s)) \quad \text{by (3.31a)} \\ &= \frac{f'(\tilde{\mathcal{X}}(s))}{2c(U_1 \circ f \circ \tilde{\mathcal{X}}(s))} x_1'(f \circ \tilde{\mathcal{X}}(s)) \quad \text{by (5.37a) and (5.37b)} \\ &= f'(\tilde{\mathcal{X}}(s)) \mathcal{V}_1(f \circ \tilde{\mathcal{X}}(s)) \quad \text{by (3.31a)} \\ &= \tilde{\mathcal{V}}_1(\tilde{\mathcal{X}}(s)) \quad \text{by (5.43c)}, \end{aligned}$$

and, by (3.31f), (5.37f) and (5.43d),

$$\bar{\mathfrak{p}}(\bar{\mathcal{X}}(s)) = \bar{H}_1(\tilde{\mathcal{X}}(s)) = f'(\tilde{\mathcal{X}}(s)) H_1(f \circ \tilde{\mathcal{X}}(s)) = f'(\tilde{\mathcal{X}}(s)) \mathfrak{p}(f \circ \tilde{\mathcal{X}}(s)) = \tilde{\mathfrak{p}}(\tilde{\mathcal{X}}(s)),$$

$$\bar{\mathfrak{q}}(\bar{\mathcal{Y}}(s)) = \bar{H}_2(\tilde{\mathcal{Y}}(s)) = g'(\tilde{\mathcal{Y}}(s)) H_2(g \circ \tilde{\mathcal{Y}}(s)) = g'(\tilde{\mathcal{Y}}(s)) \mathfrak{q}(g \circ \tilde{\mathcal{Y}}(s)) = \tilde{\mathfrak{q}}(\tilde{\mathcal{Y}}(s)).$$

This concludes the proof of (5.51e).

Step 6. We are now ready to prove (5.51f). Let

$$\Theta = \mathbf{C}(\psi), \quad (Z,p,q) = \mathbf{S}(\Theta), \quad (\bar{Z},\bar{p},\bar{q}) = \mathbf{t}_T(Z,p,q), \quad \bar{\Theta} = \mathbf{E}(\bar{Z},\bar{p},\bar{q}), \quad \bar{\psi} = \mathbf{D}(\bar{\Theta}).$$

We claim that $S_T(\psi\cdot\phi) = S_T(\psi)\cdot\phi$. This follows from Step 1-6, as

$$\begin{aligned} S_T(\psi\cdot\phi) &= \mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T\circ\mathbf{S}\circ\mathbf{C}(\psi\cdot\phi) \\ &= \mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T\circ\mathbf{S}(\mathbf{C}(\psi)\cdot\phi) \quad \text{by (5.51e)} \\ &= \mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T\circ\mathbf{S}(\Theta\cdot\phi) \\ &= \mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T(\mathbf{S}(\Theta)\cdot\phi) \quad \text{by (5.51c)} \\ &= \mathbf{D}\circ\mathbf{E}\circ\mathbf{t}_T((Z,p,q)\cdot\phi) \\ &= \mathbf{D}\circ\mathbf{E}(\mathbf{t}_T(Z,p,q)\cdot\phi) \quad \text{by (5.51b)} \\ &= \mathbf{D}\circ\mathbf{E}((\bar{Z},\bar{p},\bar{q})\cdot\phi) \\ &= \mathbf{D}(\mathbf{E}(\bar{Z},\bar{p},\bar{q})\cdot\phi) \quad \text{by (5.51a)} \\ &= \mathbf{D}(\bar{\Theta}\cdot\phi) \\ &= \mathbf{D}(\bar{\Theta})\cdot\phi \quad \text{by (5.51d)} \\ &= \bar{\psi}\cdot\phi \\ &= S_T(\psi)\cdot\phi. \end{aligned}$$

□

Definition 5.17. *We denote by $\mathcal{F}/G^2$ the quotient of $\mathcal{F}$ with respect to the action of the group G^2 on $\mathcal{F}$. More specifically, we define the equivalence relation, $\sim$, on $\mathcal{F}$ as*

$$\text{for any } \psi,\bar{\psi}\in\mathcal{F},\ \psi\sim\bar{\psi} \text{ if there exists } \phi\in G^2 \text{ such that } \bar{\psi}=\psi\cdot\phi.$$

For an element $\psi\in\mathcal{F}$, we denote the equivalence class by

$$[\psi] = \{\bar{\psi}\in\mathcal{F} \mid \bar{\psi}\sim\psi\}.$$

We define the quotient space as

$$\mathcal{F}/G^2 = \{[\psi] \mid \psi\in\mathcal{F}\}.$$

Definition 5.18. *Let*

$$\mathcal{F}_0 = \{\psi = (\psi_1,\psi_2)\in\mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \text{ and } x_2 + J_2 = \mathrm{Id}\}$$

and $\Pi:\mathcal{F}\to\mathcal{F}_0$ be the projection on $\mathcal{F}_0$ given by $\bar{\psi} = (\bar{\psi}_1,\bar{\psi}_2) = \Pi(\psi)$ where $\bar{\psi}\in\mathcal{F}_0$ is defined as follows. Let

$$f(X) = x_1(X) + J_1(X) \quad \text{and} \quad g(Y) = x_2(Y) + J_2(Y) \tag{5.54}$$

and denote $\phi = (f,g)\in G^2$. We set

$$\bar{\psi} = \psi\cdot\phi^{-1}.$$

Proof of the well-posedness of Definition 5.18. Since $x_1 + J_1$ and $x_2 + J_2$ belong to G, f and g belong to G and their inverses exist. We claim that f^{-1} and g^{-1} belong to G. This immediately follows from (3.3) and the following estimate. Since $f\in G$, Lemma 3.8 implies that there exists $\alpha \geq 0$ such that $\frac{1}{1+\alpha} \leq f' \leq 1+\alpha$ almost everywhere, and we get

$$\begin{aligned}\int_{\mathbb{R}} (f^{-1}(X)' - 1)^2 \, dX &= \int_{\mathbb{R}} \left(\frac{1 - f'(f^{-1}(X))}{f'(f^{-1}(X))} \right)^2 dX \\ &\leq (1+\alpha) \int_{\mathbb{R}} \frac{(1 - f'(f^{-1}(X)))^2}{f'(f^{-1}(X))} \, dX \\ &= (1+\alpha) \int_{\mathbb{R}} (1 - f'(X))^2 \, dX\end{aligned}$$

by a change of variables. Hence, $(f^{-1})' - 1 \in L^2(\mathbb{R})$ and $f^{-1} \in G$. The same argument shows that $g^{-1} \in G$. Then the proof of the well-posedness of Definition 5.11 implies that $\bar{\psi} = \psi \cdot \phi^{-1}$ belongs to $\mathcal{F}$. Furthermore, we have

$$\bar{x}_1(X) + \bar{J}_1(X) = (x_1 + J_1) \circ f^{-1}(X) = (x_1 + J_1) \circ (x_1 + J_1)^{-1}(X) = X.$$

By a similar calculation, we get $\bar{x}_2(Y) + \bar{J}_2(Y) = Y$ and conclude that $\bar{\psi} \in \mathcal{F}_0$. □

Lemma 5.19. *The following statements hold:*

(i) For any ψ and $\bar{\psi}$ in $\mathcal{F}$, we have

$$\psi \sim \bar{\psi} \quad \textit{if and only if} \quad \Pi(\psi) = \Pi(\bar{\psi}), \tag{5.55a}$$

so that the sets $\mathcal{F}/G^2$ and $\mathcal{F}_0$ are in bijection.

(ii) We have

$$\mathbf{M} \circ \Pi = \mathbf{M} \tag{5.55b}$$

and

$$\mathbf{L} \circ \mathbf{M}|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0} \quad \textit{and} \quad \mathbf{M} \circ \mathbf{L} = \mathrm{Id}, \tag{5.55c}$$

so that the sets $\mathcal{D}$, $\mathcal{F}_0$ and $\mathcal{F}/G^2$ are in bijection.

(iii) We have

$$\Pi \circ S_T \circ \Pi = \Pi \circ S_T. \tag{5.55d}$$

Note that the first identity in (5.55c) is equivalent to

$$\mathbf{L} \circ \mathbf{M} \circ \Pi = \Pi. \tag{5.56}$$

Before we prove the lemma we make some remarks.

Let $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$. Consider $\psi = \mathbf{L}(\xi_0)$, $\bar{\psi} = S_T(\psi)$ and $\xi_T = \mathbf{M}(\bar{\psi})$.

Let $\phi \in G^2$ and use $\hat{\psi} = \psi \cdot \phi$ as initial data for the solution operator S_T. From (5.51f) we have $S_T(\hat{\psi}) = S_T(\psi) \cdot \phi = \bar{\psi} \cdot \phi = \tilde{\psi}$. Let $\tilde{\xi}_T = \mathbf{M}(\tilde{\psi})$. Since $\phi \in G^2$ we have $\tilde{\psi} \sim \bar{\psi}$. Then, by (5.55a) we get $\Pi(\tilde{\psi}) = \Pi(\bar{\psi})$ and from (5.55b) we get

$$\mathbf{M}(\tilde{\psi}) = \mathbf{M}(\Pi(\tilde{\psi})) = \mathbf{M}(\Pi(\bar{\psi})) = \mathbf{M}(\bar{\psi}).$$

This implies

$$\tilde{\xi}_T = \mathbf{M}(\tilde{\psi}) = \mathbf{M}(\bar{\psi}) = \xi_T,$$

so the two solutions are identical.

We can think of this as follows: To each element $\xi_0 \in \mathcal{D}$ there correspond infinitely many elements in $\mathcal{F}$, all belonging to the same equivalence class. The mapping $\mathbf{L} : \mathcal{D} \to \mathcal{F}_0$ picks one member of the equivalence class, but we could also pick a different one. Applying the solution operator to all elements belonging to the same equivalence class yields infinitely many solutions in $\mathcal{F}$, which form an equivalence class. Using the mapping $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ on all of these solutions yields the same element in $\mathcal{D}$. Since we get the same solution in the end, we can think of each member of the equivalence class as a different "parametrization" of the initial data in $\mathcal{F}$, which are connected through relabeling.

The following example shows how we can use relabeling in order to get different initial curves in $\mathcal{G}_0$. We use the same notation as above. Assume that $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ belongs to $\mathcal{D}$ and satisfies the additional conditions $u_0, R_0, S_0, \rho_0, \sigma_0 \in L^\infty(\mathbb{R})$ and that μ_0 and ν_0 are absolutely continuous. We have

$$f(x_1(X)) = X \quad \text{and} \quad g(x_2(Y)) = Y,$$

where

$$f(x) = x + \frac{1}{4}\int_{-\infty}^{x} (R_0^2 + c(u_0)\rho_0^2)(z)\,dz \quad \text{and} \quad g(x) = x + \frac{1}{4}\int_{-\infty}^{x} (S_0^2 + c(u_0)\sigma_0^2)(z)\,dz.$$

Since the functions f and g are strictly increasing, they are invertible and x_1 and x_2 are given as

$$x_1(X) = f^{-1}(X) \quad \text{and} \quad x_2(Y) = g^{-1}(Y).$$

Let $\Theta = \mathbf{C}(\psi)$. The functions $\mathcal{X}$ and $\mathcal{Y}$ are given by

$$x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s)) \quad \text{and} \quad \mathcal{Y}(s) = 2s - \mathcal{X}(s).$$

From this we get that $\mathcal{X}$ and $\mathcal{Y}$ are strictly increasing functions. Furthermore, the functions f and g belong to G and can therefore be used as relabeling functions. In the above notation this means $\phi = (f, g)$. Denote $\hat{\psi} = \psi \cdot \phi$. Then we get $\hat{x}_1(X) = X$ and $\hat{x}_2(Y) = Y$. Consider $\hat{\Theta} = \mathbf{C}(\hat{\psi})$. Now we get $\hat{\mathcal{X}}(s) = s$ and $\hat{\mathcal{Y}}(s) = s$. As above, using either Θ or $\tilde{\Theta}$ yields the same solution in $\mathcal{D}$. Therefore, for this type of initial data in $\mathcal{D}$ we can without loss of generality assume that the initial curve $(\mathcal{X}, \mathcal{Y})$ is the identity, i.e., $\mathcal{X}(s) = s$ and $\mathcal{Y}(s) = s$. Note that without the additional assumptions on the initial data in $\mathcal{D}$, it is not possible to relabel the initial curve into the identity.

Proof. We decompose the proof into five steps.

Step 1. We prove (5.55a). If $\psi \sim \bar{\psi}$, there exists $\tilde{\phi} = (\tilde{f}, \tilde{g}) \in G^2$ such that $(\bar{\psi}_1, \bar{\psi}_2) = \bar{\psi} = \psi \cdot \tilde{\phi} = (\psi_1, \psi_2) \cdot \tilde{\phi}$. Let $\phi = (f, g)$ and $\bar{\phi} = (\bar{f}, \bar{g})$ be given by (5.54) for ψ and $\bar{\psi}$, respectively. We have $\bar{\phi} = \phi \circ \tilde{\phi}$ because

$$\bar{\phi} = (\bar{f}, \bar{g}) = (\bar{x}_1 + \bar{J}_1, \bar{x}_2 + \bar{J}_2) = ((x_1 + J_1) \circ \tilde{f}, (x_2 + J_2) \circ \tilde{g}) = (f \circ \tilde{f}, g \circ \tilde{g}) = \phi \circ \tilde{\phi}.$$

Then, we get

$$\begin{aligned} \Pi(\bar{\psi}) &= \bar{\psi} \cdot (\bar{\phi})^{-1} = (\psi \cdot \tilde{\phi}) \cdot (\phi \circ \tilde{\phi})^{-1} = \psi \cdot (\tilde{\phi} \circ (\phi \circ \tilde{\phi})^{-1}) \\ &= \psi \cdot (\tilde{\phi} \circ (\tilde{\phi})^{-1} \circ \phi^{-1}) = \psi \cdot \phi^{-1} = \Pi(\psi), \end{aligned}$$

where the identity $(\psi \cdot \tilde{\phi}) \cdot (\phi \circ \tilde{\phi})^{-1} = \psi \cdot (\tilde{\phi} \circ (\phi \circ \tilde{\phi})^{-1})$ follows from a straightforward calculation using Definition 5.11. For example, by (5.37a), we have

$$\bar{x}_1 \circ F(X) = x_1 \circ \tilde{f} \circ F(X) = x_1 \circ f^{-1},$$

where we denote $F=(f\circ\tilde f)^{-1}$, and

$$\begin{aligned} F'(X)\bar H_1\circ F(X) &= F'(X)(\tilde f)'\circ F(X)H_1\circ\tilde f\circ F(X)\\ &= (\tilde f\circ F(X))_X H_1\circ\tilde f\circ F(X)\\ &= (f^{-1})'H_1\circ f^{-1}(X). \end{aligned}$$

Conversely, if $\Pi(\psi)=\Pi(\bar\psi)$, then $\psi\cdot\phi^{-1}=\bar\psi\cdot(\bar\phi)^{-1}$, where $\phi=(f,g)$ and $\bar\phi=(\bar f,\bar g)$ are given by (5.54). This implies that

$$\bar\psi=\psi\cdot(\phi^{-1}\circ\bar\phi). \tag{5.57}$$

This also follows from a direct calculation. For example, we have

$$((\bar f)^{-1})'(X)\bar H_1\circ(\bar f)^{-1}(X)=(f^{-1})'(X)H_1\circ f^{-1}(X)$$

which implies that

$$((\bar f)^{-1})'\circ\bar f(X)\bar H_1(X)=(f^{-1})'\circ\bar f(X)H_1\circ f^{-1}\circ\bar f(X)$$

and since $((\bar f)^{-1})'\circ\bar f(X)=\frac{1}{(\bar f)'(X)}$, we get[5]

$$\bar H_1(X)=(f^{-1}\circ\bar f(X))_X H_1\circ f^{-1}\circ\bar f(X).$$

Then, since $\phi^{-1}\circ\bar\phi\in G^2$, (5.57) implies that ψ and $\bar\psi$ are equivalent.

Step 2. We prove (5.55b). Given $\psi=(\psi_1,\psi_2)\in\mathcal F$, let $(u,R,S,\rho,\sigma,\mu,\nu)=\mathbf M(\psi)$, $\bar\psi=(\bar\psi_1,\bar\psi_2)=\Pi(\psi)$ and $(\bar u,\bar R,\bar S,\bar\rho,\bar\sigma,\bar\mu,\bar\nu)=\mathbf M(\bar\psi)$. We want to prove that $(\bar u,\bar R,\bar S,\bar\rho,\bar\sigma,\bar\mu,\bar\nu)=(u,R,S,\rho,\sigma,\mu,\nu)$. From (5.19a), (5.37a) and (5.37b), we get

$$\bar u(\bar x_1(X))=\bar U_1(X)=U_1(f^{-1}(X))=u(x_1\circ f^{-1}(X))=u(\bar x_1(X)).$$

For any Borel set B, we have

$$\begin{aligned} \int_B\bar R(x)\,dx &= \int_{\bar x_1^{-1}(B)}2c(\bar U_1(X))\bar V_1(X)\,dX \quad\text{by (5.19c)}\\ &= \int_{\{X\in\mathbb R\ \mid\ x_1(f^{-1}(X))\in B\}}2c(U_1(f^{-1}(X)))(f^{-1}(X))'V_1(f^{-1}(X))\,dX\\ &= \int_{x_1^{-1}(B)}2c(U_1(X))V_1(X)\,dX \quad\text{by a change of variables}\\ &= \int_B R(x)\,dx \quad\text{by (5.19c)}, \end{aligned}$$

where we used (5.37a), (5.37b) and (5.37e). Hence, $\bar R=R$ almost everywhere. By (5.19e) and (5.37f), we obtain

$$\begin{aligned} \int_B\bar\rho(x)\,dx &= \int_{\bar x_1^{-1}(B)}2\bar H_1(X)\,dX\\ &= \int_{\bar x_1^{-1}(B)}2(f^{-1}(X))'H_1(f^{-1}(X))\,dX\\ &= \int_{x_1^{-1}(B)}2H_1(X)\,dX \end{aligned}$$

[5]Since $\bar f\in G$, there exists $\delta>0$ such that $\bar f'\geq\delta$ almost everywhere, see Lemma 3.8.

$$= \int_B \rho(x)\,dx.$$

Similarly, one proves that $\bar{S} = S$ and $\bar{\sigma} = \sigma$ almost everywhere. Using (5.19g) and (5.37c), we find for any Borel set $B \subset \mathbb{R}$, that

$$\begin{aligned}\bar{\mu}(B) &= \int_{\bar{x}_1^{-1}(B)} \bar{J}_1(X)\,dX \\ &= \int_{\bar{x}_1^{-1}(B)} (f^{-1}(X))' J_1'(f^{-1}(X))\,dX \\ &= \int_{x_1^{-1}(B)} J_1'(X)\,dX \\ &= \mu(B)\end{aligned}$$

and $\bar{\mu} = \mu$. One proves that $\bar{\nu} = \nu$ in a similar way.

Step 3. We prove that $\mathbf{L} \circ \mathbf{M}|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0}$. Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}_0$, let $(u, R, S, \rho, \sigma, \mu, \nu) = \mathbf{M}(\psi)$ and $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$. We want to show that $\bar{\psi} = \psi$. We first prove that $\bar{x}_1 = x_1$. Let

$$g(x) = \sup\{X \in \mathbb{R} \mid x_1(X) < x\}. \tag{5.58}$$

For all $x \in \mathbb{R}$, we have

$$x_1(g(x)) = x \tag{5.59}$$

and since x_1 is continuous and nondecreasing, $x_1^{-1}((-\infty, x)) = (-\infty, g(x))$. From (5.19g) and (3.10), we get

$$\mu((-\infty, x)) = \int_{x_1^{-1}((-\infty,x))} J_1'(X)\,dX = \int_{-\infty}^{g(x)} J_1'(X)\,dX = J_1(g(x)). \tag{5.60}$$

Since $\psi \in \mathcal{F}_0$, $x_1 + J_1 = \mathrm{Id}$ which implies, by (5.59) and (5.60), that

$$x + \mu((-\infty, x)) = g(x). \tag{5.61}$$

From the definition (3.12a) of $\bar{x}_1$, we then obtain

$$\bar{x}_1(X) = \sup\{x \in \mathbb{R} \mid g(x) < X\}. \tag{5.62}$$

This implies that, for any $X \in \mathbb{R}$, there exists an increasing sequence, z_i, such that $\lim_{i\to\infty} z_i = \bar{x}_1(X)$ and $g(z_i) < X$. Using that x_1 is nondecreasing and (5.59), we get $z_i \leq x_1(X)$. Letting i tend to infinity, we obtain $\bar{x}_1(X) \leq x_1(X)$. Assume that $\bar{x}_1(X) < x_1(X)$. Then, there exists $x \in \mathbb{R}$ such that $\bar{x}_1(X) < x < x_1(X)$ which implies, by (5.62), that $g(x) \geq X$. On the other hand, $x_1(g(x)) = x < x_1(X)$ implies that $g(x) < X$ because x_1 is nondecreasing, which gives us a contradiction. Hence, we must have $\bar{x}_1 = x_1$. Then, by (3.12c) and since $x_1 + J_1 = \mathrm{Id}$, we get

$$\bar{J}_1(X) = X - \bar{x}_1(X) = X - x_1(X) = J_1(X)$$

and from (3.12d) and (5.19a), we obtain

$$\bar{U}_1(X) = u(\bar{x}_1(X)) = u(x_1(X)) = U_1(X).$$

By (3.12e) and (5.20a), we have

$$\bar{V}_1(X) = \bar{x}_1'(X)\frac{R(\bar{x}_1(X))}{2c(\bar{U}_1(X))} = x_1'(X)\frac{R(x_1(X))}{2c(U_1(X))} = V_1(X)$$

and by (3.12f) and (3.7), we get

$$\bar{K}_1(X) = \int_{-\infty}^{X} \frac{\bar{J}_1'(\bar{X})}{c(\bar{U}_1(\bar{X}))}\, d\bar{X} = \int_{-\infty}^{X} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X} = K_1(X).$$

Using (3.12g) and (5.20c), we find that

$$\bar{H}_1(X) = \frac{1}{2}\rho(\bar{x}_1(X))\bar{x}_1'(X) = \frac{1}{2}\rho(x_1(X))x_1'(X) = H_1(X).$$

Hence, we have proved that $\bar{\psi}_1 = \psi_1$. Similarly, one proves that $\bar{x}_2 = x_2$ and $\bar{\psi}_2 = \psi_2$. For example, by (3.12g) and (5.20d), we have

$$\bar{H}_2(Y) = \frac{1}{2}\sigma(\bar{x}_2(Y))\bar{x}_2'(Y) = \frac{1}{2}\sigma(x_2(Y))x_2'(Y) = H_2(Y).$$

Step 4. Let us prove that $\mathbf{M} \circ \mathbf{L} = \mathrm{Id}$. Given $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$, let $\psi = (\psi_1, \psi_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$ and $(\bar{u}, \bar{R}, \bar{S}, \bar{\rho}, \bar{\sigma}, \bar{\mu}, \bar{\nu}) = \mathbf{M}(\psi)$. We want to prove that $(\bar{u}, \bar{R}, \bar{S}, \bar{\rho}, \bar{\sigma}, \bar{\mu}, \bar{\nu}) = (u, R, S, \rho, \sigma, \mu, \nu)$. First we show that $\bar{\mu} = \mu$. Let g be the function defined as before by (5.58). The same computation that leads to (5.61) now gives

$$x + \bar{\mu}((-\infty, x)) = g(x). \tag{5.63}$$

By (3.12a), for any $X \in \mathbb{R}$, there exists an increasing sequence, x_i, such that $\lim_{i\to\infty} x_i = x_1(X)$ and $x_i + \mu((-\infty, x_i)) < X$. Sending i to infinity, and since $x \mapsto \mu((-\infty, x))$ is lower semi-continuous, we obtain $x_1(X) + \mu((-\infty, x_1(X)))) \leq X$. We set $X = g(x)$ and get, by (5.59), that

$$x + \mu((-\infty, x)) \leq g(x). \tag{5.64}$$

From the definition of g, we have that, for any $x \in \mathbb{R}$, there exists an increasing sequence, X_i, such that $\lim_{i\to\infty} X_i = g(x)$ and $x_1(X_i) < x$. This implies, by (3.12a), that $x + \mu((-\infty, x)) \geq X_i$. Letting i tend to infinity, we obtain $x + \mu((-\infty, x)) \geq g(x)$ which, together with (5.64), yields

$$x + \mu((-\infty, x)) = g(x). \tag{5.65}$$

Comparing (5.63) and (5.65), we get that $\bar{\mu} = \mu$. Similarly, one proves that $\bar{\nu} = \nu$. By (5.19a) and (3.12d), we have

$$\bar{u}(x_1(X)) = U_1(X) = u(x_1(X)).$$

For any Borel set B, we have

$$\begin{aligned}
\int_B \bar{R}(x)\, dx &= \int_{x_1^{-1}(B)} 2c(U_1(X))V_1(X)\, dX \quad \text{by (5.19c)} \\
&= \int_{x_1^{-1}(B)} 2c(u \circ x_1(X))x_1'(X) \frac{R(x_1(X))}{2c(u \circ x_1(X))}\, dX \quad \text{by (3.12d) and (3.12e)} \\
&= \int_B R(x)\, dx \quad \text{by a change of variables,}
\end{aligned}$$

so that $\bar{R} = R$ almost everywhere. Similarly, one proves that $\bar{S} = S$ almost everywhere. From (5.19e) and (3.12g), we get

$$\int_B \bar{\rho}(x)\, dx = \int_{x_1^{-1}(B)} 2H_1(X)\, dX = \int_{x_1^{-1}(B)} \rho(x_1(X))x_1'(X)\, dX = \int_B \rho(x)\, dx.$$

Hence, $\bar\rho = \rho$ almost everywhere. Similarly, one proves that $\bar\sigma = \sigma$ almost everywhere.

Step 5. We prove (5.55d). Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, let $\phi = (f, g) \in G^2$ be defined as in (5.54) so that $\Pi(\psi) = \psi \cdot \phi^{-1}$. By (5.51f), we get

$$S_T \circ \Pi(\psi) = S_T(\psi \cdot \phi^{-1}) = S_T(\psi) \cdot \phi^{-1},$$

which implies that $S_T \circ \Pi$ and S_T are equivalent. Then, (5.55d) follows from (5.55a). □

Now we are finally in position to prove that $\bar S_T$ is a semigroup.

Theorem 5.20. *The mapping $\bar S_T$ is a semigroup.*

Proof. The proof relies on Lemma 5.19 and Theorem 5.6. From Definition 5.10 we have

$$\begin{aligned}
\bar S_T \circ \bar S_{T'} &= \mathbf{M} \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ S_{T'} \circ \mathbf{L} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ \Pi \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.55b)} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ \Pi \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.56)} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.55d)} \\
&= \mathbf{M} \circ S_T \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.55b)} \\
&= \mathbf{M} \circ S_{T+T'} \circ \mathbf{L} \quad \text{by Theorem 5.6} \\
&= \bar S_{T+T'}.
\end{aligned}$$

□

6. Existence of Weak Global Conservative Solutions

It remains to prove that the solution obtained by using the operator $\bar S_T$ is a weak solution of (1.3).

Theorem 6.1. *Let $t > 0$ and $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$. Then*

$$(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar S_t(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$$

is a weak solution of (1.3)*, meaning that*

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} \left(\big[\phi_t - c(u)\phi_x\big] R + \big[\phi_t + c(u)\phi_x\big] S + \frac{c'(u)}{c(u)} RS\phi \right) dx\, dt \\
&= \iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)} \phi\, d\mu\, dt + \iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)} \phi\, d\nu\, dt,
\end{aligned} \tag{6.1a}$$

$$\iint_{[0,\infty)\times\mathbb{R}} \big[\phi_t - c(u)\phi_x\big] \rho\, dx\, dt = 0, \tag{6.1b}$$

and

$$\iint_{[0,\infty)\times\mathbb{R}} \big[\phi_t + c(u)\phi_x\big] \sigma\, dx\, dt = 0 \tag{6.1c}$$

for all $\phi = \phi(t, x)$ in $C_0^\infty((0,\infty)\times\mathbb{R})$, where

$$R = u_t + c(u) u_x \quad \text{and} \quad S = u_t - c(u) u_x \tag{6.1d}$$

in the sense of distributions.

Moreover, the measures μ and ν satisfy the equations

$$(\mu+\nu)_t - (c(u)(\mu-\nu))_x = 0 \tag{6.2a}$$

and

$$\left(\frac{1}{c(u)}(\mu-\nu)\right)_t - (\mu+\nu)_x = 0 \tag{6.2b}$$

in the sense of distributions.

Note that if the two measures μ and ν are absolutely continuous, (6.2a) and (6.2b) coincide with (1.13) in the sense of distributions, which we derived in the smooth case. Moreover, the difference of the sign in front of μ and ν indicates the two opposite traveling directions.

Proof. We decompose the proof into two steps.

Step 1. We first show (6.1). Given $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, we consider $(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$, where $\bar{S}_t$ is given by Definition 5.10. The identities in (6.1d) follow from (5.33) in Lemma 5.9. By a change of variables, we get

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)R(t,x)\,dx\,dt \\
&= \iint_{\mathbb{R}^2} (\phi_t - c(u)\phi_x)R(t(X,Y),x(X,Y))(t_X x_Y - t_Y x_X)(X,Y)\,dX\,dY \\
&= 2\iint_{\mathbb{R}^2} \left(\frac{\phi_t - c(u)\phi_x}{c(u)}R\right)(t(X,Y),x(X,Y))x_X x_Y(X,Y)\,dX\,dY \text{ by (4.12a)} \\
&= -2\iint_{\mathbb{R}^2} \phi_Y R(t(X,Y),x(X,Y))x_X(X,Y)\,dX\,dY \\
&= -2\iint_{\mathbb{R}^2} c(U)U_X(X,Y)\phi_Y(t(X,Y),x(X,Y))\,dX\,dY \text{ by (5.30), (5.31a)} \\
&= 2\iint_{\mathbb{R}^2} (c(U)U_X)_Y(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY \\
&= 2\iint_{\mathbb{R}^2} (c'(U)U_X U_Y + c(U)U_{XY})(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY,
\end{aligned} \tag{6.3}$$

where we used integration by parts and

$$\begin{aligned}
&\phi_Y(t(X,Y),x(X,Y)) \\
&\quad= \phi_t(t(X,Y),x(X,Y))t_Y(X,Y) + \phi_x(t(X,Y),x(X,Y))x_Y(X,Y) \\
&\quad= -\left(\frac{\phi_t - c(u)\phi_x}{c(u)}\right)(t(X,Y),x(X,Y))x_Y(X,Y),
\end{aligned} \tag{6.4}$$

which follows from (4.12a).

Similarly, we find that

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t + c(u)\phi_x)S(t,x)\,dx\,dt \\
&\quad= 2\iint_{\mathbb{R}^2} (c'(U)U_X U_Y + c(U)U_{XY})(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY
\end{aligned} \tag{6.5}$$

and

$$\iint_{[0,\infty)\times\mathbb{R}} \frac{c'(u)}{c(u)} RS\phi(t,x)\,dt\,dx = -2\iint_{\mathbb{R}^2} c'(U)U_XU_Y(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY. \tag{6.6}$$

Combining (6.3), (6.5), (6.6) and (2.38c) yields

$$\iint_{[0,\infty)\times\mathbb{R}} \Big(\big[\phi_t - c(u)\phi_x\big]R + \big[\phi_t + c(u)\phi_x\big]S + \frac{c'(u)}{c(u)}RS\phi \Big)\,dx\,dt = \iint_{\mathbb{R}^2} \frac{2c'(U)}{c^2(U)}(x_Y J_X + x_X J_Y)\phi(t,x)\,dX\,dY. \tag{6.7}$$

We have[6] $(u,R,S,\rho,\sigma,\mu,\nu)(t) = \mathbf{M}\circ\mathbf{D}(\Theta(t))$ and by (5.21f)

$$\iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,x)\,d\mu(t)\,dt = \iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,\mathcal{Z}_2(t,s))\mathcal{V}_4(t,\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds\,dt,$$

where we added the t dependence in $\Theta(t)$, which gives $\mathcal{X}(t,s)$, $\mathcal{Z}_2(t,s)$ and $\mathcal{V}_4(t,s)$ in the equation above. The measures μ and ν integrate with respect to the x variable and the notation $d\mu(t)$ and $d\nu(t)$ means that they depend on t. We proceed to the change of variables $s = \frac{1}{2}(X+Y)$ and $t = t(X,Y)$ and obtain

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,\mathcal{Z}_2(t,s))\mathcal{V}_4(t,\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds\,dt \\
&= \iint_{\mathbb{R}^2} \frac{2c'(u)}{c(u)}\phi(t(X,Y),x(X,Y))J_X(X,Y) \\
&\qquad \times \mathcal{X}_s(t(X,Y),s(X,Y))\Big(\frac{t_X - t_Y}{2}\Big)(X,Y)\,dX\,dY \\
&= \iint_{\mathbb{R}^2} \frac{2c'(U)}{c^2(U)} x_Y J_X(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY \quad \text{by (4.12a)},
\end{aligned}$$

where we used that $\dot{\mathcal{Z}}_1(t,s) = 0$, which implies

$$\begin{aligned}
0 &= t_X(X,Y)\mathcal{X}_s(t(X,Y),s(X,Y)) + t_Y(X,Y)\mathcal{Y}_s(t(X,Y),s(X,Y)) \\
&= (t_X - t_Y)(X,Y)\mathcal{X}_s(t(X,Y),s(X,Y)) - 2\Big(\frac{x_Y}{c(U)}\Big)(X,Y),
\end{aligned} \tag{6.8}$$

by (2.40c) and (4.12a).

Similarly, we obtain

$$\iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,x)\,d\nu(t)\,dt$$

[6]Note that although the mapping $\mathbf{D}$ is from $\mathcal{G}_0$ to $\mathcal{F}$, the t dependence in Θ makes sense, since, for any fixed $t \geq 0$, we can consider the set $\mathcal{G}_t$, that is, the set $\mathcal{G}$ where $\mathcal{Z}_1(s) = t$. This still gives $\dot{\mathcal{Z}}_1(s) = 0$, and $\mathcal{G}_t$ can be mapped to $\mathcal{G}_0$ by $\mathbf{t}_t$.

$$= \iint_{\mathbb{R}^2} \frac{2c'(U)}{c^2(U)} x_X J_Y(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY,$$

and we get

$$\iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,x)\,d\mu(t)\,dt + \iint_{[0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)}\phi(t,x)\,d\nu(t)\,dt \\ = \iint_{\mathbb{R}^2} \frac{2c'(U)}{c^2(U)}(x_Y J_X + x_X J_Y)\phi(t,x)\,dX\,dY. \tag{6.9}$$

From (6.7) and (6.9) we conclude that (6.1a) holds.

It remains to prove (6.1b) and (6.1c). We have

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)\rho(t,x)\,dx\,dt \\
&= 2\iint_{\mathbb{R}^2} \left(\frac{\phi_t - c(u)\phi_x}{c(u)}\rho\right)(t(X,Y),x(X,Y))x_X x_Y(X,Y)\,dX\,dY \\
&= -2\iint_{\mathbb{R}^2} \phi_Y \rho(t(X,Y),x(X,Y))x_X(X,Y)\,dX\,dY \quad \text{by (6.4)} \\
&= -2\iint_{\mathbb{R}^2} p(X,Y)\phi_Y(t(X,Y),x(X,Y))\,dX\,dY \quad \text{by (5.30) and (5.31b)} \\
&= 2\iint_{\mathbb{R}^2} p_Y(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY \quad \text{by integration by parts} \\
&= 0 \qquad \text{by (2.38f)}
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t + c(u)\phi_x)\sigma(t,x)\,dx\,dt \\
&= 2\iint_{\mathbb{R}^2} \left(\frac{\phi_t + c(u)\phi_x}{c(u)}\sigma\right)(t(X,Y),x(X,Y))x_X x_Y(X,Y)\,dX\,dY \\
&= 2\iint_{\mathbb{R}^2} \phi_X \sigma(t(X,Y),x(X,Y))x_Y(X,Y)\,dX\,dY \\
&= 2\iint_{\mathbb{R}^2} q(X,Y)\phi_X(t(X,Y),x(X,Y))\,dX\,dY \quad \text{by (5.30) and (5.32b)} \\
&= -2\iint_{\mathbb{R}^2} q_X(X,Y)\phi(t(X,Y),x(X,Y))\,dX\,dY \quad \text{by integration by parts} \\
&= 0 \qquad \text{by (2.38g)}
\end{aligned}$$

because

$$\phi_X(t(X,Y),x(X,Y)) = \left(\frac{\phi_t + c(u)\phi_x}{c(u)}\right)(t(X,Y),x(X,Y))x_X(X,Y),$$

which follows from a similar calculation as in (6.4). Thus, we have proved (6.1b) and (6.1c).

Step 2. Now we prove (6.2). First we show (6.2a), that is,

$$\iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)(t,x)\,d\mu(t)\,dt + \iint_{[0,\infty)\times\mathbb{R}} (\phi_t + c(u)\phi_x)(t,x)\,d\nu(t)\,dt = 0$$

for all $\phi \in C_0^\infty((0,\infty)\times\mathbb{R})$. By a calculation as above we find

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)(t,x)\,d\mu(t)\,dt \\
&= \iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)(t,\mathcal{Z}_2(t,s))\mathcal{V}_4(t,\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds\,dt \quad \text{by (5.21f)} \\
&= \iint_{\mathbb{R}^2} (\phi_t - c(u)\phi_x)(t(X,Y),x(X,Y))J_X(X,Y) \\
&\qquad\qquad \times \mathcal{X}_s(t(X,Y),s(X,Y))\left(\frac{t_X - t_Y}{2}\right)(X,Y)\,dX\,dY \\
&= \iint_{\mathbb{R}^2} (\phi_t - c(u)\phi_x)(t(X,Y),x(X,Y)) \\
&\qquad\qquad \times J_X(X,Y)\left(\frac{x_Y}{c(U)}\right)(X,Y)\,dX\,dY \quad \text{by (6.8) and (4.12a)} \\
&= -\iint_{\mathbb{R}^2} \phi_Y(t(X,Y),x(X,Y))J_X(X,Y)\,dX\,dY \quad \text{by (6.4)},
\end{aligned}$$

where we used the change of variables $s = \frac{1}{2}(X+Y)$ and $t = t(X,Y)$.

Similarly, one proves that

$$\iint_{[0,\infty)\times\mathbb{R}} (\phi_t + c(u)\phi_x)(t,x)\,d\nu(t)\,dt = \iint_{\mathbb{R}^2} \phi_X(t(X,Y),x(X,Y))J_Y(X,Y)\,dX\,dY,$$

so that

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} (\phi_t - c(u)\phi_x)(t,x)\,d\mu(t)\,dt + \int_{[0,\infty)\times\mathbb{R}} (\phi_t + c(u)\phi_x)(t,x)\,d\nu(t)\,dt \\
&= \iint_{\mathbb{R}^2} (-\phi_Y(t(X,Y),x(X,Y))J_X(X,Y) + \phi_X(t(X,Y),x(X,Y))J_Y(X,Y))\,dX\,dY \\
&= 0
\end{aligned}$$

by integration by parts. This concludes the proof of (6.2a). In a similar way, one proves (6.2b). □

The semigroup of solutions, $\bar{S}_t$, is conservative in the following sense.

Theorem 6.2. *Given* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, *let*

$$(u,R,S,\rho,\sigma,\mu,\nu)(t) = \bar{S}_t(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0).$$

We have:

(i) For all $t \geq 0$,

$$\mu(t)(\mathbb{R}) + \nu(t)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}).$$

(ii) For almost every $t \geq 0$, *the singular parts of* $\mu(t)$ *and* $\nu(t)$ *are concentrated on the set where* $c'(u) = 0$.

Proof. We prove (i). Given $\tau \geq 0$, let

$$(u,R,S,\rho,\sigma,\mu,\nu)(\tau) = \bar{S}_\tau(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0).$$

We consider $\Theta(\tau) \in \mathcal{G}_\tau$ and $(Z,p,q) \in \mathcal{H}$ such that $(u,R,S,\rho,\sigma,\mu,\nu)(\tau) = \mathbf{M} \circ \mathbf{D}(\Theta(\tau))$ and $\Theta(\tau) = \mathbf{E}(Z,p,q)$. By Definition 5.2, we have

$\mathcal{Z}_2(\tau,s) = x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$, $\mathcal{V}_4(\mathcal{X}(\tau,s)) = J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $\mathcal{W}_4(\mathcal{Y}(\tau,s)) = J_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$. Then, from (5.21f) and (5.21g), we have for any Borel set B, that

$$\begin{aligned}\mu(\tau)(B) &= \int_{\{s\in\mathbb{R}\,|\,\mathcal{Z}_2(\tau,s)\in B\}} \mathcal{V}_4(\mathcal{X}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\\ &= \int_{\{s\in\mathbb{R}\,|\,x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in B\}} J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\end{aligned}$$

and

$$\begin{aligned}\nu(\tau)(B) &= \int_{\{s\in\mathbb{R}\,|\,\mathcal{Z}_2(\tau,s)\in B\}} \mathcal{W}_4(\mathcal{Y}(\tau,s))\mathcal{Y}_s(\tau,s)\,ds\\ &= \int_{\{s\in\mathbb{R}\,|\,x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in B\}} J_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{Y}_s(\tau,s)\,ds\end{aligned}$$

respectively. Hence,

$$\begin{aligned}&\mu(\tau)(\mathbb{R}) + \nu(\tau)(\mathbb{R})\\ &= \int_{\mathbb{R}} (J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{X}_s(\tau,s) + J_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{Y}_s(\tau,s))\,ds\\ &= \int_{\mathbb{R}} J_s(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\,ds\\ &= \lim_{s\to\infty} J(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\\ &= \lim_{s\to\infty} J(\mathcal{X}(0,s),\mathcal{Y}(0,s)) \quad \text{by Lemma 4.14}\\ &= \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}),\end{aligned}$$

where we used that $\lim_{s\to-\infty} J(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s)) = \lim_{s\to-\infty} \mathcal{Z}_4(\tau,s) = 0$.

Let us prove (ii). We decompose $\mu(\tau)$ into its absolutely continuous and singular part, that is, $\mu(\tau) = \mu(\tau)_{\text{ac}} + \mu(\tau)_{\text{sing}}$. We want to prove that, for almost every time $\tau \geq 0$,

$$\mu(\tau)_{\text{sing}}(\{x \in \mathbb{R} \mid c'(u(\tau,x)) \neq 0\}) = 0.$$

Consider the set

$$A_\tau = \{s \in \mathbb{R} \mid x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s)) > 0\}.$$

Since $x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s)) = \mathcal{V}_2(\mathcal{X}(\tau,s))$, A_τ corresponds to the set A in (5.27) in the proof of Lemma 5.8. Using $\mathcal{V}_4(\mathcal{X}(\tau,s)) = J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $\mathcal{Z}_2(\tau,s) = x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ in (5.28), we get

$$\begin{aligned}\mu_{\text{sing}}(\tau)(B) &= \int_{\{s\in\mathbb{R}\,|\,\mathcal{Z}_2(\tau,s)\in B\}\cap A_\tau^c} \mathcal{V}_4(\mathcal{X}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\\ &= \int_{\{s\in A_\tau^c\,|\,x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in B\}} J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\end{aligned} \tag{6.10}$$

for any Borel set B. Let

$$E = \{(X,Y) \in \mathbb{R}^2 \mid x_X(X,Y) = 0 \text{ and } c'(U(X,Y)) \neq 0\}.$$

For a given time τ, we consider the mapping $\Gamma_\tau : s \mapsto (\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$. From (6.10), we obtain

$$\begin{aligned}
(6.11)\qquad &\mu(\tau)_{\mathrm{sing}}(\{x\in\mathbb{R} \mid c'(u(\tau,x))\neq 0\})\\
&=\int_{\{s\in\mathbb{R}\,\mid\,(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in E\}} J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\\
&=\int_{\Gamma_\tau^{-1}(E)} J_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\mathcal{X}_s(\tau,s)\,ds\\
&\leq 2||J_X||_{W^{1,\infty}_Y(\mathbb{R})}\mathrm{meas}(\Gamma_\tau^{-1}(E)).
\end{aligned}$$

We claim that $\mathrm{meas}(\Gamma_\tau^{-1}(E))=0$. By the area formula, see Section 3.3 in [5], we obtain

$$\begin{aligned}
(6.12)\qquad \mathrm{meas}(\Gamma_\tau^{-1}(E)) &= \int_{\Gamma_\tau^{-1}(E)}\frac{1}{2}(\mathcal{X}_s(\tau,s)+\mathcal{Y}_s(\tau,s))\,ds \quad \text{since } \mathcal{X}_s+\mathcal{Y}_s=2\\
&\leq \int_{\Gamma_\tau^{-1}(E)}(\mathcal{X}_s^2(\tau,s)+\mathcal{Y}_s^2(\tau,s))^{\frac{1}{2}}\,ds\\
&=\mathcal{H}^1(\Gamma_\tau\circ\Gamma_\tau^{-1}(E))\\
&\leq \mathcal{H}^1(E\cap t^{-1}(\tau)),
\end{aligned}$$

where we used that $\mathcal{X}_s=(\mathcal{X}_s^2)^{\frac{1}{2}}\leq(\mathcal{X}_s^2+\mathcal{Y}_s^2)^{\frac{1}{2}}$ (and similarly for $\mathcal{Y}_s$). Here, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. The fact that $\Gamma_\tau\circ\Gamma_\tau^{-1}(E)\subset E\cap t^{-1}(\tau)$ follows from the following argument. If $E\cap t^{-1}(\tau)$ contains a rectangle, we have by the Definition 5.2, that the curve Γ_τ consists of the left vertical side and the upper horizontal side of the rectangle. We show that $\mathcal{H}^1(E\cap t^{-1}(\tau))=0$. We have

$$E=A_1\cup A_3,$$

where

$$A_1=\{(X,Y)\in\mathbb{R}^2 \mid x_X(X,Y)=0,\ x_Y(X,Y)>0 \text{ and } c'(U(X,Y))\neq 0\}$$

and

$$A_3=\{(X,Y)\in\mathbb{R}^2 \mid x_X(X,Y)=0,\ x_Y(X,Y)=0 \text{ and } c'(U(X,Y))\neq 0\}.$$

By an argument as in the proof of Theorem 4 in [10], we obtain $\mathrm{meas}(A_1)=0$. Hence, by the coarea formula, see Section 3.4 in [5], we get

$$\begin{aligned}
\int_{\mathbb{R}}\mathcal{H}^1(E\cap t^{-1}(\tau))\,d\tau &= \iint_E (t_X^2(X,Y)+t_Y^2(X,Y))^{\frac{1}{2}}\,dX\,dY\\
&=\iint_{A_3}(t_X^2(X,Y)+t_Y^2(X,Y))^{\frac{1}{2}}\,dX\,dY\\
&=\iint_{A_3}\frac{1}{c(U(X,Y))}(x_X^2(X,Y)+x_Y^2(X,Y))^{\frac{1}{2}}\,dX\,dY \quad \text{by (4.12a)}\\
&=0.
\end{aligned}$$

Therefore, we get from (6.12), that $\mathrm{meas}(\Gamma_\tau^{-1}(E))=0$, which inserted into (6.11) yields

$$\mu(\tau)_{\mathrm{sing}}(\{x\in\mathbb{R}\mid c'(u(\tau,x))\neq 0\})=0.$$

□

Theorem 6.3 (Finite speed of propagation). *For initial data* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ *and* $(\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0)$ *in* $\mathcal{D}$*, we consider the solutions*

$$(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$$

and

$$(\bar{u}, \bar{R}, \bar{S}, \bar{\rho}, \bar{\sigma}, \bar{\mu}, \bar{\nu})(t) = \bar{S}_t(\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0).$$

Given $\mathbf{t} > 0$ *and* $\mathbf{x} \in \mathbb{R}$*, if*

$$(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)(x) = (\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0)(x)$$

for $x \in [\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]$*, then*

$$u(\mathbf{t}, \mathbf{x}) = \bar{u}(\mathbf{t}, \mathbf{x}).$$

In the case of the linear wave equation, i.e., c is constant, one has $u(\mathbf{t}, \mathbf{x}) = \bar{u}(\mathbf{t}, \mathbf{x})$ if the initial data are equal on the interval $[\mathbf{x} - c\mathbf{t}, \mathbf{x} + c\mathbf{t}]$. If the function $c(u)$ satisfies $\frac{1}{\kappa} \leq c(u) \leq \kappa$ for some $\kappa \geq 1$ the corresponding interval is contained in $[\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]$. Thus, we require the initial data to coincide on a slightly bigger interval.

Proof. We denote $x_l = \mathbf{x} - \kappa\mathbf{t}$ and $x_r = \mathbf{x} + \kappa\mathbf{t}$. For a given $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, we define

$$(\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0)(x) = \begin{cases} (u_0, R_0, S_0, \rho_0, \sigma_0)(x) & \text{if } x \in [x_l, x_r] \\ (0, 0, 0, 0, 0) & \text{otherwise} \end{cases}$$

and

$$\bar{\mu}_0(B) = \mu_0(B \cap [x_l, x_r]), \quad \bar{\nu}_0(B) = \nu_0(B \cap [x_l, x_r])$$

for any Borel set B. It is enough to prove the theorem for the initial data $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ and $(\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0)$ in $\mathcal{D}$. We have to compute the solutions corresponding to these two initial data. We decompose the proof into five steps.

Step 1. Let $\psi = (\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ and $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) = \mathbf{L}(\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0)$. We denote $X_l = x_l$, $Y_l = x_l$, $X_r = x_r + \mu_0([x_l, x_r])$, $Y_r = x_r + \nu_0([x_l, x_r])$ and $\Omega = [X_l, X_r] \times [Y_l, Y_r]$. We claim that

$$\bar{x}_1(X) = \begin{cases} X & \text{if } X \leq X_l, \\ x_1(X + \mu_0((-\infty, x_l))) & \text{if } X_l < X \leq X_r, \\ X - \mu_0([x_l, x_r]) & \text{if } X > X_r \end{cases} \tag{6.13}$$

and

$$\bar{x}_2(Y) = \begin{cases} Y & \text{if } Y \leq Y_l, \\ x_2(Y + \nu_0((-\infty, x_l))) & \text{if } Y_l < Y \leq Y_r, \\ Y - \nu_0([x_l, x_r]) & \text{if } Y > Y_r. \end{cases} \tag{6.14}$$

From (3.12a), we have

$$\bar{x}_1(X) = \sup\{x' \in \mathbb{R} \mid x' + \bar{\mu}_0((-\infty, x')) < X\}. \tag{6.15}$$

First case: $X \leq X_l$. For any x' such that $x' + \bar{\mu}_0((-\infty, x')) < X$, we have $x' < X$, so that $x' < X \leq X_l = x_l$. Then, $\bar{\mu}_0((-\infty, x')) = \mu_0((-\infty, x') \cap [x_l, x_r]) = 0$ and $\bar{x}_1(X) = X$.

Second case: $X_l < X \leq X_r$. We have $x_l + \bar{\mu}_0((-\infty, x_l)) = x_l = X_l < X$, so that

$$\bar{x}_1(X) = \sup\{x' \in [x_l, \infty) \mid x' + \mu_0([x_l, x')) < X\}. \tag{6.16}$$

We claim that for any x' such that $x' + \bar{\mu}_0((-\infty, x')) < X$, we have $x' \leq x_r$. Assume the opposite, that is, $x' > x_r$. Then, $\bar{\mu}_0((-\infty, x')) = \mu_0((-\infty, x') \cap [x_l, x_r]) = \mu_0([x_l, x_r])$ and we get

$$x' + \mu_0([x_l, x_r]) = x' + \bar{\mu}_0((-\infty, x')) < X \leq X_r = x_r + \mu_0([x_l, x_r]),$$

which contradicts the assumption $x' > x_r$. Hence,

$$\bar{x}_1(X) = \sup\{x' \in [x_l, x_r] \mid x' + \mu_0([x_l, x')) < X\}. \tag{6.17}$$

We claim that

$$x_1(X + \mu_0((-\infty, x_l))) = \sup\{x' \in [x_l, x_r] \mid x' + \mu_0([x_l, x')) < X\}.$$

By (3.12a), we have

$$x_1(X + \mu_0((-\infty, x_l))) = \sup\{x' \in \mathbb{R} \mid x' + \mu_0((-\infty, x')) < X + \mu_0((-\infty, x_l))\}.$$

We have $x_l + \mu_0((-\infty, x_l)) = X_l + \mu_0((-\infty, x_l)) < X + \mu_0((-\infty, x_l))$, so that

$$\begin{aligned} x_1(X + \mu_0((-\infty, x_l))) &= \sup\{x' \in [x_l, \infty) \mid x' + \mu_0((-\infty, x')) < X + \mu_0((-\infty, x_l))\} \\ &= \sup\{x' \in [x_l, \infty) \mid x' + \mu_0([x_l, x')) < X\} \\ &= \bar{x}_1(X) \quad \text{by (6.16)}. \end{aligned}$$

Then, by (6.17), we get

$$x_1(X + \mu_0((-\infty, x_l))) = \sup\{x' \in [x_l, x_r] \mid x' + \mu_0([x_l, x')) < X\}.$$

Third case: $X > X_r$. Since $x_r + \bar{\mu}_0((-\infty, x_r)) = x_r + \mu_0([x_l, x_r)) \leq x_r + \mu_0([x_l, x_r]) = X_r < X$, we have

$$\bar{x}_1(X) = \sup\{x' \in [x_r, \infty) \mid x' + \bar{\mu}_0((-\infty, x')) < X\}.$$

If $x' > x_r$, $\bar{\mu}_0((-\infty, x')) = \mu_0([x_l, x_r])$, which implies that

$$\bar{x}_1(X) = X - \mu_0([x_l, x_r]).$$

This concludes the proof of (6.13). Similarly, one proves (6.14).

Let $f(X) = X + \mu_0((-\infty, x_l))$ and $g(Y) = Y + \nu_0((-\infty, x_l))$. We claim that $\phi = (f, g) \in G^2$. Since $f' = 1$, f is invertible and $f^{-1}(X) = X - \mu_0((-\infty, x_l))$. We have $||f - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} \leq \mu_0(\mathbb{R})$, $||f^{-1} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} \leq \mu_0(\mathbb{R})$, $f' - 1 = 0$ and $(f^{-1})' - 1 = 0$. Hence, f belongs to G. Similarly, one shows that $g \in G$. We denote $\tilde{\psi} = \psi \cdot \phi$. We have proved that

$$\bar{x}_1(X) = \tilde{x}_1(X) \quad \text{for } X_l < X \leq X_r \tag{6.18}$$

and

$$\bar{x}_2(Y) = \tilde{x}_2(Y) \quad \text{for } Y_l < Y \leq Y_r. \tag{6.19}$$

Step 2. Let $\bar{\Theta} = \mathbf{C}(\bar{\psi})$ and $\tilde{\Theta} = \mathbf{C}(\tilde{\psi})$. We prove that

$$\bar{\mathcal{X}}(s) = \tilde{\mathcal{X}}(s) \quad \text{and} \quad \bar{\mathcal{Y}}(s) = \tilde{\mathcal{Y}}(s) \tag{6.20}$$

for $s \in [s_l, s_r]$ where $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. First we show that

$$\tilde{\mathcal{X}}(s_l) = X_l \quad \text{and} \quad \tilde{\mathcal{Y}}(s_l) = Y_l. \tag{6.21}$$

By (3.28), we have

$$\tilde{\mathcal{X}}(s) = \sup\{X \in \mathbb{R} \mid \tilde{x}_1(X') < \tilde{x}_2(2s - X') \text{ for all } X' < X\}.$$

For any $X < X_l$, $\tilde{x}_1(X) \leq \tilde{x}_1(X_l)$ because $\tilde{x}_1$ is nondecreasing. We claim that $\tilde{x}_1(X) < \tilde{x}_1(X_l)$. Assume the opposite, that is, $\tilde{x}_1(X) = \tilde{x}_1(X_l)$. Then, since $\tilde{x}_1(X_l) = x_l$, there exists an increasing sequence x_i such that $\lim_{i\to\infty} x_i = x_l$ and $x_i + \mu_0((-\infty, x_i)) < X + \mu_0((-\infty, x_l))$. By sending i to infinity, we get, since $x \mapsto \mu_0((-\infty, x))$ is lower semi-continuous, that $x_l + \mu_0((-\infty, x_l)) \leq X + \mu_0((-\infty, x_l))$. This leads to the contradiction $x_l \leq X < X_l = x_l$. Therefore, for any $X < X_l$,

$$\begin{aligned}\tilde{x}_1(X) &< \tilde{x}_1(X_l) = x_1(x_l + \mu_0((-\infty, x_l))) = x_l \\ &= x_2(x_l + \nu_0((-\infty, x_l))) = \tilde{x}_2(Y_l) \leq \tilde{x}_2(2s_l - X),\end{aligned}$$

where we used that $\tilde{\psi} = \psi \cdot \phi$ and the fact that $\tilde{x}_2$ is nondecreasing. Hence, $\tilde{\mathcal{X}}(s_l) = X_l$ and we have proved (6.21). By using similar arguments, one obtains

$$\tilde{\mathcal{X}}(s_r) = X_r \quad \text{and} \quad \tilde{\mathcal{Y}}(s_r) = Y_r. \tag{6.22}$$

The corresponding results for $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$, which we state here for completeness, are

$$\bar{\mathcal{X}}(s_l) = X_l, \quad \bar{\mathcal{Y}}(s_l) = Y_l \quad \text{and} \quad \bar{\mathcal{X}}(s_r) = X_r, \quad \bar{\mathcal{Y}}(s_r) = Y_r.$$

In particular, we have proved (6.20) for $s = s_l$ and $s = s_r$. For $s \in (s_l, s_r)$, we have either $X_l < \bar{\mathcal{X}}(s) \leq X_r$ or $Y_l \leq \bar{\mathcal{Y}}(s) < Y_r$ by the definition of $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$. We only consider the case $X_l < \bar{\mathcal{X}}(s) \leq X_r$, as the other case can be treated similarly. By (3.28), there exists an increasing sequence X_i such that $\lim_{i\to\infty} X_i = \bar{\mathcal{X}}(s)$ and $\bar{x}_1(X_i) < \bar{x}_2(2s - X_i)$. For sufficiently large i, we have $X_l < X_i \leq X_r$ and, by (6.18), we get $\tilde{x}_1(X_i) = \bar{x}_1(X_i) < \bar{x}_2(2s - X_i)$. If $2s - X_i \leq Y_r$ then, by (6.19), $\bar{x}_2(2s - X_i) = \tilde{x}_2(2s - X_i)$, so that $\tilde{x}_1(X_i) < \tilde{x}_2(2s - X_i)$. If $2s - X_i > Y_r$ then also $\tilde{x}_1(X_i) < \tilde{x}_2(2s - X_i)$. Assume the opposite, that is, $\tilde{x}_1(X_i) \geq \tilde{x}_2(2s - X_i)$. Since $\tilde{x}_1$ and $\tilde{x}_2$ are nondecreasing, we have $\tilde{x}_1(X_r) \geq \tilde{x}_1(X_i) \geq \tilde{x}_2(2s - X_i) \geq \tilde{x}_2(Y_r)$. We have

$$\begin{aligned}\tilde{x}_1(X_r) &= x_1(X_r + \mu_0((-\infty, x_l))) = x_1(x_r + \mu_0((-\infty, x_r])) \\ &= x_1(x_r + \mu_0((-\infty, x_r))) = x_r\end{aligned}$$

and similarly, we obtain $\tilde{x}_2(Y_r) = x_r$. Thus, $\tilde{x}_1(X_r) = \tilde{x}_2(Y_r)$, which implies that $\tilde{x}_2(2s - X_i) = x_r$. By (3.12b), there exists a decreasing sequence x_j such that $\lim_{j\to\infty} x_j = \tilde{x}_2(2s - X_i)$ and

$x_j + \nu_0((-\infty, x_j)) \geq 2s - X_i + \nu_0((-\infty, x_l))$. Sending j to infinity, we get

$2s - X_i + \nu_0((-\infty, x_l)) \leq \tilde{x}_2(2s - X_i) + \nu_0((-\infty, \tilde{x}_2(2s - X_i)]) = x_r + \nu_0((-\infty, x_r])$,

which leads to the contradiction

$$2s - X_i \leq x_r + \nu_0([x_l, x_r]) = Y_r.$$

Hence, we have shown that

$$\tilde{x}_1(X_i) < \tilde{x}_2(2s - X_i). \tag{6.23}$$

If $Y_l < \bar{\mathcal{Y}}(s) < Y_r$, we get, from (6.18) and (6.19), that

$$\tilde{x}_1(\bar{\mathcal{X}}(s)) = \bar{x}_1(\bar{\mathcal{X}}(s)) = \bar{x}_2(\bar{\mathcal{Y}}(s)) = \tilde{x}_2(\bar{\mathcal{Y}}(s)). \tag{6.24}$$

If $\bar{\mathcal{Y}}(s) = Y_l$, we have $\bar{x}_2(\bar{\mathcal{Y}}(s)) = x_l = \tilde{x}_2(\bar{\mathcal{Y}}(s))$, so that (6.24) also holds. It then follows from (6.23) and (6.24) that $\bar{\mathcal{X}}(s) = \tilde{\mathcal{X}}(s)$, which concludes the proof of (6.20).

Step 3. Let $(\bar{Z}, \bar{p}, \bar{q}) = \mathbf{S}(\bar{\Theta})$ and $(\tilde{Z}, \tilde{p}, \tilde{q}) = \mathbf{S}(\tilde{\Theta})$. We prove that

$$\begin{gathered} \bar{t}(X,Y) = \tilde{t}(X,Y), \quad \bar{x}(X,Y) = \tilde{x}(X,Y), \quad \bar{U}(X,Y) = \tilde{U}(X,Y), \\ \bar{p}(X,Y) = \tilde{p}(X,Y), \quad \bar{q}(X,Y) = \tilde{q}(X,Y) \end{gathered} \tag{6.25}$$

for all $(X,Y) \in \Omega$, where $\Omega = [X_l, X_r] \times [Y_l, Y_r]$. We have $\bar{x}_1(X) = \tilde{x}_1(X)$ for $X \in [X_l, X_r]$ and $\bar{x}_2(Y) = \tilde{x}_2(Y)$ for $Y \in [Y_l, Y_r]$. Let us show that

$$\begin{gathered} \bar{U}_1(X) = \tilde{U}_1(X), \quad \bar{V}_1(X) = \tilde{V}_1(X), \quad \bar{J}_1(X) = \tilde{J}_1(X) - \tilde{J}_1(X_l), \\ \bar{K}_1(X) = \tilde{K}_1(X) - \tilde{K}_1(X_l), \quad \bar{H}_1(X) = \tilde{H}_1(X) \end{gathered}$$

for $X \in [X_l, X_r]$, and

$$\begin{gathered} \bar{U}_2(Y) = \tilde{U}_2(Y), \quad \bar{V}_2(Y) = \tilde{V}_2(Y), \quad \bar{J}_2(Y) = \tilde{J}_2(Y) - \tilde{J}_2(Y_l), \\ \bar{K}_2(Y) = \tilde{K}_2(Y) - \tilde{K}_2(Y_l), \quad \bar{H}_2(Y) = \tilde{H}_2(Y) \end{gathered} \tag{6.26}$$

for $Y \in [Y_l, Y_r]$. From (3.12d), we have

$$\bar{U}_1(X) = \bar{u}_0(\bar{x}_1(X)) = u_0(\tilde{x}_1(X)) = u_0(x_1(f(X))) = U_1(f(X)) = \tilde{U}_1(X).$$

By (3.12e), we get

$$\begin{aligned} \bar{V}_1(X) &= \bar{x}_1'(X)\frac{\bar{R}_0(\bar{x}_1(X))}{2c(\bar{U}_1(X))} \\ &= \tilde{x}_1'(X)\frac{R_0(\tilde{x}_1(X))}{2c(\tilde{U}_1(X))} \\ &= f'(X)x_1'(f(X))\frac{R_0(x_1(f(X)))}{2c(U_1(f(X)))} \\ &= f'(X)V_1(f(X)) \\ &= \tilde{V}_1(X). \end{aligned}$$

We have, by (3.12c), that

$$\begin{aligned} \bar{J}_1(X) &= X - \bar{x}_1(X) \\ &= X - \tilde{x}_1(X) \\ &= X - x_1(X + \mu_0((-\infty, x_l))) \\ &= X + \mu_0((-\infty, x_l)) - x_1(X + \mu_0((-\infty, x_l))) \\ &\quad - (X_l + \mu_0((-\infty, x_l)) + x_1(X_l + \mu_0((-\infty, x_l)))) \\ &= J_1(X + \mu_0((-\infty, x_l))) - J_1(X_l + \mu_0((-\infty, x_l))) \\ &= \tilde{J}_1(X) - \tilde{J}_1(X_l) \end{aligned}$$

since $x_1(X_l + \mu_0((-\infty, x_l))) = X_l$. From (6.13), $\bar{x}_1(X) = X$ for $X \leq X_l$, so that $\bar{J}_1(X) = X - \bar{x}_1(X) = 0$ for $X \leq X_l$. This implies, by (3.12f), that

$$\begin{aligned}
\bar{K}_1(X) &= \int_{-\infty}^{X} \frac{\bar{J}_1'(\bar{X})}{c(\bar{U}_1(\bar{X}))}\, d\bar{X} \\
&= \int_{X_l}^{X} \frac{\tilde{J}_1'(\bar{X})}{c(\tilde{U}_1(\bar{X}))}\, d\bar{X} \\
&= \int_{X_l}^{X} \frac{J_1'(\bar{X} + \mu_0((-\infty, x_l)))}{c(U_1(\bar{X} + \mu_0((-\infty, x_l))))}\, d\bar{X} \\
&= \int_{X_l+\mu_0((-\infty,x_l))}^{X+\mu_0((-\infty,x_l))} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X} \quad \text{by a change of variables} \\
&= \int_{-\infty}^{X+\mu_0((-\infty,x_l))} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X} - \int_{-\infty}^{X_l+\mu_0((-\infty,x_l))} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X} \\
&= K_1(X + \mu_0((-\infty, x_l))) - K_1(X_l + \mu_0((-\infty, x_l))) \\
&= \tilde{K}_1(X) - \tilde{K}_1(X_l).
\end{aligned}$$

By (3.12g), we have

$$\begin{aligned}
\bar{H}_1(X) &= \frac{1}{2}\bar{\rho}_0(\bar{x}_1(X))\bar{x}_1'(X) \\
&= \frac{1}{2}\rho_0(\tilde{x}_1(X))\tilde{x}_1'(X) \\
&= \frac{1}{2}\rho_0(x_1(f(X)))x_1'(f(X))f'(X) \\
&= f'(X)H_1(f(X)) \\
&= \tilde{H}_1(X).
\end{aligned}$$

In a similar way, one proves (6.26).

We have $\bar{\mathcal{X}}(s) = \tilde{\mathcal{X}}(s)$ and $\bar{\mathcal{Y}}(s) = \tilde{\mathcal{Y}}(s)$ for $s \in [s_l, s_r]$. We show that

$$\begin{aligned}
&\bar{\mathcal{Z}}_1(s) = \tilde{\mathcal{Z}}_1(s), \quad \bar{\mathcal{Z}}_2(s) = \tilde{\mathcal{Z}}_2(s), \quad \bar{\mathcal{Z}}_3(s) = \tilde{\mathcal{Z}}_3(s), \\
&\bar{\mathcal{Z}}_4(s) = \tilde{\mathcal{Z}}_4(s) - \tilde{\mathcal{Z}}_4(s_l), \quad \bar{\mathcal{Z}}_5(s) = \tilde{\mathcal{Z}}_5(s) - \tilde{\mathcal{Z}}_5(s_l)
\end{aligned}$$

for $s \in [s_l, s_r]$, and

$$\bar{\mathcal{V}}(X) = \tilde{\mathcal{V}}(X), \quad \bar{\mathcal{W}}(Y) = \tilde{\mathcal{W}}(Y), \quad \bar{\mathfrak{p}}(X) = \tilde{\mathfrak{p}}(X), \quad \bar{\mathfrak{q}}(Y) = \tilde{\mathfrak{q}}(Y) \tag{6.27}$$

for $X \in [X_l, X_r]$ and $Y \in [Y_l, Y_r]$. From (3.30a), we have $\bar{\mathcal{Z}}_1(s) = 0 = \tilde{\mathcal{Z}}_1(s)$. By (3.30b) and (3.30c), we have $\bar{\mathcal{Z}}_2(s) = \bar{x}_1(\bar{\mathcal{X}}(s)) = \tilde{x}_1(\tilde{\mathcal{X}}(s)) = \tilde{\mathcal{Z}}_2(s)$ and $\bar{\mathcal{Z}}_3(s) = \bar{U}_1(\bar{\mathcal{X}}(s)) = \tilde{U}_1(\tilde{\mathcal{X}}(s)) = \tilde{\mathcal{Z}}_3(s)$, respectively. From (3.30d), we get

$$\begin{aligned}
\bar{\mathcal{Z}}_4(s) &= \bar{J}_1(\bar{\mathcal{X}}(s)) + \bar{J}_2(\bar{\mathcal{Y}}(s)) \\
&= \tilde{J}_1(\tilde{\mathcal{X}}(s)) - \tilde{J}_1(X_l) + \tilde{J}_2(\tilde{\mathcal{Y}}(s)) - \tilde{J}_2(Y_l) \\
&= \tilde{\mathcal{Z}}_4(s) - \tilde{\mathcal{Z}}_4(s_l)
\end{aligned}$$

and similarly, we find that $\bar{\mathcal{Z}}_5(s) = \tilde{\mathcal{Z}}_5(s) - \tilde{\mathcal{Z}}_5(s_l)$. Using (3.31a)-(3.31f), a straightforward calculation shows (6.27). Hence, $\bar{\Theta}$ and $\tilde{\Theta}$ are equal in Ω, except that the energy potentials differ up to a constant. However, since the governing equations

(2.38) are invariant with respect to addition of a constant to the energy potentials, we get, by Lemma 4.10, that (6.25) holds.

Step 4. We prove that there exists $(X_0, Y_0) \in \Omega$ such that

$$\bar{t}(X_0, Y_0) = \mathbf{t} \quad \text{and} \quad \bar{x}(X_0, Y_0) = \mathbf{x}. \tag{6.28}$$

We have

$$\bar{x}(X_l, Y_l) = \bar{x}_1(X_l) = \bar{x}_2(Y_l) = x_l \quad \text{and} \quad \bar{x}(X_r, Y_r) = \bar{x}_1(X_r) = \bar{x}_2(Y_r) = x_r,$$

so that

$$\begin{aligned} \bar{x}(X_r, Y_l) - x_l &= \bar{x}(X_r, Y_l) - \bar{x}(X_l, Y_l) \\ &= \int_{X_l}^{X_r} \bar{x}_X(X, Y_l)\, dX \\ &= \int_{X_l}^{X_r} c(\bar{U}(X, Y_l))\bar{t}_X(X, Y_l)\, dX \quad \text{by (4.12a)} \\ &\leq \kappa \int_{X_l}^{X_r} \bar{t}_X(X, Y_l)\, dX \quad \text{since } \frac{1}{\kappa} \leq c \leq \kappa \text{ and } \bar{t}_X \geq 0 \\ &= \kappa\bar{t}(X_r, Y_l) \quad \text{since } \bar{t}(X_l, Y_l) = \bar{t}(\bar{\mathcal{X}}(s_l), \bar{\mathcal{Y}}(s_l)) = 0. \end{aligned} \tag{6.29}$$

In a similar way, one proves that $x_r - \bar{x}(X_r, Y_l) \leq \kappa\bar{t}(X_r, Y_l)$, which added to (6.29) yields $x_r - x_l \leq 2\kappa\bar{t}(X_r, Y_l)$, or

$$\mathbf{t} \leq \bar{t}(X_r, Y_l). \tag{6.30}$$

There exists $(X_0, Y_0) \in \mathbb{R}^2$, which may not be unique, such that

$$\bar{t}(X_0, Y_0) = \mathbf{t} \quad \text{and} \quad \bar{x}(X_0, Y_0) = \mathbf{x}.$$

Assume that

$$\bar{t}(X, Y) \neq \mathbf{t} \quad \text{or} \quad \bar{x}(X, Y) \neq \mathbf{x} \tag{6.31}$$

for all $(X, Y) \in \Omega$. We claim that we cannot have

$$X_0 > X_r \quad \text{and} \quad Y_0 < Y_l \tag{6.32}$$

or

$$X_0 < X_r \quad \text{and} \quad Y_0 > Y_l, \tag{6.33}$$

so that either $X_0 > X_r$ and $Y_0 \geq Y_l$ or $X_0 \leq X_r$ and $Y_0 < Y_l$.

If (6.32) holds, we get, since $\bar{t}_X \geq 0$ and $\bar{t}_Y \leq 0$, that $\mathbf{t} = \bar{t}(X_0, Y_0) \geq \bar{t}(X_r, Y_l) \geq \mathbf{t}$, where we used (6.30). Hence, $\bar{t}(X_r, Y_l) = \mathbf{t}$, which contradicts (6.31).

If (6.33) holds, we have either $Y_0 > Y_r$ or $Y_l < Y_0 \leq Y_r$. If $Y_0 > Y_r$, we get a contradiction, since $\mathbf{t} = \bar{t}(X_0, Y_0) \leq \bar{t}(X_r, Y_r) = \bar{t}(\bar{\mathcal{X}}(s_r), \bar{\mathcal{Y}}(s_r)) = 0$. If $Y_l < Y_0 \leq Y_r$, we have $X_0 < X_l$ because $(X_0, Y_0) \notin \Omega$, which leads to the contradiction $\mathbf{t} = \bar{t}(X_0, Y_0) \leq \bar{t}(X_l, Y_l) = \bar{t}(\bar{\mathcal{X}}(s_l), \bar{\mathcal{Y}}(s_l)) = 0$. Hence, we have either $X_0 > X_r$ and $Y_0 \geq Y_l$ or $X_0 \leq X_r$ and $Y_0 < Y_l$.

Assume that $X_0 > X_r$ and $Y_0 \geq Y_l$. If $Y_0 > Y_r$, then $\mathbf{x} = \bar{x}(X_0, Y_0) \geq \bar{x}(X_r, Y_r) = x_r = \mathbf{x} + \kappa\mathbf{t}$, that is, $\mathbf{t} \leq 0$, which is a contradiction. Therefore, $Y_l \leq Y_0 \leq Y_r$. Since $\bar{x}_X \geq 0$, we have $\bar{x}(X_r, Y_0) \leq \bar{x}(X_0, Y_0)$. We claim that $\bar{x}(X_r, Y_0) < \bar{x}(X_0, Y_0)$. Assume the opposite, that is, $\bar{x}(X_r, Y_0) = \bar{x}(X_0, Y_0)$. Then, since $\bar{x}$ is nondecreasing in the X variable, we have $\bar{x}_X(X, Y_0) = 0$ for all $X \in [X_r, X_0]$. By (4.12a), we get $\bar{t}_X(X,Y_0)=0$ for all $X\in[X_r,X_0]$, so that $\bar{t}(X_r,Y_0)=\bar{t}(X_0,Y_0)=\mathbf{t}$. This contradicts (6.31) and we must have $\bar{x}(X_r,Y_0)<\bar{x}(X_0,Y_0)$. We obtain

$$\begin{aligned}
\mathbf{x} &= \bar{x}(X_0,Y_0)\\
&> \bar{x}(X_r,Y_0)\\
&= x_r - \int_{Y_0}^{Y_r} \bar{x}_Y(X_r,Y)\,dY\\
&= x_r + \int_{Y_0}^{Y_r} c(\bar{U}(X_r,Y))\bar{t}_Y(X_r,Y)\,dY \quad \text{by (4.12a)}\\
&\geq x_r + \kappa\int_{Y_0}^{Y_r} \bar{t}_Y(X_r,Y)\,dY \quad \text{since } \frac{1}{\kappa}\leq c\leq\kappa \text{ and } \bar{t}_Y\leq 0\\
&= x_r - \kappa\bar{t}(X_r,Y_0) \quad \text{since } \bar{t}(X_r,Y_r)=\bar{t}(\bar{\mathcal{X}}(s_r),\bar{\mathcal{Y}}(s_r))=0\\
&\geq x_r - \kappa\bar{t}(X_0,Y_0) \quad \text{because } \bar{t}_X\geq 0\\
&= \mathbf{x},
\end{aligned}$$

which is a contradiction. The situation $X_0\leq X_r$ and $Y_0<Y_l$ can be treated similarly. Hence, (6.31) cannot hold and we have proved (6.28).

Step 5. We have

$$\begin{aligned}
\bar{u}(\mathbf{t},\mathbf{x}) &= \bar{U}(X_0,Y_0) \quad \text{by (6.28) and (5.30)}\\
&= \tilde{U}(X_0,Y_0) \quad \text{by (6.25)}\\
&= U(f(X_0),g(Y_0))\\
&= u(t(f(X_0),g(Y_0)),x(f(X_0),g(Y_0))) \quad \text{by (5.30)}\\
&= u(\tilde{t}(X_0,Y_0),\tilde{x}(X_0,Y_0))\\
&= u(\bar{t}(X_0,Y_0),\bar{x}(X_0,Y_0)) \quad \text{by (6.25)}\\
&= u(\mathbf{t},\mathbf{x}) \quad \text{by (6.28)}.
\end{aligned}$$

This concludes the proof. □

7. Regularity of Solutions

The theorems we prove in Section 7.1 and 7.2 are local results. The main reason for this is that we require the initial data ρ_0 and σ_0 corresponding to the equations (1.3b) and (1.3c) to be bounded from below by a strictly positive constant and to belong to L^2, which is not possible globally.

7.1. Existence of Smooth Solutions.

Theorem 7.1. *Let $-\infty<x_l<x_r<\infty$ and consider $(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0)\in\mathcal{D}$. Let $m\in\mathbb{N}$ and assume that,*

(A1) $u_0\in L^\infty([x_l,x_r])$,
(A2) $R_0,S_0,\rho_0,\sigma_0\in W^{m-1,\infty}([x_l,x_r])$,
(A3) there are constants $d>0$ and $e>0$ such that $\rho_0(x)\geq d$ and $\sigma_0(x)\geq e$ for all $x\in[x_l,x_r]$,
(A4) μ_0 and ν_0 are absolutely continuous on $[x_l,x_r]$,

(A5) $c \in C^{m-1}(\mathbb{R})$ *and* $\max_{u\in\mathbb{R}}\left|\frac{d^i}{du^i}c(u)\right| \leq k_i$ *for constants* $k_i, i = 3, 4, 5, \dots, m-1$.

For any $\tau \in \left[0, \frac{1}{2\kappa}(x_r - x_l)\right]$ *consider*

$$(u, R, S, \rho, \sigma, \mu, \nu)(\tau) = \bar{S}_\tau(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0).$$

Then

(P1) $u(\tau, \cdot) \in W^{m,\infty}([x_l + \kappa\tau, x_r - \kappa\tau])$,

(P2) $R(\tau, \cdot), S(\tau, \cdot), \rho(\tau, \cdot), \sigma(\tau, \cdot) \in W^{m-1,\infty}([x_l + \kappa\tau, x_r - \kappa\tau])$,

(P3) there are constants $\bar{d} > 0$ *and* $\bar{e} > 0$ *such that* $\rho(\tau, x) \geq \bar{d}$ *and* $\sigma(\tau, x) \geq \bar{e}$ *for all* $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$,

(P4) $\mu(\tau, \cdot)$ *and* $\nu(\tau, \cdot)$ *are absolutely continuous on* $[x_l + \kappa\tau, x_r - \kappa\tau]$.

For $\tau \in \left[-\frac{1}{2\kappa}(x_r - x_l), 0\right]$, *the solution satisfies the same properties on the interval* $\left[x_l - \kappa\tau, x_r + \kappa\tau\right]$.

Note that since $(u_0)_x = \frac{1}{2c(u_0)}(R_0 - S_0)$, it follows from assumptions (A1), (A2) and (A5) that $u_0 \in W^{m,\infty}([x_l, x_r])$.

Specifically, (A4) means that $\mu_0((-\infty, x_l)) = \mu_0((-\infty, x_l])$ and $\nu_0((-\infty, x_l)) = \nu_0((-\infty, x_l])$.

By (1.5) and (1.6), (A5) holds for $i = 0, 1, 2$.

Proof. In the following, we will consider the case $0 < \tau \leq \frac{1}{2\kappa}(x_r - x_l)$. The case $-\frac{1}{2\kappa}(x_r - x_l) \leq \tau < 0$ can be treated in the same way.

We decompose the proof into three steps.

Step 1. We first consider the case $m = 1$.

(i) Consider $(\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$. Since μ_0 is absolutely continuous on $[x_l, x_r]$ we have from (3.12a),

$$x_1(X) + \mu_0((-\infty, x_1(X))) = X$$

for all $x_1(X) \in [x_l, x_r]$. For X_l and X_r satisfying $x_1(X_l) = x_l$ and $x_1(X_r) = x_r$ we have

$$X_l = x_l + \mu_0((-\infty, x_l)) \quad \text{and} \quad X_r = x_r + \mu_0((-\infty, x_r)).$$

Therefore, since x_1 is nondecreasing, we get

$$x_1(X) + \mu_0((-\infty, x_1(X))) = X \tag{7.1}$$

for all $X \in [X_l, X_r]$. Similarly we find by using (3.12b),

$$x_2(Y) + \nu_0((-\infty, x_2(Y))) = Y$$

for all $Y \in [Y_l, Y_r]$, where $x_2(Y_l) = x_l$, $x_2(Y_r) = x_r$ and

$$Y_l = x_l + \nu_0((-\infty, x_l)) \quad \text{and} \quad Y_r = x_r + \nu_0((-\infty, x_r)).$$

We define $\Omega = [X_l, X_r] \times [Y_l, Y_r]$. From now on we only consider $(X, Y) \in \Omega$. Rewriting (7.1) yields

$$x_1(X) + \mu_0((-\infty, x_l)) + (\mu_0)_{\mathrm{ac}}((x_l, x_1(X))) = X.$$

Hence,

$$x_1(X) + \mu_0((-\infty, x_l)) + \frac{1}{4}\int_{x_l}^{x_1(X)} (R_0^2 + c(u_0)\rho_0^2)(x)\, dx = X.$$

We differentiate and obtain

$$x_1'(X) + \frac{1}{4}x_1'(X)(R_0^2 + c(u_0)\rho_0^2) \circ x_1(X) = 1,$$

which implies that

$$x_1'(X) = \frac{4}{(R_0^2 + c(u_0)\rho_0^2) \circ x_1(X) + 4}. \tag{7.2}$$

Since $R_0, \rho_0 \in L^\infty([x_l, x_r])$, we get the lower bound

$$x_1'(X) \geq \frac{4}{||R_0||^2_{L^\infty([x_l,x_r])} + \kappa||\rho_0||^2_{L^\infty([x_l,x_r])} + 4} =: d_1 > 0, \tag{7.3}$$

and since $\rho_0(x) \geq d$, we find the upper bound

$$x_1'(X) \leq \frac{4\kappa}{d^2 + 4\kappa}. \tag{7.4}$$

Similarly, we find

$$x_2'(Y) = \frac{4}{(S_0^2 + c(u_0)\sigma_0^2) \circ x_2(Y) + 4},$$

$$x_2'(Y) \geq \frac{4}{||S_0||^2_{L^\infty([x_l,x_r])} + \kappa||\sigma_0||^2_{L^\infty([x_l,x_r])} + 4} =: e_1 > 0 \tag{7.5}$$

and

$$x_2'(Y) \leq \frac{4\kappa}{e^2 + 4\kappa}. \tag{7.6}$$

(ii) Let $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) = \mathbf{C}(\psi_1, \psi_2)$. By (3.28) we have $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$, which after differentiating and using $\mathcal{X}(s) + \mathcal{Y}(s) = 2s$, yields

$$\dot{\mathcal{X}}(s) = \frac{2x_2'(\mathcal{Y}(s))}{x_1'(\mathcal{X}(s)) + x_2'(\mathcal{Y}(s))} \quad \text{and} \quad \dot{\mathcal{Y}}(s) = \frac{2x_1'(\mathcal{X}(s))}{x_1'(\mathcal{X}(s)) + x_2'(\mathcal{Y}(s))}. \tag{7.7}$$

This implies, by (7.3)-(7.6), that

$$\dot{\mathcal{X}}(s) \geq 2e_1\left(\frac{4\kappa}{d^2 + 4\kappa} + \frac{4\kappa}{e^2 + 4\kappa}\right)^{-1} \text{ and } \dot{\mathcal{Y}}(s) \geq 2d_1\left(\frac{4\kappa}{d^2 + 4\kappa} + \frac{4\kappa}{e^2 + 4\kappa}\right)^{-1} \tag{7.8}$$

for all s such that $(\mathcal{X}(s), \mathcal{Y}(s)) \in \Omega$, that is, for values of s which satisfy $X_l \leq \mathcal{X}(s) \leq X_r$ and $Y_l \leq \mathcal{Y}(s) \leq Y_r$. Using this together with the identity $\mathcal{X}(s) + \mathcal{Y}(s) = 2s$, we find that (7.8) is valid for all $s \in [s_l, s_r]$, where $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. Hence, $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are strictly increasing functions on $[s_l, s_r]$.

(iii) Consider $(Z, p, q) = \mathbf{S}(\Theta)$. By an argument as in the proof of Lemma 4.9, we obtain the inequality

$$(x_X + J_X)(X, Y) \leq (\mathcal{V}_2 + \mathcal{V}_4)(X)e^{C|Y - \mathcal{Y}(X)|}$$

for all $(X, Y) \in \Omega$, where C depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$. We have $\mathcal{V}_2 + \mathcal{V}_4 = \frac{1}{2}x_1' + J_1'$ and since $J_1' = 1 - x_1'$, $x_1' \geq 0$ and $J_1' \geq 0$, this implies that $\frac{1}{2} \leq \mathcal{V}_2 + \mathcal{V}_4 \leq 1$. Thus, we get

$$(x_X + J_X)(X, Y) \leq e^{C|Y - \mathcal{Y}(X)|}. \tag{7.9}$$

By (3.31f), (3.12g), (A3) and (7.3) we obtain

$$p(X,\mathcal{Y}(X))=\mathfrak{p}(X)=H_1(X)=\frac{1}{2}\rho_0(x_1(X))x_1'(X)\geq\frac{1}{2}dd_1=:d_2. \tag{7.10}$$

Similarly we obtain

$$q(\mathcal{X}(Y),Y)=\mathfrak{q}(Y)=H_2(Y)=\frac{1}{2}\sigma_0(x_2(Y))x_2'(Y)\geq\frac{1}{2}ee_1=:e_2.$$

Then, since $p_Y=0$, we get

$$\begin{aligned}
d_2^2&\leq p^2(X,\mathcal{Y}(X))\\
&=p^2(X,Y)\\
&\leq\left(\frac{1}{c(U)}((c(U)U_X)^2+c(U)p^2)\right)(X,Y)\\
&\leq\kappa((c(U)U_X)^2+c(U)p^2)(X,Y)\\
&=2\kappa(J_Xx_X)(X,Y)\\
&\leq 2\kappa(J_X+x_X)x_X(X,Y)\\
&\leq 2\kappa e^{C|Y-\mathcal{Y}(X)|}x_X(X,Y),
\end{aligned}$$

where we used (4.12c) and (7.9). Hence,

$$x_X(X,Y)\geq\frac{d_2^2}{2\kappa}e^{-C|Y-\mathcal{Y}(X)|}. \tag{7.11}$$

Using that $q_X=0$, we find in the same way that

$$x_Y(X,Y)\geq\frac{e_2^2}{2\kappa}e^{-\tilde{C}|X-\mathcal{X}(Y)|}, \tag{7.12}$$

where $\tilde{C}$ depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$. From (4.12a), we then get

$$t_X(X,Y)\geq\frac{d_2^2}{2\kappa^2}e^{-C|Y-\mathcal{Y}(X)|}\quad\text{and}\quad t_Y(X,Y)\leq-\frac{e_2^2}{2\kappa^2}e^{-\tilde{C}|X-\mathcal{X}(Y)|}. \tag{7.13}$$

(iv) For any $0<\tau\leq\frac{1}{2\kappa}(x_r-x_l)$ consider $\Theta(\tau)=\mathbf{E}\circ\mathbf{t}_\tau(Z,p,q)$.

We claim that $(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ lies below the curve[7] $(\mathcal{X}(s),\mathcal{Y}(s))$. For any $\bar{s}$ and s such that $\mathcal{X}(\tau,\bar{s})=\mathcal{X}(s)$ we have

$$t(\mathcal{X}(s),\mathcal{Y}(s))=0<\tau=t(\mathcal{X}(\tau,\bar{s}),\mathcal{Y}(\tau,\bar{s})),$$

so that by (4.12a) and (4.12d),

$$\mathcal{Y}(\tau,\bar{s})<\mathcal{Y}(s), \tag{7.14}$$

which proves the claim.

We prove that there exist $\bar{s}_{\min}$ and $\bar{s}_{\max}$ satisfying $s_l<\bar{s}_{\min}\leq\bar{s}_{\max}<s_r$ such that $(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ belongs to Ω for all $s\in[\bar{s}_{\min},\bar{s}_{\max}]$. First we need an estimate. Since $c(u)\leq\kappa$, $t_X(X,Y)>0$, and $t_Y(X,Y)<0$ we have

$$\begin{aligned}
\frac{1}{2\kappa}(x_r-x_l)&=\frac{1}{2\kappa}(x(X_r,Y_r)-x(X_l,Y_l))\\
&=\frac{1}{2\kappa}(x(X_r,Y_r)-x(X_r,Y_l)+x(X_r,Y_l)-x(X_l,Y_l))
\end{aligned} \tag{7.15}$$

[7] Note that this is also true outside of Ω.

$$
\begin{aligned}
&= \frac{1}{2\kappa}\bigg(\int_{Y_l}^{Y_r} x_Y(X_r,Y)\,dY + \int_{X_l}^{X_r} x_X(X,Y_l)\,dX\bigg)\\
&= \frac{1}{2\kappa}\bigg(-\int_{Y_l}^{Y_r} c(U(X_r,Y))t_Y(X_r,Y)\,dY\\
&\qquad + \int_{X_l}^{X_r} c(U(X,Y_l))t_X(X,Y_l)\,dX\bigg) \quad \text{by (4.12a)}\\
&\leq \frac{1}{2}\bigg(-\int_{Y_l}^{Y_r} t_Y(X_r,Y)\,dY + \int_{X_l}^{X_r} t_X(X,Y_l)\,dX\bigg)\\
&= t(X_r,Y_l).
\end{aligned}
$$

Consider $\bar{s}_{\max}$ such that $\mathcal{X}(\tau,\bar{s}_{\max}) = X_r$. Note that since $\mathcal{X}(\tau,\bar{s}_{\max}) = \mathcal{X}(s_r)$ we get from (7.14), $\mathcal{Y}(\tau,\bar{s}_{\max}) < \mathcal{Y}(s_r) = Y_r$. From (7.15) we get

$$t(X_r,\mathcal{Y}(\tau,\bar{s}_{\max})) = \tau \leq \frac{1}{2\kappa}(x_r - x_l) \leq t(X_r,Y_l),$$

so that $\mathcal{Y}(\tau,\bar{s}_{\max}) \geq Y_l$. In particular, we have

$$Y_l \leq \mathcal{Y}(\tau,\bar{s}_{\max}) < Y_r \tag{7.16}$$

and therefore, since $\mathcal{X}(\tau,\cdot)$ and $\mathcal{Y}(\tau,\cdot)$ are nondecreasing and continuous, there exists $\bar{s}_{\min}$ such that $\bar{s}_{\min} \leq \bar{s}_{\max}$, $\mathcal{Y}(\tau,\bar{s}_{\min}) = Y_l$ and $\mathcal{X}(\tau,\bar{s}_{\min}) \leq X_r$. Since

$$t(X_l,Y_l) = 0 < \tau = t(\mathcal{X}(\tau,\bar{s}_{\min}),Y_l)$$

we find that $\mathcal{X}(\tau,\bar{s}_{\min}) > X_l$ and we have

$$X_l < \mathcal{X}(\tau,\bar{s}_{\min}) \leq X_r. \tag{7.17}$$

For any $s \in [\bar{s}_{\min},\bar{s}_{\max}]$ we get from (7.16) and (7.17), since $\mathcal{X}(\tau,\cdot)$ and $\mathcal{Y}(\tau,\cdot)$ are nondecreasing, that

$$X_l < \mathcal{X}(\tau,\bar{s}_{\min}) \leq \mathcal{X}(\tau,s) \leq \mathcal{X}(\tau,\bar{s}_{\max}) = X_r$$

and

$$Y_l = \mathcal{Y}(\tau,\bar{s}_{\min}) \leq \mathcal{Y}(\tau,s) \leq \mathcal{Y}(\tau,\bar{s}_{\max}) < Y_r.$$

In other words, the curve $(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ lies in Ω for all $s \in [\bar{s}_{\min},\bar{s}_{\max}]$, and hence all the estimates obtained in (iii) are valid along this curve. Observe that

$$s_l = \frac{1}{2}(X_l + Y_l) < \frac{1}{2}\big(\mathcal{X}(\tau,\bar{s}_{\min}) + \mathcal{Y}(\tau,\bar{s}_{\min})\big) = \bar{s}_{\min}$$

and

$$s_r = \frac{1}{2}(X_r + Y_r) > \frac{1}{2}\big(\mathcal{X}(\tau,\bar{s}_{\max}) + \mathcal{Y}(\tau,\bar{s}_{\max})\big) = \bar{s}_{\max},$$

which implies $s_l < \bar{s}_{\min} \leq \bar{s}_{\max} < s_r$.

By differentiating $t(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s)) = \tau$ and using that $\dot{\mathcal{X}}(\tau,s) + \dot{\mathcal{Y}}(\tau,s) = 2$, we obtain

$$\dot{\mathcal{X}}(\tau,s) = \frac{-2t_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}{t_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s)) - t_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}. \tag{7.18}$$

By (4.5), we have

$$|t_X(X,Y)| \leq ||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{\kappa}{2}, \quad |t_Y(X,Y)| \leq ||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{\kappa}{2}$$

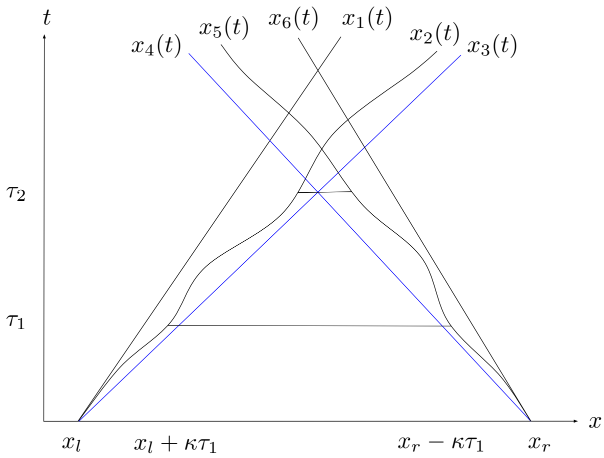

FIGURE 5. The region bounded by the characteristics $x_2(t)$ (forward) and $x_5(t)$ (backward) starting from x_l and x_r, respectively, at $t = 0$. Here, $0 < \tau_1 < \tau_2 = \frac{1}{2\kappa}(x_r - x_l)$. The remaining functions are given by $x_1(t) = x_l + \frac{t}{\kappa}$, $x_3(t) = x_l + \kappa t$, $x_4(t) = x_r - \kappa t$ and $x_6(t) = x_r - \frac{t}{\kappa}$. We have $x_3(\tau_1) = x_l + \kappa\tau_1 = x(\mathcal{X}(\tau_1, \bar{s}_1), \mathcal{Y}(\tau_1, \bar{s}_1))$ and $x_4(\tau_1) = x_r - \kappa\tau_1 = x(\mathcal{X}(\tau_1, \bar{s}_2), \mathcal{Y}(\tau_1, \bar{s}_2))$.

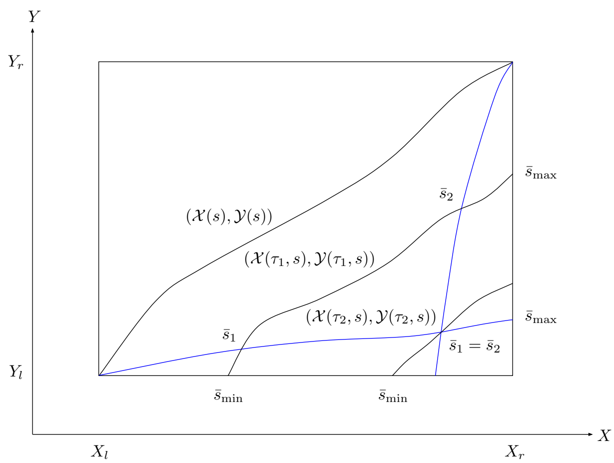

FIGURE 6. The region from Figure 5 in Lagrangian coordinates. The curves $(\mathcal{X}(s), \mathcal{Y}(s))$, $(\mathcal{X}(\tau_1, s), \mathcal{Y}(\tau_1, s))$ and $(\mathcal{X}(\tau_2, s), \mathcal{Y}(\tau_2, s))$ correspond to $t = 0$, $t = \tau_1$ and $t = \tau_2$, respectively.

for all $(X, Y) \in \Omega$. Using this and (7.13) in (7.18), we find

$$\dot{\mathcal{X}}(\tau, s) \geq \frac{e_2^2 \kappa^{-2} e^{-\tilde{C}|X - \mathcal{X}(Y)|}}{||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + ||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \kappa} =: \alpha_1 e^{-\tilde{C}|X - \mathcal{X}(Y)|} \tag{7.19}$$

for all $s \in [\bar{s}_{\min}, \bar{s}_{\max}]$. Similarly, one proves that

$$\dot{\mathcal{Y}}(\tau, s) \geq \alpha_2 e^{-C|Y-\mathcal{Y}(X)|} \tag{7.20}$$

for some positive constant α_2 that depends on d_2, κ, $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$ and $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$.

Next, we prove that there exist $\bar{s}_1$ and $\bar{s}_2$ satisfying $\bar{s}_{\min} \leq \bar{s}_1 \leq \bar{s}_2 \leq \bar{s}_{\max}$ such that

$$x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau. \tag{7.21}$$

Observe that since x is increasing with respect to both variables this will imply

$$x_l + \kappa\tau \leq x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) \leq x_r - \kappa\tau$$

for all $s \in [\bar{s}_1, \bar{s}_2]$. Furthermore, since $\bar{s}_{\min} \leq \bar{s}_1 \leq \bar{s}_2 \leq \bar{s}_{\max}$ and $\mathcal{X}(\tau, \cdot)$ and $\mathcal{Y}(\tau, \cdot)$ are increasing functions, $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ belongs to Ω for all $s \in [\bar{s}_1, \bar{s}_2]$, see Figure 5 and 6.

By (4.12a) and (1.5), we have

$$\begin{aligned}
x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) &= x(X_l, Y_l) + \int_{X_l}^{\mathcal{X}(\tau, \bar{s}_{\min})} x_X(X, Y_l)\, dX \\
&= x(X_l, Y_l) + \int_{X_l}^{\mathcal{X}(\tau, \bar{s}_{\min})} c(U(X, Y_l)) t_X(X, Y_l)\, dX \\
&\leq x(X_l, Y_l) + \kappa \int_{X_l}^{\mathcal{X}(\tau, \bar{s}_{\min})} t_X(X, Y_l)\, dX \\
&= x(X_l, Y_l) + \kappa(t(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) - t(X_l, Y_l)) \\
&= x_l + \kappa\tau.
\end{aligned}$$

Similarly, by using the lower bound on c, we get

$$x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) \geq x_l + \frac{1}{\kappa}\tau.$$

From (4.12a) and (1.5), we have

$$\begin{aligned}
x_r = x(X_r, Y_r) &= x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) + \int_{\mathcal{Y}(\tau, \bar{s}_{\max})}^{Y_r} x_Y(X_r, Y)\, dY \\
&= x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) - \int_{\mathcal{Y}(\tau, \bar{s}_{\max})}^{Y_r} c(U(X_r, Y)) t_Y(X_r, Y)\, dY \\
&\leq x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) - \kappa \int_{\mathcal{Y}(\tau, \bar{s}_{\max})}^{Y_r} t_Y(X_r, Y)\, dY \quad \text{since } t_Y < 0 \\
&= x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) + \kappa\tau,
\end{aligned}$$

so that

$$x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) \geq x_r - \kappa\tau.$$

Similarly, we obtain

$$x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) \leq x_r - \frac{1}{\kappa}\tau.$$

Hence, we end up with

$$x_l + \frac{1}{\kappa}\tau \leq x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) \leq x_l + \kappa\tau \tag{7.22}$$

and

$$x_r - \kappa\tau \leq x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) \leq x_r - \frac{1}{\kappa}\tau. \tag{7.23}$$

Since $0 < \tau \leq \frac{1}{2\kappa}(x_r - x_l)$ we have $x_l + \kappa\tau = x_l + 2\kappa\tau - \kappa\tau \leq x_l + x_r - x_l - \kappa\tau = x_r - \kappa\tau$, which implies, since x is continuous with respect to both variables, that there exist $\bar{s}_1$ and $\bar{s}_2$ such that $\bar{s}_1 \leq \bar{s}_2$ and

$$x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau. \tag{7.24}$$

From (7.22), (7.24), and the fact that x increases along the curve $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$, we have

$$x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) \leq x_l + \kappa\tau = x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)),$$

so that $\mathcal{X}(\tau, \bar{s}_{\min}) \leq \mathcal{X}(\tau, \bar{s}_1)$ which implies $\bar{s}_{\min} \leq \bar{s}_1$. By (7.23) and (7.24) we have

$$x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau \leq x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})),$$

and we get $\mathcal{Y}(\tau, \bar{s}_2) \leq \mathcal{Y}(\tau, \bar{s}_{\max})$ and $\bar{s}_2 \leq \bar{s}_{\max}$. This concludes the proof of (7.21).

We prove (P1). From (5.21a) we have

$$u(\tau, x) = \mathcal{Z}_3(\tau, s) \quad \text{if } x = \mathcal{Z}_2(\tau, s).$$

Since the function $\mathcal{Z}_2(\tau, s)$ is nondecreasing, the smallest and biggest value it can attain for $s \in [\bar{s}_1, \bar{s}_2]$ is

$$\mathcal{Z}_2(\tau, \bar{s}_1) = x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau$$

and

$$\mathcal{Z}_2(\tau, \bar{s}_2) = x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau,$$

respectively. The function $\mathcal{Z}_2(\tau, \cdot)$ is in fact strictly increasing for $s \in [\bar{s}_1, \bar{s}_2]$, as we now show. We differentiate the relation $\mathcal{Z}_2(\tau, s) = x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ and get

$$\dot{\mathcal{Z}}_2(\tau, s) = x_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s) + x_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s).$$

From (7.11), (7.12), (7.19) and (7.20) we have

$$\dot{\mathcal{Z}}_2(\tau, s) \geq \frac{1}{2\kappa}\big(\alpha_1 d_2^2 + \alpha_2 e_2^2\big)e^{-C|Y - \mathcal{Y}(X)| - \tilde{C}|X - \mathcal{X}(Y)|} > 0. \tag{7.25}$$

Hence, $s \mapsto \mathcal{Z}_2(\tau, s)$ is strictly increasing for $s \in [\bar{s}_1, \bar{s}_2]$ and therefore invertible on $[\bar{s}_1, \bar{s}_2]$. For any $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$ we get

$$u(\tau, x) = \mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \tag{7.26}$$

and since $|\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))| = |U(\mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \mathcal{Y}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))| \leq ||U||_{L^\infty(\Omega)}$ we have

$$u(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau]). \tag{7.27}$$

Next, we differentiate (7.26) and get

$$u_x(\tau, x) = \dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))\frac{d}{dx}\mathcal{Z}_2^{-1}(\tau, x) = \frac{\dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau, x))}. \tag{7.28}$$

We have

$$\begin{aligned}
|\dot{\mathcal{Z}}_3(\tau, s)| &= |\mathcal{V}_3(\tau, \mathcal{X}(\tau, s))\dot{\mathcal{X}}(\tau, s) + \mathcal{W}_3(\tau, \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s)| \\
&= |U_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s) + U_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s)| \\
&\leq 2||U_X||_{W_Y^{1,\infty}(\Omega)} + 2||U_Y||_{W_X^{1,\infty}(\Omega)},
\end{aligned}$$

so that $\dot{\mathcal{Z}}_3(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$. By (7.25) we end up with

$$|u_x(\tau,x)|\leq \frac{2\kappa||\dot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\alpha_1 d_2^2+\alpha_2 e_2^2}e^{C|Y-\mathcal{Y}(X)|+\tilde{C}|X-\mathcal{X}(Y)|},$$

which implies that

$$u_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau]). \tag{7.29}$$

From (7.27) and (7.29) we conclude that (P1) holds.

We prove (P2). From (5.22a) and (5.22b), we have

$$R(\tau,\mathcal{Z}_2(\tau,s))\mathcal{V}_2(\tau,\mathcal{X}(\tau,s))=c(\mathcal{Z}_3(\tau,s))\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))$$

and

$$\rho(\tau,\mathcal{Z}_2(\tau,s))\mathcal{V}_2(\tau,\mathcal{X}(\tau,s))=\mathfrak{p}(\tau,\mathcal{X}(\tau,s))$$

for all $s\in[\bar{s}_1,\bar{s}_2]$. We multiply these equations with $2\dot{\mathcal{X}}(\tau,s)$ and use (3.27) to get

$$R(\tau,\mathcal{Z}_2(\tau,s))\dot{\mathcal{Z}}_2(\tau,s)=2c(\mathcal{Z}_3(\tau,s))\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))\dot{\mathcal{X}}(\tau,s)$$

and

$$\rho(\tau,\mathcal{Z}_2(\tau,s))\dot{\mathcal{Z}}_2(\tau,s)=2\mathfrak{p}(\tau,\mathcal{X}(\tau,s))\dot{\mathcal{X}}(\tau,s),$$

which yields

$$R(\tau,x)=\frac{2c(\mathcal{Z}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\mathcal{V}_3(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))} \tag{7.30}$$

and

$$\rho(\tau,x)=\frac{2\mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))} \tag{7.31}$$

for all $x\in[x_l+\kappa\tau,x_r-\kappa\tau]$. Since $\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))=U_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $|\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))|\leq||U_X||_{W_Y^{1,\infty}(\Omega)}$ we have $\mathcal{V}_3(\tau,\mathcal{X}(\tau,\cdot))\in L^\infty([\bar{s}_1,\bar{s}_2])$, and since $\mathfrak{p}(\tau,\mathcal{X}(\tau,s))=p(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $|\mathfrak{p}(\tau,\mathcal{X}(\tau,s))|\leq||p||_{W_Y^{1,\infty}(\Omega)}$ we have $\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))\in L^\infty([\bar{s}_1,\bar{s}_2])$. Using (7.25) in (7.30) and (7.31) we obtain

$$|R(\tau,x)|\leq\frac{8\kappa^2||\mathcal{V}_3(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\alpha_1 d_2^2+\alpha_2 e_2^2}e^{C|Y-\mathcal{Y}(X)|+\tilde{C}|X-\mathcal{X}(Y)|}$$

and

$$|\rho(\tau,x)|\leq\frac{8\kappa||\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\alpha_1 d_2^2+\alpha_2 e_2^2}e^{C|Y-\mathcal{Y}(X)|+\tilde{C}|X-\mathcal{X}(Y)|}$$

for all $x\in[x_l+\kappa\tau,x_r-\kappa\tau]$. Therefore $R(\tau,\cdot),\rho(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$. In a similar way one shows that $S(\tau,\cdot),\sigma(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$ and we have proved (P2).

We prove (P3). By inserting

$$\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))=2x_X(\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)),\mathcal{Y}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))$$

into (7.31) we get

$$\rho(\tau,x)=\frac{\mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))}{x_X(\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)),\mathcal{Y}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))}$$

for all $s\in[\bar{s}_1,\bar{s}_2]$. Since $p_Y=0$ we get from (7.10),

$$\mathfrak{p}(\tau,X)=p(X,\mathcal{Y}(\tau,\mathcal{X}^{-1}(\tau,X)))=p(X,\mathcal{Y}(X))\geq d_2$$

for all $X \in [X_l, X_r]$. Recalling (7.9) and that $J_X(X,Y) \geq 0$ we get

$$\rho(\tau, x) \geq d_2 e^{-C|Y_l - Y_r|}$$

for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. Similarly we find

$$\sigma(\tau, x) \geq e_2 e^{-\tilde{C}|X_l - X_r|}$$

for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. This concludes the proof of (P3).

We prove (P4). Let $M \subset [x_l + \kappa\tau, x_r - \kappa\tau]$ be a Borel set. By (5.28), we have

$$\mu_{\text{sing}}(\tau, M) = \int_{\mathcal{Z}_2^{-1}(\tau, M) \cap A^c} \mathcal{V}_4(\tau, \mathcal{X}(\tau, s)) \dot{\mathcal{X}}(\tau, s)\, ds,$$

where $A = \{s \in \mathbb{R} \mid \mathcal{V}_2(\tau, \mathcal{X}(\tau, s)) > 0\}$. We have $\text{meas}(A^c) = 0$, which implies that

$$\mu_{\text{sing}}(\tau, M) \leq 2|||\Theta(\tau)|||_{\mathcal{G}(\Omega)} \,\text{meas}(A^c) = 0.$$

This proves that $\mu(\tau)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$. Similarly, one shows that $\nu(\tau)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$.

Step 2. Assume that $m = 2$. By Step 1, (P3) and (P4) hold. Moreover, (P1) and (P2) hold for $m = 1$. It remains to prove that $u_{xx}(\tau, \cdot), R_x(\tau, \cdot), S_x(\tau, \cdot), \rho_x(\tau, \cdot), \sigma_x(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$. In order to do so, we have to show that $Z_{XX} = (t_{XX}, x_{XX}, U_{XX}, J_{XX}, K_{XX})$ and $Z_{YY} = (t_{YY}, x_{YY}, U_{YY}, J_{YY}, K_{YY})$ exist and are bounded. We first consider Z_{XX}. There exists a unique solution Z_{XX} in Ω of the system

$$Z_{XXY}(X,Y) = f(X, Y, Z_{XX})$$

since f is Lipschitz continuous with respect to the Z_{XX} variable, which comes from the fact that the system is semilinear. This can be seen by differentiating the governing equations (2.38). For instance, we have

$$x_{XXY} = \frac{c'(U)}{2c(U)}(x_X U_{XY} - U_X x_{XY}) + \frac{c''(U)}{2c(U)} U_X (U_X x_Y + U_Y x_X) + \frac{c'(U)}{2c(U)}(x_Y U_{XX} + U_Y x_{XX}) \tag{7.32}$$

and

$$|x_{XX}(X,Y)| \leq |x_{XX}(X, \mathcal{Y}(X))| + \int_Y^{\mathcal{Y}(X)} |x_{XXY}(X, \tilde{Y})|\, d\tilde{Y}, \tag{7.33}$$

if we assume without loss of generality that $Y \leq \mathcal{Y}(X)$ (the other case is similar.)

Let us find a bound on x_{XX} at time $\tau = 0$. We differentiate (7.2) and get

$$x_1''(X) = -\frac{1}{4} x_1'(X)^3 (2R_0 R_{0x} + c'(u_0) u_{0x} \rho_0^2 + 2c(u_0)\rho_0 \rho_{0x}) \circ x_1(X),$$

which, by (7.4), implies that

$$\begin{aligned} |x_1''(X)| \leq \frac{1}{4}\left(\frac{4\kappa}{d^2 + 4\kappa}\right)^3 (2||R_0||_{L^\infty([x_l,x_r])} ||(R_0)_x||_{L^\infty([x_l,x_r])} \\ + k_1 ||(u_0)_x||_{L^\infty([x_l,x_r])} ||\rho_0||^2_{L^\infty([x_l,x_r])} \\ + 2\kappa ||\rho_0||_{L^\infty([x_l,x_r])} ||(\rho_0)_x||_{L^\infty([x_l,x_r])}) \end{aligned} \tag{7.34}$$

and we conclude that $x_1'' \in L^\infty([X_l, X_r])$.

By Definition 3.4 and Definition 3.7, we have

$$x_X(X, \mathcal{Y}(X)) = \mathcal{V}_2(X) = \frac{1}{2}x_1'(X).$$

We differentiate and get

$$x_{XX}(X, \mathcal{Y}(X)) + x_{XY}(X, \mathcal{Y}(X))\left(\frac{\dot{\mathcal{Y}}}{\dot{\mathcal{X}}}\right) \circ \mathcal{X}^{-1}(X) = \frac{1}{2}x_1''(X),$$

so that by (7.19),

$$\begin{aligned}|x_{XX}(X, \mathcal{Y}(X))| &\leq \frac{1}{2}|x_1''(X)| + |x_{XY}(X, \mathcal{Y}(X))|\left(\frac{\dot{\mathcal{Y}}}{\dot{\mathcal{X}}}\right) \circ \mathcal{X}^{-1}(X) \\ &\leq \frac{1}{2}||x_1''||_{L^\infty([X_l,X_r])} + \frac{k_1\kappa}{\alpha_1 e^{-\tilde{C}|X-\mathcal{X}(Y)|}}\left(|||\Theta|||^2_{\mathcal{G}(\Omega)} + \frac{1}{2}|||\Theta|||_{\mathcal{G}(\Omega)}\right)\end{aligned}$$

and $x_{XX}(\cdot, \mathcal{Y}(\cdot)) \in L^\infty([X_l, X_r])$. Here we used the estimate

$$\begin{aligned}&|x_{XY}(\mathcal{X}(s), \mathcal{Y}(s))| \\ &= \left|\frac{c'(\mathcal{Z}_3(s))}{2c(\mathcal{Z}_3(s))}(\mathcal{V}_3(\mathcal{X}(s))\mathcal{W}_2(\mathcal{Y}(s)) + \mathcal{W}_3(\mathcal{Y}(s))\mathcal{V}_2(\mathcal{X}(s)))\right| \quad \text{by (2.38)} \\ &\leq \frac{1}{2}k_1\kappa\Bigg(||\mathcal{V}_3^a||_{L^\infty([X_l,X_r])}\left(||\mathcal{W}_2^a||_{L^\infty([Y_l,Y_r])} + \frac{1}{2}\right) \\ &\qquad + ||\mathcal{W}_3^a||_{L^\infty([Y_l,Y_r])}\left(||\mathcal{V}_2^a||_{L^\infty([X_l,X_r])} + \frac{1}{2}\right)\Bigg) \\ &\leq \frac{1}{2}k_1\kappa\left(|||\Theta|||^2_{\mathcal{G}(\Omega)} + \frac{1}{2}|||\Theta|||_{\mathcal{G}(\Omega)}\right).\end{aligned}$$

We estimate x_{XXY}. Since $|Z_X^a| \leq ||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$ and $|Z_Y^a| \leq ||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, we get from (2.38) that

$$|Z_{XY}| \leq \eta, \tag{7.35}$$

where η depends on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ and k_1. We obtain from (7.32),

$$\begin{aligned}|x_{XXY}| &\leq \frac{1}{2}k_1\kappa\left[\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\eta + ||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\eta\right] \\ &\quad + \frac{1}{2}k_2\kappa||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\Bigg[||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right) \\ &\qquad + ||Z_Y^a||_{W_X^{1,\infty}(\Omega)}\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\Bigg] \\ &\quad + \frac{1}{2}k_1\kappa\left[\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right)|U_{XX}| + ||Z_Y^a||_{W_X^{1,\infty}(\Omega)}|x_{XX}|\right] \\ &\leq k_1\kappa\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\eta \\ &\quad + k_2\kappa||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right)\end{aligned} \tag{7.36}$$

$$+\frac{1}{2}k_1\kappa\bigg(||Z_Y^a||_{W_X^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)(|U_{XX}|+|x_{XX}|).$$

We insert (7.36) into (7.33) and get

$$\begin{aligned}
&|x_{XX}(X,Y)|\\
&\leq ||x_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+\bigg[k_1\kappa\bigg(||Z_X^a||_{W_Y^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)\eta\\
&\quad+k_2\kappa||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\bigg(||Z_X^a||_{W_Y^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)\bigg(||Z_Y^a||_{W_X^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)\bigg]|Y_r-Y_l|\\
&\quad+\int_Y^{\mathcal{Y}(X)}\frac{1}{2}k_1\kappa\bigg(||Z_Y^a||_{W_X^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)(|U_{XX}|+|x_{XX}|)(X,\tilde{Y})\,d\tilde{Y}.
\end{aligned}$$

Following the same lines for the other components of Z_{XX}, we obtain

$$\begin{aligned}
&(|t_{XX}|+|x_{XX}|+|U_{XX}|+|J_{XX}|+|K_{XX}|)(X,Y)\\
&\leq ||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|\\
&\quad+\int_Y^{\mathcal{Y}(X)}C_2(|t_{XX}|+|x_{XX}|+|U_{XX}|+|J_{XX}|+|K_{XX}|)(X,\tilde{Y})\,d\tilde{Y},
\end{aligned}$$

where C_1 and C_2 depend on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ, k_1 and k_2. By Gronwall's lemma, we obtain

(7.37)
$$\begin{aligned}
&(|t_{XX}|+|x_{XX}|+|U_{XX}|+|J_{XX}|+|K_{XX}|)(X,Y)\\
&\leq (||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}.
\end{aligned}$$

A similar procedure yields

(7.38)
$$\begin{aligned}
&(|t_{YY}|+|x_{YY}|+|U_{YY}|+|J_{YY}|+|K_{YY}|)(X,Y)\\
&\leq (||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])}+\tilde{C}_1|X_r-X_l|)e^{\tilde{C}_2|X-\mathcal{X}(Y)|},
\end{aligned}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ depend on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ, k_1 and k_2.

We also need estimates for $\ddot{\mathcal{X}}(\tau,s)$ and $\ddot{\mathcal{Y}}(\tau,s)$. By (3.26), we have

$$x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)=x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s).$$

We differentiate and get

(7.39)
$$\begin{aligned}
&x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2+x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s)\\
&=x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{Y}}(\tau,s).
\end{aligned}$$

Since $\mathcal{X}(\tau,s)+\mathcal{Y}(\tau,s)=2s$, we have $\ddot{\mathcal{Y}}(\tau,s)=-\ddot{\mathcal{X}}(\tau,s)$, so that

$$\ddot{\mathcal{X}}(\tau,s)=\frac{x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2-x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2}{x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}.$$

By (7.11), (7.12), (7.37) and (7.38), we find

(7.40)
$$\begin{aligned}
|\ddot{\mathcal{X}}(\tau,s)|&\leq 4\bigg(\frac{|x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))|+|x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))|}{x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}\bigg)\\
&\leq\frac{4}{\frac{d_2^2}{2\kappa}e^{-C|Y-\mathcal{Y}(X)|}+\frac{e_2^2}{2\kappa}e^{-\tilde{C}|X-\mathcal{X}(Y)|}}
\end{aligned}$$

$$\times\,(((||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}$$
$$+\,(||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])}+\tilde{C}_1|X_r-X_l|)e^{\tilde{C}_2|X-\mathcal{X}(Y)|}).$$

If we define

$$D=\big(4\max\{||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|,\\ ||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])}+\tilde{C}_1|X_r-X_l|\}\big)\bigg(\min\Big\{\frac{d_2^2}{2\kappa},\frac{e_2^2}{2\kappa}\Big\}\bigg)^{-1}$$

and

$$\bar{C}=\max\{C,\tilde{C},C_2,\tilde{C}_2\},$$

we get from (7.40),

$$|\ddot{\mathcal{X}}(\tau,s)|\leq D\bigg(\frac{e^{\bar{C}|X-\mathcal{X}(Y)|}+e^{\bar{C}|Y-\mathcal{Y}(X)|}}{e^{-\bar{C}|X-\mathcal{X}(Y)|}+e^{-\bar{C}|Y-\mathcal{Y}(X)|}}\bigg)\\ =De^{\bar{C}(|X-\mathcal{X}(Y)|+|Y-\mathcal{Y}(X)|)}. \tag{7.41}$$

We conclude that $\ddot{\mathcal{X}}(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$. Here we used that

$$\frac{e^a+e^b}{e^{-a}+e^{-b}}=e^{a+b}$$

for $a,b\in\mathbb{R}$. We differentiate $\mathcal{Z}_2(\tau,s)=x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ twice and get, by (7.39), that

$$\begin{aligned}\ddot{\mathcal{Z}}_2(\tau,s)&=x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2+x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s)\\&\quad+2x_{XY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)\dot{\mathcal{Y}}(\tau,s)\\&\quad+x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{Y}}(\tau,s)\\&=2x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2+2x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s)\\&\quad+2x_{XY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)\dot{\mathcal{Y}}(\tau,s).\end{aligned}$$

Using the estimates (7.35), (7.37), and (7.41), we get

$$|\ddot{\mathcal{Z}}_2(\tau,s)|\leq 8(||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}\\ +2\bigg(||Z_X^a||_{W_Y^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)De^{\bar{C}(|X-\mathcal{X}(Y)|+|Y-\mathcal{Y}(X)|)}+8\eta$$

and $\ddot{\mathcal{Z}}_2(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$.

We have

$$\begin{aligned}\ddot{\mathcal{Z}}_3(\tau,s)&=U_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2+U_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s)\\&\quad+2U_{XY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)\dot{\mathcal{Y}}(\tau,s)\\&\quad+U_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2+U_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{Y}}(\tau,s),\end{aligned}$$

which implies, by (7.35), (7.38) and (7.41), that

$$|\ddot{\mathcal{Z}}_3(\tau,s)|\leq 4(||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}\\ +||Z_X^a||_{W_Y^{1,\infty}(\Omega)}De^{\bar{C}(|X-\mathcal{X}(Y)|+|Y-\mathcal{Y}(X)|)}+8\eta\\ +4(||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])}+\tilde{C}_1|X_r-X_l|)e^{\tilde{C}_2|X-\mathcal{X}(Y)|}$$

$$+ ||Z_Y^a||_{W_X^{1,\infty}(\Omega)} De^{\bar{C}(|X-\mathcal{X}(Y)|+|Y-\mathcal{Y}(X)|)}.$$

Hence, $\ddot{\mathcal{Z}}_3(\tau,\cdot) \in L^\infty([\bar{s}_1, \bar{s}_2])$.

We compute u_{xx} from (7.28) and get

$$u_{xx}(\tau,x) = \frac{\ddot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))^2} - \frac{\dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))^3}.$$

By (7.25) we obtain

$$|u_{xx}(\tau,x)| \le \frac{||\ddot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\left[\frac{1}{2\kappa}\left(\alpha_1 d_2^2 + \alpha_2 e_2^2\right)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\right]^2} + \frac{||\dot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{Z}}_2(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\left[\frac{1}{2\kappa}\left(\alpha_1 d_2^2 + \alpha_2 e_2^2\right)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\right]^3},$$

and we conclude that $u_{xx}(\tau,\cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$.

By differentiating (7.30) we get

$$\begin{aligned}
&R_x(\tau,x) \\
&= \frac{2\dot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau,x))}{\left[\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))\right]^2}\Big[c'(\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x))\mathcal{V}_3(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau,x))) \\
&\qquad + c(\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\Big] \\
&\quad + \frac{2c(\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\mathcal{V}_3(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\ddot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau,x))}{\left[\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))\right]^2} \\
&\quad - \frac{2c(\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\mathcal{V}_3(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))}{\left[\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau,x))\right]^3},
\end{aligned}$$

where we denote $\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau,s)) = \frac{d}{ds}\mathcal{V}_3(\tau, \mathcal{X}(\tau,s))$. Since $\mathcal{V}_3(\tau, \mathcal{X}(\tau,s)) = U_X(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))$, we have

$$\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau,s)) = U_{XX}(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s) + U_{XY}(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s).$$

From (7.35) and (7.37) we obtain

$$|\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau,s))| \le 2\left[(||Z_{XX}(\cdot, \mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])} + C_1|Y_r - Y_l|)e^{C_2|Y-\mathcal{Y}(X)|} + \eta\right],$$

where η is a constant that depends on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ and k_1. Therefore we have $\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau,\cdot)) \in L^\infty([\bar{s}_1, \bar{s}_2])$. By (7.25) we get

$$\begin{aligned}
|R_x(\tau,x)| \le{}& \frac{2}{\left[\frac{1}{2\kappa}\left(\alpha_1 d_2^2 + \alpha_2 e_2^2\right)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\right]^2} \\
&\times \Big(2k_1||\dot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\mathcal{V}_3(\tau, \mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])} \\
&\quad + 2\kappa||\dot{\mathcal{V}}_3(\tau, \mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])} \\
&\quad + \kappa||\mathcal{V}_3(\tau, \mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{X}}(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}\Big) \\
&+ \frac{4\kappa||\mathcal{V}_3(\tau, \mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{Z}}_2(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\left[\frac{1}{2\kappa}\left(\alpha_1 d_2^2 + \alpha_2 e_2^2\right)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\right]^3},
\end{aligned}$$

which implies that $R_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau, x_r-\kappa\tau])$.

We differentiate (7.31) and get

$$\begin{aligned}\rho_x(\tau,x)&\\ =\ &\frac{2\big[\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))^2+\mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\ddot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^2}\\ &-\frac{2\mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^3},\end{aligned}$$

where we denote $\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s))=\frac{d}{ds}\mathfrak{p}(\tau,\mathcal{X}(\tau,s))$. Since $\mathfrak{p}(\tau,\mathcal{X}(\tau,s))=p(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $p_Y(X,Y)=0$ we have

$$\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s))=p_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s).$$

Furthermore, by (3.31f) and (3.12g), we have

$$p(X,\mathcal{Y}(X))=\mathfrak{p}(X)=H_1(X)=\frac{1}{2}\rho_0(x_1(X))x_1'(X). \tag{7.42}$$

We differentiate and get, since $p_Y(X,Y)=0$,

$$p_X(X,Y)=p_X(X,\mathcal{Y}(X))=\frac{1}{2}(\rho_0)_x(x_1(X))x_1'(X)^2+\frac{1}{2}\rho_0(x_1(X))x_1''(X).$$

Using (7.4), this leads to the estimate

$$|p_X(X,Y)|\leq\frac{1}{2}||(\rho_0)_x||_{L^\infty([x_l,x_r])}\left(\frac{4\kappa}{d^2+4\kappa}\right)^2+\frac{1}{2}||\rho_0||_{L^\infty([x_l,x_r])}||x_1''||_{L^\infty([X_l,X_r])},$$

which implies that

$$|\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s))|\leq||(\rho_0)_x||_{L^\infty([x_l,x_r])}\left(\frac{4\kappa}{d^2+4\kappa}\right)^2+||\rho_0||_{L^\infty([x_l,x_r])}||x_1''||_{L^\infty([X_l,X_r])}.$$

Recalling (7.34), it follows that $\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\cdot))\in L^\infty([\bar{s}_1,\bar{s}_2])$. Using (7.25), we end up with

$$\begin{aligned}|\rho_x(\tau,x)|\leq\ &\frac{2\big(2||\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}+||\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{X}}(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}\big)}{\big[\frac{1}{2\kappa}\big(\alpha_1d_2^2+\alpha_2e_2^2\big)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\big]^2}\\ &+\frac{4||\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{Z}}_2(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\big[\frac{1}{2\kappa}\big(\alpha_1d_2^2+\alpha_2e_2^2\big)e^{-C|Y-\mathcal{Y}(X)|-\tilde{C}|X-\mathcal{X}(Y)|}\big]^3}\end{aligned}$$

and we conclude that $\rho_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

In a similar way one shows that $S_x(\tau,\cdot),\sigma_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

Step 3. Assume that the result holds for $m=n$, that is, if $u_0\in L^\infty([x_l,x_r])$ and $R_0,S_0,\rho_0,\sigma_0\in W^{n-1,\infty}([x_l,x_r])$, then $u(\tau,\cdot)\in W^{n,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$, $R(\tau,\cdot)$, $S(\tau,\cdot)$, $\rho(\tau,\cdot)$, $\sigma(\tau,\cdot)\in W^{n-1,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$ and (P3) and (P4) hold. We show by induction that the result also holds for $m=n+1$, that is, we assume that R_0, S_0, ρ_0, σ_0 belong $W^{n,\infty}([x_l,x_r])$, and prove that $u(\tau,\cdot)\in W^{n+1,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$ and $R(\tau,\cdot),S(\tau,\cdot),\rho(\tau,\cdot),\sigma(\tau,\cdot)\in W^{n,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$.

Since the result holds for $m = n$ we get, following closely the argument used in Step 2 to derive (7.37) and (7.38), that

$$\frac{\partial^{\alpha+\beta}}{\partial X^\alpha \partial Y^\beta} Z \in [L^\infty(\Omega)]^5, \quad \alpha, \beta = 0, 1, \dots, n, \quad \alpha + \beta \le n. \tag{7.43}$$

Since $R_0, S_0, \rho_0, \sigma_0 \in W^{n,\infty}([x_l, x_r])$ we get by Definition 3.4 and Definition 3.7 that

$$\frac{\partial^{\alpha+\beta}}{\partial X^\alpha \partial Y^\beta} Z, \quad \alpha, \beta = 0, 1, \dots, n+1, \quad \alpha + \beta = n+1$$

is bounded on the curve $(\mathcal{X}(s), \mathcal{Y}(s))$, $s \in [s_l, s_r]$.

Since the governing equation (2.38) is semilinear, there exists a unique solution $\frac{\partial^{n+1}}{\partial X^\alpha \partial Y^\beta} Z$, $\alpha, \beta = 0, 1, \dots, n+1$, $\alpha + \beta = n+1$ in Ω of the system

$$\begin{aligned} &\frac{\partial}{\partial Y} \sum_{\substack{\alpha,\beta=0,1,\dots,n+1\\ \alpha+\beta=n+1}} \frac{\partial^{n+1}}{\partial X^\alpha \partial Y^\beta}(t + x + U + J + K) \\ &\quad = f + \sum_{\substack{\alpha,\beta=0,1,\dots,n+1\\ \alpha+\beta=n+1}} \Big\langle g_{\alpha,\beta}, \frac{\partial^{n+1}}{\partial X^\alpha \partial Y^\beta} Z \Big\rangle, \end{aligned} \tag{7.44}$$

where f and $g_{\alpha,\beta}$ depend on derivatives up to order n. By (7.43), the functions f and $g_{\alpha,\beta}$ are bounded. Here, $g_{\alpha,\beta}$ denotes $n+1$ five dimensional vectors. To clarify the notation, let us compute (7.44) for $n = 2$. We have

$$\begin{aligned} &\frac{\partial}{\partial Y} \sum_{\substack{\alpha,\beta=0,1,2,3\\ \alpha+\beta=3}} \frac{\partial^3}{\partial X^\alpha \partial Y^\beta}(t + x + U + J + K) \\ &= f + \langle g_{3,0}, Z_{XXX} \rangle + \langle g_{2,1}, Z_{XXY} \rangle + \langle g_{1,2}, Z_{XYY} \rangle + \langle g_{0,3}, Z_{YYY} \rangle. \end{aligned}$$

By Gronwall's lemma, we obtain

$$\frac{\partial^{n+1}}{\partial X^\alpha \partial Y^\beta} Z \in [L^\infty(\Omega)]^5, \quad \alpha, \beta = 0, 1, \dots, n+1, \quad \alpha + \beta = n+1.$$

This implies, since $x_X(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s) = x_Y(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)$, $\mathcal{Z}_2(\tau,s) = x(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))$ and $\mathcal{Z}_3(\tau,s) = U(\mathcal{X}(\tau,s), \mathcal{Y}(\tau,s))$, that

$$\frac{d^{n+1}}{ds^{n+1}} \mathcal{X}(\tau,\cdot), \frac{d^{n+1}}{ds^{n+1}} \mathcal{Y}(\tau,\cdot), \frac{d^{n+1}}{ds^{n+1}} \mathcal{Z}_2(\tau,\cdot), \frac{d^{n+1}}{ds^{n+1}} \mathcal{Z}_3(\tau,\cdot) \in L^\infty([s_l, s_r]).$$

It then follows from (7.25) and (7.26) that

$$\frac{\partial^{n+1}}{\partial x^{n+1}} u(\tau,\cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau]),$$

and from (7.25) and (7.30) we obtain

$$\frac{\partial^n}{\partial x^n} R(\tau,\cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau]).$$

By (7.42), $\frac{\partial^k}{\partial X^k} p(\cdot, \mathcal{Y}(\cdot))$, $k = 0, 1, \dots, n$, is bounded on the curve $(\mathcal{X}(s), \mathcal{Y}(s))$, $s \in [s_l, s_r]$. Since

$$\frac{\partial^k}{\partial X^k} p(X, \mathcal{Y}(X)) = \frac{\partial^k}{\partial X^k} p(X, Y),$$

we get from (7.25) and (7.31),

$$\frac{\partial^n}{\partial x^n}\rho(\tau,\cdot)\in L^\infty([x_l+\kappa\tau, x_r-\kappa\tau]).$$

Similarly, one proves that

$$\frac{\partial^n}{\partial x^n}S(\tau,\cdot), \frac{\partial^n}{\partial x^n}\sigma(\tau,\cdot)\in L^\infty([x_l+\kappa\tau, x_r-\kappa\tau]).$$

□

From Theorem 7.1 we obtain the following result.

Corollary 7.2. *Let* $-\infty<x_l<x_r<\infty$ *and consider* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)\in\mathcal{D}$. *Assume that*

(A1') $u_0, R_0, S_0, \rho_0, \sigma_0 \in C^\infty([x_l, x_r])$,
(A2') there are constants $d>0$ *and* $e>0$ *such that* $\rho_0(x)\geq d$ *and* $\sigma_0(x)\geq e$ *for all* $x\in[x_l,x_r]$,
(A3') μ_0 *and* ν_0 *are absolutely continuous on* $[x_l, x_r]$,
(A4') $c\in C^\infty(\mathbb{R})$ *and* $c^{(m)}\in L^\infty(\mathbb{R})$ *for* $m=3,4,5,\dots$.

For any $\tau\in\big[0,\frac{1}{2\kappa}(x_r-x_l)\big]$ *consider*

$$(u,R,S,\rho,\sigma,\mu,\nu)(\tau)=\bar{S}_\tau(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0).$$

Then

(P1') $u(\tau,\cdot), R(\tau,\cdot), S(\tau,\cdot), \rho(\tau,\cdot), \sigma(\tau,\cdot)\in C^\infty([x_l+\kappa\tau, x_r-\kappa\tau])$,
(P2') there are constants $\bar{d}>0$ *and* $\bar{e}>0$ *such that* $\rho(\tau,x)\geq\bar{d}$ *and* $\sigma(\tau,x)\geq\bar{e}$ *for all* $x\in[x_l+\kappa\tau, x_r-\kappa\tau]$,
(P3') $\mu(\tau,\cdot)$ *and* $\nu(\tau,\cdot)$ *are absolutely continuous on* $[x_l+\kappa\tau, x_r-\kappa\tau]$.

For $\tau\in\big[-\frac{1}{2\kappa}(x_r-x_l),0\big]$, *the solution satisfies the same properties on the interval* $\big[x_l-\kappa\tau, x_r+\kappa\tau\big]$.

7.2. **Approximation by Smooth Solutions.** The following theorem is our main result. In the proof we use Lemma 7.5 and Lemma 7.7 which are stated and proved in Section 7.3.

Theorem 7.3. *Let* $-\infty<x_l<x_r<\infty$. *Consider* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ *and* $(u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n, \mu_0^n, \nu_0^n)$ *in* $\mathcal{D}$. *Assume that for all* $n\in\mathbb{N}$,

(A1") $u_0, R_0, S_0, u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n\in C^\infty([x_l,x_r])$,
(A2") $\rho_0(x)=0$ *and* $\sigma_0(x)=0$ *for all* $x\in[x_l,x_r]$,
(A3") there are constants $d_n>0$ *and* $e_n>0$ *such that* $\rho_0^n(x)\geq d_n$ *and* $\sigma_0^n(x)\geq e_n$ *for all* $x\in[x_l,x_r]$,
(A4") $u_0^n\to u_0$ *in* $L^\infty([x_l,x_r])$, $R_0^n\to R_0$, $S_0^n\to S_0$, $\rho_0^n\to\rho_0$ *and* $\sigma_0^n\to\sigma_0$ *in* $L^2([x_l,x_r])$,
(A5") μ_0, ν_0, μ_0^n *and* ν_0^n *are absolutely continuous on* $[x_l,x_r]$,
(A6") $\mu_0((-\infty,x_l))=\mu_0^n((-\infty,x_l))$, $\mu_0((-\infty,x_r,))=\mu_0^n((-\infty,x_r))$, $\nu_0((-\infty,x_l))=\nu_0^n((-\infty,x_l))$ *and* $\nu_0((-\infty,x_r,))=\nu_0^n((-\infty,x_r))$,
(A7") $c\in C^\infty(\mathbb{R})$ *and* $c^{(m)}\in L^\infty(\mathbb{R})$ *for* $m=3,4,5,\dots$.

For any $\tau\in\big[0,\frac{1}{2\kappa}(x_r-x_l)\big]$ *consider*

$$(u^n,R^n,S^n,\rho^n,\sigma^n,\mu^n,\nu^n)(\tau)=\bar{S}_\tau(u_0^n,R_0^n,S_0^n,\rho_0^n,\sigma_0^n,\mu_0^n,\nu_0^n)$$

and

$$(u, R, S, \rho, \sigma, \mu, \nu)(\tau) = \bar{S}_\tau(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0).$$

Then we have

(P1") $u^n(\tau, \cdot) \to u(\tau, \cdot)$ in $L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$,
(P2") $\rho^n(\tau, \cdot) \to 0$ and $\sigma^n(\tau, \cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$.

The same conclusion holds on the interval $\big[x_l - \kappa\tau, x_r + \kappa\tau\big]$ in the case where $\tau \in \big[-\frac{1}{2\kappa}(x_r - x_l), 0\big]$.

Observe that we assume in (A4") $u_0^n \to u_0$ in $L^\infty([x_l, x_r])$, in contrast to the $L^2(\mathbb{R})$ convergence in the global case. This is because functions in $H^1(\mathbb{R})$ tend to 0 as $x \to \pm\infty$, while here we have no assumptions on the values of u_0 and u_0^n at the endpoints of $[x_l, x_r]$. Since $[x_l, x_r]$ is a bounded interval, the convergence in $L^\infty([x_l, x_r])$ implies that $u_0^n \to u_0$ in $L^2([x_l, x_r])$.

By the assumptions (A5") and (A6") we mean that

$$\mu_0([x_l, x_r]) = \mu_0^n([x_l, x_r]) \quad \text{and} \quad \nu_0([x_l, x_r]) = \nu_0^n([x_l, x_r])$$

for all n.

Note that the approximating sequence in Theorem 7.3 is smooth. Indeed, by Corollary 7.2 we have $u^n(\tau, \cdot), R^n(\tau, \cdot), S^n(\tau, \cdot), \rho^n(\tau, \cdot), \sigma^n(\tau, \cdot) \in C^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$. Furthermore, there are constants $\bar{d}_n > 0$ and $\bar{e}_n > 0$ such that $\rho^n(\tau, x) \geq \bar{d}_n$ and $\sigma^n(\tau, x) \geq \bar{e}_n$ for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$, and $\mu^n(\tau, \cdot)$ and $\nu^n(\tau, \cdot)$ are absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$ for all n. However, the limit solution does not in general satisfy these properties. Of course, we have that $(u, R, S, \rho, \sigma, \mu, \nu)(\tau)$ belongs to $\mathcal{D}$, but the functions are not necessarily smooth and the measures are not necessarily absolutely continuous. We illustrate this with an example. Consider the function

$$f_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{n} e^{-(nx)^2}.$$

We have $||f_n||_{L^2(\mathbb{R})} = 1$, $f_n(x) \to 0$ for $x \neq 0$ and $f_n(0) \to +\infty$, and $f_n \to 0$ in $L^1(\mathbb{R})$ as $n \to \infty$. By standard calculations we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(x) f_n(x)\, dx = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(x) f_n^2(x)\, dx = \phi(0)$$

for all $\phi \in C_c^\infty(\mathbb{R})$. In other words, $f_n \overset{*}{\rightharpoonup} 0$ and $f_n^2\, dx \overset{*}{\rightharpoonup} \delta_0$, where δ_0 is the Dirac delta at zero, which is a singular measure. Returning to our setting, since $\mu^n(\tau, \cdot)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$ we have

$$\mu^n(\tau, [x_l + \kappa\tau, x_r - \kappa\tau]) = \frac{1}{4} \int_{x_l + \kappa\tau}^{x_r - \kappa\tau} [(R^n)^2 + c(u^n)(\rho^n)^2](\tau, x)\, dx.$$

We can think of $\sqrt{c(u^n(\tau, x))}\rho^n(\tau, x)$ as the function f_n in the example above. We then have that $\sqrt{c(u^n(\tau, \cdot))}\rho^n(\tau, \cdot)$ is in $C^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$ and satisfies $\sqrt{c(u^n(\tau, \cdot))}\rho^n(\tau, \cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$, but $c(u^n(\tau, \cdot))(\rho^n)^2(\tau, \cdot) \overset{*}{\rightharpoonup} \delta_0$.

Proof. We will only consider the case $0 < \tau \leq \frac{1}{2\kappa}(x_r - x_l)$. The case $-\frac{1}{2\kappa}(x_r - x_l) \leq \tau < 0$ can be treated in the same way.

We split the proof into four steps.

Step 1. Set

$$(\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$$

and

$$(\psi_1^n, \psi_2^n) = \mathbf{L}(u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n, \mu_0^n, \nu_0^n),$$

where

$$\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1), \quad \psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$$

and

$$\psi_1^n = (x_1^n, U_1^n, J_1^n, K_1^n, V_1^n, H_1^n), \quad \psi_2^n = (x_2^n, U_2^n, J_2^n, K_2^n, V_2^n, H_2^n).$$

Let us find out what kind of region the interval $[x_l, x_r]$ corresponds to in Lagrangian coordinates (X, Y). Since the measures are assumed to be absolutely continuous we get from (3.12a),

$$x_1(X) + \mu_0((-\infty, x_1(X))) = X$$

for all $x_1(X) \in [x_l, x_r]$. We show which range of the X-variable this corresponds to. If $\hat{X}$ is such that $x_1(\hat{X}) = x_l$ then

$$x_l + \mu_0((-\infty, x_l)) = \hat{X}.$$

We also have

$$x_1^n(X) + \mu_0^n((-\infty, x_1^n(X))) = X$$

for all $x_1^n(X) \in [x_l, x_r]$. If $\check{X}$ is such that $x_1^n(\check{X}) = x_l$ we get

$$x_l + \mu_0^n((-\infty, x_l)) = \check{X}.$$

Using (A6") we obtain

$$\hat{X} = \check{X}$$

and we denote $X_l = \hat{X} = \check{X}$.

If $\bar{X}$ and $\tilde{X}$ are such that $x_1(\bar{X}) = x_1^n(\tilde{X}) = x_r$ then

$$x_r + \mu_0((-\infty, x_r)) = \bar{X}$$

and

$$x_r + \mu_0^n((-\infty, x_r)) = \tilde{X},$$

and by (A6") we get

$$\bar{X} = \tilde{X}$$

which we denote $X_r = \bar{X} = \tilde{X}$. We have $X_l \leq X_r$ since

$$X_l = x_l + \mu_0((-\infty, x_l)) \leq x_r + \mu_0((-\infty, x_r)) = X_r.$$

In other words, we can define the interval $[X_l, X_r]$ using either the measure μ_0 or μ_0^n, and observe that this is a consequence of the assumptions (A5") and (A6").

In a similar way we find that

$$x_2(Y) + \nu_0((-\infty, x_2(Y))) = Y \quad \text{and} \quad x_2^n(Y) + \nu_0^n((-\infty, x_2^n(Y))) = Y$$

for all $Y \in [Y_l, Y_r]$ where

$$Y_l = x_l + \nu_0((-\infty, x_l)) \quad \text{and} \quad Y_r = x_r + \nu_0((-\infty, x_r)).$$

We denote $\Omega = [X_l, X_r] \times [Y_l, Y_r]$.

Following closely the proof of Lemma 7.5, we obtain that x_1, x_1^n, x_2 and x_2^n are strictly increasing for $X \in [X_l, X_r]$ and $Y \in [Y_l, Y_r]$, respectively, and

$$\begin{gathered} x_1^n \to x_1, \quad (x_1^n)^{-1} \to x_1^{-1}, \quad U_1^n \to U_1, \quad J_1^n \to J_1 \quad \text{in } L^\infty([X_l, X_r]), \\ x_2^n \to x_2, \quad (x_2^n)^{-1} \to x_2^{-1}, \quad U_2^n \to U_2, \quad J_2^n \to J_2 \quad \text{in } L^\infty([Y_l, Y_r]), \\ V_1^n \to V_1, \quad H_1^n \to H_1, \quad (x_1^n)' \to x_1', \quad (J_1^n)' \to J_1', \quad (K_1^n)' \to K_1' \quad \text{in } L^2([X_l, X_r]), \\ V_2^n \to V_2, \quad H_2^n \to H_2, \quad (x_2^n)' \to x_2', \quad (J_2^n)' \to J_2', \quad (K_2^n)' \to K_2' \quad \text{in } L^2([Y_l, Y_r]). \end{gathered}$$

Note that (3.12f) does not imply $K_1^n \to K_1$ in $L^\infty([X_l, X_r])$ and $K_2^n \to K_2$ in $L^\infty([Y_l, Y_r])$. However, we have $K_1^n - K_1^n(X_l) \to K_1 - K_1(X_l)$ in $L^\infty([X_l, X_r])$ and $K_2^n - K_2^n(Y_l) \to K_2 - K_2(Y_l)$ in $L^\infty([Y_l, Y_r])$. The proof of this closely follows the procedure in (7.105).

Step 2. Let

$$\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) = \mathbf{C}(\psi_1, \psi_2)$$

and

$$\Theta^n = (\mathcal{X}^n, \mathcal{Y}^n, \mathcal{Z}^n, \mathcal{V}^n, \mathcal{W}^n, \mathfrak{p}^n, \mathfrak{q}^n) = \mathbf{C}(\psi_1^n, \psi_2^n).$$

From (3.28) we have

$$x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s)) \quad \text{and} \quad x_1^n(\mathcal{X}^n(s)) = x_2^n(2s - \mathcal{X}^n(s)).$$

Since we only consider the functions x_1 and x_1^n on $[X_l, X_r]$, and x_2 and x_2^n on $[Y_l, Y_r]$, the relevant values of s are those satisfying $\mathcal{X}(s) \in [X_l, X_r]$ and $2s - \mathcal{X}(s) \in [Y_l, Y_r]$. In other words, since $x_1(X_l) = x_2(Y_l) = x_l$ and $x_1(X_r) = x_2(Y_r) = x_r$, we have

$$\frac{1}{2}(X_l + Y_l) \le s \le \frac{1}{2}(X_r + Y_r)$$

and we call $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. The same conclusion holds for the s-values of $(\mathcal{X}^n(s), \mathcal{Y}^n(s))$. The points s_l and s_r correspond to the two diagonal points in the box for both curves, i.e., $(\mathcal{X}(s_l), \mathcal{Y}(s_l)) = (X_l, Y_l) = (\mathcal{X}^n(s_l), \mathcal{Y}^n(s_l))$ and $(\mathcal{X}(s_r), \mathcal{Y}(s_r)) = (X_r, Y_r) = (\mathcal{X}^n(s_r), \mathcal{Y}^n(s_r))$.

Following closely the proof of Lemma 7.7, we obtain that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}^n$ and $\mathcal{Y}^n$ are strictly increasing on $[s_l, s_r]$, and

$$\begin{gathered} \mathcal{X}^n \to \mathcal{X}, \quad \mathcal{Y}^n \to \mathcal{Y}, \quad \mathcal{Z}_j^n \to \mathcal{Z}_j \quad \text{in } L^\infty([s_l, s_r]), \\ \mathcal{V}_i^n \to \mathcal{V}_i, \quad \mathfrak{p}^n \to \mathfrak{p} \quad \text{in } L^2([X_l, X_r]), \\ \mathcal{W}_i^n \to \mathcal{W}_i, \quad \mathfrak{q}^n \to \mathfrak{q} \quad \text{in } L^2([Y_l, Y_r]) \end{gathered}$$

for $i = 1, \dots, 5$, $j = 1, \dots, 4$. From (3.31f) and (3.12g) we have

$$\mathfrak{p}(X) = H_1(X) = \frac{1}{2}\rho_0(x_1(X))x_1'(X)$$

and

$$\mathfrak{p}^n(X) = H_1^n(X) = \frac{1}{2}\rho_0^n(x_1^n(X))(x_1^n)'(X)$$

for all $X \in [X_l, X_r]$. Using (A2"), (A3") and (7.3) we get for all $X \in [X_l, X_r]$, $\mathfrak{p}(X) = 0$ and $\mathfrak{p}^n(X) \ge k_n > 0$ for some constant k_n.

Since $K_1^n - K_1^n(X_l) \to K_1 - K_1(X_l)$ in $L^\infty([X_l, X_r])$ and $K_2^n - K_2^n(Y_l) \to K_2 - K_2(Y_l)$ in $L^\infty([Y_l, Y_r])$ we get from (3.30e) that $\mathcal{Z}_5^n - \mathcal{Z}_5^n(s_l) \to \mathcal{Z}_5 - \mathcal{Z}_5(s_l)$ in $L^\infty([s_l, s_r])$.

Step 3. Consider $(Z,p,q) = \mathbf{S}(\Theta)$ and $(Z^n,p^n,q^n) = \mathbf{S}(\Theta^n)$. We prove a Gronwall type estimate. We claim that for all (X,Y) in Ω,

$$\begin{aligned}
&\Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^{5}\Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](X,Y) \\
&\leq K\Bigg\{||U_1 - U_1^n||^2_{L^\infty([X_l,X_r])} \\
&\qquad + \sum_{j=1}^{5}\Big(\big[\mathcal{V}_j(X) - \mathcal{V}_j^n(X)\big]^2 + \big[\mathcal{W}_j(Y) - \mathcal{W}_j^n(Y)\big]^2\Big) \\
&\qquad + \big[\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2 \\
&\qquad + \big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y) - \mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2\Bigg\},
\end{aligned} \tag{7.45}$$

where K depends on κ, k_1, k_2, $|||\Theta|||_{\mathcal{G}(\Omega)}$ and the size of Ω.

Let $(X,Y)\in\Omega$. Subtracting the equations

$$t_X(X,Y) = t_X(X,\mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^{Y} t_{XY}(X,\tilde{Y})\,d\tilde{Y}$$

and

$$t_X^n(X,Y) = t_X^n(X,\mathcal{Y}^n(X)) + \int_{\mathcal{Y}^n(X)}^{Y} t_{XY}^n(X,\tilde{Y})\,d\tilde{Y} \tag{7.46}$$

yields

$$\begin{aligned}
t_X(X,Y) - t_X^n(X,Y) &= \mathcal{V}_1(X) - \mathcal{V}_1^n(X) - \int_{\mathcal{Y}^n(X)}^{\mathcal{Y}(X)} t_{XY}^n(X,\tilde{Y})\,d\tilde{Y} \\
&\quad + \int_{\mathcal{Y}(X)}^{Y}(t_{XY}(X,\tilde{Y}) - t_{XY}^n(X,\tilde{Y}))\,d\tilde{Y}.
\end{aligned} \tag{7.47}$$

Using (2.38a) we get

$$\begin{aligned}
&t_{XY} - t_{XY}^n \\
&= -\frac{c'(U)}{2c(U)}(U_X t_Y + U_Y t_X) + \frac{c'(U^n)}{2c(U^n)}(U_X^n t_Y^n + U_Y^n t_X^n) \\
&= -\frac{c'(U)}{2c(U)}\Big(U_X(t_Y - t_Y^n) + t_Y^n(U_X - U_X^n) + U_Y(t_X - t_X^n) + t_X^n(U_Y - U_Y^n)\Big) \\
&\quad - \frac{1}{2}(U_X^n t_Y^n + U_Y^n t_X^n)\int_{U^n}^{U}\left(\frac{c''(V)}{c(V)} - \frac{c'(V)^2}{c(V)^2}\right)dV
\end{aligned} \tag{7.48}$$

where we used

$$\frac{d}{dV}\left(\frac{c'(V)}{c(V)}\right) = \frac{c''(V)}{c(V)} - \frac{c'(V)^2}{c(V)^2}.$$

We need a pointwise uniform bound on the components of Z_X^n and Z_Y^n. This will be done in the same way as in Lemma 4.9. We show the details here for completeness.

By (2.38a) we get

$$|t^n_{XY}| \leq \frac{1}{2}k_1\kappa\Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)\Big(|t^n_Y| + |x^n_Y| + |U^n_Y| + |J^n_Y| + |K^n_Y|\Big),$$

and by doing the same kind of estimate for the other components we get

$$\begin{aligned}\Big(|t^n_{XY}| + |x^n_{XY}| + |U^n_{XY}| + |J^n_{XY}| + |K^n_{XY}|\Big) &\\ \leq B_1\Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)&\\ \times\Big(|t^n_Y| + |x^n_Y| + |U^n_Y| + |J^n_Y| + |K^n_Y|\Big)&\end{aligned} \tag{7.49}$$

for a constant B_1 that only depends on κ and k_1. By (7.46) and the corresponding expressions for x^n_X, U^n_X, J^n_X and K^n_X we obtain

$$\begin{aligned}&\Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)(X,Y)\\ &\leq \Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)(X,\mathcal{Y}^n(X))\\ &\quad + \Big|\int_{\mathcal{Y}^n(X)}^{Y}\Big(|t^n_{XY}| + |x^n_{XY}| + |U^n_{XY}| + |J^n_{XY}| + |K^n_{XY}|\Big)(X,\tilde{Y})\,d\tilde{Y}\Big|\\ &\leq \Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)(X,\mathcal{Y}^n(X))\\ &\quad + \Big|\int_{\mathcal{Y}^n(X)}^{Y} B_1\Big\{\Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)\\ &\qquad\qquad \times\Big(|t^n_Y| + |x^n_Y| + |U^n_Y| + |J^n_Y| + |K^n_Y|\Big)\Big\}(X,\tilde{Y})\,d\tilde{Y}\Big|,\end{aligned} \tag{7.50}$$

where we used (7.49). By Gronwall's inequality,

$$\begin{aligned}&\Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)(X,Y)\\ &\leq \Big(|t^n_X| + |x^n_X| + |U^n_X| + |J^n_X| + |K^n_X|\Big)(X,\mathcal{Y}^n(X))\\ &\quad\times\exp\Big\{B_1\Big|\int_{\mathcal{Y}^n(X)}^{Y}\Big(|t^n_Y| + |x^n_Y| + |U^n_Y| + |J^n_Y| + |K^n_Y|\Big)(X,\tilde{Y})\,d\tilde{Y}\Big|\Big\}.\end{aligned}$$

By (4.12a), (4.12b), (4.12d) and (4.12e) we get

$$\begin{aligned}|t^n_Y| + |x^n_Y| + |U^n_Y| + |J^n_Y| + |K^n_Y| &= \frac{1}{c(U^n)}x^n_Y + x^n_Y + |U^n_Y| + J^n_Y + \frac{1}{c(U^n)}J^n_Y\\ &\leq (1+\kappa)(x^n_Y + J^n_Y) + |U^n_Y|.\end{aligned} \tag{7.51}$$

From (4.12c) we have

$$2J^n_Y x^n_Y = (c(U^n)U^n_Y)^2 + c(U^n)(q^n)^2 \geq (c(U^n)U^n_Y)^2,$$

and by Young's inequality,

$$|U^n_Y| \leq \kappa\sqrt{2J^n_Y x^n_Y} \leq \frac{\kappa}{\sqrt{2}}(J^n_Y + x^n_Y),$$

which by (7.51) implies

$$|t_Y^n| + |x_Y^n| + |U_Y^n| + |J_Y^n| + |K_Y^n| \leq \left[1 + \left(1 + \frac{1}{\sqrt{2}}\right)\kappa\right](x_Y^n + J_Y^n).$$

Using this in (7.50) we obtain

$$\begin{aligned}(7.52)\qquad &\Big(|t_X^n| + |x_X^n| + |U_X^n| + |J_X^n| + |K_X^n|\Big)(X,Y)\\ &\leq \Big(|t_X^n| + |x_X^n| + |U_X^n| + |J_X^n| + |K_X^n|\Big)(X,\mathcal{Y}^n(X))\\ &\quad\times \exp\left\{B_2\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n + J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right|\right\}\end{aligned}$$

for a new constant B_2 that only depends on κ and k_1. Since x^n and J^n are nondecreasing with respect to both variables, we have

$$\begin{aligned}&\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n + J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right|\\ &= \big|x^n(X,Y) - x^n(X,\mathcal{Y}^n(X)) + J^n(X,Y) - J^n(X,\mathcal{Y}^n(X))\big|\\ &\leq \big|x^n(X_r,Y_r) - x^n(X_l,Y_l)\big| + \big|J^n(X_r,Y_r) - J^n(X_l,Y_l)\big|\\ &= \big|x^n(\mathcal{X}^n(s_r),\mathcal{Y}^n(s_r)) - x^n(\mathcal{X}^n(s_l),\mathcal{Y}^n(s_l))\big|\\ &\quad + \big|J^n(\mathcal{X}^n(s_r),\mathcal{Y}^n(s_r)) - J^n(\mathcal{X}^n(s_l),\mathcal{Y}^n(s_l))\big|\\ &= \big|\mathcal{Z}_2^n(s_r) - \mathcal{Z}_2^n(s_l)\big| + \big|\mathcal{Z}_4^n(s_r) - \mathcal{Z}_4^n(s_l)\big|.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{Z}_2^n(s_r) - \mathcal{Z}_2^n(s_l) &= \mathcal{Z}_2^n(s_r) - \mathcal{Z}_2(s_r) + \mathcal{Z}_2(s_r) - \mathcal{Z}_2^n(s_l) + \mathcal{Z}_2(s_l) - \mathcal{Z}_2(s_l)\\ &= \mathcal{Z}_2^n(s_r) - \mathcal{Z}_2(s_r) + \mathcal{Z}_2^a(s_r) + s_r - \mathcal{Z}_2^n(s_l) + \mathcal{Z}_2(s_l) - \mathcal{Z}_2^a(s_l) - s_l\end{aligned}$$

and

$$\mathcal{Z}_4^n(s_r) - \mathcal{Z}_4^n(s_l) = \mathcal{Z}_4^n(s_r) - \mathcal{Z}_4(s_r) + \mathcal{Z}_4^a(s_r) - \mathcal{Z}_4^n(s_l) + \mathcal{Z}_4(s_l) - \mathcal{Z}_4^a(s_l)$$

we end up with

$$\begin{aligned}&\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n + J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right|\\ &\leq 2||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty([s_l,s_r])} + 2||\mathcal{Z}_2^a||_{L^\infty([s_l,s_r])} + s_r - s_l\\ &\quad + 2||\mathcal{Z}_4 - \mathcal{Z}_4^n||_{L^\infty([s_l,s_r])} + 2||\mathcal{Z}_4^a||_{L^\infty([s_l,s_r])}.\end{aligned}$$

The convergence $\mathcal{Z}_i^n \to \mathcal{Z}_i$ in $L^\infty([s_l,s_r])$ implies that for every $\varepsilon > 0$ we can choose n so large that $||\mathcal{Z}_i - \mathcal{Z}_i^n||_{L^\infty([s_l,s_r])} \leq \varepsilon$, $i = 1,2$. Hence,

$$(7.53)\quad \left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n + J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right| \leq 4\varepsilon + 2||\mathcal{Z}_2^a||_{L^\infty([s_l,s_r])} + 2||\mathcal{Z}_4^a||_{L^\infty([s_l,s_r])} + s_r - s_l.$$

We use (7.53) and $Z_{i,X}^n(X,\mathcal{Y}^n(X)) = \mathcal{V}_i^n(X)$ in (7.52) and obtain

$$(7.54)\qquad \Big(|t_X^n| + |x_X^n| + |U_X^n| + |J_X^n| + |K_X^n|\Big)(X,Y) \leq B_3\sum_{l=1}^{5}\big|\mathcal{V}_l^n(X)\big|$$

where B_3 only depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$. Since μ_0^n is absolutely continuous in $[x_l, x_r]$ we obtain as in (7.2) that

$$0 \leq (x_1^n)'(X) \leq 1$$

for all $X \in [X_l, X_r]$. Using (3.31a), (3.31b), (3.31d), (3.31e), (3.12c) and (3.12f) we get

$$0 \leq \mathcal{V}_1^n(X) \leq \frac{1}{2}\kappa, \quad 0 \leq \mathcal{V}_2^n(X) \leq \frac{1}{2}, \quad 0 \leq \mathcal{V}_4^n(X) \leq 1, \quad 0 \leq \mathcal{V}_5^n(X) \leq \kappa$$

for all $X \in [X_l, X_r]$. From (3.31c) and (3.8) we have

$$|\mathcal{V}_3^n(X)| \leq \frac{1}{c(U_1^n(X))}\sqrt{(x_1^n)'(X)(J_1^n)'(X)} \leq \kappa.$$

By inserting these estimates in (7.54) we get

$$\Big(|t_X^n| + |x_X^n| + |U_X^n| + |J_X^n| + |K_X^n|\Big)(X,Y) \leq B_4 \quad \text{for all } (X,Y) \in \Omega \tag{7.55}$$

for a new constant B_4, which only depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$. Similarly we can show that there exist constants B_5, B_6 and B_7, which only depend on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$, such that for all $(X,Y) \in \Omega$,

$$\Big(|t_Y^n| + |x_Y^n| + |U_Y^n| + |J_Y^n| + |K_Y^n|\Big)(X,Y) \leq B_5, \tag{7.56}$$

$$\Big(|t_X| + |x_X| + |U_X| + |J_X| + |K_X|\Big)(X,Y) \leq B_6 \tag{7.57}$$

and

$$\Big(|t_Y| + |x_Y| + |U_Y| + |J_Y| + |K_Y|\Big)(X,Y) \leq B_7. \tag{7.58}$$

From (7.49) we get for all $(X,Y) \in \Omega$ that

$$\Big(|t_{XY}^n| + |x_{XY}^n| + |U_{XY}^n| + |J_{XY}^n| + |K_{XY}^n|\Big)(X,Y) \leq D \tag{7.59}$$

for a constant D that depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$. From (7.55)-(7.58) we get in (7.48), for all $(X,Y) \in \Omega$ that

$$\begin{aligned} &\big|t_{XY} - t_{XY}^n\big|(X,Y) \\ &\quad \leq C_1\Big(|U - U^n| + |t_X - t_X^n| + |U_X - U_X^n| + |t_Y - t_Y^n| + |U_Y - U_Y^n|\Big)(X,Y), \end{aligned} \tag{7.60}$$

where C_1 depends on κ, k_1, k_2 and $|||\Theta|||_{\mathcal{G}(\Omega)}$.

Using the estimates (7.60) and (7.59) in (7.47) we get

$$\begin{aligned} &|t_X(X,Y) - t_X^n(X,Y)| \\ &\leq |\mathcal{V}_1(X) - \mathcal{V}_1^n(X)| + D|\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)| \\ &\quad + C_1\bigg|\int_{\mathcal{Y}(X)}^{Y} \big(|U - U^n| + |t_X - t_X^n| + |U_X - U_X^n| \\ &\qquad\qquad + |t_Y - t_Y^n| + |U_Y - U_Y^n|\big)(X,\tilde{Y})\,d\tilde{Y}\bigg|. \end{aligned}$$

To write the estimates more compactly, we denote $Z = (Z_1, Z_2, Z_3, Z_4, Z_5) = (t, x, U, J, K)$, and similar for Z^n. With this notation we get from the above estimate that

$$\begin{aligned}(7.61)\quad &|Z_{1,X}(X,Y) - Z^n_{1,X}(X,Y)| \\ &\leq |\mathcal{V}_1(X) - \mathcal{V}^n_1(X)| + D|\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)| \\ &\quad + C_1\bigg|\int_{\mathcal{Y}(X)}^{Y}\bigg[|Z_3 - Z^n_3| + \sum_{i=1}^{5}\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\bigg](X,\tilde{Y})\,d\tilde{Y}\bigg|.\end{aligned}$$

By the same procedure as above, we obtain

$$\begin{aligned}(7.62)\quad &|Z_{j,X}(X,Y) - Z^n_{j,X}(X,Y)| \\ &\leq |\mathcal{V}_j(X) - \mathcal{V}^n_j(X)| + D|\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)| \\ &\quad + C_j\bigg|\int_{\mathcal{Y}(X)}^{Y}\bigg[|Z_3 - Z^n_3| + \sum_{i=1}^{5}\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\bigg](X,\tilde{Y})\,d\tilde{Y}\bigg|.\end{aligned}$$

for $j = 2, 3, 4, 5$, where C_j depends on κ, k_1, k_2 and $|||\Theta|||_{\mathcal{G}(\Omega)}$, and D depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$.

Let us find similar estimates for the partial derivatives with respect to Y. Now we subtract

$$t_Y(\mathcal{X}(Y), Y) = t_Y(X,Y) + \int_X^{\mathcal{X}(Y)} t_{XY}(\tilde{X}, Y)\,d\tilde{X}$$

and

$$t^n_Y(\mathcal{X}^n(Y), Y) = t^n_Y(X,Y) + \int_X^{\mathcal{X}^n(Y)} t^n_{XY}(\tilde{X}, Y)\,d\tilde{X}$$

to get

$$\begin{aligned}t_Y(X,Y) - t^n_Y(X,Y) = \mathcal{W}_1(Y) - \mathcal{W}^n_1(Y) &+ \int_{\mathcal{X}(Y)}^{\mathcal{X}^n(Y)} t^n_{XY}(\tilde{X}, Y)\,d\tilde{X} \\ &- \int_X^{\mathcal{X}(Y)}(t_{XY}(\tilde{X},Y) - t^n_{XY}(\tilde{X},Y))\,d\tilde{X},\end{aligned}$$

and we obtain

$$\begin{aligned}&|t_Y(X,Y) - t^n_Y(X,Y)| \\ &\leq |\mathcal{W}_1(Y) - \mathcal{W}^n_1(Y)| + D|\mathcal{X}\circ\mathcal{Y}^{-1}(Y) - \mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)| \\ &\quad + C_1\bigg|\int_X^{\mathcal{X}(Y)}\big(|U - U^n| + |t_X - t^n_X| + |U_X - U^n_X| \\ &\qquad\qquad + |t_Y - t^n_Y| + |U_Y - U^n_Y|\big)(\tilde{X},Y)\,d\tilde{X}\bigg|,\end{aligned}$$

which in the alternative notation takes the form

$$\begin{aligned}(7.63)\quad &|Z_{1,Y}(X,Y) - Z^n_{1,Y}(X,Y)| \\ &\leq |\mathcal{W}_1(Y) - \mathcal{W}^n_1(Y)| + D|\mathcal{X}\circ\mathcal{Y}^{-1}(Y) - \mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)| \\ &\quad + C_1\bigg|\int_X^{\mathcal{X}(Y)}\bigg[|Z_3 - Z^n_3| + \sum_{i=1}^{5}\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\bigg](\tilde{X},Y)\,d\tilde{X}\bigg|.\end{aligned}$$

Similarly, we get

$$
\begin{aligned}
(7.64)\quad &|Z_{j,Y}(X,Y) - Z^n_{j,Y}(X,Y)| \\
&\leq |\mathcal{W}_j(Y) - \mathcal{W}^n_j(Y)| + D|\mathcal{X}\circ\mathcal{Y}^{-1}(Y) - \mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)| \\
&\quad + C_j\bigg|\int_X^{\mathcal{X}(Y)}\Big[|Z_3 - Z^n_3| + \sum_{i=1}^5\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\Big](\tilde{X},Y)\,d\tilde{X}\bigg|
\end{aligned}
$$

for $j = 2, 3, 4, 5$.

We have

$$U(X,Y) = U(X,\mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^Y U_Y(X,\tilde{Y})\,d\tilde{Y}$$

and

$$U^n(X,Y) = U^n(X,\mathcal{Y}^n(X)) + \int_{\mathcal{Y}^n(X)}^Y U^n_Y(X,\tilde{Y})\,d\tilde{Y},$$

so that

$$
\begin{aligned}
U(X,Y) - U^n(X,Y) = U(X,\mathcal{Y}(X)) - U^n(X,\mathcal{Y}^n(X)) - \int_{\mathcal{Y}^n(X)}^{\mathcal{Y}(X)} U^n_Y(X,\tilde{Y})\,d\tilde{Y} \\
+ \int_{\mathcal{Y}(X)}^Y (U_Y(X,\tilde{Y}) - U^n_Y(X,\tilde{Y}))\,d\tilde{Y}.
\end{aligned}
$$

To any X in $[X_l, X_r]$, there exist unique s and s^n in $[s_l, s_r]$ such that $X = \mathcal{X}(s)$ and $X = \mathcal{X}^n(s^n)$ and we can write

$$
\begin{aligned}
U(X,\mathcal{Y}(X)) - U^n(X,\mathcal{Y}^n(X)) &= U(\mathcal{X}(s),\mathcal{Y}(s)) - U^n(\mathcal{X}^n(s^n),\mathcal{Y}^n(s^n)) \\
&= \mathcal{Z}_3(s) - \mathcal{Z}^n_3(s^n) \\
&= U_1(\mathcal{X}(s)) - U^n_1(\mathcal{X}^n(s^n)) \\
&= U_1(X) - U^n_1(X).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|U(X,Y) - U^n(X,Y)| \\
&\leq ||U_1 - U^n_1||_{L^\infty([X_l,X_r])} + B_5|\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)| \\
&\quad + \bigg|\int_{\mathcal{Y}(X)}^Y |U_Y(X,\tilde{Y}) - U^n_Y(X,\tilde{Y})|\,d\tilde{Y}\bigg|,
\end{aligned}
$$

where we used (7.56). In the new notation this implies

$$
\begin{aligned}
(7.65)\quad &|Z_3(X,Y) - Z^n_3(X,Y)| \\
&\leq ||U_1 - U^n_1||_{L^\infty([X_l,X_r])} + B_5|\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)| \\
&\quad + \bigg|\int_{\mathcal{Y}(X)}^Y \Big[|Z_3 - Z^n_3| + \sum_{i=1}^5\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\Big](X,\tilde{Y})\,d\tilde{Y}\bigg|.
\end{aligned}
$$

From (7.61)–(7.64) we get by using Hölder's inequality[8],

$$(7.66)\quad \big[Z_{j,X}(X,Y) - Z^n_{j,X}(X,Y)\big]^2$$

[8]The factor 3 comes from that we first split the right-hand side in three terms, the factor $33 = 3\cdot 11$ comes from splitting the 11 terms in the integral.

$$\leq 3\big[\mathcal{V}_j(X)-\mathcal{V}_j^n(X)\big]^2+3D^2\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2$$
$$+33C_j^2|Y-\mathcal{Y}(X)|\bigg|\int_{\mathcal{Y}(X)}^{Y}\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](X,\tilde{Y})\,d\tilde{Y}\bigg|$$

and

$$\begin{aligned}(7.67)\quad &\big[Z_{j,Y}(X,Y)-Z_{j,Y}^n(X,Y)\big]^2\\ &\leq 3\big[\mathcal{W}_j(Y)-\mathcal{W}_j^n(Y)\big]^2+3D^2\big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2\\ &\quad+33C_j^2|\mathcal{X}(Y)-X|\bigg|\int_{X}^{\mathcal{X}(Y)}\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](\tilde{X},Y)\,d\tilde{X}\bigg|\end{aligned}$$

for $j=1,\dots,5$. Similarly, from (7.65) we get

$$\begin{aligned}(7.68)\quad &\big[Z_3(X,Y)-Z_3^n(X,Y)\big]^2\\ &\leq 3||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+3B_5^2\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2\\ &\quad+33|Y-\mathcal{Y}(X)|\bigg|\int_{\mathcal{Y}(X)}^{Y}\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](X,\tilde{Y})\,d\tilde{Y}\bigg|.\end{aligned}$$

We combine (7.66)-(7.68) and get for all $(X,Y)\in\Omega$ that

$$\begin{aligned}(7.69)\quad &\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](X,Y)\\ &\leq 3||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+3(B_5^2+5D^2)\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2\\ &\quad+15D^2\big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2\\ &\quad+3\sum_{j=1}^{5}\big[\mathcal{V}_j(X)-\mathcal{V}_j^n(X)\big]^2+3\sum_{j=1}^{5}\big[\mathcal{W}_j(Y)-\mathcal{W}_j^n(Y)\big]^2\\ &\quad+33(1+C)|Y-\mathcal{Y}(X)|\bigg|\int_{\mathcal{Y}(X)}^{Y}\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](X,\tilde{Y})\,d\tilde{Y}\bigg|\\ &\quad+33C|\mathcal{X}(Y)-X|\bigg|\int_{X}^{\mathcal{X}(Y)}\bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\bigg](\tilde{X},Y)\,d\tilde{X}\bigg|,\end{aligned}$$

where we introduced $C = \sum_{j=1}^{5} C_j^2$.

At this point we need the following Gronwall inequality, which can be found in [6], see chapter "Gronwall inequalities in higher dimensions". For completeness, we state and prove the inequality in the following lemma.

Lemma 7.4. *Consider a nonnegative function $u(x,y)$ in the region $x \geq 0$ and $y \geq 0$. Assume that*

$$u(x,y) \leq c + a\int_0^x u(r,y)\,dr + b\int_0^y u(x,s)\,ds$$

where a, b and c are nonnegative constants. Then we have

$$u(x,y) \leq ce^{2ax+2by+abxy}.$$

Proof. Set

$$G(x,y) = \int_0^y u(x,s)\,ds.$$

We have $G_y(x,y) = u(x,y)$, so that

$$G_y(x,y) \leq c + a\int_0^x u(r,y)\,dr + bG(x,y)$$

and

$$\frac{d}{dy}\Big(G(x,y)e^{-by}\Big) \leq \Big(c + a\int_0^x u(r,y)\,dr\Big)e^{-by}.$$

Integration yields

$$G(x,y) \leq \frac{c}{b}(e^{by}-1) + a\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds. \tag{7.70}$$

Therefore we get

$$\begin{aligned} u(x,y) &\leq c + a\int_0^x u(r,y)\,dr + bG(x,y) \\ &\leq ce^{by} + a\int_0^x u(r,y)\,dr + ab\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds. \end{aligned}$$

Denote

$$g(x,y) = ce^{by} + a\int_0^x u(r,y)\,dr + ab\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds$$

which implies

$$u(x,y) \leq g(x,y). \tag{7.71}$$

From (7.70) we have

$$\begin{aligned} G(x,y) &\leq \frac{c}{b}(e^{by}-1) + a\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds \\ &\leq \frac{1}{b}\Big(ce^{by} + ab\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds\Big) \\ &\leq \frac{1}{b}\Big(ce^{by} + a\int_0^x u(r,y)\,dr + ab\int_0^y\int_0^x u(r,s)e^{b(y-s)}\,dr\,ds\Big) \\ &= \frac{1}{b}g(x,y). \end{aligned} \tag{7.72}$$

We compute

$$g_x(x,y) = au(x,y) + ab\int_0^y u(x,s)e^{b(y-s)}\,ds.$$

Integration by parts yields

$$g_x(x,y) = au(x,y) + ab\int_0^y u(x,s)\,ds + ab^2\int_0^y\int_0^s u(x,l)e^{b(y-s)}\,dl\,ds. \tag{7.73}$$

In the last integral on the right-hand side we use

$$\int_0^s u(x,l)\,dl = G(x,s) \leq \frac{c}{b}(e^{bs}-1) + a\int_0^s\int_0^x u(r,t)e^{b(s-t)}\,dr\,dt$$

and get

$$\begin{aligned}
&\int_0^y\int_0^s u(x,l)e^{b(y-s)}\,dl\,ds \\
&\leq \int_0^y e^{b(y-s)}\bigg(\frac{c}{b}(e^{bs}-1) + a\int_0^s\int_0^x u(r,t)e^{b(s-t)}\,dr\,dt\bigg)ds \\
&\leq \int_0^y e^{b(y-s)}\bigg(\frac{c}{b}e^{bs} + a\int_0^y\int_0^x u(r,t)e^{b(s-t)}\,dr\,dt\bigg)ds \\
&= \int_0^y \bigg(\frac{c}{b}e^{by} + a\int_0^y\int_0^x u(r,t)e^{b(y-t)}\,dr\,dt\bigg)ds \\
&= \frac{c}{b}ye^{by} + ay\int_0^y\int_0^x u(r,t)e^{b(y-t)}\,dr\,dt.
\end{aligned} \tag{7.74}$$

Using the estimates (7.71), (7.72) and (7.74) in (7.73) gives

$$\begin{aligned}
g_x(x,y) &\leq ag(x,y) + ab\frac{1}{b}g(x,y) + abcye^{by} + a^2b^2y\int_0^y\int_0^x u(r,t)e^{b(y-t)}\,dr\,dt \\
&= 2ag(x,y) + aby\bigg(ce^{by} + ab\int_0^y\int_0^x u(r,t)e^{b(y-t)}\,dr\,dt\bigg) \\
&\leq 2ag(x,y) + aby\bigg(ce^{by} + a\int_0^x u(r,y)\,dr + ab\int_0^y\int_0^x u(r,t)e^{b(y-t)}\,dr\,dt\bigg) \\
&= 2ag(x,y) + abyg(x,y).
\end{aligned}$$

Integration leads to

$$g(x,y) \leq g(0,y)e^{2ax+abxy} = ce^{2ax+by+abxy}$$

and using (7.71) finally implies

$$u(x,y) \leq ce^{2ax+by+abxy}.$$

If we instead considered

$$\tilde{G}(x,y) = \int_0^x u(r,y)\,dr$$

we would have ended up with

$$u(x,y) \leq ce^{ax+2by+abxy}.$$

□

We return to (7.69). Using Lemma 7.4 we get

$$\begin{aligned}
&\Big[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Big](X,Y)\\
&\leq\Big\{3||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}\\
&\quad+3(B_5^2+5D^2)\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2\\
&\quad+15D^2\big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2\\
&\quad+3\sum_{j=1}^{5}\big[\mathcal{V}_j(X)-\mathcal{V}_j^n(X)\big]^2+3\sum_{j=1}^{5}\big[\mathcal{W}_j(Y)-\mathcal{W}_j^n(Y)\big]^2\Big\}\\
&\quad\times\exp\Big\{66C\big[\mathcal{X}(Y)-X\big]^2+66(1+C)\big[Y-\mathcal{Y}(X)\big]^2\\
&\qquad+33^2C(1+C)\big[\mathcal{X}(Y)-X\big]^2\big[Y-\mathcal{Y}(X)\big]^2\Big\}.
\end{aligned}\tag{7.75}$$

Since all the differences appearing in the exponential function are bounded by either X_r-X_l or Y_r-Y_l we can find a new constant K which depends on κ, k_1, k_2, $|||\Theta|||_{\mathcal{G}(\Omega)}$ and the size of Ω such that

$$\begin{aligned}
&\Big[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^{5}\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Big](X,Y)\\
&\leq K\Big\{||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+\sum_{j=1}^{5}\Big(\big[\mathcal{V}_j(X)-\mathcal{V}_j^n(X)\big]^2+\big[\mathcal{W}_j(Y)-\mathcal{W}_j^n(Y)\big]^2\Big)\\
&\qquad+\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2+\big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2\Big\},
\end{aligned}$$

and we have proved the claim (7.45).

From (7.45) we obtain an estimate for the difference $Z_i(X,2s-X)-Z_i^n(X,2s-X)$ for $i=1,\dots,4$. We have

$$\begin{aligned}
&Z_i(X,2s-X)-Z_i^n(X,2s-X)\\
&=Z_i(\mathcal{X}(s),2s-\mathcal{X}(s))+\int_{\mathcal{X}(s)}^{X}(Z_{i,X}-Z_{i,Y})(\xi,2s-\xi)\,d\xi\\
&\quad-Z_i^n(\mathcal{X}^n(s),2s-\mathcal{X}^n(s))-\int_{\mathcal{X}^n(s)}^{X}(Z_{i,X}^n-Z_{i,Y}^n)(\xi,2s-\xi)\,d\xi\\
&=\mathcal{Z}_i(s)-\mathcal{Z}_i^n(s)+\int_{\mathcal{X}(s)}^{X}\big[(Z_{i,X}-Z_{i,X}^n)-(Z_{i,Y}-Z_{i,Y}^n)\big](\xi,2s-\xi)\,d\xi\\
&\quad-\int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)}(Z_{i,X}^n-Z_{i,Y}^n)(\xi,2s-\xi)\,d\xi,
\end{aligned}$$

which implies by (7.55), (7.56), and the Cauchy–Schwarz inequality, that

$$
\begin{aligned}
(7.76)\quad &\big|Z_i(X,2s-X)-Z_i^n(X,2s-X)\big|\\
&\leq ||\mathcal{Z}_i-\mathcal{Z}_i^n||_{L^\infty([s_l,s_r])}+(B_4+B_5)||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}\\
&\quad+\big[X-\mathcal{X}(s)\big]^{\frac{1}{2}}\Bigg\{\Big|\int_{\mathcal{X}(s)}^X (Z_{i,X}-Z_{i,X}^n)^2(\xi,2s-\xi)\,d\xi\Big|^{\frac{1}{2}}\\
&\qquad\qquad+\Big|\int_{\mathcal{X}(s)}^X (Z_{i,Y}-Z_{i,Y}^n)^2(\xi,2s-\xi)\,d\xi\Big|^{\frac{1}{2}}\Bigg\}\\
&\leq ||\mathcal{Z}_i-\mathcal{Z}_i^n||_{L^\infty([s_l,s_r])}+(B_4+B_5)||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}\\
&\quad+2(X_r-X_l)^{\frac{1}{2}}\Bigg|\int_{\mathcal{X}(s)}^X\Bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^5\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2\\
&\qquad\qquad+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Bigg](\xi,2s-\xi)\,d\xi\Bigg|^{\frac{1}{2}}.
\end{aligned}
$$

From (7.45), we have

$$
\begin{aligned}
&\Bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^5\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Bigg](\xi,2s-\xi)\\
&\leq K\Bigg\{||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+\sum_{j=1}^5\Big(\big[\mathcal{V}_j(\xi)-\mathcal{V}_j^n(\xi)\big]^2+\big[\mathcal{W}_j(2s-\xi)-\mathcal{W}_j^n(2s-\xi)\big]^2\Big)\\
&\qquad+\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\\
&\qquad+\big[\mathcal{X}\circ\mathcal{Y}^{-1}(2s-\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(2s-\xi)\big]^2\Bigg\}.
\end{aligned}
$$

Integration and a change of variables leads to

$$
\begin{aligned}
(7.77)\quad &\Bigg|\int_{\mathcal{X}(s)}^X\Bigg[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^5\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Bigg](\xi,2s-\xi)\,d\xi\Bigg|\\
&\leq K\Bigg\{|X-\mathcal{X}(s)|||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}\\
&\qquad+\sum_{j=1}^5\Bigg(\Big|\int_{\mathcal{X}(s)}^X\big[\mathcal{V}_j(\xi)-\mathcal{V}_j^n(\xi)\big]^2\,d\xi\Big|+\Big|\int_{2s-X}^{\mathcal{Y}(s)}\big[\mathcal{W}_j(\xi)-\mathcal{W}_j^n(\xi)\big]^2\,d\xi\Big|\Bigg)\\
&\qquad+\Big|\int_{\mathcal{X}(s)}^X\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\Big|\\
&\qquad+\Big|\int_{2s-X}^{\mathcal{Y}(s)}\big[\mathcal{X}\circ\mathcal{Y}^{-1}(\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(\xi)\big]^2\,d\xi\Big|\Bigg\}\\
&\leq K\Bigg\{(X_r-X_l)||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}
\end{aligned}
$$

$$+\sum_{j=1}^{5}\Big(||\mathcal{V}_j-\mathcal{V}_j^n||^2_{L^2([X_l,X_r])}+||\mathcal{W}_j-\mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\Big)$$
$$+\int_{X_l}^{X_r}\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi$$
$$+\int_{Y_l}^{Y_r}\big[\mathcal{X}\circ\mathcal{Y}^{-1}(\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(\xi)\big]^2\,d\xi\Big\}.$$

From (2.40c) we have $X+\mathcal{Y}\circ\mathcal{X}^{-1}(X)=2\mathcal{X}^{-1}(X)$ and $X+\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)=2(\mathcal{X}^n)^{-1}(X)$, so that

$$\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)=2(\mathcal{X}^{-1}(X)-(\mathcal{X}^n)^{-1}(X)). \tag{7.78}$$

This leads to

$$\begin{aligned}\int_{X_l}^{X_r}&\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\\ &=4\int_{X_l}^{X_r}\big[\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\\ &\le 4(s_r-s_l)\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)|\,d\xi.\end{aligned} \tag{7.79}$$

To estimate the above integral, we need to introduce another sequence of curves on $[s_l,s_r]$ given by $(\hat{\mathcal{X}}^n(s),\hat{\mathcal{Y}}^n(s))=(\max\{\mathcal{X}(s),\mathcal{X}^n(s)\},\min\{\mathcal{Y}(s),\mathcal{Y}^n(s)\})$. This sequence satisfies that both $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{X}^n,\mathcal{Y}^n)$ lie above or are equal to $(\hat{\mathcal{X}}^n,\hat{\mathcal{Y}}^n)$. This implies that $\mathcal{X}^{-1}(X)\ge(\hat{\mathcal{X}}^n)^{-1}(X)$ and $(\mathcal{X}^n)^{-1}(X)\ge(\hat{\mathcal{X}}^n)^{-1}(X)$ for all X in $[X_l,X_r]$, and that

$$\begin{aligned}\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)|\,d\xi=&\int_{X_l}^{X_r}(\mathcal{X}^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi))\,d\xi\\ &+\int_{X_l}^{X_r}((\mathcal{X}^n)^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi))\,d\xi.\end{aligned} \tag{7.80}$$

By a change of variables and integration by parts we get

$$\int_{X_l}^{X_r}\mathcal{X}^{-1}(\xi)\,d\xi=\int_{\mathcal{X}^{-1}(X_l)}^{\mathcal{X}^{-1}(X_r)}s\dot{\mathcal{X}}(s)\,ds=X_r\mathcal{X}^{-1}(X_r)-X_l\mathcal{X}^{-1}(X_l)-\int_{\mathcal{X}^{-1}(X_l)}^{\mathcal{X}^{-1}(X_r)}\mathcal{X}(s)\,ds,$$

and similarly we find

$$\int_{X_l}^{X_r}(\hat{\mathcal{X}}^n)^{-1}(\xi)\,d\xi=X_r(\hat{\mathcal{X}}^n)^{-1}(X_r)-X_l(\hat{\mathcal{X}}^n)^{-1}(X_l)-\int_{(\hat{\mathcal{X}}^n)^{-1}(X_l)}^{(\hat{\mathcal{X}}^n)^{-1}(X_r)}\hat{\mathcal{X}}^n(s)\,ds.$$

Note that

$$\mathcal{X}^{-1}(X_l)=s_l=(\hat{\mathcal{X}}^n)^{-1}(X_l)\quad\text{and}\quad\mathcal{X}^{-1}(X_r)=s_r=(\hat{\mathcal{X}}^n)^{-1}(X_r).$$

Therefore,

$$\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi)|\,d\xi=\int_{s_l}^{s_r}(\hat{\mathcal{X}}^n(s)-\mathcal{X}(s))\,ds.$$

We obtain in a similar way,

$$\int_{X_l}^{X_r} |(\mathcal{X}^n)^{-1}(\xi) - (\hat{\mathcal{X}}^n)^{-1}(\xi)|\, d\xi = \int_{s_l}^{s_r} (\hat{\mathcal{X}}^n(s) - \mathcal{X}^n(s))\, ds.$$

Combining (7.80) and the above estimates we end up with

$$\begin{aligned}\int_{X_l}^{X_r} |\mathcal{X}^{-1}(\xi) - (\mathcal{X}^n)^{-1}(\xi)|\, d\xi &= \int_{s_l}^{s_r} (\hat{\mathcal{X}}^n(s) - \mathcal{X}(s))\, ds + \int_{s_l}^{s_r} (\hat{\mathcal{X}}^n(s) - \mathcal{X}^n(s))\, ds \\ &= \int_{s_l}^{s_r} |\mathcal{X}^n(s) - \mathcal{X}(s)|\, ds,\end{aligned}$$

which inserted in (7.79) yields

$$\int_{X_l}^{X_r} \big[\mathcal{Y} \circ \mathcal{X}^{-1}(\xi) - \mathcal{Y}^n \circ (\mathcal{X}^n)^{-1}(\xi)\big]^2 d\xi \leq 4(s_r - s_l)^2 ||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])}. \tag{7.81}$$

By similar calculations we find

$$\int_{Y_l}^{Y_r} \big[\mathcal{X} \circ \mathcal{Y}^{-1}(\xi) - \mathcal{X}^n \circ (\mathcal{Y}^n)^{-1}(\xi)\big]^2 d\xi \leq 4(s_r - s_l)^2 ||\mathcal{Y} - \mathcal{Y}^n||_{L^\infty([s_l,s_r])}. \tag{7.82}$$

Returning to (7.77) we now get

$$\begin{aligned}&\Big| \int_{\mathcal{X}(s)}^{X} \Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^{5} \Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](\xi, 2s - \xi)\, d\xi\Big| \\ &\leq K\Big\{(X_r - X_l)||U_1 - U_1^n||^2_{L^\infty([X_l,X_r])} \\ &\qquad + \sum_{j=1}^{5} \Big(||\mathcal{V}_j - \mathcal{V}_j^n||^2_{L^2([X_l,X_r])} + ||\mathcal{W}_j - \mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\Big) \\ &\qquad + 4(s_r - s_l)^2\Big(||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])} + ||\mathcal{Y} - \mathcal{Y}^n||_{L^\infty([s_l,s_r])}\Big)\Big\},\end{aligned}$$

which we insert in (7.76) and get

$$\begin{aligned}&\big|Z_i(X, 2s - X) - Z_i^n(X, 2s - X)\big| \\ &\leq ||\mathcal{Z}_i - \mathcal{Z}_i^n||_{L^\infty([s_l,s_r])} + (B_4 + B_5)||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])} \\ &\quad + 2\sqrt{K(X_r - X_l)}\Big\{(X_r - X_l)||U_1 - U_1^n||^2_{L^\infty([X_l,X_r])} \\ &\qquad + \sum_{j=1}^{5} \Big(||\mathcal{V}_j - \mathcal{V}_j^n||^2_{L^2([X_l,X_r])} + ||\mathcal{W}_j - \mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\Big) \\ &\qquad + 4(s_r - s_l)^2\Big(||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])} + ||\mathcal{Y} - \mathcal{Y}^n||_{L^\infty([s_l,s_r])}\Big)\Big\}^{\frac{1}{2}}\end{aligned} \tag{7.83}$$

for $i = 1, \dots, 4$. A similar inequality is valid for

$$\big|(Z_5(X, 2s - X) - Z_5(X_l, 2s_l - X_l)) - (Z_5^n(X, 2s - X) - Z_5^n(X_l, 2s_l - X_l))\big|,$$

the only difference from (7.83) being that we get $||(\mathcal{Z}_5-\mathcal{Z}_5(s_l))-(\mathcal{Z}_5^n-\mathcal{Z}_5^n(s_l))||_{L^\infty([s_l,s_r])}$ on the right-hand side.

Step 4. For any $0<\tau\leq\frac{1}{2\kappa}(x_r-x_l)$ consider

$$\Theta(\tau)=\mathbf{E}\circ\mathbf{t}_\tau(Z,p,q),\quad \Theta^n(\tau)=\mathbf{E}\circ\mathbf{t}_\tau(Z^n,p^n,q^n),$$
$$(u,R,S,\rho,\sigma,\mu,\nu)(\tau)=\mathbf{M}\circ\mathbf{D}(\Theta(\tau))$$

and

$$(u^n,R^n,S^n,\rho^n,\sigma^n,\mu^n,\nu^n)(\tau)=\mathbf{M}\circ\mathbf{D}(\Theta^n(\tau)).$$

We prove (P1”).

A close inspection of the proof of (7.21) reveals that there exist $\bar{s}_1$ and $\bar{s}_2$ such that $s_l<\bar{s}_1\leq\bar{s}_2<s_r$, $(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in\Omega$ for all $s\in[\bar{s}_1,\bar{s}_2]$ and

$$x(\mathcal{X}(\tau,\bar{s}_1),\mathcal{Y}(\tau,\bar{s}_1))=x_l+\kappa\tau\quad\text{and}\quad x(\mathcal{X}(\tau,\bar{s}_2),\mathcal{Y}(\tau,\bar{s}_2))=x_r-\kappa\tau.$$

Moreover, there exist $\bar{s}_1^n$ and $\bar{s}_2^n$ such that $s_l<\bar{s}_1^n\leq\bar{s}_2^n<s_r$, $(\mathcal{X}^n(\tau,s),\mathcal{Y}^n(\tau,s))\in\Omega$ for all $s\in[\bar{s}_1^n,\bar{s}_2^n]$ and

$$x^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))=x_l+\kappa\tau\quad\text{and}\quad x^n(\mathcal{X}^n(\tau,\bar{s}_2^n),\mathcal{Y}^n(\tau,\bar{s}_2^n))=x_r-\kappa\tau.$$

Consider $z\in[x_l+\kappa\tau,x_r-\kappa\tau]$. There are $s\in[\bar{s}_1,\bar{s}_2]$ and $s^n\in[\bar{s}_1^n,\bar{s}_2^n]$ such that $z=\mathcal{Z}_2(\tau,s)=\mathcal{Z}_2^n(\tau,s^n)$. Using (5.21a) we obtain

$$\begin{aligned}(7.84)\qquad u(\tau,z)-u^n(\tau,z)&=\mathcal{Z}_3(\tau,s)-\mathcal{Z}_3^n(\tau,s^n)\\&=U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n)).\end{aligned}$$

Now we can have several different scenarios depending on the order of the points s and s^n, and the intervals $[\bar{s}_1,\bar{s}_2]$ and $[\bar{s}_1^n,\bar{s}_2^n]$. We only show one of the challenging cases, the others can be treated in a similar way. Assume that $s\leq\bar{s}_1^n\leq\bar{s}_{\max}$. For the definition of $\bar{s}_{\max}$ see Step 1 (iv) in the proof of Theorem 7.1. Observe that in this case the point $(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))$ is in Ω, but the point $(\mathcal{X}^n(\tau,s),\mathcal{Y}^n(\tau,s))$ may be outside Ω. So when estimating (7.84) we have to carefully choose points on the curve so that we do not end up outside Ω, see Figure 7.

Write[9]

$$\begin{aligned}&U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n))\\&=U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))\quad(A_1^n)\\&\quad+U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))-U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\quad(A_2^n)\\&\quad+U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))-U^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\quad(A_3^n)\\&\quad+U^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n))\quad(A_4^n).\end{aligned}$$

By (3.23) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}(7.85)\qquad |A_1^n|&=|\mathcal{Z}_3(\tau,\bar{s}_1^n)-\mathcal{Z}_3(\tau,s)|\\&=\left|\int_s^{\bar{s}_1^n}\dot{\mathcal{Z}}_3(\tau,r)\,dr\right|\\&=\left|\int_s^{\bar{s}_1^n}[\mathcal{V}_3(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)+\mathcal{W}_3(\tau,\mathcal{Y}(\tau,r))\dot{\mathcal{Y}}(\tau,r)]\,dr\right|\end{aligned}$$

[9]In the case when $\bar{s}_1^n\geq\bar{s}_{\max}$ the point $(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))$ may be outside Ω. In that case the proof can be done in a similar way by replacing $\bar{s}_1^n$ with $\bar{s}_{\max}$.

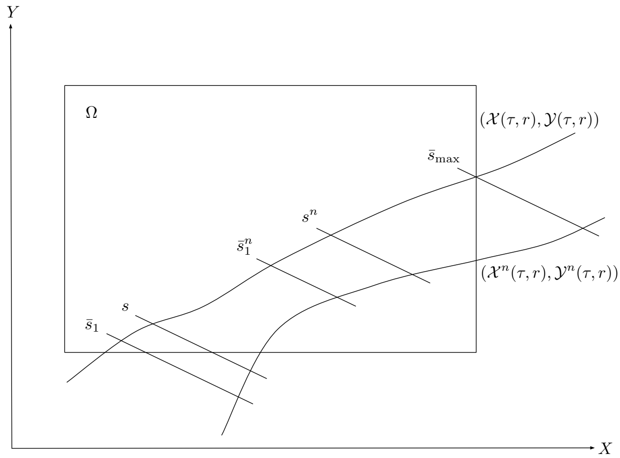

FIGURE 7. An example of the situation described in Step 4. Note that the point $(\mathcal{X}^n(\tau,s),\mathcal{Y}^n(\tau,s))$ lies outside Ω.

$$\leq \Big(\int_s^{\bar{s}_1^n} \dot{\mathcal{X}}(\tau,r)\,dr\Big)^{\frac{1}{2}}\Big(\int_s^{\bar{s}_1^n} \mathcal{V}_3^2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr\Big)^{\frac{1}{2}}$$
$$+\Big(\int_s^{\bar{s}_1^n} \dot{\mathcal{Y}}(\tau,r)\,dr\Big)^{\frac{1}{2}}\Big(\int_s^{\bar{s}_1^n} \mathcal{W}_3^2(\tau,\mathcal{Y}(\tau,r))\dot{\mathcal{Y}}(\tau,r)\,dr\Big)^{\frac{1}{2}}.$$

From (3.24c) and (3.27) we get

$$\begin{aligned}
(7.86)\qquad 0 &\leq \int_s^{\bar{s}_1^n} \mathcal{V}_3^2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr\\
&= \int_s^{\bar{s}_1^n} \Big(\frac{2\mathcal{V}_2(\tau,\mathcal{X}(\tau,r))\mathcal{V}_4(\tau,\mathcal{X}(\tau,r))}{c^2(\mathcal{Z}_3(\tau,r))} - \frac{\mathfrak{p}^2(\tau,\mathcal{X}(\tau,r))}{c(\mathcal{Z}_3(\tau,r))}\Big)\dot{\mathcal{X}}(\tau,r)\,dr\\
&\leq \int_s^{\bar{s}_1^n} \frac{2\mathcal{V}_2(\tau,\mathcal{X}(\tau,r))\mathcal{V}_4(\tau,\mathcal{X}(\tau,r))}{c^2(\mathcal{Z}_3(\tau,r))}\dot{\mathcal{X}}(\tau,r)\,dr\\
&\leq \kappa^2 B_6 \int_s^{\bar{s}_1^n} 2\mathcal{V}_2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr\\
&= \kappa^2 B_6 \int_s^{\bar{s}_1^n} \dot{\mathcal{Z}}_2(\tau,r)\,dr\\
&= \kappa^2 B_6\big[\mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2(\tau,s)\big]\\
&= \kappa^2 B_6\big[\mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,\bar{s}_1^n) + \mathcal{Z}_2^n(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,s^n)\big]\\
&\leq \kappa^2 B_6\big[\mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,\bar{s}_1^n)\big],
\end{aligned}$$

where we used that $\mathcal{Z}_2(\tau,s) = \mathcal{Z}_2^n(\tau,s^n)$ and $\mathcal{Z}_2^n(\tau,\bar{s}_1^n) \leq \mathcal{Z}_2^n(\tau,s^n)$ since $\bar{s}_1^n \leq s^n$ and $\mathcal{Z}_2^n(\tau,\cdot)$ is nondecreasing. We also used that since $(\mathcal{X}(\tau,r),\mathcal{Y}(\tau,r)) \in \Omega$ for $r \in [s,\bar{s}_1^n]$ we can use the estimate in (7.57) to obtain $|\mathcal{V}_4(\tau,\mathcal{X}(\tau,r))| = |J_X(\mathcal{X}(\tau,r),\mathcal{Y}(\tau,r))| \leq$

B_6. We have

$$
\begin{aligned}
&\mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,\bar{s}_1^n) \\
&= x(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - x^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \\
&= x(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - x(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \\
&\quad + x(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) - x^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).
\end{aligned} \tag{7.87}
$$

By (4.12a) and since $t_X \geq 0$ and $t_Y \leq 0$ we get

$$
\begin{aligned}
&|x(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - x(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))| \\
&= \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (x_X - x_Y)(X, 2\bar{s}_1^n - X)\,dX \right| \\
&= \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} [c(U)(t_X + t_Y)](X, 2\bar{s}_1^n - X)\,dX \right| \\
&\leq \kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (t_X - t_Y)(X, 2\bar{s}_1^n - X)\,dX \right| \\
&= \kappa |t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))|.
\end{aligned}
$$

Since $t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) = \tau = t^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))$ we have

$$
\begin{aligned}
&t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \\
&= t^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).
\end{aligned} \tag{7.88}
$$

Therefore,

$$
\begin{aligned}
&|x(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - x(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))| \\
&\leq \kappa |t - t^n|(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))
\end{aligned}
$$

and (7.87) yields

$$
0 \leq \mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,\bar{s}_1^n) \leq \big(|x - x^n| + \kappa|t - t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)). \tag{7.89}
$$

We insert this in (7.86) and get

$$
0 \leq \int_s^{\bar{s}_1^n} \mathcal{V}_3^2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr \leq \kappa^2 B_6\big(|x - x^n| + \kappa|t - t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).
$$

Similarly we get

$$
0 \leq \int_s^{\bar{s}_1^n} \mathcal{W}_3^2(\tau,\mathcal{Y}(\tau,r))\dot{\mathcal{Y}}(\tau,r)\,dr \leq \kappa^2 B_7\big(|x - x^n| + \kappa|t - t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).
$$

We have

$$
\int_s^{\bar{s}_1^n} \dot{\mathcal{X}}(\tau,r)\,dr = \mathcal{X}(\tau,\bar{s}_1^n) - \mathcal{X}(\tau,s) \leq X_r - X_l
$$

and

$$
\int_s^{\bar{s}_1^n} \dot{\mathcal{Y}}(\tau,r)\,dr \leq Y_r - Y_l.
$$

Using these estimates in (7.85) gives

$$
|A_1^n| \leq v_1\big(|x - x^n| + \kappa|t - t^n|\big)^{\frac{1}{2}}(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.90}
$$

for a constant v_1 that is independent of n.

By (4.12c) we have the estimates

$$|U_X| \leq \frac{\sqrt{2J_X x_X}}{c(U)} \quad \text{and} \quad |U_Y| \leq \frac{\sqrt{2J_Y x_Y}}{c(U)}$$

which leads to

$$\begin{aligned}
|A_2^n| &\leq |U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))| \\
&= \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (U_X - U_Y)(X, 2\bar{s}_1^n - X)\, dX \right| \\
&\leq \sqrt{2}\kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} \left(\sqrt{J_X x_X} + \sqrt{J_Y x_Y}\right)(X, 2\bar{s}_1^n - X)\, dX \right| \\
&\leq \sqrt{2}\kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} J_X(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}} \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} x_X(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}} \\
&\quad + \sqrt{2}\kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} J_Y(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}} \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} x_Y(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}} \\
&\leq 2\sqrt{2}\kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (J_X + J_Y)(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}} \\
&\quad \times \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (x_X + x_Y)(X, 2\bar{s}_1^n - X)\, dX \right|^{\frac{1}{2}}
\end{aligned}$$

since $x_Y \geq 0$ and $J_Y \geq 0$. From (4.12a) we get

$$\begin{aligned}
&\left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (x_X + x_Y)(X, 2\bar{s}_1^n - X)\, dX \right| \\
&= \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} \left[c(U)(t_X - t_Y)\right](X, 2\bar{s}_1^n - X)\, dX \right| \\
&\leq \kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (t_X - t_Y)(X, 2\bar{s}_1^n - X)\, dX \right| \\
&= \kappa |t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))| \\
&= \kappa |t - t^n|(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)),
\end{aligned}$$

where the last equality follows from (7.88). Using (7.57) and (7.58) yields

$$\begin{aligned}
&\left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (J_X + J_Y)(X, 2\bar{s}_1^n - X)\, dX \right| \\
&\leq (B_6 + B_7)|\mathcal{X}(\tau,\bar{s}_1^n) - \mathcal{X}^n(\tau,\bar{s}_1^n)| \\
&\leq (B_6 + B_7)(X_r - X_l).
\end{aligned}$$

Now we get

$$|A_2^n| \leq v_2|t-t^n|^{\frac{1}{2}}(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.91}$$

for a constant v_2 which is independent of n.

By estimating as we did for A_1^n we get

$$\begin{aligned}|A_4^n| \leq & \Big(\int_{\bar{s}_1^n}^{s^n} \dot{\mathcal{X}}^n(\tau,r)\,dr\Big)^{\frac{1}{2}}\Big(\int_{\bar{s}_1^n}^{s^n} (\mathcal{V}_3^n)^2(\tau,\mathcal{X}^n(\tau,r))\dot{\mathcal{X}}^n(\tau,r)\,dr\Big)^{\frac{1}{2}} \\ &+ \Big(\int_{\bar{s}_1^n}^{s^n} \dot{\mathcal{Y}}^n(\tau,r)\,dr\Big)^{\frac{1}{2}}\Big(\int_{\bar{s}_1^n}^{s^n} (\mathcal{W}_3^n)^2(\tau,\mathcal{Y}^n(\tau,r))\dot{\mathcal{Y}}^n(\tau,r)\,dr\Big)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}0 &\leq \int_{\bar{s}_1^n}^{s^n} (\mathcal{V}_3^n)^2(\tau,\mathcal{X}^n(\tau,r))\dot{\mathcal{X}}^n(\tau,r)\,dr \\ &\leq \kappa^2 B_4\big[\mathcal{Z}_2^n(\tau,s^n)-\mathcal{Z}_2^n(\tau,\bar{s}_1^n)\big] \\ &= \kappa^2 B_4\big[\mathcal{Z}_2(\tau,s)-\mathcal{Z}_2(\tau,\bar{s}_1^n)+\mathcal{Z}_2(\tau,\bar{s}_1^n)-\mathcal{Z}_2^n(\tau,\bar{s}_1^n)\big] \\ &\leq \kappa^2 B_4\big[\mathcal{Z}_2(\tau,\bar{s}_1^n)-\mathcal{Z}_2^n(\tau,\bar{s}_1^n)\big],\end{aligned}$$

where we used that $\mathcal{Z}_2^n(\tau,s^n)=\mathcal{Z}_2(\tau,s)$ and $\mathcal{Z}_2(\tau,s)\leq\mathcal{Z}_2(\tau,\bar{s}_1^n)$ since $s\leq\bar{s}_1^n$ and $\mathcal{Z}_2(\tau,\cdot)$ is nondecreasing. By using (7.89) we end up with

$$|A_4^n| \leq v_3\big(|x-x^n|+\kappa|t-t^n|\big)^{\frac{1}{2}}(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.92}$$

for a constant v_3 that is independent of n.

Using (7.90), (7.91) and (7.92) in (7.84) yields

$$\begin{aligned}&|u(\tau,z)-u^n(\tau,z)| \\ &\leq \Big[(v_1+v_3)\big(|x-x^n|+\kappa|t-t^n|\big)^{\frac{1}{2}}+v_2|t-t^n|^{\frac{1}{2}}+|U-U^n|\Big](\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\end{aligned}$$

and we conclude, after using (7.83), that $u^n(\tau,\cdot)\to u(\tau,\cdot)$ in $L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

For any $z\in[x_l+\kappa\tau,x_r-\kappa\tau]$, there are $s\in[\bar{s}_1,\bar{s}_2]$ and $s^n\in[\bar{s}_1^n,\bar{s}_2^n]$ such that $z=\mathcal{Z}_2(\tau,s)=\mathcal{Z}_2^n(\tau,s^n)$. From (5.21a) and (5.21b) we have

$$\begin{aligned}\int_{x_l+\kappa\tau}^{z} \frac{R(\tau,y)}{c(u(\tau,y))}\,dy &= \int_{\bar{s}_1}^{s} \frac{1}{c(u(\tau,\mathcal{Z}_2(\tau,r)))}2c(\mathcal{Z}_3(\tau,r))\mathcal{V}_3(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr \\ &= \int_{\bar{s}_1}^{s} 2\mathcal{V}_3(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr,\end{aligned}$$

and similarly we have

$$\int_{x_l+\kappa\tau}^{z} \frac{S(\tau,y)}{c(u(\tau,y))}\,dy = \int_{\bar{s}_1}^{s} -2\mathcal{W}_3(\tau,\mathcal{Y}(\tau,r))\dot{\mathcal{Y}}(\tau,r)\,dr.$$

Using (3.1) and (3.23) yields

$$\begin{aligned}\int_{x_l+\kappa\tau}^{z} u_x(\tau,y)\,dy &= \int_{x_l+\kappa\tau}^{z}\Big(\frac{R-S}{2c(u)}\Big)(\tau,y)\,dy \\ &= \int_{\bar{s}_1}^{s} \dot{\mathcal{Z}}_3(\tau,r)\,dr \\ &= \mathcal{Z}_3(\tau,s)-\mathcal{Z}_3(\tau,\bar{s}_1)\end{aligned}$$

$$= u(\tau, z) - u(\tau, x_l + \kappa\tau),$$

where the last equality follows from (5.21a). According to Theorem 3.35 in [7], the function $u(\tau, \cdot)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$. We also have

$$\int_{x_l+\kappa\tau}^{z} u_x^n(\tau, y)\, dy = u^n(\tau, z) - u^n(\tau, x_l + \kappa\tau).$$

Therefore

$$\left| \int_{x_l+\kappa\tau}^{z} (u_x - u_x^n)(\tau, y)\, dy \right| \leq 2||u(\tau, \cdot) - u^n(\tau, \cdot)||_{L^\infty([x_l+\kappa\tau, x_r-\kappa\tau])},$$

so that

$$\lim_{n\to\infty} \int_{x_l+\kappa\tau}^{z} u_x^n(\tau, y)\, dy = \int_{x_l+\kappa\tau}^{z} u_x(\tau, y)\, dy.$$

Since $p_Y = p_Y^n = 0$ we have

$$\mathfrak{p}(\tau, X) = p(X, \mathcal{Y}(\tau, \mathcal{X}^{-1}(\tau, X))) = p(X, \mathcal{Y}(\mathcal{X}^{-1}(X))) = \mathfrak{p}(X)$$

and

$$\mathfrak{p}^n(\tau, X) = p^n(X, \mathcal{Y}^n(\tau, (\mathcal{X}^n)^{-1}(\tau, X))) = p^n(X, \mathcal{Y}^n((\mathcal{X}^n)^{-1}(X))) = \mathfrak{p}^n(X).$$

From Step 2 we get $\mathfrak{p}(\tau, X) = 0$ and $\mathfrak{p}^n(\tau, X) \geq k_n > 0$ for all $X \in [X_l, X_r]$. Then by (5.21d) we obtain

$$\int_{x_l+\kappa\tau}^{z} \rho(\tau, y)\, dy = \int_{\bar{s}_1}^{s} 2\mathfrak{p}(\tau, \mathcal{X}(\tau, r))\dot{\mathcal{X}}(\tau, r)\, dr = 0,$$

so that by a change of variables,

$$\left| \int_{x_l+\kappa\tau}^{z} (\rho^n - \rho)(\tau, y)\, dy \right| = \int_{\mathcal{X}^n(\tau, \bar{s}_1^n)}^{\mathcal{X}^n(\tau, s^n)} 2\mathfrak{p}^n(X)\, dX \leq 2\sqrt{X_r - X_l}||\mathfrak{p}^n||_{L^2([X_l, X_r])}.$$

Since $\mathfrak{p}^n \to 0$ in $L^2([X_l, X_r])$ this implies that

$$\lim_{n\to\infty} \int_{x_l+\kappa\tau}^{z} \rho^n(\tau, y)\, dy = \int_{x_l+\kappa\tau}^{z} \rho(\tau, y)\, dy = 0. \tag{7.93}$$

As mentioned in the comment before the proof, there is for any n a constant $\bar{d}_n > 0$ such that

$$\rho^n(\tau, z) \geq \bar{d}_n$$

for all $z \in [x_l+\kappa\tau, x_r-\kappa\tau]$. Since $\rho^n(\tau, \cdot)$ is positive, (7.93) is the same as $\rho^n(\tau, \cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$.

Similarly we obtain $\sigma^n(\tau, \cdot) \to 0$ in $L^1([x_l+\kappa\tau, x_r-\kappa\tau])$. This concludes the proof of (P2"). □

7.3. **Convergence Results.**

Lemma 7.5. *Let* $(u, R, S, \rho, \sigma, \mu, \nu)$ *and* $(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)$ *belong to* $\mathcal{D}$*, and assume that* μ*,* ν*,* μ^n *and* ν^n *are absolutely continuous. Consider*

$$\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1), \quad \psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2),$$

and

$$\psi_1^n = (x_1^n, U_1^n, J_1^n, K_1^n, V_1^n, H_1^n), \quad \psi_2^n = (x_2^n, U_2^n, J_2^n, K_2^n, V_2^n, H_2^n)$$

defined by

$$(\psi_1, \psi_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$$

and

$$(\psi_1^n, \psi_2^n) = \mathbf{L}(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n).$$

Assume that

$$u^n \to u, \quad R^n \to R, \quad S^n \to S, \quad \rho^n \to \rho \quad \text{and} \quad \sigma^n \to \sigma \quad \text{in} \quad L^2(\mathbb{R}).$$

Then, x_i *and* x_i^n *are strictly increasing, and*

$$x_i^n \to x_i, \quad (x_i^n)^{-1} \to x_i^{-1}, \quad U_i^n \to U_i, \quad J_i^n \to J_i, \quad K_i^n \to K_i \quad \text{in} \quad L^\infty(\mathbb{R}),$$
$$U_i^n \to U_i, \quad V_i^n \to V_i, \quad H_i^n \to H_i \quad \text{in} \quad L^2(\mathbb{R}),$$
$$(x_i^n)' \to x_i', \quad (J_i^n)' \to J_i', \quad (K_i^n)' \to K_i' \quad \text{in} \quad L^1(\mathbb{R})$$

for $i = 1, 2$.

We mention that the functions which converge in $L^1(\mathbb{R})$ also converge in $L^2(\mathbb{R})$, because convergence in $L^1(\mathbb{R})$ and (uniform) boundedness in $L^\infty(\mathbb{R})$ imply convergence in $L^2(\mathbb{R})$.

Let us show this in detail for $(x_1^n)' \to x_1'$. Since the measures are absolutely continuous we get as in (7.2),

$$x_1'(X) = \frac{4}{(R_0^2 + c(u_0)\rho_0^2) \circ x_1(X) + 4},$$

so that $0 \le x_1'(X) \le 1$. We also have $0 \le (x_1^n)'(X) \le 1$. We get

$$\int_{\mathbb{R}} \big(x_1'(X) - (x_1^n)'(X)\big)^2 \, dX \le \int_{\mathbb{R}} \big|x_1'(X) - (x_1^n)'(X)\big| \big(x_1'(X) + (x_1^n)'(X)\big) \, dX$$
$$\le 2 \int_{\mathbb{R}} \big|x_1'(X) - (x_1^n)'(X)\big| \, dX.$$

Therefore, $(x_1^n)' \to x_1'$ in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Notice that it was important that $(x_1^n)'$ was *uniformly* bounded in $L^\infty(\mathbb{R})$, not just that $(x_1^n)' - 1 \in L^\infty(\mathbb{R})$ for all n, which is what we get from the definition of $\mathcal{F}$.

Proof. We start by proving that $u^n \to u$ in $L^\infty(\mathbb{R})$. By (3.1) and the Cauchy Schwarz inequality, we have

$$\begin{aligned}
&(u(x) - u^n(x))^2 \\
&= 2 \int_{-\infty}^{x} (u - u^n)(u_x - u_x^n)(y) \, dy \\
&= \int_{-\infty}^{x} (u - u^n) \Big(\frac{1}{c(u)}(R - S) - \frac{1}{c(u^n)}(R^n - S^n) \Big)(y) \, dy
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} \kappa|u-u^n|(|R|+|R^n|+|S|+|S^n|)(y)\,dy\\
&\leq \kappa||u-u^n||_{L^2(\mathbb{R})}\big(||R||_{L^2(\mathbb{R})}+||R^n||_{L^2(\mathbb{R})}+||S||_{L^2(\mathbb{R})}+||S^n||_{L^2(\mathbb{R})}\big)\\
&\leq \kappa||u-u^n||_{L^2(\mathbb{R})}(2||R||_{L^2(\mathbb{R})}+||R-R^n||_{L^2(\mathbb{R})}+2||S||_{L^2(\mathbb{R})}+||S-S^n||_{L^2(\mathbb{R})}),
\end{aligned}$$

which implies, since $R^n \to R$, $S^n \to S$ and $u^n \to u$ in $L^2(\mathbb{R})$, that $u^n \to u$ in $L^\infty(\mathbb{R})$.

Let

$$\psi = (\psi_1, \psi_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$$

and

$$\psi^n = (\psi_1^n, \psi_2^n) = \mathbf{L}(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n).$$

We will only prove the results for the components of ψ_1, the proof for ψ_2 can be done in a similar way.

Let $f = \frac{1}{4}(R^2 + c(u)\rho^2)$ and $f^n = \frac{1}{4}((R^n)^2 + c(u^n)(\rho^n)^2)$. From (3.2) we have $\mu((-\infty, x)) = \int_{-\infty}^x f(z)\,dz$ and $\mu^n((-\infty,x)) = \int_{-\infty}^x f^n(z)\,dz$, since μ and μ^n are absolutely continuous measures. Consider the functions

$$F(x) = x + \int_{-\infty}^x f(z)\,dz \quad \text{and} \quad F^n(x) = x + \int_{-\infty}^x f^n(z)\,dz,$$

which are strictly increasing and continuous. By (3.12a), we have that $F(x_1(X)) = X$ and $F^n(x_1^n(X)) = X$ for all $X \in \mathbb{R}$, which implies that x_1 and x_1^n are strictly increasing. Since $x_1^{-1} = F$ and $(x_1^n)^{-1} = F^n$, the inverses x_1^{-1} and $(x_1^n)^{-1}$ exist and are strictly increasing. We prove that $(x_1^n)^{-1} \to x_1^{-1}$ in $L^\infty(\mathbb{R})$. Since

$$|x_1^{-1}(x) - (x_1^n)^{-1}(x)| \leq ||f - f^n||_{L^1(\mathbb{R})}$$

we have to show that $f^n \to f$ in $L^1(\mathbb{R})$. By using the estimate

$$\left|\frac{1}{c(u)} - \frac{1}{c(u^n)}\right| = \left|\int_{u^n}^u -\frac{c'(w)}{c^2(w)}\,dw\right| \leq \kappa^2 k_1|u-u^n| \tag{7.94}$$

and the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
||f-f^n||_{L^1(\mathbb{R})} &= \frac{1}{4}\int_{\mathbb{R}} |R^2-(R^n)^2+\rho^2(c(u)-c(u^n))+c(u^n)(\rho^2-(\rho^n)^2)|(x)\,dx\\
&\leq \frac{1}{4}\int_{\mathbb{R}}\big(|R^2-(R^n)^2|+k_1\rho^2|u-u^n|+\kappa|\rho^2-(\rho^n)^2|\big)(x)\,dx\\
&\leq \frac{1}{4}||R+R^n||_{L^2(\mathbb{R})}||R-R^n||_{L^2(\mathbb{R})}+\frac{k_1}{4}||\rho||^2_{L^2(\mathbb{R})}||u-u^n||_{L^\infty(\mathbb{R})}\\
&\quad+\frac{\kappa}{4}||\rho+\rho^n||_{L^2(\mathbb{R})}||\rho-\rho^n||_{L^2(\mathbb{R})}\\
&\leq \frac{1}{4}\big(2||R||_{L^2(\mathbb{R})}+||R-R^n||_{L^2(\mathbb{R})}\big)||R-R^n||_{L^2(\mathbb{R})}\\
&\quad+\frac{k_1}{4}||\rho||^2_{L^2(\mathbb{R})}||u-u^n||_{L^\infty(\mathbb{R})}\\
&\quad+\frac{\kappa}{4}\big(2||\rho||_{L^2(\mathbb{R})}+||\rho-\rho^n||_{L^2(\mathbb{R})}\big)||\rho-\rho^n||_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $R^n \to R$ and $\rho^n \to \rho$ in $L^2(\mathbb{R})$, and $u^n \to u$ in $L^\infty(\mathbb{R})$, it follows that $f^n \to f$ in $L^1(\mathbb{R})$. Therefore $(x_1^n)^{-1} \to x_1^{-1}$ in $L^\infty(\mathbb{R})$.

We prove that $x_1^n \to x_1$ in $L^\infty(\mathbb{R})$. We first consider the case $x_1^n(X) \leq x_1(X)$. By direct calculations we get

$$x_1(X) - x_1^n(X) = -\int_{-\infty}^{x_1(X)} f(x)\,dx + \int_{-\infty}^{x_1^n(X)} f^n(x)\,dx \tag{7.95}$$
$$= -\int_{-\infty}^{x_1(X)} (f(x) - f^n(x))\,dx - \int_{x_1^n(X)}^{x_1(X)} f^n(x)\,dx$$
$$\leq -\int_{-\infty}^{x_1(X)} (f(x) - f^n(x))\,dx,$$

since $f^n \geq 0$. If $x_1(X) \leq x_1^n(X)$, we get in a similar way that

$$x_1^n(X) - x_1(X) \leq \int_{-\infty}^{x_1(X)} (f(x) - f^n(x))\,dx. \tag{7.96}$$

Combining (7.95) and (7.96), we end up with

$$|x_1(X) - x_1^n(X)| \leq ||f - f^n||_{L^1(\mathbb{R})},$$

which implies, since $f^n \to f$ in $L^1(\mathbb{R})$, that $x_1^n \to x_1$ in $L^\infty(\mathbb{R})$. Using (3.12c) we get $J_1^n \to J_1$ in $L^\infty(\mathbb{R})$.

We prove that $(x_1^n)' \to x_1'$ in $L^1(\mathbb{R})$. As in (7.2) we get

$$(x_1^n)' = \frac{1}{f^n \circ x_1^n + 1} \quad \text{and} \quad x_1' = \frac{1}{f \circ x_1 + 1},$$

so that

$$(x_1^n)' - x_1' = (f \circ x_1 - f^n \circ x_1^n)(x_1^n)' x_1'. \tag{7.97}$$

For every $\varepsilon > 0$, there exists a function l in $C_c(\mathbb{R})$ such that $||f - l||_{L^1(\mathbb{R})} \leq \varepsilon$. Here $C_c(\mathbb{R})$ denotes the space of continuous functions with compact support. Applying the triangle inequality in (7.97) yields

$$\int_{\mathbb{R}} |x_1'(X) - (x_1^n)'(X)|\,dX \tag{7.98}$$
$$\leq \int_{\mathbb{R}} |f \circ x_1(X) - l \circ x_1(X)| x_1'(X)\,dX + \int_{\mathbb{R}} |l \circ x_1(X) - l \circ x_1^n(X)|\,dX$$
$$+ \int_{\mathbb{R}} |l \circ x_1^n(X) - f \circ x_1^n(X)|(x_1^n)'(X)\,dX$$
$$+ \int_{\mathbb{R}} |f \circ x_1^n(X) - f^n \circ x_1^n(X)|(x_1^n)'(X)\,dX$$
$$= 2||f - l||_{L^1(\mathbb{R})} + ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} + ||f - f^n||_{L^1(\mathbb{R})},$$

by a change of variables. Furthermore we used that $0 \leq x_1' \leq 1$ and $0 \leq (x_1^n)' \leq 1$. It remains to show that $\lim_{n\to\infty} ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} = 0$. We have $l \circ x_1^n \to l \circ x_1$ pointwise almost everywhere. In order to use the dominated convergence theorem we have to show that $l \circ x_1^n$ can be uniformly bounded by a function which belongs to $L^1(\mathbb{R})$. We prove a slightly more general result which will be used many times throughout the text.

Lemma 7.6. *Assume that $g \in C_c(\mathbb{R})$, and that h and h_n satisfy $h - \mathrm{Id}, h_n - \mathrm{Id} \in L^\infty(\mathbb{R})$ and $h_n \to h$ in $L^\infty(\mathbb{R})$. Then there exists a constant $0 < K < \infty$ that is independent of n such that*

$$|g \circ h_n| \leq ||g||_{L^\infty(\mathbb{R})}\chi_{[-K,K]}, \tag{7.99}$$

where $\chi_{[-K,K]}$ denotes the indicator function of the interval $[-K,K]$.

Proof. Since g has compact support there is a constant $0 < k < \infty$ such that $\mathrm{supp}(g) \subset [-k,k]$. Writing $h_n(x) = h_n(x) - h(x) + h(x) - x + x$, we get $\{x \mid h_n(x) \in [-k,k]\} \subset [-K_n, K_n]$ where $K_n = k + ||h - h_n||_{L^\infty(\mathbb{R})} + ||h - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$. Since $h_n \to h$ in $L^\infty(\mathbb{R})$ we can find a constant M such that $||h - h_n||_{L^\infty(\mathbb{R})} \leq M$ for all n. If we set $K = k + M + ||h - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$ we get

$$|g \circ h_n(x)| \leq |g \circ h_n(x)|\chi_{[-K,K]} \leq ||g||_{L^\infty(\mathbb{R})}\chi_{[-K,K]},$$

which proves (7.99). □

From (7.99) we conclude that $l \circ x_1^n$ can be uniformly bounded by an $L^1(\mathbb{R})$ function. By the dominated convergence theorem we obtain $\lim_{n\to\infty} ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} = 0$. We conclude that the right-hand side of (7.98) can be made arbitrarily small, so that $(x_1^n)' \to x_1'$ in $L^1(\mathbb{R})$ and as an immediate consequence $(J_1^n)' \to J_1'$ in $L^1(\mathbb{R})$.

We show that $U_1^n \to U_1$ in $L^\infty(\mathbb{R})$. By (3.12d) and the Cauchy–Schwarz inequality we get

$$\begin{aligned}
\big|U_1(X) - U_1^n(X)\big| &\leq \big|u(x_1(X)) - u(x_1^n(X))\big| + \big|u(x_1^n(X)) - u^n(x_1^n(X))\big| \\
&\leq \bigg|\int_{x_1^n(X)}^{x_1(X)} u_x(x)\,dx\bigg| + \big|u(x_1^n(X)) - u^n(x_1^n(X))\big| \\
&\leq ||u_x||_{L^2(\mathbb{R})}||x_1 - x_1^n||^{\frac{1}{2}}_{L^\infty(\mathbb{R})} + ||u - u^n||_{L^\infty(\mathbb{R})}.
\end{aligned}$$

From (3.1) we have $||u_x||_{L^2(\mathbb{R})} \leq \frac{\kappa}{2}(||R||_{L^2(\mathbb{R})} + ||S||_{L^2(\mathbb{R})})$, and since $x_1^n \to x_1$ and $u^n \to u$ in $L^\infty(\mathbb{R})$ we conclude that $U_1^n \to U_1$ in $L^\infty(\mathbb{R})$.

Let us prove that $U_1^n \to U_1$ in $L^2(\mathbb{R})$. Since $u \in H^1(\mathbb{R})$ there is for every $\varepsilon > 0$ a continuous function η with compact support such that $||u - \eta||_{L^2(\mathbb{R})} \leq \varepsilon$ and $||u - \eta||_{L^\infty(\mathbb{R})} \leq \varepsilon$. We have

$$\begin{aligned}
||U_1 - U_1^n||_{L^2(\mathbb{R})} &\leq ||u \circ x_1 - \eta \circ x_1||_{L^2(\mathbb{R})} + ||\eta \circ x_1 - \eta \circ x_1^n||_{L^2(\mathbb{R})} \\
&\quad + ||\eta \circ x_1^n - u \circ x_1^n||_{L^2(\mathbb{R})} + ||u \circ x_1^n - u^n \circ x_1^n||_{L^2(\mathbb{R})}.
\end{aligned} \tag{7.100}$$

Let

$$D_1 = \left\{X \in \mathbb{R} \mid x_1'(X) < \frac{1}{2}\right\} \quad \text{and} \quad D_2 = \left\{Y \in \mathbb{R} \mid x_2'(Y) < \frac{1}{2}\right\}.$$

Observe that

$$\mathrm{meas}(D_1) \leq \int_{\mathbb{R}} 2J_1'(X)\,dX \quad \text{and} \quad \mathrm{meas}(D_2) \leq \int_{\mathbb{R}} 2J_2'(Y)\,dY,$$

since $J_i' = 1 - x_i'$, $i = 1, 2$. Since $\lim_{X\to-\infty} J_1(X) = \lim_{Y\to-\infty} J_2(Y) = 0$ we get

$$\mathrm{meas}(D_1) \leq 2||J_1||_{L^\infty(\mathbb{R})} \quad \text{and} \quad \mathrm{meas}(D_2) \leq 2||J_2||_{L^\infty(\mathbb{R})}.$$

In a similar way we find that the measures of the sets

$$D_1^n = \left\{X \in \mathbb{R} \mid (x_1^n)'(X) < \frac{1}{2}\right\} \quad \text{and} \quad D_2^n = \left\{Y \in \mathbb{R} \mid (x_2^n)'(Y) < \frac{1}{2}\right\}$$

have the bounds

$$\operatorname{meas}(D_1^n) \leq 2||J_1^n||_{L^\infty(\mathbb{R})} \quad \text{and} \quad \operatorname{meas}(D_2^n) \leq 2||J_2^n||_{L^\infty(\mathbb{R})}.$$

Since $J_i^n \to J_i$ in $L^\infty(\mathbb{R})$, we can find constants E_1 and E_2 that are independent of n such that $\operatorname{meas}(D_1^n) \leq E_1$ and $\operatorname{meas}(D_2^n) \leq E_2$ for all n. We have

$$\begin{aligned}
(7.101)\quad & ||u \circ x_1 - \eta \circ x_1||^2_{L^2(\mathbb{R})} \\
&= \int_{D_1} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2 \, dX + \int_{D_1^c} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2 \, dX \\
&\leq \operatorname{meas}(D_1)||u - \eta||^2_{L^\infty(\mathbb{R})} + 2\int_{D_1^c} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2 x_1'(X) \, dX \\
&\leq \operatorname{meas}(D_1)||u - \eta||^2_{L^\infty(\mathbb{R})} + 2\int_{\mathbb{R}} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2 x_1'(X) \, dX \\
&= \operatorname{meas}(D_1)||u - \eta||^2_{L^\infty(\mathbb{R})} + 2||u - \eta||^2_{L^2(\mathbb{R})}
\end{aligned}$$

by a change of variables. Similarly,

$$(7.102)\qquad ||\eta \circ x_1^n - u \circ x_1^n||^2_{L^2(\mathbb{R})} \leq E_1||u - \eta||^2_{L^\infty(\mathbb{R})} + 2||u - \eta||^2_{L^2(\mathbb{R})}$$

and

$$||u \circ x_1^n - u^n \circ x_1^n||^2_{L^2(\mathbb{R})} \leq E_2||u - u^n||^2_{L^\infty(\mathbb{R})} + 2||u - u^n||^2_{L^2(\mathbb{R})}.$$

Since $u_n \to u$ in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ we get that

$$(7.103)\qquad \lim_{n\to\infty} ||u \circ x_1^n - u^n \circ x_1^n||_{L^2(\mathbb{R})} = 0.$$

For the remaining term in (7.100) we use the dominated convergence theorem. We have $\eta \circ x_1^n \to \eta \circ x_1$ pointwise almost everywhere, and by Lemma 7.6 we find that $\eta \circ x_1^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function. By the dominated convergence theorem we get

$$(7.104)\qquad \lim_{n\to\infty} ||\eta \circ x_1 - \eta \circ x_1^n||_{L^2(\mathbb{R})} = 0.$$

From (7.103) and (7.104), and since the right-hand sides of (7.101) and (7.102) can be made arbitrarily small, we conclude that $U_1^n \to U_1$ in $L^2(\mathbb{R})$.

We prove that $K_1^n \to K_1$ in $L^\infty(\mathbb{R})$. By (3.12f) we get

$$\begin{aligned}
(7.105)\qquad K_1(X) - K_1^n(X) &= \int_{-\infty}^{X} \left[\frac{J_1'(\bar{X})}{c(U_1(\bar{X}))} - \frac{(J_1^n)'(\bar{X})}{c(U_1^n(\bar{X}))}\right] d\bar{X} \\
&= \int_{-\infty}^{X} J_1'(\bar{X})\left[\frac{1}{c(U_1(\bar{X}))} - \frac{1}{c(U_1^n(\bar{X}))}\right] d\bar{X} \\
&\quad + \int_{-\infty}^{X} \frac{1}{c(U_1^n(\bar{X}))}\left[J_1'(\bar{X}) - (J_1^n)'(\bar{X})\right] d\bar{X}.
\end{aligned}$$

For the first term on the right-hand side we use the Cauchy–Schwarz inequality and get

$$\begin{aligned}\Big|\int_{-\infty}^{X} & J_1'(\bar X)\Big[\frac{1}{c(U_1(\bar X))}-\frac{1}{c(U_1^n(\bar X))}\Big]\,d\bar X\Big|\\ &\le \|J_1'\|_{L^2(\mathbb{R})}\Big|\int_{-\infty}^{X}\Big(\int_{U_1^n(\bar X)}^{U_1(\bar X)}-\frac{c'(U)}{c^2(U)}\,dU\Big)^2\,d\bar X\Big|^{\frac12}\\ &\le k_1\kappa^2\|J_1'\|_{L^2(\mathbb{R})}\|U_1-U_1^n\|_{L^2(\mathbb{R})},\end{aligned}$$

and for the second term we have

$$\Big|\int_{-\infty}^{X}\frac{1}{c(U_1^n(\bar X))}\big[J_1'(\bar X)-(J_1^n)'(\bar X)\big]\,d\bar X\Big|\le\kappa\|J_1'-(J_1^n)'\|_{L^1(\mathbb{R})},$$

which implies that $K_1^n\to K_1$ in $L^\infty(\mathbb{R})$.

The above proof in fact also shows that $(K_1^n)'\to K_1'$ in $L^1(\mathbb{R})$. This is because

$$\int_{\mathbb{R}}\big|K_1'(X)-(K_1^n)'(X)\big|\,dX=\int_{\mathbb{R}}\Big|\frac{J_1'(X)}{c(U_1(X))}-\frac{(J_1^n)'(X)}{c(U_1^n(X))}\Big|\,dX,$$

which is very similar to the term we estimated above.

We prove that $H_1^n\to H_1$ in $L^2(\mathbb{R})$. Since ρ and ρ^n belong to $L^2(\mathbb{R})$ there exist for every $\varepsilon>0$ functions ϕ and ϕ^n in $C_c(\mathbb{R})$ such that $\|\rho-\phi\|_{L^2(\mathbb{R})}\le\varepsilon$ and $\|\rho^n-\phi^n\|_{L^2(\mathbb{R})}\le\varepsilon$. Since $\rho^n\to\rho$ in $L^2(\mathbb{R})$ we can for every $\varepsilon>0$ choose n so large that $\|\rho-\rho^n\|_{L^2(\mathbb{R})}\le\varepsilon$. This implies that for large n we have $\|\phi-\phi^n\|_{L^2(\mathbb{R})}\le 3\varepsilon$. From (3.12g) we have

$$\begin{aligned}H_1-H_1^n&=\frac12 x_1'(\rho\circ x_1-\phi\circ x_1)+\frac12((\phi\circ x_1)x_1'-(\phi^n\circ x_1^n)(x_1^n)')\\ &\quad+\frac12(x_1^n)'(\phi^n\circ x_1^n-\rho^n\circ x_1^n),\end{aligned}$$

so that

$$\begin{aligned}\|H_1-H_1^n\|_{L^2(\mathbb{R})}&\le\frac12\|x_1'(\rho\circ x_1-\phi\circ x_1)\|_{L^2(\mathbb{R})}\\ &\quad+\frac12\|(\phi\circ x_1)x_1'-(\phi^n\circ x_1^n)(x_1^n)'\|_{L^2(\mathbb{R})}\\ &\quad+\frac12\|(x_1^n)'(\phi^n\circ x_1^n-\rho^n\circ x_1^n)\|_{L^2(\mathbb{R})}.\end{aligned}\tag{7.106}$$

Since $0\le x_1'\le 1$ we get for the first term on the right-hand side by a change of variables,

$$\begin{aligned}\|x_1'(\rho\circ x_1-\phi\circ x_1)\|_{L^2(\mathbb{R})}^2&\le\int_{\mathbb{R}}x_1'(X)(\rho(x_1(X))-\phi(x_1(X)))^2\,dX\\ &=\|\rho-\phi\|_{L^2(\mathbb{R})}^2.\end{aligned}\tag{7.107}$$

Similarly we get for the third term,

$$\|(x_1^n)'(\phi^n\circ x_1^n-\rho^n\circ x_1^n)\|_{L^2(\mathbb{R})}\le\|\rho^n-\phi^n\|_{L^2(\mathbb{R})}.\tag{7.108}$$

For the second term we have

$$\|(\phi\circ x_1)x_1'-(\phi^n\circ x_1^n)(x_1^n)'\|_{L^2(\mathbb{R})}\tag{7.109}$$

$$\begin{aligned}&\leq ||(\phi\circ x_1)(x_1'-(x_1^n)')||_{L^2(\mathbb{R})}+||(x_1^n)'(\phi\circ x_1-\phi\circ x_1^n)||_{L^2(\mathbb{R})}\\&\quad+||(x_1^n)'(\phi\circ x_1^n-\phi^n\circ x_1^n)||_{L^2(\mathbb{R})}\\&\leq\sqrt{2}||\phi||_{L^\infty(\mathbb{R})}||x_1'-(x_1^n)'||^{\frac{1}{2}}_{L^1(\mathbb{R})}+||\phi\circ x_1-\phi\circ x_1^n||_{L^2(\mathbb{R})}\\&\quad+||\phi-\phi^n||_{L^2(\mathbb{R})},\end{aligned}$$

where we used that $0\leq x_1'\leq 1$, $0\leq (x_1^n)'\leq 1$, and a change of variables for the last term on the right-hand side. We have $\phi\circ x_1^n\to\phi\circ x_1$ pointwise almost everywhere, and by Lemma 7.6 we get that $\phi\circ x_1^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function. By the dominated convergence theorem we have $\lim\limits_{n\to\infty}||\phi\circ x_1-\phi\circ x_1^n||_{L^2(\mathbb{R})}=0$.

Using the estimates (7.107)-(7.109) in (7.106) we observe that all terms on the right-hand side can be made arbitrarily small, which implies that $H_1^n\to H_1$ in $L^2(\mathbb{R})$.

We can prove in more or less the same way that $V_1^n\to V_1$ in $L^2(\mathbb{R})$, where we also have to use that $U_1^n\to U_1$ in $L^\infty(\mathbb{R})$ and the boundedness of c and c'. □

Lemma 7.7. *Let (ψ_1,ψ_2) and (ψ_1^n,ψ_2^n) belong to $\mathcal{F}$, where $\psi_i=(x_i,U_i,J_i,K_i,V_i,H_i)$ and $\psi_i^n=(x_i^n,U_i^n,J_i^n,K_i^n,V_i^n,H_i^n)$. Assume that x_i and x_i^n are strictly increasing, and that $x_i+J_i=\mathrm{Id}$, and $x_i^n+J_i^n=\mathrm{Id}$. Consider*

$$(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})=\mathbf{C}(\psi_1,\psi_2)$$

and

$$(\mathcal{X}^n,\mathcal{Y}^n,\mathcal{Z}^n,\mathcal{V}^n,\mathcal{W}^n,\mathfrak{p}^n,\mathfrak{q}^n)=\mathbf{C}(\psi_1^n,\psi_2^n).$$

Assume

$$\begin{gathered}x_i^n\to x_i,\quad (x_i^n)^{-1}\to x_i^{-1},\quad U_i^n\to U_i,\quad J_i^n\to J_i,\quad K_i^n\to K_i\quad in\quad L^\infty(\mathbb{R}),\\ U_i^n\to U_i,\quad V_i^n\to V_i,\quad H_i^n\to H_i\quad in\quad L^2(\mathbb{R}),\\ (x_i^n)'\to x_i',\quad (J_i^n)'\to J_i',\quad (K_i^n)'\to K_i'\quad in\quad L^2(\mathbb{R})\end{gathered}$$

for $i=1,2$. Then $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}^n$ and $\mathcal{Y}^n$ are strictly increasing, and

$$\mathcal{X}^n\to\mathcal{X},\quad \mathcal{Y}^n\to\mathcal{Y},\quad \mathcal{Z}_i^n\to\mathcal{Z}_i\quad in\quad L^\infty(\mathbb{R}),$$

$$\mathcal{Z}_3^n\to\mathcal{Z}_3,\quad \mathcal{V}_i^n\to\mathcal{V}_i,\quad \mathcal{W}_i^n\to\mathcal{W}_i,\quad \mathfrak{p}^n\to\mathfrak{p},\quad \mathfrak{q}^n\to\mathfrak{q}\quad in\quad L^2(\mathbb{R})$$

for $i=1,\dots,5$.

Proof. Since x_i and x_i^n are continuous and strictly increasing for $i=1,2$, we have by (3.28) that for every $s\in\mathbb{R}$ there exist *unique* points $\mathcal{X}(s)$ and $\mathcal{X}^n(s)$ such that

$$x_1(\mathcal{X}(s))=x_2(2s-\mathcal{X}(s))\quad\text{and}\quad x_1^n(\mathcal{X}^n(s))=x_2^n(2s-\mathcal{X}^n(s)).$$

Moreover, $\mathcal{X}$ and $\mathcal{X}^n$ are strictly increasing and continuous. It follows that $\mathcal{Y}$ and $\mathcal{Y}^n$ are strictly increasing and continuous. Thus, the inverse functions $\mathcal{X}^{-1}$, $\mathcal{Y}^{-1}$, $(\mathcal{X}^n)^{-1}$ and $(\mathcal{Y}^n)^{-1}$ exist, and they are continuous and strictly increasing.

We prove that $\mathcal{X}^n\to\mathcal{X}$ in $L^\infty(\mathbb{R})$. To begin with we show that $J_i^n\circ(x_i^n)^{-1}\to J_i\circ x_i^{-1}$ in $L^\infty(\mathbb{R})$. Write

$$J_i\circ x_i^{-1}(X)-J_i^n\circ(x_i^n)^{-1}(X)=\int_{(x_i^n)^{-1}(X)}^{x_i^{-1}(X)}J_i'(Z)\,dZ+J_i\circ(x_i^n)^{-1}(X)-J_i^n\circ(x_i^n)^{-1}(X)$$

to get

$$|J_i\circ x_i^{-1}(X)-J_i^n\circ(x_i^n)^{-1}(X)|\leq||x_i^{-1}-(x_i^n)^{-1}||_{L^\infty(\mathbb{R})}+||J_i-J_i^n||_{L^\infty(\mathbb{R})}.$$

We used that $0 \leq J_i' \leq 1$, where the upper bound comes from the identity $x_i'+J_i'=1$, and $x_i' \geq 0$. Using the convergence assumptions implies that $J_i^n \circ (x_i^n)^{-1} \to J_i \circ x_i^{-1}$ in $L^\infty(\mathbb{R})$.

We insert $X = \mathcal{X}(s)$ in $x_1(X) + J_1(X) = X$ and get

$$\mathcal{X}(s) = x_1 \circ \mathcal{X}(s) + J_1 \circ \mathcal{X}(s) = x_1 \circ \mathcal{X}(s) + J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s). \tag{7.110}$$

Similarly we get

$$\mathcal{Y}(s) = x_2 \circ \mathcal{Y}(s) + J_2 \circ \mathcal{Y}(s) = x_1 \circ \mathcal{X}(s) + J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s), \tag{7.111}$$

where we used that $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$. The expressions for $\mathcal{X}^n$ and $\mathcal{Y}^n$ are defined in a similar way. By (2.40c),

$$\mathcal{X}(s) + \mathcal{Y}(s) - \mathcal{X}^n(s) - \mathcal{Y}^n(s) = 0,$$

which combined with the above expressions yields

$$\begin{aligned} 2\big(x_1 \circ \mathcal{X}(s) - x_1^n \circ \mathcal{X}^n(s)\big) + J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s) - J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) \\ + J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s) - J_2^n \circ (x_2^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) = 0. \end{aligned}$$

Since $J_i^n \circ (x_i^n)^{-1}$ is increasing we get

$$\begin{aligned} &2\big|x_1 \circ \mathcal{X}(s) - x_1^n \circ \mathcal{X}^n(s)\big| \\ &\quad + |J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s) - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\ &\quad + |J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s) - J_2 \circ x_2^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\ &\qquad \leq |J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\ &\qquad\quad + |J_2^n \circ (x_2^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) - J_2 \circ x_2^{-1} \circ x_1^n \circ \mathcal{X}^n(s)|, \end{aligned}$$

Therefore

$$\begin{aligned} ||x_1 \circ \mathcal{X} - x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &\leq \frac{1}{2}||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\ &\quad + \frac{1}{2}||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})}, \end{aligned}$$

$$\begin{aligned} ||J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X} - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &\leq ||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\ &\quad + ||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})}, \end{aligned}$$

and

$$\begin{aligned} ||J_1 \circ \mathcal{X} - J_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &= ||J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X} - J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} \\ &\leq 2||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\ &\quad + ||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})}. \end{aligned}$$

Thus, we showed that $x_1^n \circ \mathcal{X}^n \to x_1 \circ \mathcal{X}$ and $J_1^n \circ \mathcal{X}^n \to J_1 \circ \mathcal{X}$ in $L^\infty(\mathbb{R})$ since $J_i^n \circ (x_i^n)^{-1} - J_i \circ x_i^{-1}$ in $L^\infty(\mathbb{R})$.

From (7.110), (2.40c), (7.111), (3.30b) and (3.30d) it immediately follows that $\mathcal{X}^n \to \mathcal{X}$, $\mathcal{Y}^n \to \mathcal{Y}$, $\mathcal{Z}_2^n \to \mathcal{Z}_2$ and $\mathcal{Z}_4^n \to \mathcal{Z}_4$ in $L^\infty(\mathbb{R})$.

We show that $\mathcal{Z}_5^n \to \mathcal{Z}_5$ in $L^\infty(\mathbb{R})$. By (3.30e) we have

$$\begin{aligned} \mathcal{Z}_5(s) - \mathcal{Z}_5^n(s) &= K_1(\mathcal{X}(s)) - K_1(\mathcal{X}^n(s)) + K_1(\mathcal{X}^n(s)) - K_1^n(\mathcal{X}^n(s)) \\ &\quad + K_2(\mathcal{Y}(s)) - K_2(\mathcal{Y}^n(s)) + K_2(\mathcal{Y}^n(s)) - K_2^n(\mathcal{Y}^n(s)) \end{aligned}$$

and for the first line we get

$$\begin{aligned}&\big|K_1(\mathcal{X}(s)) - K_1(\mathcal{X}^n(s)) + K_1(\mathcal{X}^n(s)) - K_1^n(\mathcal{X}^n(s))\big|\\&= \left|\int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)} K_1'(X)\,dX + K_1(\mathcal{X}^n(s)) - K_1^n(\mathcal{X}^n(s))\right|\\&\leq ||K_1'||_{L^\infty(\mathbb{R})}||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})} + ||K_1 - K_1^n||_{L^\infty(\mathbb{R})}.\end{aligned}$$

A similar estimate for the second line yields $\mathcal{Z}_5^n \to \mathcal{Z}_5$ in $L^\infty(\mathbb{R})$. By (3.30c) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\mathcal{Z}_3(s) - \mathcal{Z}_3^n(s)| &\leq \left|\int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)} U_1'(X)\,dX\right| + |U_1(\mathcal{X}^n(s)) - U_1^n(\mathcal{X}^n(s))|\\&\leq ||U_1'||_{L^2(\mathbb{R})}||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}} + ||U_1 - U_1^n||_{L^\infty(\mathbb{R})},\end{aligned}$$

which shows that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^\infty(\mathbb{R})$.

We prove that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^2(\mathbb{R})$. We have

$$||\mathcal{Z}_3 - \mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2 = ||\mathcal{Z}_3||_{L^2(\mathbb{R})}^2 - 2\langle \mathcal{Z}_3, \mathcal{Z}_3^n\rangle + ||\mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2, \tag{7.112}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(\mathbb{R})$. Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$ we get from (3.30c) and a change of variables,

$$\begin{aligned}||\mathcal{Z}_3||_{L^2(\mathbb{R})}^2 &= \frac{1}{2}\int_{\mathbb{R}} U_1^2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds + \frac{1}{2}\int_{\mathbb{R}} U_2^2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds\\&= \frac{1}{2}||U_1||_{L^2(\mathbb{R})}^2 + \frac{1}{2}||U_2||_{L^2(\mathbb{R})}^2,\end{aligned} \tag{7.113}$$

and similarly

$$||\mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2 = \frac{1}{2}||U_1^n||_{L^2(\mathbb{R})}^2 + \frac{1}{2}||U_2^n||_{L^2(\mathbb{R})}^2. \tag{7.114}$$

Using that $\dot{\mathcal{X}}^n + \dot{\mathcal{Y}}^n = 2$ we have

$$2\langle \mathcal{Z}_3, \mathcal{Z}_3^n\rangle = \int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds + \int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{Y}}^n(s)\,ds.$$

Since $\mathcal{Z}_3 \in L^2(\mathbb{R})$ there exists for every $\varepsilon > 0$ a function $\phi \in C_c^\infty(\mathbb{R})$ such that $||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})} \leq \varepsilon$. Write

$$\begin{aligned}&\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds\\&= \int_{\mathbb{R}} \big[\mathcal{Z}_3(s) - \phi(s)\big]\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds + \int_{\mathbb{R}} \phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds - \int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds\\&\quad + \int_{\mathbb{R}} \mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\big[\phi(s) - \mathcal{Z}_3(s)\big]\,ds + \int_{\mathbb{R}} \mathcal{Z}_3^2(s)\dot{\mathcal{X}}(s)\,ds\\&= T_1^n + \int_{\mathbb{R}} \mathcal{Z}_3^2(s)\dot{\mathcal{X}}(s)\,ds.\end{aligned}$$

By a change of variables we get

$$\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds = T_1^n + ||U_1||_{L^2(\mathbb{R})}^2 \tag{7.115}$$

and in a similar way we obtain

$$\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{Y}}^n(s)\,ds = T_2^n + ||U_2||^2_{L^2(\mathbb{R})}, \tag{7.116}$$

where T_2^n is equal to T_1^n with $\mathcal{X}(s)$ and $\mathcal{X}^n(s)$ replaced by $\mathcal{Y}(s)$ and $\mathcal{Y}^n(s)$, respectively.

Using (7.113)-(7.116) in (7.112) we get

$$\begin{aligned} ||\mathcal{Z}_3 - \mathcal{Z}_3^n||^2_{L^2(\mathbb{R})} = \frac{1}{2}\Big(&||U_1^n||^2_{L^2(\mathbb{R})} - ||U_1||^2_{L^2(\mathbb{R})} + ||U_2^n||^2_{L^2(\mathbb{R})} - ||U_2||^2_{L^2(\mathbb{R})}\Big) \\ &- T_1^n - T_2^n. \end{aligned} \tag{7.117}$$

The strong convergence $U_i^n \to U_i$ in $L^2(\mathbb{R})$ implies that $||U_i^n||_{L^2(\mathbb{R})} \to ||U_i||_{L^2(\mathbb{R})}$ for $i = 1, 2$. Thus, it remains to show that $T_i^n \to 0$ for $i = 1, 2$.

Using (3.30c) and $0 \leq \dot{\mathcal{X}}^n \leq 2$ we get by the Cauchy–Schwarz inequality and a change of variables,

$$\left| \int_{\mathbb{R}} \mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\big[\phi(s) - \mathcal{Z}_3(s)\big]\,ds \right| \leq \sqrt{2}||U_1||_{L^2(\mathbb{R})}||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})}, \tag{7.118}$$

and similarly

$$\begin{aligned} &\left| \int_{\mathbb{R}} \big[\mathcal{Z}_3(s) - \phi(s)\big]\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds \right| \\ &\leq \sqrt{2}\Big(||U_1^n - U_1||_{L^2(\mathbb{R})} + ||U_1||_{L^2(\mathbb{R})}\Big)||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})}. \end{aligned} \tag{7.119}$$

Since ϕ has compact support, there exists $k > 0$ such that $\text{supp}(\phi) \subset [-k, k]$. Integration by parts yields

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds = \int_{-k}^{k} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds = -\int_{-k}^{k} \phi'(s)\int_{-k}^{s} \mathcal{Z}_3(t)\dot{\mathcal{X}}(t)\,dt\,ds$$

where the first term in the integration by parts equals zero because ϕ has compact support and the second integral is finite, since

$$\left| \int_{-k}^{s} \mathcal{Z}_3(t)\dot{\mathcal{X}}(t)\,dt \right| = \left| \int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\,dX \right| \leq \big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2k\big)^{\frac{1}{2}}||U_1||_{L^2(\mathbb{R})}$$

for $s \in [-k, k]$. Here we used a change of variables and the estimate

$$\begin{aligned} \big|\mathcal{X}(k) - \mathcal{X}(-k)\big| &= \big|(\mathcal{X}(k) - k) - (\mathcal{X}(-k) - (-k)) + 2k\big| \\ &\leq 2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2k. \end{aligned} \tag{7.120}$$

We have

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds = -\int_{-k}^{k} \phi'(s)\int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\,dX\,ds$$

and similarly we obtain

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds = -\int_{-k}^{k} \phi'(s)\int_{\mathcal{X}^n(-k)}^{\mathcal{X}^n(s)} U_1^n(X)\,dX\,ds.$$

By the Cauchy–Schwarz inequality we get

$$\left| \int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\,dX - \int_{\mathcal{X}^n(-k)}^{\mathcal{X}^n(s)} U_1^n(X)\,dX \right| \tag{7.121}$$

$$\begin{aligned}&\leq \left| \int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} \big[U_1(X) - U_1^n(X)\big]\, dX \right| \\ &\quad + \left| \int_{\mathcal{X}^n(-k)}^{\mathcal{X}(-k)} U_1^n(X)\, dX \right| + \left| \int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)} U_1^n(X)\, dX \right| \\ &\leq ||U_1 - U_1^n||_{L^2(\mathbb{R})} \big|\mathcal{X}(k) - \mathcal{X}(-k)\big|^{\frac{1}{2}} + 2||U_1^n||_{L^2(\mathbb{R})} ||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}},\end{aligned}$$

which implies by (7.120) that

$$\begin{aligned}&\left| \int_{\mathbb{R}} \phi(s) \mathcal{Z}_3^n(s) \dot{\mathcal{X}}^n(s)\, ds - \int_{\mathbb{R}} \phi(s) \mathcal{Z}_3(s) \dot{\mathcal{X}}(s)\, ds \right| \\ &\leq 2k||\phi'||_{L^\infty(\mathbb{R})} \Big[||U_1 - U_1^n||_{L^2(\mathbb{R})} \big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2k\big)^{\frac{1}{2}} \\ &\qquad + 2\big(||U_1^n - U_1||_{L^2(\mathbb{R})} + ||U_1||_{L^2(\mathbb{R})}\big) ||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \Big].\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} \phi(s) \mathcal{Z}_3^n(s) \dot{\mathcal{X}}^n(s)\, ds - \int_{\mathbb{R}} \phi(s) \mathcal{Z}_3(s) \dot{\mathcal{X}}(s)\, ds \right| = 0. \tag{7.122}$$

From (7.118), (7.119) and (7.122) we conclude that $T_1^n \to 0$ as $n \to \infty$. In a similar way we can prove that $T_2^n \to 0$. This implies by (7.117) that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^2(\mathbb{R})$.

Using (3.31a) we get

$$\big|\big|\mathcal{V}_1 - \mathcal{V}_1^n\big|\big|_{L^2(\mathbb{R})} \leq \left|\left| x_1' \left(\frac{1}{2c(U_1)} - \frac{1}{2c(U_1^n)} \right) \right|\right|_{L^2(\mathbb{R})} + \left|\left| \frac{1}{2c(U_1^n)} \Big(x_1' - (x_1^n)' \Big) \right|\right|_{L^2(\mathbb{R})},$$

and by inserting the estimates

$$\begin{aligned}&\left|\left| x_1' \left(\frac{1}{2c(U_1)} - \frac{1}{2c(U_1^n)} \right) \right|\right|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{1}{4} \Big(||x_1' - 1||_{L^\infty(\mathbb{R})} + 1 \Big)^2 \int_{\mathbb{R}} \left(\int_{U_1^n(X)}^{U_1(X)} -\frac{c'(U)}{c^2(U)}\, dU \right)^2 dX \\ &\leq \frac{1}{4} k_1^2 \kappa^4 \Big(||x_1' - 1||_{L^\infty(\mathbb{R})} + 1 \Big)^2 \big|\big|U_1 - U_1^n\big|\big|_{L^2(\mathbb{R})}^2\end{aligned}$$

and

$$\left|\left| \frac{1}{2c(U_1^n)} \Big(x_1' - (x_1^n)' \Big) \right|\right|_{L^2(\mathbb{R})} \leq \frac{1}{2} \kappa \big|\big|x_1' - (x_1^n)'\big|\big|_{L^2(\mathbb{R})}$$

we see that $\mathcal{V}_1^n \to \mathcal{V}_1$ in $L^2(\mathbb{R})$.

From (3.31b)-(3.31f) and the assumptions we immediately get that $\mathcal{V}_i^n \to \mathcal{V}_i$, $i = 2, \ldots, 5$ and $\mathfrak{p}^n \to \mathfrak{p}$ in $L^2(\mathbb{R})$.

The corresponding results for $\mathcal{W}$ and $\mathfrak{q}$ can be proved in a similar way. □

Theorem 7.3 deals with convergence of the elements u, ρ and σ for $(u, R, S, \rho, \sigma, \mu, \nu)$ in $\mathcal{D}$, see (P1") and (P2"). The following result indicates the type of convergence we have to assume for the elements of the set $\mathcal{G}_0$ in order to get weak-star convergence of the remaining elements in $\mathcal{D}$.

Lemma 7.8. *Let* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ *and* $\Theta^n = (\mathcal{X}^n, \mathcal{Y}^n, \mathcal{Z}^n, \mathcal{V}^n, \mathcal{W}^n, \mathfrak{p}^n, \mathfrak{q}^n)$ *belong to* $\mathcal{G}_0$. *Consider*

$$(u, R, S, \rho, \sigma, \mu, \nu) = \mathbf{M} \circ \mathbf{D}(\Theta)$$

and

$$(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n) = \mathbf{M} \circ \mathbf{D}(\Theta^n).$$

Assume that

$$\mathcal{X}^n \to \mathcal{X}, \quad \mathcal{Y}^n \to \mathcal{Y}, \quad \mathcal{Z}_i^n \to \mathcal{Z}_i \quad in \quad L^\infty(\mathbb{R}),$$
$$\mathcal{Z}_3^n \to \mathcal{Z}_3, \quad \mathcal{V}_i^n \to \mathcal{V}_i, \quad \mathcal{W}_i^n \to \mathcal{W}_i, \quad \mathfrak{p}^n \to \mathfrak{p}, \quad \mathfrak{q}^n \to \mathfrak{q} \quad in \quad L^2(\mathbb{R}),$$

for $i = 1, \dots, 5$. *Then*

$$u^n \to u \quad in \quad L^\infty(\mathbb{R})$$

and

$$R^n \overset{*}{\rightharpoonup} R, \quad S^n \overset{*}{\rightharpoonup} S, \quad \rho^n \overset{*}{\rightharpoonup} \rho, \quad \sigma^n \overset{*}{\rightharpoonup} \sigma, \quad \mu^n \overset{*}{\rightharpoonup} \mu \quad and \quad \nu^n \overset{*}{\rightharpoonup} \nu.$$

Observe that there are no assumptions on the monotonicity of $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{X}^n, \mathcal{Y}^n)$. This means that as functions of s they are nondecreasing, but not necessarily strictly increasing.

From these results it follows immediately that $u_x^n \overset{*}{\rightharpoonup} u_x$ due to (3.1).

Proof. We will use Lemma 5.8. For any x, there exist s and s^n, which are not necessarily unique, such that $x = \mathcal{Z}_2(s)$ and $x = \mathcal{Z}_2^n(s^n)$. By (5.21a), we have $u(x) = \mathcal{Z}_3(s)$ and $u^n(x) = \mathcal{Z}_3^n(s^n)$. We have

$$u(x) - u^n(x) = \mathcal{Z}_3(s) - \mathcal{Z}_3^n(s^n) = \mathcal{Z}_3(s) - \mathcal{Z}_3(s^n) + \mathcal{Z}_3(s^n) - \mathcal{Z}_3^n(s^n). \tag{7.123}$$

We estimate the difference $\mathcal{Z}_3(s) - \mathcal{Z}_3(s^n)$. We assume that $s^n \le s$, the other case can be treated similar. We have

$$\begin{aligned} |\mathcal{Z}_3(s) - \mathcal{Z}_3(s^n)| &= \left| \int_{s^n}^{s} \dot{\mathcal{Z}}_3(\bar{s})\, d\bar{s} \right| \\ &= \left| \int_{s^n}^{s} (\mathcal{V}_3(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s}) + \mathcal{W}_3(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s}))\, d\bar{s} \right| \\ &\le \left(\int_{s^n}^{s} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \left(\int_{s^n}^{s} \mathcal{V}_3^2(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{s^n}^{s} \dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \left(\int_{s^n}^{s} \mathcal{W}_3^2(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \end{aligned} \tag{7.124}$$

by the Cauchy–Schwarz inequality. From (3.24c), we get

$$\begin{aligned} &\int_{s^n}^{s} \mathcal{V}_3^2(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s})\, d\bar{s} \\ &= \int_{s^n}^{s} \left(\frac{2\mathcal{V}_2(\mathcal{X}(\bar{s}))\mathcal{V}_4(\mathcal{X}(\bar{s}))}{c^2(\mathcal{Z}_3(\bar{s}))} - \frac{\mathfrak{p}^2(\mathcal{X}(\bar{s}))}{c(\mathcal{Z}_3(\bar{s}))} \right) \dot{\mathcal{X}}(\bar{s})\, d\bar{s} \\ &\le \int_{s^n}^{s} \frac{2\mathcal{V}_2(\mathcal{X}(\bar{s}))\mathcal{V}_4(\mathcal{X}(\bar{s}))}{c^2(\mathcal{Z}_3(\bar{s}))} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} \\ &\le \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} \int_{s^n}^{s} 2\mathcal{V}_2(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s})\, d\bar{s} \end{aligned} \tag{7.125}$$

$$
\begin{aligned}
&= \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} \int_{s^n}^{s} \dot{\mathcal{Z}}_2(\bar{s})\, d\bar{s} \quad \text{by (3.27)} \\
&= \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} (\mathcal{Z}_2(s) - \mathcal{Z}_2(s^n)) \\
&= \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} (\mathcal{Z}_2^n(s^n) - \mathcal{Z}_2(s^n)) \quad \text{since } \mathcal{Z}_2(s) = \mathcal{Z}_2^n(s^n) \\
&\leq \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} ||\mathcal{Z}_2^n - \mathcal{Z}_2||_{L^\infty(\mathbb{R})}.
\end{aligned}
$$

In a similar way, we obtain

$$
\int_{s^n}^{s} \mathcal{W}_3^2(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \leq \kappa^2 ||\mathcal{W}_4^a||_{L^\infty(\mathbb{R})} ||\mathcal{Z}_2^n - \mathcal{Z}_2||_{L^\infty(\mathbb{R})}. \tag{7.126}
$$

Using $\mathcal{Z}_2(s) = \mathcal{Z}_2^n(s^n)$ once more we get

$$
\begin{aligned}
\int_{s^n}^{s} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} &= \mathcal{X}(s) - \mathcal{X}(s^n) \\
&= \big(\mathcal{X}(s) - s\big) - \big(\mathcal{X}(s^n) - s^n\big) - \big(\mathcal{Z}_2(s) - s\big) + \big(\mathcal{Z}_2(s) - s^n\big) \\
&= \big(\mathcal{X}(s) - s\big) - \big(\mathcal{X}(s^n) - s^n\big) - \big(\mathcal{Z}_2(s) - s\big) + \big(\mathcal{Z}_2^n(s^n) - s^n\big) \\
&= \big(\mathcal{X}(s) - s\big) - \big(\mathcal{X}(s^n) - s^n\big) - \big(\mathcal{Z}_2(s) - s\big) \\
&\quad + \big(\mathcal{Z}_2^n(s^n) - \mathcal{Z}_2(s^n)\big) + \big(\mathcal{Z}_2(s^n) - s^n\big) \\
&\leq 2||\mathcal{X} - \operatorname{Id}||_{L^\infty(\mathbb{R})} + 2||\mathcal{Z}_2 - \operatorname{Id}||_{L^\infty(\mathbb{R})} + ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}
\end{aligned} \tag{7.127}
$$

Similarly, we get

$$
\int_{s^n}^{s} \dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \leq 2||\mathcal{Y} - \operatorname{Id}||_{L^\infty(\mathbb{R})} + 2||\mathcal{Z}_2 - \operatorname{Id}||_{L^\infty(\mathbb{R})} + ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}. \tag{7.128}
$$

Combining (7.125)-(7.128) in (7.124) and using that $\mathcal{Z}_2^n \to \mathcal{Z}_2$ in $L^\infty(\mathbb{R})$ we find that $\mathcal{Z}_3(s^n) \to \mathcal{Z}_3(s)$. Using this and that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^\infty(\mathbb{R})$ in (7.123) we conclude that $u^n \to u$ in $L^\infty(\mathbb{R})$.

We prove that $\mu^n \overset{*}{\rightharpoonup} \mu$, that is,

$$
\lim_{n\to\infty} \int_{\mathbb{R}} \phi(x)\, d\mu^n = \int_{\mathbb{R}} \phi(x)\, d\mu
$$

for all $\phi \in C_0(\mathbb{R})$. Here $C_0(\mathbb{R})$ is the space of continuous functions that vanish at infinity. Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ it suffices to consider test functions ϕ in $C_c^\infty(\mathbb{R})$. By (5.21f) we have

$$
\int_{\mathbb{R}} \phi(x)\, d\mu = \int_{\mathbb{R}} \phi(\mathcal{Z}_2(s))\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds. \tag{7.129}
$$

By (3.23) and (3.24b) we have

$$
\begin{aligned}
2\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) &= \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \\
&\quad + \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) - \mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \\
&= \dot{\mathcal{Z}}_4(s) + c(\mathcal{Z}_3(s))\big[\mathcal{V}_5(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_5(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\big] \\
&= \dot{\mathcal{Z}}_4(s) + c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s).
\end{aligned}
$$

Inserting this in (7.129) and using integration by parts yields

$$
\int_{\mathbb{R}} \phi(x)\, d\mu = \frac{1}{2}\int_{\mathbb{R}} \big[\phi(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_4(s) + \phi(\mathcal{Z}_2(s))c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s)\big]\, ds
$$

$$= -\frac{1}{2}\int_{\mathbb{R}} \phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\mathcal{Z}_4(s)\,ds$$
$$- \frac{1}{2}\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s)) + \phi(\mathcal{Z}_2(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\big]\mathcal{Z}_5(s)\,ds.$$

Similarly we find

$$\int_{\mathbb{R}} \phi(x)\,d\mu^n = -\frac{1}{2}\int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)\mathcal{Z}_4^n(s)\,ds$$
$$- \frac{1}{2}\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s)) + \phi(\mathcal{Z}_2^n(s))c'(\mathcal{Z}_3^n(s))\dot{\mathcal{Z}}_3^n(s)\big]\mathcal{Z}_5^n(s)\,ds.$$

We subtract and get

$$\begin{aligned}
\int_{\mathbb{R}} \phi(x)\,d\mu - \int_{\mathbb{R}} \phi(x)\,d\mu^n
&= -\frac{1}{2}\bigg(\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\mathcal{Z}_4(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)\mathcal{Z}_4^n(s)\big]\,ds \\
&\quad + \int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) \\
&\qquad - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s))\mathcal{Z}_5^n(s)\big]\,ds \\
&\quad + \int_{\mathbb{R}} \big[\phi(\mathcal{Z}_2(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\mathcal{Z}_5(s) \\
&\qquad - \phi(\mathcal{Z}_2^n(s))c'(\mathcal{Z}_3^n(s))\dot{\mathcal{Z}}_3^n(s)\mathcal{Z}_5^n(s)\big]\,ds\bigg).
\end{aligned} \tag{7.130}$$

The three integrals on the right-hand side of (7.130) can be treated in more or less the same way, and we only consider the second one. We have

$$\begin{aligned}
&\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s))\mathcal{Z}_5^n(s)\big]\,ds \\
&= \int_{\mathbb{R}} \dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s)\big[\phi'(\mathcal{Z}_2(s)) - \phi'(\mathcal{Z}_2^n(s))\big]\,ds \quad (I_1^n) \\
&\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))c(\mathcal{Z}_3(s))\mathcal{Z}_5(s)\big[\dot{\mathcal{Z}}_2(s) - \dot{\mathcal{Z}}_2^n(s)\big]\,ds \quad (I_2^n) \\
&\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3(s))\big[\mathcal{Z}_5(s) - \mathcal{Z}_5^n(s)\big]\,ds \quad (I_3^n) \\
&\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)\mathcal{Z}_5^n(s)\big[c(\mathcal{Z}_3(s)) - c(\mathcal{Z}_3^n(s))\big]\,ds \quad (I_4^n).
\end{aligned} \tag{7.131}$$

We have

$$\big|I_1^n\big| \leq 4\kappa\bigg(||\mathcal{V}_2^a||_{L^\infty(\mathbb{R})} + \frac{1}{2}\bigg)||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}||\phi'\circ\mathcal{Z}_2 - \phi'\circ\mathcal{Z}_2^n||_{L^1(\mathbb{R})} \tag{7.132}$$

since

$$0 \leq \dot{\mathcal{Z}}_2(s) = 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) \leq 4\mathcal{V}_2(\mathcal{X}(s)) = 4\bigg(\mathcal{V}_2^a(\mathcal{X}(s)) + \frac{1}{2}\bigg).$$

We have $\phi'\circ\mathcal{Z}_2^n \to \phi'\circ\mathcal{Z}_2$ pointwise almost everywhere, and by Lemma 7.6 we find that $\phi'\circ\mathcal{Z}_2^n$ can be uniformly bounded by an $L^1(\mathbb{R})$ function. By the dominated convergence theorem we get $\lim_{n\to\infty} ||\phi'\circ\mathcal{Z}_2 - \phi'\circ\mathcal{Z}_2^n||_{L^1(\mathbb{R})} = 0$, which from (7.132) implies $\lim_{n\to\infty} I_1^n = 0$.

Integration by parts yields

$$I_2^n = \int_{\mathbb{R}} \big[\phi''(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) + \phi'(\mathcal{Z}_2^n(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\mathcal{Z}_5(s) \\ + \phi'(\mathcal{Z}_2^n(s))c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s)\big]\big[\mathcal{Z}_2(s) - \mathcal{Z}_2^n(s)\big]\,ds$$

which implies

$$\begin{aligned}|I_2^n| \le \Bigg(&\kappa||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\big|\phi''(\mathcal{Z}_2^n(s))\big|\dot{\mathcal{Z}}_2^n(s)\,ds \\ &+ \Big[2k_1||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}\big(||\mathcal{V}_3^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_3^a||_{L^\infty(\mathbb{R})}\big) \\ &\quad + 2\kappa\big(||\mathcal{V}_5^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_5^a||_{L^\infty(\mathbb{R})}\big)\Big] \\ &\quad \times \int_{\mathbb{R}}\big|\phi'(\mathcal{Z}_2^n(s))\big|\,ds\Bigg)||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}.\end{aligned}$$

By a change of variables and an estimate as in the proof of Lemma 7.6 we get

$$\begin{aligned}|I_2^n| \le \Bigg(&\kappa||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}||\phi''||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi'')\big) \\ &+ \Big[2k_1||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}\big(||\mathcal{V}_3^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_3^a||_{L^\infty(\mathbb{R})}\big) \\ &\quad + 2\kappa\big(||\mathcal{V}_5^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_5^a||_{L^\infty(\mathbb{R})}\big)\Big] \\ &\quad \times \tilde{C}||\phi'||_{L^\infty(\mathbb{R})}\Bigg)||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}\end{aligned}$$

for a constant $\tilde{C}$ that is independent of n. Since $\mathcal{Z}_2^n \to \mathcal{Z}_2$ in $L^\infty(\mathbb{R})$ we get $\lim_{n\to\infty} I_2^n = 0$.

For the third integral we use a change of variables and get

$$\begin{aligned}|I_3^n| &\le \kappa||\mathcal{Z}_5 - \mathcal{Z}_5^n||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\big|\phi'(\mathcal{Z}_2^n(s))\big|\dot{\mathcal{Z}}_2^n(s)\,ds \\ &\le \kappa\,\mathrm{meas}\big(\mathrm{supp}(\phi')\big)||\phi'||_{L^\infty(\mathbb{R})}||\mathcal{Z}_5 - \mathcal{Z}_5^n||_{L^\infty(\mathbb{R})},\end{aligned}$$

which implies since $\mathcal{Z}_5^n \to \mathcal{Z}_5$ in $L^\infty(\mathbb{R})$ that $\lim_{n\to\infty} I_3^n = 0$.

By the Cauchy–Schwarz inequality we get

$$\begin{aligned}|I_4^n| &\le ||c(\mathcal{Z}_3) - c(\mathcal{Z}_3^n)||_{L^\infty(\mathbb{R})}||\mathcal{Z}_5^n||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\big|\phi'(\mathcal{Z}_2^n(s))\big|\dot{\mathcal{Z}}_2^n(s)\,ds \\ &\le ||c(\mathcal{Z}_3) - c(\mathcal{Z}_3^n)||_{L^\infty(\mathbb{R})}\big(||\mathcal{Z}_5^n - \mathcal{Z}_5||_{L^\infty(\mathbb{R})} + ||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}\big)||\phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n||_{L^1(\mathbb{R})}\end{aligned}$$

and by inserting the estimates

$$\begin{aligned}||\phi'(\mathcal{Z}_2^n)\dot{\mathcal{Z}}_2^n||_{L^1(\mathbb{R})} &= ||\phi'||_{L^1(\mathbb{R})} \\ &\le ||\phi'||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi')\big)\end{aligned}$$

and

$$||c(\mathcal{Z}_3) - c(\mathcal{Z}_3^n)||_{L^\infty(\mathbb{R})} \le k_1||\mathcal{Z}_3 - \mathcal{Z}_3^n||_{L^\infty(\mathbb{R})}$$

we obtain

$$|I_4^n| \leq k_1||\phi'||_{L^\infty(\mathbb{R})}\text{meas}\big(\text{supp}(\phi')\big)\big(||\mathcal{Z}_5^n-\mathcal{Z}_5||_{L^\infty(\mathbb{R})}+||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}\big)||\mathcal{Z}_3-\mathcal{Z}_3^n||_{L^\infty(\mathbb{R})}$$

Since $\mathcal{Z}_5^n \to \mathcal{Z}_5$ and $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^\infty(\mathbb{R})$ we obtain $\lim_{n\to\infty} I_4^n = 0$.

We return to (7.131) and obtain

$$\lim_{n\to\infty}\bigg|\int_{\mathbb{R}}\big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s))\mathcal{Z}_5^n(s)\big]\,ds\bigg| = 0. \tag{7.133}$$

By using (7.133) in (7.130) we get

$$\lim_{n\to\infty}\bigg|\int_{\mathbb{R}}\phi(x)\,d\mu - \int_{\mathbb{R}}\phi(x)\,d\mu^n\bigg| = 0 \tag{7.134}$$

for all $\phi \in C_c^\infty(\mathbb{R})$, and we conclude that $\mu^n \overset{*}{\rightharpoonup} \mu$. Similarly we prove that $\nu^n \overset{*}{\rightharpoonup} \nu$.

Next we show that $\rho^n \overset{*}{\rightharpoonup} \rho$, that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\rho^n(x)\phi(x)\,dx = \int_{\mathbb{R}}\rho(x)\phi(x)\,dx$$

for all $\phi \in L^2(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it suffices to consider test functions ϕ in $C_c^\infty(\mathbb{R})$.

Consider a test function ϕ in $C_c^\infty(\mathbb{R})$. By (5.21d), we have

$$\int_{\mathbb{R}}\phi(x)\rho(x)\,dx = 2\int_{\mathbb{R}}\phi(\mathcal{Z}_2(s))\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds$$

and

$$\int_{\mathbb{R}}\phi(x)\rho^n(x)\,dx = 2\int_{\mathbb{R}}\phi(\mathcal{Z}_2^n(s))\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\,ds.$$

By subtracting we get

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)\big[\rho(x)-\rho^n(x)\big]\,dx &\\ = 2\int_{\mathbb{R}}\phi(\mathcal{Z}_2(s))\big[\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)-\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big]\,ds &\quad (A_1^n)\\ + 2\int_{\mathbb{R}}\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[\phi(\mathcal{Z}_2(s))-\phi(\mathcal{Z}_2^n(s))\big]\,ds &\quad (A_2^n).\end{aligned} \tag{7.135}$$

Since ϕ has compact support, there exists $k > 0$ such that $\text{supp}(\phi) \subset [-k,k]$. Thus, we only integrate over s such that $-k \leq \mathcal{Z}_2(s) \leq k$ in A_1^n. Since the quantity $\mathcal{Z}_2 - \text{Id}$ belongs to $L^\infty(\mathbb{R})$, the region we integrate over is contained in

$$-k - ||\mathcal{Z}_2 - \text{Id}\,||_{L^\infty(\mathbb{R})} \leq s \leq k + ||\mathcal{Z}_2 - \text{Id}\,||_{L^\infty(\mathbb{R})},$$

or written more compactly, $-M \leq s \leq M$ with $M = k + ||\mathcal{Z}_2 - \text{Id}\,||_{L^\infty(\mathbb{R})}$. Integration by parts yields

$$\begin{aligned}A_1^n &= 2\int_{-M}^{M}\phi(\mathcal{Z}_2(s))\big[\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)-\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big]\,ds\\ &= 2\bigg[\phi(\mathcal{Z}_2(s))\int_{-M}^{s}\big[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)-\mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\big]\,d\tau\bigg]_{s=-M}^{s=M}\end{aligned} \tag{7.136}$$

$$- 2\int_{-M}^{M} \phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s) \times \left(\int_{-M}^{s} \left[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\right] d\tau\right) ds.$$

By a change of variables we have

$$\int_{-M}^{s} \left[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\right] d\tau = \int_{\mathcal{X}(-M)}^{\mathcal{X}(s)} \mathfrak{p}(X)\, dX - \int_{\mathcal{X}^n(-M)}^{\mathcal{X}^n(s)} \mathfrak{p}^n(X)\, dX.$$

Using an estimate as in (7.121) yields

$$\begin{aligned}&\left|\int_{-M}^{s} \left[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\right] d\tau\right| \\ &\leq ||\mathfrak{p} - \mathfrak{p}^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2M\big)^{\frac{1}{2}} \\ &\quad + 2\big(||\mathfrak{p}^n - \mathfrak{p}||_{L^2(\mathbb{R})} + ||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}}.\end{aligned}$$

This implies that the first term on the right-hand side of (7.136) equals zero. Moreover, we get

$$\begin{aligned}|A_1^n| &\leq 2\Big[||\mathfrak{p} - \mathfrak{p}^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2M\big)^{\frac{1}{2}} \\ &\qquad + 2\big(||\mathfrak{p}^n - \mathfrak{p}||_{L^2(\mathbb{R})} + ||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\Big] \int_{-M}^{M} |\phi'(\mathcal{Z}_2(s))|\dot{\mathcal{Z}}_2(s)\, ds \\ &\leq 2\Big[||\mathfrak{p} - \mathfrak{p}^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2M\big)^{\frac{1}{2}} \\ &\qquad + 2\big(||\mathfrak{p}^n - \mathfrak{p}||_{L^2(\mathbb{R})} + ||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\Big] ||\phi'||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi')\big)\end{aligned}$$

where we used that

$$\begin{aligned}\int_{-M}^{M} |\phi'(\mathcal{Z}_2(s))|\dot{\mathcal{Z}}_2(s)\, ds &\leq \int_{\mathbb{R}} |\phi'(\mathcal{Z}_2(s))|\dot{\mathcal{Z}}_2(s)\, ds \\ &= \int_{\mathbb{R}} |\phi'(x)|\, dx \leq ||\phi'||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi')\big).\end{aligned}$$

Since $\mathcal{X}^n \to \mathcal{X}$ in $L^\infty(\mathbb{R})$ and $\mathfrak{p}^n \to \mathfrak{p}$ in $L^2(\mathbb{R})$ we find that $\lim_{n\to\infty} A_1^n = 0$.

The second integral in (7.135) is estimated as follows. Using the Cauchy–Schwarz inequality and that $0 \leq \dot{\mathcal{X}}^n \leq 2$ we obtain

$$\begin{aligned}|A_2^n| &\leq 2\bigg(\int_{\mathbb{R}} \big(\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big)^2\, ds\bigg)^{\frac{1}{2}} ||\phi \circ \mathcal{Z}_2 - \phi \circ \mathcal{Z}_2^n||_{L^2(\mathbb{R})} \\ &\leq 2\sqrt{2}||\mathfrak{p}^n||_{L^2(\mathbb{R})}||\phi \circ \mathcal{Z}_2 - \phi \circ \mathcal{Z}_2^n||_{L^2(\mathbb{R})} \\ &\leq 2\sqrt{2}\big(||\mathfrak{p}^n - \mathfrak{p}||_{L^2(\mathbb{R})} + ||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\phi \circ \mathcal{Z}_2 - \phi \circ \mathcal{Z}_2^n||_{L^2(\mathbb{R})}\end{aligned}$$

by a change of variables. We have $\phi \circ \mathcal{Z}_2^n \to \phi \circ \mathcal{Z}_2$ pointwise almost everywhere, and by Lemma 7.6 we get that $\phi \circ \mathcal{Z}_2^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function, so the dominated convergence theorem implies that $\lim_{n\to\infty} ||\phi \circ \mathcal{Z}_2 - \phi \circ \mathcal{Z}_2^n||_{L^2(\mathbb{R})} = 0$. Since also $\mathfrak{p}^n \to \mathfrak{p}$ in $L^2(\mathbb{R})$ we find that $\lim_{n\to\infty} A_2^n = 0$.

We return to (7.135), and conclude that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}}\phi(x)\big[\rho(x)-\rho^n(x)\big]\,dx\right|=0.$$

Therefore, $\rho^n \overset{*}{\rightharpoonup} \rho$. In a similar way one shows that $\sigma^n \overset{*}{\rightharpoonup} \sigma$.

We prove that $R^n \overset{*}{\rightharpoonup} R$. Consider a test function ϕ in $C_c^\infty(\mathbb{R})$. By (5.21b) we can write

$$\begin{aligned}
&\int_{\mathbb{R}}\phi(x)\big[R(x)-R^n(x)\big]\,dx\\
&=2\int_{\mathbb{R}}\phi(\mathcal{Z}_2(s))c(\mathcal{Z}_3(s))\big[\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)-\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big]\,ds\\
&\quad+2\int_{\mathbb{R}}c(\mathcal{Z}_3(s))\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[\phi(\mathcal{Z}_2(s))-\phi(\mathcal{Z}_2^n(s))\big]\,ds\\
&\quad+2\int_{\mathbb{R}}\phi(\mathcal{Z}_2^n(s))\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[c(\mathcal{Z}_3(s))-c(\mathcal{Z}_3^n(s))\big]\,ds.
\end{aligned}$$

The first and second integral can be treated more or less like A_1^n and A_2^n in (7.135), respectively. The last integral is estimated as follows

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\phi(\mathcal{Z}_2^n(s))\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[c(\mathcal{Z}_3(s))-c(\mathcal{Z}_3^n(s))\big]\,ds\right|\\
&\le ||\phi||_{L^\infty(\mathbb{R})}||(\mathcal{V}_3^n\circ\mathcal{X}^n)\dot{\mathcal{X}}^n||_{L^2(\mathbb{R})}||c\circ\mathcal{Z}_3-c\circ\mathcal{Z}_3^n||_{L^2(\mathbb{R})}\\
&\le \sqrt{2}k_1||\phi||_{L^\infty(\mathbb{R})}\big(||\mathcal{V}_3^n-\mathcal{V}_3||_{L^2(\mathbb{R})}+||\mathcal{V}_3||_{L^2(\mathbb{R})}\big)||\mathcal{Z}_3-\mathcal{Z}_3^n||_{L^2(\mathbb{R})}.
\end{aligned}$$

This implies that $R^n \overset{*}{\rightharpoonup} R$. In a similar way one shows that $S^n \overset{*}{\rightharpoonup} S$. □

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References

[1] L. Ambrosio, N. Fusco, and D. Pallara: Functions of bounded variation and free discontinuity problems. Oxford University Press, New York (2000).

[2] Y. Brenier: L^2 formulation of multidimensional scalar conservation laws. Arch. Ration. Mech. Anal. **193**, 1–19 (2009).

[3] A. Bressan and T. Huang: Representation of dissipative solutions to a nonlinear variational wave equation. Commun. Math. Sci. **14**, 31–53 (2016).

[4] A. Bressan and Y. Zheng: Conservative solutions to a nonlinear variational wave equation. Comm. Math. Phys. **266**, 471–497 (2006).

[5] L. C. Evans and R. F. Gariepy: Measure theory and fine properties of functions. CRC Press, Boca Raton (1992).

[6] A. M. Fink, D. S. Mitrinović, and J. E. Pečarić: Inequalities involving functions and their integrals and derivatives. Mathematics and its Applications (East European Series), vol. 53., Kluwer Academic Publishers Group, Dordrecht (1991).

[7] G. B. Folland: Real analysis, second edition. Wiley, New York (1999).

[8] R. T. Glassey, J. K. Hunter, and Y. Zheng: Singularities of a variational wave equation. J. Differential Equations **129**, 49–78 (1996).

[9] H. Holden and X. Raynaud: Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. Comm. Partial Differential Equations **32**, 1511–1549 (2007).

[10] H. Holden and X. Raynaud: Global semigroup of conservative solutions of the nonlinear variational wave equation. Arch. Ration. Mech. Anal. **201**, 871–964 (2011).

[11] R. A. Saxton: Dynamic instability of the liquid crystal director. In: W. B. Lindquist (ed.) Current Progress in Hyperbolic Systems: Riemann Problems and Computations (Brunswick, ME, 1988), 325–330. Contemp. Math., Vol. 100, Amer. Math. Soc., Providence, RI (1989).

[12] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation. Arch. Rat. Mech. Anal. **166**, 303–319 (2003).

[13] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation with general data. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 207–226 (2005).

[14] P. Zhang and Y. Zheng: On the global weak solutions to a variational wave equation. In: C. M. Dafermos and E. Feireisl (eds.) Handbook of Differential Equations. Evolutionary Equations. Volume 2, 561–648. Elsevier, Amsterdam (2005).

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Paper II

Traveling Waves for the Nonlinear Variational Wave Equation

K. Grunert and A. Reigstad

TRAVELING WAVES FOR THE NONLINEAR VARIATIONAL WAVE EQUATION

KATRIN GRUNERT AND AUDUN REIGSTAD

Abstract. We study traveling wave solutions of the nonlinear variational wave equation. In particular, we show how to obtain global, bounded, weak traveling wave solutions from local, classical ones. The resulting waves consist of monotone and constant segments, glued together at points where at least one one-sided derivative is unbounded.

Applying the method of proof to the Camassa–Holm equation, we recover some well-known results on its traveling wave solutions.

1. Introduction

We consider the nonlinear variational wave (NVW) equation

$$u_{tt} - c(u)(c(u)u_x)_x = 0, \tag{1.1}$$

with initial data

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1. \tag{1.2}$$

Here, $u = u(t, x)$ where $t \geq 0$ and $x \in \mathbb{R}$.

The NVW equation was introduced by Saxton in [17], where it is derived by applying the variational principle to the functional

$$\int_0^\infty \int_{-\infty}^\infty (u_t^2 - c^2(u)u_x^2)\,dx\,dt.$$

The equation appears in the study of liquid crystals, where it describes the director field of a nematic liquid crystal, and where the function c is given by

$$c^2(u) = \lambda_1 \sin^2(u) + \lambda_2 \cos^2(u), \tag{1.3}$$

where λ_1 and λ_2 are positive physical constants. We refer to [14] and [17] for information about liquid crystals, and the derivation of the equation.

It is well known that derivatives of solutions of the NVW equation can develop singularities in finite time even for smooth initial data, see [8]. A singularity means that either u_x or u_t becomes unbounded pointwise while u remains continuous. The continuation past singularities is highly nontrivial, and allows for various distinct solutions. The most common way of continuing the solution is to require that the energy is non-increasing, which naturally leads to the following two notions of solutions: Dissipative solutions for which the energy is decreasing in time, see [1, 18, 19, 20], and conservative solutions for which the energy is constant in time. In the latter case a semigroup of solutions has been constructed in [2, 12].

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We are interested in traveling wave solutions of (1.1) with wave speed $s \in \mathbb{R}$, i.e., solutions of the form $u(t,x) = w(x - st)$ for some bounded and continuous function w.

A bounded traveling wave was constructed in [8], corresponding to the function c given in (1.3). The constructed wave is a weak solution, which is continuous and piecewise smooth. In particular, the smooth parts are monotone and at their endpoints cusp singularities might turn up. By the latter we mean points where the derivative is unbounded while the solution itself is bounded.

In this paper we consider local, classical traveling wave solutions of (1.1), and study whether these can be glued together to produce globally bounded traveling waves. The approach we use is similar to the derivation of the Rankine–Hugoniot condition for hyperbolic conservation laws, see e.g. [13].

We assume that the function c belongs to $C^2(\mathbb{R})$ and that there exists $0 < \alpha < \beta < \infty$, such that

$$\alpha = \min_{u\in\mathbb{R}} c(u) \quad \text{and} \quad \beta = \max_{u\in\mathbb{R}} c(u). \tag{1.4}$$

Moreover, we assume that

$$\max_{u\in\mathbb{R}} |c'(u)| \leq K_1 \quad \text{and} \quad \max_{u\in\mathbb{R}} |c''(u)| \leq K_2 \tag{1.5}$$

for positive constants K_1 and K_2.

The following theorem is our main result, and will be proved in the next section.

Theorem 1.1. *Let $c \in C^2(\mathbb{R})$ such that α and β defined in (1.4) satisfy $0 < \alpha < \beta < \infty$. Consider the continuous function $w : \mathbb{R} \mapsto \mathbb{R}$ composed of local, classical traveling wave solutions of (1.1) with wave speed $s \in \mathbb{R}$.*

If $|s| \notin [\alpha, \beta]$, then w is a monotone, classical solution, which is globally unbounded.

If $|s| \in [\alpha, \beta]$, we have the following two possibilities:

1. If for some ξ, $|s| \neq c(w(\xi))$ and c has a local maximum or minimum at $w(\xi)$, the wave w is a monotone, classical solution near ξ, which has an inflection point at ξ.

2. If for some ξ, $|s| = c(w(\xi))$ and $c'(w(\xi)) \neq 0$, the wave w has a singularity at ξ, meaning that the derivative is unbounded at ξ while w is continuous. Near the singularity, the wave is a monotone, classical solution on both sides of ξ. The following scenarios are possible:

i) The derivative has the same sign (nonzero) on both sides of ξ, and the wave has an inflection point at ξ.

ii) The derivative has opposite sign (nonzero) on each side of ξ. Then, the wave is either convex or concave on both sides, and the singularity is a cusp.

iii) The wave can be constant on one side of the singularity and strictly monotone on the other side.

For $|s| \in [\alpha, \beta]$, a weak bounded traveling wave solution of (1.1) can be constructed.

We observe that case 2 of Theorem 1.1 allows for globally bounded waves w. Excluding the trivial case of w constant on the whole real line, we then see that the wave consists of increasing, constant, and decreasing parts, and that it has at least two singularities. The simplest nontrivial traveling wave consists of two constant

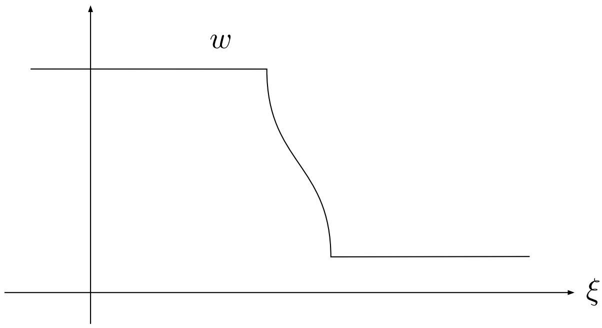

FIGURE 1. A traveling wave solution $w(\xi)$ consisting of two constant values joined together by a strictly decreasing part.

values joined together by a monotone segment, which has two singularities, see Figure 1. This corresponds to the case that $c(u)$ satisfies $c^2(u) = s^2$ for at least two values of u.

In Section 3, we consider the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.6}$$

which was introduced in [5]. The CH equation has been studied intensively within the last three decades. There are too many interesting results to mention here, and we refer to [3, 4, 5, 6, 7, 9, 10, 11] and the citations therein for more information. We point out that the peakon solution, which was already observed in [5], is a weak traveling wave solution of (1.6). This is in contrast to the NVW equation, where there are no known non-constant explicit weak solutions. Moreover, like the NVW equation, singularity formation in the derivatives of solutions to (1.6) may occur, see [7].

In [15], Lenells derives criteria for gluing together local, classical traveling wave solutions of (1.6) to obtain global, bounded traveling waves, see also [16]. By doing so, all weak, bounded traveling wave solutions of the CH equation are classified. Some of these traveling waves have discontinuous derivatives, such as peakons, cuspons, stumpons, and composite waves. These waves have, except for the peakons, singularities in their derivatives.

We apply the aforementioned method to the CH equation and reproduce the criteria derived by Lenells.

2. Proof of Theorem 1.1

Let $\xi = x - st$ and denote the derivative of w with respect to ξ by w_ξ. Assume for the moment that $w \in C^2(\mathbb{R})$. Inserting the derivatives

$$u_t(t,x) = -sw_\xi(x-st), \qquad u_{tt}(t,x) = s^2 w_{\xi\xi}(x-st),$$
$$u_x(t,x) = w_\xi(x-st), \qquad u_{xx}(t,x) = w_{\xi\xi}(x-st)$$

into (1.1) yields

$$\big[s^2 - c^2(w)\big] w_{\xi\xi} - c(w)c'(w)w_\xi^2 = 0. \tag{2.1}$$

Observe that (2.1) is satisfied at all points ξ such that $|s| = c(w(\xi))$, at which either $w_\xi(\xi) = 0$, leading to constant solutions, or $c'(w(\xi)) = 0$. We multiply (2.1) by $2w_\xi$ and get

$$2\big[s^2 - c^2(w)\big]w_{\xi\xi}w_\xi - 2c(w)c'(w)w_\xi^3 = 0,$$

which is the same as

$$\frac{d}{d\xi}\Big(w_\xi^2\big[s^2 - c^2(w)\big]\Big) = 0.$$

Integration leads to

$$w_\xi^2\big[s^2 - c^2(w)\big] = k \tag{2.2}$$

for some integration constant k. Observe that we derived (2.2) assuming that $w \in C^2(\mathbb{R})$, but for (2.2) to make sense it suffices that w is in $C^1(\mathbb{R})$.

We say that u is a local, classical traveling wave solution of (1.1) if $u(t,x) = w(x - st)$, for some w in $C^2(I)$, where I denotes some interval, and satisfies (2.1).

If $|s| \notin [\alpha, \beta]$, then $|s| \neq c(w(\xi))$ for all ξ and we have

$$w_\xi(\xi) = \pm\frac{\sqrt{|k|}}{\sqrt{|s^2 - c^2(w(\xi))|}}. \tag{2.3}$$

The right-hand side of (2.3) is Lipschitz continuous with respect to w and there exists a unique local solution w which is continuously differentiable and monotone. For these solutions we see from (2.3) that the derivatives are bounded. In particular, the solutions are bounded locally, but not globally.

In the case $|s| \in [\alpha, \beta]$, Lipschitz continuity fails, and the standard existence and uniqueness result for ordinary differential equations does not apply. In this case we show, under some specific conditions, that if there is a local solution, it is Hölder continuous. Let w be a bounded and strictly monotone solution of (2.3) on an interval $[\xi_0, \xi_1]$ such that $c(w(\xi_0)) \neq |s|$, $c(w(\xi_1)) \neq |s|$, and $|s| = c(w(\eta))$ for some $\eta \in (\xi_0, \xi_1)$. Then, by assumption, the derivative w_ξ is bounded at ξ_0 and ξ_1. We claim that the solution is Hölder continuous on $[\xi_0, \xi_1]$ if $c'(w(\eta)) \neq 0$. From (2.3) and a change of variables we get

$$\int_{\xi_0}^{\xi_1} w_\xi^2(\xi)\,d\xi = \sqrt{|k|}\bigg|\int_{w(\xi_0)}^{w(\xi_1)} \frac{1}{\sqrt{|s^2 - c^2(z)|}}\,dz\bigg| \tag{2.4}$$

and $c(w(\eta)) = |s|$ yields

$$\int_{\xi_0}^{\xi_1} w_\xi^2(\xi)\,d\xi = \sqrt{|k|}\bigg|\int_{w(\xi_0)}^{w(\xi_1)} \frac{1}{\sqrt{|c^2(w(\eta)) - c^2(z)|}}\,dz\bigg|.$$

The integrand is finite everywhere except at $z = w(\eta)$. For z near $w(\eta)$ we replace $c(z)$ by its Taylor approximation and get

$$\begin{aligned}
|c^2(w(\eta)) - c^2(z)| &= |c(w(\eta)) + c(z)| \cdot |c(w(\eta)) - c(z)| \\
&\geq 2\alpha|c(w(\eta)) - c(z)| \\
&= 2\alpha\Big|c'(w(\eta))(z - w(\eta)) + \frac{1}{2}c''(p)(z - w(\eta))^2\Big|,
\end{aligned}$$

for some p between z and $w(\eta)$. The expression

$$\Big|c'(w(\eta))(z - w(\eta)) + \frac{1}{2}c''(p)(z - w(\eta))^2\Big|^{-\frac{1}{2}}$$

is integrable if $c'(w(\eta)) \neq 0$ and not integrable if $c'(w(\eta)) = 0$. Therefore, the integral

$$\int_{\xi_0}^{\xi_1} w_\xi^2(\xi)\,d\xi$$

is finite if for all $\xi \in (\xi_0, \xi_1)$ such that $|s| = c(w(\xi))$ we have $c'(w(\xi)) \neq 0$. In particular, by the Cauchy–Schwarz inequality we have

$$|w(\xi_1) - w(\xi_0)| \leq ||w_\xi||_{L^2([\xi_0,\xi_1])}|\xi_1 - \xi_0|^{\frac{1}{2}}$$

and w is Hölder continuous with exponent $\frac{1}{2}$. This continuity will be important later in the text when we discuss which traveling waves can be glued together.

We illustrate the above result with an example.

Example 2.1. *Let*

$$A = \frac{\beta - \alpha}{\pi} \quad \textit{and} \quad B = \alpha + \beta,$$

where $0 < \alpha < \beta < \infty$. *Consider the function*

$$c(u) = A\arctan(u) + \frac{B}{2}, \tag{2.5}$$

which is strictly increasing and satisfies

$$\lim_{u\to-\infty} c(u) = \alpha \quad \textit{and} \quad \lim_{u\to+\infty} c(u) = \beta.$$

Consider the wave speed

$$s = \frac{B}{2},$$

where we have $\alpha < s < \beta$. *Let* $f(u) = s^2 - c^2(u)$. *We have*

$$f(u) = -A\arctan(u)(A\arctan(u) + B).$$

We compute the derivative and get

$$f'(u) = -\frac{A}{1+u^2}(2A\arctan(u) + B)$$

and since

$$2A\arctan(u) + B \geq 2\alpha > 0$$

for all u *we have* $f'(u) < 0$. *The only point satisfying* $f(u) = 0$ *is* $u = 0$. *In other words,* $s = c(0)$.

Denote by w *the strictly increasing solution to* (2.3) *and* (2.5). *We assume that* $w(\xi_0) < 0 < w(\xi_1)$, *so that* $c(w(\xi_0)) \neq s$ *and* $c(w(\xi_1)) \neq s$, *which implies that the derivative* w_ξ *is bounded at* ξ_0 *and* ξ_1. *From* (2.4) *we get*

$$\int_{\xi_0}^{\xi_1} w_\xi^2(\xi)\,d\xi = \sqrt{|k|}\int_{w(\xi_0)}^{0} \frac{1}{\sqrt{-A\arctan(z)(A\arctan(z)+B)}}\,dz + \sqrt{|k|}\int_{0}^{w(\xi_1)} \frac{1}{\sqrt{A\arctan(z)(A\arctan(z)+B)}}\,dz, \tag{2.6}$$

By a change of variables we have

$$\int_0^{w(\xi_1)} \frac{1}{\sqrt{A\arctan(z)(A\arctan(z)+B)}}\,dz$$

$$\begin{aligned}
&\leq \int_0^{w(\xi_1)} \frac{1}{\sqrt{AB\arctan(z)}}\,dz \\
&\leq (1+w^2(\xi_1))\int_0^{w(\xi_1)} \frac{1}{\sqrt{AB\arctan(z)}}\frac{1}{1+z^2}\,dz \\
&= (1+w^2(\xi_1))\frac{1}{\sqrt{AB}}\int_0^{\arctan(w(\xi_1))} y^{-\frac{1}{2}}\,dy \\
&= 2(1+w^2(\xi_1))\frac{1}{\sqrt{AB}}\sqrt{\arctan(w(\xi_1))}
\end{aligned}$$

and since $w(\xi_1)$ is finite, the integral converges. Note that this only holds locally. The first integral in (2.6) *can be treated in the same way, showing that $w_\xi \in L^2([\xi_0,\xi_1])$ and we conclude that w is Hölder continuous on $[\xi_0,\xi_1]$.*

Let us focus on weak traveling wave solutions. To derive the weak form of (1.1) we first assume that we have a bounded solution $u \in C^2((0,\infty)\times\mathbb{R})$. We multiply (1.1) by a smooth test function $\phi \in C_c^\infty((0,\infty)\times\mathbb{R})$ and integrate. We have

$$\int_0^\infty\int_{-\infty}^\infty \big[u_{tt} - c(u)c'(u)u_x^2 - c^2(u)u_{xx}\big]\phi\,dx\,dt = 0.$$

Integrating by parts yields

$$\int_0^\infty\int_{-\infty}^\infty \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x\big]\,dx\,dt = 0. \tag{2.7}$$

We say that a function u satisfying $u(t,\cdot) \in L^\infty(\mathbb{R})$ and $u_t(t,\cdot), u_x(t,\cdot) \in L^2(\mathbb{R})$ for all $t \geq 0$ is a weak solution of (1.1) if (2.7) holds for all test functions ϕ in $C_c^\infty((0,\infty)\times\mathbb{R})$. We observe that if there exists a piecewise smooth traveling wave solution satisfying these conditions, it is Hölder continuous with exponent $\frac{1}{2}$.

In the case of a traveling wave, (2.7) reads

$$\int_0^\infty\int_{-\infty}^\infty \big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt = 0.$$

Now we want to glue together two classical solutions to produce a weak traveling wave solution. At the points where we glue them together the derivatives may be discontinuous. Thus, we consider the following situation: assume that u_t and u_x have discontinuities that move along a smooth curve $\Gamma : x = \gamma(t)$, where we assume that γ is a smooth and strictly increasing function. Moreover, we assume that there exists a sufficiently small neighborhood of $\gamma(t)$ such that u is a classical solution of (1.1) on each side of $\gamma(t)$.

Lemma 2.2. *Given a curve $\Gamma : x = \gamma(t) = st + \gamma_0$, where γ_0 is a constant, denote by D a neighborhood of $(\bar{t},\gamma(\bar{t})) \in \Gamma$. Furthermore, let $D = D_1 \cup \Gamma|_D \cup D_2$, where D_1 and D_2 are the parts of D to the left and to the right of Γ, respectively, see Figure 2. Consider two local, classical traveling wave solutions u_1 and u_2 of* (1.1) *in D_1 and D_2, respectively. Assume that we glue these waves at Γ to obtain a continuous traveling wave $u(t,x) = w(x - st)$ in D, which satisfies*

$$\iint_D \big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt = 0 \quad \textit{for any } \phi \in C_c^\infty(D).$$

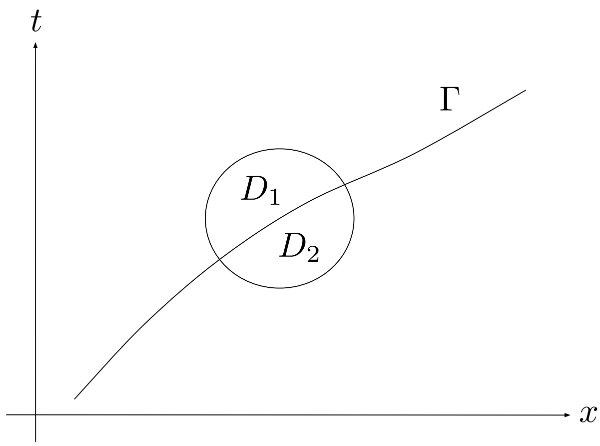

FIGURE 2. Some strictly increasing curve $\Gamma : x = \gamma(t)$ and the neighborhoods D_1 and D_2.

If $|s| \notin [\alpha, \beta]$*, then*

$$w_\xi(\gamma_0-) = w_\xi(\gamma_0+). \tag{2.8a}$$

If $|s| \in (\alpha, \beta)$ *and* $c'(w(\xi)) \neq 0$ *for all* $\xi = x - st$*, such that* $(t, x) \in D$ *and* $|s| = c(w(\xi))$*, then*

$$\begin{aligned} &\Big[\sqrt{|k_1|}\,\mathrm{sign}\big(\big((c^2(w(\gamma_0)) - s^2)w_\xi(\gamma_0)\big) - \big) \\ &\quad - \sqrt{|k_2|}\,\mathrm{sign}\big(\big((c^2(w(\gamma_0)) - s^2)w_\xi(\gamma_0)\big) + \big)\Big]\sqrt{|c^2(w(\gamma_0)) - s^2|} = 0, \end{aligned} \tag{2.8b}$$

where k_1 *and* k_2 *denote the constants in* (2.2) *corresponding to the classical solutions* u_1 *and* u_2 *in* D_1 *and* D_2*, respectively.*

Proof. Let

$$I = \{t \in [0, \infty) \mid (t, \gamma(t)) \in D\}.$$

For any $\varepsilon > 0$ consider

$$D_i^\varepsilon = \{(t, x) \in D_i \mid \mathrm{dist}((t, x), \Gamma) > \varepsilon\}$$

for $i = 1, 2$. We have

$$\begin{aligned} &\iint_D \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x\big]\,dx\,dt \\ &= \lim_{\varepsilon\to 0}\Big(\sum_{i=1}^2 \iint_{D_i^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x\big]\,dx\,dt\Big). \end{aligned} \tag{2.9}$$

Since u is a classical solution in D_1^ε we have

$$\iint_{D_1^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x\big]\,dx\,dt$$

$$= \iint_{D_1^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x - u_{tt}\phi + c(u)c'(u)u_x^2\phi + c^2(u)u_{xx}\phi \big]\, dx\, dt$$
$$= \iint_{D_1^\varepsilon} \big[-(u_t\phi)_t + (c^2(u))_x u_x\phi + c^2(u)(u_x\phi)_x \big]\, dx\, dt$$
$$= \iint_{D_1^\varepsilon} \big[(c^2(u)u_x\phi)_x - (u_t\phi)_t \big]\, dx\, dt.$$

By Green's theorem we get

$$\iint_{D_1^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x \big]\, dx\, dt = \int_{\partial D_1^\varepsilon} \big[u_t\phi\, dx + c^2(u)u_x\phi\, dt \big]$$
$$= \int_{\Gamma_1^\varepsilon} \big[u_t\phi\, dx + c^2(u)u_x\phi\, dt \big],$$

where the last equality follows since ϕ is zero everywhere on ∂D_1^ε except on Γ_1^ε, where Γ_1^ε is the part of ∂D_1^ε which does not coincide with the boundary D. We can parametrize the curve by $\Gamma_1^\varepsilon : x = \gamma_1^\varepsilon(t)$ for $t \in I_1^\varepsilon$, where γ_1^ε is a smooth and strictly increasing function and I_1^ε is an interval. Now we have

$$\iint_{D_1^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x \big]\, dx\, dt$$
$$= \int_{I_1^\varepsilon} \big[(u_t\phi)(t, \gamma_1^\varepsilon(t))(\gamma_1^\varepsilon)'(t) + (c^2(u)u_x\phi)(t, \gamma_1^\varepsilon(t)) \big]\, dt. \tag{2.10}$$

By assumption $u(t, x) = w(x - st)$ is a classical traveling wave solution in D_1^ε. It follows that $\gamma_1^\varepsilon(t) = \gamma(t) - \varepsilon\sqrt{s^2 + 1}$,

$$I_1^\varepsilon = \{ t \in [0, \infty) \mid (t, \gamma_1^\varepsilon(t)) \in D_1 \}$$

and

$$\Gamma_1^\varepsilon = \{ (t, \gamma_1^\varepsilon(t)) \mid t \in I_1^\varepsilon \}.$$

From (2.10) we get

$$\iint_{D_1^\varepsilon} \big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x \big]\, dx\, dt$$
$$= \int_{I_1^\varepsilon} \big[c^2(w(\gamma_1^\varepsilon(t) - st)) - s^2 \big] w_\xi(\gamma_1^\varepsilon(t) - st)\phi(t, \gamma_1^\varepsilon(t))\, dt. \tag{2.11}$$

By similar computations, as above, for D_2^ε we get

$$\iint_{D_2^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x \big]\, dx\, dt = \int_{\Gamma_2^\varepsilon} \big[u_t\phi\, dx + c^2(u)u_x\phi\, dt \big],$$

where Γ_2^ε is the part of ∂D_2^ε which does not coincide with the boundary ∂D. We parametrize the curve by $\Gamma_2^\varepsilon : x = \gamma_2^\varepsilon(t)$ for $t \in I_2^\varepsilon$, where γ_2^ε is a smooth and strictly increasing function and I_2^ε is an interval. We obtain

$$\iint_{D_2^\varepsilon} \big[-u_t\phi_t + c(u)c'(u)u_x^2\phi + c^2(u)u_x\phi_x \big]\, dx\, dt$$
$$= -\int_{I_2^\varepsilon} \big[(u_t\phi)(t, \gamma_2^\varepsilon(t))(\gamma_2^\varepsilon)'(t) + (c^2(u)u_x\phi)(t, \gamma_2^\varepsilon(t)) \big]\, dt \tag{2.12}$$

where the negative sign comes from the fact that we are integrating counterclockwise around the boundary in Green's theorem. Inserting $u(t,x) = w(x-st)$ yields

$$\begin{aligned}\iint_{D_2^\varepsilon} \big[s w_\xi \phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt \\ = -\int_{I_2^\varepsilon} \big[c^2(w(\gamma_2^\varepsilon(t)-st)) - s^2\big] w_\xi(\gamma_2^\varepsilon(t)-st)\phi(t,\gamma_2^\varepsilon(t))\,dt,\end{aligned} \tag{2.13}$$

where $\gamma_2^\varepsilon(t) = \gamma(t) + \varepsilon\sqrt{s^2+1}$,

$$I_2^\varepsilon = \{t \in [0,\infty) \mid (t,\gamma_2^\varepsilon(t)) \in D_2\}$$

and

$$\Gamma_2^\varepsilon = \{(t,\gamma_2^\varepsilon(t)) \mid t \in I_2^\varepsilon\}.$$

Consider $|s| \notin [\alpha,\beta]$. Then $|s| \neq c(w(\xi))$ for all ξ, and by (2.2) the derivative w_ξ is bounded at all points in D. From (2.11) and (2.13) we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\iint_{D_1^\varepsilon} \big[s w_\xi \phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt \\ = \int_I \big[c^2(w(\gamma_0)) - s^2\big] w_\xi(\gamma_0-)\phi(t,\gamma(t))\,dt\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\iint_{D_2^\varepsilon} \big[s w_\xi \phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt \\ = -\int_I \big[c^2(w(\gamma_0)) - s^2\big] w_\xi(\gamma_0+)\phi(t,\gamma(t))\,dt,\end{aligned}$$

respectively. Here, $w_\xi(\gamma_0-)$ and $w_\xi(\gamma_0+)$ denote the left and right limit of w_ξ at γ_0, respectively. We insert these expressions in (2.9) and get

$$\begin{aligned}\iint_D \big[s w_\xi \phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt \\ = \big[c^2(w(\gamma_0)) - s^2\big]\big[w_\xi(\gamma_0-) - w_\xi(\gamma_0+)\big]\int_I \phi(t,\gamma(t))\,dt.\end{aligned}$$

For w to be a weak solution this expression has to be zero for every test function ϕ, and we must have

$$\big[c^2(w(\gamma_0)) - s^2\big]\big[w_\xi(\gamma_0-) - w_\xi(\gamma_0+)\big] = 0$$

which implies

$$w_\xi(\gamma_0-) = w_\xi(\gamma_0+).$$

This proves (2.8a).

Now we consider $|s| \in [\alpha,\beta]$. In this case w_ξ may be unbounded on the curve Γ and we have to eliminate the derivatives from (2.11) and (2.13). Recall that we only consider continuous waves.

Since w is a classical solution in $\overline{D_1^\varepsilon}$ we get from (2.2),

$$w_\xi^2(\xi)\big[s^2 - c^2(w(\xi))\big] = k_1 \tag{2.14}$$

where k_1 is a constant. We have

$$\begin{aligned}(c^2(w)-s^2)w_\xi &= \operatorname{sign}\big((c^2(w)-s^2)w_\xi\big)\sqrt{|c^2(w)-s^2|}\sqrt{|c^2(w)-s^2|w_\xi^2}\\ &= \operatorname{sign}\big((c^2(w)-s^2)w_\xi\big)\sqrt{|c^2(w)-s^2|}\sqrt{|k_1|}.\end{aligned}$$

In (2.11) we now get

$$\begin{aligned}&\iint_{D_1^\varepsilon}\big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt\\ &= \int_{I_1^\varepsilon}\sqrt{|k_1|}\Big(\operatorname{sign}\big((c^2(w)-s^2)w_\xi\big)\sqrt{|c^2(w)-s^2|}\Big)(\gamma_1^\varepsilon(t)-st)\phi(t,\gamma_1^\varepsilon(t))\,dt\end{aligned}$$

and we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}&\iint_{D_1^\varepsilon}\big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt\\ &= \int_I\sqrt{|k_1|}\operatorname{sign}\Big(\big((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0)\big)-\Big)\\ &\qquad\times\sqrt{|c^2(w(\gamma_0))-s^2|}\phi(t,\gamma(t))\,dt.\end{aligned}\tag{2.15}$$

In a similar way we get by using that

$$w_\xi^2(\xi)\big[s^2-c^2(w(\xi))\big]=k_2\tag{2.16}$$

in $\overline{D_2^\varepsilon}$ for some constant k_2,

$$\begin{aligned}&\iint_{D_2^\varepsilon}\big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt\\ &= -\int_{I_2^\varepsilon}\sqrt{|k_2|}\Big(\operatorname{sign}\big((c^2(w)-s^2)w_\xi\big)\sqrt{|c^2(w)-s^2|}\Big)(\gamma_2^\varepsilon(t)-st)\phi(t,\gamma_2^\varepsilon(t))\,dt,\end{aligned}$$

where we used that

$$(c^2(w)-s^2)w_\xi = \operatorname{sign}\big((c^2(w)-s^2)w_\xi\big)\sqrt{|c^2(w)-s^2|}\sqrt{|k_2|}.$$

We have

$$\begin{aligned}\lim_{\varepsilon\to 0}&\iint_{D_2^\varepsilon}\big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt\\ &= -\int_I\sqrt{|k_2|}\operatorname{sign}\Big(\big((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0)\big)+\Big)\\ &\qquad\times\sqrt{|c^2(w(\gamma_0))-s^2|}\phi(t,\gamma(t))\,dt.\end{aligned}\tag{2.17}$$

Combining (2.15) and (2.17) in (2.9) we get

$$\begin{aligned}&\iint_D\big[sw_\xi\phi_t + c(w)c'(w)w_\xi^2\phi + c^2(w)w_\xi\phi_x\big]\,dx\,dt\\ &= \int_I\Big[\sqrt{|k_1|}\operatorname{sign}\Big(\big((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0)\big)-\Big)\\ &\quad-\sqrt{|k_2|}\operatorname{sign}\Big(\big((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0)\big)+\Big)\Big]\sqrt{|c^2(w(\gamma_0))-s^2|}\phi(t,\gamma(t))\,dt.\end{aligned}$$

For w to be a weak solution this expression has to be zero for every test function, and we must have

$$\Big[\sqrt{|k_1|}\,\mathrm{sign}\big(((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0))-\big) - \sqrt{|k_2|}\,\mathrm{sign}\big(((c^2(w(\gamma_0))-s^2)w_\xi(\gamma_0))+\big)\Big]\sqrt{|c^2(w(\gamma_0))-s^2|}=0.$$

This concludes the proof of (2.8b). □

Remark 2.3. *Note that* (2.10) *and* (2.12) *hold for any solution* u *and curve* $x=\gamma(t)$ *as described before the lemma, not just for traveling wave solutions, where* $\gamma(t)=st+\gamma_0$.

Using Lemma 2.2 we prove Theorem 1.1.

Proof of Theorem 1.1. Consider $|s|\notin[\alpha,\beta]$. From (2.8a), w and its derivative w_ξ are continuous at γ_0. In particular, w is monotone and coincides with the global solution for (2.3) for a fixed k. Thus, the resulting wave will be unbounded and hence not a weak solution to the NVW equation. We will not discuss this case further.

Now consider $|s|\in[\alpha,\beta]$. First we study the case $|s|\neq c(w(\gamma_0))$. For (2.8b) to be satisfied we must have

$$\sqrt{|k_1|}\,\mathrm{sign}\big(w_\xi(\gamma_0)-\big)-\sqrt{|k_2|}\,\mathrm{sign}\big(w_\xi(\gamma_0)+\big)=0.$$

If $\mathrm{sign}\big(w_\xi(\gamma_0)-\big)$ and $\mathrm{sign}\big(w_\xi(\gamma_0)+\big)$ have opposite sign we get $k_1=k_2=0$ and w is constant in D.

If $\mathrm{sign}\big(w_\xi(\gamma_0)-\big)$ and $\mathrm{sign}\big(w_\xi(\gamma_0)+\big)$ have the same sign then $|k_1|=|k_2|$. Then the solution w is monotone in D and is given by

$$w_\xi(\xi)=\pm\frac{\sqrt{|k_1|}}{\sqrt{|s^2-c^2(w(\xi))|}}.$$

Since w is a classical solution in D_1^ε and D_2^ε, and $|s|\neq c(w(\gamma_0))$, we have $|s|\neq c(w(\xi))$ in D. Both w and its derivative w_ξ are continuous at γ_0. In particular, w is monotone and coincides with the local solution in D of the above differential equation for a fixed k_1.

Thus, we showed that gluing solutions at points γ_0 so that $c(w(\gamma_0))\neq|s|$, does not yield a new solution. In particular, one can possibly only glue two solutions with different k together at a point γ_0 to obtain a new solution, if $c(w(\gamma_0))=|s|$. This means in particular, for bounded, non-constant waves, that c must have at least one extremal point and hence w must have at least one inflection point.

Next we consider $|s|\in[\alpha,\beta]$ such that $|s|=c(w(\gamma_0))$. As discussed before, at points ξ where $|s|=c(w(\xi))$ and $c'(w(\xi))=0$, $w_\xi(\xi)$ is unbounded and w_ξ does not belong to $L^2_{\mathrm{loc}}(\mathbb{R})$. Therefore, by the definition of a weak solution, we cannot use such waves as building blocks. This immediately excludes the cases $|s|=\alpha$ and $|s|=\beta$. Thus, we consider $|s|\in(\alpha,\beta)$ and assume that all points ξ such that $|s|=c(w(\xi))$ satisfy $c'(w(\xi))\neq 0$.

The remaining case to be treated is $|s|\in(\alpha,\beta)$ such that $|s|=c(w(\gamma_0))$ and $c'(w(\gamma_0))\neq 0$. Using the same notation as in the proof of Lemma 2.2, denote by $u_1(t,x)=w_1(x-st)$ and $u_2(t,x)=w_2(x-st)$ the classical solutions to (1.1) in D_1^ε and D_2^ε, respectively. Then w_1 and w_2 are locally Hölder continuous. Furthermore, we see from (2.14) and (2.16) that k_1 and k_2 are finite and (2.8b) is satisfied for any values of k_1 and k_2. In particular, the functions w_1 and w_2 satisfy

$$w_{1,\xi}(\xi) = \pm \frac{\sqrt{|k_1|}}{\sqrt{|s^2 - c^2(w_1(\xi))|}} \quad \text{and} \quad w_{2,\xi}(\xi) = \pm \frac{\sqrt{|k_2|}}{\sqrt{|s^2 - c^2(w_2(\xi))|}} \tag{2.18}$$

in D_1^ε and D_2^ε, respectively.

We study the case $w_{1,\xi}(\xi) \to \pm\infty$ as $\xi \to \gamma_0-$ and $w_{2,\xi}(\xi) \to \pm\infty$ as $\xi \to \gamma_0+$. It remains to show which solutions, if any, can be glued together.

Let $s > 0$. Assume that $s = c(w_1(\gamma_0)) = c(w_2(\gamma_0))$ and $c'(w_1(\gamma_0)) = c'(w_2(\gamma_0)) < 0$. Since c', w_1 and w_2 are continuous we have $c'(w_1(\xi)) < 0$ for $\xi < \gamma_0$ near γ_0 and $c'(w_2(\xi)) < 0$ for $\xi > \gamma_0$ near γ_0. We have the following four possibilities:

1. If $w_{1,\xi}(\xi) > 0$ for $\xi < \gamma_0$ near γ_0 then $c(w_1(\xi)) > s$ and from (2.1) we get $w_{1,\xi\xi}(\xi) > 0$.

2. If $w_{1,\xi}(\xi) < 0$ for $\xi < \gamma_0$ near γ_0 then $c(w_1(\xi)) < s$ and (2.1) implies that $w_{1,\xi\xi}(\xi) < 0$.

3. If $w_{2,\xi}(\xi) > 0$ for $\xi > \gamma_0$ near γ_0 then $c(w_2(\xi)) < s$ and by (2.1) we have $w_{2,\xi\xi}(\xi) < 0$.

4. If $w_{2,\xi}(\xi) < 0$ for $\xi > \gamma_0$ near γ_0 then $c(w_2(\xi)) > s$ and by (2.1) we have $w_{2,\xi\xi}(\xi) > 0$.

We have now 4 possibilities for gluing waves at γ_0: 1. and 4., 1. and 3., 2. and 3., and 2. and 4. In all cases the derivatives are unbounded at the gluing point. For instance, combining 1. and 4. results in a wave with a cusp at γ_0. Since the constants k_1 and k_2 may differ, w_1 and w_2 may have different slope away from γ_0.

Another possibility, due to (2.1), is that either w_1 or w_2 is constant. We can combine constant solutions with singular waves. For instance, let $w_1(\xi) = w_2(\gamma_0)$ for $\xi \leq \gamma_0$, and w_2 be as in 3.

We can also combine the wave in 1. with the constant solution where $w_2(\xi) = w_1(\gamma_0)$ for $\xi \geq \gamma_0$.

A similar analysis can be done in the case $c'(w_1(\gamma_0)) = c'(w_2(\gamma_0)) > 0$.

Note that the resulting waves may be unbounded. This is for example the case if $c(u) = |s|$ for exactly one $u \in \mathbb{R}$. On the other hand, the resulting waves belong to $L^2(D)$ and are locally Hölder continuous.

Finally we study how we can glue local waves to get a bounded traveling wave. Let us consider the wave composed of 1. and 3. For $\xi < \gamma_0$ near γ_0, it is given by $w_1(\xi)$ which is strictly increasing and convex. For $\xi > \gamma_0$ near γ_0, it is given by $w_2(\xi)$ which is strictly increasing and concave. In this case we assumed that the function c is strictly decreasing at the point $w_1(\gamma_0) = w_2(\gamma_0)$. Now we assume that c has a local minimum to the right of this point. More precisely, we assume that there exists $E_1 > \gamma_0$ such that $c'(w_2(E_1)) = 0$, $c'(w_2(\xi)) < 0$ for all $\gamma_0 \leq \xi < E_1$, and $c'(w_2(\xi)) > 0$ for all $E_1 < \xi < \xi_1$ for ξ_1 near E_1, so that $c(w_2(\xi)) < s$ for all $E_1 < \xi \leq \xi_1$.

The function $w_2(\xi)$ is a strictly increasing classical solution for all $\gamma_0 < \xi < \xi_1$. Furthermore, $w_2(\xi)$ has an inflection point at $\xi = E_1$ and is concave for $\gamma_0 \leq \xi < E_1$ and convex for $E_1 < \xi \leq \xi_1$.

If $c'(w_2(\xi)) > 0$ for all $\xi_1 \leq \xi \leq \gamma_1$ where γ_1 satisfies $c(w_2(\gamma_1)) = s$, we can continue the wave after γ_1 either by a singular wave or by setting w equal to $w(\gamma_1)$ for $\gamma_1 < \xi$. The situation is illustrated in Figure 3 and 4.

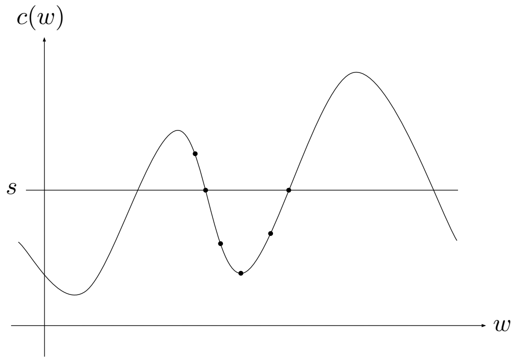

FIGURE 3. The points are from left to right: $w_1(\xi)$ for $\xi < \gamma_0$, $w_1(\gamma_0) = w_2(\gamma_0)$, $w_2(\xi)$ for $\gamma_0 < \xi < E_1$, $w_2(E_1)$, $w_2(\xi_1)$ and $w_2(\gamma_1)$.

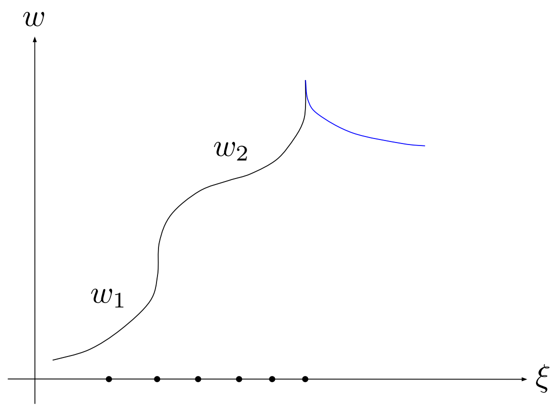

FIGURE 4. The points are from left to right: $\xi < \gamma_0$, γ_0, $\gamma_0 < \xi < E_1$, E_1, ξ_1 and γ_1. The blue part to the right shows one of the three ways of continuing the wave for $\xi > \gamma_1$.

Depending on the function c, we can continue this gluing procedure to produce a wave w consisting of decreasing, increasing, and constant segments. Note that to get a non-constant bounded traveling wave we have to use both increasing and decreasing parts. The derivative of the composite wave belongs to $L^2_{\text{loc}}(\mathbb{R})$. If $w_\xi \in L^2(\mathbb{R})$, then w is not only a global traveling wave solution, but also a weak solution. $\square$

3. The Camassa–Holm Equation

Now we study weak traveling wave solutions of the CH equation (1.6). We insert for a traveling wave solution $u(t, x) = w(x - st)$, and get

$$-sw_\xi + sw_{\xi\xi\xi} + 3ww_\xi - 2w_\xi w_{\xi\xi} - ww_{\xi\xi\xi} = 0. \tag{3.1}$$

We say that u is a local, classical traveling wave solution of (1.6) if $u(t,x) = w(x-st)$ for some w in $C^3(I)$, where I denotes some interval, and satisfies (3.1).

Rewriting (3.1) yields

$$-sw_\xi + sw_{\xi\xi\xi} + \frac{3}{2}(w^2)_\xi - \frac{1}{2}(w_\xi^2)_\xi - (ww_{\xi\xi})_\xi = 0.$$

Integration leads to

$$-sw + sw_{\xi\xi} + \frac{3}{2}w^2 - \frac{1}{2}w_\xi^2 - ww_{\xi\xi} = a, \tag{3.2}$$

where a is some integration constant. Multiplying (3.2) by $2w_\xi$ leads to

$$-s(w^2)_\xi + (w^3)_\xi + \big(w_\xi^2(s-w)\big)_\xi = 2aw_\xi$$

and integrating once more we get

$$-sw^2 + w^3 + w_\xi^2(s-w) = 2aw + b, \tag{3.3}$$

where b is some integration constant.

We study if we can glue together local, classical wave solutions like we did in the previous section for the NVW equation. We are interested in the situation where the composite wave has a discontinuous derivative at the gluing points.

First we derive a weak form of the CH equation. Rewrite the equation as

$$u_t - u_{txx} + \frac{3}{2}(u^2)_x - \frac{1}{2}(u_x^2)_x - (uu_{xx})_x = 0.$$

Assume that we have a bounded solution $u \in C^3((0,\infty)\times\mathbb{R})$. We multiply with a smooth test function $\phi \in C_c^\infty((0,\infty)\times\mathbb{R})$, integrate by parts, and get

$$\int_0^\infty \int_{-\infty}^\infty \big(-u\phi_t + u\phi_{txx} - \frac{3}{2}u^2\phi_x + \frac{1}{2}u_x^2\phi_x + uu_{xx}\phi_x\big)\,dx\,dt = 0.$$

Writing

$$\frac{1}{2}u_x^2 + uu_{xx} = (uu_x)_x - \frac{1}{2}u_x^2,$$

and integrating by parts again yields

$$\int_0^\infty \int_{-\infty}^\infty \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt = 0, \tag{3.4}$$

which serves as a basis for defining weak solutions.

A function u satisfying $u(t,\cdot) \in L^\infty(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\mathbb{R})$ for all $t \geq 0$ is said to be a weak solution of (1.6) if (3.4) holds for all test functions ϕ in $C_c^\infty((0,\infty)\times\mathbb{R})$. In the case of a traveling wave solution, $u(t,x) = w(x-st)$, (3.4) reads

$$\int_0^\infty \int_{-\infty}^\infty \big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt.$$

Assume that u_t and u_x have discontinuities that move along a smooth curve $\Gamma : x = \gamma(t)$, where γ is a strictly increasing function, and that u is a local, classical solution of (1.6) on each side of $\gamma(t)$.

Lemma 3.1. *Given a curve $\Gamma : x = \gamma(t) = st + \gamma_0$, where γ_0 is a constant, denote by D a neighborhood of $(\bar{t}, \gamma(\bar{t})) \in \Gamma$. Furthermore, let $D = D_1 \cup \Gamma|_D \cup D_2$, where D_1 and D_2 are the parts of D to the left and to the right of Γ, respectively, see Figure 2. Consider two local, classical traveling wave solution u_1 and u_2 of* (1.6) *in D_1 and D_2, respectively. Assume that we glue these waves at Γ to obtain a continuous traveling wave $u(t,x) = w(x - st)$ in D, which satisfies*

(3.5)
$$\iint_D \left[w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\right] dx\, dt = 0 \quad \text{for any } \phi \in C_c^\infty(D).$$

Then we have $a_1 = a_2$, where a_1 and a_2 denote the constants in (3.2) *corresponding to the local, classical solutions $u_1(t,x) = w_1(x - st)$ and $u_2(t,x) = w_2(x - st)$ in D_1 and D_2, respectively.*

If w_ξ and $w_{\xi\xi}$ are bounded on the curve $x = \gamma(t)$, then

$$(w(\gamma_0) - s)(w_\xi(\gamma_0-) - w_\xi(\gamma_0+)) = 0. \tag{3.6a}$$

If w_ξ and $w_{\xi\xi}$ may be unbounded on the curve $x = \gamma(t)$, then

$$\begin{aligned}
&\operatorname{sign}\big(\big((w(\gamma_0) - s)w_\xi(\gamma_0)\big) - \big) \\
&\times \sqrt{(w^2(\gamma_0)(w(\gamma_0) - s) - 2a_1 w(\gamma_0) - b_1)(w(\gamma_0) - s)} \\
&- \operatorname{sign}\big(\big((w(\gamma_0) - s)w_\xi(\gamma_0)\big) + \big) \\
&\times \sqrt{(w^2(\gamma_0)(w(\gamma_0) - s) - 2a_1 w(\gamma_0) - b_2)(w(\gamma_0) - s)} = 0,
\end{aligned} \tag{3.6b}$$

where b_1 and b_2 are the constants in (3.3) *corresponding to the local, classical solution $u_1(t,x) = w_1(x - st)$ and $u_2(t,x) = w_2(x - st)$ in D_1 and D_2, respectively.*

Proof. Let

$$I = \{t \in [0, \infty) \mid (t, \gamma(t)) \in D\}.$$

For any $\varepsilon > 0$ consider

$$D_i^\varepsilon = \{(t,x) \in D_i \mid \operatorname{dist}((t,x), \Gamma) > \varepsilon\}$$

for $i = 1, 2$. We have

$$\begin{aligned}
&\iint_D \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\, dx\, dt \\
&= \lim_{\varepsilon \to 0} \bigg(\sum_{i=1}^{2} \iint_{D_i^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\, dx\, dt\bigg).
\end{aligned}$$

Since u is a classical solution in D_1^ε we get

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\, dx\, dt \\
&= \iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x \\
&\qquad + u_t\phi - u_{txx}\phi + \frac{3}{2}(u^2)_x\phi - \frac{1}{2}(u_x^2)_x\phi - (uu_{xx})_x\phi\big)\, dx\, dt.
\end{aligned}$$

We integrate by parts and get

$$\iint_{D_1^\varepsilon} u\phi_{txx}\,dx\,dt = \int_{I_1^\varepsilon} (u\phi_{tx})(t,\gamma_1^\varepsilon(t))\,dt - \iint_{D_1^\varepsilon} u_x\phi_{tx}\,dx\,dt$$

$$\iint_{D_1^\varepsilon} u_{txx}\phi\,dx\,dt = \int_{I_1^\varepsilon} (u_{tx}\phi)(t,\gamma_1^\varepsilon(t))\,dt - \iint_{D_1^\varepsilon} u_{tx}\phi_x\,dx\,dt,$$

and

$$\iint_{D_1^\varepsilon} (uu_{xx})_x\phi\,dx\,dt = \int_{I_1^\varepsilon} (uu_{xx}\phi)(t,\gamma_1^\varepsilon(t))\,dt - \iint_{D_1^\varepsilon} uu_{xx}\phi_x\,dx\,dt,$$

where Γ_1^ε denotes the part of ∂D_1^ε, which does not coincide with the boundary of D. We can parametrize this curve by $\Gamma_1^\varepsilon : x = \gamma_1^\varepsilon(t)$ for $t \in I_1^\varepsilon$. Here γ_1^ε is a smooth and strictly increasing function and I_1^ε is an interval. Now we get

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= \iint_{D_1^\varepsilon} \big((u\phi)_t + u_x\phi_{tx} + \frac{3}{2}(u^2\phi)_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x \\
&\qquad + u_{tx}\phi_x - \frac{1}{2}(u_x^2)_x\phi + uu_{xx}\phi_x\big)\,dx\,dt \\
&\quad - \int_{I_1^\varepsilon} \big(uu_{xx}\phi + u\phi_{tx} + u_{tx}\phi\big)(t,\gamma_1^\varepsilon(t))\,dt.
\end{aligned} \tag{3.7}$$

Using

$$\begin{aligned}
&uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x - \frac{1}{2}(u_x^2)_x\phi + uu_{xx}\phi_x \\
&= uu_x\phi_{xx} - \frac{1}{2}u_x^2\phi_x - \frac{1}{2}(u_x^2)_x\phi + u_x^2\phi_x + uu_{xx}\phi_x \\
&= uu_x\phi_{xx} - \frac{1}{2}(u_x^2\phi)_x + (uu_x)_x\phi_x \\
&= -\frac{1}{2}\big(u_x^2\phi\big)_x + (uu_x\phi_x)_x
\end{aligned}$$

we obtain

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= \iint_{D_1^\varepsilon} \big((u\phi)_t + (u_x\phi_x)_t + \frac{3}{2}(u^2\phi)_x - \frac{1}{2}\big(u_x^2\phi\big)_x + (uu_x\phi_x)_x\big)\,dx\,dt \\
&\quad - \int_{I_1^\varepsilon} \big(uu_{xx}\phi + u\phi_{tx} + u_{tx}\phi\big)(t,\gamma_1^\varepsilon(t))\,dt \\
&= \iint_{D_1^\varepsilon} \Big[\frac{\partial}{\partial x}\Big(\frac{3}{2}u^2\phi - \frac{1}{2}u_x^2\phi + uu_x\phi_x\Big) + \frac{\partial}{\partial t}\Big(u\phi + u_x\phi_x\Big)\Big]\,dx\,dt \\
&\quad - \int_{I_1^\varepsilon} \big(uu_{xx}\phi + u\phi_{tx} + u_{tx}\phi\big)(t,\gamma_1^\varepsilon(t))\,dt.
\end{aligned}$$

By Green's theorem we get

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= \int_{\partial D_1^\varepsilon} \Big[-\Big(u\phi + u_x\phi_x\Big)\,dx + \Big(\frac{3}{2}u^2\phi - \frac{1}{2}u_x^2\phi + uu_x\phi_x\Big)\,dt\Big] \\
&\quad - \int_{I_1^\varepsilon} \big(uu_{xx}\phi + u\phi_{tx} + u_{tx}\phi\big)(t,\gamma_1^\varepsilon(t))\,dt,
\end{aligned}$$

and since the integrand is zero everywhere on ∂D_1^ε except on the part corresponding to $\gamma_1^\varepsilon(t)$ we have

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= \int_{I_1^\varepsilon} \Big[-(u\phi + u_x\phi_x)(t,\gamma_1^\varepsilon(t))(\gamma_1^\varepsilon)'(t) \\
&\qquad + \Big(\frac{3}{2}u^2\phi - \frac{1}{2}u_x^2\phi + uu_x\phi_x - uu_{xx}\phi - u\phi_{tx} - u_{tx}\phi\Big)(t,\gamma_1^\varepsilon(t))\Big]\,dt.
\end{aligned} \tag{3.8}$$

From this point on we assume that $u(t,x) = w(x - st)$ is a classical traveling wave solution of (1.6). Then (3.8) rewrites as

$$\begin{aligned}
&\iint_{D_1^\varepsilon} \big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt \\
&= \int_{I_1^\varepsilon} \Big(\Big[-ws + \frac{3}{2}w^2 - \frac{1}{2}w_\xi^2 - (w-s)w_{\xi\xi}\Big](\gamma_1^\varepsilon(t) - st)\phi(t,\gamma_1^\varepsilon(t)) \\
&\qquad + \big[(w-s)w_\xi\big](\gamma_1^\varepsilon(t) - st)\phi_x(t,\gamma_1^\varepsilon(t)) \\
&\qquad - w(\gamma_1^\varepsilon(t) - st)\phi_{tx}(t,\gamma_1^\varepsilon(t))\Big)\,dt.
\end{aligned} \tag{3.9}$$

In a similar way as above we obtain

$$\begin{aligned}
&\iint_{D_2^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= \int_{\partial D_2^\varepsilon} \Big[-\Big(u\phi + u_x\phi_x\Big)\,dx + \Big(\frac{3}{2}u^2\phi - \frac{1}{2}u_x^2\phi + uu_x\phi_x\Big)\,dt\Big] \\
&\quad + \int_{I_2^\varepsilon} \big(uu_{xx}\phi + u\phi_{tx} + u_{tx}\phi\big)(t,\gamma_2^\varepsilon(t))\,dt,
\end{aligned}$$

where I_2^ε and $\gamma_2^\varepsilon(t)$ are defined in the same way as their counterparts in D_1^ε. Note that the sign in front of the second integral has changed compared to (3.7), which comes from the fact that D_2^ε is to the right of the curve Γ. Proceeding as above we find

$$\begin{aligned}
&\iint_{D_2^\varepsilon} \big(u\phi_t - u\phi_{txx} + \frac{3}{2}u^2\phi_x + uu_x\phi_{xx} + \frac{1}{2}u_x^2\phi_x\big)\,dx\,dt \\
&= -\int_{I_2^\varepsilon} \Big[-\Big(u\phi + u_x\phi_x\Big)(t,\gamma_2^\varepsilon(t))(\gamma_2^\varepsilon)'(t) + \Big(\frac{3}{2}u^2\phi - \frac{1}{2}u_x^2\phi + uu_x\phi_x\Big)(t,\gamma_2^\varepsilon(t))\Big]
\end{aligned}$$

$$+\int_{I_2^\varepsilon}\big(uu_{xx}\phi+u\phi_{tx}+u_{tx}\phi\big)(t,\gamma_2^\varepsilon(t))\,dt,$$

where the negative sign in front of the first integral comes from the fact that we are integrating counterclockwise around the boundary in Green's theorem. Now we assume that u is a traveling wave solution, i.e., $u(t,x)=w(x-st)$ and get

$$\begin{aligned}(3.10)\qquad &\iint_{D_2^\varepsilon}\big(w\phi_t-w\phi_{txx}+\frac{3}{2}w^2\phi_x+ww_\xi\phi_{xx}+\frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\ &=\int_{I_2^\varepsilon}\bigg(-\Big[-ws+\frac{3}{2}w^2-\frac{1}{2}w_\xi^2-(w-s)w_{\xi\xi}\Big](\gamma_2^\varepsilon(t)-st)\phi(t,\gamma_2^\varepsilon(t))\\ &\qquad-\big[(w-s)w_\xi\big](\gamma_2^\varepsilon(t)-st)\phi_x(t,\gamma_2^\varepsilon(t))\\ &\qquad+w(\gamma_2^\varepsilon(t)-st)\phi_{tx}(t,\gamma_2^\varepsilon(t))\bigg)\,dt.\end{aligned}$$

From (3.2) and (3.3) we have

$$(3.11)\qquad -sw+sw_{\xi\xi}+\frac{3}{2}w^2-\frac{1}{2}w_\xi^2-ww_{\xi\xi}=a_i$$

and

$$(3.12)\qquad -sw^2+w^3+w_\xi^2(s-w)=2a_iw+b_i$$

in $\overline{D_i^\varepsilon}$, $i=1,2$. We insert (3.11) in (3.9) and obtain

$$\begin{aligned}(3.13)\qquad &\iint_{D_1^\varepsilon}\big(w\phi_t-w\phi_{txx}+\frac{3}{2}w^2\phi_x+ww_\xi\phi_{xx}+\frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\ &=\int_{I_1^\varepsilon}\bigg(a_1\phi(t,\gamma_1^\varepsilon(t))+\big[(w-s)w_\xi\big](\gamma_1^\varepsilon(t)-st)\phi_x(t,\gamma_1^\varepsilon(t))\\ &\qquad-w(\gamma_1^\varepsilon(t)-st)\phi_{tx}(t,\gamma_1^\varepsilon(t))\bigg)\,dt.\end{aligned}$$

Inserting (3.12) into (3.10) yields

$$\begin{aligned}(3.14)\qquad &\iint_{D_2^\varepsilon}\big(w\phi_t-w\phi_{txx}+\frac{3}{2}w^2\phi_x+ww_\xi\phi_{xx}+\frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\ &=\int_{I_2^\varepsilon}\bigg(-a_2\phi(t,\gamma_2^\varepsilon(t))-\big[(w-s)w_\xi\big](\gamma_2^\varepsilon(t)-st)\phi_x(t,\gamma_2^\varepsilon(t))\\ &\qquad+w(\gamma_2^\varepsilon(t)-st)\phi_{tx}(t,\gamma_2^\varepsilon(t))\bigg)\,dt.\end{aligned}$$

Assume that w_ξ and $w_{\xi\xi}$ are bounded on the curve $x=\gamma(t)$. Since w, ϕ and the derivatives of ϕ are continuous we get from (3.13),

$$\begin{aligned}(3.15)\qquad &\lim_{\varepsilon\to0}\iint_{D_1^\varepsilon}\big(w\phi_t-w\phi_{txx}+\frac{3}{2}w^2\phi_x+ww_\xi\phi_{xx}+\frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\ &=\int_I\bigg(a_1\phi(t,\gamma(t))+\big[(w(\gamma_0)-s)w_\xi(\gamma_0-)\big]\phi_x(t,\gamma(t))\end{aligned}$$

$$- w(\gamma_0)\phi_{tx}(t,\gamma(t))\Big)\, dt.$$

From (3.14) we obtain

$$\lim_{\varepsilon\to 0}\iint_{D_2^\varepsilon} \big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\, dx\, dt \tag{3.16}$$
$$= \int_I \Big(- a_2\phi(t,\gamma(t)) - \big[(w(\gamma_0) - s)w_\xi(\gamma_0+)\big]\phi_x(t,\gamma(t)) + w(\gamma_0)\phi_{tx}(t,\gamma(t))\Big)\, dt.$$

We combine (3.15) and (3.16), and obtain

$$\iint_D \big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\, dx\, dt \tag{3.17}$$
$$= \int_I \Big((a_1 - a_2)\phi(t,\gamma(t)) + (w(\gamma_0) - s)(w_\xi(\gamma_0-) - w_\xi(\gamma_0+))\phi_x(t,\gamma(t))\Big)\, dt$$

for all $\phi \in C_c^\infty(D)$. Now we derive conditions that ensure that the integral in (3.17) is equal to zero for all test functions $\phi \in C_c^\infty(D)$. For a positive number d and constants k and l satisfying $k < l$, consider the domain $\tilde{D} \subset D$ bounded by the lines $t = k$, $t = l$ and $x = \gamma(t) \pm d$. We define the test function $\tilde{\phi} \in C_c^\infty(D)$ by

$$\tilde{\phi}(t,x) = \exp\left\{-\frac{1}{d - (x-\gamma(t))^2} - \frac{1}{\frac{1}{4}(l-k)^2 - (t - \frac{k+l}{2})^2}\right\}, \tag{3.18}$$

which is positive and smooth in $\tilde{D}$ and equals zero on the boundary $\partial\tilde{D}$. In particular, $\tilde{\phi}_x(t,\gamma(t)) = 0$ for all $t \in [k,l]$. From (3.5) and (3.17), we then get

$$\iint_{\tilde{D}} \big(w\tilde{\phi}_t - w\tilde{\phi}_{txx} + \frac{3}{2}w^2\tilde{\phi}_x + ww_\xi\tilde{\phi}_{xx} + \frac{1}{2}w_\xi^2\tilde{\phi}_x\big)\, dx\, dt = \int_a^b \big[a_1 - a_2\big]\tilde{\phi}(t,\gamma(t))\, dt = 0,$$

and since $\tilde{\phi}$ is positive in $\tilde{D}$ this implies that $a_1 = a_2$. Furthermore, since (3.17) should be equal to zero for all test functions ϕ, we must have

$$\iint_D \big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\, dx\, dt \tag{3.19}$$
$$= \int_I (w(\gamma_0) - s)(w_\xi(\gamma_0-) - w_\xi(\gamma_0+))\phi_x(t,\gamma(t))\, dt = 0$$

for all $\phi \in C_c^\infty(D)$, which implies

$$(w(\gamma_0) - s)(w_\xi(\gamma_0-) - w_\xi(\gamma_0+)) = 0.$$

This proves (3.6a).

Now assume that w_ξ and $w_{\xi\xi}$ may be unbounded on the curve $x = \gamma(t)$. Recall that w is a classical solution in D_i^ε, $i = 1, 2$.

Using (3.12) we write

$$(w - s)w_\xi = \operatorname{sign}\big((w-s)w_\xi\big)\sqrt{(w-s)^2 w_\xi^2}$$
$$= \operatorname{sign}\big((w-s)w_\xi\big)\sqrt{(w^2(w-s) - 2a_1 w - b_1)(w-s)}$$

in $\overline{D_1^\varepsilon}$, which inserted into (3.13) yields

$$\begin{aligned}&\iint_{D_1^\varepsilon}\big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\&= \int_{I_1^\varepsilon}\bigg(a_1\phi(t,\gamma_1^\varepsilon(t)) - w(\gamma_1^\varepsilon(t)-st)\phi_{tx}(t,\gamma_1^\varepsilon(t))\\&\qquad + \Big[\operatorname{sign}\big((w-s)w_\xi\big)\sqrt{(w^2(w-s)-2a_1w-b_1)(w-s)}\Big](\gamma_1^\varepsilon(t)-st)\\&\qquad\qquad \times\phi_x(t,\gamma_1^\varepsilon(t))\bigg)\,dt.\end{aligned}$$

Letting ε tend to zero we obtain

$$\begin{aligned}&\lim_{\varepsilon\to0}\iint_{D_1^\varepsilon}\big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\&= \int_I\bigg(a_1\phi(t,\gamma(t)) - w(\gamma_0)\phi_{tx}(t,\gamma(t)) + \operatorname{sign}\big(((w(\gamma_0)-s)w_\xi(\gamma_0))-\big)\\&\qquad\times\sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_1)(w(\gamma_0)-s)}\phi_x(t,\gamma(t))\bigg)\,dt.\end{aligned}$$

In a similar way we get by using (3.12) in (3.14),

$$\begin{aligned}&\lim_{\varepsilon\to0}\iint_{D_2^\varepsilon}\big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\&= \int_I\bigg(-a_2\phi(t,\gamma(t)) + w(\gamma_0)\phi_{tx}(t,\gamma(t)) - \operatorname{sign}\big(((w(\gamma_0)-s)w_\xi(\gamma_0))+\big)\\&\qquad\times\sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_2w(\gamma_0)-b_2)(w(\gamma_0)-s)}\phi_x(t,\gamma(t))\bigg)\,dt.\end{aligned}$$

This implies that

$$\begin{aligned}&\iint_{D}\big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\&= \int_I\bigg((a_1-a_2)\phi(t,\gamma(t)) + \Big(\operatorname{sign}\big(((w(\gamma_0)-s)w_\xi(\gamma_0))-\big)\\&\qquad\times\sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_1)(w(\gamma_0)-s)}\\&\qquad-\operatorname{sign}\big(((w(\gamma_0)-s)w_\xi(\gamma_0))+\big)\\&\qquad\times\sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_2w(\gamma_0)-b_2)(w(\gamma_0)-s)}\Big)\phi_x(t,\gamma(t))\bigg)\,dt\end{aligned}$$

for all $\phi\in C_c^\infty(D)$.

As before, we choose the test function $\tilde\phi\in C_c^\infty(D)$ given by (3.18) to obtain $a_1=a_2$. Thus, we have

$$\begin{aligned}&\iint_{D}\big(w\phi_t - w\phi_{txx} + \frac{3}{2}w^2\phi_x + ww_\xi\phi_{xx} + \frac{1}{2}w_\xi^2\phi_x\big)\,dx\,dt\\&= \int_I\Big(\operatorname{sign}\big(((w(\gamma_0)-s)w_\xi(\gamma_0))-\big)\end{aligned}$$

$$\times \sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_1)(w(\gamma_0)-s)}$$
$$-\operatorname{sign}\big(\big((w(\gamma_0)-s)w_\xi(\gamma_0)\big)+\big)$$
$$\times \sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_2)(w(\gamma_0)-s)}\Big)\phi_x(t,\gamma(t))\,dt$$

for all $\phi \in C_c^\infty(D)$. For this integral to be equal to zero for all test functions, we must have

$$\operatorname{sign}\big(\big((w(\gamma_0)-s)w_\xi(\gamma_0)\big)-\big)$$
$$\times \sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_1)(w(\gamma_0)-s)}$$
$$-\operatorname{sign}\big(\big((w(\gamma_0)-s)w_\xi(\gamma_0)\big)+\big)$$
$$\times \sqrt{(w^2(\gamma_0)(w(\gamma_0)-s)-2a_1w(\gamma_0)-b_2)(w(\gamma_0)-s)}=0.$$

This concludes the proof of (3.6b). □

We apply Lemma 3.1 to study which local, classical traveling waves we can glue together.

Bounded derivatives

Case 1. Consider $s \neq w(\gamma_0)$. For (3.6a) to be satisfied we must have $w_\xi(\gamma_0-) = w_\xi(\gamma_0+)$, i.e., the derivative is continuous at γ_0. From (3.12) we then get that $b_1 = b_2$, and w coincides with the local solution in D of the differential equation (3.12).

Case 2. If $s = w(\gamma_0)$, (3.6a) is satisfied. Since $u(t,\gamma(t)) = w(\gamma(t)-st) = w(\gamma_0)$ we get $u(t,\gamma(t)) = s$ for all $t \geq 0$. We denote the solution in D_1 and D_2 by w_1 and w_2, respectively. Then $w_1(\gamma_0) = w_2(\gamma_0) = s$. We let ξ tend to γ_0 in (3.11) and obtain

$$a_1 = \frac{1}{2}w_1^2(\gamma_0) - \frac{1}{2}w_{1,\xi}^2(\gamma_0)$$

and

$$a_2 = \frac{1}{2}w_2^2(\gamma_0) - \frac{1}{2}w_{2,\xi}^2(\gamma_0).$$

Since $a_1 = a_2$ this implies that $w_{1,\xi}^2(\gamma_0) = w_{2,\xi}^2(\gamma_0)$.

If $w_{1,\xi}(\gamma_0) = w_{2,\xi}(\gamma_0)$, one has $b_1 = b_2$ and as in Case 1 w coincides with the local solution in D of the differential equation (3.12).

If

$$w_{1,\xi}(\gamma_0) = -w_{2,\xi}(\gamma_0). \tag{3.20}$$

We get, from (3.12) that

$$2a_1s + b_1 = 0 \quad \text{and} \quad 2a_1s + b_2 = 0,$$

which implies $b_1 = b_2$. Furthermore, (3.12) takes the form

$$\big(w_{1,\xi}^2(\xi) - w_1^2(\xi) + 2a_1\big)\big(s - w_1(\xi)\big) = 0$$

and

$$\big(w_{2,\xi}^2(\xi) - w_2^2(\xi) + 2a_1\big)\big(s - w_2(\xi)\big) = 0$$

in $\overline{D_1^\varepsilon}$ and $\overline{D_2^\varepsilon}$, respectively.

Assuming that w_1 and w_2 are not constant and equal to s near γ_0, we have

$$w_{1,\xi}^2 - w_1^2 + 2a_1 = 0 \quad \text{and} \quad w_{2,\xi}^2 - w_2^2 + 2a_2 = 0. \tag{3.21}$$

For this to be well-defined we require

$$w_1^2 - 2a_1 \geq 0 \quad \text{and} \quad w_2^2 - 2a_2 \geq 0,$$

and in particular,

$$w_{1,\xi} = \pm\sqrt{w_1^2 - 2a_1} \quad \text{and} \quad w_{2,\xi} = \pm\sqrt{w_2^2 - 2a_2}. \tag{3.22}$$

Note that w_1 and w_2 can only change from increasing to decreasing or the other way round if $w_1^2 = 2a_1$. We differentiate (3.21) and get

$$w_{1,\xi}(w_{1,\xi\xi} - w_1) = 0 \quad \text{and} \quad w_{2,\xi}(w_{2,\xi\xi} - w_2) = 0,$$

and since the solutions are not constant we have

$$w_{1,\xi\xi} = w_1 \quad \text{and} \quad w_{2,\xi\xi} = w_2. \tag{3.23}$$

Letting ξ tend to γ_0 in (3.23) yields

$$w_{1,\xi\xi}(\gamma_0) = w_1(\gamma_0) \quad \text{and} \quad w_{2,\xi\xi}(\gamma_0) = w_2(\gamma_0),$$

so that if s is positive then w_1 and w_2 are convex. Since the functions are not constant and (3.20) holds, this implies that w_1 is increasing and w_2 is decreasing near γ_0. Otherwise the resulting function w would be globally unbounded. Thus, the maximum value of w_1 and w_2 near γ_0 is attained at γ_0 where $w_1(\gamma_0) = w_2(\gamma_0) = s$.

If s is negative, w_1 and w_2 are concave, and w_1 is decreasing and w_2 is increasing. Otherwise the resulting function w would be globally unbounded. The minimum value of w_1 and w_2 near γ_0 is attained at γ_0 where $w_1(\gamma_0) = w_2(\gamma_0) = s$.

Example 3.2. *Let $\gamma_0 = 0$. Then*

$$w_1(\xi) = c_1 e^{\xi} + c_2 e^{-\xi} \quad \textit{and} \quad w_2(\xi) = c_2 e^{\xi} + c_1 e^{-\xi},$$

where c_1 and c_2 are constants satisfying $\frac{c_1}{c_2} > e^2$, solve the differential equations (3.23) *in* $[-1, 0]$ *and* $[0, 1]$*, respectively. Observe that w_1 is increasing in $[-1, 0]$, w_2 is decreasing in $[0, 1]$, $w_1(0) = w_2(0)$ and $w_{1,\xi}(0) = -w_{2,\xi}(0)$.*

Remark 3.3. *Note that the so-called multipeakon solutions are of the form $u(t, x) = \sum_{i=1}^n p_i(t) e^{-|x-q_i(t)|}$. Thus if one only glues together local, traveling waves, which have bounded derivatives, one ends up with a multipeakon solution due to* (3.23)*.*

Unbounded derivatives

Since we have from (3.12),

$$w_\xi^2 = w^2 - \frac{2a_1 w + b_i}{w - s} \tag{3.24}$$

in $\overline{D_i^\varepsilon}$, it follows that $s = w(\gamma_0)$ at the possible glueing point. Furthermore, due to (3.6b) the constants b_1 and b_2 do not have to be identical.

Note that (3.6b) implies that it is possible to glue together both constant and non-constant local, classical solutions as long as $s = w(\gamma_0)$. This means in particular that one can insert constant parts by gluing.

In [15], Lenells presents a complete classification of weak, bounded traveling waves for the CH equation. He shows that there exists a wide range of waves, such as smooth waves, but also peakons, cuspons, stumpons, and composite waves which might have singularities.

Lenells proves that two traveling waves w_1 and w_2 can only be glued together at a point ξ if the wave height equals the wave speed, i.e., $w_1(\xi) = w_2(\xi) = s$, and if the constants a_1 and a_2 from (3.12) are equal. We remark that the constants a and c in [15] corresponds to $2a$ and s here, respectively, and that we assume $k = 0$.

Our main objective was to recover these conditions by using the method presented above. Showing other important features of traveling wave solutions of the CH equation requires the machinery used by Lenells, which we outline next. For a detailed description we refer to [15]. A key property is that the maximum value of the wave equals s for $s > 0$ and the minimum value equals s for $s < 0$.

In particular, we highlight the role the constant b_i plays in obtaining a bounded wave. Assume that we are in our usual setting where we have classical solutions w_1 and w_2 in D_1^ε and D_2^ε, respectively. We want to glue these waves together. Thus, we must have $w_1(\gamma_0) = w_2(\gamma_0) = s$ and $a_1 = a_2$. Hence, we can write (3.12) as

$$w_\xi^2(s - w) = -w^3 + sw^2 + 2a_1 w + b_i$$

in $\overline{D_i^\varepsilon}$, $i = 1, 2$. Introducing

$$f(w) = -w^3 + sw^2 + 2a_1 w,$$

we can write these equations as

$$w_\xi^2(s - w) = f(w) + b_i = g_i(w) \tag{3.25}$$

in $\overline{D_i^\varepsilon}$, $i = 1, 2$. Note that

$$g_1'(w) = g_2'(w) = f'(w) = -3w^2 + 2sw + 2a_1.$$

In what follows we assume that $s > 0$.

If $s^2 + 6a_1 \leq 0$, then

$$g_1'(w) \leq -3w^2 + 2sw - \frac{s^2}{3} = -3\Big(w - \frac{s}{3}\Big)^2,$$

which is strictly negative provided that w is not identically equal to $\frac{s}{3}$. This means that $g_1(w)$ is strictly decreasing and $f(w) + b_1$ has exactly one zero. Assume that b_1 is such that $f(s) + b_1 < 0$. By continuity we have $f(w) + b_1 < 0$ for w near s. Then (3.25) implies that $w > s$. Since $f(w) + b_1 < 0$ for all $w > s$, (3.25) shows that $w_\xi \neq 0$ for all $w > s$. Thus, w is strictly monotone and unbounded. Next, let us set b_1 larger so that $f(s) + b_1 > 0$. Then $f(w) + b_1 > 0$ for w near s and from (3.25) we get $w < s$. We have $f(w) + b_1 > 0$ for all $w < s$, so (3.25) implies that $w_\xi \neq 0$ for all $w < s$. Hence, w is strictly monotone and unbounded. The situation $f(s) + b_1 = 0$ can be treated similarly, showing that there are no bounded solutions. Thus, if $s^2 + 6a_1 \leq 0$, $f(w) + b_1$ has one zero and there exist no bounded solutions to (3.25).

If $s^2 + 6a_1 > 0$, then $g_1(w) = f(w) + b_1$ has at least one zero, but the number of zeros is dependent on the choice of b_1. To be more precise the function $g_1(w)$ has a local minimum and maximum at

$$w_{\min} = \frac{s - \sqrt{s^2 + 6a_1}}{3} \quad \text{and} \quad w_{\max} = \frac{s + \sqrt{s^2 + 6a_1}}{3},$$

respectively. It is strictly decreasing for $w < w_{\min}$ and $w > w_{\max}$.

If g_1 has only one zero, we can show as before (i.e. in the case $s^2 + 6a_1 \leq 0$), that there only exist unbounded solutions.

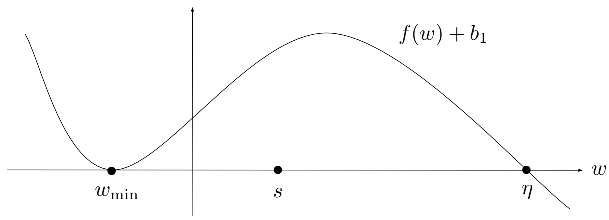

FIGURE 5. Sketch of the function g_1 with a double zero at $w = w_{\min}$ and a simple zero at $w = \eta$. Furthermore, $w_{\min} < s < \eta$.

Consider the case where g_1 has three zeros. Moreover, we consider s between two of the zeros of g_1, since any other case yields unbounded solutions.

First we treat the case where g_1 has a double zero and a simple zero. The double zero is either the local minimum or maximum of g_1. We only consider the case where the double zero is the local minimum of g_1, see Figure 5, since the other one follows the same lines. Denoting the simple zero by η, we write

$$g_1(w) = -(w - w_{\min})^2(w - \eta). \tag{3.26}$$

Expanding this expression and comparing it with $g_1(w) = f(w) + b_1$ we get the relations

$$\eta + 2w_{\min} = s, \quad -2\eta w_{\min} - w_{\min}^2 = 2a_1, \quad \text{and} \quad \eta w_{\min}^2 = b_1.$$

By assumption $w_{\min} < s < \eta$. Then $g_1(s) > 0$, so that $g_1(w) > 0$ for w near s. By (3.25), $w < s$. Note that $w_\xi = 0$ for the value $w = w_{\min}$. We have

$$w_\xi = \pm(w - w_{\min})\sqrt{\frac{\eta - w}{s - w}} \tag{3.27}$$

and w exponentially decays to $w_{\min}$ as $\xi \to \pm\infty$. With the notation above, we see that one possibility is to choose w_1 and w_2 to be solutions to (3.27), which yields the cuspon with decay. In particular, w_1 is the strictly increasing part, and w_2 the strictly decreasing part. Note that the derivatives are unbounded at $w_1 = w_2 = s$.

Let us investigate if we can glue waves w_i, $i = 1, 2$ given by (3.27) to constant solutions. Consider w_1 given by (3.27). From (3.25) we have

$$w_{2,\xi}^2 = \frac{g_2(w_2)}{s - w_2}.$$

We are looking for solutions satisfying $w_{2,\xi} = 0$. Since at the gluing point we have $w_1 = w_2 = s$, we require that $g_2(w) = (d - w)(s - w)^2$ for some constant d. Comparing the coefficients, yields the relations

$$d = -s, \quad 2a_1 = s^2, \quad \text{and} \quad b_2 = -s^3.$$

Hence, if $s^2 = 2a_1$ we can glue w_i as given by (3.27) with the constant solution $w_{i\pm1} = s$, which are the building blocks for so-called stumpons.

Remark 3.4. *Note that the condition $s^2 = 2a_1$ is related to* (3.22)*, which describes all local, classical traveling waves that have a bounded derivative at points where $w = s$. In particular,* (3.22) *implies that peakons can only turn up in bounded,*

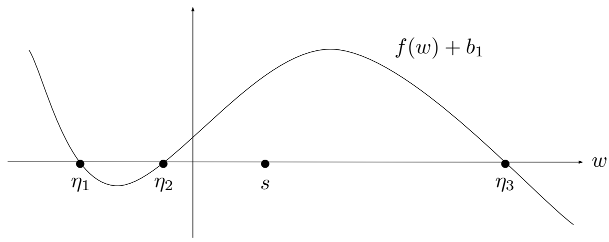

FIGURE 6. Sketch of the function g_1 with three simple zeros $\eta_1 < \eta_2 < \eta_3$. Furthermore, $\eta_2 < s < \eta_3$.

composite waves such that $s^2 \geq 2a_1$ *and the case* $s^2 = 2a_1$ *corresponds to the constant solution.*

Let us turn back to (3.26). If $s = \eta$, then

$$w_\xi = \pm(w - w_{\min}). \tag{3.28}$$

In particular, w is monotone and decays to $w_{\min}$ as $\xi \to \pm\infty$. Choosing w_1 and w_2 to be solutions to (3.28) yields the peakon with decay. In particular, w_1 is the strictly increasing part and w_2 the strictly decreasing one.

Let us see if we can glue waves given by (3.28) to constant solutions. Let w_1 be the strictly increasing function given by (3.28). The graph of the function $g_2(w) = f(w) + b_2$ is equal to the one for g_1 up to a vertical shift, i.e., there is a constant α such that $g_2(w) = -(w - w_{\min})^2(w - s) + \alpha$. From (3.25) we get

$$w_{2,\xi}^2 = (w_2 - w_{\min})^2 + \frac{\alpha}{s - w}.$$

Observe that the only choice of the constant α which gives a solution with bounded derivative is $\alpha = 0$. Then we get

$$w_{2,\xi}^2 = (w_2 - w_{\min})^2,$$

and the only possibility for $w_{2,\xi} = 0$ is if $w_2 = w_{\min}$. But then w_2 can not be glued to w_1, as $w_{\min} \neq s$. We conclude that waves given by (3.28) cannot be glued to constant solutions.

Next we treat the case where g_1 has three simple zeros $\eta_1 < \eta_2 < \eta_3$, i.e.,

$$g_1(w) = -(w - \eta_1)(w - \eta_2)(w - \eta_3),$$

see Figure 6.

Let s be such that $\eta_2 < s < \eta_3$. Then $g_1(s) > 0$, so that $g_1(w) > 0$ for w near s. By (3.25), $w < s$. Observe that $w_\xi = 0$ at $w = \eta_2$. We have

$$w_\xi = \pm\sqrt{w - \eta_2}\, h(w), \tag{3.29}$$

where

$$h(w) = \sqrt{-\frac{(w - \eta_1)(w - \eta_3)}{s - w}} > 0$$

for all $\eta_2 < w < s$, so that w attains the value η_2 at some finite point $\bar{\xi}$. Note that the solution to (3.29) is not unique. Thus, $w \in C^1$ can be defined in such a way that w attains its minimum at $\bar{\xi}$, is strictly decreasing to the left of $\bar{\xi}$, and strictly increasing to the right of $\bar{\xi}$. Gluing countably many of these waves together yields a periodic cuspon.

If $s = \eta_3$, then

$$w_\xi = \pm\sqrt{w-\eta_1}\sqrt{w-\eta_2}, \tag{3.30}$$

whose solutions, following the same lines as above, serves as building blocks for a periodic peakon.

In a similar way as above, we can study if the waves given by (3.29) and (3.30) can be glued to constant solutions. We find that we can only glue waves given by (3.29) to constant solutions which are equal to s. Then we obtain stumpons, which consist of monotone segments glued at points where the derivative is unbounded to piecewise constants parts.

References

[1] A. Bressan and T. Huang: Representation of dissipative solutions to a nonlinear variational wave equation. Commun. Math. Sci. **14**, 31–53 (2016).

[2] A. Bressan and Y. Zheng: Conservative solutions to a nonlinear variational wave equation. Comm. Math. Phys. **266**, 471–497 (2006).

[3] A. Bressan and A. Constantin: Global conservative solutions of the Camassa–Holm equation. Arch. Ration. Mech. Anal. **183**, 215–239 (2007).

[4] A. Bressan and A. Constantin: Global dissipative solutions of the Camassa–Holm equation. Anal. Appl. (Singap.) **5**, 1–27 (2007).

[5] R. Camassa and D. Holm: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993).

[6] A. Constantin and J. Escher: Global existence and blow-up for a shallow water equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26**, 303–328 (1998).

[7] A. Constantin and J. Escher: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**, 229–243 (1998).

[8] R. T. Glassey, J. K. Hunter, and Y. Zheng: Singularities of a variational wave equation. J. Differential Equations **129**, 49–78 (1996).

[9] K. Grunert, H. Holden, and X. Raynaud: Global dissipative solutions of the two-component Camassa–Holm system for initial data with nonvanishing asymptotics. Nonlinear Anal. Real World Appl. **17**, 203–244 (2014).

[10] H. Holden and X. Raynaud: Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. Comm. Partial Differential Equations **32**, 1511–1549 (2007).

[11] H. Holden and X. Raynaud: Dissipative solutions for the Camassa–Holm equation. Discrete Contin. Dyn. Syst. **24**, 1047–1112 (2009).

[12] H. Holden and X. Raynaud: Global semigroup of conservative solutions of the nonlinear variational wave equation. Arch. Ration. Mech. Anal. **201**, 871–964 (2011).

[13] H. Holden and N. H. Risebro: Front tracking for hyperbolic conservation laws. *Appl. Math. Sci.* vol. 152, Springer-Verlag, New York (2002).

[14] J. K. Hunter and R. Saxton: Dynamics of director fields. SIAM J. Appl. Math. **51**, 1498–1521 (1991).

[15] J. Lenells: Traveling wave solutions of the Camassa–Holm equation. J. Differential Equations **217**, 393–430 (2005).

[16] J. Lenells: Classification of all travelling-wave solutions for some nonlinear dispersive equations. Phil. Trans. R. Soc. A **365**, 2291–2298 (2007).

[17] R. A. Saxton: Dynamic instability of the liquid crystal director. In: W. B. Lindquist (ed.) Current Progress in Hyperbolic Systems: Riemann Problems and Computations (Brunswick, ME, 1988), 325–330. Contemp. Math., Vol. 100, Amer. Math. Soc., Providence, RI (1989).

[18] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation. Arch. Rat. Mech. Anal. **166**, 303–319 (2003).

[19] P. Zhang and Y. Zheng: Weak solutions to a nonlinear variational wave equation with general data. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 207–226 (2005).
[20] P. Zhang and Y. Zheng: On the global weak solutions to a variational wave equation. In: C. M. Dafermos and E. Feireisl (eds.) Handbook of Differential Equations. Evolutionary Equations. Volume 2, 561–648. Elsevier, Amsterdam (2005).

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Paper III

Competition Models for Plant Stems

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Competition models for plant stems

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Abstract

The models introduced in this paper describe a uniform distribution of plant stems competing for sunlight. The shape of each stem, and the density of leaves, are designed in order to maximize the captured sunlight, subject to a cost for transporting water and nutrients from the root to all the leaves. Given the intensity of light, depending on the height above ground, we first solve the optimization problem determining the best possible shape for a single stem. We then study a competitive equilibrium among a large number of similar plants, where the shape of each stem is optimal given the shade produced by all others. Uniqueness of equilibria is proved by analyzing the two-point boundary value problem for a system of ODEs derived from the necessary conditions for optimality.

MSC: 34B15; 49N90; 91A40; 92B05

Keywords: Optimal shape; Competitive equilibrium; Nonlinear boundary value problem

1. Introduction

Optimization problems for tree branches have recently been studied in [3,5]. In these models, optimal shapes maximize the total amount of sunlight gathered by the leaves, subject to a cost for

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building a network of branches that will bring water and nutrients from the root to all the leaves. Following [2,8,11,13,14], this cost is defined in terms of a ramified transport.

In the present paper we consider a competition model, where a large number of similar plants compete for sunlight. To make the problem tractable, instead of a tree-like structure we assume that each plant consists of a single stem. As a first step, assuming that the intensity of light $I(\cdot)$ depends only on the height above ground, we determine the corresponding optimal shape of the stem. This will be a curve $\gamma(\cdot)$ which can be found by classical techniques of the Calculus of Variations or optimal control [4,6,7]. In turn, given the density of plants (i.e., the average number of plants growing per unit area), if all stems have the same shape $\gamma(\cdot)$ one can compute the intensity of light $I(h)$ that reaches a point at height h.

An equilibrium configuration is now defined as a fixed point of the composition of the two maps $I(\cdot) \mapsto \gamma(\cdot)$ and $\gamma(\cdot) \mapsto I(\cdot)$. A major goal of this paper is to study the existence and properties of these equilibria, where the shape of each stem is optimal subject to the presence of all other competing plants.

In Section 2 we introduce our two basic models. In the first model, the length ℓ of the stems and the thickness (i.e., the density of leaves along each stem) are assigned a priori. The only function to optimize is thus the curve $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ describing the shape of the stems. In the second model, also the length and the thickness of the stems are allowed to vary, and optimal values for these variables need to be determined.

In Section 3, given a light intensity function $I(\cdot)$, we study the optimization problem for Model 1, proving the existence of an optimal solution and deriving necessary conditions for optimality. We also give a condition which guarantees the uniqueness of the optimal solution. A counterexample shows that, in general, if this condition is not satisfied multiple solutions can exist. In Section 4 we consider the competition of a large number of stems, and prove the existence of an equilibrium solution. In this case, the common shape of the plant stems can be explicitly determined by solving a particular ODE.

The subsequent sections extend the analysis to a more general setting (Model 2), where both the length and the thickness of the stems are to be optimized. In Section 5 we prove the existence of optimal stem configurations, and derive necessary conditions for optimality, while in Section 6 we establish the existence of a unique equilibrium solution for the competitive game, assuming that the density (i.e., the average number of stems growing per unit area) is sufficiently small. The key step in the proof is the analysis of a two-point boundary value problem, for a system of ODEs derived from the necessary conditions.

In the above models, the density of stems was assumed to be uniform on the whole space. As a consequence, the light intensity $I(h)$ depends only of the height h above ground. Section 7, on the other hand, is concerned with a family of stems growing only on the positive half line. In this case the light intensity $I = I(h, x)$ depends also on the spatial location x, and the analysis becomes considerably more difficult. Here we only derive a set of equations describing the competitive equilibrium, and sketch what we conjecture should be the corresponding shape of stems.

The final section contains some concluding remarks. In particular, we discuss the issue of phototropism, i.e. the tendency of plant stems to bend in the direction of the light source. Devising a mathematical model, which demonstrates phototropism as an advantageous trait, remains a challenging open problem. For a biological perspective on plant growth we refer to [9]. A recent mathematical study of the stabilization problem for growing stems can be found in [1].

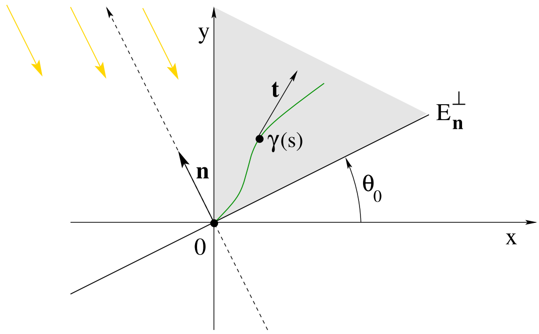

Fig. 1. By a reflection argument, it is not restrictive to assume that the tangent vector $\mathbf{t}(s)$ to the stem satisfies (2.4), i.e., it lies in the shaded cone.

2. Optimization problems for a single stem

We shall consider plant stems in the x-y plane, where y is the vertical coordinate. We assume that sunlight comes from the direction of the unit vector

$$\mathbf{n} = (n_1, n_2), \qquad n_2 < 0 < n_1.$$

As in Fig. 1, we denote by $\theta_0 \in]0, \pi/2[$ the angle such that

$$(-n_2, n_1) = (\cos\theta_0, \sin\theta_0). \tag{2.1}$$

Moreover, we assume that the light intensity $I(y) \in [0, 1]$ is a non-decreasing function of the height y. This is due to the presence of competing vegetation: close to the ground, less light can get through.

Model 1 (a stem with fixed length and constant thickness). We begin by studying a simple model, where each stem has a fixed length ℓ. Let $s \mapsto \gamma(s) = (x(s), y(s))$, $s \in [0, \ell]$, be an arc-length parameterization of the stem. As a first approximation, we assume that the leaves are uniformly distributed along the stem, with density κ. The total distribution of leaves in space is thus by a measure μ, with

$$\mu(A) = \kappa \cdot \text{meas}\Big(\big\{s \in [0, \ell]\,;\;\; \gamma(s) \in A\big\}\Big) \tag{2.2}$$

for every Borel set $A \subseteq \mathbb{R}^2$.

Among all stems with given length ℓ, we seek the shape which will collect the most sunlight. This can be formulated as an optimal control problem. Since γ is parameterized by arc-length, the map $s \mapsto \gamma(s)$ is Lipschitz continuous with constant 1. Hence the tangent vector

$$\mathbf{t}(s) = \dot\gamma(s) = (\cos\theta(s), \sin\theta(s))$$

is well defined for a.e. $s \in [0, \ell]$. The map $s \mapsto \theta(s)$ will be regarded as a control function.

According to the model in [5], calling $\Phi(\cdot)$ the density of the projection of μ on the space $E_{\mathbf{n}}^{\perp}$ orthogonal to $\mathbf{n}$, the total sunlight captured by the stem is

$$\mathcal{S}(\gamma) = \int \Big(1 - \exp\{-\Phi(z)\}\Big)\, dz$$
$$= \int_0^\ell I(y(s)) \cdot \left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta(s) - \theta_0)}\right\}\right) \cos(\theta(s) - \theta_0)\, ds. \tag{2.3}$$

In order to maximize (2.3), we claim that it is not restrictive to assume that the angle satisfies

$$\theta_0 \;\le\; \theta(s) \;\le\; \frac{\pi}{2} \qquad \text{for all } s \in [0, \ell]. \tag{2.4}$$

Indeed, for any measurable map $s \mapsto \theta(s) \in]-\pi, \pi]$, we can define a modified angle function $\theta^\sharp(\cdot)$ by setting

$$\theta^\sharp(s) \;=\; \begin{cases} \theta(s) & \text{if} \quad \theta(s) \in]0, \theta_0 + \pi/2], \\ -\theta(s) & \text{if} \quad \theta(s) \in]-\pi,\, \theta_0 - \pi/2], \\ 2\theta_0 + \pi - \theta(s) & \text{if} \quad \theta(s) \in]\theta_0 + \pi/2, \pi], \\ 2\theta_0 - \theta(s) & \text{if} \quad \theta(s) \in]\theta_0 - \pi/2, 0]. \end{cases} \tag{2.5}$$

Calling $\gamma^\sharp : [0, \ell] \mapsto \mathbb{R}^2$ the curve whose tangent vector is $\dot\gamma^\sharp(s) = (\cos\theta^\sharp(s), \sin\theta^\sharp(s))$, since the light intensity function $y \mapsto I(y)$ is nondecreasing, we have $\mathcal{S}(\gamma^\sharp) \ge \mathcal{S}(\gamma)$.

By this first step, without loss of generality we can now assume $\theta(s) \in]0,\, \theta_0 + \pi/2]$. To proceed further, consider the piecewise affine map

$$\varphi(\theta) \;=\; \begin{cases} \theta & \text{if} \quad \theta \in]\theta_0, \pi/2], \\ \pi - \theta & \text{if} \quad \theta \in [\pi/2,\, \theta_0 + \pi/2], \\ 2\theta_0 - \theta & \text{if} \quad \theta \in [0, \theta_0]. \end{cases} \tag{2.6}$$

Call γ^φ the curve whose tangent vector is $\dot\gamma^\varphi(s) = \Big(\cos(\varphi(\theta(s))),\, \sin(\varphi(\theta(s)))\Big)$. Since $I(\cdot)$ is nondecreasing, we again have $\mathcal{S}(\gamma^\varphi) \;\ge\; \mathcal{S}(\gamma)$. We now observe that, since $0 < \theta_0 < \pi/2$, there exists an integer $m \ge 1$ such that the m-fold composition $\varphi^m \doteq \varphi \circ \cdots \circ \varphi$ maps $[0, \theta_0 + \pi/2]$ into $[\theta_0, \pi/2]$. An inductive argument now yields $\mathcal{S}(\gamma^{\varphi^m}) \;\ge\; \mathcal{S}(\gamma)$, completing the proof of our claim.

As shown in Fig. 2, left, we call z the coordinate along the space $E_{\mathbf{n}}^\perp$ perpendicular to $\mathbf{n}$, and let y be the vertical coordinate. Hence

$$dz(s) \;=\; \cos(\theta(s) - \theta_0)\, ds, \qquad\qquad dy(s) \;=\; \sin(\theta(s))\, ds. \tag{2.7}$$

In view of (2.4), one can express both γ and θ as functions of the variable y. Introducing the function

$$g(\theta) \;\doteq\; \left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta - \theta_0)}\right\}\right) \frac{\cos(\theta - \theta_0)}{\sin\theta}\,, \tag{2.8}$$

the problem can be equivalently formulated as follows.

(OP1) *Given a length $\ell > 0$, find $h > 0$ and a control function $y \mapsto \theta(y) \in [\theta_0, \pi/2]$ which maximizes the integral*

$$\int_0^h I(y)\, g(\theta(y))\, dy \tag{2.9}$$

subject to

$$\int_0^h \frac{1}{\sin\theta(y)}\, dy = \ell. \tag{2.10}$$

Model 2 (stems with variable length and thickness). Here we still assume that the plant consists of a single stem, parameterized by arc-length: $s \mapsto \gamma(s)$, $s \in [0, \ell]$. However, now we give no constraint on the length ℓ of the stem, and we allow the density of leaves to be variable along the stem.

Call $u(s)$ the density of leaves at the point $\gamma(s)$. In other words, μ is now the measure which is absolutely continuous w.r.t. arc-length measure on γ, with density u. Instead of (2.2) we thus have

$$\mu(A) = \int_{\{s\,;\ \gamma(s)\in A\}} u(s)\, ds\,. \tag{2.11}$$

Calling $I(y)$ the intensity of light at height y, the total sunlight gathered by the stem is now computed by

$$\mathcal{S}(\mu) = \int_0^\ell I(y(s)) \cdot \left(1 - \exp\left\{\frac{-u(s)}{\cos(\theta(s) - \theta_0)}\right\}\right) \cos(\theta(s) - \theta_0)\, ds. \tag{2.12}$$

As in [5], we consider a cost for transporting water and nutrients from the root to the leaves. This is measured by

$$\mathcal{I}^\alpha(\mu) = \int_0^\ell \left(\int_s^\ell u(t)\, dt\right)^\alpha ds, \tag{2.13}$$

for some $0 < \alpha < 1$. Notice that, in Model 1, this cost was the same for all stems and hence it did not play a role in the optimization.

For a given constant $c > 0$, we now consider a second optimization problem:

$$\text{maximize:} \quad \mathcal{S}(\mu) - c\mathcal{I}^\alpha(\mu), \tag{2.14}$$

subject to:

$$y(0) = 0, \qquad \dot{y}(s) = \sin\theta(s). \tag{2.15}$$

The maximum is sought over all controls $\theta : \mathbb{R}_+ \mapsto [0, \pi]$ and $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Calling

$$z(t) \doteq \int_t^{+\infty} u(s)\, ds, \tag{2.16}$$

$$G(\theta, u) \doteq \left(1 - \exp\left\{\frac{-u}{\cos(\theta - \theta_0)}\right\}\right)\cos(\theta - \theta_0)\,, \tag{2.17}$$

this leads to an optimal control problem in a more standard form.

(OP2) *Given a sunlight intensity function $I(y)$, and constants $0 < \alpha < 1$, $c > 0$, find controls $\theta : \mathbb{R}_+ \mapsto [\theta_0, \pi/2]$ and $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which maximize the integral*

$$\int_0^{+\infty} \Big[I(y)\, G(\theta, u) - c\, z^\alpha\Big]\, dt, \tag{2.18}$$

subject to

$$\begin{cases} \dot{y}(t) = \sin\theta, \\ \dot{z}(t) = -u, \end{cases} \qquad \begin{cases} y(0) = 0, \\ z(+\infty) = 0. \end{cases} \tag{2.19}$$

3. Optimal stems with fixed length and thickness

3.1. Existence of an optimal solution

Let $I(y)$ be the light intensity, which we assume is a non-decreasing function of the vertical component y. For a given $\kappa > 0$ (the thickness of the stem), we seek a curve $s \mapsto \gamma(s)$, starting at the origin and with a fixed length ℓ, which maximizes the sunlight functional defined at (2.9).

Theorem 3.1. *For any non-decreasing function $y \mapsto I(y) \in [0, 1]$ and any constants $\ell, \kappa > 0$ and $\theta_0 \in]0, \pi/2[$, the optimization problem* **(OP1)** *has at least one solution.*

Proof. 1. Let M be the supremum among all admissible payoffs in (2.9). By the analysis in [5] it follows that

$$0 \leq M \leq \kappa\, \mu(\mathbb{R}^2) = \kappa\, \ell.$$

Hence there exists a maximizing sequence of control functions $\theta_n : [0, h_n] \mapsto [\theta_0, \pi/2]$, so that

$$\int_0^{h_n} \frac{1}{\sin\theta_n(y)}\, dy = \ell \qquad \text{for all } n \geq 1, \tag{3.1}$$

$$\int_0^{h_n} I(y)g(\theta_n(y))\,dy \;\to\; M. \tag{3.2}$$

2. For each n, let $\theta_n^\sharp$ be the non-increasing rearrangement of the function θ_n. Namely, $\theta_n^\sharp$ is the unique (up to a set of zero measure) non-increasing function such that, for every $c \in \mathbb{R}$

$$\text{meas}\Big(\{s\,;\;\; \theta_n^\sharp(s) < c\}\Big) \;=\; \text{meas}\Big(\{s\,;\;\; \theta_n(s) < c\}\Big). \tag{3.3}$$

This can be explicitly defined as

$$\theta_n^\sharp(y) \;=\; \sup\Big\{\xi\,;\;\; \text{meas}\Big(\{\sigma\in[0,h_n]\,;\;\; \theta_n(\sigma)\ge\xi\}\Big) > y\Big\}.$$

For every $n \ge 1$ we claim that

$$\int_0^{h_n} \frac{1}{\sin\theta_n^\sharp(y)}\,dy \;=\; \int_0^{h_n} \frac{1}{\sin\theta_n(y)}\,dy \;=\; \ell, \tag{3.4}$$

$$\int_0^{h_n} I(y)g(\theta_n^\sharp(y))\,dy \;\ge\; \int_0^{h_n} I(y)g(\theta_n(y))\,dy. \tag{3.5}$$

Indeed, to prove the first identity we observe that, by (3.3), there exists a measure-preserving map $y\mapsto\zeta(y)$ from $[0,h_n]$ into itself such that $\theta_n^\sharp(y) \;=\; \theta_n(\zeta(y))$. Using ζ as new variable of integration, one immediately obtains (3.4).

To prove (3.5) we observe that the function g introduced at (2.8) is smooth and satisfies

$$g'(\theta) \;\le\; 0 \qquad \text{for all } \theta\in[\theta_0,\,\pi/2]. \tag{3.6}$$

Therefore, the map $y\mapsto g(\theta_n^\sharp(y))$ coincides with the non-decreasing rearrangement of $y\mapsto g(\theta_n(y))$. On the other hand, since $I(\cdot)$ is non-decreasing, it trivially coincides with the non-decreasing rearrangement of itself. Therefore, (3.5) is an immediate consequence of the Hardy-Littlewood inequality [10].

3. Since all functions $\theta_n^\sharp$ are non-increasing, they have bounded variation. Using Helly's compactness theorem, by possibly extracting a subsequence, we can find $h>0$ and a non-increasing function $\theta^*:[0,h]\mapsto[\theta_0,\pi/2]$ such that

$$\lim_{n\to\infty} h_n \;=\; h, \qquad\qquad \lim_{n\to\infty}\theta_n^\sharp(y) \;=\; \theta^*(y) \qquad \text{for a.e. } y\in[0,h]. \tag{3.7}$$

This implies

$$\int_0^{h} \frac{1}{\sin\theta^*(y)}\,dy \;=\; \ell, \qquad\qquad \int_0^{h} I(y)g(\theta^*(y))\,dy \;=\; M,$$

proving the optimality of θ^*. □

3.2. Necessary conditions for optimality

Let $y \mapsto \theta^*(y)$ be an optimal solution. By the previous analysis we already know that the function $\theta^*(\cdot)$ is non-increasing. Otherwise, its non-increasing rearrangement achieves a better payoff. In particular, this implies that the left limit at the terminal point $y = h$ is well defined:

$$\theta^*(h) \;=\; \lim_{y\to h-} \theta^*(y). \tag{3.8}$$

Consider an arbitrary perturbation

$$\theta_\epsilon \;=\; \theta^* + \epsilon\Theta, \qquad\qquad h_\epsilon \;=\; h + \epsilon\eta.$$

The constraint (2.10) implies

$$\int_0^{h+\epsilon\eta} \frac{1}{\sin\theta_\epsilon(y)}\, dy \;=\; \ell. \tag{3.9}$$

Differentiating (3.9) w.r.t. ϵ one obtains

$$\frac{1}{\sin\theta^*(h)}\,\eta - \int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\,\Theta(y)\,dy \;=\; 0. \tag{3.10}$$

Next, calling

$$J_\epsilon \;\doteq\; \int_0^{h_\epsilon} I(y) g(\theta_\epsilon(y)) dy$$

and assuming that $I(\cdot)$ is continuous at least at $y = h$, by (3.10) we obtain

$$\begin{aligned} 0 \;=\; \frac{d}{d\epsilon} J_\epsilon\Big|_{\epsilon=0} \;&=\; \int_0^h I(y) g'(\theta^*(y))\Theta(y)\,dy \\ &\quad + I(h) g(\theta^*(h))\cdot \sin\theta^*(h) \int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\,\Theta(y)\,dy. \end{aligned} \tag{3.11}$$

Since (3.11) holds for arbitrary perturbations $\Theta(\cdot)$, the optimal control $\theta^*(\cdot)$ should satisfy the identity

$$I(y) g'\big(\theta^*(y)\big) + \lambda\cdot \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)} \;=\; 0, \qquad \text{for a.e. } y \in [0, h], \tag{3.12}$$

where

$$\lambda = I(h)g(\theta^*(h)) \cdot \sin\theta^*(h). \tag{3.13}$$

It will be convenient to write

$$g(\theta) = \frac{G(\theta)}{\sin\theta}, \qquad G(\theta) \doteq \left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta-\theta_0)}\right\}\right)\cos(\theta-\theta_0). \tag{3.14}$$

Inserting (3.14) in (3.12) one obtains the pointwise identities

$$I(y)\Big(G'(\theta^*(y))\sin\theta^*(y) - G(\theta^*(y))\cos\theta^*(y)\Big) + \lambda\cdot\cos\theta^*(y) = 0. \tag{3.15}$$

At $y = h$, the identities (3.13) and (3.15) yield

$$G'(\theta^*(h))\,\tan\theta^*(h) - G(\theta^*(h)) = -\frac{I(h)G(\theta^*(h))}{I(h)}.$$

Hence

$$G'(\theta^*(h))\,\tan\theta^*(h) = 0,$$

which implies

$$\theta^*(h) = \theta_0, \qquad \lambda = I(h)g(\theta_0)\sin\theta_0 = \left(1-e^{-\kappa}\right)I(h). \tag{3.16}$$

Notice that (3.15) corresponds to

$$\theta^*(y) = \arg\max_{\theta\in[0,\pi]} \left\{I(y)\frac{G(\theta)}{\sin\theta} - \frac{\lambda}{\sin\theta}\right\}. \tag{3.17}$$

Equivalently, $\theta = \theta^*(y)$ is the solution to

$$G'(\theta)\tan\theta - G(\theta) = -\frac{\lambda}{I(y)}, \tag{3.18}$$

where G is the function at (3.14).

Lemma 3.2. *Let G be the function at (3.14). Then for every* $z \in\,]-\infty, e^{-\kappa}-1]$ *the equation*

$$F(\theta) \doteq G'(\theta)\tan\theta - G(\theta) = z \tag{3.19}$$

has a unique solution $\theta = \varphi(z) \in [\theta_0, \pi/2[$.

Proof. Observing that

$$\begin{cases} G(\theta_0) = 1-e^{-\kappa}, \\ G'(\theta_0) = 0, \end{cases} \qquad \begin{cases} G'(\theta) < 0 \\ G''(\theta) < 0 \end{cases} \quad \text{for } \theta\in\,]\theta_0,\pi/2[, \tag{3.20}$$

we obtain $F(\theta_0) = e^{-\kappa} - 1$ and

$$F'(\theta) = G''(\theta)\tan\theta + G'(\theta)\tan^2\theta < 0 \qquad \text{for } \theta \in [\theta_0, \pi/2[\,.$$

Therefore, for $\theta \in [\theta_0, \pi/2[$, the left hand side of (3.19) is monotonically decreasing from $e^{-\kappa} - 1$ to $-\infty$. We conclude that (3.19) has a unique solution $\theta = \varphi(z)$ for any $z \in]-\infty, e^{-\kappa} - 1]$. □

The optimal control $\theta^*(\cdot)$ determined by the necessary condition (3.18) is thus recovered by

$$\theta^*(y) = \varphi\left(\frac{-\lambda}{I(y)}\right) = \varphi\left(\frac{(e^{-\kappa} - 1)I(h)}{I(y)}\right). \tag{3.21}$$

Next, we need to determine h so that the constraint

$$L(h) \doteq \int_0^h \frac{1}{\sin(\theta^*(y))}\,dy = \ell \tag{3.22}$$

is satisfied. As shown by Example 3.4 below, the solution of (3.21)-(3.22) may not be unique.

In the following, we seek a condition on I which implies that L is monotone, i.e.,

$$L'(h) = \frac{1}{\sin(\theta_0)} + \int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\,\frac{1}{F'(\theta^*(y))}\,\frac{I'(h)}{I(y)}G(\theta_0)\,dy > 0\,. \tag{3.23}$$

This will guarantee that (3.22) has a unique solution. To get an upper bound for $F'(\theta)$, observe that, for $\theta \in [\theta_0, \pi/2[$,

$$\begin{aligned}
F'(\theta) &\le \tan(\theta)G''(\theta)\\
&= -\tan(\theta)\Bigg[\cos(\theta-\theta_0)\left(1 - \left(\frac{\kappa}{\cos(\theta-\theta_0)} + 1\right)\exp\left\{\frac{-\kappa}{\cos(\theta-\theta_0)}\right\}\right)\\
&\qquad + \tfrac{\tan^2(\theta-\theta_0)}{\cos(\theta-\theta_0)}\kappa^2\exp\left\{\tfrac{-\kappa}{\cos(\theta-\theta_0)}\right\}\Bigg]\\
&= -\tan(\theta)\,\cos(\pi/2-\theta_0)\big(1-(\kappa+1)e^{-\kappa}\big).
\end{aligned}$$

Since $\theta^*(y) \in [\theta_0, \pi/2]$ and $G(\theta_0) = 1 - e^{-\kappa}$, using the above inequality one obtains

$$\begin{aligned}
&\int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)} \cdot \frac{1}{|F'(\theta^*(y))|}\,\frac{I'(h)}{I(y)}G(\theta_0)\,dy\\
&\le \frac{\cos^2\theta_0}{\sin^3\theta_0} \cdot \frac{1-e^{-\kappa}}{\cos(\pi/2-\theta_0)\big(1-(\kappa+1)e^{-\kappa}\big)} \int_0^h \frac{I'(h)}{I(y)}\,dy\,.
\end{aligned}$$

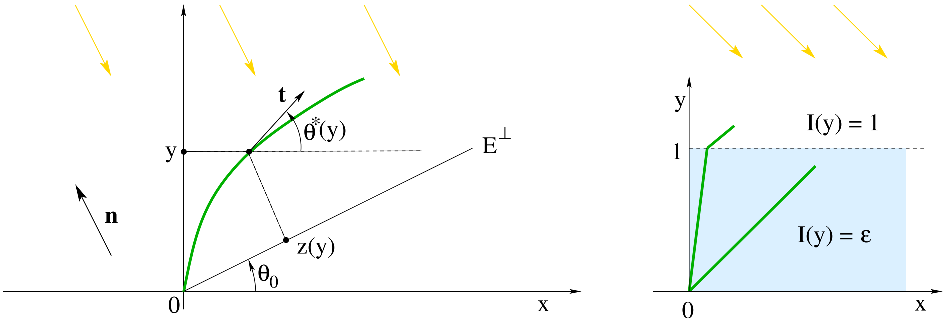

Fig. 2. Left: the optimal shape of a stem, as described in Theorem 3.3. Right: if the light intensity I changes abruptly as a function of the hight, the optimal shape may not be unique, as shown in Example 3.4.

Hence (3.23) is satisfied provided that

$$\int_0^h \frac{I'(h)}{I(y)}\,dy \;<\; \tan^2\theta_0 \cdot \frac{\cos(\pi/2-\theta_0)\big(1-(\kappa+1)e^{-\kappa}\big)}{1-e^{-\kappa}}\,. \tag{3.24}$$

From the above analysis, we conclude

Theorem 3.3. *Assume that the light intensity function I is Lipschitz continuous and satisfies the strict inequality (3.24) for a.e. $h \in [0,\ell]$. Then the optimization problem* **(OP1)** *has a unique optimal solution $\theta^* : [0,h^*] \mapsto [\theta_0, \pi/2]$. The function θ^* is non-increasing, and satisfies*

$$\theta^*(y) \;=\; \varphi\left((e^{-\kappa}-1)\frac{I(h^*)}{I(y)}\right), \tag{3.25}$$

where $z \mapsto \varphi(z) = \theta$ is the function implicitly defined by (3.19).

The following example shows that, without the bound (3.24) on the sunlight intensity function $I(\cdot)$, the conclusion of Theorem 3.3 can fail.

Example 3.4 *(non-uniqueness).* Choose $\mathbf{n} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\ell = 6/5 < \sqrt{2}$, $\kappa = 1$,

$$I(y) \;=\; \begin{cases} \varepsilon & \text{if} \quad y \in [0,1], \\ 1 & \text{if} \quad y > 1, \end{cases}$$

with $\varepsilon > 0$l.

By Theorem 3.1 at least one optimal solution exists. By the previous analysis, any optimal solution $\theta^* : [0,h^*] \mapsto [\theta_0, \pi/2]$ satisfies the necessary conditions (3.25). In this particular case, this implies that $\theta^*(y)$ is constant separately for $y < 1$ and for $y > 1$. As shown in Fig. 2, right, these necessary conditions can have two solutions.

Solution 1. If $h^* < 1$, then $I(y) = \varepsilon$ for all $y \in [0,h^*]$ and the necessary conditions (3.25) yield

$$\theta_1^*(y) \;=\; \varphi(e^{-1}-1) \;=\; \theta_0 \;=\; \pi/4 \qquad \text{for all } y \in [0, h^*].$$

The total sunlight collected is

$$\mathcal{S}_\varepsilon(\theta_1^*) \;=\; \frac{6}{5}(1-e^{-1})\,. \tag{3.26}$$

Solution 2. If $h^* > 1$, then $I(h^*) = 1$ and the necessary conditions (3.25) yield

$$\theta_2^*(y) \;=\; \varphi\left((e^{-1}-1)\frac{I(h^*)}{I(y)}\right) \;=\; \begin{cases} \varphi\left((e^{-1}-1)\varepsilon^{-1}\right) & \text{if} \quad y \in [0,1], \\ \pi/4 & \text{if} \quad y > 1. \end{cases}$$

Calling $\alpha = \alpha(\varepsilon) \doteq \varphi\left((e^{-1}-1)\varepsilon^{-1}\right)$, the total sunlight collected in this case is

$$\mathcal{S}_\varepsilon(\theta_2^*) \;=\; \left(1-\exp\left\{-\frac{1}{\cos(\alpha-\pi/4)}\right\}\right)\cos(\alpha-\pi/4)\,\varepsilon + \left(\frac{6}{5}-\frac{1}{\sin\alpha}\right)\left(1-e^{-1}\right). \tag{3.27}$$

We claim that, for a suitable choice of $\varepsilon \in \,]0,1[\,$, the two quantities in (3.26) and (3.27) become equal. Indeed, as $\varepsilon \to 0+$ we have

$$\alpha(\varepsilon) \;\doteq\; \varphi\left(\frac{e^{-1}-1}{\varepsilon}\right) \;\to\; \frac{\pi}{2}\,,$$

$$\mathcal{S}_\varepsilon(\theta_1^*) \;\to\; 0, \qquad\qquad \mathcal{S}_\varepsilon(\theta_2^*) \;\to\; \frac{1-e^{-1}}{5}\,. \tag{3.28}$$

On the other hand, as $\varepsilon \to 1$ we have $\alpha(\varepsilon) \to \pi/4$. By continuity, there exists $\varepsilon_1 \in \,]0,1[\,$ such that

$$\sin\alpha(\varepsilon_1) \;=\; \frac{5}{6}\,.$$

As $\varepsilon \to \varepsilon_1+$, we have

$$\mathcal{S}_\varepsilon(\theta_2^*) \;\to\; \left(1-\exp\left\{-\frac{1}{\cos(\alpha(\varepsilon_1)-\pi/4)}\right\}\right)\cos(\alpha(\varepsilon_1)-\pi/4)\,\varepsilon_1 \;<\; \mathcal{S}_{\varepsilon_1}(\theta_1^*). \tag{3.29}$$

Comparing (3.28) with (3.29), by continuity we conclude that there exists some $\widehat{\varepsilon} \in \,]0,\varepsilon_1[\,$ such that $\mathcal{S}_{\widehat{\varepsilon}}(\theta_1^*) \;=\; \mathcal{S}_{\widehat{\varepsilon}}(\theta_2^*)$. Hence for $\varepsilon = \widehat{\varepsilon}$ the optimization problem has two distinct solutions.

We remark that in this example the light intensity $I(y)$ is discontinuous at $y = 1$. However, by a mollification one can still construct a similar example with two optimal configurations, also for $I(\cdot)$ smooth. Of course, in this case the derivative $I'(h)$ will be extremely large for $h \approx 1$, so that the assumption (3.24) fails.

4. A competition model

In the previous analysis, the light intensity function $I(\cdot)$ was a priori given. We now consider a continuous distribution of stems, and determine the average sunlight $I(y)$ available at height y above ground, depending on the density of vegetation above y.

Let the constants $\ell, \kappa > 0$ be given, specifying the length and thickness of each stem. We now introduce another constant $\rho > 0$ describing the density of stems, i.e. how many stems grow per unit area. Assume that all stems have the same height and shape, described by the function $\theta : [0, h] \mapsto [\theta_0, \pi/2]$. For any $y \in [0, h]$, the total amount of vegetation at height $\geq y$, per unit length, is then measured by

$$\rho \cdot \int_y^h \frac{\kappa}{\sin\theta(y)}\, dy.$$

The corresponding light intensity function is defined as

$$I(y) \doteq \exp\left\{-\rho \cdot \int_y^h \frac{\kappa}{\sin\theta(y)}\, dy\right\} \qquad \text{for} \quad y \in [0, h], \tag{4.1}$$

while $I(y) = 1$ for $y \geq h$. We are interested in equilibrium configurations, where the shape of the stems is optimal for the light intensity $I(\cdot)$. We recall that θ_0 is the angle of incoming light rays, as in (2.1), while the constants $\ell, \kappa > 0$ denote the length and thickness of the stems.

Definition 4.1. Given an angle $\theta_0 \in]0, \pi/2]$ and constants $\ell, \kappa, \rho > 0$, we say that a light intensity function $I^* : \mathbb{R}_+ \mapsto [0, 1]$ and a stem shape function $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$ yield a **competitive equilibrium** if the following holds.

(i) The stem shape function $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$ provides an optimal solution to the optimization problem **(OP1)**, with light intensity function $I = I^*$.
(ii) For all $y \geq 0$, the light intensity at height y satisfies

$$I^*(y) = \exp\left\{-\rho \cdot \int_{\min\{y,h^*\}}^{h^*} \frac{\kappa}{\sin\theta^*(y)}\, dy\right\}. \tag{4.2}$$

If the density of vegetation is sufficiently small, we now show that an equilibrium configuration exists.

Theorem 4.2. *Let the light angle $\theta_0 \in]0, \pi/2]$ be given, together with the constants $\ell, \kappa > 0$ determining the common length and thickness of all the stems. Then there exists a constant $c_0 > 0$ such that, for all $0 < \rho \leq c_0$, an equilibrium configuration exists.*

Proof. 1. Consider the set of stem configurations

$$\mathcal{K} \doteq \Big\{\Theta : [0, \ell] \mapsto [\theta_0, \pi/2], \quad \Theta \text{ is nonincreasing}\Big\}, \tag{4.3}$$

and the set of light intensity functions

$$\mathcal{J} \doteq \Big\{ I : [0, +\infty[\mapsto [0, 1]\,; \quad I \text{ is nondecreasing}, \quad I(y) = 1 \ \text{ for } y \geq \ell, \\ I \text{ is Lipschitz continuous with constant } \frac{\rho\kappa}{\sin\theta_0} \Big\}. \tag{4.4}$$

We observe that $\mathcal{K}$ is a compact, convex subset of $\mathbf{L}^1([0, \ell])$, while $\mathcal{J}$ is a compact, convex subset of $\mathcal{C}^0([0, +\infty[)$.

If $\Theta(\cdot) \in \mathcal{K}$ describes the common configuration of all stems, we denote by $I^\Theta(\cdot)$ the corresponding light intensity function. Moreover, for a given function $I(\cdot)$, we denote by $\Theta^*(I)$ the corresponding optimal configuration of plant stems.

In the following steps we shall prove that:

(i) The map $\Theta \mapsto I^\Theta$ is continuous from $\mathcal{K}$ into $\mathcal{J}$.
(ii) The map $I \mapsto \Theta^*(I)$ is continuous from $\mathcal{J}$ into $\mathcal{K}$.

As a consequence, the composed map $\Theta \mapsto \Theta^*(I^\Theta)$ is continuous from $\mathcal{K}$ into itself. By Schauder's theorem, it has a fixed point, which provides an equilibrium solution.

2. Given $\Theta \in \mathcal{K}$, define the constant

$$\bar{h} \doteq \int_0^\ell \sin\Theta(t)\, dt\,. \tag{4.5}$$

More generally, for $s \in [0, \ell]$, set

$$y(s) \doteq \int_0^s \sin\Theta(t)\, dt \ \in \ [0, \bar{h}]. \tag{4.6}$$

We observe that, since $\Theta(t) \in [\theta_0, \pi/2]$, the inverse function $y \mapsto s(y)$ from $[0, \bar{h}]$ into $[0, \ell]$ is a strictly increasing bijection, with Lipschitz constant $L = \frac{1}{\sin\theta_0}$. The corresponding light intensity function is determined by

$$I^\Theta(y) = \begin{cases} \exp\big\{-\rho\kappa(\ell - s(y))\big\} & \text{if } \ y \in [0, \bar{h}], \\ 1 & \text{if } \ y > \ell. \end{cases} \tag{4.7}$$

From the above definitions it follows that $\Theta \mapsto I^\Theta$ is continuous from $\mathcal{K}$ into $\mathcal{J}$.

3. Next, let $I \in \mathcal{J}$. Given the constants ℓ, κ, by choosing $\rho > 0$ small enough, any Lipschitz continuous function $I : [0, \ell] \mapsto [0, 1]$ with Lipschitz constant $L = \frac{\rho\kappa}{\sin\theta_0}$ will satisfy the inequality (3.24). Hence, by Theorem 3.3, the optimization problem **(OP1)** has a unique optimal solution $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$.

Notice that in Theorem 3.3 this solution is written in terms of the variable $y \in [0, h^*]$, and satisfies the optimality condition (3.25). In terms of the arc-length parameter $s \in [0, \ell]$, this corresponds to

$$\Theta^*(s) \;=\; \theta^*(h(s))$$

where the variable $y(s) \in [0, h^*]$ is implicitly defined by

$$\int_0^{y(s)} \frac{1}{\sin\theta^*(z)}\,dz \;=\; s.$$

In view of (2.3), given $I \in \mathcal{J}$ and $\Theta \in \mathcal{K}$, the total sunlight collected by the stem is computed by

$$\mathcal{S}(I,\Theta) \;=\; = \int_0^{\ell} I(y(s)) \cdot \left(1 - \exp\left\{\frac{-\kappa}{\cos(\Theta(s)-\theta_0)}\right\}\right) \cos(\Theta(s)-\theta_0)\,ds, \tag{4.8}$$

where

$$y(s) \;\doteq\; \int_0^{s} \sin\Theta(s)\,ds.$$

From the above formulas it follows that the map $(I,\Theta) \mapsto \mathcal{S}(I,\Theta)$ is continuous on the compact set $\mathcal{J} \times \mathcal{K}$. In particular, the function

$$I \;\mapsto\; \max_{\Theta\in\mathcal{K}} \mathcal{S}(I,\Theta) \tag{4.9}$$

is continuous on the compact set $\mathcal{J}$.

Given a light intensity function $I \in \mathcal{J}$, call $\Theta^*(I) \in \mathcal{K}$ the unique optimal stem shape. We claim that the map $I \mapsto \Theta^*(I)$ is continuous.

Indeed, this is a straightforward consequence of continuity and compactness. If continuity fails, there exists a convergent sequence $I_n \to I$ such that $\Theta(I_n)$ does not converge to $\Theta(I)$. By the compactness of $\mathcal{K}$, we can extract a subsequence such that

$$\Theta_{n_k} \;\to\; \Theta^\sharp \;\neq\; \Theta(I).$$

By continuity, one obtains

$$\begin{aligned}\mathcal{S}(I,\Theta(I)) = \sup_{\Theta\in\mathcal{K}} \mathcal{S}(I,\Theta) \;&=\; \lim_{k\to\infty} \sup_{\Theta\in\mathcal{K}} \mathcal{S}(I_{n_k},\Theta)\\ &= \lim_{k\to\infty} \mathcal{S}(I_{n_k},\Theta(I_{n_k}))) \;=\; \mathcal{S}(I,\Theta^\sharp).\end{aligned}$$

This contradicts the uniqueness of the optimal stem configuration, stated in Theorem 3.3. We thus conclude that the map $I \mapsto \Theta^*(I)$ is continuous, completing the proof. □

4.1. Uniqueness and representation of equilibrium solutions

By (3.21) and (4.2), this equilibrium configuration (h^*, θ^*) must satisfy the necessary condition

$$\theta^*(y) = \varphi\left((e^{-\kappa} - 1)\exp\left\{\int_y^{h^*} \frac{\rho\kappa}{\sin\theta^*(y)}\,dy\right\}\right), \qquad y \in [0, h^*], \tag{4.10}$$

where φ is the function defined in Lemma 3.2. Here the constant h^* must be determined so that

$$\int_0^{h^*} \frac{1}{\sin\theta^*(y)}\,dy = \ell. \tag{4.11}$$

Based on (4.10), one obtains a simple representation of all equilibrium configurations, for any length $\ell > 0$. Indeed, for $t \in]-\infty, 0]$, let $t \mapsto \widehat{\zeta}(t)$ be the solution of the Cauchy problem

$$\zeta' = -\frac{\rho\kappa}{\sin\theta}, \qquad \text{where} \qquad \theta = \varphi\Big((e^{-\kappa} - 1)\, e^{\zeta}\Big),$$

with terminal condition $\zeta(0) = 0$.

Notice that the corresponding function $t \mapsto \widehat{\theta}(t) = \varphi\Big((e^{-\kappa} - 1)\, e^{\widehat{\zeta}(t)}\Big)$ satisfies

$$\widehat{\theta}(0) = \varphi(e^{-\kappa} - 1) = \theta_0 .$$

For any length ℓ of the stem, choose $h^* = h^*(\ell)$ so that

$$\int_{-h^*}^{0} \frac{1}{\sin\widehat{\theta}(t)}\,dt = \ell . \tag{4.12}$$

The shape of the stem that achieves the competitive equilibrium is then provided by

$$\theta^*(y) = \widehat{\theta}(y - h^*), \qquad y \in [0, h^*]. \tag{4.13}$$

Since the backward Cauchy problem

$$\zeta' = -\frac{\rho\kappa}{\sin\Big(\varphi\Big((e^{-\kappa} - 1)\, e^{\zeta}\Big)\Big)}, \qquad \zeta(0) = 0, \tag{4.14}$$

has a unique solution, we conclude that, if an equilibrium solution exists, by the representation (4.13) it must be unique. (See Fig. 3.)

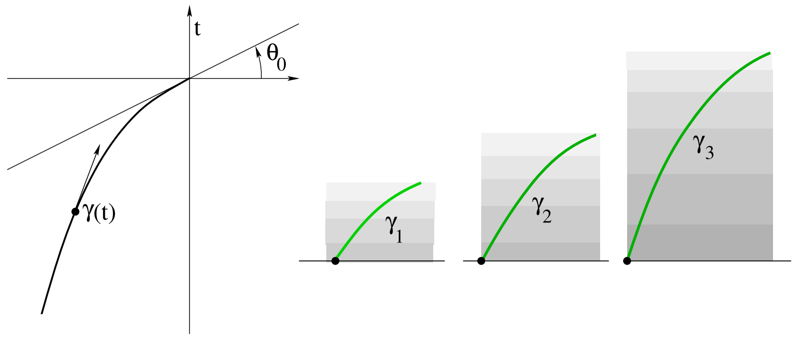

Fig. 3. Left: the curve γ, parameterized by the coordinate t. For $t < 0$, the tangent vector is $\frac{d\gamma}{dt} = (\tan\theta(t), 1)$, where $\theta(t)$ is obtained by solving the Cauchy problem (4.14). Right: for different lengths $0 < \ell_1 < \ell_2 < \ell_3$, the equilibrium configuration is obtained by taking the upper portion of the same curve γ, up to the length ℓ_i, $i = 1, 2, 3$.

5. Stems with variable length and thickness

We now consider the optimization problem **(OP2)**, allowing for stems of different lengths and with variable density of leaves.

5.1. Existence of an optimal solution

Theorem 5.1. *For any bounded, non-decreasing function $y \mapsto I(y) \in [0, 1]$ and any constants $0 < \alpha < 1$, $c > 0$ and $\theta_0 \in]0, \pi/2[$, the optimization problem* **(OP2)** *has at least one solution.*

Proof. 1. Consider a maximizing sequence of couples $(\theta_k, u_k) : \mathbb{R}_+ \mapsto [\theta_0, \pi/2] \times \mathbb{R}_+$. For $k \geq 1$, let

$$s \mapsto \gamma_k(s) = \left(\int_0^s \cos\theta_k(s)\, ds\,,\ \int_0^s \sin\theta_k(s)\, ds \right)$$

be the arc-length parameterization of the stem γ_k. Call μ_k the Radon measure on $\mathbb{R}^2$ describing the distribution of leaves along γ_k. For every Borel set $A \subseteq \mathbb{R}^n$, we thus have

$$\mu_k(A) = \int_{\{s\,;\ \gamma_k(s) \in A\}} u_k(s)\, ds. \tag{5.1}$$

For a given radius $\rho > 0$, we have the decomposition

$$\mu_k = \mu_k^\flat + \mu_k^\sharp,$$

where $\mu_k^\flat$ is the restriction of μ_k to the ball $B(0, \rho)$, while $\mu_k^\sharp$ is the restriction of μ_k to the complement $\mathbb{R}^2 \setminus B(0, \rho)$. By the same arguments used in steps **1-2** of the proof of Theorem 3.1 in [3], if the radius ρ is sufficiently large, then

$$\mathcal{S}(\mu_k^\flat) - c\mathcal{I}^\alpha(\mu_k^\flat) \geq \mathcal{S}(\mu_k) - c\mathcal{I}^\alpha(\mu_k) \tag{5.2}$$

for all $k \geq 1$. Here $\mathcal{S}$ and $\mathcal{I}^\alpha$ are the functionals defined at (2.12)-(2.13). According to (5.2), we can replace the measure μ_k with $\mu_k^\flat$ without decreasing the objective functional.

Without loss of generality we can thus choose $\ell > 0$ sufficiently large and assume that

$$u_k(s) = 0 \qquad \text{for all } s > \ell, \quad k \geq 1.$$

In turn, since $\mathcal{S}(\mu_k) - c\mathcal{I}^\alpha(\mu_k) \geq 0$, we obtain the uniform bound

$$\mathcal{I}^\alpha(\mu_k) \;\leq\; \kappa_1 \;\doteq\; \frac{1}{c}\mathcal{S}(\mu_k) \;\leq\; \frac{\ell}{c}. \tag{5.3}$$

2. In this step we show that the measures μ_k can be taken with uniformly bounded mass. Consider a measure μ_k for which (5.3) holds. By (2.13), for every $r \in [0, \ell]$ one has

$$\mathcal{I}^\alpha(\mu_k) \;\geq\; r \cdot \left(\int_r^\ell u_k(t)\, dt \right)^\alpha .$$

In view of (5.3), this implies

$$\int_r^\ell u_k(s)\, ds \;\leq\; \left(\frac{\kappa_1}{r}\right)^{1/\alpha} . \tag{5.4}$$

It thus remains to prove that, in our maximizing sequence, the functions u_k can be replaced with functions $\tilde{u}_k$ having a uniformly bounded integral over $[0, r]$, for some fixed $r > 0$.

Toward this goal we fix $0 < \varepsilon < \beta < 1$, and, for $j \geq 1$, we define $r_j = 2^{-j}$, and the interval $V_j =]r_{j+1}, r_j]$. Given $u = u_k$, if $\int_{V_j} u(s)\, ds > r_j^\varepsilon$, we introduce the functions

$$u_j(s) \;\doteq\; \chi_{V_j}(s) u(s), \qquad\qquad \tilde{u}_j(s) \;\doteq\; \min\{u_j(s), c_j\}, \tag{5.5}$$

choosing the constant $c_j \geq 2r_j^{\beta-1}$ so that

$$\int_{V_j} \tilde{u}_j(s)\, ds \;=\; r_j^\beta. \tag{5.6}$$

We then let $\mu_j = u_j\mu$ and $\tilde{\mu}_j = \tilde{u}_j\mu$ be the measures supported on V_j, corresponding to these densities.

For a fixed integer j^*, whose precise value will be chosen later, consider the set of indices

$$J \;\doteq\; \left\{ j \geq j^* \;\middle|\; \int_{V_j} u(s)\, ds > r_j^\varepsilon \right\} \tag{5.7}$$

and the modified density

$$\tilde{u}(s) \doteq u(s) + \sum_{j\in J} (\tilde{u}_j(s) - u_j(s)). \tag{5.8}$$

Moreover, call $\tilde{\mu}$ the measure obtained by replacing u with $\tilde{u}$ in (2.11). By (5.4) and (5.5) the total mass of $\tilde{\mu}$ is bounded. Indeed

$$\tilde{\mu}(\mathbb{R}^2) = \int_{r_{j^*}}^{\ell} \tilde{u}(s)\,ds + \int_0^{r_{j^*}} \tilde{u}(s)\,ds \le \left(\frac{\kappa_1}{r_{j^*}}\right)^{1/\alpha} + \sum_{j\ge j^*} r_j^{\varepsilon} \le \left(\frac{\kappa_1}{r_{j^*}}\right)^{1/\alpha} + \sum_{j\ge 1} 2^{-j\varepsilon} < +\infty. \tag{5.9}$$

We now claim that

$$\mathcal{S}(\tilde{\mu}) - c\mathcal{I}^{\alpha}(\tilde{\mu}) \ \ge\ \mathcal{S}(\mu) - c\mathcal{I}^{\alpha}(\mu). \tag{5.10}$$

Toward a proof of (5.10), we estimate

$$\begin{aligned} \mathcal{S}(\mu) - \mathcal{S}(\tilde{\mu}) \le \sum_{j\in J} \Bigg(& \int_{V_j} I(y(t)) \cos(\theta(t)-\theta_0)\,dt \\ & - \int_{V_j} I(y(t)) \left(1 - \exp\left\{ - \frac{\tilde{u}_j(t)}{\cos(\theta(t)-\theta_0)} \right\}\right) \cos(\theta(t)-\theta_0)\,dt \Bigg) \\ \le \sum_{j\in J} \int_{r_{j+1}}^{r_j} & \exp\{-\tilde{u}_j(t)\}dt \ \le\ \sum_{j\in J} r_{j+1} \exp\left\{-2r_j^{\beta-1}\right\}. \end{aligned} \tag{5.11}$$

To estimate the difference in the irrigation cost, we first observe that the inequality

$$\left(\int_r^{\ell} u(t)\,dt\right)^{\alpha} \ \le\ \frac{1}{r}\mathcal{I}^{\alpha}(\mu) \ =\ \frac{\kappa_1}{r}$$

implies

$$\left(\int_r^{\ell} u(t)\,dt\right)^{\alpha-1} \ \ge\ \left(\frac{\kappa_1}{r}\right)^{\frac{\alpha-1}{\alpha}}. \tag{5.12}$$

Since $\tilde{u}(s) \le u(s)$ for every $s \in [0,\ell]$, using (5.12) we now obtain

$$\begin{aligned} \mathcal{I}^{\alpha}(\mu) - \mathcal{I}^{\alpha}(\tilde{\mu}) &= \int_0^1 \frac{d}{d\lambda} \mathcal{I}^{\alpha}\big(\lambda\mu + (1-\lambda)\tilde{\mu}\big)\,d\lambda \\ &= \int_0^1 \int_0^{\ell} \frac{d}{d\lambda} \left(\int_s^{\ell} [\lambda u(t) + (1-\lambda)\tilde{u}(t)]\,dt \right)^{\alpha} ds\,d\lambda \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^\ell \left\{ \alpha \left(\int_s^\ell [\lambda u(t) + (1-\lambda)\tilde{u}(t)]\,dt \right)^{\alpha-1} \int_s^\ell [u(t)-\tilde{u}(t)]\,dt \right\} ds\,d\lambda \\
&\geq \int_0^\ell \left\{ \alpha \left(\int_s^\ell u(t)\,dt \right)^{\alpha-1} \int_s^\ell [u(t)-\tilde{u}(t)]\,dt \right\} ds \\
&\geq \sum_{j\in J} \int_{r_{j+2}}^{r_{j+1}} \left[\alpha \left(\int_s^\ell u(t)\,dt \right)^{\alpha-1} \int_{r_{j+1}}^{r_j} (u_j(t)-\tilde{u}_j(t))\,dt \right] ds \\
&\geq \sum_{j\in J} \alpha \left(\frac{\kappa_1}{r_{j+2}} \right)^{\frac{\alpha-1}{\alpha}} \cdot (r_j^\varepsilon - r_j^\beta) \cdot r_{j+2} \\
&= \sum_{j\in J} \kappa_2 r_j^{1/\alpha} (r_j^\varepsilon - r_j^\beta),
\end{aligned} \tag{5.13}
$$

where $\kappa_2 = \alpha(4\kappa_1)^{\frac{\alpha-1}{\alpha}}$. Combining (5.11) with (5.13) we obtain

$$
c[\mathcal{I}^\alpha(\mu) - \mathcal{I}^\alpha(\tilde{\mu})] - [\mathcal{S}(\mu) - \mathcal{S}(\tilde{\mu})] \;\geq\; \sum_{j\in J} \left(c\kappa_2 r_j^{1/\alpha} (r_j^\varepsilon - r_j^\beta) - r_{j+1} \exp\left\{ -2 r_j^{\beta-1} \right\} \right). \tag{5.14}
$$

By choosing the integer j^* large enough in (5.7), for $j \geq j^*$ all terms in the summation on the right hand side of (5.14) are ≥ 0. This implies (5.10).

3. By the two previous steps, w.l.o.g. we can assume that the measures μ_k have uniformly bounded support and uniformly bounded total mass. Otherwise, we can replace the sequence $(u_k)_{k\geq 1}$ with a new maximizing sequence $(\tilde{u}_k)_{k\geq 1}$ having these properties.

By taking a subsequence, we can thus assume the weak convergence $\mu_k \rightharpoonup \overline{\mu}$. The upper semicontinuity of the functional $\mathcal{S}$, proved in [5], yields

$$
\mathcal{S}(\overline{\mu}) \;\geq\; \limsup_{k\to\infty} \mathcal{S}(\mu_k). \tag{5.15}
$$

In addition, since all maps $s \mapsto \gamma_k(s)$ are 1-Lipschitz, by taking a further subsequence we can assume the convergence

$$
\gamma_k(s) \;\to\; \overline{\gamma}(s) \tag{5.16}
$$

for some limit function $\overline{\gamma}$, uniformly for $s \in [0, \ell]$.

Since each measure μ_k is supported on γ_k, the weak limit $\overline{\mu}$ is a measure supported on the curve $\overline{\gamma}$.

4. Since $\theta_k(s) \in [\theta_0, \pi/2]$, we can re-parameterize each stem γ_k in terms of the vertical variable

$$
y_k(s) \;=\; \int_0^s \sin\theta_k(t)\,dt.
$$

Calling $s = s_k(y)$ the inverse function, we thus obtain a maximizing sequence of couples

$$y \mapsto (\widehat{\theta}_k(y), \widehat{u}_k(y)) \doteq \Big(\theta_k(s_k(y)),\, u_k(s_k(y))\Big), \qquad y \in [0, h_k].$$

Moreover, the stem γ_k can be described as the graph of the Lipschitz function

$$x = x_k(y) = \int_0^{s_k(y)} \cos\theta_k(s)\, ds.$$

Since all functions $x_k(\cdot)$ satisfy $x_k(0) = 0$ and are non-decreasing, uniformly continuous with Lipschitz constant $L = \cos\theta_0 / \sin\theta_0$, by possibly extracting a further subsequence, we obtain the convergence $h_k \to \bar{h}$ and $x_k(\cdot) \to \bar{x}(\cdot)$. Here $\bar{x} : [0, \bar{h}] \mapsto \mathbb{R}$ is a nondecreasing continuous function with Lipschitz constant L, such that $\bar{x}(0) = 0$. More precisely, the convergence $x_k \to \bar{x}$ is uniform on every compact subinterval $[0, h]$ with $h < \bar{h}$.

5. We claim that the irrigation cost of $\overline{\mu}$ is no greater that the lim-inf of the irrigation costs for μ_k. Let $\sigma \mapsto \gamma(\sigma)$ be an arc-length parameterization of $\overline{\gamma}$. Since $s \mapsto \overline{\gamma}(s)$ is 1-Lipschitz, one has $d\sigma/ds \le 1$. We now compute

$$\begin{aligned} \mathcal{I}^\alpha(\overline{\mu}) &= \int_0^{\sigma(\ell)} \left(\int_\sigma^{\sigma(\ell)} \overline{u}(t)\, dt \right)^\alpha d\sigma = \int_0^{\sigma(\ell)} \left(\lim_{k\to\infty} \int_s^\ell u_k(t)\, dt \right)^\alpha d\sigma(s) \\ &\le \lim_{k\to\infty} \int_0^\ell \left(\int_s^\ell u_k(t)\, dt \right)^\alpha ds = \lim_{k\to\infty} \mathcal{I}^\alpha(\mu_k). \end{aligned} \tag{5.17}$$

6. Combining (5.15) with (5.17) we conclude that the measure $\overline{\mu}$, supported on the stem $\overline{\gamma}$, is optimal.

Let $\bar{u}$ be the density of the absolutely continuous part of $\overline{\mu}$ w.r.t. the arc-length measure on $\bar{\gamma}$, and call μ^* the measure that has density $\bar{u}$ w.r.t. arc-length measure. Since $\mathcal{S}(\mu^*) = \mathcal{S}(\overline{\mu})$, it follows that $\mu^* = \overline{\mu}$. Otherwise $\mathcal{I}^\alpha(\mu^*) < \mathcal{I}^\alpha(\overline{\mu})$ and $\overline{\mu}$ is not optimal. This argument shows that the optimal measure $\overline{\mu}$ is absolutely continuous w.r.t. the arc-length measure on $\overline{\gamma}$.

Calling $\sigma \mapsto \gamma(\sigma)$ the arc-length parameterization of $\overline{\gamma}$, the optimal solution to **(OP2)** is now provided by $\sigma \mapsto (\overline{\theta}(\sigma), \bar{u}(\sigma))$, where $\overline{\theta}$ is the orientation of the tangent vector:

$$\frac{d}{d\sigma}\overline{\gamma}(\sigma) = \big(\cos\overline{\theta}(\sigma),\, \sin\overline{\theta}(\sigma)\big). \quad \square$$

5.2. *Necessary conditions for optimality*

Let $t \mapsto (\theta^*(t), u^*(t))$ be an optimal solution to the problem **(OP2)**. The necessary conditions for optimality [4,6,7] yield the existence of dual variables p, q satisfying

$$\begin{cases} \dot{p} = -I'(y)\, G(\theta, u), \\ \dot{q} = c\alpha\, z^{\alpha-1}, \end{cases} \qquad \begin{cases} p(+\infty) = 0, \\ q(0) = 0, \end{cases} \tag{5.18}$$

and such that the maximality condition

$$(\theta^*(t), u^*(t)) \;=\; \arg\max_{\theta\in[0,\pi],\; u\geq 0} \Big\{ p(t)\,\sin\theta - q(t)u + I(y(t))\,G(\theta,u) - cz^\alpha \Big\}. \tag{5.19}$$

We recall that $G(\theta, u)$ is the function defined at (2.17). An intuitive interpretation of the quantities on the right-hand side of (5.19) goes as follows:

- $p(t)$ is the rate of increase in the gathered sunlight, if the upper portion of stem $\{\gamma(s)\,;\;\; s > t\}$ is raised higher.
- $q(t)$ is the rate at which the irrigation cost increases, adding mass at the point $\gamma(t)$.
- $I(y(t))\,G(\theta, u)$ is the sunlight captured by the leaves at the point $\gamma(t)$.

6. Uniqueness of the optimal stem configuration

Aim of this section is to show that, if the light intensity $I(y)$ remains sufficiently close to 1 for all $y \geq 0$, then the shape of the optimal stem is uniquely determined. This models a case where the density of external vegetation is small.

Theorem 6.1. *Let $h \mapsto I(h) \in [0, 1]$ be a non-decreasing, absolutely continuous function which satisfies*

$$I'(y) \leq Cy^{-\beta} \qquad \text{for a.e. } y > 0, \tag{6.1}$$

for some constants $C > 0$ and $0 < \beta < 1$. If

$$I(0) \geq 1 - \delta \tag{6.2}$$

for some $\delta > 0$ sufficiently small, then the optimal solution to **(OP2)** *is unique.*

Proof. We will show that the necessary conditions for optimality have a unique solution. This will be achieved in several steps. **1.** Given I, p, q, define the functions Θ, U by setting

$$\Big(\Theta(I,p,q),\, U(I,p,q)\Big) \;\doteq\; \arg\max_{\theta\in[0,\pi],\; u\geq 0} \Big\{ p\cdot\sin\theta - q\,u + I\cdot G(\theta,u) - cz^\alpha \Big\}. \tag{6.3}$$

We recall that G is the function defined at (2.17). Notice that one can write

$$G(\theta, u) = u\,\widetilde{G}\left(\frac{\cos(\theta-\theta_0)}{u}\right)$$

with

$$\widetilde{G}(x) \doteq \left(1 - \exp\left\{-\frac{1}{x}\right\}\right) x \;>\; 0, \qquad \widetilde{G}'(x) \leq 1, \qquad \widetilde{G}''(x) \leq 0, \qquad \text{for all } x > 0. \tag{6.4}$$

Denote by

$$\mathcal{H}(\theta, u) \;\doteq\; p\cdot\sin\theta - q\,u + I(y)\,G(\theta,u) - cz^\alpha \tag{6.5}$$

the quantity to be maximized in (6.3). Differentiating $\mathcal{H}$ w.r.t. θ and imposing that the derivative is zero, we obtain

$$\begin{aligned}\frac{p}{I} &= -\frac{G_\theta(\theta,u)}{\cos\theta}\\ &= \frac{\sin(\theta-\theta_0)}{\cos\theta}\left[1-\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\}-\frac{u}{\cos(\theta-\theta_0)}\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\}\right].\end{aligned} \tag{6.6}$$

Similarly, differentiating $\mathcal{H}$ w.r.t. u, we find

$$-q+IG_u(\theta,u) \;=\; -q+I\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\} \;=\; 0.$$

This yields

$$u \;=\; -\ln\left(\frac{q}{I}\right)\cos(\theta-\theta_0). \tag{6.7}$$

A lengthy but elementary computation shows that the Hessian matrix of second derivatives of $\mathcal{H}$ w.r.t. θ, u is negative definite, and the critical point is indeed the point where the global maximum is attained. By (6.7) it follows

$$U(I,p,q) \;=\; -\ln\left(\frac{q}{I}\right)\cos\big(\Theta(I,p,q)-\theta_0\big). \tag{6.8}$$

Inserting (6.8) in (6.6) and using the identity

$$\frac{\sin(\theta-\theta_0)}{\cos\theta} \;=\; \cos\theta_0\tan\theta-\sin\theta_0$$

we obtain

$$\Theta(I,p,q) \;=\; \arctan\left(\tan\theta_0+\frac{\frac{1}{\cos\theta_0}\,\frac{p}{I}}{1-\frac{q}{I}+\frac{q}{I}\ln\left(\frac{q}{I}\right)}\right). \tag{6.9}$$

Introducing the function

$$w(I,p,q) \;\doteq\; \frac{p/I}{1-\frac{q}{I}+\frac{q}{I}\ln\left(\frac{q}{I}\right)}, \tag{6.10}$$

by (6.9) one has the identities

$$\begin{cases} \sin\big(\Theta(I,p,q)\big) \;=\; \dfrac{\sin\theta_0+w}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}},\\[2ex] \cos\big(\Theta(I,p,q)-\theta_0\big) \;=\; \dfrac{1+w\sin\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}.\end{cases} \tag{6.11}$$

Note that $w \geq 0$, because $p,q,I \geq 0$. In turn, from (6.11) it follows

$$\begin{cases} \cos\big(\Theta(I,p,q)\big) &= \dfrac{\cos\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}, \\ \sin\left(\Theta(I,p,q)-\theta_0\right) &= \dfrac{w\cos\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}. \end{cases} \tag{6.12}$$

2. The necessary conditions for the optimality of a solution to **(OP2)** yield the boundary value problem

$$\begin{cases} \dot y(t) &= \sin\Theta, \\ \dot z(t) &= -U, \\ \dot p(t) &= -I'(y)G\big(\Theta,U\big), \\ \dot q(t) &= c\alpha z^{\alpha-1}, \end{cases} \qquad \begin{cases} y(0) &= 0, \\ z(T) &= 0, \\ p(T) &= 0, \\ q(T) &= I(y(T)), \\ q(0) &= 0. \end{cases} \tag{6.13}$$

Here $[0,T[$ is the interval where $u>0$, while

$$\Theta = \Theta(I(y),p,q), \qquad U = U(I(y),p,q) \tag{6.14}$$

are the functions introduced at (6.3), or more explicitly at (6.8)-(6.9). Notice that the length T of the stem is a quantity to be determined, using the boundary conditions in (6.13).

3. Since the control system (2.19) and the running cost (2.18) do not depend explicitly on time, the Hamiltonian function

$$H(y,z,p,q) \doteq \max_{\theta\in[0,\pi],\ u\geq 0}\Big\{p\cdot\sin\theta - q\,u + I(y)\,G(\theta,u) - cz^\alpha\Big\} \tag{6.15}$$

is constant along trajectories of (6.13). Observing that the terminal conditions in (6.13) imply $H(y(T),z(T),p(T),q(T))=0$, one has the first integral

$$H(y(t),z(t),p(t),q(t)) = 0 \qquad \text{for all } t\in[0,T]. \tag{6.16}$$

This yields

$$\begin{aligned} 0 &= p\sin\Theta + \left[I(y) - q + q\ln\left(\frac{q}{I(y)}\right)\right]\cos\left(\Theta-\theta_0\right) - cz^\alpha \\ &= \frac{p\left[\sin\theta_0+w\right] + \left[I(y)-q+q\ln\left(\frac{q}{I(y)}\right)\right]\left[1+w\sin\theta_0\right]}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}} - cz^\alpha \\ &= I(y)\left[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2} - cz^\alpha. \end{aligned}$$

We can use this identity to express z as a function of the other variables:

$$
\begin{aligned}
z\big(I(y),p,q\big) &= \left\{\frac{I(y)}{c}\left[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}\right\}^{1/\alpha} \\
&= c^{-1/\alpha}\left\{\left(\left[I(y)-q+q\ln\left(\frac{q}{I(y)}\right)\right]\cos\theta_0\right)^2\right. \\
&\qquad\left. +\left(p+\left[I(y)-q+q\ln\left(\tfrac{q}{I(y)}\right)\right]\sin\theta_0\right)^2\right\}^{1/2\alpha}.
\end{aligned}
\tag{6.17}
$$

4. Since I is given as a function of the height y, it is convenient to rewrite the equations (6.13) using y as an independent variable. Using the identity (6.17), we obtain a system of two equations for the variables p, q:

$$
\begin{aligned}
\frac{d}{dy}p(y) &= -I'(y)\left[1-\frac{q(y)}{I(y)}\right]\frac{\cos\big(\Theta\big(I(y),p(y),q(y)\big)-\theta_0\big)}{\sin\Theta\big(I(y),p(y),q(y)\big)} \\
&= -I'(y)\left[1-\frac{q(y)}{I(y)}\right]\frac{1+w\sin\theta_0}{w+\sin\theta_0} \\
&\doteq -I'(y)\,f_1\big(I(y),p(y),q(y)\big),
\end{aligned}
\tag{6.18}
$$

$$
\begin{aligned}
\frac{d}{dy}q(y) &= \frac{c\alpha\big[z\big(I(y),p(y),q(y)\big)\big]^{\alpha-1}}{\sin\Theta\big(I(y),p(y),q(y)\big)} \\
&= \frac{\alpha c^{1/\alpha}}{w+\sin\theta_0}\left[\cos^2\theta_0+(\sin\theta_0+w)^2\right]^{1-\frac{1}{2\alpha}} \\
&\qquad\times\left[I(y)\left(1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right)\right]^{1-\frac{1}{\alpha}} \\
&\doteq f_2\big(I(y),p(y),q(y)\big),
\end{aligned}
\tag{6.19}
$$

where $w = w(I, p, q)$ is the function introduced at (6.10). Note that under our assumptions, f_1 remains bounded, while f_2 diverges as $q(y) \to I(y)$. The system (6.13) can now be equivalently formulated as

$$
\begin{cases} p'(y) = -I'(y)\,f_1\big(I(y),p,q\big), \\ q'(y) = f_2\big(I(y),p,q\big), \end{cases}
\qquad
\begin{cases} p(h) = 0, \\ q(h) = I(h), \end{cases}
\qquad q(0) = 0.
\tag{6.20}
$$

5. To prove uniqueness of the solution to the boundary value problem (6.13), it thus suffices to prove the following (see Fig. 4, right).

(U) *Call*

$$
y \mapsto \big(p(y,h),\, q(y,h)\big)
\tag{6.21}
$$

the solution to the system (6.20), with the two terminal conditions given at $y = h$. Then there is a unique choice of $h > 0$ which satisfies also the third boundary condition

$$
q(0,h) = 0.
\tag{6.22}
$$

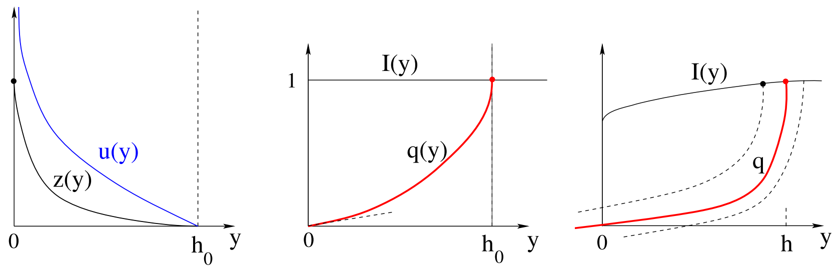

Fig. 4. Left and center: sketch of the solution of the system (5.18) in the case where $I(y) \equiv 1$. Left: the graphs of the functions z in (6.25) and $u = -\ln q$. Center: the graph of the function q at (6.26). The figure on the right shows the case where $I(\cdot)$ is not constant. As before, h must be determined so that $q(0, h) = 0$.

To make the argument more clear, the uniqueness property **(U)** will be proved in two steps.

(i) When $I(y) \equiv 1$, the map

$$h \mapsto q(0,h) \tag{6.23}$$

is strictly decreasing, hence it vanishes at a unique point h_0.

(ii) For all functions $I(\cdot)$ sufficiently close to the constant map $\equiv 1$, the map (6.23) is strictly decreasing in a neighborhood of h_0.

In the case $I(y) \equiv 1$, recalling (6.9) we obtain (see Fig. 4)

$$\begin{aligned} &I'(y) = 0, \qquad p(y,h) = 0, \qquad \Theta(I,0,q) = \theta_0, \qquad G(\theta_0, U) = 1 - e^{-U}, \\ &U(1,0,q) = \operatorname*{argmax}_u \left\{-qu + G(\theta_0, U)\right\} = \operatorname*{argmax}_u \{-qu + 1 - e^{-u}\} = -\ln q, \end{aligned}$$

The system (6.13) can now be written as

$$\begin{cases} p'(y) = 0, \\ q'(y) = \dfrac{c\alpha z^{\alpha-1}}{\sin\theta_0}, \\ z'(y) = \dfrac{\ln q}{\sin\theta_0}, \end{cases} \qquad \begin{cases} p(h) = 0, \\ q(h) = 1, \\ z(h) = 0, \end{cases} \qquad q(0) = 0. \tag{6.24}$$

From (6.24) it follows $p(y) \equiv 0$, while

$$\frac{dz}{dq} = \frac{\ln q}{c\alpha z^{\alpha-1}}.$$

Integrating the above ODE with terminal conditions $q = 1$, $z = 0$, one obtains

$$z = c^{-1/\alpha}\Big[1 + q\ln q - q\Big]^{1/\alpha}. \tag{6.25}$$

The second equation in (6.24) thus becomes

$$q'(y) \;=\; \frac{\alpha c^{1/\alpha}}{\sin\theta_0}\Big[1+q\ln|q|-q\Big]^{\frac{\alpha-1}{\alpha}}. \tag{6.26}$$

Notice that here the right hand side is strictly positive for all $q \in]-1, 1[$. Of course, only positive values of q are relevant for the optimization problem, but for the analysis it is convenient to extend the definition also to negative values of q. The solution of (6.26) with terminal condition $q(h) = 1$ is implicitly determined by

$$h - y \;=\; \frac{\sin\theta_0}{\alpha c^{1/\alpha}} \int_{q(y)}^{1} \Big[1+s\ln|s|-s\Big]^{\frac{1-\alpha}{\alpha}}\,ds\,. \tag{6.27}$$

The map $h \mapsto q(0, h)$ thus vanishes at the unique point

$$h_0 \;=\; \frac{\sin\theta_0}{\alpha c^{1/\alpha}} \int_0^1 \Big[1+s\ln|s|-s\Big]^{\frac{1-\alpha}{\alpha}}\,ds. \tag{6.28}$$

As expected, the height h_0 of the optimal stem decreases as we increase the constant c in the transportation cost. A straightforward computation yields

$$\frac{\partial}{\partial h} q(0,h) \;=\; -\,\frac{\alpha c^{1/\alpha}}{\sin\theta_0}\Big[1+q(0,h)\ln|q(0,h)|-q(0,h)\Big]^{\frac{1-\alpha}{\alpha}}. \tag{6.29}$$

In particular, at $h = h_0$ we have $q^{(h_0)}(0) = 0$ and hence

$$\frac{d}{dh} q(0,h)\Big|_{h=h_0} \;=\; -\,\frac{\alpha c^{1/\alpha}}{\sin\theta_0} \;<\; 0. \tag{6.30}$$

6. We will show that a strict inequality as in (6.30) remains valid for a more general function $I(\cdot)$, provided that the assumptions (6.1)-(6.2) hold.

Toward this goal, we need to determine how p and q vary w.r.t. the parameter h. Denoting by

$$P(y) \;\doteq\; \frac{\partial p(y,h)}{\partial h}, \qquad Q(y) \;\doteq\; \frac{\partial q(y,h)}{\partial h} \tag{6.31}$$

their partial derivatives, by (6.20) one obtains the linear system

$$\begin{pmatrix} P(y) \\ Q(y) \end{pmatrix}' \;=\; \begin{pmatrix} -I'(y) f_{1,p} & -I'(y) f_{1,q} \\ f_{2,p} & f_{2,q} \end{pmatrix} \begin{pmatrix} P(y) \\ Q(y) \end{pmatrix}. \tag{6.32}$$

The boundary conditions at $y = h$ require some careful consideration. As $y \to h-$, we expect $f_2(I(y), p(y), q(y)) \to +\infty$ and $Q(y) \to -\infty$. To cope with this singularity we introduce the new variable

$$\widetilde{Q}(y) \;\doteq\; \frac{Q(y)}{f_2\big(I(y), p(y), q(y)\big)}. \tag{6.33}$$

The system (6.32), together with the new boundary conditions for P, $\widetilde{Q}$, can now be written as

$$\begin{cases} P'(y) = -I'(y)\big[f_{1,p}P + f_{1,q}f_2\widetilde{Q}\big], \\ \widetilde{Q}'(y) = \dfrac{f_{2,p}}{f_2}P - \dfrac{I'(y)[f_{2,I} - f_{2,p}f_1]}{f_2}\widetilde{Q}, \end{cases} \qquad \begin{cases} P(h) = 0, \\ \widetilde{Q}(h) = -1. \end{cases} \tag{6.34}$$

To analyze this system we must compute the partial derivatives of f_1 and f_2. From the definition (6.10) it follows

$$\frac{\partial w}{\partial I} = \frac{w^2}{p}\left[1 - \frac{q}{I}\right], \qquad \frac{\partial w}{\partial p} = \frac{w}{p}, \qquad \frac{\partial w}{\partial q} = -\frac{w^2}{p}\ln\left(\frac{q}{I}\right). \tag{6.35}$$

Using (6.35), from (6.18), (6.19) we obtain

$$\begin{cases} f_{1,p}\big(I(y),p,q\big) = \dfrac{1 - \frac{q}{I(y)}}{I(y)\tan^2\Theta\left[1 - \frac{q}{I(y)} + \frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]}, \\ f_{1,q}\big(I(y),p,q\big) = \dfrac{1}{I(y)}\dfrac{\cos(\Theta-\theta_0)}{\sin\Theta} - \dfrac{\sin(\Theta-\theta_0)\cos\Theta\left[1 - \frac{q}{I(y)}\right]\ln\left(\frac{q}{I}\right)}{I(y)\sin^2\Theta\left[1 - \frac{q}{I(y)} + \frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]}, \\ f_{2,p}\big(I(y),p,q\big) = -\left[1 + \dfrac{\alpha}{\sin^2\Theta} - 2\alpha\right]\dfrac{1}{z\big(I(y),p,q\big)}, \\ f_{2,q}\big(I(y),p,q)\big) = -\left[\dfrac{(1-\alpha)\sin\theta_0}{\sin^2\Theta} - \dfrac{\sin(\Theta-\theta_0)}{\cos\Theta}\left(1 + \dfrac{\alpha}{\sin^2\Theta} - 2\alpha\right)\right]\dfrac{\ln\left(\frac{q}{I(y)}\right)}{z\big(I(y),p,q\big)}, \\ f_{2,I}\big(I(y),p,q\big) = -\left[\dfrac{(1-\alpha)\sin\theta_0}{\sin^2\Theta} + \dfrac{\sin(\Theta-\theta_0)}{\cos\Theta}\left(1 + \dfrac{\alpha}{\sin^2\Theta} - 2\alpha\right)\right]\dfrac{1 - \frac{q}{I(y)}}{z\big(I(y),p,q\big)}. \end{cases} \tag{6.36}$$

At this stage, the strategy of the proof is straightforward. When $I'(y) \equiv 0$, the solution to (6.34) is trivially given by $P(y) \equiv 0$, $\widetilde{Q}(y) \equiv -1$. This implies

$$\frac{\partial}{\partial h}q(0,h) = \widetilde{Q}(0)\cdot f_2(I(0),p(0),q(0)) < 0.$$

We need to show that the same strict inequality holds when $\delta > 0$ in (6.2) is small enough. Notice that, if the right hand sides of the equations in (6.34) were bounded, letting $\|I'\|_{\mathbf{L}^\infty} \to 0$ a continuity argument would imply the uniform convergence $P(y) \to 0$ and $\widetilde{Q}(y) \to -1$. The same conclusion can be achieved provided that the right hand sides in (6.34) are uniformly integrable. This is precisely what will be proved in the next two steps, relying on the identities (6.36).

7. In this step we prove an inequality of the form

$$0 < \theta_0 \le \Theta(I,p,q) \le \theta^+ < \frac{\pi}{2}. \tag{6.37}$$

As a consequence, this implies that all terms in (6.36) involving $\sin\Theta$ or $\cos\Theta$ remain uniformly positive.

The lower bound $\Theta \geq \theta_0$ is an immediate consequence of (6.9). To obtain an upper bound on Θ, we set

$$q^\sharp \doteq \frac{q(y)}{I(y)}.$$

By (6.13), a differentiation yields

$$\dot{q}^\sharp = \frac{c\alpha z^{\alpha-1} - q^\sharp I' \sin(\Theta)}{I}.$$

Next, we observe that, by (6.13), one has

$$\frac{dz}{dq^\sharp} = \ln q^\sharp \cdot \cos(\Theta - \theta_0) \cdot \frac{I}{c\alpha z^{\alpha-1} - q^\sharp I' \sin(\Theta)} = \varphi_1(q^\sharp) \cdot \ln q^\sharp \cdot \alpha z^{\alpha-1}, \qquad \begin{cases} z(h) = 0, \\ q^\sharp(h) = 1. \end{cases}$$

In (6.2) we can now choose $\delta \leq c\alpha M^{\alpha-1}$, where $M \geq z(0)$ is an a priori bound on the mass of the stem, derived in Section 5. This ensures that φ_1 is a bounded, uniformly positive function for y close enough to h, say

$$0 < c^- \leq \varphi_1 \leq c^+,$$

for some constants c^-, c^+. Integrating, we obtain

$$z^\alpha = \int_0^z \alpha\zeta^{\alpha-1}\, d\zeta = -\int_{q^\sharp}^1 \varphi_1(s) \ln s\, ds = -\varphi_2(q^\sharp) \int_{q^\sharp}^1 \ln s\, ds = \varphi_3(q^\sharp) \cdot (1 - q^\sharp)^2, \tag{6.38}$$

and

$$\frac{dq^\sharp}{dy} = \frac{c\alpha}{\sin\Theta} \left(-\int_{q^\sharp}^1 \varphi_1(s)\, \ln s\, ds \right)^{\frac{\alpha-1}{\alpha}} = \varphi_4(q^\sharp) \cdot \left(-\int_{q^\sharp}^1 \ln s\, ds \right)^{\frac{\alpha-1}{\alpha}}$$
$$= \varphi_5(q^\sharp) \cdot (1 - q^\sharp)^{\frac{2(\alpha-1)}{\alpha}}. \tag{6.39}$$

Here the φ_k are uniformly positive, bounded functions. Integrating (6.39) we obtain

$$\int_{q^\sharp}^1 \frac{1}{\varphi_5(s)} (1 - s)^{\frac{2(1-\alpha)}{\alpha}}\, ds = h - y. \tag{6.40}$$

To fix the ideas, assume

$$0 < c_3 \leq \varphi_5(s) \leq C_3.$$

Then

$$\frac{1}{c_3}\int_{q^\sharp}^{1}(1-s)^{\frac{2(1-\alpha)}{\alpha}}\,ds \;=\; \frac{\alpha}{(2-\alpha)c_3}(1-q^\sharp)^{\frac{2-\alpha}{\alpha}}\,ds \;\geq\; h-y.$$

$$1-q^\sharp(y) \;\geq\; \left(\frac{(2-\alpha)c_3}{\alpha}\right)^{\frac{\alpha}{2-\alpha}}(h-y)^{\frac{\alpha}{2-\alpha}}. \tag{6.41}$$

A similar argument yields

$$1-q^\sharp(y) \;\leq\; \left(\frac{(2-\alpha)C_3}{\alpha}\right)^{\frac{\alpha}{2-\alpha}}(h-y)^{\frac{\alpha}{2-\alpha}}. \tag{6.42}$$

Using (6.1) and (6.42) in the equation (6.18) we obtain a bound of the form

$$-p'(y) \;\leq\; C_1(1-q(y)) \;\leq\; C_2(h-y)^{\frac{\alpha}{2-\alpha}} \tag{6.43}$$

for y in a left neighborhood of h, which yields

$$p(y) \;\leq\; \frac{C_2}{\alpha+1}(h-y)^{\frac{2}{2-\alpha}}. \tag{6.44}$$

Since $\alpha<1$, using (6.41) and (6.44) in (6.9) we obtain the limit $\Theta(y)\to\theta_0$ as $y\to h-$.

On the other hand, when y is bounded away from h, the denominator in (6.10) is strictly positive and the quantity $w=w(I,p,q)$ remains uniformly bounded. By (6.9), we obtain the upper bound $\Theta\leq\theta^+$, for some $\theta^+<\pi/2$.

8. Relying on (6.36), in this step we prove that all terms on the right hand sides of the ODEs in (6.34) are uniformly integrable.

(i) We first consider the terms appearing in the ODE for $P(y)$. Concerning $f_{1,p}$, as $y\to h-$ one has

$$f_{1,p} \;=\; \mathcal{O}(1)\cdot\left(1-\frac{q}{I}\right)^{-1} \;=\; \mathcal{O}(1)\cdot(h-y)^{\frac{-\alpha}{2-\alpha}}, \tag{6.45}$$

because of (6.41). Since $\alpha<1$, this implies that $f_{1,p}$ is an integrable function of y.

(ii) By the second equation in (6.36), as $y\to h-$ one has

$$f_{1,q} \;=\; \mathcal{O}(1)\cdot\frac{(1-q^\sharp)\ln(q^\sharp)}{1-q^\sharp+q^\sharp\ln(q^\sharp)} \;=\; \mathcal{O}(1). \tag{6.46}$$

(iii) The term f_2 blows up as $y\to h-$, due to the factor $z^{\alpha-1}$. However, this factor is integrable in y because, by (6.38), (6.41) and (6.42)

$$z^\alpha\big(I(y),p(y),q(y)\big) \;=\; \mathcal{O}(1)\cdot(h-y)^{\frac{2\alpha}{2-\alpha}}. \tag{6.47}$$

This implies

$$f_2\big(I(y), p(y), q(y)\big) = \mathcal{O}(1)\cdot z^{\alpha-1}\big(I(y), p(y), q(y)\big)$$
$$= \mathcal{O}(1)\cdot (h-y)^{-1+\frac{\alpha}{2-\alpha}}, \tag{6.48}$$

showing that f_2 is integrable, because $\alpha > 0$.

(iv) We now solve the linear ODE for P in (6.34) with terminal condition $P(h) = 0$. By the estimates (6.45)-(6.46) and (6.48) one obtains a bound of the form

$$P(y) \;=\; \mathcal{O}(1)\cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.49}$$

valid in a left neighborhood of $y = h$.

(v) In a neighborhood of the origin, the function $f_{1,q}$ contains a logarithm which blows up as $y \to 0+$. However, this is integrable because, for $y \approx 0$, we have

$$\frac{q(y)}{I(y)} \;\approx\; \left(\frac{d}{dy}\frac{q(y)}{I(y)}\right)\bigg|_{y=0}\cdot y \;=\; \frac{c\alpha}{(z(0))^{1-\alpha} I(0)\sin(\Theta(0))}\, y,$$

and $\ln y$ is integrable in y. Recalling (6.1), as y ranges in a right neighborhood of the origin, i.e. for $y > 0$, we conclude

$$\begin{cases} I'(y)\cdot f_{1,q} f_2 \;=\; \mathcal{O}(1)\cdot I'(y) f_{1,q} \;=\; \mathcal{O}(1)\cdot y^{-\beta}\ln y, \\[2mm] I'(y)\cdot f_{1,p} \;=\; \mathcal{O}(1)\cdot I'(y) \;=\; \mathcal{O}(1)\cdot y^{-\beta}. \end{cases} \tag{6.50}$$

This shows that, in (6.34), the coefficients in first equation are uniformly integrable in a right neighborhood of the origin.

(vi) It remains to consider the terms appearing in the ODE for $\widetilde{Q}(y)$. We first observe that

$$\frac{f_{2,p}}{f_2} \;=\; -\frac{\sin\Theta}{c\alpha}\left[1 + \frac{\alpha}{\sin^2\Theta} - 2\alpha\right] z^{-\alpha}\big(I(y), p(y), q(y)\big).$$

As $y \to h-$, by (6.47) and (6.49) this implies

$$\frac{f_{2,p}}{f_2}\cdot P \;=\; \mathcal{O}(1)\cdot (h-y)^{\frac{-2\alpha}{2-\alpha}}\cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.51}$$

which is integrable for $\alpha < 1$.

(vii) Finally, as $y \to h-$, we consider

$$\begin{aligned} \frac{f_{2,I}}{f_2} \;&=\; -\frac{\sin\Theta}{c\alpha}\left[\frac{(1-\alpha)\sin\theta_0}{\sin^2\Theta} + \frac{\sin(\Theta-\theta_0)}{\cos\Theta}\left(1 + \frac{\alpha}{\sin^2\Theta} - 2\alpha\right)\right] \\ &\quad\times \frac{1 - \frac{q}{I(y)}}{z^{\alpha}\big(I(y), p(y), q(y)\big)} \\ &= \mathcal{O}(1)\cdot (1-q^{\sharp}) z^{-\alpha}\big(I(y), p(y), q(y)\big) \;=\; \mathcal{O}(1)\cdot (h-y)^{\frac{\alpha}{2-\alpha}}\cdot (h-y)^{\frac{-2\alpha}{2-\alpha}}, \end{aligned} \tag{6.52}$$

which is integrable in y since $\alpha < 1$. Similarly, by (6.51), (6.18), and (6.42), it follows

$$\frac{f_{2,p}}{f_2} \cdot f_1 \;=\; \mathcal{O}(1) \cdot (h-y)^{\frac{-2\alpha}{2-\alpha}} \cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.53}$$

which is again integrable.

9. The proof can now be accomplished by a contradiction argument. If the conclusion of the theorem were not true, one could find a sequence of absolutely continuous, non-decreasing functions $I_n : \mathbb{R}_+ \mapsto [0,1]$, all satisfying (6.1), with $I_n(0) \to 1$, and such that, for each $n \geq 1$, the optimization problem **(OP2)** has two distinct solutions, say $(\check{\theta}_n, \check{u}_n)$ and $(\hat{\theta}_n, \hat{u}_n)$. As a consequence, for each $n \geq 1$ the system (6.13) has two solutions. To fix the ideas, let the first solution be defined on $[0, \check{h}_n]$ and the second on $[0, \hat{h}_n]$, with $\check{h}_n < \hat{h}_n$. These two solutions will be denoted by $(\check{p}_n, \check{q}_n, \check{z}_n)$ and $(\hat{p}_n, \hat{q}_n, \hat{z}_n)$. They both satisfy the boundary conditions

$$\check{p}_n(\check{h}_n) = \hat{p}_n(\hat{h}_n) = 0, \qquad \check{q}_n(\check{h}_n) = I(\check{h}_n), \qquad \hat{q}_n(\hat{h}_n) = I(\hat{h}_n), \qquad \check{q}_n(0) = \hat{q}_n(0) = 0. \tag{6.54}$$

As a preliminary, we observe that, for $\delta > 0$ small, the heights $\hat{h}$, $\check{h}$ of optimal stems must remain uniformly positive. Indeed, by (2.3) the sunlight gathered by a stem γ of length ℓ is bounded by

$$\mathcal{S}(\gamma) \;\leq\; \ell.$$

Hence, for a sequence of stems γ_n with heights $\hat{h}_n \to 0$, the total sunlight satisfies

$$\mathcal{S}(\gamma_n) \;\leq\; \ell_n \;\leq\; \frac{\hat{h}_n}{\sin\theta_0} \;\to\; 0.$$

Therefore, for n large, none of these stems can be optimal.

Thanks to the last identity in (6.54), by the mean value theorem there exists some intermediate point $k_n \in [\check{h}_n, \hat{h}_n]$ such that, with the notation introduced at (6.21),

$$\frac{\partial q_n}{\partial h}(0, k_n) \;=\; 0. \tag{6.55}$$

For each $n \geq 1$ consider the corresponding system

$$\begin{cases} P_n'(y) \;=\; -I_n'(y)\big[f_{1,p} P_n + f_{1,q} f_2 \widetilde{Q}_n\big], \\[2mm] \widetilde{Q}'(y) \;=\; \dfrac{f_{2,p}}{f_2} P_n - \dfrac{I_n'(y)[f_{2,I} - f_{2,p} f_1]}{f_2} \widetilde{Q}_n, \end{cases} \qquad \begin{cases} P_n(k_n) \;=\; 0, \\[2mm] \widetilde{Q}_n(k_n) \;=\; -1. \end{cases} \tag{6.56}$$

Since $f_2\big(I_n(0), p_n(0,k_n), 0\big) > 0$, by (6.55) it follows

$$\widetilde{Q}_n(0) \;=\; \frac{1}{f_2\big(I_n(0), p_n(0,k_n), 0\big)} \cdot \frac{\partial q_n}{\partial h}(0, k_n) \;=\; 0. \tag{6.57}$$

Let

$$P_n(y) \doteq \frac{\partial p(y,k_n)}{\partial h}, \qquad \widetilde{Q}_n(y) \doteq \frac{1}{f_2\big(I_n(y), p_n(y,k_n), q_n(y,k_n)\big)} \cdot \frac{\partial q(y,k_n)}{\partial h},$$

be the solutions to (6.56). By the previous steps, their derivatives $\big(P_n', \widetilde{Q}_n'\big)_{n\geq 1}$ form a sequence of uniformly integrable functions defined on the intervals $[0, k_n]$. Note that the existence of an upper bound $\sup_n k_n \doteq h^+ < +\infty$ follows from the existence proof.

Thanks to the uniform integrability, by possibly taking a subsequence, we can assume the convergence $k_n \to \overline{h} \in [0, h^+]$, the weak convergence of derivatives $P_n' \rightharpoonup P'$, $\widetilde{Q}_n' \rightharpoonup \widetilde{Q}'$ in $\mathbf{L}^1$, and the convergence

$$P_n \to P, \qquad \widetilde{Q}_n \to \widetilde{Q},$$

uniformly on every subinterval $[0, h]$ with $h < \bar{h}$.

Recalling that every I_n' satisfies the uniform bounds (6.1), since $I_n(y) \to I(y) \equiv 1$ uniformly for all $y \geq 0$, we conclude that $(P, \widetilde{Q})$ provides a solution to the linear system (6.34) on $[0, \bar{h}]$, corresponding to the constant function $I(y) \equiv 1$. We now observe that, when $I(y) \equiv 1$, the solution to (6.34) is $P(y) \equiv 0$ and $\widetilde{Q}(y) \equiv -1$. On the other hand, our construction yields

$$\widetilde{Q}(0) = \lim_{n\to\infty} \widetilde{Q}_n(0) = 0.$$

This contradiction achieves the proof of Theorem 6.1. □

7. Existence of an equilibrium solution

Given a nondecreasing light intensity function $I : \mathbb{R}_+ \mapsto [0, 1]$, in the previous section we proved the existence of an optimal solution (θ^*, u^*) for the maximization problem **(OP2)**.

Conversely, let $\rho_0 > 0$ be the constant density of stems, i.e. the number of stems growing per unit area. If all stems have the same configuration, described by the couple of functions $y \mapsto (\theta(y), u(y))$ as in (2.18), then the corresponding intensity of light at height y above ground is computed as

$$I^{(\theta,u)}(y) \doteq \exp\left\{-\frac{\rho_0}{\cos\theta_0}\int_y^{+\infty} \frac{u(\zeta)}{\sin\theta(\zeta)}\, d\zeta\right\}. \tag{7.1}$$

The main goal of this section is to find a competitive equilibrium, i.e. a fixed point of the composition of the two maps $I \mapsto (\theta^*, u^*)$ and $(\theta, u) \mapsto I^{(\theta,u)}$.

Definition 7.1. Given an angle $\theta_0 \in]0, \pi/2[$ and a constant $\rho_0 > 0$, we say that the light intensity function $I^* : \mathbb{R}_+ \mapsto [0, 1]$ and the stem configuration $(\theta^*, u^*) : \mathbb{R}_+ \mapsto [\theta_0, \pi/2] \times \mathbb{R}_+$ yield a **competitive equilibrium** if the following holds.

(i) The couple (θ^*, u^*) provides an optimal solution to the optimization problem **(OP2)**, with light intensity function $I = I^*$.

(ii) The identity $I^* = I^{(\theta^*,u^*)}$ holds.

The main result of this section provides the existence of a competitive equilibrium, assuming that the density ρ_0 of stems is sufficiently small.

Theorem 7.2. *Let an angle $\theta_0 \in]0, \pi/2[$ be given. Then, for all $\rho_0 > 0$ sufficiently small, a unique competitive equilibrium (I^*, θ^*, u^*) exists.*

Proof. 1. Setting $C = 1$ and $\beta = 1/2$ in (6.1), we define the family of functions

$$\mathcal{F} \doteq \Big\{ I : \mathbb{R}_+ \mapsto [1-\delta,\, 1]; \quad I \text{ is absolutely continuous,} \\ I'(y) \in \big[0,\, y^{-1/2}\big] \quad \text{for a.e. } y > 0 \Big\}, \tag{7.2}$$

where $\delta > 0$ is chosen small enough so that the conclusion of Theorem 6.1 holds.

2. For each $I \in \mathcal{F}$, let $(\theta^{(I)}, u^{(I)})$ describe the corresponding optimal stem. Calling

$$h^{(I)} = \sup\, \big\{ y \geq 0; \;\; u^{(I)}(y) > 0 \big\}$$

the height of this stem, by the a priori bounds proved in Section 6 we have a uniform bound

$$h^{(I)} \leq h^+$$

for all $I \in \mathcal{F}$. Let $p^{(I)}, q^{(I)} : [0, h^{(I)}] \mapsto \mathbb{R}_+$ be the corresponding solutions of (6.20). For convenience, we extend all these functions to the larger interval $[0, h^+]$ by setting

$$p^{(I)}(y) \doteq p^{(I)}\big(h^{(I)}\big), \qquad q^{(I)}(y) \doteq q^{(I)}\big(h^{(I)}\big), \qquad \text{for all } y \in [h^{(I)}, h^+].$$

3. By the analysis in Section 6, for any $I \in \mathcal{F}$, the solution to the system of optimality conditions (6.13) satisfies

$$\theta_0 \leq \Theta(I(y), p(y), q(y)) \leq \theta^+, \qquad c_0\, y \leq \frac{q(y)}{I(y)} \leq 1, \tag{7.3}$$

for some $\theta^+ < \pi/2$ and $c_0 > 0$ sufficiently small. In view of (6.8), this implies

$$U(I(y), p(y), q(y)) \doteq -\ln\left(\frac{q(I)}{I(y)}\right) \cos\big(\Theta(I(y), p(y), q(y)) - \theta_0\big) \leq -\ln(c_0 y). \tag{7.4}$$

Note that $\Theta(I(y), p^{(I)}(y), q^{(I)}(y)) = \theta^{(I)}(y)$ and $U(I(y), p^{(I)}(y), q^{(I)}(y)) = u^{(I)}(y)$. Thus, if we choose $\rho_0 > 0$ small enough, it follows that the corresponding light intensity function $I^{(\theta,u)}$ at (7.1) is again in $\mathcal{F}$. A competitive equilibrium will be obtained by constructing a fixed point of the composition of the two maps

$$\Lambda_1 : I \mapsto \big(\theta^{(I)},\, u^{(I)}\big), \qquad \Lambda_2 : (\theta, u) \mapsto I^{(\theta,u)}. \tag{7.5}$$

In order to use Schauder's theorem, we need to check the continuity of these maps, in a suitable topology.

We start by observing that $\mathcal{F} \subset \mathcal{C}^0([0,h^+])$ is a compact, convex set. Again by the analysis in Section 6, as I varies within the domain $\mathcal{F}$, the corresponding functions $\theta^{(I)}$ are uniformly bounded in $\mathbf{L}^\infty([0,h^+])$, while $u^{(I)}$ is uniformly bounded in $\mathbf{L}^1([0,h^+])$.

From the estimate (6.43) it follows that the functions $p^{(I)}$ are equicontinuous on $[0,h^+]$. Recalling that $q = q^\sharp \cdot I$, by (6.39) we conclude that the functions $q^{(I)}$ are equicontinuous as well.

4. Motivated by (7.3)-(7.4), we consider the set of functions

$$\mathcal{U} \doteq \Big\{(\theta,u) \in \mathbf{L}^1\big([0,h^+];\ \mathbb{R}^2\big), \quad \theta(y) \in [\theta_0,\theta^+],\ \ 0 \le u(y) \le -\ln(c_0 y)\Big\}. \tag{7.6}$$

Thanks to the uniform bounds imposed on θ and u in the definition (7.6), the continuity of the map $\Lambda_2 : \mathcal{U} \mapsto \mathcal{C}^0$, defined at (7.1) is now straightforward.

5. To prove the continuity of the map Λ_1, consider a sequence of functions $I_n \in \mathcal{F}$, with $I_n \to I$ uniformly on $[0,h^+]$. Let $(\theta_n, u_n) : [0,h^+] \mapsto \mathbb{R}^2$ be the corresponding unique optimal solutions.

We claim that $(\theta_n, u_n) \to (\theta, u)$ in $\mathbf{L}^1([0,h^+])$, where (θ,u) is the unique optimal solution, given the light intensity I.

To prove the claim, let (p_n, q_n) be the corresponding solutions of the system (6.20). By the estimates on p', q' proved in Section 6, the functions (p_n, q_n) are equicontinuous. From any subsequence we can thus extract a further subsequence and obtain the convergence

$$p_{n_j} \to \widehat{p}, \qquad q_{n_j} \to \widehat{q}, \qquad I_{n_j} \to I, \tag{7.7}$$

for some functions $\widehat{p}, \widehat{q}$, uniformly on $[0,h^+]$.

For every $j \ge 1$ we now have

$$\theta_{n_j}(y) \;=\; \Theta\big(I_{n_j}(y), p_{n_j}(y), q_{n_j}(y)\big), \qquad\qquad u_{n_j}(y) \;=\; U\big(I_{n_j}(y), p_{n_j}(y), q_{n_j}(y)\big),$$

where U and Θ are the functions in (6.8)-(6.9). By the dominated convergence theorem, the convergence (7.7) together with the uniform integrability of θ_{n_j} and u_{n_j} yields the $\mathbf{L}^1$ convergence

$$\|\theta_{n_j} - \widehat{\theta}\|_{\mathbf{L}^1} \;\to\; 0, \qquad\qquad \|u_{n_j} - \widehat{u}\|_{\mathbf{L}^1} \;\to\; 0. \tag{7.8}$$

In turn this implies that $(\widehat{p}, \widehat{q})$ provide a solution to the problem (6.20), in connection with the light intensity I. By uniqueness, $\widehat{p} = p$ and $\widehat{q} = q$. Therefore, $\widehat{\theta} = \theta$ and $\widehat{u} = u$ as well.

The above argument shows that, from any subsequence, one can extract a further subsequence so that the $\mathbf{L}^1$-convergence (7.8) holds. Therefore, the entire sequence $(\theta_n, u_n)_{n\ge 1}$ converges to (θ, u) in $\mathbf{L}^1([0,h^+])$. This establishes the continuity of the map Λ_1.

6. The map $\Lambda_2 \circ \Lambda_1$ is now a continuous map of the compact, convex domain $\mathcal{F} \subset \mathcal{C}^0([0,h^+])$ into itself. By Schauder's theorem it admits a fixed point $I^*(\cdot)$. By construction, the optimal stem configuration $\big(\theta^{(I^*)}, u^{(I^*)}\big)$ yields a competitive equilibrium, in the sense of Definition 7.1.

7. To prove uniqueness, we derive a set of necessary conditions satisfied by the equilibrium solution, and show that this system has a unique solution.

Using (6.8) and (6.11), we can rewrite the light intensity function (7.1) as

$$I(y) \;=\; \exp\left\{ \frac{\rho_0}{\cos\theta_0} \int_y^\infty \ln\left(\frac{q}{I}\right) \frac{1 + w\sin\theta_0}{\sin\theta_0 + w}\, d\zeta \right\},$$

where $w = w(I, p, q)$ is the function introduced at (6.10). Differentiating w.r.t. y one obtains

$$I'(y) \;=\; -\frac{\rho_0}{\cos\theta_0} \ln\left(\frac{q}{I}\right) \frac{1 + w\sin\theta_0}{\sin\theta_0 + w} \cdot I \;\doteq\; f_3(I, p, q). \tag{7.9}$$

Combining (7.9) with (6.20), we conclude that the competitive equilibrium satisfies the system of equations and boundary conditions

$$\begin{cases} p'(y) = -f_1\big(I(y), p(y), q(y)\big) \cdot f_3(I(y), p(y), q(y)), \\ q'(y) \;=\; f_2\big(I(y), p(y), q(y)\big), \\ I'(y) \;=\; f_3(I(y), p(y), q(y)), \end{cases} \qquad \begin{cases} p(h) \;=\; 0, \\ q(h) \;=\; 1, \\ I(h) \;=\; 1, \end{cases} \tag{7.10}$$

together with

$$q(0) \;=\; 0. \tag{7.11}$$

Here the common height of the stems $h > 0$ is a constant to be determined.

8. The uniqueness of solutions to (7.10) will be achieved by a contradiction argument. Since this is very similar to the one used in the proof of Theorem 6.1, we only sketch the main steps.

In analogy with (6.31), (6.33), denote by $p(y, h), q(y, h), I(y, h)$ the unique solution to the Cauchy problem (7.10), with terminal conditions given at $y = h$. Consider the functions

$$P(y) \;\doteq\; \frac{\partial p(y, h)}{\partial h}, \qquad \widetilde{Q}(y) \;\doteq\; \frac{1}{f_2(I, p, q)} \frac{\partial q(y, h)}{\partial h}, \qquad J(y) \;\doteq\; \frac{\partial I(y, h)}{\partial h}.$$

By (7.10), these functions satisfy

$$\begin{cases} P'(y) \;=\; -\big[f_{3,I} f_1 + f_3 f_{1,I}\big] J - \big[f_{3,p} f_1 + f_3 f_{1,p}\big] P - \big[f_{3,q} f_1 + f_3 f_{1,q}\big] f_2 \widetilde{Q}, \\ \widetilde{Q}'(y) \;=\; \frac{f_{2,I}}{f_2} J + \frac{f_{2,p}}{f_2} P - \frac{f_3}{f_2}\big[f_{2,I} - f_{2,p} f_1\big] \widetilde{Q}, \\ J'(y) \;=\; f_{3,I} J + f_{3,p} P + f_{3,q} f_2 \widetilde{Q}, \end{cases} \tag{7.12}$$

with boundary conditions

$$P(h) = 0, \qquad \widetilde{Q}(h) = -1, \qquad J(h) = 0.$$

Set $d_0 = \frac{\rho_0}{\cos\theta_0}$. Several of the partial derivatives on the right-hand side of (7.12) were computed in (6.36). The remaining ones are

$$f_{1,I}(I,p,q) = \frac{q}{I^2}\cdot\frac{1+w\sin\theta_0}{\sin\theta_0+w} - \frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\left[1-\frac{q}{I}\right],$$

$$f_{3,I}(I,p,q) = -d_0\left[\left(\ln\left(\frac{q}{I}\right)-1\right)\frac{1+w\sin\theta_0}{\sin\theta_0+w} - I\ln\left(\frac{q}{I}\right)\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\left(1-\frac{q}{I}\right)\right],$$

$$f_{3,p}(I,p,q) = d_0 I\ln\left(\frac{q}{I}\right)\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w}{p},$$

$$f_{3,q}(I,p,q) = -d_0 I\left[\frac{1}{q}\cdot\frac{1+w\sin\theta_0}{\sin\theta_0+w} + \left[\ln\left(\frac{q}{I}\right)\right]^2\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\right].$$

By the same arguments used in step **8** of the proof of Theorem 6.1, we conclude that the right-hand side of (7.12) is uniformly integrable.

9. Let a density $\rho_0 > 0$ be given. Assume that the problem (7.10)-(7.11) has two distinct solutions $(\hat p, \hat q, \hat I)$ and $(\check p, \check q, \check I)$, defined on $[0, \hat h]$ and $[0, \check h]$ say with $\hat h < \check h$. Since $\hat q(0) = \check q(0) = 0$, by the mean value theorem there exists $k \in [\hat h, \check h]$ such that $\frac{\partial q}{\partial h}(0, k) = 0$.

Next, if multiple solutions exist for arbitrarily small values of the density ρ_0, we can find a decreasing sequence $\rho_{0,n} \downarrow 0$ and corresponding solutions P_n, Q_n, I_n of (7.12), defined for $y \in [0, k_n]$, such that

$$P_n(k_n) = 0, \qquad \widetilde Q_n(k_n) = -1, \qquad J_n(k_n) = 0, \qquad \widetilde Q_n(0) = 0. \tag{7.13}$$

Thanks to the uniform integrability of the right hand sides of (7.12), by possibly extracting a subsequence we can achieve the convergence $k_n \to \bar h \in [0, h^+]$, the weak convergence $P_n' \rightharpoonup P'$, $\widetilde Q_n' \rightharpoonup \widetilde Q'$, $J_n' \rightharpoonup J'$ in $\mathbf{L}^1$, and the strong convergence

$$P_n \to P, \qquad \widetilde Q_n \to \widetilde Q, \qquad J_n \to J,$$

uniformly on every subinterval $[0, h]$ with $h < \bar h$.

To reach a contradiction, we observe that

$$J_n(y) \;=\; -\int_y^{k_n} J_n'(z)\,dz$$

and the right-hand side of J_n' in (7.12) consists of uniformly integrable terms which are multiplied by $\rho_{0,n}$. This implies $J(y) \equiv 0$. This corresponds to the case of an intensity function $I(y) \equiv 1$. But in this case we know that $\widetilde Q(y) \equiv -1$, contradicting the fact that, by (7.13),

$$\widetilde Q(0) \;=\; \lim_{n\to\infty} \widetilde Q_n(0) \;=\; 0. \quad \square$$

8. Stem competition on a domain with boundary

We consider here the same model introduced in Section 2, where all stems have fixed length ℓ and constant thickness κ. But we now allow the sunlight intensity $I = I(x, y)$ to vary w.r.t. both variables x, y. As shown in Fig. 5, left, we denote by

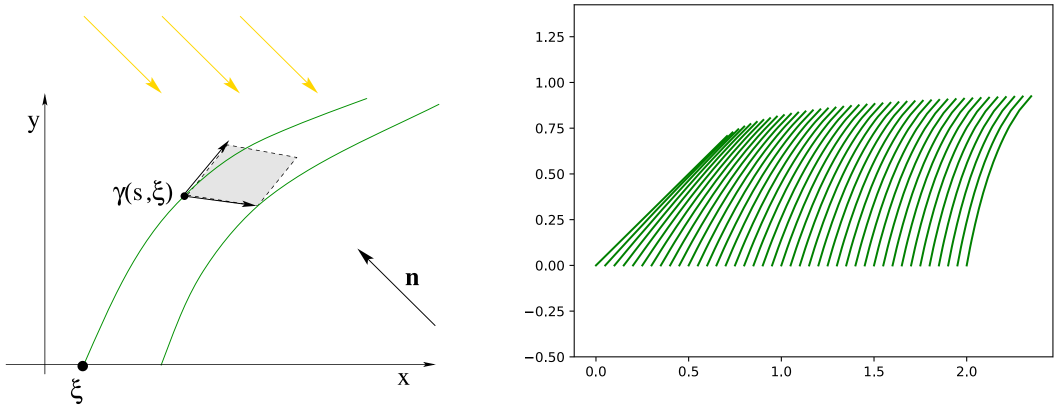

Fig. 5. Left: to leading order, the amount of vegetation in the shaded region is proportional to $\kappa\,\bar\rho(\xi)d\xi ds$. Since the area is computed in terms of the cross product $\frac{\partial\gamma}{\partial\xi}\times\frac{\partial\gamma}{\partial s}$, this motivates the formula (8.4). Right: a possible competitive equilibrium, where the light rays come from the direction $\mathbf{n}=(\frac{-1}{\sqrt2},\frac{1}{\sqrt2})$ and stems are distributed along the positive half line, with density as in (8.9). In this case, stems originating from points close to the origin have no incentive to grow upward, because they already receive a nearly maximum light intensity. Hence they bend to the right, almost perpendicularly to the light rays.

$$s\ \mapsto\ \gamma(s,\xi)\ =\ (x(s),y(s)),\qquad s\in[0,\ell],\tag{8.1}$$

the arc-length parameterization of the stem whose root is located at $(\xi,0)$, and write g for the function introduced at (2.8). This leads to the optimization problem

(OP3) *Given a light intensity function $I=I(x,y)$, find a control $s\mapsto\theta(s)\in[0,\pi]$ which maximizes the integral*

$$\int_0^\ell I(x(s),y(s))\,g(\theta(s))\,ds\tag{8.2}$$

subject to

$$\frac{d}{ds}(x(s),y(s))\ =\ (\cos\theta(s),\sin\theta(s)),\qquad (x(0),y(0))\ =\ (\xi,0).\tag{8.3}$$

Next, consider a function $\bar\rho(\xi)\geq 0$ describing the density of stems which grow near $\xi\in\mathbb{R}$. At any point in space reached by a stem, i.e. such that

$$(x,y)=\gamma(s,\xi)\qquad\text{for some }\ \xi\in\mathbb{R},\quad s\in[0,\ell],$$

the density of vegetation is

$$\rho(x,y)\ =\ \rho(\gamma(s,\xi))\ =\ \kappa\,\bar\rho(\xi)\cdot\left[\frac{\partial\gamma}{\partial\xi}\times\frac{\partial\gamma}{\partial s}\right]^{-1}.\tag{8.4}$$

The light intensity at a point $P = (x, y) \in \mathbb{R}^2$ is now given by

$$I(P) \;=\; \exp\left\{-\int_0^{+\infty} \rho(P + t\mathbf{n})\, dt\right\}. \tag{8.5}$$

Definition 8.1. Given the constants ℓ, κ and the density $\bar\rho \in \mathbf{L}^\infty(\mathbb{R})$, we say that the maps $\gamma : [0, \ell] \times \mathbb{R}$ and $I : \mathbb{R} \times \mathbb{R}_+ \mapsto [0, 1]$ yield a **competitive equilibrium** if the following holds:

(i) For each $\xi \in \mathbb{R}$, the stem $\gamma(\cdot, \xi)$ provides an optimal solution to **(OP3)**.
(ii) The function $I(\cdot)$ coincides with the light intensity determined by (8.4)-(8.5).

We shall not analyze the existence or uniqueness of the competitive equilibrium, in the case where the distribution of stem roots is not uniform. We only observe that, if the stem $\gamma(\cdot, \xi)$ in (8.1) is optimal, the necessary conditions yield the existence of a dual vector $s \mapsto \mathbf{p}(s)$ satisfying

$$\dot{\mathbf{p}}(s) \;=\; -\nabla I\big(x(s), y(s)\big)\, g(\theta(s)), \qquad\qquad \mathbf{p}(\ell) \;=\; (0, 0), \tag{8.6}$$

and such that, for a.e. $s \in [0, \ell]$, the optimal angle $\theta^*(s)$ satisfies

$$\theta^*(s) \;=\; \underset{\theta}{\operatorname{argmax}} \Big\{\mathbf{p}(s) \cdot (\cos\theta, \sin\theta) + I(x(s), y(s))g(\theta)\Big\}. \tag{8.7}$$

Differentiating the expression on the right hand side of (8.7) one obtains an implicit equation for $\theta^*(s)$, namely

$$I\big(x(s), y(s)\big)) g'(\theta^*(s)) + \mathbf{p}(s) \cdot \mathbf{n}(s) \;=\; 0 \tag{8.8}$$

for a.e. $s \in [0, \ell]$. Here $\mathbf{n}(s) \doteq \big(-\sin\theta(s), \cos\theta(s)\big)$ is the unit vector perpendicular to the stem. Moreover, by (8.6) one has

$$\mathbf{p}(s) \;=\; \int_s^\ell \nabla I\big(x(\sigma), y(\sigma))\, g(\theta^*(\sigma)\big)\, d\sigma.$$

An interesting case is where stems grow only on the half line $\{\xi \geq 0\}$. For example, one can take

$$\bar\rho(\xi) \;=\; \begin{cases} 0 & \text{if } \xi < 0, \\ b^{-1}\xi & \text{if } \xi \in [0, b], \\ 1 & \text{if } \xi > b. \end{cases} \tag{8.9}$$

In this case, we conjecture that the competitive equilibrium has the form illustrated in Fig. 5, right.

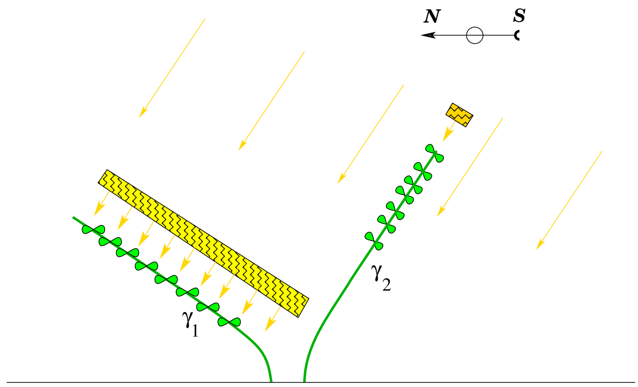

Fig. 6. The stem γ_1, oriented perpendicularly to the sun rays, collects much more sunlight than γ_2. Indeed, γ_1 would give the best orientation for solar panels. Notice that γ_2 minimizes the sunlight gathered because the upper leaves put the lower ones in shade.

9. Concluding remarks

A motivation for the present study was to understand whether competition for sunlight could explain phototropism, i.e. the tendency of plant stems to bend toward the light source. A naive approach may suggest that, if a stem bends in the direction of the light rays, the leaves will be closer to the sun and hence gather more light. However, since the average distance of the earth from the sun is approximately 90 million miles, getting a few inches closer cannot make a difference.

As shown in Fig. 6, if a single stem were present, to maximize the collected sunlight it should be perpendicular to the light rays, not parallel. In the presence of competition among several plant stems, our analysis shows that the best configuration is no longer perpendicular to light rays: the lower part of the stems should grow in a nearly vertical direction, while the upper part bends away from the sun.

Still, our competition models do not predict the tilting of stems in the direction of the sun rays. This may be due to the fact that these models are "static", i.e., they do not describe how plants grow in time. This leaves open the possibility of introducing further models that can explain phototropism in a time-dependent framework. As suggested in [12], the preemptive conquering of space, in the direction of the light rays, can be an advantageous strategy. We leave these issues for future investigation.

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References

[1] F. Ancona, A. Bressan, O. Glass, W. Shen, Feedback stabilization of stem growth, J. Dyn. Differ. Equ. 31 (2019) 1079–1106.

[2] M. Bernot, V. Caselles, J.M. Morel, Optimal Transportation Networks. Models and Theory, Springer Lecture Notes in Mathematics, vol. 1955, Berlin, 2009.

[3] A. Bressan, M. Palladino, Q. Sun, Variational problems for tree roots and branches, Calc. Var. Partial Differ. Equ. 59 (2020) 7.

[4] A. Bressan, B. Piccoli, Introduction to the Mathematical Theory of Control, AIMS Series in Applied Mathematics, AIMS, Springfield, MO, 2007.

[5] A. Bressan, Q. Sun, On the optimal shape of tree roots and branches, Math. Models Methods Appl. Sci. 28 (2018) 2763–2801.

[6] L. Cesari, Optimization - Theory and Applications, Springer-Verlag, 1983.

[7] W.H. Fleming, R.W. Rishel, Deterministic and Stochastic Optimal Control, Springer, 1975.

[8] E.N. Gilbert, Minimum cost communication networks, Bell Syst. Tech. J. 46 (1967) 2209–2227.

[9] O. Leyser, S. Day, Mechanisms in Plant Development, Blackwell Publishing, 2003.

[10] E. Lieb, M. Loss, Analysis, second edition, American Mathematical Society, Providence, 2001.

[11] F. Maddalena, J.M. Morel, S. Solimini, A variational model of irrigation patterns, Interfaces Free Bound. 5 (2003) 391–415.

[12] A. Runions, B. Lane, P. Prusinkiewicz, Modeling trees with a space colonization algorithm, in: Eurographics Workshop on Natural Phenomena, 2007.

[13] Q. Xia, Optimal paths related to transport problems, Commun. Contemp. Math. 5 (2003) 251–279.

[14] Q. Xia, Motivations, ideas and applications of ramified optimal transportation, ESAIM Math. Model. Numer. Anal. 49 (2015) 1791–1832.