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C^*-uniqueness Results for Groupoids

Are Austad and Eduard Ortega*

Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim 7034, Norway

**Correspondence to be sent to: e-mail: eduard.ortega@ntnu.no*

For a 2nd-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with a continuous 2-cocycle σ we find conditions that guarantee that $\ell^1(\mathcal{G}, \sigma)$ has a unique C^*-norm.

1 Introduction

Given a reduced (Banach) $*$-algebra $\mathcal{A}$, the enveloping C^*-algebra $C^*(\mathcal{A})$ plays a fundamental role in the representation theory of $\mathcal{A}$. However, any faithful $*$-representation of $\mathcal{A}$ will yield a C^*-completion of $\mathcal{A}$, and one may ask if this completion is isomorphic to the enveloping C^*-algebra. In the particular case of a locally compact group G, we may for example consider the $*$-algebras $C_c(G)$ or $L^1(G)$. There are then two canonical C^*-norms, namely the one arising from the left regular representation and the maximal C^*-norm. It is well known that G is an amenable group if and only if these two C^*-norms coincide. However, even for amenable groups we can not rule out that there are C^*-norms on $C_c(G)$ and $L^1(G)$ that are properly dominated by the norm induced by the left regular representation. Examples of this are given in [8, p. 230]. This invites the notion of *C^*-uniqueness*. A reduced $*$-algebra $\mathcal{A}$ is called C^*-unique if $C^*(\mathcal{A})$ is the unique C^*-completion of $\mathcal{A}$ up to isomorphism. This was extensively studied in [6] for $*$-algebras. Moreover, a more specialized study for convolution algebras of locally compact groups was conducted in [8], where C^*-uniqueness of $L^1(G)$ was studied by considering properties of the underlying group G. These two papers spawned

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investigations on C^*-uniqueness in the following decades, see for example [5, 9, 11, 13]. In later years, algebraic C^*-uniqueness of discrete groups has garnered some attention [1, 10, 16]. This is the study of C^*-uniqueness of the group ring $\mathbb{C}[\Gamma]$ for a discrete group Γ and is not equivalent to the study of C^*-uniqueness of $\ell^1(\Gamma)$, see Remark 2.8.

We will in this paper study the C^*-uniqueness of certain Banach $*$-algebras associated to groupoids. To be more precise, given a 2nd-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with a normalized continuous 2-cocycle σ, we will study the C^*-uniqueness of the I-norm completion of $C_c(\mathcal{G},\sigma)$, which will be denoted by $\ell^1(\mathcal{G},\sigma)$, see (3). Here, $C_c(\mathcal{G},\sigma)$ denotes the space $C_c(\mathcal{G})$ equipped with σ-twisted convolution and involution, see (1) and (2), and similarly for $\ell^1(\mathcal{G},\sigma)$. Associated to $\ell^1(\mathcal{G},\sigma)$ are two canonical C^*-norms, namely the one coming from the σ-twisted left regular representation, see (6), and the full C^*-norm. If these coincide, we say $\mathcal{G}$ twisted by σ has the weak containment property. The technicalities will be postponed to Section 2.3. Letting $\mathrm{Iso}(\mathcal{G})^\circ$ denote the interior of the isotropy subgroupoid of $\mathcal{G}$, we will first find that for $\ell^1(\mathcal{G},\sigma)$ to be C^*-unique, it is sufficient that $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is C^*-unique. If we further let $\mathrm{Iso}(\mathcal{G})^\circ_x$ denote the fiber of $\mathrm{Iso}(\mathcal{G})^\circ$ in the point $x \in \mathcal{G}^{(0)}$, and let σ_x denote the restriction of σ to this fiber, we have the following main result.

Theorem 1.1 (cf. Theorem 3.1). Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid with a continuous 2-cocycle σ. Suppose that $\mathcal{G}$ twisted by σ has the weak containment property. Then $\ell^1(\mathcal{G},\sigma)$ is C^*-unique if all the twisted convolution algebras $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x,\sigma_x)$, $x \in \mathcal{G}^{(0)}$, are C^*-unique.

The theorem allows us to deduce C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ by considering C^*-uniqueness of the (twisted) convolution algebras of the discrete groups $\mathrm{Iso}(\mathcal{G})^\circ_x$, $x \in \mathcal{G}^{(0)}$. The latter has been studied earlier, the untwisted case in [8] and the twisted case in [5]. Using this we obtain several examples of groupoids $\mathcal{G}$ for which $\ell^1(\mathcal{G},\sigma)$ is C^*-unique in Section 4. Additionally, we are able to deduce C^*-uniqueness of some wreath products using our groupoid approach, see Example 4.4.

We will proceed in the following manner. In Section 2, we will collect all results we will need regarding C^*-uniqueness of Banach $*$-algebras, C^*-algebra bundles, as well as cocycle-twisted convolution algebras associated to 2nd-countable locally compact Hausdorff étale groupoids. In Section 3, we first present our main theorem, Theorem 3.1. The remainder of the section will be dedicated to its proof. Lastly, in Section 4 we present examples of C^*-unique convolution algebras coming from groupoids, as well as deducing C^*-uniqueness of some wreath products.

2 Preliminaries

2.1 C^*-uniqueness for Banach $*$-algebras

A $*$-representation of a Banach $*$-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$, where $B(\mathcal{H})$ are the bounded linear operators on a Hilbert space $\mathcal{H}$. We say $\mathcal{A}$ is *reduced* if $\mathcal{A}_{\mathcal{R}} = \{a \in \mathcal{A} : \pi(a) = 0 \text{ for every} * -\text{representation} \pi \text{ of} \mathcal{A}\} = \{0\}$. All Banach $*$-algebras we consider in the sequel will be reduced. The *enveloping C^*-algebra* of a reduced Banach $*$-algebra $\mathcal{A}$ is the unique C^*-algebra $C^*(\mathcal{A})$ that admits the following universal property: there exists an injective $*$-homomorphism $\Phi : \mathcal{A} \to C^*(\mathcal{A})$ with dense range so that for every $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$, there exists a unique $*$-representation $\hat{\pi} : C^*(\mathcal{A}) \to B(\mathcal{H})$ so that $\pi = \hat{\pi} \circ \Phi$. In order to ease notation in the sequel we will identify $\mathcal{A}$ with the Banach $*$-subalgebra $\Phi(\mathcal{A})$ of $C^*(\mathcal{A})$ whenever it is natural to do so. The enveloping C^*-algebra of a Banach $*$-algebra always exists [15, Section 10.1].

Definition 2.1. Let $\mathcal{A}$ be a reduced Banach $*$-algebra. We say that $\mathcal{A}$ is C^*-*unique* if the C^*-norm given by

$$\|a\| := \sup\{\|\pi(a)\| : \pi : \mathcal{A} \to B(\mathcal{H}) \text{ is a *-representation}\}$$

for every $a \in \mathcal{A}$, is the unique C^*-norm on $\mathcal{A}$. In other words, $\mathcal{A}$ is C^*-unique if $C^*(\mathcal{A})$ is the unique C^*-completion of $\mathcal{A}$ up to isomorphism.

We will make repeated use of the following result on C^*-uniqueness of Banach $*$-algebras, see [15, Proposition 10.5.19].

Proposition 2.2. Let $\mathcal{A}$ be a reduced Banach $*$-algebra with enveloping C^*-algebra $C^*(\mathcal{A})$. Then $\mathcal{A}$ is C^*-unique if and only if for every nonzero two-sided closed ideal $I \lhd C^*(\mathcal{A})$ we have $\mathcal{A} \cap I \neq \{0\}$.

2.2 C^*-algebra bundles

The notion of a $C_0(X)$-algebra will be of importance in the proof of the main theorem. Hence, we briefly revise some basic notions and results on $C_0(X)$-algebras and C^*-bundles.

Definition 2.3. Let X be a locally compact Hausdorff space. A $C_0(X)$-*algebra* is a C^*-algebra A together with a non-degenerate injection $\iota : C_0(X) \to \mathcal{Z}(M(A))$, where the latter denotes the center of the multiplier algebra of A.

We shall also need to consider (upper semi-continuous) C^*-bundles.

Definition 2.4. Let X be a locally compact Hausdorff space and let $\{B_x\}_{x\in X}$ be a family of C^*-algebras. A map f defined on X such that $f(x) \in B_x$ for all $x \in X$, is called a *section*. An *upper semi-continuous C^*-bundle* $\mathbf{B}$ *over* X is a triple $(X, \{B_x\}_{x\in X}, \Gamma_0(\mathbf{B}))$, where $\Gamma_0(\mathbf{B})$ is a family of sections, such that the following conditions are satisfied:

1. $\Gamma_0(\mathbf{B})$ is a C^*-algebra under pointwise operations and supremum norm,
2. for each $x \in X$, $B_x = \{f(x) : f \in \Gamma_0(\mathbf{B})\}$,
3. for each $f \in \Gamma_0(\mathbf{B})$ and each $\varepsilon > 0$, $\{x \in X : \|f(x)\| \geq \varepsilon\}$ is compact,
4. $\Gamma_0(\mathbf{B})$ is closed under multiplication by $C_0(X)$, that is, for each $g \in C_0(X)$ and $f \in \Gamma_0(\mathbf{B})$, the section gf defined by $gf(x) = g(x)f(x)$ is in $\Gamma_0(\mathbf{B})$.

The two above concepts can be combined to obtain the main theorem of [14], which we present shortly for the reader's convenience. Suppose X is a locally compact Hausdorff space, and suppose A is a $C_0(X)$-algebra with map $\iota\colon C_0(X) \to \mathcal{Z}(M(A))$. For $x \in X$, denote by $J_x := C_0(X \setminus \{x\})$ and realize $J_x \subseteq C_0(X)$ in the natural way. Moreover, we define $I_x := \iota(J_x)A$, which is a closed two-sided ideal of A. We then have the following result that will play a major role in the proof of Theorem 3.1.

Proposition 2.5 ([14, Theorem 2.3]). Let X be a locally compact Hausdorff space and let A be a $C_0(X)$-algebra. Then there exists a unique upper semi-continuous C^*-bundle $\mathbf{B}$ over X such that

i) the fibers $B_x = A/I_x$, and
ii) there is an isomorphism $\phi\colon A \to \Gamma_0(\mathbf{B})$ satisfying $\phi(a)(x) = a + I_x$.

2.3 Groupoids, cocycle twists and associated algebras

Given a groupoid $\mathcal{G}$ we will denote by $\mathcal{G}^{(0)}$ its unit space and write $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ for the range and source maps, respectively. We will also denote by $\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} : s(\alpha) = r(\beta)\}$ the set of *composable elements*. In this paper, we will only consider groupoids $\mathcal{G}$ equipped with a 2nd-countable locally compact Hausdorff topology making all the structure maps continuous. A groupoid $\mathcal{G}$ is called *étale* if the range map, and hence also the source map, is a local homeomorphism. A subset B of an étale groupoid $\mathcal{G}$ is called a *bisection* if there is an open set $U \subseteq \mathcal{G}$ containing B such that $r\colon U \to r(U)$ and $s\colon U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Second-countable locally compact Hausdorff étale groupoids have countable bases consisting of open bisections.

Given $x \in \mathcal{G}^{(0)}$ we define by $\mathcal{G}_x := \{\gamma \in \mathcal{G} : s(\gamma) = x\}$ and $\mathcal{G}^x := \{\gamma \in \mathcal{G} : r(\gamma) = x\}$. Observe that if $\mathcal{G}$ is étale the sets $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete for every $x \in \mathcal{G}^{(0)}$. The *isotropy group of* x is given by $\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$, and the *isotropy subgroupoid of* $\mathcal{G}$ is the subgroupoid $\mathrm{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x$ with the relative topology from $\mathcal{G}$. Let $\mathrm{Iso}(\mathcal{G})^\circ$ denote the interior of $\mathrm{Iso}(\mathcal{G})$. We then say that $\mathcal{G}$ is *topologically principal* if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$.

We will consider groupoid twists where the twist is implemented by a continuous 2-cocycle. To be more precise, let $\mathcal{G}$ be a 2nd-countable locally compact étale groupoid. A *normalized continuous* 2*-cocycle* is then a continuous map $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying

$$\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$$

for all $\gamma \in \mathcal{G}$, and

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$$

whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$. The set of non-normalized continuous 2-cocycles on $\mathcal{G}$ will be denoted $Z^2(\mathcal{G}, \mathbb{T})$. Note that this is not the most general notion of a twist of a groupoid (see [17, Chapter 5]).

Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid. We will define the σ-twisted convolution algebra $C_c(\mathcal{G}, \sigma)$ as follows: As a set it is just

$$C_c(\mathcal{G}, \sigma) = \{f : \mathcal{G} \to \mathbb{C} : f \text{ is continuous with compact support}\},$$

but equipped with σ-twisted convolution product

$$(f *_\sigma g)(\gamma) = \sum_{\mu \in \mathcal{G}_{s(\gamma)}} f(\gamma\mu^{-1}) g(\mu) \sigma(\gamma\mu^{-1}, \mu), \quad f, g \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G}, \tag{1}$$

and σ-twisted involution

$$f^{*_\sigma}(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma) f(\gamma^{-1})}, \quad f \in C_c(\mathcal{G}, \sigma), \gamma \in \mathcal{G}. \tag{2}$$

We complete $C_c(\mathcal{G}, \sigma)$ in the "fiberwise 1-norm", also known as the I-norm, given by

$$\|f\|_I = \sup_{x \in \mathcal{G}^{(0)}} \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\} \tag{3}$$

for $f \in C_c(\mathcal{G}, \sigma)$. Denote by $\ell^1(\mathcal{G}, \sigma)$ the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the I-norm. This is a Banach $*$-algebra with the natural extensions of (1) and (2). For later use we record the following lemma.

Lemma 2.6. Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid. Then for any $f \in \ell^1(\mathcal{G})$, the map defined by

$$\mathcal{G}^{(0)} \ni x \mapsto \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\}, \tag{4}$$

is continuous.

Proof. By density it is enough to show this for $f \in C_c(\mathcal{G})$. It is well-known that $C_c(\mathcal{G}) = \text{span}\{g \in C_c(\mathcal{G}) : g \text{ is supported on a bisection}\}$. Hence, we may assume f is supported on a bisection U, that is, $\text{supp}(f) \subseteq U$. Furthermore, for f we denote the assignment of (4) by F. We thus wish to show that $F \in C(\mathcal{G}^{(0)})$.

To this end, fix $x \in \mathcal{G}^{(0)}$. As $f(x) = 0$ if $x \notin s(U)$, we assume $x \in s(U)$. Since $s(x) = x$ and $s \colon U \to s(U)$ is a homeomorphism, we therefore have $x \in U$. Moreover, let $(x_i)_i \subseteq \mathcal{G}^{(0)}$ be such that $x_i \to x$. Then eventually $x_i \in s(U)$ for all i large enough. For such i we have $F(x_i) = |f(\gamma_i)|$, where γ_i is the unique element of U with $s(\gamma_i) = x_i$. Now, as $s \colon U \to s(U)$ is a homeomorphism and $x_i \to x$, we have $\gamma_i \to \gamma \in U$, where γ is the unique element of U such that $s(\gamma) = x$. As $f \in C_c(\mathcal{G})$, it follows that $f(\gamma_i) \to f(\gamma)$, and hence $F(x_i) \to F(x)$. Hence, $F \in C(\mathcal{G}^{(0)})$, and the result follows. ∎

We wish to understand when $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique, that is, when it only permits one separating C^*-norm. To do this it will be of importance to use Proposition 2.2.

The *(full) twisted groupoid C^*-algebra* $C^*(\mathcal{G}, \sigma)$ is the completion of $C_c(\mathcal{G}, \sigma)$ in the norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm bounded *-representation}\}, \tag{5}$$

for $f \in C_c(\mathcal{G}, \sigma)$. It was observed in [4, Lemma 3.3.19] that if $\mathcal{G}$ is étale, then every $*$-representation of $C_c(\mathcal{G}, \sigma)$ is bounded by the I-norm. Then, since we are completing with respect to a supremum over $*$-representations, $C^*(\mathcal{G}, \sigma)$ is just the C^*-envelope of $\ell^1(\mathcal{G}, \sigma)$.

Now we will construct a faithful representation of $\ell^1(\mathcal{G}, \sigma)$ called the *σ-twisted left regular representation*. In particular, we have that $\ell^1(\mathcal{G}, \sigma)$ is reduced. The completion of the image of $\ell^1(\mathcal{G}, \sigma)$ under the σ-twisted left regular representation is called the

σ-twisted reduced groupoid C^*-algebra of $\mathcal{G}$ and will be denoted $C_r^*(\mathcal{G},\sigma)$. Let $x \in \mathcal{G}^{(0)}$. Then there is a representation $L^{\sigma,x}\colon C_c(\mathcal{G},\sigma) \to B(\ell^2(\mathcal{G}_x))$ that is given by

$$L^{\sigma,x}(f)\delta_\gamma = \sum_{\mu \in \mathcal{G}^{r(\gamma)}} \sigma(\mu,\mu^{-1}\gamma)f(\mu)\delta_{\mu\gamma}, \quad \text{for } f \in C_c(\mathcal{G},\sigma) \text{ and } \gamma \in \mathcal{G}_x. \tag{6}$$

We then obtain a faithful I-norm bounded $*$-representation of $C_c(\mathcal{G},\sigma)$ given by

$$\bigoplus_{x\in\mathcal{G}^{(0)}} L^{\sigma,x}\colon C_c(\mathcal{G},\sigma) \to \bigoplus_{x\in\mathcal{G}^{(0)}} B(\ell^2(\mathcal{G}_x)) \subset B(\bigoplus_{x\in\mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)). \tag{7}$$

$C_r^*(\mathcal{G},\sigma)$ is then the completion of the image of $C_c(\mathcal{G},\sigma)$ under the σ-twisted left regular representation. As the $*$-representation is I-norm bounded, $C_r^*(\mathcal{G},\sigma)$ is also the completion of $\ell^1(\mathcal{G},\sigma)$ in the same norm. Therefore, since $C^*(\mathcal{G},\sigma)$ is the C^*-envelope of $\ell^1(\mathcal{G},\sigma)$, by universality, there exists a natural (surjective) $*$-homomorphism λ : $C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$.

Definition 2.7. Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff groupoid and let $\sigma \in Z^2(G,\mathbb{T})$. We say that $\mathcal{G}$ twisted by σ has the *weak containment property* when the natural map $\lambda\colon C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$ is an isomorphism.

If $\mathcal{G}$ is an amenable groupoid [3], we have that $C_r^*(\mathcal{G},\sigma) = C^*(\mathcal{G},\sigma)$ for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$ [3, Proposition 6.1.8], and hence $\mathcal{G}$ twisted by σ has the weak containment property for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. In [18] it was proved that amenability is not equivalent to having the weak containment property. On the other hand, it is not known to the authors whether the weak containment property is equivalent to the weak containment property with respect every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$.

Remark 2.8. While both $\ell^1(\mathcal{G},\sigma)$ and $C_c(\mathcal{G},\sigma)$ complete to the same C^*-algebras $C^*(\mathcal{G},\sigma)$ and $C_r^*(\mathcal{G},\sigma)$ in the above setup, the question of C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ is not equivalent to C^*-uniqueness of the $*$-algebra $C_c(\mathcal{G},\sigma)$. To see this, let $\mathcal{G} = \mathbb{Z}$, the group of integers and consider the trivial twist $\sigma = 1$. Then $\ell^1(\mathbb{Z},1) = \ell^1(\mathbb{Z})$ is C^*-unique by [7], while $C_c(\mathbb{Z}) = \mathbb{C}[\mathbb{Z}]$ is not C^*-unique by [1, Proposition 2.4].

Denoting the restriction of σ to $\mathrm{Iso}(G)^\circ \subseteq \mathcal{G}$ also by σ, we define the Banach $*$-subalgebra $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ of $\ell^1(\mathcal{G},\sigma)$. We then have the following result.

Proposition 2.9 ([4, Proposition 5.3.1]). Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. There is a $*$-homomorphism

$$\iota\colon C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to C^*(\mathcal{G}, \sigma)$$

such that

$$\iota(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \mathrm{Iso}(\mathcal{G})^\circ, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. This homomorphism descends to an injective $*$-homomorphism

$$\iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to C_r^*(\mathcal{G}, \sigma).$$

We observe that the homomorphism ι is an isometry at the ℓ^1-level, that is, that $\iota\colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to \ell^1(\mathcal{G}, \sigma)$ is an isometric $*$-homomorphism.

We then also have the following result from [4], which will be key to our approach to study C^*-uniqueness of twisted groupoid convolution algebras in Section 3.

Proposition 2.10 ([4, Theorem 5.3.13]). Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Let $\iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to C_r^*(\mathcal{G}, \sigma)$ be the injective $*$-homomorphism of Proposition 2.9. Suppose A is a C^*-algebra and that $\Psi\colon C_r^*(\mathcal{G}, \sigma) \to A$ is a homomorphism. Then Ψ is injective if and only if $\Psi \circ \iota_r\colon C_r^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to A$ is and injective homomorphism.

3 C^*-uniqueness for Cocycle-Twisted Groupoid Convolution Algebras

We begin this section by presenting our main theorem. The remainder of the section will be dedicated to proving it.

Given a 2nd-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$, denote the restriction of σ to the fiber $\mathrm{Iso}(\mathcal{G})_x^\circ$ by σ_x. Note that σ_x is continuous as $\mathrm{Iso}(\mathcal{G})_x^\circ$ is discrete, that is, $\sigma_x \in Z^2(\mathrm{Iso}(\mathcal{G})_x^\circ, \mathbb{T})$. The following then constitutes our main theorem.

Theorem 3.1. Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Suppose that $\mathcal{G}$ twisted by σ has the weak containment property. Then

$\ell^1(\mathcal{G},\sigma)$ is C^*-unique if all the twisted convolution algebras $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x,\sigma_x)$, $x\in\mathcal{G}^{(0)}$, are C^*-unique.

As a 1st step towards proving Theorem 3.1 we relate C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$.

Proposition 3.2. Suppose $\mathcal{G}$ is a 2nd-countable locally compact Hausdorff étale groupoid with the weak containment property when twisted by $\sigma\in Z^2(\mathcal{G},\mathbb{T})$. If $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is C^*-unique, then $\ell^1(\mathcal{G},\sigma)$ is C^*-unique.

Proof. Suppose $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is C^*-unique. Then in particular $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)=C^*_r(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. Let $\{0\}\neq J\lhd C^*(\mathcal{G},\sigma)=C^*_r(\mathcal{G},\sigma)$ be a closed two-sided ideal. By Proposition 2.2 it suffices to show that $J\cap\ell^1(\mathcal{G},\sigma)\neq\{0\}$. By Proposition 2.10 we have $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)\cap J\neq\{0\}$ as the $*$-homomorphism $C^*(\mathcal{G},\sigma)\to C^*(\mathcal{G},\sigma)/J$ is not injective. Now define $I:=J\cap C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. It is straightforward to verify that I is a two-sided ideal in $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$, and as both J and $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ are closed in $C^*(\mathcal{G},\sigma)$, I is also closed in $C^*(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. By C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ it then follows that $I\cap\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)\neq\{0\}$. From this we get

$$\{0\}\neq I\cap\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)=J\cap\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)\subset J\cap\ell^1(\mathcal{G},\sigma),$$

from which we deduce by Proposition 2.2 that $\ell^1(\mathcal{G},\sigma)$ is C^*-unique. ■

Having related the question of C^*-uniqueness of $\ell^1(\mathcal{G},\sigma)$ to a question regarding C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$, we proceed to further relate this to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x,\sigma_x)$ for $x\in\mathcal{G}^{(0)}$. To do this we will show that for any $*$-representation $\pi\colon\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)\to B(\mathcal{H})$, the resulting C^*-algebra $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ is a $C_0(\mathcal{G}^{(0)})$-algebra. This is the content of Lemma 3.3. However, we first do some preparatory work.

First observe that there exists a $*$-homomorphism $\phi: C_0(\mathcal{G}^{(0)})\to\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma))$, the latter meaning the center of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$. Indeed, as $\mathcal{G}^{(0)}$ is open in $\mathrm{Iso}(\mathcal{G})^\circ$, we may take ϕ to be the inclusion where we extend functions in $C_0(\mathcal{G}^{(0)})$ by zero. The map ϕ is clearly isometric. As ϕ can be viewed as an inclusion, we omit writing it from now on to ease notation. Then given $g\in C_0(\mathcal{G}^{(0)})$ and $f\in C_c(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ we have that

$$\begin{aligned}(g*_\sigma f)(\gamma)&=g(r(\gamma))f(\gamma)\sigma(r(\gamma),\gamma)=g(r(\gamma))f(\gamma)\\&=f(\gamma)g(s(\gamma))\sigma(\gamma,s(\gamma))=(f*_\sigma g)(\gamma)\,,\end{aligned}$$

for every $\gamma \in \mathrm{Iso}(\mathcal{G})^\circ$. The resulting action of $C_0(\mathcal{G}^{(0)})$ on $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ can then be viewed as pointwise multiplication in the fibers of $\mathcal{G}^{(0)}$. By continuity we can extend ϕ to a continuous $*$-homomorphism from $C_0(\mathcal{G}^{(0)})$ to $\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))$. Let $\pi : \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to B(\mathcal{H})$ be a faithful $*$-representation and let $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ denote the completion in the operator norm of $B(\mathcal{H})$. Define the map $\iota := \pi \circ \phi : C_0(\mathcal{G}^{(0)}) \to \pi(\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)))$. We have that

$$\pi(\mathcal{Z}(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))) = \mathcal{Z}(\pi(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))) \subseteq \mathcal{Z}(M(C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))).$$

The following is then immediate.

Lemma 3.3. Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Let π be a $*$-representation of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Then $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a $C_0(\mathcal{G}^{(0)})$-algebra.

Now fix $x \in \mathcal{G}^{(0)}$ and denote by $J_x = C_0(\mathcal{G}^{(0)} \setminus \{x\})$ the space of continuous functions of $\mathcal{G}^{(0)}$ vanishing at both infinity and x. As $C_0(\mathcal{G}^{(0)})$ is central in $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ and J_x is a closed two-sided ideal of $C_0(\mathcal{G}^{(0)})$, the space $I_x := J_x \cdot \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a closed two-sided ideal in $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Recall that we denote by σ_x the restriction of σ to the fiber $\mathrm{Iso}(\mathcal{G})^\circ_x$. We then have the following result.

Lemma 3.4. Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. For every $x \in \mathcal{G}^{(0)}$ the map $\psi_x \colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ given by restriction of functions is a continuous $*$-homomorphism inducing an isometric $*$-isomorphism between $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x$ and $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$.

Proof. For $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ we have

$$\|\psi_x(f)\|_{\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x)} = \sum_{\gamma \in \mathrm{Iso}(\mathcal{G})^\circ_x} |f(\gamma)| \le \sup_{y \in \mathcal{G}^{(0)}} \sum_{\mu \in \mathrm{Iso}(\mathcal{G})^\circ_y} |f(\mu)| = \|f\|_I$$

for all $f \in C_c(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Thus, ψ_x is a I-norm decreasing map, so it extends to a continuous $*$-homomorphism $\psi_x : \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$. It is surjective by Tietze's extension theorem.

Next we want to show that $\ker\psi_x = I_x$. First observe that given $g \in C_0(\mathcal{G}^{(0)})$ and $h \in C_c(\mathcal{G},\sigma)$ we have that

$$\psi_x(g *_\sigma h)(\gamma) = (\psi_x(g) *_\sigma \psi_x(h))(\gamma) = \sum_{\mu \in \mathrm{Iso}(\mathcal{G})_x^\circ} g(\mu)h(\mu^{-1}\gamma)\sigma(\mu,\mu^{-1}\gamma)$$
$$= g(x)h(x\gamma)\sigma(x,\gamma) = g(x)h(\gamma),$$

for every $\gamma \in \mathrm{Iso}(\mathcal{G})_x^\circ$.

Now let $f \in I_x$. We may then assume that f is the norm limit of elements f_n of the form $f_n = \sum_{i=1}^n g_i *_\sigma h_i$, where $g_i \in J_x$ and $h_i \in C_c(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)$ for all $i \in \mathbb{N}$. It suffices to prove that $\psi_x(g_i *_\sigma h_i) = 0$ for all $i \in \mathbb{N}$. For any $\gamma \in \mathrm{Iso}(\mathcal{G})_x^\circ$ we then have $\psi_x(g_i *_\sigma h_i)(\gamma) = g_i(x)h_i(\gamma) = 0$ since $g_i(x) = 0$. Then it follows that $\psi_x(f_n) = 0$ for every $n \in \mathbb{N}$, and by continuity $\psi_x(f) = 0$. Thus, $I_x \subset \ker\psi_x$.

Conversely, suppose $f \in \ker\psi$. Then $f = \lim f_n$ for some $f_n \in C_c(\mathcal{G},\sigma) \cap \ker\psi_x$, and hence $f_n(x) = 0$ for every $n \in \mathbb{N}$. Let $\{\rho_\lambda\}_{\lambda\in\Lambda} \subset C_0(\mathcal{G}^{(0)} \setminus \{x\})$ be a partition of the unit of $\mathcal{G}^{(0)} \setminus \{x\}$. Then given $n \in \mathbb{N}$ there exists a finite subset Λ_n of Λ, such that $g_n := \sum_{\lambda\in\Lambda_n} \rho_n \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) = J_x$ and $g_n(y) = 1$ for every $y \in r(\mathrm{supp}(f_n)) = s(\mathrm{supp}(f_n))$, and hence

$$f_n(\gamma) = g_n(r(\gamma))f_n(\gamma)\sigma(r(\gamma),\gamma) = (g_n *_\sigma f_n)(\gamma)$$

for every $\gamma \in \mathcal{G}$. Therefore, we have that

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty}(g_n *_\sigma f_n) \in \overline{J_x \cdot \ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)} = I_x,$$

as we wanted. We would like to see that the isomorphism $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ,\sigma)/I_x \cong \ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ,\sigma_x)$ is isometric. To do that, it is enough to check that

$$\inf\{\|f+h\| : h \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) \cdot C_c(\mathcal{G},\sigma)\} = \|\psi_x(f)\|$$

for every $f \in C_c(\mathcal{G},\sigma)$. Observe that by continuity of ψ_x we have $\|f+h\| \geq \|\psi_x(f)\|$ for every $h \in C_0(\mathcal{G}^{(0)} \setminus \{x\}) \cdot C_c(\mathcal{G},\sigma)$. As $\mathcal{G}$ is 2nd-countable locally compact Hausdorff, so is $\mathcal{G}^{(0)} \setminus \{x\}$. Hence, it is paracompact, and we can guarantee that there is a countable partition of unity $\{\rho_i\}_{i=1}^{\infty}$ for $\mathcal{G}^{(0)} \setminus \{x\}$. For $n \in \mathbb{N}$ let $U_n := \mathcal{G}^{(0)} \setminus \bigcup_{i=1}^{n} \operatorname{supp}(\rho_i)$. Then we have

$$\|f - (\sum_{i=0}^{n} \rho_i)f\| \leq \max_{y \in U_n} \|\psi_y(f)\| .$$

By Lemma 2.6 the assignment $\mathcal{G}^{(0)} \ni x \mapsto \max\{\sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)|\}$ is continuous. It follows that for every $\varepsilon > 0$ there exists n such that $|\|\psi_y(f)\| - \|\psi_x(f)\|| < \varepsilon$ for every $y \in U_n$. As $U_k \supset U_{k-1}$ for all k, it follows that $\|f - (\sum_{i=0}^{k} \rho_i)f\| \leq \|\psi_x(f)\| + \varepsilon$ for all $k \geq n$. As ε was arbitrary, this finishes the proof. ■

We may finally prove Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.2 it suffices to show that the condition implies that $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is C^*-unique. As above, denote by $J_x = C_0(\mathcal{G}^{(0)} \setminus \{x\})$ and by $I_x := \overline{J_x \cdot \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)}$ the resulting closed two-sided ideal in $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. Let $\pi \colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \to B(\mathcal{H})$ be a faithful $*$-representation and denote by $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ the completion of $\pi(\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))$. Moreover, let I_x^π denote the closure of $\pi(I_x)$ in $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$. By Proposition 2.5 and Lemma 3.3 there is an isomorphism $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \cong \Gamma_0(\mathbf{B}^\pi)$, where the fibers B_x^π, $x \in \mathcal{G}^{(0)}$, are given by

$$B_x^\pi = C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x^\pi .$$

We will show that there is an injective $*$-homomorphism

$$\Psi_x \colon \ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x) \to B_x^\pi$$

for every $x \in \mathcal{G}^{(0)}$. To do this, fix $x \in \mathcal{G}^{(0)}$. First, we show that the composition

$$\ell^1(\mathrm{Iso}(\mathcal{G})_x^\circ, \sigma_x) \cong \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x \to C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/I_x^\pi \cong B_x^\pi$$

given by first applying the isomorphism of Lemma 3.4 and then applying the map $f + I_x \mapsto f + I_x^\pi$ for $f \in \ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is a well-defined continuous $*$-homomorphism. This is our candidate for the map Ψ_x. Denote by I_x^π also the image of the ideal $I_x^\pi \lhd C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ in $\Gamma_0(\mathbf{B}^\pi)$. It then suffices to show that if $F \in I_x^\pi$, then $F(x) = 0$.

To see this, note that we can let $C_0(\mathcal{G}^{(0)} \setminus \{x\})$ act on $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ by pointwise multiplication to obtain a have a continuous $*$-homomorphism

$$C_0(\mathcal{G}^{(0)} \setminus \{x\}) = J_x \to \mathcal{Z}(M(C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma))),$$

which leaves I^π_x invariant, and as a result I^π_x becomes a Banach J_x-module. It is even non-degenerate as

$$\overline{J_x I^\pi_x} = \overline{J_x \overline{J_x C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)}} \supset \overline{J_x J_x C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)} = \overline{J_x C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)} = I^\pi_x,$$

since J_x, being a C^*-algebra, has an approximate identity. It then follows by Cohen–Hewitt factorization that if $F \in I^\pi_x$, then $F = f \cdot H$, where $f \in J_x$ and $H \in I^\pi_x$. Then $F(x) = f(x)H(x) = 0$, and the map Ψ_x is a well-defined $*$-homomorphism.

As $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ is dense in its C^*-completion $C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, it follows that the image of Ψ_x is dense.

Lastly, if $\Psi_x(f) = 0$, then $\Psi_x(f) \in I^\pi_x$, and so $f|_{\mathrm{Iso}(\mathcal{G})^\circ_x} = 0$ by the above argument. Thus, ψ_x is injective. Hence, we have a continuous dense embedding

$$\Psi_x \colon \ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x) \hookrightarrow C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J^\pi_x.$$

Now $C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J^\pi_x$ becomes a C^*-completion of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$. Since π is an arbitrary faithful $*$-representation of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, we deduce that this holds for all faithful $*$-representations. But as $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$ is assumed C^*-unique, we may then deduce

$$C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J^\pi_x \cong C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J^{\mathrm{full}}_x, \tag{8}$$

where $C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$ and J^{full}_x denotes the completions in the maximal C^*-norm. As $x \in \mathcal{G}^{(0)}$ was arbitrary, we deduce that this holds for all $x \in \mathcal{G}^{(0)}$. Now let $B^{\mathrm{full}}_x = C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)/J^{\mathrm{full}}_x$. By Proposition 2.5 and (8) we then have

$$C^*_\pi(\mathrm{Iso}(\mathcal{G})^\circ, \sigma) \cong \Gamma_0(\mathbf{B}^\pi) \cong \Gamma_0(\mathbf{B}^{\mathrm{full}}) \cong C^*(\mathrm{Iso}(\mathcal{G})^\circ, \sigma).$$

From this we deduce that $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma)$, and hence also $\ell^1(\mathcal{G}, \sigma)$, is C^*-unique. ∎

4 Examples

In this section we present some (classes of) examples of C^*-unique groupoids. Due to the nature of our main result, Theorem 3.1, our examples will draw upon previously proved results on C^*-uniqueness of locally compact groups. We begin with a class of examples in the case of trivial cocycle twists.

Example 4.1 (The untwisted case). If we consider a 2nd-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ with the trivial 2-cocycle $\sigma = 1$, then C^*-uniqueness of $\ell^1(\mathcal{G}, 1) = \ell^1(\mathcal{G})$ can by Theorem 3.1 be deduced by C^*-uniqueness of the Banach $*$-algebras $\ell^1(\text{Iso}(\mathcal{G})^\circ_x, \sigma_x) = \ell^1(\text{Iso}(\mathcal{G})^\circ_x)$ for $x \in \mathcal{G}^{(0)}$. C^*-uniqueness of untwisted convolution algebras has been studied before, and it is known that for a locally compact group G, the Banach $*$-algebra $\ell^1(G)$ is C^*-unique if G is a semidirect product of abelian groups, or a group where every compactly generated subgroup is of polynomial growth [8, p. 224]. Hence, if for every $x \in \mathcal{G}^{(0)}$ the discrete group $\text{Iso}(\mathcal{G})^\circ_x$ is of one of these types, $\ell^1(\mathcal{G})$ will be C^*-unique.

In the case of locally compact groups it is well-known that amenability of the group is equivalent to the group having the weak containment property. Indeed, amenability is even equivalent to the weak containment property when twisted for all continuous 2-cocycles σ of the group. Moreover, it is easy to see that if a group is C^*-unique, then it is amenable. The converse is however not true [8, p. 230]. In stark contrast to the case of locally compact groups, the following example shows that groupoids can be C^*-unique without even being amenable.

Example 4.2 (Non-amenable C^*-unique groupoid). In [2, Theorem 2.7] the authors constructed a 2nd-countable, locally compact, Hausdorff non-amenable étale groupoid $\mathcal{G}$ such that $\text{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$ and $C^*_r(\mathcal{G}) = C^*(\mathcal{G})$. Then since $\ell^1(\text{Iso}(\mathcal{G})^\circ) = C_0(\mathcal{G}^{(0)}) \subseteq \ell^1(\mathcal{G})$, we have by Proposition 2.10 that every nonzero two-sided ideal I of $C^*(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}^{(0)})$, and hence with $\ell^1(\mathcal{G})$. Therefore, by Proposition 2.2 we have that $\ell^1(\mathcal{G})$ is C^*-unique.

In this particular case we may also deduce C^*-uniqueness of $\ell^1(\mathcal{G})$ in another way. Namely, as $\text{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, we have that $\text{Iso}(\mathcal{G})^\circ_x$ is the trivial group for every $x \in \mathcal{G}^{(0)}$. Hence, $\ell^1(\text{Iso}(\mathcal{G})^\circ_x)$ is C^*-unique by Example 4.1. This argument of course carries over to any topologically principal groupoid. Indeed, this approach shows that whenever $\mathcal{G}$ is a 2nd-countable, locally compact, Hausdorff topologically principal étale groupoid, then $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique for any $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$.

We also have classes of examples that includes more general cocycle twists.

Example 4.3 (The twisted case). Let $\mathcal{G}$ be a 2nd-countable locally compact Hausdorff étale groupoid, and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. By Theorem 3.1 C^*-uniqueness of $\ell^1(\mathcal{G}, \sigma)$ can be deduced by C^*-uniqueness of the Banach $*$-algebras $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_x, \sigma_x)$, for $x \in \mathcal{G}^{(0)}$, where σ_x as before denotes the restriction of σ to $\mathrm{Iso}(\mathcal{G})^\circ_x$. C^*-uniqueness of twisted convolution algebras of locally compact groups was studied in [5]. In [5, Theorem 3.1] it was found that if G is a locally compact group and $c \in Z^2(G, \mathbb{T})$, then $L^1(G, c)$ is C^*-unique if $L^1(G_c)$ is C^*-unique, where G_c denotes the Mackey group associated to G and c. As a topological space G_c is just $G \times \mathbb{T}$, but the binary operation is given by

$$(x, \tau) \cdot (y, \eta) = (xy, \tau\eta\overline{c(x, y)}).$$

Thus, we may relate C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ, \sigma_x)$ to C^*-uniqueness of $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x})$, where $\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x}$ denotes the Mackey group associated to $\mathrm{Iso}(\mathcal{G})^\circ_x$ and σ_x, and we deduce that $\ell^1(\mathcal{G}, \sigma)$ is C^*-unique if $\ell^1(\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x})$ is C^*-unique for every $x \in \mathcal{G}^{(0)}$. This happens if, for example, $\mathrm{Iso}(\mathcal{G})^\circ_{\sigma_x}$ is a group of one of the types discussed in Example 4.1.

In the following example we are able to deduce C^*-uniqueness of a locally compact group not of the form discussed in Example 4.1 by relating the question to C^*-uniqueness of a groupoid.

Example 4.4 (The wreath product). Let Γ denote the wreath product $H \wr G := \left(\bigoplus_G H\right) \rtimes G$ where H is a finite abelian group and where G is a countable discrete amenable group. We will show that $\ell^1(\Gamma)$ is C^*-unique.

To do this, let $\mathcal{G} = X \rtimes_\varphi G$ be the transformation groupoid where $X = \prod_G \hat{H}$, and φ is the shift homeomorphism of X by G. $\mathcal{G}$ is amenable since G is amenable. Then we have that

$$C^*(\Gamma) \cong C^*(\bigoplus_G H) \rtimes_\varphi G \cong C(X) \rtimes_\varphi G\,.$$

Now recall that by the Fourier transform $\ell^1(\bigoplus_G H) \cong A(X)$, where $A(X)$ is a dense subalgebra of $C(X)$. Indeed, it becomes a Banach $*$-subalgebra of $C(X)$ when equipped with the induced ℓ^1-norm through the Fourier transform, and then the isomorphism is also an isometry. It also follows that $C(X)$ is the completion of $\ell^1(\bigoplus_G H)$ with respect to some C^*-norm. We have that $\ell^1(\Gamma) \cong \ell^1(\ell^1\left(\bigoplus_G H\right), G) \cong \ell^1(A(X), G)$ (see for example [13, Remark and Notation 2.4]). Then there exists an isometric embedding $\iota : \ell^1(A(X), G) \hookrightarrow \ell^1(\mathcal{G})$ defined as follows. If $F \in \ell^1(A(X), G)$, we define $\iota(F)$ to be

$$\iota(F)(x, g) = \widehat{f_g}(x),$$

for $x \in X = \prod_G \hat{H}$ and $g \in G$, where f_g is the unique element of $\ell^1(\bigoplus_G H)$ with $\widehat{f_g} = F(g)$. Therefore, by the isomorphisms $C^*(\ell^1(\Gamma)) \cong C^*(\ell^1(A(X), G)) \cong C^*(\ell^1(\mathcal{G}))$ it would be enough to check that any nonzero two-sided ideal I of $C^*(\mathcal{G})$ has a nontrivial intersection with the image of $\ell^1(A(X), G)$ by the inclusion ι. Observe that then $\ell^1(\bigoplus_G H) \subseteq \ell^1(\Gamma)$ can be identified with $\iota(A(X))$ in $C(X) \subseteq C^*(\mathcal{G})$. The groupoid $\mathcal{G}$ is clearly topologically principal, and hence $\ell^1(\mathcal{G})$ is C^*-unique. Moreover, for every closed two-sided ideal $\{0\} \neq I \trianglelefteq C^*(\mathcal{G})$ we have that $\{0\} \neq J := I \cap C(X)$ [12, Theorem 4.1]. But since $\bigoplus_G H$ is locally finite, then $\ell^1(\bigoplus_G H)$, and hence $A(X)$, are C^*-unique by [10]. Thus, $J \cap A(X) \neq \{0\}$, which further implies $J \cap \ell^1(A(X), G) \neq \{0\}$. It follows that $\ell^1(\Gamma)$ is C^*-unique.

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