On the Comparison of Performance-per-cost for Coherent and Mixed Systems

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Abstract

The present paper is concerned with reliability economics, considering a certain performance-per-cost criterion for coherent and mixed systems, as introduced in *[M. R. Dugas and F. J. Samaniego, On optimal system designs in reliability-economics frameworks, Naval Research Logistics, vol. 54, pp. 568-582, 2007]*. We first present a new comparison result for performance-per-cost of systems with independent and identically distributed component lifetimes under certain stochastic orderings. We then consider optimization of the performance-per-cost criterion, first reconsidering and refining results from the above cited paper, and then considering mixtures of given subsets of coherent systems.

Keywords: coherent system, system signature, mixed system, ordering of random variables, performance-per-cost.

1 Introduction

Complex engineering systems play an ever increasing role in today's industry and society. Reliability and availability of systems are hence becoming increasingly important, and are in most applications interdependent with economic considerations. Typically, decision makers are faced with questions of whether a certain new system design can be defended on economic grounds, or whether improvements of certain components or subsystems should be undertaken. This demonstrates the need for a common framework for the simultaneous treatment of reliability and economics and the attendant analyses of costs versus benefits. While there appears to be a large literature on reliability optimization and related topics, the literature on comprehensive economic evaluations, possibly from the design process to system retirement, is somewhat sparse. There is, however, excellent published work in the area, including the book by Billinton and Allan [3] on reliability of power systems. Here the economic aspect of reliability both from the supplier and consumer's point of view, in the latter case named reliability worth, are considered.

In the same area of applications, [9] and [19] consider economic evaluations of power systems involving renewal energy. The latter article provides a review of different studies, emphasizing the necessity of evaluating reliability and economics simultaneously. It is noted in particular that the optimal system should have low initial cost but high reliability in order to maintain the system with minimum life-cycle cost per unit of energy delivered.

[17] and [8] provide other examples of the kinds of problems that can be treated within the framework of reliability and economics. The former paper considers wind farm reliability optimization using multistate systems in order to find the best system topology under constraints on performance, e.g., availability, costs and the Global Warming Potential (GWP) of the system. The latter article considers coastal flood defense systems, which may consist of multiple lines of defense. In case of a system with a front and a rear defense, the front defense can improve the reliability of the rear defense by reducing the load on the latter. The authors develop a framework for an economic cost-benefit analysis in order to assess the effect of the load reduction obtained by including the front defense.

That the study of the economic aspects of reliability is not new becomes moreover clear from two early references, [4] and [18]. The authors of these articles identified both the need for probabilistic evaluation of reliability and for relating economics to reliability.

Dugas and Samaniego [7] defined the field of *reliability economics* to mean "the collection of problems and frameworks in which there is tension between the performance of a group of systems of interest and their cost". Their stated goal was to identify system designs which are optimal according to some reasonable criterion depending on both performance and cost. They formulated the problem in terms of classical system reliability, and they further utilized the flexible tool of system signatures as introduced by Samaniego [13] (see also the monograph of Samaniego [14]). In this framework, they constructed a reasonable criterion function for measuring performance-per-cost for systems, and considered its optimization in the collection of systems of an arbitrary but fixed size.

In this paper, we shall follow their path, and complement their work in certain directions. More specifically, we shall in the first part consider the problem roughly described as follows: Suppose there are given two different designs of systems for performing a specific task, where one system is known to perform better than the other, although at a higher price. For any given criterion function a question naturally arises regarding the conditions under which the more expensive system is to be preferred. In the second part we shall consider the optimization of the criterion function, when a given set of coherent systems is available. We first reconsider results from [14], and then establish new and more general results.

The paper is organized as follows. In Section 2, we review the basic definitions and properties of coherent and mixed systems and their signatures, as well as the basic ingredients of the theory for reliability economics as introduced in [7]. For ease of reference, we shall mostly refer to Chapter 7 in [14], the chapter which is based on the paper [7]. In Section 3 we state and discuss a series of results which give sufficient conditions for improved performance-per-cost under orderings of the system signatures. The proof is relegated to the Appendix. In Section 4, we reconsider the optimization of the performance-per-cost criterion of [14, Chapter 7] and give, in particular, a strengthening of one of the optimization results. In Section 5 we explore the possibility of improving performance among a given set of coherent systems by means of mixtures. An application is given in Section 6, while in Section 7 we offer some concluding remarks. The paper is ended by an Appendix giving some key results and proofs.

2 Definitions and Basic Theory

2.1 Coherent and Mixed Systems and their Signatures

Consider a coherent system with n binary components as studied, for example, in the monograph by Barlow and Proschan [2]. In the following developments we shall refer to a system with n components as an n-system. Suppose that the components' lifetimes, $X_1, \ldots, X_n$, are independent and identically distributed (i.i.d.) with continuous distribution F, and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be their ordered values. Let T be the lifetime of the system. The *signature vector* $\mathbf{s} = (s_1, \ldots, s_n)$ of the system, introduced in [13], is defined by $s_i = P(T = X_{i:n})$ for $i = 1, \ldots, n$. A key property of $\mathbf{s}$ is that it depends only on the system structure and does not depend on the distribution F of component lifetimes. Moreover [14, Theorem 3.1], the survival function of the lifetime T of the system can be represented as

$$\begin{aligned} P(T > t) &= \sum_{i=1}^{n} s_i P(X_{i:n} > t) \\ &= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}, \end{aligned} \tag{1}$$

which further implies that

$$E(T) = \sum_{i=1}^{n} s_i E(X_{i:n}). \tag{2}$$

This shows in particular that the system's lifetime distribution depends on the system structure only through the signature vector $\mathbf{s}$.

Standard examples of coherent systems are series systems, which work if and only if all components are working, and parallel systems, which work if and only if at least one component is working. These are special cases of the so-called i-out-of-n systems which fail upon the i-th component failure, for $1 \le i \le n$. It is easy to see that the signature vector of an i-out-of-n system is

$$(0, \ldots, 1_i, \ldots, 0), \tag{3}$$

where the subindex i refers to the ith element of the vector.

For practical as well as mathematical reasons, it has been proven useful to extend the class of systems to include so-called *mixed* systems [14, page 28-31]. A mixed system can be considered as the result of selecting a system at random from a given finite set of coherent systems. To be more precise, suppose that a k-dimensional probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_k)$ gives positive weight to k distinct coherent n-systems with signature vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$, respectively, each assumed to have components whose lifetimes are i.i.d with a common distribution F. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the ordered lifetimes of the n components. Then, assuming that the jth system is chosen with probability p_j $(j = 1, \ldots, k)$ and letting T be the resulting lifetime, we have

$$\begin{aligned} P(T = X_{i:n}) &= \sum_{j=1}^{k} P(T = X_{i:n} | \; j\text{th system is chosen}) P(\; j\text{th system is chosen}) \\ &= \sum_{j=1}^{k} p_j s_{ij}, \end{aligned}$$

where $\mathbf{s}_j = (s_{1j}, \ldots, s_{nj})$. It follows that the signature $\mathbf{s}^*$ associated with the process of selecting among the m systems according to the probability distribution $\boldsymbol{p}$ is the vector equal to the mixture of the signature vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$, that is, $\mathbf{s}^* = \sum_{j=1}^{k} p_j \mathbf{s}_j$. It is easily verified that the results (1) and (2) continue to hold for mixed systems [14, page 30]. Note that any probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ can serve as the signature of a mixed system. One possible representation of such a mixed system is the one which gives weight s_i to an i-out-of-n system, for $i = 1, \ldots, n$.

Computation of signatures may be a complicated task, particularly for large systems. A comprehensive review of methods for computation of signatures, as well as a new approach, are given in the recent paper [6].

2.2 Ordering of Random Variables

Signature vectors have proven particularly useful in the comparison of lifetimes of different systems. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be signature vectors of two mixed n-systems, and let T_1 and T_2 be their lifetimes. As shown in [14, Section 4.2], certain ordering properties of signature vectors are preserved for the corresponding lifetime distributions. This applies to the three orders presented in Definition 1 below, which are involved in the main result of Section 3. Note that here and hereafter we will use *increasing* to mean *non-decreasing* and *decreasing* to mean *non-increasing*

Definition 1. *Let X_1 and X_2 be random variables, either discrete or absolutely continuous, with corresponding cumulative distribution functions F_1 and F_2, and let $\bar{F}_i = 1 - F_i$ for $i = 1, 2$. Then X_1 is smaller than X_2 in the* ***stochastic order****, denoted $X_1 \leq_{st} X_2$, if and only if $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x; in the* ***hazard rate order****, denoted $X_1 \leq_{hr} X_2$, if and only if $\bar{F}_2(x)/\bar{F}_1(x)$ is increasing in x; and in the* ***likelihood ratio order****, denoted $X_1 \leq_{lr} X_2$, if and only if $f_2(x)/f_1(x)$ is increasing in x (where the f_i are either probability mass functions or density functions, according to whether the X_i are discrete or absolutely continuous).*

In the above definitions, and elsewhere in the paper, $a/0 = \infty$ for $a > 0$.

Theorems 4.3-4.5 of [14] state that when g denotes any of the three orders in Definition 1, that is st, hr, lr, we have that

$$\mathbf{s}_1 \leq_g \mathbf{s}_2 \Rightarrow T_1 \leq_g T_2. \tag{4}$$

It is furthermore well known [16, Chapter 1] that

$$X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2.$$

2.3 Representation of Performance of a Coherent System

In describing system performance, two particular measures seem the most natural: the expected system lifetime $E(T)$ and the system's reliability function $R_T(t) = P(T > t)$. If the system has been designed to survive beyond a predetermined mission time t_0, then the reliability $R_T(t_0)$ at this mission time might be taken as the appropriate measure of performance. It follows from (1) and (2) that both $E(T)$ and $R_T(t_0)$ can be expressed in terms of the system's signature $\mathbf{s}$ as

$$\sum_{i=1}^{n} h_i s_i, \tag{5}$$

where the h_i are increasing positive constants. More generally, for any increasing function $\Phi(T)$, we can write $E[\Phi(T)] = \sum_{i=1}^n s_i E[\Phi(X_{i:n})]$, so the class of performance measures of the form $\sum_{i=1}^n h_i s_i$, where the h_i are increasing, is a natural and flexible class to consider.

Note that since an i-out-of-n system has the signature vector displayed in (3), the coefficient h_i in (5) can be interpreted as measuring the performance of an i-out-of-n system for $i = 1, 2, \ldots, n$.

Since monotonicity of the h_i is not needed for all our results, we shall generally assume only that $h_i > 0$ for each i and state further assumptions when appropriate.

2.4 Representation of Cost of a Coherent System

Similarly, one may let the expected cost of a system with signature vector $\mathbf{s}$ be represented as $\sum_{i=1}^n c_i s_i$, where c_i is interpreted as the cost of an i-out-of-n system, $i = 1, \ldots, n$. The justification of this is that a system with signature vector $\mathbf{s}$ can be represented as a mixture of i-out-of-n systems, see Section 2.1. Thus, the cost of producing a system with signature $\mathbf{s}$ can reasonably be considered to be the cost of producing the equivalent mixture of i-out-of-n systems. This latter system is produced using the probability vector $\mathbf{s}$ as the mixing distribution for i-out-of-n systems, and its expected cost would be exactly $\sum_{i=1}^n c_i s_i$.

[14, page 95] gives an example of how the linear representation $\sum_{i=1}^n c_i s_i$ of expected cost of an n-system might occur in practice, in the so called "salvage model". Here it is assumed that the cost c_i of an i-out-of-n system can be written as

$$c_i = C + nA - (n-i)B \text{ for } i = 1, \ldots, n, \tag{6}$$

where C is the initial fixed cost of manufacturing the system, A is the cost of an individual component, and B is the salvage value of a used but working component which is removed after system failure.

As for the h_i, although it is natural from the setting that the c_i are increasing, we will only assume that $c_i > 0$ for each i, unless otherwise stated.

2.5 The Performance-per-cost Criterion

Having established measures of, respectively, performance and cost of a system, Dugas and Samaniego [7] proposed the following measure of the relative value of performance and cost for a mixed n-system with signature vector $\mathbf{s}$:

$$m_r(\mathbf{s}, \mathbf{h}, \mathbf{c}) = \frac{\sum_{i=1}^{n} h_i s_i}{(\sum_{i=1}^{n} c_i s_i)^r}. \tag{7}$$

As explained in [14, page 97], the power parameter $r > 0$ serves as a calibration parameter, determining the weight to be put on cost relative to performance in the criterion function (7). Thus $r = 1$ is the natural choice if equal weight is put on performance and cost, while $r < 1$ ($r > 1$) correspond to cases where the most weight is put on performance (cost).

The optimality problem considered in [7] was the problem of maximizing the performance-per-cost criterion (7) with respect to the signature vector $\mathbf{s}$ among all mixed systems of a given size. The authors were then able to give several explicit optimality results for the given criterion, see also Section 4 below.

The class of criterion functions given by (7) is of course by no means uniquely appropriate to the problem of interest. The particular form has been discussed in [14, Chapter 7.2], where it is pointed to at least two good reasons for the choice of functions; their desirable monotonicity properties with respect to performance and cost, and their amenability to analytical treatment.

3 On the Comparison of Performance-per-cost for Ordered Systems

Consider two n-systems, where the signature vectors are ordered with respect to one of the orders considered in Definition 1.

Assume that there are given a performance vector $\mathbf{h} = (h_1, \dots, h_n)$ and a cost vector $\mathbf{c} = (c_1, \dots, c_n)$ where we now may think of these as having increasing entries. It follows that for two n-systems with respective signature vectors $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{t} = (t_1, \dots, t_n)$ which are stochastically ordered, i.e., $\mathbf{s} \leq_{st} \mathbf{t}$, both the performance and cost will be larger for the system with signature $\mathbf{t}$. Since both vectors appear, respectively, in the numerator and denominator of the performance-per-cost measure (7) for each system, it remains undetermined which of the two systems has the highest value in terms of performance-per-cost. Intuitively, if in some sense the performance increases relatively more than the cost when going to a stochastically better system, then the value of the criterion function should rise. The following theorem makes this issue more precise for the case $r = 1$. It presents sufficient conditions under which two n-systems with signature vectors $\mathbf{s}$ and $\mathbf{t}$, where $\mathbf{s} \leq_g \mathbf{t}$ according to one of the orders of random variables considered in Definition 1, are ordered in the same direction by the performance per cost criterion, i.e.,

$$\frac{\sum_{i=1}^{n} h_i s_i}{\sum_{i=1}^{n} c_i s_i} \leq \frac{\sum_{i=1}^{n} h_i t_i}{\sum_{i=1}^{n} c_i t_i}. \tag{8}$$

The proof of the theorem is given in the Appendix, where we first present and prove a slightly more general result based on the monograph by Shaked and Shanthikumar [16, Chapter 1].

Theorem 1. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be signatures of two mixed* n*-systems. Let* $\mathbf{c} = (c_1, \dots, c_n)$ *and* $\mathbf{h} = (h_1, \dots, h_n)$ *be, respectively, the cost vector and the performance vector. Consider the follwing conditions:*

C1: $c_i \geq 0$ *for all* i

C2: h_i/c_i *is increasing in* i

C3: c_i *is increasing in* i

C4: $\frac{h_{i+1}-h_i}{c_{i+1}-c_i} \geq \frac{h_n}{c_n}$ *for* $i = 1, 2, \dots, n-1$

Then the following implications hold:

(i) *If* $\mathbf{s} \leq_{lr} \mathbf{t}$*, then C1,C2 together imply (8).*

(ii) *If* $\mathbf{s} \leq_{hr} \mathbf{t}$*, then C1,C2,C3 together imply (8).*

(iii) *If* $\mathbf{s} \leq_{st} \mathbf{t}$*, then C1,C2,C3,C4 together imply (8).*

Theorem 1 makes explicit the fact that the weakening of the ordering between the signatures $\mathbf{s}$ and $\mathbf{t}$ results in the need for stronger conditions on the cost and performance vectors in order to establish the inequality in (8). For example, if $\mathbf{s} \leq_{lr} \mathbf{t}$, then the increasing property of the c_i is not needed, provided C2 holds, i.e., the ratios h_i/c_i are increasing. This expresses the intuition that performance needs to increase more, relative to the cost, when going from an i-out-of-n system to an $(i+1)$-out-of-n system. An interpretation of condition C4 is that the ratios of the increase in performance and increase in cost when going from an i-out-of-n to an $(i+1)$-out-of-n system are bounded below by the ratio of performance and cost for a parallel system of size n.

Some aspects of Theorem 1 are illustrated and discussed in the example below.

Example 1. Let $n = 4$ and suppose costs are given by the salvage model (6), where the constants A, B, C are $A = 4/10,\ B = 5/10$ and $C = 3/10$. It follows that the cost vector for the system are given by $\mathbf{c} = (0.4, 0.9, 1.4, 1.9)$.

Now suppose that the component lifetimes X_i are exponentially distributed with mean 1. As is well known, for $i = 1, 2, \dots, n$,

$$E(X_{i:n}) = \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{n-i+1}. \tag{9}$$

Thus, letting $n = 4$ and assuming that performance is measured by expected lifetime, we obtain the performance vector $\mathbf{h} = (1/4, 7/12, 13/12, 25/12) = (0.250, 0.583, 1.083, 2.083)$.

It is now straightforward to see that conditions C1-C3 of Theorem 1 hold. It is easily confirmed, however, that C4 does not hold. Indeed, we calculate

$$\frac{h_2 - h_1}{c_2 - c_1} = 0.667 < \frac{h_4}{c_4} = 1.096.$$

Still, in order for a system with signature $\mathbf{t}$ to outperform a system with signature $\mathbf{s}$ relative to the criterion function (7) with $r = 1$, it is sufficient that $\mathbf{s} \leq_{hr} \mathbf{t}$. It is, on the other hand, not sufficient to have $\mathbf{s} \leq_{st} \mathbf{t}$. Suppose, for example, that $\mathbf{s} = (1/4, 1/4, 1/2, 0)$ and $\mathbf{t} = (0, 1/2, 1/2, 0)$. These are both signatures of coherent systems [14, Table 3.2], and the fact that $\mathbf{s} \leq_{st} \mathbf{t}$ but $\mathbf{s} \not\leq_{hr} \mathbf{t}$ is easily verified. Thus, Theorem 1 does not guarantee that the system with signature $\mathbf{t}$ has the highest performance-per-cost. In fact, substitution of the relevant values into (8), gives $0.75/1.025 = 0.732$ for the left hand side, while for the right hand, one obtains $0.833/1.15 = 0.725$. Hence, the system with signature $\mathbf{t}$ does not have the best performance-per-cost.

Changing the signature from $\mathbf{t}$ to $\mathbf{t}' = (0, 1/3, 2/3, 0)$ (which also corresponds to a coherent system), it is seen that $\mathbf{s} \leq_{hr} \mathbf{t}'$. Hence, since C1-C3 hold, the theorem guarantees a better performance-per-cost for the latter system. This is also seen by recalculating the right hand side of (8), now for $\mathbf{t}'$, which gives $0.917/1.233 = 0.743$.

The following corollary to Theorem 1 considers the case when the performance per cost criterion in (7) is used with $r < 1$.

Corollary 1. *Implications (ii) and (iii) of Theorem 1 will continue to hold in the case $r < 1$ if (8) in the statements (ii) and (iii) is replaced by*

$$\frac{\sum_{i=1}^n h_i s_i}{(\sum_{i=1}^n c_i s_i)^r} \leq \frac{\sum_{i=1}^n h_i t_i}{(\sum_{i=1}^n c_i t_i)^r}. \tag{10}$$

Proof. Note first that (10) can be written

$$\frac{\sum_{i=1}^n h_i s_i}{\sum_{i=1}^n c_i s_i} \leq \frac{\sum_{i=1}^n h_i t_i}{\sum_{i=1}^n c_i t_i} \cdot \left(\frac{\sum_{i=1}^n c_i s_i}{\sum_{i=1}^n c_i t_i}\right)^{r-1}.$$

Since (8) holds under the given conditions, the above inequality holds if the last factor is at least 1. By condition C3, the c_i are increasing in i. Since $\mathbf{s} \leq_{st} \mathbf{t}$, this implies that $\sum_{i=1}^n c_i s_i \leq \sum_{i=1}^n c_i t_i$. The result then follows by the assumption $r < 1$. $\square$

The implication (i) in Theorem 1 will not hold in general for $r < 1$, however, as shown by the following example.

Example 2. Let $n = 4$ and consider two systems with signatures, respectively, $\mathbf{u} = (0, 1/2, 1/2, 0)$ and $\boldsymbol{v} = (0, 1/3, 2/3, 0)$. Both of these signatures correspond to coherent systems [14, Table 3.2], and it is also easy to see that $\mathbf{u} \leq_{lr} \boldsymbol{v}$. Next let the cost and performance vectors be given by, respectively, $\mathbf{c} = (4, 3, 2, 1)$ and $\mathbf{h} = (4, 4, 3, 2)$, so that conditions C1 and C2 (but not C3) are satisfied. Now (10) corresponds to

$$\frac{7/2}{(5/2)^r} \leq \frac{10/3}{(7/3)^r}$$

which, however, holds if and only if $r > 0.707$ ($r = 1$ is of course guaranteed by Theorem 1). Thus the corollary can not be extended to include implication (i).

Example 1 (continued). Corollary 1 implies that the system with signature $\mathbf{t}'$ has higher performance-per-cost than $\mathbf{s}$ also for any $r < 1$. If one calculates the two sides of (10) with $\mathbf{s}$ and $\mathbf{t}'$ also for some $r > 1$, one finds that the system with signature $\mathbf{t}'$ is the better one for $r < 1.085$, but that the system with signature $\mathbf{s}$ is the better one for higher values of r. The reason for this behavior is, of course, the lower cost of the system with signature $\mathbf{s}$, which is $\sum c_i s_i = 1.025$, while the cost of the system with signature $\mathbf{t}'$ is 1.233.

4 Optimization of the Criterion Function

In this section, we reconsider the optimization of the criterion function (7) as treated in [14]. First, as a second corollary to our Theorem 1, we obtain the result of Corollary 7.1 of [14] for the case $r = 1$. We present the result and the new proof as a theorem below:

Theorem 2 (Samaniego, 2007)**.** *Consider the class of mixed n-systems. Assume that $r = 1$ in the criterion function (7), with $\mathbf{h}$ and $\mathbf{c}$ fixed. Let*

$$K^* = \left\{ k \mid k = \mathit{argmax}_i \left\{ \frac{h_i}{c_i}, i = 1, \ldots, n \right\} \right\} \tag{11}$$

Then (7) is maximized by any mixture of i-out-of-n systems with $i \in K^*$.

Proof. Let $\mathbf{s}$ be a given signature vector. Let $\{i_1, \ldots, i_n\}$ be a reordering of the components $\{1, 2, \ldots, n\}$ such that

$$\frac{h_{i_n}}{c_{i_n}} \geq \frac{h_{i_{n-1}}}{c_{i_{n-1}}} \geq \cdots \geq \frac{h_{i_1}}{c_{i_1}}$$

Let $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_n})$. Then $\mathbf{s}' \leq_{lr} (0, \ldots, 0, 1)$ and it follows from (i) of Theorem 1 that

$$\frac{\sum_{j=1}^{n} h_{i_j} s_{i_j}}{\sum_{j=1}^{n} c_{i_j} s_{i_j}} \leq \frac{h_{i_n}}{c_{i_n}}$$

Here the left hand side equals $m_1(\mathbf{s}, \mathbf{h}, \mathbf{c})$, while the right hand side equals

$$\frac{\sum_{i=1}^{n} h_i t_i}{\sum_{i=1}^{n} c_i t_i}$$

for any signature (i.e. probability) vector $\mathbf{t} = (t_1, \ldots, t_n)$ for which $t_k > 0$ only if $k \in K^*$. Since $\mathbf{s}$ was chosen arbitrarily, this proves the corollary.

$\square$

Remark 1. In view of Corollary 1 one might think that the result of Theorem 2 would hold also for $r < 1$. The following example shows, however, that this is not the case. Suppose $n = 3$ and

$$m_r(\mathbf{s}, \mathbf{h}, \mathbf{c}) = \frac{2s_1 + 3s_2 + 5.7s_3}{(s_1 + 2s_2 + 3s_3)^r}.$$

For $r = 1$, this is uniquely maximized by $\mathbf{s}_0 = (1, 0, 0)$ by Theorem 2. Now for $\mathbf{t} = (0, 0, 1)$, the criterion $m_r(\mathbf{t}, \mathbf{h}, \mathbf{c})$ equals $5.7/3^r$, while $m_r(\mathbf{s}_0, \mathbf{h}, \mathbf{c})$ equals 2 for any r. Hence for $r \leq 0.9533$, $\mathbf{t}$ gives a higher value of the criterion function than $\mathbf{s}_0$.

This suggests that for $r < 1$, one should replace the h_i/c_i in (11) by h_i/c_i^r. Now Theorem 3 below takes care of this, and in addition strengthens the corresponding result of [14, Chapter 7] for $r < 1$, which states (Theorem 7.2) that the optimal signature for $r \neq 1$ has no more than two positive entries. Our Theorem 3 shows that, when $r < 1$, the optimum is always a single i-out-of n system. The proof uses Lemma 1 which is found in the Appendix, and which will be basic in the next section.

Theorem 3. *Assume that $r < 1$ in the criterion function (7), with $\mathbf{h}$ and $\mathbf{c}$ fixed with $0 < h_1 < h_2 < \ldots < h_n$ and $0 < c_1 < c_2 < \ldots < c_n$. Let*

$$K^* = \left\{ k \mid k = argmax_i \left\{ \frac{h_i}{c_i^r}, i = 1, \ldots, n \right\} \right\}.$$

Then (7) is maximized by a system with signature vector $\mathbf{s}$ if and only if $\mathbf{s}$ puts mass 1 on one of the components $i \in K^*$, i.e., can be represented as an i-out-of-n system with $i \in K^*$.

Proof. Theorem 7.2 of [14] states that, under the requirements on $\mathbf{h}$ and $\mathbf{c}$ as given above, a signature vector maximizing (7) can be given having at most two non-zero elements. Without loss of generality we can assume that $s_3 = \cdots = s_n = 0$, and thus, denoting s_1 by s, we may consider the function

$$m(s) = \frac{h_1 s + h_2(1 - s)}{(c_1 s + c_2(1 - s))^r}$$

for $0 \leq s \leq 1$. Since $r < 1$, Lemma 1 shows that the function $m(s)$ cannot have a local maximum in the interval $(0, 1)$, and hence that only $s = 0$ or $s = 1$ are possible for its maximum. This proves the theorem. $\square$

Remark 2. It should be remarked that in Theorem 2, any *mixture* of i-out-of-n systems with $i \in K^*$ is optimal, while in Theorem 3, the optimal systems are only the ones that put mass 1 in single i-out-of-n systems with $i \in K^*$.

Example 7.1 in [14], shows that the result of Theorem 3 does not hold in general for $r > 1$. In this case, there is not necessarily a single i-out-of-n system that is optimal. The reason for the difference between the cases $r < 1$ and $r > 1$ is related to the fact that the denominator of (7) for $r < 1$ is a concave function, while for $r > 1$, it is convex. The difference between the cases $r < 1$, $r = 1$ and $r > 1$ is further studied in the next section. Note finally that the monotonicity requirements of $\mathbf{h}$ and $\mathbf{c}$ are not necessary for Theorem 3 to hold, as will be clear from Section 5.

5 Comparing Coherent Systems and their Mixtures

In this section we study the case where there is given a set of k candidate coherent n-systems, assigned for a specific task, with respective signatures $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$. The task to be considered is the search for the optimal system with respect to performance-per-cost, among all mixtures of the k coherent systems. The main result (Theorem 5) states that if $r \leq 1$, then the optimal system is always one of the original systems, i.e., no mixture of several systems can outperform all the single systems with respect to the criterion (7), while for $r > 1$ the optimal system is either a single system or a mixture of just two of the original systems.

The performance vector $\mathbf{h}$ and the cost vector $\mathbf{c}$ are assumed fixed in this section, and they will therefore be dropped in the notation of the m_r-function (7). Unless otherwise stated, we will only assume that $h_i > 0,\ c_i > 0$ for each i. Note also that although we state the theorems for given sets of *coherent* systems, the statements will hold also if we start from a set of mixed systems which are not necessarily coherent.

Consider first the case $k = 2$, and let $\boldsymbol{s}$ and $\boldsymbol{t}$ be the signature vectors of two coherent n-systems. A mixed system which selects the first system with probability p and the second with probability $1-p$ has signature vector $p\boldsymbol{s}+(1-p)\boldsymbol{t}$. The criterion function (7) for this mixture is given by

$$\begin{aligned} m_r(p\boldsymbol{s} + (1-p)\boldsymbol{t}) &= \frac{p\sum_{i=1}^n h_i s_i + (1-p)\sum_{i=1}^n h_i t_i}{\left(p\sum_{i=1}^n c_i s_i + (1-p)\sum_{i=1}^n c_i t_i\right)^r} \\ &= \frac{p\ \mathbf{h}'\mathbf{s} + (1-p)\ \mathbf{h}'\mathbf{t}}{\left(p\ \mathbf{c}'\mathbf{s} + (1-p)\ \mathbf{c}'\mathbf{t}\right)^r}. \end{aligned}$$

By using Lemma 1 (see Appendix), we have the following result:

Theorem 4. *Let there be given two coherent systems with signatures $\boldsymbol{s}$ and $\boldsymbol{t}$, respectively. If $r = 1$, then*

$$m_1(p\boldsymbol{s} + (1-p)\boldsymbol{t}) < \max\{m_1(\boldsymbol{s}), m_1(\boldsymbol{t})\} \quad \text{for all } 0 < p < 1 \tag{12}$$

when $m_1(\mathbf{s}) \neq m_1(\mathbf{t})$, while $m_1(p\boldsymbol{s}+(1-p)\boldsymbol{t})$ is constant for $0 \leq p \leq 1$ if $m_1(\mathbf{s}) = m_1(\mathbf{t})$.

If $r < 1$, then

$$m_r(p\boldsymbol{s} + (1-p)\boldsymbol{t}) < \max\{m_r(\boldsymbol{s}), m_r(\boldsymbol{t})\} \quad \text{for all } 0 < p < 1.$$

If $r > 1$, then if

$$\mathbf{h}'(\mathbf{s}-\mathbf{t})\mathbf{c}'(\mathbf{s}-\mathbf{t}) > 0 \tag{13}$$

and

$$0 < p_r \equiv \frac{\mathbf{h}'(\mathbf{s}-\mathbf{t})\mathbf{c}'\mathbf{t} - r\mathbf{h}'\mathbf{t}\mathbf{c}'(\mathbf{s}-\mathbf{t})}{(r-1)\mathbf{h}'(\mathbf{s}-\mathbf{t})\mathbf{c}'(\mathbf{s}-\mathbf{t})} < 1, \tag{14}$$

we have

$$m_r(p\boldsymbol{s} + (1-p)\boldsymbol{t}) \leq m_r(p_r\boldsymbol{s} + (1-p_r)\boldsymbol{t}) \quad \text{for all } 0 \leq p \leq 1;$$

with equality if and only if $p = p_r$.

Otherwise, if $r > 1$ *and* $\mathbf{s} \neq \mathbf{t}$, *then*

$$m_r(p\boldsymbol{s} + (1-p)\boldsymbol{t}) < \max\{m_r(\boldsymbol{s}), m_r(\boldsymbol{t})\} \quad \textit{for all } 0 < p < 1.$$

Remark 3. Suppose that $\mathbf{s} \leq_{st} \mathbf{t}$. It was demonstrated in Example 1 that this does not necessarily imply that $m_r(\mathbf{s}) \leq m_r(\mathbf{t})$, even when $r = 1$. It should be noted, however, that this fact does not contradict (12), which in Example 1 would merely state that, when $r = 1$, no mixture of $\mathbf{s}$ and $\mathbf{t}$ can beat $\mathbf{s}$ with the given $\mathbf{h}$ and $\mathbf{c}$.

Remark 4. Note that if $\mathbf{s} \leq_{st} \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$, while both $\mathbf{h}$ and $\mathbf{c}$ are increasing, then $\mathbf{h}'(\mathbf{s} - \mathbf{t})$ and $\mathbf{c}'(\mathbf{s} - \mathbf{t})$ are both negative (see [14, Proposition 7.1]). Hence (13) holds, so if $r > 1$, there might be a strict mixture of the two systems which is optimal. In Example 1, a numerical search shows that this is not the case for the given $\mathbf{s}$ and $\mathbf{t}$ for any $r > 1$. Let therefore $\mathbf{t}$ be replaced by the $\mathbf{t}'$ which is given in the example. Then $\mathbf{s} \leq_{st} \mathbf{t}'$, and a calculation shows that for $r = 1.08$ we have $p_r = 0.25$ from (14). A strict mixture of $\mathbf{s}$ and $\mathbf{t}'$ is hence optimal. In fact we get

$$m_{1.08}(0.25\boldsymbol{s} + 0.75\boldsymbol{t}') = 0.73094$$

while $m_{1.08}(\boldsymbol{s}) = 0.73026$ and $m_{1.08}(\boldsymbol{t}') = 0.73088$.

Suppose now that there are $k > 2$ systems at hand. Theorem 5 below extends the results for $k = 2$ in Theorem 4 to this case. In fact, the conclusion for $r \leq 1$ extends easily by induction. When $r > 1$, the extension of Theorem 4 to $k > 2$ does at first sight look rather unclear. It turns out, however, that the optimization of the criterion function for $k > 2$ can be done by considering pairs of systems, which hence enables the application of Theorem 4. The proof of Theorem 5 for the case $r > 1$ is given in the Appendix.

Theorem 5. *Let there be given* k *coherent* n*-systems with signatures, respectively,* $\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_k$. *Consider the mixed system with signature* $\sum_{j=1}^k p_j \boldsymbol{s}_j$, *where* $\sum_{j=1}^k p_j = 1$.

If $r = 1$, *then*

$$m_1\left(\sum_{j=1}^k p_j \boldsymbol{s}_j\right) \leq \max\{m_1(\boldsymbol{s}_1), \cdots, m_1(\boldsymbol{s}_k)\},$$

with equality if and only if $p_j > 0$ *implies* $m_1(\mathbf{s}_j) = \max_{j'} m_1(\mathbf{s}_{j'})$.

If $r < 1$, *then*

$$m_r\left(\sum_{j=1}^k p_j \boldsymbol{s}_j\right) \leq \max\{m_r(\boldsymbol{s}_1), \cdots, m_r(\boldsymbol{s}_k)\}.$$

with equality if and only if $p_j = 1$ *for some* j *with* $m_r(\mathbf{s}_j) = \max_{j'} m_r(\mathbf{s}_{j'})$.

If $r > 1$, *then the criterion function* $m_r\left(\sum_{j=1}^k p_j \boldsymbol{s}_j\right)$ *is maximized by a vector* $(p_1, \ldots, p_k)$ *for which* $p_j + p_{j'} = 1$ *for some* $j, j' \in \{1, 2, \ldots, k\}$.

Remark 5. For $r \leq 1$, Theorem 5 tells us that the maximum performance-per-cost among all mixtures of a set of coherent n-systems is always attained by a single system. (It is seen, moreover, that in the case $r = 1$, systems with the same value of the criterion function may be mixed withouot losing the optimal property. This is not the case if $r < 1$.)

When $r > 1$, the maximization can be done by considering all the $\binom{k}{2}$ different pairs of systems. The optimal mixture is then given by the pair that gives the maximal performance-per-cost, based on the result of Theorem 4. Section 6 provides an illustration.

Remark 6. If $k = n$, and the set of coherent systems in question consists of the i-out-of-n systems for $i = 1, \ldots, n$, then the mixed systems considered in Theorem 5 correspond exactly to the collection of all mixed n-systems, see Section 2.1. It is then not difficult to see that Theorem 5 implies the previous results Theorem 2 and Theorem 3, as well as the result of [14, Theorem 7.2] which states that the maximum of the criterion function is attained by a signature vector with at most two non-zero entries (see the proof of Theorem 3). It should be remarked, however, that the results of the present section are proven under the weaker assumption that the vectors $\mathbf{h}$ and $\mathbf{c}$ have positive entries, while strict monotonicity of the entries of $\mathbf{h}$ and $\mathbf{c}$ are assumed in [14].

6 Application

As mentioned in the Introduction, Dupuits et al. [8] considered economic optimization of coastal flood defence systems. Such systems can consist of multiple lines of defence. The cited paper studied in particular systems consisting of a front and rear defence with the aim of finding the economically optimal heights of the two barriers, when taking into account both risk cost and investment cost.

Motivated by this, we will redefine the problem in the setting of the present paper to that of choosing between different system configurations under certain constraints. In the present example we consider three systems, each consisting of three components, but which are structured differently. The three systems, named A, B, C, are illustrated in Figure 1. Their signature vectors are, respectively, $\mathbf{s}_A = (1/3, 2/3, 0)$, $\mathbf{s}_B = (0, 1, 0)$, $\mathbf{s}_C = (0, 2/3, 1/3)$, which are seen to be stochastically ordered as $\mathbf{s}_A \geq_{st} \mathbf{s}_B \geq_{st} \mathbf{s}_C$.

The choice between these structures, or possibly mixtures of them (if $r > 1$), will be based on a performance vector $\mathbf{h}$ and cost vector $\mathbf{c}$. In their application, [8] seeks to minimize total costs, consisting of risk costs plus investment costs, with precise definitions given in the paper. In our setting, this would correspond to letting *minus risk cost* serve as the performance measure, while investment cost corresponds to our cost.

For ease of exposition, we shall in our example of choosing between the systems A, B and C, let the performance vector $\mathbf{h}$ be defined in the same manner as in Example 1, i.e., by the expected values of the order statistics of the unit exponential distribution. Letting $n = 3$ in (9) we obtain the vector $\mathbf{h} = (1/3, 5/6, 11/6)$. Somewhat arbitrarily, we define the cost vector $\mathbf{c}$ by letting $c_i = i$ for $i = 1, 2, 3$.

With the given values, we maximized the criterion function

$$m_r(p_A\mathbf{s}_A + p_B\mathbf{s}_B + p_C\mathbf{s}_C)$$

with respect to the mixture probability vector (p_A, p_B, p_C), for r in steps of 0.1 from 0 and upwards, using Theorem 5. Hence, for $r \leq 1$, the optimum was simply obtained as the maximum of $m_r(\mathbf{s}_A)$, $m_r(\mathbf{s}_B)$ and $m_r(\mathbf{s}_C)$, while for $r > 1$ we had to maximize over each system plus all pairs of systems. Table 1 shows the resulting optimal systems for each value of r. It is seen that for the smallest r, i.e., when cost is less emphasized, the optimal system is the one with the highest performance (stochastically largest signature), C. As r increases, and cost hence gets a larger impact, the less costly system, A, will be more and more part of the optimal solution. The example shows, moreover, that for $r > 1$, the optimal system needs not be a single system, in contrast to the case $r \leq 1$. Thus the special treatment of the case $r > 1$ in Theorem 5 is needed. It is also noticeable from Table 1 that system B is never part of the optimal solution.

r	*Optimal system*
0-1.5	C
1.6	0.389 A + 0.611 C
1.7	0.667 A + 0.333 C
1.8	0.875 A + 0.6 C
1.9 –	A

Table 1: Optimal system or mixed system

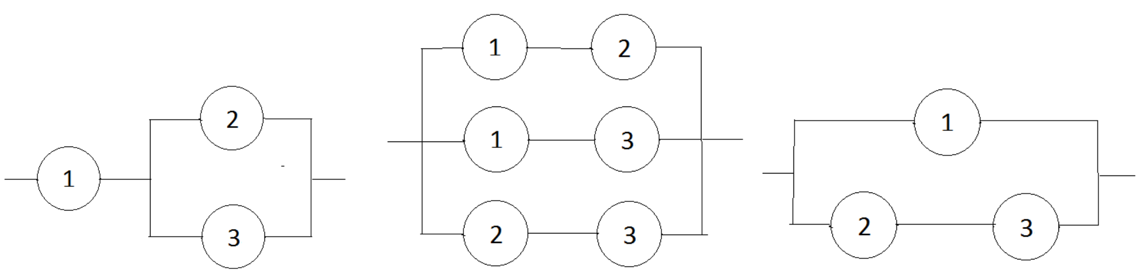

Figure 1: Systems A (left), B (middle) and C (right)

7 Concluding Remarks

In the present paper we have obtained new results for the reliability economics framework introduced by Dugas and Samaniego [7]. In particular, we have obtained comparison results for the performance-per-cost measure for different stochastic orders. Our Theorem 1 may throw new light on both the performance-per cost measure and on the nature of the various orders that are considered.

Next we have studied, under various settings, the optimization of the performance-per-cost function of [7]. In particular we have improved the optimization result of [7] for $r < 1$.

We have further studied the problem of maximizing the criterion function for mixtures of arbitrary sets of coherent systems. Using the powerful, but simple, Lemma 1, we have proved in particular that no mixture of coherent systems can have better performance-per-cost than the best single system in the set when $r \leq 1$. By an example we have shown that such a result does not hold in general when $r > 1$. The optimal mixture may, however, in this case be described by a mixture of at most two systems, and an explicit formula is given for the mixture probabilities.

In our study, we have restricted attention to comparison of systems with the same number of components, that is, with signature vectors of the same dimension. For comparison of systems with different sizes, [14, page 32] shows how to "convert" the smaller of two systems into an equivalent system of the same size as the larger one. This conversion allows, for example, the extension of results like (4) to comparison of systems of different sizes in a fairly straightforward manner. This suggests, therefore, that the comparison results of the present paper can be extended in a similar manner. Some work in this connection was done in [11]. In the setting of the present paper, [11] considered the problem of whether one may increase performance-per-cost by building smaller and hence less costly systems, i.e., systems with fewer components.

In this paper we have restricted the attention to coherent systems with independent and identically distributed component lifetimes. There is by now an increasing literature where the study of coherent systems in the i.i.d. case is generalized in various directions. A successful extension of the traditional signature of systems as described in the present article, is the so-called survival signature introduced by Coolen and Coolen-Maturi [5], which allows several (finitely many) types of components. Survival signatures have recently been applied in various directions. For example, Aslett et al. [1] presented an application to networks; Huang et al. [10] used the survival signature in an analysis of phased mission systems, while Samaniego and Navarro [15] investigated methods for comparison of coherent systems with heterogeneous components. In another direction, Navarro et al. [12] introduced the notion of identically distributed, but dependent, component lifetimes (d.i.d.), with dependency modeled by a copula.

It would certainly be of interest to adapt the present work to these new and more general approaches, and this is an obvious topic for further research.

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Appendix

Proof of Theorem 1.

We prove Theorem 1 by first stating and proving the following theorem, which follows from results presented in the monograph by Shaked and Shanthikumar [16].

Theorem 6. *Let X and Y be two independent random variables and let $\alpha(\cdot)$ and $\beta(\cdot)$ be arbitrary real-valued functions. Consider the following conditions:*

D1: $\beta(x) \geq 0$ *for all* x

D2: $\alpha(x)/\beta(x)$ *is increasing in* x

D3: $\beta(x)$ *is increasing in* x

D4: $\alpha(y)\beta(x) - \alpha(x)\beta(y)$ *is decreasing in* x *on* $\{x \leq y\}$

D5: $E[\alpha(X)]E[\beta(Y)] \leq E[\alpha(Y)]E[\beta(X)]$ *(assuming that the expectations exist)*

The following implications hold:

(i) *If $X \leq_{lr} Y$, then D1,D2 together imply D5.*

(ii) *If $X \leq_{hr} Y$, then D1,D2,D3 together imply D5.*

(iii) *If $X \leq_{st} Y$, then D1,D2,D3,D4 together imply D5.*

Proof. Statement (ii) is part of Theorem 1.B.12 in [16]. In their proof, (ii) is obtained by application of their Theorem 1.B.10 to a particular pair of functions ϕ_1, ϕ_2. The results (i) and (iii) are not stated in [16], but it is straightforward to check that (i) is obtained by using the same pair of functions as in their Theorem 1.C.22, while (iii) is obtained similarly from their Theorem 1.A.10.

□

Remark: Regarding (ii) above, Theorem 1.B.12 in [16] in fact shows that if D5 holds for all functions α, β satisfying D1,D2,D3, then $X \leq_{hr} Y$.

Proof of Theorem 1: We consider the special case of Theorem 6 where X, Y are defined on $\{1, \ldots, n\}$ with probability distributions respectively given by the signature vectors $\mathbf{s}$ and $\mathbf{t}$. Further, we let the functions α and β be defined on $\{1, \ldots, n\}$ with values, respectively, $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$. Then it is seen that condition D5 can be written as

$$\frac{\sum_{i=1}^n h_i s_i}{\sum_{i=1}^n c_i s_i} \leq \frac{\sum_{i=1}^n h_i t_i}{\sum_{i=1}^n c_i t_i},$$

which is (8).

The conditions D1 and D3 now become, respectively, $c_i \geq 0$ and c_i is increasing in i, which are C1 and C3. Further, D2 can be written

$$\frac{h_1}{c_1} \leq \frac{h_2}{c_2} \leq \cdots \leq \frac{h_n}{c_n}, \tag{15}$$

which is C2.

Next, D4 states that $h_j c_i - h_i c_j$ is decreasing in i for $i \leq j$, which means that $h_j c_i - h_i c_j \geq h_j c_{i+1} - h_{i+1} c_j$ or, equivalently,

$$\frac{h_{i+1} - h_i}{c_{i+1} - c_i} \geq \frac{h_j}{c_j} \text{ for } j = 2, \ldots, n; \ i = 1, 2, \ldots, j-1.$$

Thus it follows from (15) that if we assume both D3 and D4, then in addition to (15) we have the condition

$$\frac{h_{i+1} - h_i}{c_{i+1} - c_i} \geq \frac{h_n}{c_n} \text{ for } i = 1, 2, \ldots, n-1,$$

which is C4 in Theorem 1.

Lemma 1 and its proof

Lemma 1. *Let A, B, C and D be positive numbers and, for $0 \leq p \leq 1$ and $r > 0$, let*

$$H_r(p) = \frac{pA + (1-p)B}{(pC + (1-p)D)^r}. \tag{16}$$

If $r = 1$, then

$$H_1(p) < \max\{H_1(0), H_1(1)\} \quad \textit{for all } 0 < p < 1$$

when $H_1(0) \neq H_1(1)$, while $H_1(p)$ is constant for $0 \leq p \leq 1$ when $H_1(0) = H_1(1)$.

If $r < 1$, then

$$H_r(p) < \max\{H_r(0), H_r(1)\} \quad \textit{for all } 0 < p < 1. \tag{17}$$

If $r > 1$, then if

$$(A-B)(C-D) > 0$$

and

$$0 < p_r \equiv \frac{(A-B)D - rB(C-D)}{(r-1)(A-B)(C-D)} < 1, \tag{18}$$

we have

$$H_r(p) \le H_r(p_r), \quad \textit{for all } 0 \le p \le 1, \tag{19}$$

with equality if and only if $p = p_r$.
Otherwise, if $r > 1$,

$$H_r(p) < \max\{H_r(0), H_r(1)\} \quad \textit{for all } 0 < p < 1, \tag{20}$$

unless $A = B$ *and* $C = D$ *in which case* $H_r(p)$ *is constant in* $0 \le p \le 1$.

Proof. Let $H_r(p)$ be given by (16). The derivative of $H_r(p)$ at a fixed $p \in [0,1]$ is

$$\begin{aligned} H_r'(p) &= \frac{(A-B)\,(pC+(1-p)D)^r - r(pA+(1-p)B)(C-D)\,(pC+(1-p)D)^{r-1}}{(pC+(1-p)D)^{2r}} \\ &= \frac{(A-B)\,(pC+(1-p)D) - r(pA+(1-p)B)(C-D)}{(pC+(1-p)D)^{r+1}} \\ &= \frac{(1-r)(A-B)(C-D)p + (AD - rBC - (1-r)BD)}{(pC+(1-p)D)^{r+1}}. \end{aligned}$$

The sign of $H_r'(p)$ for $0 \le p \le 1$ thus equals the sign of the linear function

$$p \mapsto (1-r)(A-B)(C-D)p + (AD - rBC - (1-r)BD). \tag{21}$$

Thus, if $(1-r)(A-B)(C-D) = 0$, $H_r(p)$ is either increasing or decreasing or constant. This means that, unless $H_r(p)$ is constant, we have

$$H_r(p) < \max\{H_r(0), H_r(1)\} \text{ for all } 0 < p < 1.$$

This proves the lemma for the case when $p = 1$.

If $(1-r)(A-B)(C-D) \ne 0$, then an interior extreme point of the function $H_r(p)$ will necessarily be at p_r as given by (18). This is a maximum point only if

$$H_r'(0) > 0 \text{ and } H_r'(1) < 0. \tag{22}$$

In such a case, putting $p = 0$ and $p = 1$ in (21), it is necessary that

$$(1-r)(A-B)(C-D) < 0. \tag{23}$$

This proves (19), since (23) implies that, if $r > 1$, an interior point can be a maximum only if $(A-B)(C-D) > 0$. Also, (20) follows from this.

Assume finally that $r < 1$. Note first that (22) implies, using (21), that

$$\begin{cases} AD - rBC - (1-r)BD > 0, \\ (1-r)(A-B)(C-D) + AD - rBC - (1-r)BD < 0, \end{cases}$$

or, by reordering terms, (22) implies that

$$\begin{cases} (A-B)D - r(C-D)B > 0, \\ (A-B)C - r(C-D)A < 0. \end{cases} \tag{24}$$

On the other hand, it follows from (23) that, when $r < 1$, there can be an interior maximum point only if

$$(A-B)(C-D) < 0. \tag{25}$$

From this, if $C-D > 0$, we must have $A-B < 0$. But this contradicts the first inequality in (24). Similarly, if $C - D < 0$, then $A - B > 0$ by (25), and the second inequality in (24) is violated. It follows that no interior point can be a maximum in the case $r < 1$, so (17) holds. □

Proof of Theorem 5 for the case $r > 1$

Proof. Let $\mathbf{s}_j = (s_{1j}, s_{2j}, \ldots, s_{nj})$ for $j = 1, 2, \ldots, k$. Then by (7) we have

$$\begin{aligned} m_r\left(\sum_{j=1}^{k} p_j \mathbf{s}_j\right) &= \frac{\sum_{j=1}^k p_j \sum_{i=1}^n h_i s_{ij}}{\left(\sum_{j=1}^k p_j \sum_{i=1}^n c_i s_{ij}\right)^r} = \frac{\sum_{j=1}^k p_j \alpha_j}{\left(\sum_{j=1}^k p_j \beta_j\right)^r} \\ &= \frac{\alpha_k + \sum_{j=1}^{k-1}(\alpha_j - \alpha_k)p_j}{\left(\beta_k + \sum_{j=1}^{k-1}(\beta_j - \beta_k)p_j\right)^r} = \frac{\alpha_k + \sum_{j=1}^{k-1} \tilde{\alpha}_j p_j}{\left(\beta_k + \sum_{j=1}^{k-1} \tilde{\beta}_j p_j\right)^r}. \end{aligned} \tag{26}$$

Here $\alpha_j = \sum_{i=1}^n h_i s_{ij}$, $\beta_j = \sum_{i=1}^n c_i s_{ij}$, $\tilde{\alpha}_j = \alpha_j - \alpha_k$ and $\tilde{\beta}_j = \beta_j - \beta_k$ for $j = 1, \ldots, k-1$. We consider the task of maximizing (26) on the convex set R of $\boldsymbol{p} = (p_1, \ldots, p_{k-1})$ with each $p_i \geq 0$ and $\sum_{j=1}^{k-1} p_j \leq 1$.

Let us first exclude the case when $c_i = a h_i$ for $i = 1, \ldots, n$ for some constant $a > 0$. Then clearly $m_r\left(\sum_{j=1}^k p_j \mathbf{s}_j\right)$ is constant, so the result of the theorem is trivially satisfied.

We now claim that the maximum of (26) occurs on the boundary of R. The following argument is similar to the one of [11, page 346]. Assume for contradiction that the maximum occurs at an interior point $\hat{\boldsymbol{p}}$ of R. Consider the two hyperplanes of $(p_1, \ldots, p_{k-1})$ for which, respectively, the linear functions $\sum_{i=1}^{n-1} \tilde{\alpha}_j p_j$ and $\sum_{i=1}^{n-1} \tilde{\beta}_j p_j$ have the same value as in the optimum point $\hat{\boldsymbol{p}}$. (These hyperplanes are not parallel when the proportionality case considered above has been excluded). Let N be an open neighborhood of $\hat{\boldsymbol{p}}$ which is included in R, and consider the intersection N_0 of N and the above hyperplane defined by the $\tilde{\beta}_j$. On this set, the denominator of (26) is constant. The set N_0 will, however, contain points on both sides of the hyperplane defined by the $\tilde{\alpha}_j$, meaning that $\sum_{j=1}^{k-1} \tilde{\alpha}_j p_j$ on N_0 will have values both larger and smaller than its value at $\hat{\boldsymbol{p}}$. But then

(26) will in N_0 take values larger than the value at $\hat{\boldsymbol{p}}$, which gives a contradiction since the maximum is assumed to be at $\hat{\boldsymbol{p}}$. Thus the maximum point of (26) is a boundary point of R.

It follows that the maximum of $m_r\left(\sum_{j=1}^{k} p_j \mathbf{s}_j\right)$ is obtained either for a $\boldsymbol{p}$ for which $p_j = 0$ for some $j \in \{1, 2, \ldots, k-1\}$ or for a $\boldsymbol{p}$ for which $\sum_{j=1}^{k-1} p_j = 1$, in which case $p_k = 0$. Hence the problem of maximizing the criterion function is reduced to a case of $k-1$ systems. The above approach can then be continued by first checking the possible proportionality of the remaining h_i and c_i, in which case we are done, and then in case of non-proportionality, reducing the set of systems by one. This process will end when only two systems are left, in which case we are done and Theorem 4 applies. $\square$