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Fredrik Arbo Høeg

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Thesis for the Degree of
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Faculty of Information Technology and Electrical
Engineering
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Trondheim, September 2020

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# Preface

My interest and enjoyment of mathematics started early. After nine years in Trondheim at NTNU this has further developed and there are many to thank for helping me complete my studies.

First of all, I would like to thank my supervisor Peter Lindqvist, who has helped me for many years during my studies. For the mathematics, he somehow always manages to find the interesting calculations that intrigue me. He has always been supportive and has thought me many things both personally and in mathematics.

During my third year as a PhD student I got to work with Eero Ruosteenoja who was staying in Trondheim for a research period of one year. We shared similar view of mathematics and it was a pleasure to work with him. I would like to thank him for that.

There are many other colleagues that I am grateful to, and here I mention a few of them. Karl Kristian Brustad and Erik Lindgren both invited me to give a talk at their respectful universities. My co-supervisor Katrin Grunert saw something in me and gave me the opportunity to teach early in my studies, which is something that I will do more of in the future. Amal Attouchi helped and gave me very useful input on my papers.

I would also like to thank my fellow PhD students who helped me with mathematics and for making my years in Trondheim enjoyable. The same goes for my friends from Fysmat and my friends from Larvik.

I would like to thank my parents, my sisters Annikken and Camilla (with family) for encouraging me to start my PhD and for being supporting along the way.

Finally, I would like to thank my girlfriend Ingvild. She brightens my days with laughter, encouraging words and a terrible sense of humor. Thank you for being there for me.

Fredrik Arbo Høeg, Trondheim, June 3, 2020

# Contents

# Notation

Throughout the thesis, we will use the following notation without reference. Here, $u, v$ are functions, $a, b$ are vectors, $A, B$ are matrices with elements $a_{ij}, b_{ij}$ and $\Omega$ is a domain.

- $\Omega_T = \Omega \times (0, T)$.
- $B_r(x)$ is the ball centered at $x$ with radius $r$.
- $u_t = \frac{\partial u}{\partial t}$.
- $u_i = u_{x_i} = \frac{\partial u}{\partial x_i}$.
- $\nabla u = (u_{x_1}, u_{x_2}, ..., u_{x_n}) \in \mathbb{R}^n$.
- $D^2 u$ is the *Hessian matrix* of $u$ with elements

$$(D^2 u)_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

  and eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. We will often denote $\lambda_1 = \lambda_{\min}$ and $\lambda_n = \lambda_{\max}$.
- $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.
- $u \in C^m(\Omega)$ if $u$ and its partial derivatives up to order $m$ are continuous in $\Omega$.
- $u \in L^P(\Omega)$ if $\int_\Omega |u|^p \, dx$ is finite.
- $||u||_{p,\Omega} = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}$.
- $u \in L^P_{\text{loc}}(\Omega)$ if $u \in L^p(K)$ for each compact $K \subset \Omega$.
- $u = o(v)$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{u(x)}{v(x)} = 0$.
- $⨍_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega u \, dx$.
- The divergence of a vector valued function $F$ is denoted by

$$\text{div} F = \frac{\partial F_1}{\partial x_1} + ... + \frac{\partial F}{\partial x_n}.$$

- $|a| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$.

- $||a||_\infty = \max_i |a_i|$.
- $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + ... + a_n b_n$.
- $\mathrm{dist}(a, b) = |a - b|$.
- $\mathrm{dist}(a, \Omega) = \min_{y \in \Omega} |a - y|$.
- $A \in S^n$ if $A$ is symmetric, $a_{ij} = a_{ji}$.
- $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$.
- $A \le B$ if $\langle x, Ax \rangle \le \langle x, Bx \rangle$ for any $x \in \mathbb{R}^n$.
- $|A|^2 = \sum_{i,j=1}^n a_{ij}^2$.
- The expected value of a random variable $X$ is denoted $\mathbb{E}(X)$.

# Introduction

The Laplace equation,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial u^2}{\partial x_n^2} = 0$$

is a widely studied linear second order partial differential equation. It is the first equation one encounters when studying partial differential equations of second order. Solutions to Laplace's equation are called *harmonic functions* and they occur in many branches of physics, for example in electric and gravitational potentials. The parabolic version,

$$\frac{\partial u}{\partial t} = \Delta u$$

is used in the study of heat conduction. It is called the heat equation and was studied already in the early 1800's by Joseph Fourier.

Laplace's equation in a domain $\Omega \in \mathbb{R}^n$ is the Euler-Lagrange equation of the Dirichlet integral

$$\int_\Omega |\nabla u|^2 \, dx.$$

If we instead look at the variational integral

$$\int_\Omega |\nabla u|^p \, dx, \quad 1 \le p \le \infty,$$

the Euler-Lagrange equation is the $p$-Laplace equation. It can be written

$$\Delta_p u = \operatorname{div}\left( |\nabla u|^{p-2} \nabla u \right) = 0, \quad 1 \le p \le \infty. \tag{1.1}$$

In contrast to Laplace's equation, it is nonlinear. Its solutions are called $p$-harmonic functions. The equation is *singular* when $p \in [1,2)$ and *degenerate* for $p > 2$. Due to this, solutions to the above equation are not always smooth.

However, the equation is in divergence form, which allows us to define a particular notion of a weak solution. Namely, we say that $u$ is a weak solution of equation (2.5) in a domain $\Omega \subset \mathbb{R}^n$ if

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = 0$$

for all smooth test functions $\phi$ with compact support in $\Omega$.

The normalized $p$-Laplace equation

$$\Delta_p^N u = |\nabla u|^{2-p} \Delta_p u = |\nabla u|^{2-p} \mathrm{div}\left( |\nabla u|^{p-2} \nabla u \right) = 0$$

where $p \in [1, \infty]$, arises in game theory [MPR],[PS] and is used in image processing [KD]. The word "normalized" is perhaps more visible if we write out the divergence,

$$\Delta_p^N u = \Delta u + (p-2) \left\langle \frac{\nabla u}{|\nabla u|}, D^2 u \frac{\nabla u}{|\nabla u|} \right\rangle \equiv \Delta u + (p-2) \Delta_\infty^N u = 0$$

The equation is singular unless $p = 2$. When $p = 2$ it is the Laplace operator. For $p > 1$ the equation is *uniformly elliptic*.

The main difference to the ordinary $p$-Laplace equation is that the equation is no longer in divergence form. Again, solutions may not be smooth, which is why we need a different notion of what it means to be a solution. We use the *viscosity* solutions.

After the game interpretation of the equation was made it got more attention. We refer to [JS] for Hölder gradient estimates and [APR] for $C^{1,\alpha}$ regularity of viscosity solutions.

The Dominative $p$-Laplace equation,

$$\mathcal{D}_p u = \Delta u + (p-2) \lambda_{\max}(D^2 u) = 0,$$

for $p \geq 2$, was introduced by Brustad in [B] where he used the equation to explain a superposition principle for $p$-superharmonic functions. See [CZ] and [LM] for more about this property. The equation has a stochastic game associated to it, which was studied first in [BLM] and later in [HR]. Here, we also use viscosity solutions.

Throughout the thesis, we discuss properties of viscosity solutions to the normalized and the Dominative $p$-Laplace equations. The main focus for the normalized $p$-Laplace equation will be the regularity of the time derivative, [HL]. Finally, we discuss the game associated to the Dominative $p$-Laplace equation [HR] and a particular concavity problem for the same equation [H].

## 1.1 The Normalized $p$-Laplace equation

There has been a growing interest on the properties of the normalized $p$-Laplacian over the last ten years. One of the reasons is that it can be used as a model to describe a stochastic game with two players. For the equation

$$\Delta_p^N u = \Delta u + (p-2)\left\langle \frac{\nabla u}{|\nabla u|}, D^2 u \frac{\nabla u}{|\nabla u|} \right\rangle = 0,$$

we see that a problem arises when the gradient $\nabla u$ vanishes. As mentioned earlier, the equation is not in divergence form, which is why we use *viscosity solutions* as a notion of a weak solution to the equation. See Appendix A for an overview of viscosity solutions.

The equation behaves differently when $p$ varies. For $p = 2$, the operator is reduced to the linear Laplace operator. For the equation $\Delta_p^N u = 0$, we may divide by $p$ and send $p \to \infty$ to obtain the infinity Laplace equation,

$$\Delta_\infty u = \left\langle \nabla u, D^2 u \nabla u \right\rangle = 0.$$

The equation was derived by Aronsson [A] in 1967. The function

$$u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$$

is a viscosity solution to the problem in two dimensions, but note that some of the second derivatives do not exist along the axes. The equation also describes a two player Tug-of-war game, see [PSSW].

For $p = 1$, the equation becomes

$$\Delta_1^N u = |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = -H|\nabla u| = 0$$

where $H$ is the mean curvature of the level sets of the function $u$. The mean curvature flow equation,

$$u_t = |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

is the parabolic normalized 1-Laplace equation. We follow the level set of a function $u$,

$$\Gamma_t = \{x \in \mathbb{R} : u(x, t) = 0\}$$

which has an inward pointing normal $\nu$ provided $\nabla u \neq 0$. Each point $x \in \Gamma_t$ is required to move according to the rule

$$\frac{dx}{dt} = H\nu = -\frac{\nabla u}{|\nabla u|} \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Since $\frac{\partial}{\partial t} u(x(t), t) = 0$, the equation for mean curvature flow is obtained. The equation is geometric, which means that if $u$ is a solution to the equation, then any reasonable function $f(u)$ is also a solution.

In the paper [Bra], Brakke studied motion of grain boundaries, in which he introduced motion by mean curvature for surfaces. Other physical phenomena that can be described by mean curvature flow are surface tension, horizons of black holes and soap films stretched across a wire frame. Using methods from differential geometry, Huisken [Hui] showed that convex surfaces in $\mathbb{R}^3$ remain convex under the mean curvature flow. Evans and Spruck [ES] used the level set formulation to prove uniqueness of viscosity solutions.

The fundamental solution to the equation $\Delta_p^N u = 0$ is

$$u(x) = \begin{cases} -\frac{p-1}{p-n}|x|^{\frac{p-n}{p-1}}, & \text{if } p \geq n \\ -\log|x|, & \text{if } p = n \\ |x|, & \text{if } p = \infty. \end{cases}$$

For the parabolic case, $u_t = \Delta_p^N u$, the ansatz $u(x,t) = t^\alpha u(\frac{|x|^2}{t})$ for some constant $\alpha$, gives an ordinary differential equation which can be solved. The solution is, for $p > 1$,

$$u(x,t) = t^{-\frac{n+p-2}{2(p-1)}} \exp\left\{-\frac{|x|^2}{4(p-1)t}\right\}. \tag{1.2}$$

In this formula, one may plug in $p = 2$ to discover the Heat kernel. Note that

$$\int_{\mathbb{R}^n} t^{-\frac{n+p-2}{2(p-1)}} \exp\left\{-\frac{|x|^2}{4(p-1)t}\right\} dx = (4(p-1)\pi)^{\frac{n}{2}} t^{\frac{(p-2)(n-1)}{2(p-1)}}.$$

This is independent of time if $p = 2$ or $n = 1$. In these cases, the solution can be written up in $\mathbb{R}^n$ given an initial data $u_0(x)$. The two cases correspond to the heat equation. For $p = 1$, the solutions are many due to the equation being geometric, and we here list a few of them:

$$|x|^2 + 4t, \quad \exp\left\{|x|^2 + 4t\right\}, \quad e^{x_1}, \quad \cosh\{x_1\}, \quad \cosh\{|x|^2 + 4t\}.$$

Finally, we mention another type of solution to the equations $\Delta_p^N u = 0$ and $u_t = \Delta_p^N u$, namely the mean value solutions. They are useful for studying the qualitative properties of the equation, and the underlying stochastic game.

Let

$$\fint_{B_\epsilon(x)} u(y)\, dy = \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y)\, dy$$

denote the average of $u$ over the ball $B_\epsilon(x)$. It turns out that if $u$ is a solution to $\Delta_p^N u = 0$ with non vanishing gradient in a domain $\Omega$, then

$$u(x) = \frac{n+2}{p+n} \fint_{B_\epsilon(x)} u(y)\,dy + \frac{p-2}{2(p+n)} \left\{ \max_{\overline{B}_\epsilon(x)} u + \min_{\overline{B}_\epsilon(x)} u \right\} + o(\epsilon^2).$$

If $p = 2$, we rediscover the mean value property for harmonic functions. The calculation is given in Appendix B. Similarly, for the parabolic case in a scaled form, $2(n+p)u_t = \Delta_p^N u$, the solution satisfies

$$\begin{aligned} u(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\epsilon(x)} u(y, t-\epsilon^2)\,dy \\ & + \frac{p-2}{2(p+n)} \left( \max_{y \in \overline{B}_\epsilon(x)} u(y, t-\epsilon^2) + \min_{y \in \overline{B}_\epsilon(x)} u(y, t-\epsilon^2) \right) + o(\epsilon^2). \end{aligned}$$

Note that the constants add up to 1. If $p \geq 2$, they are in fact probabilities in the stochastic game. A similar result for viscosity solutions can be found in [MPR].

## 1.2 The Dominative $p$-Laplace equation

The Dominative $p$-Laplace equation

$$\mathcal{D}_p u = \Delta u + (p-2)\lambda_{\max}(D^2 u) = 0,$$

was introduced to explain a superposition principle for superharmonic functions. The operator is sublinear and convex. To the naked eye, the equation may seem easy to handle compared to the normalized $p$-Laplace equation. However, the equation is nonlinear in the second derivatives. [1]

The Dominative $p$-Laplace equation is however closely related to the normalized $p$-Laplacian,

$$\Delta_p^N u = \Delta u + (p-2) \left\langle \frac{\nabla u}{|\nabla u|}, D^2 u \frac{\nabla u}{|\nabla u|} \right\rangle \leq \Delta u + (p-2)\lambda_{\max}(D^2 u) = \mathcal{D}_p u.$$

in the sense that it dominates the normalized $p$-Laplacian provided $\nabla u \neq 0$. For viscosity solutions, this means that if $u$ is a viscosity subsolution to $-\Delta_p^N u = 1$, it is also a viscosity subsolution to $-\mathcal{D}_p u = 1$.

[1] For $n = 2$, the equation reads

$$\mathcal{D}_p u = \frac{p}{2}\Delta u + \frac{p-2}{2}\sqrt{\left(u_{xx} - u_{yy}\right)^2 + 4u_{xy}^2} = 0.$$

They are also connected in the sense that radial solutions are the same. For $u(x) = u(|x|)$,

$$\lambda_{\max}(D^2u) = u_{rr} = \left\langle \frac{\nabla u}{|\nabla u|}, D^2u\frac{\nabla u}{|\nabla u|} \right\rangle.$$

The radial solutions are therefore the same as those listed in section 1.1.

The underlying stochastic game is a one-player game. Here, a token is placed at $x_0$ inside a domain. Each turn, the controller tosses a biased coin with probabilities $\alpha$ and $\beta$. With probability $\beta$, the token will move according to uniform probability density. With probability $\alpha$, the controller chooses a unit vector $\sigma$ and moves the token to either $x_0 + \epsilon\sigma$ or $x_0 - \epsilon\sigma$ with equal probability. Here $\epsilon > 0$ is a given small number. The game stops when the token reaches the boundary, and the controller is paid an amount described by the boundary data. The value function $u$, or the expected income for the controller, can be written

$$u(x) = \beta \fint_{B_\epsilon(x)} u(y)\,dy + \alpha \sup_{|\sigma|=1}\left[\frac{u(x+\epsilon\sigma) + u(x-\epsilon\sigma)}{2}\right] + o(\epsilon^2).$$

We refer to [HR] and [BLM] for more investigation of this game. For equations involving other eigenvalues of the Hessian matrix, see [BER].

## 1.3 Summary of papers

The scientific contribution of this thesis is presented in the following papers. No alterations to the scientific content has been made, however the layout has been changed to fit the thesis format.

**Paper 1: Regularity of solutions of the parabolic normalized $p$-Laplace equation**

*Fredrik Arbo Høeg and Peter Lindqvist*
Published in Advances in Nonlinear Analysis 9(1), pp. 7-15 (2019).

In this paper, viscosity solutions of

$$\frac{\partial u}{\partial t} = |\nabla u|^{2-p}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad 1 < p < \infty \tag{1.3}$$

are studied. In particular, it is shown that the partial time derivative $\frac{\partial u}{\partial t}$ exists in the sense of Sobolev for some values of $p$. The same holds true for the spatial second derivatives. A fundamental identity is derived for viscosity solutions to a regularized version of equation (1.3),

$$\frac{\partial u^\epsilon}{\partial t} = \left(|\nabla u^\epsilon|^2 + \epsilon^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^\epsilon|^2 + \epsilon^2\right)^{\frac{p-2}{2}} \nabla u^\epsilon\right).$$

We are able to obtain a uniform bound for the $L^2$-norm of the second derivatives for solutions of the regularized equation, which is preserved when we extract a convergent subsequence. Using this bound we get weak convergence in $L^2$ for a sequence of functions involving the second derivatives of $u_\epsilon$. Uniqueness of viscosity solutions shows that the time derivative exists in the sense of Sobolev. With our method, we had to restrict the values of $p$ to a certain range. Some time after this paper was published, it was shown in [DFZZ] that in the plane, the time derivative exists for all $p$.

**Paper 2: A control problem related to the parabolic dominative $p$-Laplace equation**

*Fredrik Arbo Høeg and Eero Ruosteenoja*
To appear in Nonlinear Analysis.

In this paper, a stochastic game associated with the equation

$$2(n+p)\frac{\partial u}{\partial t} = \mathcal{D}_p u \tag{1.4}$$

is studied. The elliptic version of the game was studied by Brustad, Lindqvist and Manfredi [BLM]. We show that the unique viscosity solution of equation (1.4) is the uniform limit of functions $u_\varepsilon$ that satisfy a dynamic programming principle,

$$\begin{aligned} u_\varepsilon(x,t) = &\frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y, t-\varepsilon^2)\, dy \\ &+ \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x+\varepsilon\sigma, t-\varepsilon^2) + u_\varepsilon(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right]. \end{aligned}$$

The solution $u^\varepsilon$ is the value function for a time-dependent control problem. To show that we can pass to the limit $\varepsilon \to 0$ we use an Arzelá-Ascoli-type lemma. The main difficulty in the proof is to show that the family $\{u_\varepsilon\}_\varepsilon$ is equicontinuous. Once this is established one is able to pass to the limit. Then uniqueness of viscosity solutions guarantees that the limit is the unique viscosity solution of equation (1.4).

**Paper 3: Concave power solutions of the Dominative $p$-Laplace equation**

*Fredrik Arbo Høeg*
Published in Nonlinear Differential Equations and Applications 27(2), pp. 1-12 (2020).

In this paper, viscosity solutions of

$$\begin{cases} -\mathcal{D}_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are studied. We show that $u^\alpha$ is concave for $\alpha = \frac{1}{2}$, given that $u$ is a viscosity solution to the above problem.

Power concavity problems have been studied since the 70's for the Laplace, the $p$-Laplace and the normalized $p$-Laplace equations. We mention [K], [Ka], [S], [M], [Ke], [Ko], [CF] and [ALL] for some of this work. For the two-dimensional Laplace equation, an interesting calculation gives the power concavity, see appendix C.

To show that the square root is a concave function, we first look at what problem $v = -\sqrt{u}$ solves in the viscosity sense. It is a viscosity supersolution to some PDE. It turns out that the *convex envelope*, which is the largest convex function lying below $v$, is a supersolution to the same equation that $v$ solves in the viscosity sense. The methods used to show this relies on *Ishii's Lemma* or the *Theorem on sums* which is a useful technical tool in the theory of viscosity solutions. Finally, the comparison principle is used to show that the function $v$ is convex, making $\sqrt{u}$ a concave function.

## Bibliography

[A] G. ARONSSON: Extension of functions satisfying Lipshitz conditions. *Arkix för Matematik*, 6(28):551-561, 1967.

[ALL] O. ALVAREZ, J. LASRY, P. LIONS: Convex viscosity solutions and state constraints. *Journal de Mathématiques Pures et Appliquées*, 76, 265-288, 1997.

[APR] A. ATTOUCHI, M. PARVIAINEN AND E. RUOSTEENOJA: $C^{1,\alpha}$ regularity for the normalized p-Poisson problem. *Journal de Mathématiques Pures et Appliquées*, 108(4):553-591, 2017.

[B] K. BRUSTAD: Superposition of p-superharmonic functions. *Advances in Calculus of Variations*, Advanced online publication. https://doi.org/10.1515/acv-2017-0030

[Bra] K. A. BRAKKE: The motion of a surface by its mean curvature. *Princeton Univ. Press, Princeton, NJ*, 1978.

[BLM] K. BRUSTAD, P. LINDQVIST, J.J. MANFREDI: A discrete stochastic interpretation of the Dominative $p$-Laplacian. *ArXiv preprint:1809.00714* 2018.

[BER] P. BLANC, C. ESTEVE, J. ROSSI: The evolution problem associated with eigenvalues of the Hessian. *ArXiv preprint:1901.01052*, 2019.

[CF] G. CRASTA AND I. FRAGALÁ: On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: Regularity and geometric results. *Archive for Rational Mechanics and Analysis*, 218(3):1577-1607, 2015.

[CZ] M.G. CRANDALL, J. ZHANG: Another way to say harmonic. *Trans. Amer. Math. Soc.* 355(1):241–263, 2003.

[D] K. DOES: An evolution equation involving the normalized $p$-Laplacian. *Communications on Pure & Applied Analysis*, 10(1):361-396, 2011.

[DFZZ] H. DONG, P. FA, Y.R-Y ZHANG AND Y. ZHOU: Second order regularity for elliptic and parabolic equations involving $p$-Laplacian via a fundamental inequality. *ArXiv preprint:1908.01547*, 2019.

[ES] L. C. EVANS AND J. SPRUCK: Motion of level sets by mean curvature. i. *J. Differential Geom.*, 33(3): 635–681, 1991.

[Hui] G. HUISKEN: Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1): 237–266, 1984.

[H] F. A. HØEG: Concave power solutions of the dominative p-Laplace equation. *Nonlinear Differential Equations and Applications NoDEA*, 27(2):1-12, 2020.

[HL] F. A. HØEG AND P. LINDQVIST: Regularity of solutions of the parabolic normalized p-Laplace equation. *Advances in Nonlinear Analysis*, 9(1):7-15, 2019.

[HR] F. A. HØEG AND E. RUOSTEENOJA: A control problem related to the parabolic dominative $p$-Laplace equation. *Nonlinear Analysis*, 195(111721), 2020.

[JS] T. JIN AND L. SILVESTRE: Hölder gradient estimates for parabolic homogeneous p-Laplacian equations. *Journal de Mathématiques Pures et Appliquées*, 108(1):63-87, 2017.

[K] M. KÜHN: Power- and Log-concavity of viscosity solutions to some elliptic Dirichlet problems. *Communications on Pure & Applied Analysis*, 17, 2018.

[Ka] B. KAWOHL: Rearrangements and convexity of level sets in PDE. *Springer, Lecture Notes in Math.*, 1150,1985.

[Ke] A. KENNINGTON: Power concavity and boundary value problems. *Indiana University Mathematics Journal*, 34(3):687-704, 1985.

[Ko] N. KOREVAAR: Capillary surface convexity above convex domains. *Indiana University Mathematics Journal*, 32(1):73-81, 1983.

[LM] P. LINDQVIST, J.J. MANFREDI: Note on a remarkable superposition for a nonlinear equation. *Proc. Amer. Math. Soc.* 136(1):133–140 (electronic), 2008.

[M] L. G. MAKAR-LIMANOV: Solution of Dirichlet's problem for the equation $\Delta u = -1$ in a convex region. *Mathematical Notes of the Academy of Sciences of the USSR*, 9(1):52-53, 1971.

[MPR] J. MANFREDI, M. PARVIAINEN AND J. ROSSI: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM Journal on Mathematical Analysis*, 42(5):2058-2081, 2010.

[PS] Y. PERES AND S. SHEFFIELD: Tug-of-war with noise: a game theoretic view of the $p$-Laplacian. *Duke Math. J.*, 145:91–210, 2008.

[PSSW] Y. PERES, O. SCHRAMM, S. SHEFFIELD, D. B. WILSON: Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.*, 22:167–210, 2009.

[S] S. SAKAGUCHI: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 14(3):403-421, 1987.

# Regularity of solutions of the parabolic normalized $p$-Laplace equation

*Fredrik Arbo Høeg and Peter Lindqvist*

**Published in Advances in Nonlinear Analysis 9(1), pp. 7-15 (2019)**

# Regularity of solutions of the parabolic normalized $p$-Laplace equation

**Abstract.**The parabolic normalized p-Laplace equation is studied. We prove that a viscosity solution has a time derivative in the sense of Sobolev belonging locally to $L^2$.

## 2.1 Introduction

We consider viscosity solutions of the *normalized p-Laplace* equation

$$\frac{\partial u}{\partial t} = |\nabla u|^{2-p}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad 1 < p < \infty, \tag{2.5}$$

in $\Omega_T = \Omega \times (0, T)$, $\Omega$ being a domain in $\mathbb{R}^n$. Formally, the equation reads

$$\frac{\partial u}{\partial t} = \Delta u + (p-2)|\nabla u|^{-2}\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

In the linear case $p = 2$ we have the Heat Equation $u_t = \Delta u$ and also for $n = 1$ the equation reduces to the Heat Equation $u_t = (p-1)u_{xx}$. At the limit $p = 1$ we obtain the equation for motion by mean curvature. We aim at showing that the time derivative $\frac{\partial u}{\partial t}$ exists in the Sobolev sense and belongs to $L^2_{\text{loc}}(\Omega_T)$. We also study the second derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

There has been some recent interest in connexion with Stochastic Game Theory, where the equation appears, cf. [MPR]. From our point of view the work [D] is of actual interest, because there it is shown that the time derivative $u_t$ of the viscosity solutions exists and is locally bounded, provided that the lateral boundary values are smooth. Thus the boundary values control the time regularity. If no such assumptions about the behaviour at the lateral boundary $\partial\Omega \times (0, T)$ are made, a conclusion like $u_t \in L^\infty_{\text{loc}}(\Omega_T)$ is in doubt. Our main result is the following, where we unfortunately have to restrict $p$:

**Theorem 2.1.1.** *Suppose that $u = u(x, t)$ is a viscosity solution of the normalized p-Laplace equation in $\Omega_T$. If $\frac{6}{5} < p < \frac{14}{5}$, then the Sobolev derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist and belong to $L^2_{\text{loc}}(\Omega_T)$.*

We emphasize that no assumptions on boundary values are made for this interior estimate. Our method of proof is based on a verification of the identity

$$\int_0^T\int_\Omega u\phi_t \, dx dt = -\int_0^T\int_\Omega U\phi \, dx dt, \qquad \phi \in C_0^\infty(\Omega_T),$$

where we have to prove that the function $U$, which is the right-hand side of equation (2.5), belongs to $L^2_{\mathrm{loc}}(\Omega_T)$. Thus the second spatial derivatives $D^2u$ are crucial (local boundedness of $\nabla u$ was proven in [D], [BG] and interior Hölder estimates for the gradient in [JS]). The elliptic case has been studied in [APR].

In the range $1 < p < 2$ one can bypass the question of second derivatives.

**Theorem 2.1.2.** *Suppose that $u = u(x,t)$ is a viscosity solution of the normalized p-Laplace equation in $\Omega_T$. If $1 < p < 2$, then the Sobolev derivative $\frac{\partial u}{\partial t}$ exists and belongs to $L^2_{\mathrm{loc}}(\Omega_T)$.*

To avoid the problem of vanishing gradient, we first study the regularized equation

$$\frac{\partial u^\epsilon}{\partial t} = (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{2-p}{2}} \operatorname{div}\left((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \nabla u^\epsilon\right). \tag{2.6}$$

Here the classical parabolic regularity theory is applicable. The equation was studied by K. Does in [D], where an estimate of the gradient $\nabla u^\epsilon$ was found with Bernstein's method. We shall prove a maximum principle for the gradient. Further, we differentiate equation (2.6) with respect to the space variables and derive estimates for $u^\epsilon$ which are passed over to the solution $u$ of (2.5).

Analogous results seem to be possible to reach through the *Cordes condition*. It also restricts the range of valid exponents $p$. We have refrained from this approach, mainly since the absence of zero (lateral) boundary values produces many undesired terms to estimate. Finally, we mention that the limits $\frac{6}{5}$ and $\frac{14}{5}$ in Theorem 2.1.1 are evidently an artifact of the method. It would be interesting to know whether the theorem is valid in the whole range $1 < p < \infty$. In any case, our method is not capable to reach all exponents.

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## 2.2 Preliminaries

**Notation.** The gradient of a function $f : \Omega_T \to \mathbb{R}$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

and its Hessian matrix is

$$\left(D^2 f\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \qquad |D^2 f|^2 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2.$$

We shall, occasionally, use the abbreviation

$$u_j = \frac{\partial u}{\partial x_j}, \quad u_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k}$$

for partial derivatives. Young's inequality

$$|ab| \le \delta \frac{|a|^p}{p} + \left(\frac{1}{\delta}\right)^{q-1} \frac{|b|^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

is often referred to. Finally, the summation convention is used when convenient.

**Viscosity solutions.** The normalized $p$-Laplace Equation is not in divergence form. Thus the concept of weak solutions with test functions under the integral sign is problematic. Fortunately, the modern concept of viscosity solutions works well. Existence and uniqueness of viscosity solutions of the normalized $p$-Laplace equation was established in [BG]. We recall the definition.

**Definition 2.2.1.** *We say that an upper semi-continuous function $u$ is a* viscosity subsolution *of equation* (2.5) *if for all $\phi \in C^2(\Omega_T)$ we have*

$$\phi_t \le \left(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}$$

*at any interior point $(x,t)$ where $u - \phi$ attains a local maximum, provided $\nabla\phi(x,t) \neq 0$. Further, at any interior point $(x,t)$ where $u-\phi$ attains a local maximum and $\nabla\phi(x,t) = 0$ we require*

$$\phi_t \le \left(\delta_{ij} + (p-2)\eta_i\eta_j\right)\phi_{x_i x_j}$$

*for some $\eta \in \mathbb{R}^n$ with $|\eta| \le 1$.*

**Definition 2.2.2.** *We say that a lower semi-continuous function $u$ is a* viscosity supersolution *of equation* (2.5) *if for all $\phi \in C^2(\Omega_T)$ we have*

$$\phi_t \ge \left(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}$$

*at any interior point $(x,t)$ where $u-\phi$ attains a local minimum, provided $\nabla\phi(x,t)\neq 0$. Further, at any interior point $(x,t)$ where $u-\phi$ attains a local minimum and $\nabla\phi(x,t)=0$ we require*

$$\phi_t \geq \Big(\delta_{ij}+(p-2)\eta_i\eta_j\Big)\phi_{x_i x_j}$$

*for some $\eta\in\mathbb{R}^n$ with $|\eta|\leq 1$.*

**Definition 2.2.3.** *A continuous function $u$ is a* viscosity solution *if it is both a viscosity subsolution and a viscosity supersolution.*

For a detailed discussion on the definition at critical points we refer to Evans and Spruck [ES]. The reason behind the choice of $\eta\in\mathbb{R}^n$ is given in [ES] section 2. Viscosity solutions of equation (2.6) are defined in a similar manner, except that now $\nabla\phi(x,t)=0$ is not a problem.

**Maximum Principle for the Gradient.** In order to estimate the time derivative we need bounds on the second derivatives of $u^\epsilon$ (and also on its gradient). If we first assume that $u^\epsilon$ is $C^1$ on the parabolic boundary $\partial_{\text{par}}\Omega_T$, we get bounds on the gradient in all of $\Omega_T$. This follows from the following maximum principle.

**Proposition 2.2.4.** *Let $u^\epsilon$ be a solution of equation* (2.6)*. If $\nabla u^\epsilon\in C^1(\overline{\Omega}_T)$, then*

$$\max_{\overline{\Omega}_T}\left\{|\nabla u^\epsilon|\right\} = \max_{\partial_{\text{par}}\Omega_T}\left\{|\nabla u^\epsilon|\right\}.$$

*Proof.* With some modifications a proof can be extracted from [D]. We give a direct proof. To this end, consider

$$V^\epsilon(x,t)=|\nabla u^\epsilon|^2+\epsilon^2,$$

To find the partial differential equation satisfied by $V^\epsilon$, we calculate[1]

$$V_i^\epsilon = 2u_\nu^\epsilon u_{i\nu}^\epsilon, \quad V_{ij}^\epsilon = 2u_{\nu j}^\epsilon u_{i\nu}^\epsilon + 2u_\nu^\epsilon u_{ij\nu}^\epsilon$$
$$u_i^\epsilon u_j^\epsilon V_{ij}^\epsilon = \frac{1}{2}|\nabla V^\epsilon|^2 + 2u_i^\epsilon u_j^\epsilon u_\nu^\epsilon u_{ij\nu}^\epsilon.$$

Writing equation (2.5) in the form

$$u_t^\epsilon = \Big(\delta_{ij}+(p-2)\frac{u_i^\epsilon u_j^\epsilon}{|\nabla u^\epsilon|^2+\epsilon^2}\Big)u_{ij}^\epsilon,$$

[1] Sum over repeated indices.

we find

$$\frac{1}{2}V_t^\epsilon = u_\nu^\epsilon \frac{\partial}{\partial x_\nu} u_t^\epsilon = u_\nu^\epsilon \Delta u_\nu^\epsilon - \frac{p-2}{2(V^\epsilon)^2}\left|\left\langle \nabla u^\epsilon, \nabla V^\epsilon \right\rangle\right|^2$$

$$+\frac{p-2}{V^\epsilon}\Big(\frac{1}{4}|\nabla V^\epsilon|^2 + \frac{1}{2}u_\nu^\epsilon u_\mu^\epsilon V_{\nu\mu}^\epsilon\Big).$$

Rearranging and using

$$\Delta V^\epsilon = 2|D^2 u^\epsilon|^2 + 2\langle \nabla u^\epsilon, \nabla \Delta u^\epsilon \rangle$$

we arrive at the following differential equation for $V^\epsilon$:

$$V_t^\epsilon = \Delta V^\epsilon - 2|D^2 u^\epsilon|^2 - \frac{p-2}{(V^\epsilon)^2}\left|\left\langle \nabla u^\epsilon, \nabla V^\epsilon \right\rangle\right|^2 + \frac{p-2}{V^\epsilon}\Big\{\frac{1}{2}|\nabla V^\epsilon|^2 + u_\nu^\epsilon u_\mu^\epsilon V_{\nu\mu}^\epsilon\Big\}. \tag{2.7}$$

Let

$$w(x,t) = |\nabla u^\epsilon(x,t)|^2 + \epsilon^2 - \alpha t = V^\epsilon(x,t) - \alpha t \quad \text{for} \quad \alpha > 0.$$

Suppose that $w^\epsilon$ has an *interior* maximum point at $(x_0, t_0)$. At this point $V^\epsilon(x_0, t_0) > 0$, otherwise we would have $V^\epsilon(x,t) \equiv 0$ in $\Omega_T$ in which case there is nothing to prove. By the infinitesimal calculus,

$$\nabla w(x_0, t_0) = 0, \leq 0 \quad \text{and} \quad w_t(x_0, t_0) \geq 0,$$

where we have included the case $t_0 = T$. Further, the matrix $D^2 w(x_0, t_0)$ is negative semidefinite. Using equation (2.7) and noting that $\nabla w = \nabla V^\epsilon$ and $D^2 w = D^2 V^\epsilon$, we get at $(x_0, t_0)$

$$\begin{aligned} 0 \leq w_t &= V_t^\epsilon - \alpha \\ &= \Delta V^\epsilon - 2|D^2 u^\epsilon|^2 - \frac{p-2}{(V^\epsilon)^2}\left|\left\langle \nabla u^\epsilon, \nabla V^\epsilon \right\rangle\right|^2 \\ &+ \frac{p-2}{V^\epsilon}\left\{\frac{1}{2}|\nabla V^\epsilon|^2 + u_\nu^\epsilon u_\mu^\epsilon V_{\nu\mu}^\epsilon\right\} - \alpha \\ &= \left(\delta_{ij} + (p-2)\frac{u_i^\epsilon u_j^\epsilon}{V^\epsilon}\right) w_{ij}^\epsilon - 2|D^2 u^\epsilon|^2 - \alpha \leq -\alpha \end{aligned}$$

since the matrix $A$ with elements $A_{ij} = \delta_{ij} + (p-2)\frac{u_i^\epsilon u_j^\epsilon}{V^\epsilon}$ is positive semidefinite.

To avoid the contradiction $\alpha \leq 0$, $w$ must attain its maximum on the parabolic boundary.

Hence, for any $(x,t) \in \Omega_T$ we have

$$V^{\epsilon}(x,t) - \alpha t \leq \max_{\partial_{\text{par}}\Omega_T} \left\{ V^{\epsilon}(x,t) - \alpha t \right\} \leq \max_{\partial_{\text{par}}\Omega_T} V^{\epsilon}(x,t).$$

We finish the proof by sending $\alpha \to 0^+$. □

With no assumptions for $u^{\epsilon}$ on the parabolic boundary, we need a stronger result taken from [D] p.381.

**Theorem 2.2.5.** *Let $u^{\epsilon}$ be a solution of equation* (2.6)*, with $u^{\epsilon}(x,0) = u_0(x)$. Then*

$$|\nabla u^{\epsilon}(x,t)| \leq C_{n,p} ||u_0||_{L_\infty(\Omega_T)} \left\{ 1 + \left( \frac{1}{\text{dist}((x,t), \partial_{\text{par}}\Omega_T)} \right)^2 \right\}.$$

Note that no condition on the lateral boundary $\partial\Omega \times [0,T]$ was used. By continuity,

$$|\nabla u^{\epsilon}(x,t)| \leq C_{n,p} ||u^{\epsilon}(\cdot, t_0)||_{\infty} \left\{ 1 + \left( \frac{1}{\text{dist}((x,t), \partial_{\text{par}}\Omega_T)} \right)^2 \right\}$$

for $x \in D \subset\subset \Omega$ and $0 < t_0 \leq t \leq T - t_0$. The estimate

$$||\nabla u^{\epsilon}||_{L^\infty(D\times[t_0, T-t_0])} \leq C||u^{\epsilon}||_{L^\infty(\Omega_T)} \left\{ 1 + \left( \frac{1}{\text{dist}(D, \partial_{\text{par}}\Omega_T)} \right)^2 \right\} \tag{2.8}$$

follows. (Here one can pass to the limit as $\epsilon \to 0$.)

The proof of the lemma below, a simple special case of the Miranda - Talenti lemma, can be found for smooth functions in [E] p. 308. If $f$ is not smooth, we perform a *strictly* interior approximation, so that no boundary inegrals appear (which is possible since $\xi \in C_0^\infty$).

**Lemma 2.2.6** (Miranda - Talenti)**.** *Let $\xi \in C_0^\infty(\Omega_T)$ and $f \in L^2(0,T,W^{2,2}(\Omega))$. Then*

$$\int_0^T \int_\Omega |\Delta(\xi f)|^2 \, dx dt = \int_0^T \int_\Omega |D^2(\xi f)|^2 \, dx dt.$$

## 2.3 Regularization

The next lemma tells us that solutions of (2.6) converge locally uniformly to the viscosity solution of (2.5).

**Lemma 2.3.1.** *Let $u$ be a viscosity solution of equation* (2.5) *and let $u^\epsilon$ be the classical solution of the regularized equation* (2.6) *with boundary values*

$$u = u^\epsilon \quad \text{on} \quad \partial_{par}\Omega_T.$$

*Then $u^\epsilon \to u$ uniformly on compact subsets of $\Omega_T$.*

*Proof.* By Theorem 2.2.5 we can use Ascoli's Theorem to extract a convergent subsequence $u^{\epsilon_j}$ converging locally uniformly to some continuous function: $u^{\epsilon_j} \to v$. We claim that $v$ is a viscosity solution of equation (2.5). The lemma then follows by uniqueness.

We demonstrate that $v$ is a viscosity subsolution. (A symmetric proof shows that $v$ is a viscosity supersolution.) Assume that $v - \phi$ attains a strict local maximum at $z_0 = (x_0, t_0)$. Since $u^\epsilon \to v$ locally uniformly, there are points

$$z_\epsilon \to z_0$$

such that $u^\epsilon - \phi$ attains a local maximum at $z_\epsilon$. If $\nabla\phi(z_0) \neq 0$, then $\nabla\phi(z_\epsilon) \neq 0$ for all $\epsilon > 0$ small enough, and at $z_\epsilon$ we have

$$\phi_t \leq \left(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2 + \epsilon^2}\right)\phi_{x_i x_j}. \tag{2.9}$$

Letting $\epsilon \to 0$, we see that $v$ satisfies Definition 2.2.3 when $\nabla\phi(z_0) \neq 0$. If $\nabla\phi(z_0) = 0$, let

$$\eta_\epsilon = \frac{\nabla\phi(z_\epsilon)}{\sqrt{|\nabla\phi(z_\epsilon)|^2 + \epsilon^2}}.$$

Since $|\eta_\epsilon| \leq 1$, there is a subsequence so that $\eta_{\epsilon_k} \to \eta$ when $k \to \infty$ for some $\eta \in \mathbb{R}^n$ with $|\eta| \leq 1$. Passing to the limit $\epsilon_k \to 0$ in equation (2.9), we see that $v$ is a viscosity subsolution. □

Our proof of Theorem 2.1.1 consists in showing that the second derivatives $D^2 u^\epsilon$ belong locally to $L^2$ with a bound independent of $\epsilon$. Once this is established, we see that

$$\begin{aligned}&\left(|\nabla u^\epsilon|^2 + \epsilon^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^\epsilon|^2 + \epsilon^2\right)^{\frac{p-2}{2}} \nabla u^\epsilon\right)\\ &= \Delta u^\epsilon + \frac{p-2}{|\nabla u^\epsilon|^2 + \epsilon^2}\left\langle \nabla u^\epsilon, D^2 u^\epsilon \nabla u^\epsilon\right\rangle \leq C_{p,n}|D^2 u^\epsilon|.\end{aligned}$$

Hence, for any bounded subdomain $D \subset\subset \Omega_T$

$$\left\| \left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p-2}{2}} \nabla u^\epsilon\right)\right\|_{L^2(D)} \leq C,$$

with $C$ independent of $\epsilon$. By this uniform bound, there exists a subsequence such that, as $j \to \infty$,

$$\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{p-2}{2}} \nabla u^{\epsilon_j}\right) \to U \qquad \text{weakly in } L^2(D).$$

In particular, this means that $U \in L^2(D)$ and for any $\phi \in C_0^\infty(D)$ we have

$$\lim_{j\to\infty}\int_0^T\int_D \phi\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{p-2}{2}} \nabla u^{\epsilon_j}\right) dxdt = \int_0^T\int_D \phi U\,dxdt.$$

If $u$ is the unique viscosity solution of (2.5), we invoke Lemma 2.3.1 and the calculations above to find, for any test function $\phi \in C_0^\infty(D)$,

$$\begin{aligned}
\int_0^T\int_D u\frac{\partial\phi}{\partial t}\,dxdt &= \lim_{j\to\infty}\int_0^T\int_D u^{\epsilon_j}\frac{\partial\phi}{\partial t}\,dxdt\\
&= -\lim_{j\to\infty}\int_0^T\int_D \phi\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{2-p}{2}} \operatorname{div}\left(\left(|\nabla u^{\epsilon_j}|^2+\epsilon_j^2\right)^{\frac{p-2}{2}} \nabla u^{\epsilon_j}\right) dxdt\\
&= -\int_0^T\int_D \phi U\,dxdt.
\end{aligned}$$

This shows that the Sobolev derivative $u_t$ exists and, since the previous equation holds for any subdomain $D \subset\subset \Omega_T$, we conclude that $\frac{\partial u}{\partial t} = U \in L^2_{\text{loc}}(\Omega_T)$. — To finish the proof of Theorem 2.1.1 it remains to establish the missing local bound of $\|D^2 u^\epsilon\|_{L^2}$ uniformly in $\epsilon$.

## 2.4 The differentiated equation

We shall derive a fundamental identity. Let

$$v^\epsilon = |\nabla u^\epsilon|^2, \quad V^\epsilon = |\nabla u^\epsilon|^2+\epsilon^2.$$

Differentiating equation (2.6) with respect to the variable $x_j$ we obtain

$$\frac{\partial}{\partial t}u_j^\epsilon = \frac{2-p}{2}\left(V^\epsilon\right)^{-\frac{p}{2}} v_j^\epsilon \operatorname{div}\left(\left(V^\epsilon\right)^{\frac{p-2}{2}}\nabla u^\epsilon\right) \quad + \left(V^\epsilon\right)^{\frac{2-p}{2}} \operatorname{div}\left[\left(\left(V^\epsilon\right)^{\frac{p-2}{2}}\nabla u^\epsilon\right)_j\right].$$

Take $\xi \in C_0^\infty(\Omega_T)$, with $\xi \geq 0$. Multiply both sides of the equation by $\xi^2 V^\epsilon u_j^\epsilon$ and sum $j$ from 1 to $n$. Integrate over $\Omega_T$, using integration by parts and keeping in mind that $\xi$ is compactly supported in $\Omega_T$, to obtain

$$-\frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon\,dx\,dt = \frac{2-p}{2}\int_0^T\int_\Omega \xi^2 (V^\epsilon)^{-\frac{p}{2}}\left\langle \nabla u^\epsilon, \nabla v^\epsilon\right\rangle \operatorname{div}\left((V^\epsilon)^{\frac{p-2}{2}}\nabla u^\epsilon\right)dx\,dt$$

$$-\int_0^T\int_\Omega \frac{\partial}{\partial x_j}\left\{(V^\epsilon)^{\frac{p-2}{2}}u_k^\epsilon\right\}\frac{\partial}{\partial x_k}\left\{\xi^2(V^\epsilon)^{\frac{2-p}{2}}u_j^\epsilon\right\}dx\,dt.$$

Writing out the derivatives gives the fundamental formula

$$\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dx\,dt \qquad \text{Main Term} \qquad (I)$$

$$+\frac{p-2}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi^2\left\langle\nabla u^\epsilon,\nabla v^\epsilon\right\rangle \Delta u^\epsilon\,dx\,dt \qquad (II)$$

$$=\frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon\,dx\,dt \qquad (III)$$

$$+(2-p)\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\left\langle\nabla u^\epsilon,\nabla v^\epsilon\right\rangle\left\langle\nabla u^\epsilon,\nabla\xi\right\rangle dx\,dt \qquad (IV)$$

$$-\int_0^T\int_\Omega \xi\left\langle\nabla v^\epsilon,\nabla\xi\right\rangle dx\,dt \qquad (V).$$

In the next section we shall bound the Main Term (I) uniformly with respect to $\epsilon$.

## 2.5 Estimate of the second derivatives

We shall provide an estimate of the main term (I). First, we record the elementary inequality

$$|\nabla v^\epsilon|^2 = \left|2D^2u^\epsilon\nabla u^\epsilon\right|^2 \leq 4|D^2u^\epsilon|^2 v^\epsilon. \tag{2.10}$$

**One Dimension.** As an exercise, we show that in this case the second derivatives are locally bounded in $L^2$ for any $1 < p < \infty$. In one dimension, equation (2.5) reads

$$u_t = |u_x|^{2-p} \frac{\partial}{\partial x} \left\{ |u_x|^{p-2} u_x \right\} = (p-1) u_{xx}.$$

We absorb the terms (IV) and (V), using Young's inequality and inequality (2.10). For any $\delta > 0$,

$$\int_0^T \int_\Omega \xi^2 \Big( \frac{\partial^2 u^\epsilon}{\partial x^2} \Big)^2 \left( 1 + (p-2) \frac{(\frac{\partial u^\epsilon}{\partial x})^2}{(\frac{\partial u^\epsilon}{\partial x})^2 + \epsilon^2} - \delta \big( |p-2| + 1 \big) \right) dx dt$$

$$\leq \frac{1}{2} \int_0^T \int_\Omega \xi \xi_t V^\epsilon \, dx dt + \frac{|p-2|+1}{\delta} \int_0^T \int_\Omega V^\epsilon |\nabla \xi|^2 \, dx dt.$$

Applying Theorem 2.2.5 we see that the right-hand side is bounded by a constant independent of $\epsilon > 0$. We have

$$1 + (p-2) \frac{(\frac{\partial u^\epsilon}{\partial x})^2}{(\frac{\partial u^\epsilon}{\partial x})^2 + \epsilon^2} \geq \min\{1, p-1\} > 0.$$

It follows that $\frac{\partial^2 u^\epsilon}{\partial x^2} \in L^2$ locally for any $p \in (1, \infty)$.

**General $n$.** We assume for the moment that $1 < p < 2$. We rewrite the term (II) involving the Laplacian as

$$\tfrac{2-p}{2} \frac{1}{V^\epsilon} \xi^2 \big\langle \nabla u^\epsilon, \nabla v^\epsilon \big\rangle \Delta u^\epsilon = \tfrac{2-p}{2} \frac{1}{V^\epsilon} \xi \big\langle \nabla u^\epsilon, \nabla v^\epsilon \big\rangle \Big\{ \Delta(\xi u^\epsilon) - 2 \big\langle \nabla u^\epsilon, \nabla \xi \big\rangle - u^\epsilon \Delta \xi \Big\}.$$

Upon this rewriting the term (IV) disappears from the equation. We focus our attention on the term involving $\Delta(\xi u^\epsilon)$. By Lemma 2.2.6

$$\int_0^T \int_\Omega |D^2(\xi u^\epsilon)|^2 \, dx dt = \int_0^T \int_\Omega |\Delta(\xi u^\epsilon)|^2 \, dx dt.$$

Differentiating, we see that

$$\begin{aligned} (\xi u^\epsilon)_i &= \xi_i u^\epsilon + \xi u^\epsilon_i \\ (\xi u^\epsilon)_{ij} &= \xi_{ij} u^\epsilon + u^\epsilon_i \xi_j + \xi_i u^\epsilon_j + \xi u^\epsilon_{ij}. \end{aligned}$$

It follows that

$$|D^2(\xi u^\epsilon)|^2 = \xi^2 |D^2 u^\epsilon|^2 + f(u^\epsilon, \nabla u^\epsilon, D^2 u^\epsilon),$$

where $f(u^\epsilon, \nabla u_i^\epsilon, D^2 u^\epsilon)^2$ depends only linearly on the second derivatives $u_{ij}^\epsilon$. By Young's inequality we obtain

$$\begin{aligned}\frac{2-p}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\left\langle \nabla u^\epsilon, \nabla v^\epsilon\right\rangle \Delta(\xi u^\epsilon)\,dxdt \leq \frac{5}{4}(2-p)\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dxdt\\ + \frac{2-p}{4}\int_0^T\int_\Omega f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon)\,dxdt.\end{aligned}$$

Inserting this into the main equation gives

$$\begin{aligned}&\left(1-\frac{5}{4}(2-p)\right)\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dxdt && (I*)\\ &\leq \frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon\,dxdt && (III)\\ &\quad -\int_0^T\int_\Omega \xi\left\langle \nabla v^\epsilon, \nabla\xi\right\rangle\,dxdt && (V)\\ &\quad +\frac{2-p}{2}\int_0^T\int_\Omega f(u^\epsilon, u_i^\epsilon, u_{ij}^\epsilon)\,dxdt && (VI)\\ &\quad +\frac{2-p}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\left\langle \nabla u^\epsilon, \nabla v^\epsilon\right\rangle u^\epsilon\Delta\xi\,dxdt && (VII).\end{aligned}$$

All terms containing $D^2u^\epsilon$ can be absorbed by the new main term $(I*)$. To this end, we use Young's inequality with a small parameter $\delta > 0$ to balance[3]

[2]

$$\begin{aligned}f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon) = (u^\epsilon)^2|D^2\xi|^2 + 4u^\epsilon\left\langle \nabla\xi, D^2\xi\nabla u^\epsilon\right\rangle + 4\xi\left\langle \nabla\xi, D^2u^\epsilon\nabla u^\epsilon\right\rangle\\ + 2|\nabla\xi|^2|\nabla u^\epsilon|^2 + 2\left|\left\langle \nabla u^\epsilon, \nabla\xi\right\rangle\right|^2 + 2u^\epsilon\xi\,\mathrm{trace}\left\{\left(D^2\xi\right)\left(D^2u^\epsilon\right)\right\}.\end{aligned}$$

[3]The parameter $\delta$ is to be made so small that terms like $\delta\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dxdt$ can be moved over to the left-hand side.

the terms. For term (V), we have

$$\int_0^T\int_\Omega \xi\left\langle \nabla v^\epsilon, \nabla \xi\right\rangle dxdt \le \delta \int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\, dxdt + \frac{1}{\delta}\int_0^T\int_\Omega V^\epsilon|\nabla\xi|^2\, dxdt.$$

Similarly, for term (VII)

$$\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\left\langle \nabla u^\epsilon, \nabla v^\epsilon\right\rangle u^\epsilon \Delta\xi\, dxdt \le 2\delta_1 \int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2 + \frac{1}{\delta_1}\int_0^T\int_\Omega |u^\epsilon|^2|\Delta\xi|^2\, dxdt.$$

Using similar inequalities for the term involving $f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon)$ and chosing the parameters small enough in Young's inequality, we find,

$$\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\, dxdt \le C\iint_{\{\xi\neq 0\}} ((u^\epsilon)^2 + |\nabla u^\epsilon|^2)\, dxdt \qquad (2.11)$$

where $C$ is independent of $\epsilon$ but depends on $\|\xi\|_{C^2}$, provided that

$$1 - \frac{5}{4}(2-p) > 0, \quad \text{i.e.} \quad p > \frac{6}{5}.$$

This is now a decisive restriction. Invoking Lemma 2.3.1 and the estimate (2.8), we deduce that that the majorant in (2.11) is independent of $\epsilon$.

A symmetric proof when $p > 2$ shows that equation (2.11) holds when

$$p < \frac{14}{5}.$$

## 2.6 The case $1 < p < 2$

In this section, we give a proof of Theorem 2.1.2. To this end, let $\xi \in C_0^\infty(\Omega_T)$, with $0 \le \xi \le 1$. We claim that

$$\int_0^T\int_\Omega \xi^2\Big(\frac{\partial u^\epsilon}{\partial t}\Big)^2 dxdt \le 4||V^\epsilon||_\infty^2 \left\{\int_0^T\int_\Omega \left|\nabla\xi\right|^2 dxdt + \frac{1}{p}\int_0^T\int_\Omega \xi|\xi_t|\, dxdt\right\} \qquad (2.12)$$

where the supremum norm of $V^\epsilon = \left|\nabla u^\epsilon\right|^2 + \epsilon^2$ is taken locally, over the support of $\xi$. Here, $u^\epsilon$ is the solution of the regularized equation (2.6). This is enough to complete the proof of Theorem 2.1.2, in virtue of Theorem 2.2.5.

Multiplying the regularized equation (2.6) by $\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p-2}{2}}\xi^2 u_t^\epsilon$ yields

$$\xi^2\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p-2}{2}}\left(u_t^\epsilon\right)^2=\xi^2 u_t^\epsilon \operatorname{div}\left((|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p-2}{2}}\nabla u^\epsilon\right)$$

$$=\operatorname{div}\left(\xi^2 u_t^\epsilon(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p-2}{2}}\nabla u^\epsilon\right)-(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla\left(\xi^2 u_t^\epsilon\right)\right\rangle.$$

The integral of the divergence term vanishes by Gauss's Theorem and, upon integration, we have

$$\int_0^T\int_\Omega \xi^2\left(V^\epsilon\right)^{\frac{p-2}{2}}\left(u_t^\epsilon\right)^2\,dxdt$$

$$=-\int_0^T\int_\Omega \left(V^\epsilon\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla\left(\xi^2 u_t^\epsilon\right)\right\rangle\,dxdt$$

$$=-2\int_0^T\int_\Omega \xi\left(V^\epsilon\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla\xi\right\rangle u_t^\epsilon\,dxdt-\int_0^T\int_\Omega \xi^2\left(V^\epsilon\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla u_t^\epsilon\right\rangle\,dxdt.$$

The first integral on the right-hand side can be absorbed by the left-hand side by choosing $\sigma=\frac{1}{2}$ in

$$\left|2\xi\left(V^\epsilon\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla\xi\right\rangle u_t^\epsilon\right|\le\sigma\xi^2\left(V^\epsilon\right)^{\frac{p-2}{2}}\left(u_t^\epsilon\right)^2+\frac{1}{\sigma}\left(V^\epsilon\right)^{\frac{p-2}{2}}|\nabla u^\epsilon|^2|\nabla\xi|^2,$$

and integrating.

For the last term, the decisive observation is that

$$\frac{1}{p}\frac{\partial}{\partial t}\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p}{2}}=\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla u_t^\epsilon\right\rangle=\left(V^\epsilon\right)^{\frac{p-2}{2}}\left\langle \nabla u^\epsilon,\nabla u_t^\epsilon\right\rangle.$$

We use this in the last integral on the right-hand side to obtain

$$-\int_0^T\int_\Omega \xi^2 \left(V^\epsilon\right)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla u_t^\epsilon \rangle \, dxdt$$

$$= -\int_0^T\int_\Omega \frac{\partial}{\partial t}\left\{\frac{\xi^2}{p}\left(V^\epsilon\right)^{\frac{p}{2}}\right\} dxdt + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t \left(V^\epsilon\right)^{\frac{p}{2}} \, dxdt$$

$$= -\int_\Omega \left[\frac{\xi^2}{p}\left(V^\epsilon\right)^{\frac{p}{2}}\right]_{t=0}^{t=T} dx + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t \left(V^\epsilon\right)^{\frac{p}{2}} \, dxdt$$

$$= \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t \left(V^\epsilon\right)^{\frac{p}{2}} \, dxdt.$$

To sum up, we have now the final estimate

$$\frac{1}{2}\int_0^T\int_\Omega \xi^2 \left(V^\epsilon\right)^{\frac{p-2}{2}} \left(u_t^\epsilon\right)^2 \, dxdt$$

$$\leq 2\int_0^T\int_\Omega \left(V^\epsilon\right)^{\frac{p-2}{2}} |\nabla u^\epsilon|^2 |\nabla \xi|^2 \, dxdt + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t \left(V^\epsilon\right)^{\frac{p}{2}} \, dxdt$$

$$\leq 2\int_0^T\int_\Omega \left(V^\epsilon\right)^{\frac{p}{2}} |\nabla \xi|^2 \, dxdt + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t \left(V^\epsilon\right)^{\frac{p}{2}} \, dxdt.$$

So far, our calculations are valid in the full range $1 < p < \infty$. For $1 < p < 2$, we have

$$\left(V^\epsilon\right)^{\frac{p-2}{2}} \geq ||V^\epsilon||_\infty^{\frac{p-2}{2}},$$

where the supremum norm is taken over the support of $\xi$. Hence, equation (2.12) holds for $1 < p < 2$ and the proof of Theorem 2.1.2 is complete.

# Bibliography


[APR] A. Attouchi, M. Parviainen and E. Ruosteenoja: $C^{1,\alpha}$ regularity for the normalized p-Poisson problem, *Journal de Mathématiques Pures et Appliquées*, 108(4):553-591, 2017.

[BG] A. Banerjee and N. Garofalo: Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations, *Indiana University Mathematics Journal*, pages 699-736, 2013.

[D] K. Does: An evolution equation involving the normalized $p$-Laplacian, *Communications on Pure & Applied Analysis*, 10(1):361-396, 2011.

[E] L. EVANS: *Partial Differential Equations*, volume 19 of *Graduate students in Mathematics*. Provindence, Rhode Island, 1998.

[ES] L. EVANS AND J. SPRUCK: Motion of level sets by mean curvature. *Journal of Differential Geometry*, 33(3):635-681, 1991.

[JS] T. JIN AND L. SILVESTRE: Hölder gradient estimates for parabolic homogeneous p-Laplacian equations. *Journal de Mathématiques Pures et Appliquées*, 108(1):63-87, 2017.

[MPR] J. MANFREDI, M. PARVIAINEN AND J. ROSSI: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM Journal on Mathematical Analysis*, 42(5):2058-2081, 2010.

# A control problem related to the parabolic dominative $p$-Laplace equation

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# A control problem related to the parabolic dominative $p$-Laplace equation

**Abstract.**We show that value functions of a certain time-dependent control problem in $\Omega \times (0,T)$, with a continuous payoff $F$ on the parabolic boundary, converge uniformly to the viscosity solution of the parabolic dominative $p$-Laplace equation

$$2(n+p)u_t = \Delta u + (p-2)\lambda_n(D^2u),$$

with the boundary data $F$. Here $2 < p < \infty$, and $\lambda_n(D^2u)$ is the largest eigenvalue of the Hessian $D^2u$.

## 3.1 Introduction

In this paper we give a control problem interpretation for the parabolic dominative $p$-Laplace equation

$$2(n+p)u_t = \mathcal{D}_p u \qquad \text{in} \quad \Omega_T. \tag{3.13}$$

Here $\Omega_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying a uniform exterior sphere condition, and

$$\mathcal{D}_p u := (\lambda_1 + ... + \lambda_{n-1}) + (p-1)\lambda_n = \Delta u + (p-2)\lambda_n,$$

where $2 < p < \infty$, and $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ are the eigenvalues of the Hessian $D^2u$. The operator $\mathcal{D}_p$ is called *the dominative p-Laplacian*, introduced by Brustad [Bru17, Bru18] and later studied by Brustad, Lindqvist and Manfredi [BLM18] and Høeg [Hoe19] in the elliptic case. The dominative $p$-Laplacian explains the superposition principle of the $p$-Laplace equation, see [CZ03, LM08] for more about this property. The operator $\mathcal{D}_p$ is sublinear, so it is convex, and equation (3.13) is uniformly parabolic. By Theorem 3.2 in [Wan92], viscosity solutions of (3.13) are in $C^{2+\alpha,\frac{2+\alpha}{2}}(\Omega_T)$ for some $\alpha > 0$.

Let $u$ be a viscosity solution of (3.13) with a given continuous boundary data $F$ on $\partial_p\Omega_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$. By [CIL92], the solution is unique. In Section 3.3 we see that for $\varepsilon > 0$ and the boundary data $F$, there is a unique Borel-measurable function $u_\varepsilon$ satisfying a *dynamic programming principle* (hereafter DPP)

$$\begin{aligned} u_\varepsilon(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y, t-\varepsilon^2)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x+\varepsilon\sigma, t-\varepsilon^2) + u_\varepsilon(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] \quad \text{in } \Omega_T. \end{aligned} \tag{3.14}$$

Here $B_\varepsilon(x) \subset \mathbb{R}^n$ is a ball centered at $x$ with the radius $\varepsilon$, in the first term we have an average integral, and in the second term the supremum is taken over all unit vectors in $\mathbb{R}^n$. In Theorem 3.4.3 we show that $u_\varepsilon \to u$ uniformly when $\varepsilon \to 0$. The idea of the proof is to first show that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded and asymptotically equicontinuous, and use a variant of the Arzelá-Ascoli theorem to see that solutions of the DPP converge uniformly to some continuous function. To show that the uniform limit is the viscosity solution of (3.13), we make use of an asymptotic mean value formula

$$\begin{aligned} &\frac{n+2}{p+n} \fint_{B_\varepsilon(x)} v(y, t-\varepsilon^2)\, dy \\ &\quad + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] \\ &= v(x,t) + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p v(x,t) - 2(n+p) v_t(x,t)) + o(\varepsilon^2), \end{aligned} \tag{3.15}$$

which is valid for *all* functions $v \in C^{2,1}(\Omega_T)$, see Theorem 3.2.1.

It turns out that the solution $u_\varepsilon$ of DPP (3.14) is the value of the following time-dependent control problem. Let us denote $\alpha = \frac{p-2}{p+n}, \beta = \frac{n+2}{p+n}$, and place a token at $(x_0, t_0) \in \Omega_T$. The controller tosses a biased coin with probabilities $\alpha$ and $\beta$. If she gets tails (with probability $\beta$), the game state moves according to the uniform probability density to a point $x_1 \in B_\varepsilon(x_0)$. If the coin toss is heads (with probability $\alpha$), the controller chooses a unitary vector $\sigma \in \mathbb{R}^n$. The position of the token is then moved to $x_1 = x_0 + \varepsilon\sigma$ or $x_1 = x_0 - \varepsilon\sigma$ with equal probabilities. After this step, the position of the token is now at $(x_1, t_1)$, where $t_1 = t_0 - \varepsilon^2$. The game continues from $(x_1, t_1)$ according to the same rules yielding a sequence of game states

$$(x_0, t_0), (x_1, t_1), (x_2, t_2), \ldots$$

The game is stopped when the token is moved outside of $\Omega_T$ for the first time and we denote this point by $(x_\tau, t_\tau)$. The controller is then paid the amount $F(x_\tau, t_\tau)$. Naturally, the controller aims to maximize her payoff, and heuristically, the rules of the game can be read from the DPP (3.14).

We remark that the scaling of the time derivative in equation (3.13) is just a matter of convenience. For the equation $u_t = \mathcal{D}_p u$ we would define a game with the same rules as before, except that we would have $t_{j+1} = t_j - \frac{\varepsilon^2}{2(n+p)}$ for every step in the game, see also Remark 3.2.4.

This control problem has some similarities with two-player zero-sum *tug-of-war* games, which were introduced by Peres, Schramm, Sheffield and Wilson [PSSW09, PS08] and later studied from different perspectives, see e.g. [AS12, MPR12, Lew18]. Time-dependent tug-of-war games, having connections to parabolic equations with the normalized $p$-Laplacian, were studied in

[MPR10,PR16,Han18], whereas two-player games for equations $u_t = \lambda_j(D^2u)$, $j \in \{1,...,n\}$, were recently formulated in [BER19]. For a deterministic game-theoretic approach to parabolic equations, we refer to [KS10].

This paper is organized as follows. In Section 3.2 we prove the asymptotic mean value formula (3.15). In Section 3.3 we show that the value of the control problem satisfies the DPP (3.14). Finally, in Section 3.4 we show that value functions converge uniformly to the viscosity solution of (3.13) when $\varepsilon \to 0$.

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## 3.2 Asymptotic mean value formula

**Theorem 3.2.1.** *Let $v : \Omega_T \to \mathbb{R}$ be in $C^{2,1}(\Omega_T)$. Then it satisfies the asymptotic mean value formula* (3.15).

*Proof.* Averaging the Taylor expansion

$$\begin{aligned} v(y,t-\varepsilon^2) = v(x,t) + \langle Dv(x,t),(y-x)\rangle + \frac{1}{2}\langle D^2v(x,t)(y-x),(y-x)\rangle \\ -\varepsilon^2 v_t(x,t) + o(|y-x|^2+\varepsilon^2) \end{aligned}$$

over the ball $B_\varepsilon(x)$ and calculating

$$\fint_{B_\varepsilon(x)} \langle Dv(x,t),(y-x)\rangle\, dy = 0$$

and

$$\fint_{B_\varepsilon(x)} \langle D^2v(x,t)(y-x),(y-x)\rangle\, dy = \frac{\varepsilon^2}{n+2}\Delta v(x,t),$$

we obtain

$$\begin{aligned} &\fint_{B_\varepsilon(x)} v(y,t-\varepsilon^2)\,\mathrm{d}y \\ &\quad = v(x,t) + \frac{\varepsilon^2}{2(n+2)}\Delta v(x,t) - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \end{aligned} \tag{3.16}$$

Next we take an arbitrary unit vector $\sigma$ and write the Taylor expansions for $v(x+h,t-\varepsilon^2)$ with $h=\varepsilon\sigma$ and $h=-\varepsilon\sigma$ to obtain

$$\begin{aligned} v(x+\varepsilon\sigma,t-\varepsilon^2) = v(x,t) + \langle Dv(x,t),\varepsilon\sigma\rangle + \frac{1}{2}\langle D^2v(x,t)\varepsilon\sigma,\varepsilon\sigma\rangle \\ -\varepsilon^2 v_t(x,t) + o(\varepsilon^2), \end{aligned}$$

$$
\begin{aligned}
v(x-\varepsilon\sigma, t-\varepsilon^2) = v(x,t) - \langle Dv(x,t), \varepsilon\sigma\rangle + \frac{1}{2}\langle D^2 v(x,t)(-\varepsilon\sigma), (-\varepsilon\sigma)\rangle \\
- \varepsilon^2 v_t(x,t) + o(\varepsilon^2),
\end{aligned}
$$

which yield

$$
\begin{aligned}
&\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \\
&\quad = v(x,t) + \frac{\varepsilon^2}{2}\langle D^2 v(x,t)\sigma, \sigma\rangle - \varepsilon^2 v_t(x,t) + o(\varepsilon^2).
\end{aligned}
$$

Taking the supremum over all $|\sigma| = 1$ gives

$$
\begin{aligned}
&\sup_{|\sigma|=1}\left[\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2}\right] \\
&\quad = v(x,t) + \frac{\varepsilon^2}{2}\lambda_n - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \qquad (3.17)
\end{aligned}
$$

By multiplying equations (3.16) and (3.17) by $\frac{n+2}{p+n}$ and $\frac{p-2}{p+n}$ respectively, we get

$$
\begin{aligned}
&\frac{n+2}{p+n}\fint_{B_\varepsilon(x)} v(y, t-\varepsilon^2)\,dy \\
&\quad + \frac{p-2}{p+n}\sup_{|\sigma|=1}\left[\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2}\right] \\
&= v(x,t) + \frac{\varepsilon^2}{2(n+p)}\left(\mathcal{D}_p v(x,t) - 2(n+p) v_t(x,t)\right) + o(\varepsilon^2). \qquad \square
\end{aligned}
$$

Next we define viscosity solutions for equation (3.13).

**Definition 3.2.2.** *An upper semicontinuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity subsolution to equation $2(n+p)u_t = \mathcal{D}_p u$ in $\Omega_T$ if for all $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ such that*

*i)* $u(x_0, t_0) = \phi(x_0, t_0)$,

*ii)* $\phi(x,t) > u(x,t)$ *for* $(x,t) \in \Omega_T$, $(x,t) \neq (x_0, t_0)$,

*it holds* $2(n+p)\phi_t(x_0, t_0) \le \mathcal{D}_p\phi(x_0, t_0)$.

*A lower semicontinuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity supersolution to equation $2(n+p)u_t = \mathcal{D}_p u$ in $\Omega_T$ if for all $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ such that*

*i)* $u(x_0, t_0) = \phi(x_0, t_0)$,

*ii)* $\phi(x,t) < u(x,t)$ *for* $(x,t) \in \Omega_T$, $(x,t) \neq (x_0, t_0)$, *it holds* $2(n+p)\phi_t(x_0,t_0) \geq \mathcal{D}_p\phi(x_0,t_0)$.

*A continuous function* $u : \Omega_T \to \mathbb{R}$ *is a viscosity solution to equation* $2(n+p)u_t = \mathcal{D}_p u$ *in* $\Omega_T$ *if it is both a subsolution and a supersolution.*

Because viscosity solutions of (3.13) are in $C^{2+\alpha,\frac{2+\alpha}{2}}(\Omega_T)$ for some $\alpha > 0$ (see Section 3.1), we get the following corollary.

**Corollary 3.2.3.** *Let* $u$ *be a viscosity solutions of* (3.13). *Then it satisfies an asymptotic mean value formula*

$$\begin{aligned} u(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u(y, t-\varepsilon^2)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u(x+\varepsilon\sigma, t-\varepsilon^2) + u(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] + o(\varepsilon^2). \end{aligned} \tag{3.18}$$

**Remark 3.2.4.** *Our scaling of the time variable is for convenience. The same idea would give for viscosity solutions of*

$$u_t = \mathcal{D}_p u$$

*an asymptotic mean value formula*

$$\begin{aligned} u(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u\Big(y, t - \frac{\varepsilon^2}{2(n+p)}\Big)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u(x+\varepsilon\sigma, t-\frac{\varepsilon^2}{2(n+p)}) + u(x-\varepsilon\sigma, t-\frac{\varepsilon^2}{2(n+p)})}{2} \right] + o(\varepsilon^2). \end{aligned}$$

## 3.3 Control problem formulation

In this section we show that the value of the control problem described in Section 3.1 satisfies the DPP (3.14). Since the game token may be placed outside of $\overline{\Omega}_T$, we denote the compact parabolic boundary strip of width $\varepsilon > 0$ by

$$\Gamma_\varepsilon = \Big(S_\varepsilon \times \big[-\varepsilon^2, 0\big]\Big) \cup \Big(\Omega \times \big[-\varepsilon^2, 0\big]\Big),$$

where

$$S_\varepsilon = \big\{x \in \mathbb{R}^n \setminus \Omega : \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\big\}.$$

Throughout this section, we are given a continuous function

$$F : \Gamma_\varepsilon \to \mathbb{R}.$$

Our control problem with the payoff $F$ was formulated in Section 3.1. The process is stopped when the token hits the boundary strip $\Gamma_\varepsilon$ for the first time at, say $(x_\tau, t_\tau) \in \Gamma_\varepsilon$, and then the controller earns the amount $F(x_\tau, t_\tau)$.

Next we define the stochastic vocabulary for the control problem. A *strategy* is a rule which gives, at each step of the game, a direction $\sigma$,

$$S(t_0, x_0, x_1, ..., x_k) = \sigma \in \mathbb{R}^n, \quad |\sigma| = 1.$$

Here, $S$ is a Borel measurable function. Let $A \subset \Omega_T \cup \Gamma_\varepsilon$ be a measurable set. Given a sequence of token positions $(x_0, t_0), (x_1, t_1), ..., (x_k, t_k)$ and a strategy $S$, the next position of the token is distributed according to the transition probability

$$\begin{aligned}\pi_S\big((x_0, t_0), (x_1, t_1), ..., (x_k, t_k), A\big) &= \beta \frac{\Big|A \cap \Big(B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\}\Big)\Big|}{\big|B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\}\big|} \\ &\quad + \frac{\alpha}{2}\delta_{(x_k + \varepsilon\sigma, t_k - \varepsilon^2)}(A) + \frac{\alpha}{2}\delta_{(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}(A)\end{aligned}$$

where in the first term we use the $n$-dimensional Lebesgue measure, and in the last terms $\delta_{(y,s)}(B) = 1$ if $(y, s) \in B$ and 0 otherwise.

For a starting point $(x_0, t_0)$, a strategy $S$ and the corresponding transition probabilities, we can use Kolmogorov's extension theorem to determine a unique probability measure $\mathbb{P}_S^{(x_0,t_0)}$ in the space of all game sequences denoted $H^\infty$. The expected payoff is then

$$\mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] = \int_{H^\infty} F(x_\tau, t_\tau)\, d\mathbb{P}_S^{(x_0,t_0)},$$

and the value of the game for the controller is

$$u^\varepsilon(x_0, t_0) = \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)].$$

Since $F$ is bounded and

$$\tau \leq \frac{T}{\varepsilon^2} + 1,$$

the value of the game is well defined. From the definition we immediately get the following comparison principle.

**Proposition 3.3.1.** *Fix $\varepsilon > 0$. Let $u^\varepsilon$ be the value of the game with the payoff $F_1$, and $v^\varepsilon$ the value of the game with the payoff $F_2$. Assume that $F_1 \geq F_2$ on $\Gamma_\varepsilon$. Then $u^\varepsilon \geq v^\varepsilon$ in $\Omega_T$.*

Our aim is to show that the value function $u^\varepsilon$ satisfies the DPP with the boundary data $F$.

**Definition 3.3.2.** *A Borel measurable function $u_\varepsilon$ satisfies the dynamic programming principle, abbreviated DPP, in $\Omega_T$, with the boundary data $F$, if*

$$
\begin{aligned}
u_\varepsilon(x,t) &= \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y, t-\varepsilon^2)\,dy \\
&\quad + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x+\varepsilon\sigma, t-\varepsilon^2) + u_\varepsilon(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] \qquad \text{in } \Omega_T \\
u_\varepsilon(x,t) &= F(x,t) \quad \text{on } \Gamma_\varepsilon.
\end{aligned}
$$

**Lemma 3.3.3.** *There is a unique Borel measurable function $u_\varepsilon$ satisfying the DPP. Moreover, $u_\varepsilon$ is lower semi-continuous.*

*Proof.* The existence and uniqueness of such a function $u_\varepsilon$ can be seen from the following argument. Given $F$ on $\Gamma_\varepsilon$, we can determine $u_\varepsilon(x,t)$ for all $x \in \Omega$ and $0 < t < \varepsilon^2$. We want to continue this process, but we need to make sure that the function is lower semi-continuous or at least Borel measurable. The following argument is from personal communication with Brustad, Lindqvist, and Manfredi. In general, when $u$ is any bounded and lower semi-continuous function, then by using Fatou's lemma,

$$
\begin{aligned}
&\frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u(y, t-\varepsilon^2)\,dy \\
&\quad + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u(x+\varepsilon\sigma, t-\varepsilon^2) + u(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right]
\end{aligned}
$$

is again bounded and lower semi-continuous. This gives a lower semi-continuous function $u_\varepsilon$ defined for all $x \in \Omega$ and $0 < t < \varepsilon^2$. Continuing this process until $t = T$ gives the desired function.

□

**Lemma 3.3.4.** *Let $u_\varepsilon$ be the unique function satisfying the DPP of definition 3.3.2 with the boundary data $F$ on $\Gamma_\varepsilon$, and let $u^\varepsilon$ be the value of the game with the payoff $F$. Then*

$$u_\varepsilon = u^\varepsilon.$$

*Proof.* Let $(x_0, t_0) \in \Omega_T$. We aim to show that $u_\varepsilon(x_0, t_0) = u^\varepsilon(x_0, t_0)$. Assume that the game starts at $(x_0, t_0) \in \Omega_T$.

First we assume that the controller uses an arbitrary strategy $S$. Then we have for the function $u_\varepsilon$ satisfying the DPP,

$$\begin{aligned}
&\mathbb{E}_S^{(x_0,t_0)}[u_\varepsilon(x_{k+1},t_{k+1})|(t_0,x_0,x_1,...,x_k)] = \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y,t_k-\varepsilon^2)\,dy \\
&\quad + \alpha \frac{u_\varepsilon(x_k+\varepsilon\sigma,t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma,t_k-\varepsilon^2)}{2} \\
&\le \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y,t_k-\varepsilon^2)\,dy \\
&\quad + \alpha \sup_{|\sigma|=1}\left[\frac{u_\varepsilon(x_k+\varepsilon\sigma,t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma,t_k-\varepsilon^2)}{2}\right] \\
&= u_\varepsilon(x_k,t_k).
\end{aligned}$$

This shows that $M_k := u_\varepsilon(x_k,t_k)$ is a supermartingale, so

$$\mathbb{E}_S^{(x_0,t_0)}[F(x_\tau,t_\tau)|(t_0,x_0,x_1,...,x_{\tau-1})] \le u_\varepsilon(x_0,t_0)$$

by the optimal stopping theorem. Hence

$$u^\varepsilon(x_0,t_0) = \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau,t_\tau)] \le u_\varepsilon(x_0,t_0).$$

To prove the reverse inequality, we choose a strategy $S_0$ giving a corresponding $\sigma(x,t)$ for the controller that *almost* maximizes $u_\varepsilon(x,t)$. To be more precise, for arbitrary $\eta>0$, the controller chooses

$$\begin{aligned}
&\frac{u_\varepsilon(x_k+\varepsilon\sigma(x_k,t_k),t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma(x_k,t_k),t_k-\varepsilon^2)}{2} \\
&\ge \sup_{|\sigma|=1}\left[\frac{u_\varepsilon(x_k+\varepsilon\sigma,t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma,t_k-\varepsilon^2)}{2}\right] - \eta 2^{-(k+1)}.
\end{aligned}$$

The function $S_0$ can be taken to be a Borel function, see Lemma 3.4 in [LM17].

We obtain

$$\begin{aligned}
&\mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_{k+1},t_{k+1}) - \eta 2^{-(k+1)}|(t_0,x_0,x_1,...,x_k)] \\
&\ge \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y,t_k-\varepsilon^2)\,dy \\
&\quad + \alpha \sup_{|\sigma|=1}\left[\frac{u_\varepsilon(x_k+\varepsilon\sigma,t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma,t_k-\varepsilon^2)}{2}\right] \\
&\quad - \alpha\eta 2^{-(k+1)} - \eta 2^{-(k+1)} \\
&\ge u_\varepsilon(x_k,t_k) - \eta 2^{-k}.
\end{aligned}$$

Hence

$$M_k = u_\varepsilon(x_k, t_k) - \eta 2^{-k}$$

is a submartingale. Using the optimal stopping theorem for this submartingale we find

$$\begin{aligned} u^\varepsilon(x_0, t_0) &= \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] \geq \mathbb{E}_{S_0}^{(x_0,t_0)}[F(x_\tau, t_\tau)] \\ &\geq \mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_\tau, t_\tau) - \eta 2^{-k}] \\ &\geq \mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_0, t_0) - \eta 2^{-0}] = u_\varepsilon(x_0, t_0) - \eta. \end{aligned}$$

Since $\eta > 0$ was arbitrary, this proves the lemma. □

## 3.4 Convergence to the viscosity solution

In this section, we are given a continuous payoff function $F : \Gamma_1 \to \mathbb{R}$. Our goal is to show that with this payoff, value functions of our game converge uniformly to the unique viscosity solution of

$$\begin{cases} 2(n+p)u_t = \mathcal{D}_p u & \text{in } \quad \Omega_T, \\ u = F & \text{on } \quad \partial_p \Omega_T. \end{cases} \tag{3.19}$$

We will make use of the following Arzelá-Ascoli-type lemma, which has been previously used e.g. in [MPR10,PR16,BER19]. We omit the proof, which is a modification of [MPR12, Lemma 4.2].

**Lemma 3.4.1.** *Let* $\left\{ f_\varepsilon : \overline{\Omega}_T \to \mathbb{R} \right\}_{\varepsilon \in (0,1)}$ *be a uniformly bounded family of functions such that for a given* $\eta > 0$*, there are constants* $r_0$ *and* $\varepsilon_0$ *such that for every* $\varepsilon < \varepsilon_0$ *and any* $(x,t), (y,s) \in \overline{\Omega}_T$ *with*

$$|(x,t) - (y,s)| < r_0,$$

*it holds*

$$\left| f_\varepsilon(x,t) - f_\varepsilon(y,s) \right| < \eta.$$

*Then there exists a uniformly continuous function* $f : \overline{\Omega}_T \to \mathbb{R}$ *and a subsequence, still denoted by* $(f_\varepsilon)$*, such that* $f_\varepsilon \to f$ *uniformly in* $\overline{\Omega}_T$ *as* $\varepsilon \to 0$.

For the next lemma, we assume that the domain $\Omega$ satisfies a *uniform exterior sphere condition*. That is, we assume that there is $\delta > 0$ such that for any $y \in \partial\Omega$, there is an open ball $B_\delta \subset \mathbb{R}^n \setminus \Omega$ with the radius $\delta$ so that $\overline{B}_\delta \cap \overline{\Omega} = \{y\}$.

**Lemma 3.4.2.** *The family* $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ *of value functions of the game satisfies the assumptions of Lemma 3.4.1.*

*Proof.* Since $|u_\varepsilon(x,t)| \le \max_{\Gamma_1}|F|$ for all $(x,t)\in\overline{\Omega}_T$ and $\varepsilon\in(0,1)$, the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded.

Fix $\eta>0$. Since the payoff function $F$ is uniformly continuous on $\Gamma_1$, there is $\gamma>0$ so that when $(x,t),(y,s)\in\Gamma_1$ with $|(x,t)-(y,s)|<\gamma$, it holds $|F(x,t)-F(y,s)|<\frac{\eta}{2}$. We prove the asymptotic equicontinuity of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ in four steps. In all steps we have $\varepsilon<\varepsilon_0$ and $|(x,t)-(y,s)|<r_0$. The precise choices of $\varepsilon_0$ and $r_0$ clarify during the proof. We will denote by $C_1, C_2, \ldots$ constants larger than 1 which may depend only on $n, \delta$, and the diameter of $\Omega$.

## Step 1

If $(x,t),(y,s)\in\partial_p\Omega_T$, then

$$|u_\varepsilon(x,t)-u_\varepsilon(y,s)| = |F(x,t)-F(y,s)| < \eta$$

when $r_0<\gamma$.

## Step 2

Suppose that $(x,t)\in\Omega_T$ and $(y,0)\in\Gamma_\varepsilon$. Let us start the game from $(x_0,t_0)=(x,t)$ with an arbitrary strategy $S$. We obtain

$$\begin{aligned}
&\mathbb{E}_S^{(x_0,t_0)}[|x_k-x_0|^2\,|\,(t_0,x_0,\ldots,x_{k-1})]\\
&= \frac{\alpha}{2}(|(x_{k-1}+\sigma\varepsilon)-x_0|^2+|(x_{k-1}-\sigma\varepsilon)-x_0|^2)+\beta\fint_{B_\varepsilon(x_{k-1})}|y-x_0|^2\,dy\\
&\le \alpha(|x_{k-1}-x_0|^2+\varepsilon^2)+\beta(|x_{k-1}-x_0|^2+C_1\varepsilon^2)\\
&\le |x_{k-1}-x_0|^2+C_1\varepsilon^2.
\end{aligned}$$

Hence,

$$M_k := |x_k-x_0|^2 - C_1 k\varepsilon^2$$

is a supermartingale, and the optimal stopping theorem gives

$$\mathbb{E}_S^{(x_0,t_0)}[|x_\tau-x_0|^2] \le |x_0-x_0|^2 + C_1\varepsilon^2\mathbb{E}_S^{(x_0,t_0)}[\tau] \le C_1(r_0+\varepsilon_0^2).$$

Here, we used the fact that the stopping time $\tau\le\frac{t_0}{\varepsilon^2}+1$ for a game starting at $t_0$ and in this case $t_0\le r_0$. Since this is true for all strategies, it holds

$$\sup_S \mathbb{E}_S^{(x_0,t_0)}[|x_\tau-x_0|^2] \le C_1(r_0+\varepsilon_0^2),$$

which yields

$$|u_\varepsilon(x_0,t_0) - u_\varepsilon(x_0,0)| = |\sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau,t_\tau)] - F(x_0,0)| < \frac{\eta}{2},$$

when $r_0, \varepsilon_0$ are chosen so that $C_1(r_0+\varepsilon_0^2) < \gamma^2$.

The triangle inequality finishes the argument. Recalling that $(x_0,t_0) = (x,t)$, we have

$$|u_\varepsilon(x,t) - u_\varepsilon(y,0)| \le |u_\varepsilon(x,t) - F(x,0)| + |F(x,0) - F(y,0)| < \eta.$$

## Step 3

Suppose that $(x,t) \in \Omega_T$ and $(y,s) \in \partial_p\Omega_T$ with $y \in \partial\Omega$. Since the domain $\Omega$ satisfies the uniform exterior sphere condition with $\delta$, there is a ball $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ with $\partial B_\delta(z) \cap \overline{\Omega} = \{y\}$.

We use a barrier argument. In an annulus of $\mathbb{R}^n$, define a function $w$ as

$$\begin{cases} w(x) = -a|x-z|^2 - b|x-z|^{-\xi} + c & \text{in } B_R(z) \setminus \overline{B}_\delta(z), \\ w = 0 & \text{on } \partial B_\delta(z), \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial B_R(z), \end{cases}$$

where $\frac{\partial w}{\partial \nu}$ is the normal derivative, and $R$ is chosen so that $\Omega \subset B_R(z)$. The exponent $\xi = n+p-4 > 0$, since $p > 2$ and we may assume that $n \ge 2$ (1-dimensional case is essentially a random walk in an open interval). The positive constants $a, b, c$ are specified below. The function $w$ satisfies

$$\Delta w(x) = -2an + b\xi n|x-z|^{-\xi-2} - b\xi(\xi+2)|x-z|^{-\xi-2},$$

$$\lambda_n(D^2 w(x)) = -2a + b\xi|x-z|^{-\xi-2},$$

hence

$$\mathcal{D}_p w = -2a(n+p-2) \quad \text{in } B_R(z) \setminus \overline{B}_\delta(z), \tag{3.20}$$

and it can be extended as a solution to the same equations in $B_{R+\varepsilon}(z) \setminus \overline{B}_{\delta-\varepsilon}(z)$ so that equation (3.20) holds also near the boundaries. It satisfies an estimate

$$w(x) \le C_2(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x) + o(1)$$

for any $x \in B_R(z) \setminus B_\delta(z)$. Here $o(1) \to 0$ when $\varepsilon \to 0$.

Let us consider for a moment an elliptic game starting at $x_0 = x$ and played by the rules of our game without a time-dependence in the annulus $B_R(z) \setminus \overline{B}_\delta(z)$, with a special rule that if we are at, say $x_k$, a possible random move is chosen from $B_\varepsilon(x_k) \cap B_R(z)$ according to the uniform probability density, and also the controller cannot exit $B_R(z)$. The game ends when the token enters the ball

$\overline{B}_\delta(z)$. Because of the random moves, the game ends almost surely in a finite time. Define a stopping time for this game as $\tau^*$,

$$\tau^* = \inf\{k \, : \, x_k \in \overline{B}_\delta(z)\}.$$

Let $S$ be an arbitrary strategy for the controller. The Taylor expansion for $w$ gives

$$\begin{aligned}
&\frac{1}{2}(w(x_{k-1}+\varepsilon\sigma) + w(x_{k-1}-\varepsilon\sigma)) \\
&\quad = w(x_{k-1}) + \frac{1}{2}\varepsilon^2\langle D^2 w(x_{k-1})\sigma, \sigma\rangle + o(\varepsilon^2) \\
&\quad \le w(x_{k-1}) + \frac{1}{2}\varepsilon^2 \lambda_n(D^2 w(x_{k-1})) + o(\varepsilon^2),
\end{aligned}$$

since the first order terms vanish,

$$\langle Dw(x_{k-1}), \varepsilon\sigma\rangle + \langle Dw(x_{k-1}), -\varepsilon\sigma\rangle = 0.$$

Moreover, since $w$ is radially increasing, it holds

$$\fint_{B_\varepsilon(x_{k-1})\cap B_R(z)} w(y)\,\mathrm{d}y \le w(x_{k-1}) + \frac{\varepsilon^2}{2(n+2)}\Delta w(x_{k-1}) + o(\varepsilon^2).$$

By choosing the constant $a$ properly,

$$M_k := w(x_k) + k\varepsilon^2$$

is a supermartingale. Indeed, we have

$$\begin{aligned}
\mathbb{E}_S^{x_0}[M_k \,|\, x_0, ..., x_{k-1}] &= \frac{\alpha}{2}(w(x_{k-1}+\varepsilon\sigma) + w(x_{k-1}-\varepsilon\sigma)) \\
&\quad + \beta \fint_{B_\varepsilon(x_{k-1})\cap B_R(z)} w(y)\,dy + k\varepsilon^2 \\
&\le w(x_{k-1}) + \frac{\varepsilon^2}{2(p+n)}\mathcal{D}_p w(x_{k-1}) + k\varepsilon^2 + o(\varepsilon^2) \\
&= w(x_{k-1}) - \frac{n+p-2}{n+p} a\varepsilon^2 + k\varepsilon^2 + o(\varepsilon^2) \\
&\le w(x_{k-1}) + (k-1)\varepsilon^2,
\end{aligned}$$

by choosing for example $a = 2\frac{n+p}{n+p-2}$ and assuming that $o(\varepsilon^2) < \varepsilon^2$. The choice of $a$ determines the other constants $b$ and $c$: The Neumann and Dirichlet boundary conditions of the barrier function $w$ are satisfied by choosing $b = (2a/\xi)R^{\xi+2}$ and $c = a\delta^2 + b\delta^{-\xi}$.

By the optimal stopping theorem, we have

$$\mathbb{E}_S^{x_0}[w(x_{\tau *}) + \tau^* \varepsilon^2] \le w(x_0),$$

that is,

$$\mathbb{E}_S^{x_0}[\tau^*] \leq \frac{w(x_0)}{\varepsilon^2} \leq \frac{C_2(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

where we used $|\mathbb{E}_S^{x_0}[w(x_{\tau*})]| \leq o(1)$.

Now we come back to our game, starting at $(x_0, t_0) = (x, t)$, again with an arbitrary strategy $S$. Since it holds $|x_0 - y| \geq \mathrm{dist}(\partial B_\delta(z), x_0)$, for the stopping time of our game we now have an estimate

$$\begin{aligned}\mathbb{E}_S^{(x_0,t_0)}[\tau] &\leq \mathbb{E}_S^{(x_0,t_0)}[\tau^*]\\ &\leq \frac{C_2(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2}\\ &\leq \frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2}.\end{aligned}$$

By using the same martingale argument as in Step 2 but replacing $x_0$ by $y$, we have

$$\begin{aligned}\mathbb{E}_S^{(x_0,t_0)}[|x_\tau - y|^2] &\leq |x_0 - y|^2 + C_1\varepsilon^2\mathbb{E}_S^{(x_0,t_0)}[\tau]\\ &\leq |x_0 - y|^2 + C_1\varepsilon^2\frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2}\\ &\leq |x_0 - y|^2 + C_3(|x_0 - y| + o(1))\\ &< r_0^2 + C_3(r_0 + o(1)) < \left(\frac{\gamma}{2}\right)^2,\end{aligned}$$

when $\varepsilon_0, r_0$ are chosen so that $C_3(r_0 + o(1)) < \left(\frac{\gamma}{4}\right)^2$ and $r_0^2 < \left(\frac{\gamma}{4}\right)^2$. This also gives

$$|\mathbb{E}_S^{(x_0,t_0)}[t_\tau] - t_0| < \left(\frac{\gamma}{4}\right)^2.$$

Hence, we have

$$|u_\varepsilon(x_0, t_0) - u_\varepsilon(y, t_0)| = |\sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] - F(y, t_0)| < \frac{\eta}{2},$$

and recalling that $(x_0, t_0) = (x, t)$ the triangle inequality gives

$$|u_\varepsilon(x, t) - u_\varepsilon(y, s)| \leq |u_\varepsilon(x, t) - F(y, t)| + |F(y, t) - F(y, s)| < \eta.$$

### Step 4

Finally, suppose that $(x, t), (y, s) \in \Omega_T$. This is an argument based on translation invariance and comparison principle. Let $r_0, \varepsilon_0$ satisfy the conditions of the previous steps. Define an inner $\varepsilon$-strip $I_\varepsilon$ by

$$I_\varepsilon := \{(z, r) \in \overline{\Omega}_T \;:\; \mathrm{dist}((z, r), \partial_p\Omega_T) \leq r_0\}.$$

If $(x,t) \in I_\varepsilon$, there is a point $(x',t') \in \partial_p \Omega_T$ such that $|(x,t)-(x',t')| \le r_0$. Then from the conclusions of the previous steps we obtain

$$|u_\varepsilon(x,t)-u_\varepsilon(y,s)| \le |u_\varepsilon(x,t)-F(x',t')| + |F(x',t')-u_\varepsilon(y,s)| < \eta.$$

The argument is identical if $(y,s) \in I_\varepsilon$, so it remains to study the case $(x,t),(y,s) \in \Omega_T \setminus I_\varepsilon$. We may assume that $t \le s$. Define functions $F_1, F_2$ on the strip $I_\varepsilon$ as follows,

$$F_1(z,r) = u_\varepsilon(z-x+y, r-t+s) - \eta, \quad F_2(z,r) = u_\varepsilon(z-x+y, r-t+s) + \eta.$$

Then

$$F_1(z,r) \le u_\varepsilon(z,r) \le F_2(z,r)$$

for all $(z,r) \in I_\varepsilon$. Let $u_\varepsilon^1$ be the value function of the game in $\Omega_T \setminus I_\varepsilon$ with the payoff $F_1$ on $I_\varepsilon$, and $u_\varepsilon^2$ the value function of the game in $\Omega_T \setminus I_\varepsilon$ with the payoff $F_2$ on $I_\varepsilon$. By the uniquess of the value function, we have for all $(z,r) \in \Omega_T \setminus I_\varepsilon$

$$\begin{aligned} u_\varepsilon^1(z,r) &= u_\varepsilon(z-x+y, r-t+s) - \eta, \\ u_\varepsilon^2(z,r) &= u_\varepsilon(z-x+y, r-t+s) + \eta. \end{aligned}$$

By the comparison principle, see Proposition 3.3.1, we have

$$\begin{aligned} u_\varepsilon(x,t) &\ge u_\varepsilon^1(x,t) = u_\varepsilon(y,s) - \eta, \\ u_\varepsilon(x,t) &\le u_\varepsilon^2(x,t) = u_\varepsilon(y,s) + \eta. \end{aligned} \qquad \square$$

From the previous lemmas it follows that if $(u_{\varepsilon_j})$ is a sequence of value functions with $\varepsilon_j \to 0$ and $(u_{\varepsilon_{j_k}})$ is an arbitrary subsequence, then this subsequence has a subsequence converging uniformly to $v$. Hence, the sequence $(u_{\varepsilon_j})$ converges to $v$ uniformly, and we write $u_\varepsilon \to v$ to simplify the notation. It remains to show that the function $v$ is the solution of (3.19).

**Theorem 3.4.3.** *The uniform limit $v = \lim_{\varepsilon \to 0} u_\varepsilon$ is the unique viscosity solution of* (3.19).

*Proof.* By uniqueness of viscosity solutions (see [CIL92]), it is sufficient to show that $v$ is a viscosity solution of (3.19). To this end, let $\phi \in C^2$ touch $v$ from above at $(x_0,t_0) \in \Omega_T$,

$$0 = (v-\phi)(x_0,t_0) > (v-\phi)(x,t)$$

for all $(x,t)$ close to $(x_0,t_0)$. From the definition of supremum, given $\delta_\varepsilon > 0$, there are points $(x_\varepsilon, t_\varepsilon)$ close to $(x_0,t_0)$ such that

$$u_\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon) \ge u_\varepsilon(y,s) - \phi(y,s) - \delta_\varepsilon$$

for all $(y,s)$ in a neighborhood of $(x_\varepsilon, t_\varepsilon)$. Using the fact that $u_\varepsilon \to v$ uniformly and $v-\phi$ is a continuous function with a maximum point at $(x_0, t_0)$, we see that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ as $\varepsilon \to 0$.

Since $\phi \in C^2(\Omega_T)$, Theorem 3.2.1 gives

$$
\begin{aligned}
\beta & \fint_{B_\varepsilon(x_\varepsilon)} \phi(y, t_\varepsilon - \varepsilon^2)\, dy \\
& + \alpha \sup_{|\sigma|=1} \left[ \frac{\phi(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
& = \phi(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\end{aligned}
$$

We can now estimate

$$
\begin{aligned}
\beta & \fint_{B_\varepsilon(x_\varepsilon)} u_\varepsilon(y, t_\varepsilon - \varepsilon^2)\, dy \\
& + \alpha \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + u_\varepsilon(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
\le\ & u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \beta \fint_{B_\varepsilon(x)} \phi(y, t_\varepsilon - \varepsilon^2)\, dy \\
& + \alpha \sup_{|\sigma|=1} \left[ \frac{\phi(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
=\ & u_\varepsilon(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\end{aligned}
$$

As the function $u_\varepsilon$ satisfies the DPP, we are left with

$$
0 < \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
$$

Choose now $\delta_\varepsilon = o(\varepsilon^2)$. Dividing by $\varepsilon^2$ and letting $\varepsilon \to 0$ gives

$$
2(n+p)\phi_t(x_0, t_0) \le \mathcal{D}_p \phi(x_0, t_0),
$$

which shows that $v$ is a viscosity subsolution. To show that $v$ is a viscosity supersolution is analogous. □

# Bibliography


[AS12] S. Armstrong, C. Smart. A finite difference approach to the infinity Laplace equation and tug-of-war games. *Trans. Amer. Math. Soc.* 364(2):595–636, 2012.

[BER19] P. Blanc, C. Esteve, J.D. Rossi. The evolution problem associated with eigenvalues of the Hessian. ArXiv preprint:1901.01052, 2019.

[Bru17] K. Brustad. Superposition of $p$-superharmonic functions. To appear in *Adv. Calc. Var.*

[Bru18] K. Brustad. Sublinear elliptic operators. ArXiv preprint:1802.04297, 2018.

[BLM18] K. Brustad, P. Lindqvist, J.J. Manfredi. A discrete stochastic interpretation of the Dominative $p$-Laplacian. ArXiv preprint:1809.00714, 2018.

[CIL92] M.G. Crandall, H. Ishii, P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.,* 27:1–67, 1992.

[CZ03] M.G. Crandall, J. Zhang. Another way to say harmonic. *Trans. Amer. Math. Soc.* 355(1):241–263, 2003.

[Han18] J. Han. Local Lipschitz regularity for functions satisfying a time-dependent dynamic programming principle. ArXiv preprint:1812.00646, 2018.

[Hoe19] F.A. Høeg. Concave power solutions of the Dominative $p$-Laplace equation. ArXiv preprint:1901.07053, 2019.

[KS10] R. V. Kohn, S. Serfaty. A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. *Comm. Pure Appl. Math.* 63(10):1298–1350, 2010.

[Lew18] M. Lewicka. Random tug of war games for the $p$-Laplacian: $1 < p < \infty$. ArXiv preprint:1810.03413, 2018.

[LM08] P. Lindqvist, J.J. Manfredi. Note on a remarkable superposition for a nonlinear equation. *Proc. Amer. Math. Soc.* 136(1):133–140 (electronic), 2008.

[LM17] M. Lewicka, J.J. Manfredi.The obstacle problem for the p-laplacian via optimal stopping of tug-of-war games. *Probability Theory and Related Fields.* 162(1-2):349-378,2017

[MPR10] J.J. Manfredi, M. Parviainen, J.D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM J. Math. Anal.* 42(5):2058-2081, 2010.

[MPR12] J.J. Manfredi, M. Parviainen, J.D. Rossi. On the definition and properties of p-harmonious functions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 11(2):215–241, 2012.

[PR16] M. Parviainen, E. Ruosteenoja. Local regularity for time-dependent tug-of-war games with varying probabilities. *J. Differential Equations*, 261(2):1357–1398, 2016.

[PS08] Y. Peres, S. Sheffield. Tug-of-war with noise: a game-theoretic view of the $p$-Laplacian. *Duke Math. J.*, 145(1):91–120, 2008.

[PSSW09] Y. Peres, O. Schramm, S. Sheffield, D. B. Wilson. Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.*, 22(1):167–210, 2009.

[Wan92] L. Wang. On the regularity theory of fully nonlinear parabolic equations. II. *Comm. Pure Appl. Math.*, 45(2):141–178, 1992.

# Concave power solutions of the Dominative p-Laplace equation

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# Concave power solutions of the Dominative $p$-Laplace equation

**Abstract.**In this paper, we study properties of solutions of the Dominative $p$-Laplace equation with homogeneous Dirichlet boundary conditions in a bounded convex domain $\Omega$. For the equation $-\mathcal{D}_p u = 1$, we show that $\sqrt{u}$ is concave, and for the eigenvalue problem $\mathcal{D}_p u + \lambda u = 0$, we show that $\log u$ is concave.

## 4.1 Introduction

The *Dominative* $p$-Laplace operator was defined as

$$\mathcal{D}_p u := \Delta u + (p-2)\lambda_{\max}(D^2 u), \quad p \geq 2,$$

by Brustad in [B1] and later studied in [B2]. See also [BLM] for a stochastic interpretation and a game-theoretic approach of the equation. Here, $\lambda_{\max}$ denotes the largest eigenvalue of the Hessian matrix

$$D^2 u = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{ij}.$$

We shall study the two equations

$$-\mathcal{D}_p u = 1 \quad \text{and} \quad \mathcal{D}_p u + \lambda u = 0$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$. The positive solutions with zero boundary values have the property for $-\mathcal{D}_p u = 1$ that $\sqrt{u}$ is concave, see Theorem 4.1.1 below. In Theorem 4.1.2 we show that for $\mathcal{D}_p u + \lambda u = 0$, $\log u$ is concave. Problems related to concave solutions have been studied for $p$-Laplace type equations, and we give a quick review of the results. The operator is closely related to the *normalized* $p$-Laplace operator,

$$\Delta_p^N u = |\nabla u|^{2-p} \operatorname{div}\left( |\nabla u|^{p-2} \nabla u \right),$$

which describes a Tug-of-war game with noise, see [MPR]. Due to this, the operator has been studied extensively over the last 15 years, and we refer to [D],[HL], [APR] for an introduction and some regularity results. The solutions are weak and appear in the form of *viscosity solutions* and we refer to [CIL] for an introduction of viscosity solutions. If $u$ is a solution of the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$, one can show that $\sqrt{u}$ is concave. This problem, including more complex right-hand sides, was studied in the 1970's and 1980's by [Ka],[Ke],[Ko] and [M]. For $n = 1$ and $n = 2$, a brute force calculation shows that $\sqrt{u}$ is concave. For $n \geq 3$ the proofs are more complicated. For the ordinary $p$-Laplacian, [S] showed that $u^{\frac{p-1}{p}}$ is concave. One should note that simply setting $p = 2$ does not simplify the proof. Thus, the papers [Ke] and [Ko] are still of great value. For the infinity Laplacian, $\Delta_\infty u = \left\langle D^2 u \nabla u, \nabla u \right\rangle$, [CF] showed that $u^{\frac{3}{4}}$ is concave. Our result for the Dominative $p$-Laplace equation can be formulated in the following theorem. We say that $\Omega$ satisfies the interior sphere condition if for all $y \in \partial\Omega$ there is an $x \in \Omega$ and an open ball $B_r(x)$ such that $B_r(x) \subset \Omega$ and $y \in \partial B_r(x)$.

**Theorem 4.1.1.** *Let $u \in C(\bar{\Omega})$ be a viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = 1 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

*in a bounded convex domain $\Omega \subset \mathbb{R}^n$ which satisfies the interior sphere condition. Then $\sqrt{u}$ is concave.*

Further, we study the eigenvalue problem and give the following result.

**Theorem 4.1.2.** *Let $u \in C(\bar{\Omega})$ be a positive viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*with $\lambda > 0$ in a bounded convex domain $\Omega \subset R^n$. Then $\log u$ is concave.*

**Remark:** We give a remark on what happens when $p \to \infty$ in Theorem 4.1.1. After dividing the equation by $p$ and letting $p$ approach infinity, the following equation is obtained

$$\begin{cases} -\lambda_{\max}(D^2 u) = 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation has the solution $u = 0$, which is obviously already concave. This is better than the square root being concave, so for $p = \infty$ a stronger result is obtained. (For a less trivial result, another normalization with $p$ is needed.)

For the Helmholtz equation $\Delta u + \lambda u = 0$, the problem related to concave logarithmic solutions has been studied in [BL], [Ko] and [CS]. The nonlinear eigenvalue problem associated with the $p$-Laplace equation has been studied for example in [L] and [S]. In [S], Sakaguchi showed that $\log u$ is a concave function.

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## 4.2 Preliminaries and notation

The gradient of a function $f:\Omega_T \to \mathbb{R}$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

and its Hessian matrix is

$$\left(D^2 f\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

We will use the operator

$$\mathcal{D}_p u = \Delta u + (p-2)\lambda_{\max}(D^2 u)$$

and if applied to a matrix $X \in S^n$, we use

$$D_p X = \operatorname{tr}(X) + (p-2)\lambda_{\max}(X).$$

Also, the *normalized* $p$-Laplace operator is referred to,

$$\Delta_p^N u = \Delta u + (p-2) \cdot \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{N} u_{x_i} u_{x_j} u_{x_i x_j}.$$

**Viscosity solutions.** The Dominative $p$-Laplace operator is uniformly elliptic. Therefore, it is convenient to use viscosity solutions as a notion of weak solutions. Throughout the text, we always keep $p \geq 2$. In the definition below, $g$ is assumed to be continuous in all variables.

**Definition 4.2.1.** *A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution to $-\mathcal{D}_p u = g(x,u,\nabla u)$ if, for all $\phi \in C^2(\Omega)$,*

$$-\mathcal{D}_p \phi(x) \leq g(x,u,\nabla\phi).$$

*at any point $x \in \Omega$ where $u-\phi$ attains a local maximum. A function $u \in LSC(\bar{\Omega})$ is a viscosity supersolution to $-\mathcal{D}_p u = g(x,u,\nabla u)$ if, for all $\phi \in C^2(\Omega)$,*

$$-\mathcal{D}_p \phi(x) \geq g(x,u,\nabla\phi).$$

*at any point $x \in \Omega$ where $u - \phi$ attains a local minimum.*

*A function $u \in C(\bar{\Omega})$ is a viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = g(x,u,\nabla u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

*if it is a viscosity sub- and supersolution of $-\mathcal{D}_p u = g(x,u,\nabla u)$ and $u = 0$ on $\partial\Omega$.*

When defining viscosity solutions to $\Delta_p^N u = g(x,u,\nabla u)$, one has to be careful at points where the gradient vanishes.

**Definition 4.2.2.** *A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution of $-\Delta_p^N u = 1$ if, for all $\phi \in C^2(\Omega)$,*

$$\begin{cases} -\Delta_p^N \phi(x) \le 1, & \text{if } \nabla\phi(x) \neq 0 \\ -\mathcal{D}_p \phi(x) \le 1, & \text{if } \nabla\phi(x) = 0. \end{cases}$$

*at any point $x \in \Omega$ where $u - \phi$ attains a local minimum. A function $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of $-\Delta_p^N u = 1$ if, for all $\phi \in C^2(\Omega)$,*

$$\begin{cases} -\Delta_p^N \phi(x) \ge 1, & \text{if } \nabla\phi(x) \neq 0 \\ \mathcal{D}_p(-\phi(x)) \ge 1, & \text{if } \nabla\phi(x) = 0. \end{cases}$$

*at any point $x \in \Omega$ where $u - \phi$ attains a local minimum. A function $u \in C(\bar{\Omega})$ is a viscosity solution of*

$$\begin{cases} -\Delta_p^N u = 1 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

*if it is a viscosity sub- and supersolution of $-\Delta_p^N u = 1$ and $u = 0$ on $\partial\Omega$.*

We also need an equivalent definition of viscosity solutions using the *sub-* and *superjets*. For functions $u : \Omega \to \mathbb{R}^n$ they are given by

$$J^{2,+}u(x) = \Big\{(q,X) \in \mathbb{R}^n \times S^n : u(y) \le u(x) + \langle q, y-x \rangle \\ + \frac{1}{2}\langle X(y-x), y-x \rangle + o(|y-x|^2|) \text{ as } y \to x\Big\}$$

and

$$J^{2,-}u(x) = \Big\{(q,X) \in \mathbb{R}^n \times S^n : u(y) \ge u(x) + \langle q, y-x \rangle \\ + \frac{1}{2}\langle X(y-x), y-x \rangle + o(|y-x|^2|) \text{ as } y \to x\Big\}.$$

**Definition 4.2.3.** *A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution to $-\mathcal{D}_p u = g(x,u,\nabla u)$ if $(q,X) \in J^{2,+}u(x)$ implies*

$$-D_p X \leq g(x,u,q).$$

*A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution of $-\mathcal{D}_p u = g(x,u,\nabla u)$ if $(q,X) \in J^{2,-}u(x)$ implies*

$$-D_p X \geq g(x,u,q).$$

*A function $u \in C(\bar{\Omega})$ is a viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = g(x,u,\nabla u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

*if it is a viscosity sub- and supersolution of $-\mathcal{D}_p u = g(x,u,\nabla u)$ and $u = 0$ on $\partial\Omega$.*

We mention some results in [K] obtained for the normalized $p$-Laplace equation, which we will use together with the relationship between the normalized $p$-Laplace equation and the Dominative $p$-Laplace equation.

**Lemma 4.2.4.** *A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\Delta_p^N u = 1$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\sqrt{u} \in LSC(\bar{\Omega})$ is a negative viscosity supersolution of*

$$-\Delta_p^N v = \frac{1}{v}\left((p-1)|\nabla v|^2 + \frac{1}{2}\right).$$

**Lemma 4.2.5.** *Let $\lambda > 0$. A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\Delta_p^N u = \lambda u$ if and only if $v = -\ln u \in LSC(\bar{\Omega})$ is a negative viscosity supersolution of*

$$-\Delta_p^N v = -(p-1)|\nabla v|^2 - \lambda.$$

**Properties of the operator.** We give some properties of viscosity solutions of the Dominative $p$-Laplace equation.

- **Comparison principle:** Let $u \in USC(\bar{\Omega})$ be a viscosity subsolution of $-\mathcal{D}_p u = 1$ and let $v \in LSC(\bar{\Omega})$ be a viscosity supersolution of $-\mathcal{D}_p v = 1$. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in $\Omega$. For a proof, see [Theorem 3.3, CIL].

- **Positive supersolutions:** If $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$, then $u > 0$ in $\Omega$. To see this, note that $w = 0$ is a viscosity subsolution, and $u \geq w$ by the comparison principle. This inequality must be strict. If $u(x_0) = 0$, then $x_0$ is a minimum for $u$. Let $\phi(x) = u(x_0)$ be a test function. Then $u - \phi$ has a local minimum at $x_0$. But $-\mathcal{D}_p \phi = 0 < 1$, which contradicts $u$ being a supersolution.

The Dominative $p$-Laplace operator has many of the same properties that the normalized $p$-Laplace operator possess. Here, we give some connections for viscosity solutions.

**Lemma 4.2.6.** *If $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of*

$$-\mathcal{D}_p u = g(x, u, \nabla u),$$

*then $u$ is a viscosity supersolution of*

$$-\Delta_p^N u = g(x, u, \nabla u).$$

*Here, $g$ is assumed to be continuous in all variables. Similarly, if $u \in USC(\bar{\Omega})$ is a viscosity subsolution of $-\Delta_p^N u = g(x, u, \nabla u)$, then $u$ is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$.*

*Proof.* Assume $u$ is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$. If $u - \phi$ obtains a minimum at $x \in \Omega$, we have, provided $\nabla\phi(x) \neq 0$,

$$-\Delta_p^N \phi \geq -\mathcal{D}_p \phi \geq g(x, u, \nabla\phi).$$

If $\nabla\phi(x) = 0$,

$$-\Delta\phi - (p-2)\lambda_{\min}(D^2\phi) \geq -\mathcal{D}_p \phi \geq g(x, u, \nabla\phi).$$

Hence, $u$ is a viscosity supersolution of $-\Delta_p^N u = g(x, u, \nabla u)$. If $u$ is a viscosity subsolution of $-\Delta_p^N u = g(x, u, \nabla u)$ and $u - \phi$ obtains a maximum at $x \in \Omega$,

$$-\mathcal{D}_p \phi \leq -\Delta_p^N \phi \leq g(x, u, \nabla\phi), \quad \text{provided } \nabla\phi(x) \neq 0.$$

On the other hand, if $\nabla\phi(x) = 0$, $-\mathcal{D}_p \phi \leq g(x, u, 0)$ by definition. Hence, $u$ is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$. $\square$

The following Lemma will be applied in the proof of the concavity, and it relies on the fact that the mapping $(q, A) \to \left\langle q, A^{-1} q \right\rangle$ is convex in $S^+$ for each $q \in \mathbb{R}^n$. Here, $S^+$ consists of the symmetric positive definite matrices.

**Lemma 4.2.7.** *Let $X_i \in S^+, \nu_i \in [0,1], i = 1,...,k$, with $\sum_{i=1}^k \nu_i = 1$. Then*

$$\frac{1}{D_p\left(\sum_{i=1}^k \nu_i X_i\right)^{-1}} \geq \sum_{i=1}^k \frac{\nu_i}{D_p X_i^{-1}}.$$

*Proof.* In the appendix of [ALL] it was shown that $(q, A) \to \left\langle q, A^{-1}q\right\rangle$ is convex,

$$\left\langle q, (\mu A_1 + (1-\mu)A_2)^{-1} q\right\rangle \leq \mu \left\langle q, A^{-1}q\right\rangle + (1-\mu)\left\langle q, A_2^{-1}q\right\rangle,$$

for $q \in \mathbb{R}^n, A_1, A_2 \in S^+$ and $\mu \in [0,1]$. Consequently,

$$D_p\left(\mu A_1 + (1-\mu)A_2\right)^{-1} \leq \mu D_p A_1^{-1} + (1-\mu) D_p A_2^{-1}. \tag{4.21}$$

We label $c_1 = D_p(X_1^{-1}), c_2 = D_p(X_2^{-1})$ and choose

$$A_1 = \frac{X_1}{c_2},\ A_2 = \frac{X_2}{c_1},\ \mu = \frac{\nu c_2}{\nu c_2 + (1-\nu)c_1}.$$

With these choices,

$$D_p\left(\nu X_1 + (1-\nu)X_2\right)^{-1} = \frac{D_p\left(\mu A_1 + (1-\mu)A_2\right)^{-1}}{\nu c_2 + (1-\nu)c_1}.$$

Using inequality (4.21) we find

$$\begin{aligned}
\frac{1}{D_p\left(\nu X_1 + (1-\nu)X_2\right)^{-1}} &= \frac{\nu c_2 + (1-\nu)c_1}{D_p\left(\mu A_1 + (1-\mu)A_2\right)^{-1}} \\
&\geq \frac{\nu c_2 + (1-\nu)c_1}{\mu D_p(A_1^{-1}) + (1-\mu)D_p(A_2^{-1})} \\
&= \frac{\nu c_2 + (1-\nu)c_1}{\mu c_1 c_2 + (1-\mu)c_1 c_2} \\
&= \frac{\nu}{c_1} + \frac{1-\nu}{c_2} \\
&= \frac{\nu}{D_p(X_1^{-1})} + \frac{1-\nu}{D_p(X_2^{-1})}.
\end{aligned}$$

By induction, the inequality in Lemma 4.2.7 holds.

□

**Convex envelope**

The *convex envelope* of a function $u : \Omega \to \mathbb{R}^n$ is defined as

$$u_{**}(x) = \inf\left\{\sum_{i=1}^k \mu_i u(x_i) : x_i \in \Omega, \sum \mu_i x_i = x, \sum \mu_i = 1, k \leq n+1, \mu_i \geq 0\right\}.$$

We are interested in the convex envelope of the square root, $v = -\sqrt{u}$, and we have the following result on what happens near the boundary of $\Omega$.

**Lemma 4.2.8.** *Let $u$ be a viscosity solution to $-\mathcal{D}_p u = 1$ in a convex domain $\Omega$ that satisfies the interior sphere condition. Further let $x \in \Omega$, $x_1, \dots, x_k \in \bar{\Omega}$, $\sum_{i=1}^k \mu_i = 1$ with*

$$x = \sum_{i=1}^k \mu_i x_i, \quad u_{**}(x) = \sum_{i=1}^k \mu_i u(x_i).$$

*Then $x_1, \dots, x_k \in \Omega$.*

*Proof.* Since $u$ is, in particular a viscosity supersolution to $-\Delta_p^N u = 1$, Lemma 3.2 in [K] gives the result. □

## 4.3 Concave square-root solutions.

First, we examine which equation $v = -\sqrt{u}$ solves in the viscosity sense.

**Lemma 4.3.1.** *A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\sqrt{u} \in LSC(\bar{\Omega})$, with $v = 0$ on $\partial\Omega$, is a negative viscosity supersolution of*

$$-\mathcal{D}_p v = \frac{1}{v}\left((p-1)|\nabla v|^2 + \frac{1}{2}\right).$$

*Proof.* Let $u$ be a viscosity subsolution of $-\mathcal{D}_p u = 1$. Take $\phi \in C^2(\Omega)$ such that for some $r > 0$,

$$0 = (v - \phi)(x_0) < (v - \phi)(x), \quad \text{for all } x \in B_r(x_0),$$

so that $v - \phi$ has a strict local minimum point at $x_0 \in \Omega$. Let $\psi(x) = \phi(x)^2$. Then, since $v(x), \phi(x) < 0$,

$$\begin{aligned}
(u - \psi)(x_0) &= \big(v(x_0) - \phi(x_0)\big)\big(v(x_0) + \phi(x_0)\big) = 0 \\
(u - \psi)(x) &= \big(v(x) - \phi(x)\big)\big(v(x) + \phi(x)\big) < 0.
\end{aligned}$$

Hence, $u - \psi$ has a strict local *maximum* at $x_0$. We see that $\psi_{x_i} = 2\phi\phi_{x_i}$, $\psi_{x_i x_j} = 2\phi_{x_i}\phi_{x_j} + 2\phi\phi_{x_i x_j}$. Since $u$ is a viscosity subsolution we have at $x_0$,

$$\begin{aligned}
1 &\geq -\mathcal{D}_p \psi \\
&= -2\mathrm{tr}\left(\nabla\phi \otimes \nabla\phi + \phi D^2\phi\right) - 2(p-2)\lambda_{\max}\left(\nabla\phi \otimes \nabla\phi + \phi D^2\phi\right) \\
&\geq -2|\nabla\phi|^2 - 2\phi\Delta\phi - 2(p-2)\lambda_{\max}\left(\nabla\phi \otimes \nabla\phi\right) - 2\phi(p-2)\lambda_{\max}(D^2\phi) \\
&= -2(p-1)|\nabla\phi|^2 - 2\phi\mathcal{D}_p\phi.
\end{aligned}$$

Dividing by $\frac{1}{2\phi(x_0)}$ gives $-\mathcal{D}_p\phi(x_0) \geq \frac{1}{2\phi(x_0)}\left((p-1)|\nabla\phi(x_0)|^2 + \frac{1}{2}\right)$, which shows that $v$ is a viscosity supersolution of

$$-\mathcal{D}_p v = \frac{1}{v}\left((p-1)|\nabla v|^2 + \frac{1}{2}\right).$$

On the other hand, suppose $v \in LSC(\Omega)$ is a negative viscosity supersolution of $-\mathcal{D}_p v = \frac{1}{v}\left((p-1)|\nabla v|^2 + \frac{1}{2}\right)$. By Lemma 4.2.6, $v$ is a viscosity supersolution of

$$-\Delta_p^N v \geq \frac{1}{v}\left((p-1)|\nabla v|^2 + \frac{1}{2}\right).$$

Applying Lemma 4.2.4 we see that $u = v^2$ is a positive viscosity subsolution of

$$-\Delta_p^N u = 1.$$

A second application of Lemma 4.2.6 shows that $u$ is a viscosity subsolution of

$$-\mathcal{D}_p u = 1.$$

□

We now focus our attention on the convex envelope, $v_{**}$. It turns out that $v_{**}$ is a viscosity supersolution to the same equation as $v$.

**Lemma 4.3.2.** *Let $u \in USC(\bar{\Omega})$ be a positive viscosity subsolution to $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$ in a convex domain $\Omega$ that satisfies the interior sphere condition. If $v = -\sqrt{u}$, then $v_{**}$ is a negative viscosity supersolution to*

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}}\left((p-1)|\nabla v_{**}|^2 + \frac{1}{2}\right)$$

*with $v_{**} = 0$ on $\partial\Omega$.*

*Proof.* According to [ALL, Lemma 4] we have $v_{**} = v = 0$ on $\partial\Omega$ so we only have to show that $v_{**}$ is a viscosity supersolution. To this end, let $(q, A) \in J^{2,-}v_{**}(x)$. By Lemma 4.2.8 we can decompose $x$ in a convex combination of interior points,

$$x = \sum_{i=1}^{k} \mu_i x_i, \quad v_{**}(x) = \sum_{i=1}^{k} \mu_i v(x_i), \quad \sum_{i=1}^{k} \mu_i = 1,$$

with $x_1, ..., x_k \in \Omega$. By Proposition 1 in [ALL] there are $A_1, ..., A_k \in S^+$ such that $(q, A_i) \in \bar{J}^{2,-}v(x_i)$ and

$$A - \epsilon A^2 \leq \left(\mu_1 A_1^{-1} + ... + \mu_k A_k^{-1}\right)^{-1}$$

for all $\epsilon > 0$ small enough. Since $v$ is a viscosity supersolution,

$$-D_p(A_i) \geq \frac{1}{v(x_i)}\left((p-1)|q|^2 + \frac{1}{2}\right)$$

Multiplying both sides with $\mu_i v(x_i)$ and a summation $i = 1,...,k$ yields

$$-v_{**}(x) \leq \left((p-1)|q|^2 + \frac{1}{2}\right)\sum_{i=1}^{k}\frac{\mu_i}{D_p(A_i)}.$$

Using this inequality we find

$$\begin{aligned}
&-D_p(A-\epsilon A^2) - \frac{1}{v_{**}(x)}\left((p-1)|q|^2 + \frac{1}{2}\right)\\
&\geq -D_p(A-\epsilon A^2) + \left(\sum_{i=1}^{k}\frac{\mu_i}{D_p(A_i)}\right)^{-1}.
\end{aligned}$$

Lemma 4.2.7 then gives

$$\begin{aligned}
&-D_p(A-\epsilon A^2) - \frac{1}{v_{**}(x)}\left((p-1)|q|^2 + \frac{1}{2}\right)\\
&\geq -D_p(A-\epsilon A^2) + D_p\left(\sum_{i=1}^{k}\mu_i X_i^{-1}\right)^{-1} \geq 0
\end{aligned}$$

since $A - \epsilon A^2 \leq \left(\sum_{i=1}^{k}\mu_i X_i^{-1}\right)^{-1}$. Letting $\epsilon \to 0$ we see that

$$-D_p(A) \geq \frac{1}{v_{**}(x)}\left((p-1)|q|^2 + \frac{1}{2}\right)$$

which shows that $v_{**}$ is a viscosity supersolution to

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}}\left((p-1)|\nabla v_{**}|^2 + \frac{1}{2}\right).$$

□

**Proof of Theorem 4.1.1.**

*Proof.* We have to show that $v = -\sqrt{u}$ is convex, making $\sqrt{u}$ concave, if $u$ is a viscosity solution of

$$\begin{aligned}
-\mathcal{D}_p u &= 1 \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{4.22}$$

Since $u$ is, in particular, a supersolution, it is positive. By Lemma 4.3.2, $v_{**}$ is a negative supersolution of

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}}\left((p-1)|\nabla v_{**}|^2 + \frac{1}{2}\right).$$

By Lemma 4.3.1

$$-\mathcal{D}_p \left(v_{**}\right)^2 \leq 1.$$

We have found a subsolution of equation (4.22). The comparison principle allows us to conclude that

$$v_{**}^2 \leq u = v^2, \quad \text{in } \Omega.$$

But $v_{**} \leq v < 0$. Thus we must have $v_{**} = v$, showing that $v$ is convex. □

## 4.4 Log-concavity for the eigenvalue problem

We proceed in the same manner as in section 4.3. The proofs of the following two Lemmas are similar to the proofs of Lemma 4.3.1 and 4.3.2. We note that the interior sphere condition is not needed here, since $v = -\ln u$ converges to infinity on the boundary. This makes a similar version of Lemma 4.2.8 redundant.

**Lemma 4.4.1.** *Assume that $\Omega$ is a convex domain in $\mathbb{R}^n$ and let $\lambda > 0$. A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution to $-\mathcal{D}_p u = \lambda u$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\ln u \in LSC(\bar{\Omega})$ is a negative viscosity supersolution to*

$$-\mathcal{D}_p v = -(p-1)|\nabla v|^2 - \lambda.$$

**Lemma 4.4.2.** *Assume that $\Omega$ is a convex domain in $\mathbb{R}^n$ and let $\lambda > 0$. Let $u \in USC(\bar{\Omega})$ be a positive viscosity subsolution to $-\mathcal{D}_p u = \lambda u$ with $u = 0$ on $\partial\Omega$. If $v = -\ln u$, then $v_{**}$ is a viscosity supersolution to*

$$-\mathcal{D}_p v_{**} = -(p-1)|\nabla v_{**}|^2 - \lambda.$$

**Proof of Theorem 4.1.2.**

*Proof.* Let $u$ be a positive viscosity solution of $-\mathcal{D}_p u = \lambda u$. Denoting $v = -\ln u$, Lemma 4.4.2 gives that $v_{**}$ is a viscosity supersolution to

$$-\mathcal{D}_p v_{**} = -(p-1)|\nabla v_{**}|^2 - \lambda.$$

Lemma 4.4.1 gives

$$-\mathcal{D}_p e^{v_{**}} \leq \lambda e^{-v_{**}}$$

in the viscosity sense. By the comparison principle,

$$e^{-v_{**}} \leq u = e^{-v}, \quad \text{in } \Omega.$$

This together with the fact that $v_{**} \leq v$ shows that $v_{**} = v$, making $v$ a convex function and $\log u$ a concave function. □

## 4.5 Conclusion and further problems

In this paper, we showed certain concavity properties of power functions for solutions of the homogeneous Dirichlet problem for the Dominative $p$-Laplace equation. This was due to the structure of the equation and its relation to the normalized $p$-Laplace operator. An interesting question is whether the parabolic version, $u_t = \mathcal{D}_p u$ has similar concavity properties and in what way it depends on the initial data. Further, for $n = 2$, the equation can be explicitly written out, and it would be interesting to see a simple proof of the same result.

## Bibliography


[ALL] O. ALVAREZ, J. LASRY, P. LIONS: Convex viscosity solutions and state constraints. *Journal de Mathématiques Pures et Appliquées*, 76, 265-288, 1997.

[APR] A. ATTOUCHI, M. PARVIAINEN, E. RUOSTEENOJA : $C^{1,\alpha}$ regularity for the normalized $p$-Poisson problem. *Journal de Mathématiques Pures et Appliquées*, 108(4):553-591, 2017.

[B1] K. BRUSTAD: Superposition of $p$-superharmonic functions, *Preprint*, 2018.

[B2] K. BRUSTAD: Sublinear elliptic operators, *Preprint*, 2018.

[BL] H. J. BRASCAMP AND E. H. LIEB: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *Journal of Functional Analysis*, 22(4):366-389, 1976.

[BLM] K. Brustad, P. Lindqvist and J. Manfredi: A discrete stochastic interpretation of the Dominative $p$-Laplacian, *Preprint*, 2018.

[CF] G. Crasta and I. Fragalá: On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: Regularity and geometric results. *Archive for Rational Mechanics and Analysis*, 218(3):1577-1607, 2015.

[CIL] M. Crandall, H. Ishii and P. Lions: User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc (N.S.)*, 27(1):1-67,1992.

[CS] L. Caffarelli and J. Spruck: Convexity properties of solutions to some classical variational problems. *Communications in Partial Differential Equations*, 7(11):1337-1379, 1982.

[D] K. Does: An evolution equation involving the normalized $p$-Laplacian, *Communications on Pure & Applied Analysis*, 10(1):361-396, 2011.

[HL] F. A. Høeg and P. Lindqvist: Regularity of solutions of the parabolic normalized $p$-Laplace equation, *Preprint*, *Advances in Nonlinear Analysis.* Advance online publication. `https://www.degruyter.com/view/j/anona.ahead-of-print/anona-2018-0091/anona-2018-0091.xml?format=INT`

[K] M. Kühn: Power- and Log-concavity of viscosity solutions to some elliptic Dirichlet problems, *Communications on Pure & Applied Analysis*, 17, 2018.

[Ka] B. Kawohl: When are superharmonic functions concave-applications to the St-Venant torsion problem and to the fundamental mode of the clamped membrane. *Zeitschrift fur Angewandte Mathematik und Mechanik*, 64(5):364-366, 1984.

[Ke] A. Kennington: Power concavity and boundary value problems. *Indiana University Mathematics Journal*, 34(3):687-704, 1985.

[Ko] N. Korevaar: Capillary surface convexity above convex domains. *Indiana University Mathematics Journal*, 32(1):73-81, 1983.

[L] P. Lindqvist: A nonlinear eigenvalue problem. Topics in Mathematical Analysis, 175-203, 2008.

[M] L. G. Makar-Limanov: Solution of Dirichlet's problem for the equation $\Delta u=-1$ in a convex region. *Mathematical Notes of the Academy of Sciences of the USSR*, 9(1):52-53, 1971.

[MPR] J. Manfredi, M. Parviainen and J. Rossi: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM Journal on Mathematical Analysis*, 42(5):2058-2081, 2010.

[S] S. SAKAGUCHI: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 14(3):403-421, 1987.

# Viscosity solutions

## A.1 Viscosity solutions

Viscosity solutions are weak solutions. They were introduced in [CIL] in the early 90's. The name has its origin in an old method, namely the *method of vanishing viscosity*. For an introduction, we also refer to [K]. Here, we specifically look at viscosity solutions for the parabolic normalized $p$-Laplace equation, but the definitions and remarks are similar for many other second order PDEs.

## A.2 Definition of viscosity solutions

Suppose first $u$ is a classical subsolution to

$$u_t = \Delta_p^N u \quad \text{in } \Omega_T, \tag{A.23}$$

and $\nabla u \neq 0$ in $\Omega_T$. That is, at each point in $\Omega_T$, $u_t \leq \Delta_p^N u$. Now, we take a test function $\phi \in C^2(\Omega_T)$ such that $u - \phi$ has a local maximum at the point $(x_0, t_0)$ inside $\Omega_T$. By the infinitesimal calculus, we have at $(x_0, t_0)$

$$\begin{aligned} &u_t = \phi_t, \quad \nabla u = \nabla \phi, \\ &D^2(u - \phi) \leq 0. \end{aligned}$$

The last inequality can be used to see that $\Delta_p^N(u - \phi) \leq 0$ at $(x_0, t_0)$. Then

$$\phi_t = u_t \leq \Delta_p^N u \leq \Delta_p^N \phi$$

at $(x_0, t_0)$. We see that $\phi$ is a classical subsolution at the point $(x_0, t_0)$. This motivates the definition of viscosity solutions. At points where $\nabla u = 0$, we have to adjust the definition.

**Definition A.2.1.** *Assume $1 \leq p < \infty$ and let $u$ be an upper semi-continuous function in $\Omega_T$. We say that $u$ is a* viscosity subsolution *of equation* (A.23) *in*

$\Omega_T$, *if*

$$\begin{cases} \phi_t(x_0,t_0) \le \Delta_p^N \phi(x_0,t_0), & \text{if } \nabla\phi(x_0,t_0) \neq 0 \\ \phi_t(x_0,t_0) \le \Delta\phi(x_0,t_0) + (p-2)\lambda_{\max}(D^2\phi(x_0,t_0)), & \text{if } \nabla\phi(x_0,t_0) = 0, \text{and } p \ge 2 \\ \phi_t(x_0,t_0) \le \Delta\phi(x_0,t_0) + (p-2)\lambda_{\min}(D^2\phi(x_0,t_0)), & \text{if } \nabla\phi(x_0,t_0) = 0, \text{and } 1 \le p < 2, \end{cases}$$

*whenever* $(x_0,t_0) \in \Omega_T$ *and* $\phi \in C^2(\Omega_T)$ *are such that* $\phi(x_0,t_0) = u(x_0,t_0)$ *and* $\phi(x,t) < u(x,t)$ *for* $(x,t) \in \Omega_T \setminus \{(x_0,t_0)\}$*, that is,* $\phi$ *touches* $u$ *from below at* $(x_0,t_0)$.
*For a lower semi-continuous function* $u$*, we say that* $u$ *is a* viscosity supersolution *of equation* (A.23) *in* $\Omega_T$*, if*

$$\begin{cases} \phi_t(x_0,t_0) \ge \Delta_p^N \phi(x_0,t_0), & \text{if } \nabla\phi(x_0,t_0) \neq 0 \\ \phi_t(x_0,t_0) \ge \Delta\phi(x_0,t_0) + (p-2)\lambda_{\min}(D^2\phi(x_0,t_0)), & \text{if } \nabla\phi(x_0,t_0) = 0, \text{and } p \ge 2 \\ \phi_t(x_0,t_0) \ge \Delta\phi(x_0,t_0) + (p-2)\lambda_{\max}(D^2\phi(x_0,t_0)), & \text{if } \nabla\phi(x_0,t_0) = 0, \text{and } 1 \le p < 2, \end{cases}$$

*whenever* $(x_0,t_0) \in \Omega_T$ *and* $\phi \in C^2(\Omega_T)$ *are such that* $\phi(x_0,t_0) = u(x_0,t_0)$ *and* $\phi(x,t) > u(x,t)$ *for* $(x,t) \in \Omega_T \setminus \{(x_0,t_0)\}$*, that is,* $\phi$ *touches* $u$ *from above at* $(x_0,t_0)$.
*Finally,* $u \in C(\Omega_T)$ *is a* viscosity solution *if it is both a viscosity subsolution and a viscosity supersolution.*

There are several equivalent ways to define viscosity solutions of equation (A.23). We mention one here involving the parabolic semijets $P^{2,\pm}$. The semijet $P^{2,+}u(x,t)$ consists of all scalars $r$, vectors $q$ and symmetric matrices $X$ such that

$$u(y,\tau) \le u(x,t) + r(\tau - t) + \langle q, y-x \rangle + \frac{1}{2}\langle y-x, X(y-x)\rangle + o(|y-x|^2 + |\tau - t|)$$

as $y \to x$ and $\tau \to t$. Similarly, $P^{2,-}u(x,t)$ contains scalars $r$, vectors $q$ and symmetric matrices $X$ such that

$$u(y,\tau) \ge u(x,t) + r(\tau - t) + \langle q, y-x \rangle + \frac{1}{2}\langle y-x, X(y-x)\rangle + o(|y-x|^2 + |\tau - t|)$$

as $y \to x$ and $\tau \to t$. Note that $r = \phi_t(x,t), q = \nabla\phi(x,t)$ and $X = D^2\phi(x,t)$ if $u$ is a $C^2$ function. We now state an equivalent definition of viscosity solutions.

**Definition A.2.2.** *Assume* $1 \le p < \infty$ *and let* $u$ *be an upper semi-continuous function in* $\Omega_T$*. We say that* $u$ *is a viscosity subsolution of equation* (A.23) *in* $\Omega_T$*, if* $(x,t) \in \Omega_T$ *and* $(r,q,X) \in P^{2,+}u(x,t)$ *implies*

$$\begin{cases} r \le \operatorname{tr}(X) + \frac{p-2}{|q|^2}\langle q, Xq\rangle, & \text{if } q \neq 0 \\ r \le \operatorname{tr}(X) + (p-2)\lambda_{\max}(X), & \text{if } q = 0, \text{and } p \ge 2 \\ r \le \operatorname{tr}(X) + (p-2)\lambda_{\min}(X), & \text{if } q = 0, \text{and } 1 \le p < 2, \end{cases}$$

*For a lower semi-continuous function $u$, we say that $u$ is a viscosity supersolution of equation* (A.23) *in $\Omega_T$, if $(x,t) \in \Omega_T$ and $(r,q,X) \in P^{2,-}u(x,t)$ implies*

$$\begin{cases} r \geq \operatorname{tr}(X) + \frac{p-2}{|q|^2}\langle q, Xq\rangle, & \text{if } q \neq 0 \\ r \geq \operatorname{tr}(X) + (p-2)\lambda_{\min}(X), & \text{if } q = 0, \text{and } p \geq 2 \\ r \geq \operatorname{tr}(X) + (p-2)\lambda_{\max}(X), & \text{if } q = 0, \text{and } 1 \leq p < 2, \end{cases}$$

*Finally, $u \in C(\Omega_T)$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.*

For proof of the equivalence of definition A.2.1 and definition A.2.2, we refer to [K]. The equivalence leads to the following strange example.

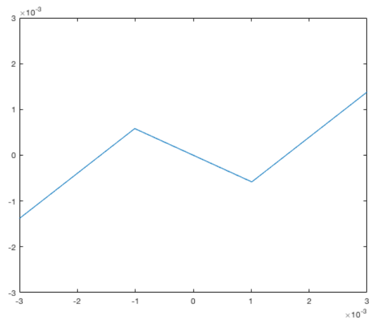
**Example: "Harry Potter"**

Consider the function

$$f(x) = x \sin\left(\ln|x|\right)$$

in one variable. See figure A.1 for a plot for small values of $x$. There are *no* test-functions touching $f$ from below or above at $x = 0$. Hence, $f$ is a viscosity solution of *any* ordinary differential equation in one variable at $x = 0$. The point passes for free. This could potentially be a problem in the theory of viscosity solutions. If one could create functions to fit any differential equation, there would be no value in finding viscosity solutions. However, for any continuous function $f$, the points at which there exists a test-function $\phi$ touching $u$ from below or above, are dense in $\mathbb{R}$, see [K] for a proof.

**Figure A.1:** $f(x) = x\sin\left(\ln|x|\right)$



## A.3 The method of vanishing viscosity

The name viscosity solution originally stems from the method of vanishing viscosity. For a general second order PDE,

$$F(x,u,\nabla u,D^2u)=0, \tag{A.24}$$

we add the term $\epsilon\Delta u$ to the right hand side. Usually the solution of

$$F(x,u_\epsilon,\nabla u_\epsilon,D^2u_\epsilon)=\epsilon\Delta u_\epsilon$$

will be more smooth. One can then send $\epsilon$ to zero in hope to find a solution. Under natural assumptions on $F$, the process yields the correct viscosity solution to equation (A.24), see [E] chapter 10.

The method can be seen in the following example:

$$\begin{aligned}&\frac{\partial u_\epsilon}{\partial t}+\frac{1}{2}\left(\frac{\partial u_\epsilon}{\partial x}\right)^2=\epsilon\frac{\partial^2 u_\epsilon}{\partial x^2}, \quad x\in\mathbb{R}^n, 0\le t<\infty\\ &u_\epsilon(x,0)=x^2.\end{aligned}$$

The solution may be found by a transformation of the equation to the heat equation. The solution is

$$u_\epsilon(x,t)=\epsilon\ln(1+2t)+\frac{x^2}{1+2t}.$$

Sending $\epsilon$ to zero, gives us $u(x,t)=\frac{x^2}{1+2t}$, which is the viscosity solution to

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2=0, \quad x\in\mathbb{R}^n, 0\le t<\infty\\ &u(x,0)=x^2.\end{aligned}$$

# Bibliography

[CIL] M.G. CRANDALL, H. ISHII, P.L. LIONS: User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.,* 27:1–67, 1992.

[E] L. EVANS: Partial Differential Equations. *American Mathematical Soc.* vol 19, 2010.

[K] S. KOIKE: A Beginner's Guide to the Theory of Viscosity Solutions. *MSJ Memoirs, vol. 13, Math. Soc. Japan, Tokyo*, 2004.

# Mean value property

We show that the value function $u$ satisfies

$$u(x) = \frac{n+2}{p+n} \fint_{B_\epsilon(x)} u(y)\,dy + \frac{p-2}{2(p+n)} \left\{ \max_{\overline{B}_\epsilon(x)} u + \min_{\overline{B}_\epsilon(x)} u \right\} + o(\epsilon^2)$$

if and only if $\Delta_p^N u = 0$. Here, $u$ is assumed to be a $C^2$ function with non-vanishing gradient. First, by the classical Taylor expansion, we claim that

$$u(x) - \fint_{B_\epsilon(x)} u(y)\,dy = -\frac{\epsilon^2}{2(n+2)} \Delta u(x) + o(\epsilon^2). \tag{B.25}$$

To see this, let $\epsilon > 0$ be given. We use a Taylor expansion for $u$ and integrate over $B_\epsilon(x)$:

$$\begin{aligned}\int_{B_\epsilon(x)} u(y)\,dy = &\int_{B_\epsilon(x)} u(x)\,dy + \int_{B_\epsilon(x)} \langle \nabla u(x), y-x\rangle\,dy \\ &+ \frac{1}{2}\int_{B_\epsilon(x)} \Big\langle y-x, D^2u(x)(y-x)\Big\rangle\,dy + o(\epsilon^2)\end{aligned}$$

The third integral is zero, because $\int_{B_\epsilon(x)} \left(y_i - x_i\right) dy = 0$ for $i = 1,...,n$ by symmetry. For the last integral we have

$$\sum_{i,j=1}^n u_{ij} \int_{B_\epsilon(x)} (y_i - x_i)(y_j - x_j)\,dy = \sum_{i,j=1}^n u_{ij}\delta_{ij} \int_{B_\epsilon(x)} (y_i - x_i)^2\,dy.$$

Let $V_n(R)$ denote the volume of the $n$-ball with radius $R$. Then

$$\int_{B_\epsilon(x)} (y_i - x_i)^2\,dy = \epsilon^{n+2} \int_{B_1(0)} z_i^2\,dz = \frac{\epsilon^{n+2}}{n} \int_{B_1(0)} |z|^2\,dz = \frac{\epsilon^{n+2}}{n+2} V_n(1).$$

Returning to the Taylor expansion we find

$$u(x) - \fint_{B_\epsilon(x)} u(y)\,dy = -\frac{\epsilon^{n+2}}{2(n+2)} \cdot \frac{V_n(1)}{V_n(\epsilon)} \Delta u(x) + o(\epsilon^2) = -\frac{\epsilon^2}{2(n+2)} \Delta u(x) + o(\epsilon^2).$$

Hence, equation (B.25) holds.

Next, we want to relate the minimum and maximum of a function to the $\infty-$Laplacian. The gradient direction is close to the maximizing direction,

$$\max_{\overline{B}_\epsilon(x)} u \approx u\left(x+\epsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right).$$

Writing out the Taylor expansion for both the maximum and minimum gives

$$u(x)-\frac{1}{2}\left(\max_{\overline{B}_\epsilon(x)} u+\min_{\overline{B}_\epsilon(x)} u\right)\approx -\frac{\epsilon^2}{2}\Delta_\infty^N u(x)+o(\epsilon^2).$$

The approximation in the above equation can be estimated with an error of $o(\epsilon^2)$, see Lemma 13 in [L]. Multiplying equation (B.25) by $(n+2)$ and the above equation by $(p-2)$ gives

$$u(x)-\frac{p-2}{2(n+p)}\left\{\max_{\overline{B}_\epsilon(x)} u+\min_{\overline{B}_\epsilon(x)} u\right\}-\frac{n+2}{n+p}\fint_{B_\epsilon(x)} u(y)\,dy=-\frac{\epsilon^2}{2(n+p)}\Delta_p^N u+o(\epsilon^2).$$

We see that

$$u(x)=\frac{n+2}{n+p}\fint_{B_\epsilon(x)} u(y)\,dy+\frac{p-2}{2(n+p)}\left\{\max_{\overline{B}_\epsilon(x)} u+\min_{\overline{B}_\epsilon(x)} u\right\}+o(\epsilon^2)$$

if and only if $\Delta_p^N u=0$.

# Bibliography

[L] P. LINDQVIST: Notes on the infinity Laplace equation. *Springer*, 2016.

# Concave square root for the two-dimensional Laplacian

The problem

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega$ is a convex domain, has the property that $\sqrt{u}$ is concave. The problem was studied in [M] and here we discuss the calculations hidden in this paper. Using complex notation for a function $f(x,y)$ of two variables we list some identities.

1. $\frac{\partial f}{\partial z} = f_z = \frac{1}{2}\left(f_x - i f_y\right), \quad \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = \frac{1}{2}\left(f_x + i f_y\right)$
2. $f_z f_{\bar{z}} = \frac{1}{4}|\nabla f|^2$
3. $f_{zz} = \frac{1}{4}\left(f_{xx} - f_{yy}\right) - \frac{i}{2} f_{xy}, \quad f_{\bar{z}\bar{z}} = \frac{1}{4}\left(f_{xx} - f_{yy}\right) + \frac{i}{2} f_{xy}$
4. $f_{z\bar{z}} = \frac{1}{4}\Delta f$
5. $f_{zz} f_{\bar{z}\bar{z}} - f_{z\bar{z}}^2 = \frac{1}{4}\left(f_{xy}^2 - f_{xx} f_{yy}\right)$

Solutions to the problem above are always *positive* due to the comparison principle. Hence, we may study the function $v = \sqrt{u}$ in $\Omega$. It has the following second derivatives.

$$\begin{aligned} v_{zz} &= -\frac{1}{4u^{\frac{3}{2}}} u_z^2 + \frac{1}{2\sqrt{u}} u_{zz} \\ v_{\bar{z}\bar{z}} &= -\frac{1}{4u^{\frac{3}{2}}} u_{\bar{z}}^2 + \frac{1}{2\sqrt{u}} u_{\bar{z}\bar{z}} \\ v_{z\bar{z}} &= -\frac{1}{4u^{\frac{3}{2}}} u_z u_{\bar{z}} - \frac{1}{8\sqrt{u}}. \end{aligned}$$

We wish to study the quantity $\det(D^2 v) = v_{xx} v_{yy} - v_{xy}^2$. The sign of this quantity can tell us whether $v$ is concave or not. We have, using the identities (1)-(5),

$$\begin{aligned} 4u^2 \det(D^2 v) &= 16u^2 \left( v_{z\bar{z}} - v_{zz} v_{\bar{z}\bar{z}} \right) \\ &= \frac{u}{4} + u_z u_{\bar{z}} - 4u u_{zz} u_{\bar{z}\bar{z}} + 2\left( u_{\bar{z}}^2 u_{zz} + u_z^2 u_{\bar{z}\bar{z}} \right). \end{aligned}$$

Note that on the boundary, $\partial\Omega$ we have

$$\left(4u^2 \det(D^2 v)\right)_{\partial\Omega} = -\frac{1}{2}\left( u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} \right).$$

Makar-Limanov showed that this quantity, which is close to the mean curvature for the level set of $u$, is strictly positive on the boundary, provided $\Omega$ is a convex domain. We now have a function that is positive on the boundary. If we can show that it solves some elliptic PDE, there might be hope to use a comparison principle to show that it must be positive everywhere. In the following calculations, we will use the fact that $u_{z\bar{z}} = -\frac{1}{4}$ and higher derivatives are zero.

$$\begin{aligned} \frac{1}{4}\Delta\left(4u^2 \det(D^2 v)\right) &= \left(4u^2 \det(D^2 v)\right)_{z\bar{z}} \\ &= \left(2u_{\bar{z}}^2 u_{zzz} - 4u u_{\bar{z}\bar{z}} u_{zzz}\right)_{\bar{z}} \\ &= -4u|u_{zzz}|^2 \le 0. \end{aligned}$$

Thus, $4u^2\det(D^2 v)$ is a superharmonic function and cannot obtain a strict minimum inside $\Omega$. Since $u > 0$, we must have $\det(D^2 v) > 0$ everywhere in $\Omega$, which means that both eigenvalues of the Hessian matrix $D^2 v$ are of the same sign. Note that

$$\Delta v = 4v_{z\bar{z}} = -\frac{1}{4u^{\frac{3}{2}}}|\nabla u|^2 - \frac{1}{8\sqrt{u}} < 0.$$

This means that both eigenvalues must be negative making $v = \sqrt{u}$ a concave function.

# Bibliography

[M] L. G. MAKAR-LIMANOV: Solution of Dirichlet's problem for the equation Δu=- 1 in a convex region. *Mathematical Notes of the Academy of Sciences of the USSR*, 9(1):52-53, 1971.