

# A LIPSCHITZ METRIC FOR THE CAMASSA–HOLM EQUATION

JOSÉ A. CARRILLO <sup>1</sup>, KATRIN GRUNERT <sup>2</sup> and HELGE HOLDEN <sup>2</sup>

<sup>1</sup> Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK;

email: carrillo@maths.ox.ac.uk

<sup>2</sup> Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, NO-7491 Trondheim, Norway;

email: katrin.grunert@ntnu.no, helge.holden@ntnu.no

Received 30 November 2019; accepted 7 April 2020

## Abstract

We analyze stability of conservative solutions of the Cauchy problem on the line for the Camassa–Holm (CH) equation. Generically, the solutions of the CH equation develop singularities with steep gradients while preserving continuity of the solution itself. In order to obtain uniqueness, one is required to augment the equation itself by a measure that represents the associated energy, and the breakdown of the solution is associated with a complicated interplay where the measure becomes singular. The main result in this paper is the construction of a Lipschitz metric that compares two solutions of the CH equation with the respective initial data. The Lipschitz metric is based on the use of the Wasserstein metric.

2010 Mathematics Subject Classification: 35Q53, 35B35 (primary); 35B60 (secondary)

## 1. Introduction

We study the Cauchy problem for weak conservative solutions of the Camassa–Holm (CH) equation, which reads as

$$u_t + uu_x + p_x = 0, \quad (1.1a)$$

$$\mu_t + (u\mu)_x = (u^3 - 2pu)_x, \quad (1.1b)$$

where  $p(t, x)$  is given by

$$p(t, x) = \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} u^2(t, y) dy + \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} d\mu(t, y), \quad (1.2)$$

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with initial data  $(u, \mu)|_{t=0} = (u_0, \mu_0)$ . In this case, the natural solution space consists of all pairs  $(u, \mu)$  such that

$$u(t, \cdot) \in H^1(\mathbb{R}), \quad \mu(t, \cdot) \in \mathcal{M}_+(\mathbb{R}), \quad \text{and} \quad d\mu_{ac} = (u^2 + u_x^2) dx,$$

where  $\mathcal{M}_+(\mathbb{R})$  denotes the set of all positive and finite Radon measures on  $\mathbb{R}$ . Our main goal is to prove the existence of a metric  $d$  such that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq \alpha(t)d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions  $(u_j(t), \mu_j(t))$   $j = 1, 2$  of (1.1) with initial data  $(u_{j,0}, \mu_{j,0})$   $j = 1, 2$ . Here  $\alpha(t)$  depends on the total energy of the solutions and  $\alpha(0) = 1$ .

The CH equation has been introduced in the seminal paper [10]; see also [11]. Originally derived in the context of models for shallow water (see also [27, 58]), it also turns up in models for hyperelastic rods [24, 28, 55]. Since it captures the nonlinear effects that give insight into important phenomena such as breaking waves and breaking rods, the CH equation has been intensively studied. The intricate behavior of solutions of the Cauchy problem will be the focus of this paper. We will not discuss further properties of the CH equation, for example, the fact that the equation is completely integrable and allows for a geometric interpretation. With the latter, we mean that the CH equation is a re-expression of a geodesic flow on the diffeomorphism group of the line or the circle [26, 31, 60]. Several extensions and generalizations exist, but we will focus on (1.1).

The intriguing aspect of solutions to the Cauchy problem is the generic development of singularities in finite time, irrespective of the smoothness of the initial data. A solution may develop steep gradients, but in contrast to, say, hyperbolic conservation laws, the solution itself remains continuous. A finer analysis reveals that the energy density  $u^2 + u_x^2$  develops singularities, and at breakdown, which is often referred to as wave breaking, energy concentrates on sets of measure zero. Thus it becomes useful to introduce a measure, here denoted by  $\mu$ , that encodes the energy, and away from breakdowns this measure should coincide with the energy density  $u^2 + u_x^2$ . In technical terms, we consider a nonnegative Radon measure  $\mu$  with an absolutely continuous part  $d\mu_{ac} = (u^2 + u_x^2) dx$ . This measure  $\mu$  satisfies equation (1.1b) (and as can easily be verified,  $\mu = u^2 + u_x^2$  will satisfy the same equation in the case of smooth solutions). An illustrating example of how intricate the structure of the points of wave breaking may be can be found in [40]. The behavior in the proximity of the point of wave breaking, and, in particular, the prolongation of the solution past wave breaking, has been extensively studied. See, for example, [6, 7, 21, 22, 25, 32, 33, 39, 43–54, 56, 57] and references therein. The key point here is that past wave breaking

uniqueness fails, and there is a continuum of distinct solutions [47], with two extreme cases called dissipative and conservation solutions. To understand this conundrum, it turns out to be advantageous to rewrite the equation in a different set of variables where the solution remains smooth.

The explicit peakon–antipeakon solution [41], which illustrates this problem, is given by

$$u(t, x) = \begin{cases} -\alpha(t)e^x, & x \leq -\gamma(t), \\ \beta(t) \sinh(x), & -\gamma(t) < x < \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leq x, \end{cases} \quad (1.3)$$

where

$$\alpha(t) = \frac{E}{2} \sinh\left(\frac{E}{2}t\right), \quad \beta(t) = E \frac{1}{\sinh(\frac{E}{2}t)}, \quad \gamma(t) = \ln\left(\cosh\left(\frac{E}{2}t\right)\right),$$

where  $E = \|u(t)\|_{H^1}$  for all  $t \neq 0$ . This function is a weak conservative solution which consists of a ‘peak’ moving to the right and an ‘antipeak’ moving to the left; see Figure B.3. At  $t = 0$  the ‘peak’ and ‘antipeak’ collide, and the solution vanishes, yet the solution is highly nontrivial before and after the collision time. Clearly the trivial solution will coincide with this solution at  $t = 0$ , yet the trivial solution and (1.3) are very different at any other time. Thus it is not clear how to derive a metric comparing two solutions, that is stable under the time evolution. This is the task of the present work.

To be more precise, we are here presenting a metric  $d$  with the property that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq \alpha(t)d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions  $(u_j(t), \mu_j(t))$ ,  $j = 1, 2$ , of (1.1) with initial data  $(u_{j,0}, \mu_{j,0})$ ,  $j = 1, 2$ . Here  $\alpha(t)$  is a continuous function with  $\alpha(0) = 1$ , which may depend on the total energies involved, but not on the particular solutions. We stress that no standard Sobolev norm nor Lebesgue space norm will work. There exist alternative metrics for solutions of the CH equation; see [8, 42, 43]. In [8] the periodic case is treated by approximating the solution by multipeakons. The metric is defined by optimizing over a class of functions. The approach in [42, 43] depends on a reformulation of the CH equation in terms of Lagrangian variables. An intrinsic problem in this formulation is that of *relabeling*, where there will be many different parametrizations in Lagrangian coordinates, corresponding to one and the same solution  $(u(t), \mu(t))$  in Eulerian variables. Thus one has to compute the distance between equivalence classes, which is not transparent. In the present approach, the key idea is to introduce a new set of variables, where one variable plays a role similar to a characteristic, while the remaining variables are linked to  $(u, \mu)$  with the help of the ‘characteristic’. As we will outline next, there is no

need to resort to equivalence classes or to optimize over classes of functions, and in spite of this proof being longer, we consider this approach to be more natural.

Our approach is based on the fact that a natural metric for measuring distances between Radon measures (with the same total mass) is given through the Wasserstein (or Monge–Kantorovich) distance  $d_W$ , which in one dimension is defined with the help of pseudoinverses; see [67]. Given a nonnegative measure  $\mu$  of finite mass  $M > 0$ , we define the cumulative distribution function associated to  $\mu$  as

$$F(x) = \mu((-\infty, x)),$$

which is a nondecreasing function from  $\mathbb{R}$  onto  $[0, M]$ , left continuous and with limit from the right at any point  $x \in \mathbb{R}$ . The pseudoinverse associated to  $\mu$  denoted by  $\mathcal{X}$  is the function from  $[0, M]$  onto  $\mathbb{R}$  given by

$$\mathcal{X}(\eta) = \sup\{x \mid F(x) < \eta\}.$$

The pseudoinverse of  $F$  is a nondecreasing function from  $[0, M]$  onto  $\mathbb{R}$ , left continuous and with limit from the right (caglad) at any point  $x \in \mathbb{R}$ . Notice the different convention adopted here with respect to the usual one in probability theory defining cumulative distribution functions continuous from the right and with limits from the left (cadlag) at every point. We prefer to have caglad instead of cadlag functions due to the use of the methods from [54] developed under the present convention. Wasserstein distances between nonnegative measures with the same mass can be defined via  $L^p$ -norms of the difference between their associated pseudoinverses; see [19, 20, 61, 67] and the references therein.

The approach of using Wasserstein distances to control the expansion of solutions of evolutionary PDEs leading to curves of probability measures goes back to the proofs of the mean-field limit of McKean–Vlasov and Vlasov equations in the late seventies and eighties of the last century. We refer to the classical references [5, 29, 63–65] proving these large particle limits by means of the bounded Lipschitz distance and the coupling method. See the recent results and surveys in [3, 12, 13, 35, 36]. The optimal transport viewpoint for one-dimensional models was developed using pseudoinverse distributions for nonlinear aggregation and diffusion equations in [15, 19, 20, 61, 66] and the references therein, showing the contractivity of the Wasserstein distance in one dimension without the heavy machinery of optimal transport developed for general gradient flows in [1]. More recently, these metrics have been used with success to show uniqueness past the blow-up time for multidimensional aggregation equations [14] using gradient flow solutions. It is also interesting to point out that gradient flow solutions of the aggregation equation in one dimension with particular potentials are equivalent to entropy solutions of the Burgers

equation as proven in [4]. Another strategy using unbalanced optimal transport tools has been recently analyzed in [34] with the objective of understanding the relation between the incompressible Euler equations and the CH equations.

Finally, it is worth mentioning that there have been several works [2, 17, 18, 30, 37, 38, 59, 62, 68] making use of this change of variables to produce numerical schemes capable of going over blow-up of solutions to nonlinear aggregations and being able to capture the blow-up of solutions of aggregation–diffusion models in one dimension such as toy versions of the Keller–Segel model for chemotaxis. It is a nice avenue of research to use this approach to produce numerical schemes for conservative solutions of the CH equation; see [23] for related particle methods.

In the present work, we will adapt this strategy of defining suitable distances between measures to the present problem of finding good metrics for solutions of the CH equation. Let  $(u(t, \cdot), \mu(t, \cdot))$  be a weak conservative solution to the CH equation with total energy  $\mu(t, \mathbb{R}) = C > 0$  (which for simplicity here is assumed to be smooth). Let

$$F(t, x) = \mu(t, (-\infty, x)) = \int_{-\infty}^x d\mu(t)$$

due to the smoothness, and introduce the basic quantity

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x) = 2p_x(t, x) + 2F(t, x).$$

The key function here is the (spatial) inverse of the strictly increasing function  $G$  for fixed time  $t$ . To that end, we define

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}.$$

Formally we have that  $G(t, \mathcal{Y}(t, \eta)) = \eta$  for all  $\eta \in (0, 2C)$  and  $\mathcal{Y}(t, G(t, x)) = x$  for all  $x \in \mathbb{R}$ . Here it is important to note that the domain of  $\mathcal{Y}$  depends on the total energy  $C$ . Next, we want to determine the time evolution of  $\mathcal{Y}$ . Direct formal calculations yield that

$$\mathcal{Y}_t(t, G(t, x)) + \mathcal{Y}_\eta(t, G(t, x))G_t(t, x) = 0, \quad (1.4a)$$

$$\mathcal{Y}_\eta(t, G(t, x))G_x(t, x) = 1. \quad (1.4b)$$

Thus we need to compute the time evolution of  $G(t, x)$  before being able to compute the time evolution of  $\mathcal{Y}(t, \eta)$ . To that end, we find, after some computations, that

$$G_t(t, x) + uG_x(t, x) = \frac{2}{3}u^3(t, x) + S(t, x), \quad (1.5)$$

where

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{2}{3} u^3 - u_x p_x - 2pu \right) (t, y) dy. \quad (1.6)$$

Introducing  $\eta = G(t, x)$ ,  $\mathcal{S}(t, \eta) = S(t, \mathcal{Y}(t, \eta))$ , and

$$\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta)), \quad (1.7)$$

we find by combining (1.4) and (1.5) that

$$\mathcal{Y}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) (t, \eta) \mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta),$$

where we used that  $\mathcal{Y}(t, G(t, x)) = x$  for all  $x \in \mathbb{R}$ . As far as the time evolution of  $\mathcal{U}(t, \eta)$  is concerned, we find

$$\mathcal{U}_t(t, \eta) = -\mathcal{Q}(t, \eta) - \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta),$$

where we introduced  $\mathcal{Q}(t, \eta) = p_x(t, \mathcal{Y}(t, \eta))$ . Thus, formally we end up with the system

$$\begin{aligned} \mathcal{Y}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{Y}_\eta(t, \eta) &= \mathcal{U}(t, \eta), \\ \mathcal{U}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta) &= -\mathcal{Q}(t, \eta). \end{aligned}$$

However, this system is not closed, and we need to introduce the function

$$\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta)), \quad (1.8)$$

and determine its time evolution. We find, after some computations, that

$$\mathcal{P}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta),$$

where

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned}$$

To summarize, we have established the following system of differential equations

$$\mathcal{Y}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (1.9a)$$

$$\mathcal{U}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (1.9b)$$

$$\mathcal{P}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (1.9c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (1.10a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - \mathcal{U}_\eta \mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta, \quad (1.10b)$$

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned} \quad (1.10c)$$

Derived under assumptions of smoothness of the functions involved, the same system is valid also in the general case of weak conservative solutions. However, that requires considerable analysis, and Section 3 is devoted to that. The next step is to estimate the time evolution of these quantities  $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$ . It turns out that the natural functional space is the space of square integrable functions for the unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ . For this reason, we prefer to work with  $\mathcal{P}^{1/2}$  rather than  $\mathcal{P}$ . Section 3 focuses on the first qualitative properties of the time evolution of the solutions of (1.9) for weak conservative solutions of the CH equation (1.1) as well as the propagation in time of the  $L^2$ -norm of the unknowns.

The main aim of our work is to identify the right distance between two general conservative solutions of the CH equation (1.1), or equivalently, between two general  $L^2$  solutions  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  of system (1.9) with possibly different energies. In order to compare solutions with different energies, we need to rescale the solutions of (1.9) in such a way that they are defined on the same interval. Since the natural functional space for our unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  was identified as the  $L^2$ -functional space, it seems natural to do a scaling conserving the  $L^2$ -norms of the unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ , but leading to the domain being independent of the total energy  $C$ .

Let us define the scaled unknowns  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$  associated to a conservative solution  $(u(t), \mu(t))$  with energy  $C = \mu(t, \mathbb{R})$  of the CH equation (1.1) as  $\tilde{\mathcal{Y}}(t, \eta) = \sqrt{2C} \mathcal{Y}(t, 2C\eta)$ ,  $\tilde{\mathcal{U}}(t, \eta) = \sqrt{2C} \mathcal{U}(t, 2C\eta)$ , and  $\tilde{\mathcal{P}}^{1/2}(t, \eta) = \sqrt{2C} \mathcal{P}^{1/2}(t, 2C\eta)$ , where  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  is the solution of (1.9). This scaling allows also for the zero solution to (1.1) to be included in our considerations, as outlined in Section 4. A similar system to (1.9) can be written for  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ , but this is postponed to Section 4. With this new set of unknowns in place, we can now define a metric to compare two general conservative solutions  $(u_i, \mu_i)$ ,  $i = 1, 2$ , of (1.1) with total

energy  $C_i = \mu_i(\mathbb{R})$ . We define it as

$$d((u_1, \mu_1), (u_2, \mu_2)) = \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|_{L^2((0,1))} + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|_{L^2((0,1))} \\ + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|_{L^2((0,1))} + |\sqrt{2C_1} - \sqrt{2C_2}|.$$

Our main result reads as follows.

**THEOREM 1.1.** *Consider initial data  $u_{i,0} \in H^1(\mathbb{R})$ ,  $\mu_{i,0} \in \mathcal{M}_+(\mathbb{R})$  such that  $d(\mu_{ac})_{i,0} = (u_i^2 + u_{i,x}^2) dx$  and  $C_i = \mu_i(\mathbb{R})$ , and let  $(u_i, \mu_i)$  for  $i = 1, 2$  denote the corresponding weak conservative solutions of the CH equation (1.1). Then we have that*

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq e^{\mathcal{O}(1)t} d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

where  $\mathcal{O}(1)$  denotes a constant depending only on  $\max_j(C_j)$  remaining bounded as  $\max_j(C_j) \rightarrow 0$ .

The main core of this work lies in estimating the Lipschitz property of the right-hand side of the equivalent system to (1.9) in the  $L^2$ -sense for the unknowns  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ . This is much easier in case we compare to the zero solution as it coincides with the propagation of the  $L^2$ -norms of the unknowns. Due to the intricate nonlinearities of the right-hand sides of (1.9), this leads in the general case to long detailed technical estimates that are displayed in full in Subsections 5.1, 5.2, and 5.3. In the case of peakon–antipeakon solutions, as solution (1.3) is denoted, all quantities described in this paper can be computed explicitly. The details are to be found in Appendix B.

A notational comment is in order. We decided to denote by  $\mathcal{O}(1)$  constants depending on  $\max_j(C_j)$  that may change from line to line along the proofs, but remain bounded as  $\max_j(C_j) \rightarrow 0$ . Explicit tracking of the constants could be possible but it is highly cumbersome and avoided for the sake of the reader.

## 2. Formal ideas: transformations with smoothness

Let us start by explaining all the mathematical details for the transformation in the case of smooth solutions as outlined in the introduction. Let  $(u(t, \cdot), \mu(t, \cdot))$  be a weak conservative solution to the CH equation with total energy  $\mu(t, \mathbb{R}) = C > 0$ . We assume that  $F(t, x)$ , given by

$$F(t, x) = \int_{-\infty}^x d\mu(t), \tag{2.1}$$



is increasing and smooth, and, in particular, that  $\mu = \mu_{ac} = (u^2 + u_x^2) dx$  for all  $t$ . Introduce the function

$$\begin{aligned} G(t, x) &= \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x) \\ &= 2p_x(t, x) + 2F(t, x), \end{aligned}$$

where we used integration by parts and (1.2). First of all, note that the function  $G(t, x)$  satisfies

$$\lim_{x \rightarrow -\infty} G(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} G(x) = 2C,$$

since  $|p_x(t, x)| \leq p(t, x)$  and  $p$  is an  $H^1$  function on the line due to (1.2). Moreover, the function  $(2p - u^2)(t, x) \geq 0$  for all  $(t, x) \in \mathbb{R}^2$  as the following computation shows,

$$\begin{aligned} (2p - u^2)(t, x) &= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (u^2 + F_x)(t, y) dy - u^2(t, x) \\ &= \frac{1}{2} \int_{-\infty}^x e^{y-x} u^2(t, y) dy + \frac{1}{2} \int_x^{\infty} e^{x-y} u^2(t, y) dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} F_x(t, y) dy - u^2(t, x) \\ &= \frac{1}{2} u^2(t, x) - \frac{1}{2} \int_{-\infty}^x e^{y-x} 2uu_x(t, y) dy \\ &\quad + \frac{1}{2} u^2(t, x) + \frac{1}{2} \int_x^{\infty} e^{x-y} 2uu_x(t, y) dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} F_x(t, y) dy - u^2(t, x) \\ &\geq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (F_x(t, y) - 2|uu_x|(t, y)) dy \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (|u(t, y)| - |u_x(t, y)|)^2 dy \geq 0. \end{aligned} \quad (2.2)$$

Thus the function  $G(t, x)$  is nondecreasing and, in our case, since the function  $F(t, x)$  is smooth, also  $G(t, x)$  is smooth.

REMARK 2.1. Estimate (2.2), that is,  $2p - u^2 \geq 0$ , remains valid also in the case where the functions are nonsmooth.

Last but not least, we want to make sure that  $G(t, x)$  is strictly increasing, so that its pseudoinverse will have no jumps.  $F(t, x)$  is constant if and only if  $d\mu$ ,

$u$ , and  $u_x$  are equal to zero. Therefore assume that there exists (for fixed  $t$ ) some interval  $[b, c]$  such that  $d\mu(t, x) = u(t, x) = u_x(t, x) = 0$  for all  $x \in [b, c]$ . Then the only term that can save us is  $p(t, x)$ , which in general satisfies  $p(t, x) \geq 0$  for all  $(t, x) \in \mathbb{R}^2$ . However, whenever  $\mu(t, \mathbb{R}) \neq 0$ , one has by its definition in (1.2) that  $p(t, x) > 0$  and the claim follows.

Thus the function  $G(t, x)$  is strictly increasing and continuous, and we can consider its pseudoinverse  $\mathcal{Y}: [0, 2C] \rightarrow \mathbb{R}$ , which in this case coincides with its inverse and which is given by

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}. \quad (2.3)$$

Since  $G(t, x)$  is strictly increasing and continuous, we have that  $G(t, \mathcal{Y}(t, \eta)) = \eta$  for all  $\eta \in (0, 2C)$  and  $\mathcal{Y}(t, G(t, x)) = x$  for all  $x \in \mathbb{R}$ . By the smoothness assumption on  $F$ , direct calculations yield that

$$\mathcal{Y}_t(t, G(t, x)) + \mathcal{Y}_\eta(t, G(t, x))G_t(t, x) = 0, \quad (2.4a)$$

$$\mathcal{Y}_\eta(t, G(t, x))G_x(t, x) = 1. \quad (2.4b)$$

Thus we need to compute the time evolution of  $G(t, x)$  before being able to compute the time evolution of  $\mathcal{Y}(t, \eta)$ . The following calculations are only valid in the case of smooth solutions, but we will show in the next section how to overcome this issue for weak conservative solutions. Since  $e^{-|x-y|}/2$  is the integral kernel of  $(-\partial_x^2 + 1)^{-1}$ , we observe from (1.2) that  $p$  is the solution to

$$p - p_{xx} = \frac{1}{2}u^2 + \frac{1}{2}\mu$$

and hence

$$\begin{aligned} p_t - p_{txx} &= uu_t + \frac{1}{2}F_{xt} \\ &= -u^2u_x - up_x - \frac{1}{2}(uF_x)_x + \frac{1}{2}(u^3)_x - (pu)_x \\ &= \frac{1}{6}(u^3)_x - (pu)_x - \frac{1}{2}(uF_x)_x - up_x, \end{aligned}$$

where we used the abbreviation  $\mu = F_x$ . Thus we end up with

$$\begin{aligned} p_t(t, x) &= -\frac{1}{2} \int_{\mathbb{R}} \text{sign}(x-y) e^{-|x-y|} \left( \frac{1}{6}u^3 - pu - \frac{1}{2}uF_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} up_x(t, y) dy. \end{aligned}$$

Similar calculations yield that

$$p_{xt}(t, x) = -\frac{1}{6}u^3 + pu + \frac{1}{2}uF_x + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{2}{3}u^3 - u_x p_x - 2pu \right) (t, y) dy.$$

Thus we get for the time evolution of  $G(t, x)$  that

$$G_t(t, x) + uG_x(t, x) = \frac{2}{3}u^3(t, x) + S(t, x), \quad (2.5)$$

where

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{2}{3}u^3 - u_x p_x - 2pu \right)(t, y) dy.$$

Combining (2.4) and (2.5), we end up with

$$\mathcal{Y}_t(t, G(t, x)) + (\frac{2}{3}u^3(t, x) + S(t, x))\mathcal{Y}_\eta(t, G(t, x)) = u(t, x).$$

Introducing  $\eta = G(t, x)$ , we deduce

$$\mathcal{Y}_t(t, \eta) + (\frac{2}{3}u^3 + S)(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = u(t, \mathcal{Y}(t, \eta)),$$

where we used that  $\mathcal{Y}(t, G(t, x)) = x$  for all  $x \in \mathbb{R}$ . As far as the time evolution of

$$\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta)) \quad (2.6)$$

is concerned, we have

$$\begin{aligned} \mathcal{U}_t(t, \eta) &= u_t(t, \mathcal{Y}(t, \eta)) + u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_t(t, \eta) \\ &= u_t(t, \mathcal{Y}(t, \eta)) + uu_x(t, \mathcal{Y}(t, \eta)) - (\frac{2}{3}u^3 + S)u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) \\ &= -p_x(t, \mathcal{Y}(t, \eta)) - (\frac{2}{3}u^3 + S)u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) \\ &= -\mathcal{Q}(t, \eta) - (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t, \eta), \end{aligned}$$

where we introduced  $\mathcal{Q}(t, \eta) = p_x(t, \mathcal{Y}(t, \eta))$  and  $\mathcal{S}(t, \eta) = S(t, \mathcal{Y}(t, \eta))$ . Thus, formally we end up with the system

$$\mathcal{Y}_t(t, \eta) + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (2.7a)$$

$$\mathcal{U}_t(t, \eta) + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (2.7b)$$

where  $\mathcal{Q}(t, \eta)$  and  $\mathcal{S}(t, \eta)$  can be written as

$$\begin{aligned} \mathcal{Q}(t, \eta) &= -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} (u^2(t, y) + F_x(t, y)) dy \\ &= -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} (2(u^2(t, y) - p(t, y)) + G_x(t, y)) dy \\ &= -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}(t, \eta) &= \int_{\mathbb{R}} e^{-|\mathcal{Y}(t, \eta) - y|} \left( \frac{2}{3} u^3 - u_x p_x - 2pu \right) (t, y) dy \\ &= \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - \mathcal{U}_\eta \mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta, \end{aligned}$$

with

$$\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta)). \quad (2.8)$$

It is then natural, in order to close system (2.7), that besides the quantities  $u$  and  $\mu$ ,  $p$  in the new variables must be considered. One main reason is that these three quantities turn up in the definition of  $G$ . We already computed before that

$$\begin{aligned} p_t(t, x) &= -\frac{1}{2} \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x - y|} \left( \frac{1}{6} u^3 - pu - \frac{1}{2} u F_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x - y|} u p_x(t, y) dy \\ &= \frac{1}{2} \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x - y|} \left( \frac{1}{3} u^3 + \frac{1}{2} u G_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x - y|} u p_x(t, y) dy, \end{aligned}$$

where we used that

$$G_x(t, x) = 2p(t, x) - u^2(t, x) + F_x(t, x).$$

Thus direct computations yield the additional equation

$$\mathcal{P}_t(t, \eta) + \left( \frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta),$$

where

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} \left( \frac{2}{3} u^3 + u G_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|\mathcal{Y}(t, \eta) - y|} u p_x(t, y) dy \\ &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned}$$

We summarize the result in the following proposition.

**PROPOSITION 2.2.** *Let  $(u, \mu)$  denote a smooth solution of (1.1). Define  $\mathcal{Y}$  by (2.3),  $\mathcal{U}$  by (2.6), and  $\mathcal{P}$  by (2.8). Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (2.9a)$$

$$\mathcal{U}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (2.9b)$$

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (2.9c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (2.10a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta, \quad (2.10b)$$

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned} \quad (2.10c)$$

In the next section we will derive this system of equations also in the general case without assuming smoothness of the quantities involved; see (3.28).

Let us finish this section by checking some properties of system (2.9), which will also hold in the case of weak conservative solutions as we will see in the next section.

The quantity  $\frac{2}{3}\mathcal{U}^3 + \mathcal{S}$  is the velocity field of the three equations in (2.9). Instead of applying a characteristic method to estimate the solutions to this system, we will perform integration by parts by which the  $\eta$ -derivative of this quantity will naturally appear.

**LEMMA 2.3.** *Given  $(u, \mu)$  a smooth solution of (1.1), then the solution to (2.9) satisfies*

$$|\left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)_\eta| \leq \mathcal{O}(1).$$

*Proof.* Since  $G(t, \mathcal{Y}(t, \eta)) = 2p_x(t, \mathcal{Y}(t, \eta)) + 2F(t, \mathcal{Y}(t, \eta)) = \eta$ , we have due to the smoothness that

$$G_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = (2p - u^2 + F_x)(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = 1.$$

Since  $(2p - u^2)(t, x) \geq 0$  due to (2.2) and  $(F_x - u^2)(t, x) \geq 0$  due to (2.1), we have that

$$\begin{aligned} 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) &\leq (2p - u^2 + F_x)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) = 1, \\ \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) &\leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1, \\ 2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| &= 2|uu_x(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta)| \leq (u^2 + u_x^2)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) \\ &= F_x(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) \leq (2p - u^2 + F_x)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) = 1. \end{aligned}$$

We conclude that

$$\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq \frac{1}{2}, \quad |\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq \frac{1}{2}, \quad \text{and} \quad \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq 1. \quad (2.11)$$

From the fact that the energy is conserved, it follows that  $u(t) \in H^1(\mathbb{R})$  for all  $t \geq 0$  and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{C}.$$

Therefore, the term  $\mathcal{U}^2\mathcal{U}_\eta$  is bounded by  $\mathcal{O}(1)$ .

Furthermore,  $\mathcal{S}_\eta(t, \eta)$  is bounded. In particular, one can establish that

$$\mathcal{S}_\eta(t, \eta) \leq \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t, \eta),$$

which is going to play a key role. Indeed, by definition one has

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{2}{3}u^3 - u_x p_x - 2pu \right) (t, y) dy,$$

and hence

$$S_x(t, x) = - \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x-y|} \left( \frac{2}{3}u^3 - u_x p_x - 2pu \right) (t, y) dy.$$

Our aim is to show that

$$S_x(t, x) \leq \mathcal{O}(1)p(t, x).$$

First of all, note that we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x-y|} \frac{2}{3}u^3(t, y) dy \right| &\leq \frac{8}{3} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} u^2(t, y) dy \\ &\leq \mathcal{O}(1)p(t, x). \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \text{sign}(x-y) e^{-|x-y|} (u_x p_x + 2pu)(t, y) dy \right| \\ & \leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (2u^2 + u_x^2 + 3p^2)(t, y) dy \\ & \leq 2p(t, x) + \frac{3}{2} \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy, \end{aligned}$$

since  $|p_x(t, x)| \leq p(t, x)$ . Thus it remains to show that the last term can be bounded by a multiple of  $p(t, x)$ . Our reasoning will be based on integration by parts and the fact that

$$p(t, x) - p_{xx}(t, x) = \frac{1}{2}u^2(t, x) + \frac{1}{2}F_x(t, x).$$

Indeed, first direct computations yield

$$\begin{aligned} & \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy \\ & = \int_{\mathbb{R}} e^{-|x-y|} p \left( \frac{1}{2}u^2 + \frac{1}{2}F_x \right) (t, y) dy + \int_{\mathbb{R}} e^{-|x-y|} pp_{xx}(t, y) dy \\ & = I_1(t, x) + I_2(t, x). \end{aligned}$$

Since  $p(t, x) \leq \frac{1}{2} \int_{\mathbb{R}} F_x(t, y) dy \leq \frac{1}{2}C$ , we have that

$$I_1(t, x) \leq \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (u^2 + F_x)(t, y) dy \leq 2\|p(t, x)\|_{L^\infty(\mathbb{R})} p(t, x).$$

As far as  $I_2$  is concerned, we have

$$\begin{aligned} & \int_{-\infty}^x e^{y-x} pp_{xx}(t, y) dy \\ & = pp_x(t, x) - \int_{-\infty}^x e^{y-x} (pp_x + p_x^2)(t, y) dy \\ & = pp_x(t, x) - \frac{1}{2}p^2(t, x) + \int_{-\infty}^x e^{y-x} \left( \frac{1}{2}p^2 - p_x^2 \right) (t, y) dy, \end{aligned}$$

and

$$\begin{aligned} & \int_x^\infty e^{x-y} pp_{xx}(t, y) dy \\ & = -pp_x(t, x) - \int_x^\infty e^{x-y} (p_x^2 - pp_x)(t, y) dy \\ & = -pp_x(t, x) - \frac{1}{2}p^2(t, x) + \int_x^\infty e^{x-y} \left( \frac{1}{2}p^2 - p_x^2 \right) (t, y) dy. \end{aligned}$$

Thus

$$I_2(t, x) = -p^2(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{1}{2} p^2 - p_x^2 \right) (t, y) dy$$

and subsequently

$$\begin{aligned} & \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) - p^2(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{1}{2} p^2 - p_x^2 \right) (t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left( \frac{1}{2} p^2 - p_x^2 \right) (t, y) dy. \end{aligned}$$

Reshuffling the terms, we end up with

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy & \leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (p^2 + p_x^2)(t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) \leq Cp(t, x), \end{aligned} \quad (2.12)$$

showing the desired estimate.  $\square$

Next, we show that all properties seen in this section for smooth solutions remain true for weak conservative solutions to (1.1).

### 3. Rigorous transformation: weak conservative solutions

To accommodate for the wave breaking of the solutions, it has turned out to be advantageous to rewrite the CH equation from the original Eulerian variables into Lagrangian variables; see [6, 54]. We will show that the system of equations obtained in Proposition 2.2 holds for the weak conservative solutions introduced in [54]. With this aim in mind let us start by summarizing their approach, which uses the different adopted convention followed in this work for cumulative distribution functions to be continuous from the left and with limit from the right (caglad) at all points  $x \in \mathbb{R}$ . Therefore, both  $F(t, \cdot)$  and  $G(t, \cdot)$  are nondecreasing and caglad functions.

Given some initial data  $(u_0, \mu_0)$ , the corresponding initial data in Lagrangian coordinates is then given by

$$y(0, \xi) = \sup\{x \mid x + F_0(x) < \xi\}, \quad (3.1a)$$

$$H(0, \xi) = \xi - y(0, \xi), \quad (3.1b)$$

$$U(0, \xi) = u(0, y(0, \xi)), \quad (3.1c)$$



and  $(y(t, \cdot), U(t, \cdot), H(t, \cdot))$  are the solutions of

$$y_t(t, \xi) = U(t, \xi), \quad (3.2a)$$

$$U_t(t, \xi) = -Q(t, \xi), \quad (3.2b)$$

$$H_t(t, \xi) = (U^3 - 2PU)(t, \xi), \quad (3.2c)$$

where

$$P(t, \xi) = \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_\xi + H_\xi)(t, \sigma) d\sigma, \quad (3.3a)$$

$$Q(t, \xi) = -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_\xi + H_\xi)(t, \sigma) d\sigma. \quad (3.3b)$$

Moreover, the relation between  $P$  and  $Q$  is given by

$$P_\xi(t, \xi) = Q(t, \xi) y_\xi(t, \xi), \quad (3.4a)$$

$$Q_\xi(t, \xi) = (P - \frac{1}{2}U^2) y_\xi(t, \xi) - \frac{1}{2}H_\xi(t, \xi). \quad (3.4b)$$

Introduce the function

$$\begin{aligned} I(t, \xi) &= \int_{-\infty}^{\xi} (2P - U^2) y_\xi(t, \sigma) d\sigma = \int_{-\infty}^{\xi} (2Q_\xi + H_\xi)(t, \sigma) d\sigma \\ &= 2Q(t, \xi) + H(t, \xi), \end{aligned}$$

where we used that

$$\lim_{\xi \rightarrow -\infty} Q(t, \xi) = 0 = \lim_{\xi \rightarrow -\infty} H(t, \xi),$$

which follows from the definition of  $H(t, \xi)$ . The relation between  $H$  and  $F$  is given by

$$F(t, y(t, \xi)) \leq H(t, \xi) \leq F(t, y(t, \xi) +).$$

Here we have introduced the common notation

$$\Phi(x \pm) = \lim_{\epsilon \downarrow 0} \Phi(x \pm \epsilon). \quad (3.5)$$

Notice that  $I(t, \xi) + H(t, \xi)$  is the Lagrangian counterpart to the function  $G(t, x)$ . To convince oneself that this is really the case, one should take a quick look back first. The function  $G(t, x)$  was defined as

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x).$$

Thus, whenever  $F(t, x)$  has a jump of height  $\alpha$  at a point  $\bar{x}$ , that is,  $\mu(t, \{\bar{x}\}) = \alpha$ , then also  $G(t, x)$  has a jump of height  $\alpha$  at  $\bar{x}$ , since the function  $2p - u^2$  is continuous. Furthermore, the point  $\bar{x}$  in Eulerian coordinates is mapped to some maximal interval  $[\xi_l, \xi_r]$  in Lagrangian coordinates, on which  $y_\xi(t, \xi) = 0$  and  $H_\xi(t, \xi) = 1$  for a specific choice of a relabeling function. In fact,  $H_\xi(t, \xi) > 0$  for all  $\xi \in [\xi_l, \xi_r]$  as proven in [54, Theorem 4.2] and [54, Definition 2.6]. Thus a close look at  $Q(t, \xi)$  reveals that  $Q(t, \xi) = Q(t, \xi_l) - \frac{1}{2} \int_{\xi_l}^\xi H_\xi(t, \sigma) d\sigma$  for all  $\xi \in [\xi_l, \xi_r]$  and hence

$$\begin{aligned} I(t, \xi) &= 2Q(t, \xi) + H(t, \xi) \\ &= 2Q(t, \xi_l) - \int_{\xi_l}^\xi H_\xi(t, \sigma) d\sigma + H(t, \xi_l) + \int_{\xi_l}^\xi H_\xi(t, \sigma) d\sigma \\ &= 2Q(t, \xi_l) + H(t, \xi_l) = I(t, \xi_l). \end{aligned}$$

In short, we have that  $I(t, \xi) = I(t, \xi_l)$  for all  $\xi \in [\xi_l, \xi_r]$ . This allows us now to follow a similar approach as for the Hunter–Saxton (HS) equation in [9, 16]. Therefore introduce

$$J(t, \xi) = I(t, \xi) + H(t, \xi) = \int_{-\infty}^\xi (2P - U^2)y_\xi(t, \sigma) d\sigma + H(t, \xi), \quad (3.6)$$

and observe that for all solutions except the zero solution  $J(t, \xi)$  is strictly increasing and continuous. In more detail, one has for all solutions except the zero solution that  $P(t, \xi) \neq 0$  for all  $\xi \in \mathbb{R}$ , since  $y_\xi + H_\xi > 0$  almost everywhere due to [54, Definition 2.6]. Moreover, if  $H_\xi(t, \bar{\xi}) = 0$  for some  $\bar{\xi}$  one has that  $U(t, \bar{\xi}) = 0$  and  $y_\xi(t, \bar{\xi}) \neq 0$ , since  $y_\xi H_\xi(t, \bar{\xi}) = U^2 y_\xi(t, \bar{\xi}) + U_\xi^2(t, \bar{\xi})$  almost everywhere due to [54, Definition 2.6], and hence the  $\xi$ -derivative of  $J(t, \xi)$  is strictly positive at the point  $\bar{\xi}$ .

LEMMA 3.1. *Given  $\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}$ , then*

$$\mathcal{Y}(t, \eta) = y(t, l(t, \eta)), \quad (3.7)$$

where we have introduced  $l(t, \cdot) : [0, 2C] \rightarrow \mathbb{R}$  by

$$l(t, \eta) = \sup\{\xi \mid J(t, \xi) < \eta\}. \quad (3.8)$$

*Proof.* For each time  $t$  we have

$$y(t, \xi) = \sup\{x \mid x + F(t, x) < y(t, \xi) + H(t, \xi)\},$$

which implies that

$$y(t, \xi) + F(t, y(t, \xi)) \leq y(t, \xi) + H(t, \xi) \leq y(t, \xi) + F(t, y(t, \xi) +).$$

Moreover, one has that  $G(t, x) - F(t, x)$  is continuous, and, in particular,

$$\begin{aligned} G(t, y(t, \xi)) - F(t, y(t, \xi)) &= \int_{-\infty}^{y(t, \xi)} (2p - u^2)(t, y) dy \\ &= \int_{-\infty}^{\xi} (2P - U^2)y_{\xi}(t, \sigma) d\sigma = I(t, \xi). \end{aligned}$$

Thus one has

$$y(t, \xi) + G(t, y(t, \xi)) \leq y(t, \xi) + J(t, \xi) \leq y(t, \xi) + G(t, y(t, \xi) +).$$

Subtracting  $y(t, \xi)$  in the above inequality, we end up with

$$G(t, y(t, \xi)) \leq J(t, \xi) \leq G(t, y(t, \xi) +) \quad \text{for all } \xi \in \mathbb{R}.$$

Comparing the last equation and (3.8), we have

$$G(t, y(t, l(t, \eta))) \leq J(t, l(t, \eta)) = \eta \leq G(t, y(t, l(t, \eta) +).$$

Since  $y(t, \cdot)$  is surjective and nondecreasing, we end up with

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\} = y(t, l(t, \eta)), \quad (3.9)$$

thereby proving (3.7). □

In the next step we want to establish rigorously the corresponding system of differential equations. Hence we first have to establish that the function  $l(t, \eta)$  is differentiable, both with respect to time and space.

**3.1. The differentiability of  $Q(t, \xi)$ .** The differentiability of  $Q(t, \xi)$  with respect to  $\xi$  has been proven in [54] and  $Q_{\xi}(t, \xi)$  is given by (3.4b). Thus it is left to establish the differentiability with respect to time of  $Q(t, \xi)$ . To be more precise, we are going to establish the Lipschitz continuity of  $Q(t, \xi)$  with respect to time.

Let us recall that since  $U(t, \cdot) \in H^1(\mathbb{R})$ , one has in particular that  $U(t, \cdot) \in L^{\infty}(\mathbb{R})$ . Moreover, direct calculations yield

$$\begin{aligned} U^2(t, \xi) &= U^2(t, \xi) - U^2(t, -\infty) = \int_{-\infty}^{\xi} 2UU_{\xi}(t, \sigma) d\sigma \\ &\leq \int_{-\infty}^{\xi} H_{\xi}(t, \sigma) d\sigma \leq H(t, \infty) = C, \end{aligned} \quad (3.10)$$

and we end up with

$$\|U(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \leq \sqrt{C} \quad \text{for all } t \in \mathbb{R}.$$

Here we used *Xavier's relation* from [54] which asserts that

$$U^2 y_\xi^2(t, \xi) + U_\xi^2(t, \xi) = y_\xi H_\xi(t, \xi), \quad (3.11)$$

and hence

$$|U_\xi(t, \xi)| \leq \sqrt{y_\xi H_\xi}(t, \xi) \leq \frac{1}{2}(y_\xi + H_\xi)(t, \xi), \quad (3.12a)$$

$$|U y_\xi(t, \xi)| \leq \sqrt{y_\xi H_\xi}(t, \xi) \leq \frac{1}{2}(y_\xi + H_\xi)(t, \xi), \quad (3.12b)$$

$$|U U_\xi(t, \xi)| \leq \frac{1}{2} H_\xi(t, \xi). \quad (3.12c)$$

Similar considerations apply for  $P(t, \xi)$ . Indeed one has

$$\begin{aligned} 0 \leq P(t, \xi) &= \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \sigma) - y(t, \xi)|} (U^2 y_\xi + H_\xi)(t, \sigma) d\sigma \\ &\leq \frac{1}{4} \int_{\mathbb{R}} 2H_\xi(t, \sigma) d\sigma = \frac{1}{2} C. \end{aligned} \quad (3.13)$$

Since  $|Q(t, \xi)| \leq P(t, \xi)$ , we end up with

$$\|Q(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} C, \quad \|P(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} C \quad \text{for all } t \in \mathbb{R}. \quad (3.14)$$

Direct calculations, using (3.11) and (3.12), yield

$$\begin{aligned} &-\mathcal{O}(1)(y_\xi(t, \xi) + H_\xi(t, \xi)) \\ &\leq (y_\xi(t, \xi) + H_\xi(t, \xi))_t \\ &= U_\xi(t, \xi) + 3U^2 U_\xi(t, \xi) - 2QU y_\xi(t, \xi) - 2PU_\xi(t, \xi) \\ &\leq \mathcal{O}(1)(y_\xi(t, \xi) + H_\xi(t, \xi)), \end{aligned}$$

since  $U(t, \xi)$ ,  $P(t, \xi)$ , and  $Q(t, \xi)$  are uniformly bounded with respect to both space and time due to (3.10), (3.13), and (3.14). Thus, we have for  $s < t$  that

$$\begin{aligned} &(y_\xi(s, \xi) + H_\xi(s, \xi))e^{-\mathcal{O}(1)(t-s)} \\ &\leq y_\xi(t, \xi) + H_\xi(t, \xi) \leq (y_\xi(s, \xi) + H_\xi(s, \xi))e^{\mathcal{O}(1)(t-s)} \end{aligned}$$

or equivalently

$$\begin{aligned} (y_\xi(t, \xi) + H_\xi(t, \xi))e^{-\mathcal{O}(1)(t-s)} &\leq (y_\xi(s, \xi) + H_\xi(s, \xi)) \\ &\leq (y_\xi(t, \xi) + H_\xi(t, \xi))e^{\mathcal{O}(1)(t-s)}. \end{aligned} \quad (3.15)$$

Recall that

$$\begin{aligned} Q(t, \xi) &= -\frac{1}{4} \int_{-\infty}^{\xi} e^{y(t,\sigma)-y(t,\xi)} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\quad + \frac{1}{4} \int_{\xi}^{\infty} e^{y(t,\xi)-y(t,\sigma)} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &=: \frac{1}{4} Q_1(t, \xi) + \frac{1}{4} Q_2(t, \xi). \end{aligned}$$

We are only establishing the Lipschitz continuity with respect to time for  $Q_1(t, \xi)$ , since the argument for  $Q_2(t, \xi)$  follows the same lines. Let  $t < \hat{t}$ . Then one has

$$\begin{aligned} &|Q_1(t, \xi) - Q_1(\hat{t}, \xi)| \\ &\leq \int_{-\infty}^{\xi} |e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}| (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\quad + \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} |(U^2 y_{\xi} + H_{\xi})(t, \sigma) - (U^2 y_{\xi} + H_{\xi})(\hat{t}, \sigma)| d\sigma \\ &=: I_1 + I_2. \end{aligned}$$

As far as  $I_1$  is concerned, we observe, using Lemma A.1(i), that

$$\begin{aligned} |e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}| &\leq |y(t, \sigma) - y(\hat{t}, \sigma)| + |y(t, \xi) - y(\hat{t}, \xi)| \\ &\leq \int_t^{\hat{t}} (|U(s, \sigma)| + |U(s, \xi)|) ds \leq \mathcal{O}(1)|t - \hat{t}|, \end{aligned}$$

where we used that both  $e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}$  and  $e^{y(t,\sigma)-y(t,\xi)}$  are bounded from above by one, and that  $U$  can be uniformly bounded both with respect to space and time due to (3.10). Thus we end up with

$$\begin{aligned} I_1 &= \int_{-\infty}^{\xi} |e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}| (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\leq \mathcal{O}(1)|t - \hat{t}| \int_{-\infty}^{\xi} 2H_{\xi}(t, \sigma) d\sigma \leq \mathcal{O}(1)|t - \hat{t}|, \end{aligned}$$

where we used that  $U^2 y_{\xi} \leq H_{\xi}$ .

Thus it remains to establish a similar estimate for  $I_2$ . The idea is to use a similar strategy combined with (3.12a) and (3.15). We have

$$\begin{aligned} I_2 &= \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} |(U^2 y_{\xi} + H_{\xi})(t, \sigma) - (U^2 y_{\xi} + H_{\xi})(\hat{t}, \sigma)| d\sigma \\ &\leq \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} \int_t^{\hat{t}} |4U^2 U_{\xi} - 4QU y_{\xi} - 2PU_{\xi}|(s, \sigma) ds d\sigma \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{-\infty}^{\hat{\xi}} e^{y(\hat{t},\sigma)-y(\hat{t},\hat{\xi})} \mathcal{O}(1) \int_t^{\hat{t}} (y_{\xi} + H_{\xi})(s, \sigma) ds d\sigma \\
 &\leq \mathcal{O}(1) \int_{-\infty}^{\hat{\xi}} e^{y(\hat{t},\sigma)-y(\hat{t},\hat{\xi})} \int_t^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)} (y_{\xi} + H_{\xi})(\hat{t}, \sigma) ds d\sigma \\
 &\leq \mathcal{O}(1) \int_{-\infty}^{\hat{\xi}} e^{y(\hat{t},\sigma)-y(\hat{t},\hat{\xi})} (y_{\xi} + H_{\xi})(\hat{t}, \sigma) \int_t^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)} ds d\sigma \\
 &\leq \mathcal{O}(1)e^{\mathcal{O}(1)(\hat{t}-t)} |\hat{t} - t| \leq \mathcal{O}(1)|\hat{t} - t|,
 \end{aligned}$$

under the assumption that  $|\hat{t} - t| \leq 1$  (or in general bounded). Thus we established that

$$|Q_1(t, \xi) - Q_1(\hat{t}, \xi)| \leq \mathcal{O}(1)|\hat{t} - t|$$

for all pairs  $t, \hat{t}$ , such that  $|\hat{t} - t| \leq 1$ . Furthermore, one has

$$|Q(t, \xi) - Q(\hat{t}, \xi)| \leq \mathcal{O}(1)|\hat{t} - t|$$

for all pairs  $t, \hat{t}$ , such that  $|\hat{t} - t| \leq 1$ . Thus for fixed  $\xi$ , the function  $Q$  is locally Lipschitz with respect to time, and hence differentiable almost everywhere by Rademacher’s theorem.

A close look at the above argument reveals that we cannot only differentiate  $Q(t, \xi)$  with respect to time, but also that we can apply the dominated convergence theorem, which yields

$$\begin{aligned}
 Q_t(t, \xi) &= -\frac{1}{4} \int_{\mathbb{R}} \frac{d}{dt} (\text{sign}(\xi - \sigma) e^{-|y(t,\xi)-y(t,\sigma)|} (U^2 y_{\xi} + H_{\xi})(t, \sigma)) d\sigma \\
 &= -\frac{2}{3} U^3(t, \xi) + 2PU(t, \xi) \\
 &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|} \left( \frac{2}{3} U^3 y_{\xi} - 2PU y_{\xi} - QU_{\xi} \right) (t, \sigma) d\sigma \quad (3.16)
 \end{aligned}$$

after some integrations by parts.

Next we want to show that  $Q_t(t, \xi)$  can be uniformly bounded by a constant  $\mathcal{O}(1)$ . Observe that (3.10) and (3.13) imply

$$\begin{aligned}
 &\left| \int_{\mathbb{R}} e^{-|y(t,\sigma)-y(t,\xi)|} \left( \frac{2}{3} U^3 y_{\xi} - 2PU y_{\xi} - QU y_{\xi} \right) (t, \sigma) d\sigma \right| \\
 &\leq \mathcal{O}(1) \left( \int_{\mathbb{R}} e^{-|y(t,\sigma)-y(t,\xi)|} y_{\xi}(t, \sigma) d\sigma \right) \leq \mathcal{O}(1). \quad (3.17)
 \end{aligned}$$

Thus recalling (3.16) and combining (3.10), (3.13), and (3.17), we finally end up with

$$\|Q_t(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \mathcal{O}(1) \quad \text{for all } t \in \mathbb{R}. \quad (3.18)$$

This completes the argument regarding the differentiability of  $Q$ . Notice that a direct application of the dominated convergence theorem using formula (3.16) shows that  $Q_t$  is a continuous function in time, under the assumption that  $P_t$  exists and satisfies an estimate similar to (3.18).

Next we focus on the differentiability with respect to time of  $P$ . An analogous argument to the one for  $Q$  leads to the differentiability in time of  $P$ . Let us show that the derivative of  $P$  with respect to time exists by applying the dominated convergence theorem to

$$\lim_{s \rightarrow t} \frac{P(t, \xi) - P(s, \xi)}{t - s}, \quad (3.19)$$

where  $\xi$  is chosen such that  $y$  is differentiable with respect to time. Since

$$\frac{d}{dt} (e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_\xi + H_\xi)(t, \sigma))$$

exists almost everywhere, it is left to show that we can find a function in  $L^1(\mathbb{R})$ , which bounds the integrand of (3.19) uniformly in  $s$ . Therefore observe that we can write, using (3.2), for  $s < t$ , that

$$\begin{aligned} & P(t, \xi) - P(s, \xi) \\ &= \frac{1}{4} \int_{-\infty}^{\xi} \int_s^t e^{y(l, \sigma) - y(l, \xi)} (U(l, \sigma) - U(l, \xi)) (U^2 y_\xi + H_\xi)(l, \sigma) \, dl \, d\sigma \\ &+ \frac{1}{4} \int_{\xi}^{\infty} \int_s^t e^{y(l, \xi) - y(l, \sigma)} (U(l, \xi) - U(l, \sigma)) (U^2 y_\xi + H_\xi)(l, \sigma) \, dl \, d\sigma \\ &+ \frac{1}{4} \int_{\mathbb{R}} \int_s^t e^{-|y(l, \sigma) - y(l, \xi)|} (4U^2 U_\xi - 4U Q y_\xi - 2P U_\xi)(l, \sigma) \, dl \, d\sigma \\ &= I_1 + I_2 + I_3. \end{aligned}$$

As far as the first term is concerned, observe that (3.15) implies that

$$\begin{aligned} & \left| \int_s^t e^{y(l, \sigma) - y(l, \xi)} (U(l, \sigma) - U(l, \xi)) (U^2 y_\xi + H_\xi)(l, \sigma) \, dl \right| \\ & \leq \mathcal{O}(1) \int_s^t e^{y(l, \sigma) - y(l, \xi)} |(U^2 y_\xi + H_\xi)(l, \sigma)| \, dl \\ & \leq \mathcal{O}(1) e^{\mathcal{O}(1)|t-s|} e^{y(t, \sigma) - y(t, \xi)} (y_\xi + H_\xi)(t, \sigma) |t - s| \\ & \leq \mathcal{O}(1) e^{y(t, \sigma) - y(t, \xi)} (y_\xi + H_\xi)(t, \sigma) |t - s|, \end{aligned}$$

if we assume that  $|t - s| \leq 1$  or any other fixed constant. Moreover, the function in the last line belongs to  $L^1(\mathbb{R})$ . The remaining two terms can be bounded by a function, which is of the same form and belongs to  $L^1(\mathbb{R})$  uniformly in  $s$ .

Thus all assumptions of the dominated convergence theorem are fulfilled, and we end up with

$$\begin{aligned}
 P_t(t, \xi) &= UQ(t, \xi) + \frac{1}{4} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t,\xi) - y(t,\sigma)|} U(U^2 y_\xi + H_\xi)(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} (4U^2 U_\xi - 4UQy_\xi - 2PU_\xi)(t, \sigma) d\sigma \\
 &= UQ(t, \xi) + \frac{1}{2} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t,\xi) - y(t,\sigma)|} \\
 &\quad \times U \left( \frac{1}{3} U^2 y_\xi + Q_\xi + H_\xi \right) (t, \sigma) d\sigma \\
 &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} UQy_\xi(t, \sigma) d\sigma, \tag{3.20}
 \end{aligned}$$

where we used integration by parts together with (3.4b) in the last step. Moreover, we have that

$$\begin{aligned}
 |P_t(t, \xi)| &\leq |UQ(t, \xi)| + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} |U|(U^2 y_\xi + H_\xi)(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} (4U^2 |U_\xi| + 4|UQ|y_\xi + 2P|U_\xi|)(t, \sigma) d\sigma \\
 &\leq \|U(t, \cdot)\|_{L^\infty} P(t, \xi) \\
 &\quad + \frac{1}{4} \|U(t, \cdot)\|_{L^\infty} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} (U^2 y_\xi + H_\xi)(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} (2U^4 y_\xi + 3H_\xi + 2U^2 y_\xi + 3P^2 y_\xi)(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1)P(t, \xi) + \frac{3}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} P^2 y_\xi(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} (2\|U(t, \cdot)\|_{L^\infty}^2 U^2 y_\xi + 3H_\xi + 2U^2 y_\xi)(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1)P(t, \xi) + \frac{3}{4} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} P^2 y_\xi(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1) \left( P(t, \xi) + \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} P^2 y_\xi(t, \sigma) d\sigma \right)
 \end{aligned}$$

due to (3.10). It remains to show that

$$\int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\sigma)|} P^2 y_\xi(t, \sigma) d\eta \leq \mathcal{O}(1)P(t, \xi).$$

The proof follows the same lines as the one of (2.12) in Eulerian coordinates.



**3.2. The differentiability of  $l(t, \eta)$ .** Finally, we can start thinking about the differentiability with respect to space and time of  $l(t, \eta)$ . Therefore, recall that the function  $l(t, \eta)$  is defined as

$$l(t, \eta) = \sup\{\xi \in \mathbb{R} \mid J(t, \xi) < \eta\}.$$

*Differentiability with respect to  $\eta$ :* For every  $t \in \mathbb{R}$ , the function  $J(t, \cdot) = 2Q(t, \cdot) + 2H(t, \cdot)$  is strictly increasing and continuous, and hence differentiable almost everywhere with respect to  $\xi$ . Its inverse  $J^{-1}(t, \cdot): [0, 2C] \rightarrow \mathbb{R}$  is therefore also strictly increasing and continuous, and hence differentiable almost everywhere. Since we have by definition that  $J(t, l(t, \eta)) = \eta$  for all  $\eta$ , it follows immediately that  $J(t, l(t, \cdot))$  is Lipschitz continuous with Lipschitz constant one and hence differentiable almost everywhere with respect to  $\eta$ . Since

$$J^{-1}(t, J(t, l(t, \eta))) = l(t, \eta) \quad \text{for all } \eta,$$

we can finally conclude that  $l(t, \eta)$  is differentiable almost everywhere with respect to  $\eta$ . In particular, one has (cf. (3.6)) that

$$J_\xi(t, l(t, \eta))l_\eta(t, \eta) = 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) - \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) = 1, \quad (3.21)$$

with

$$\begin{aligned} \mathcal{H}(t, \eta) &= H(t, l(t, \eta)), & \mathcal{P}(t, \eta) &= P(t, l(t, \eta)), \\ & \text{and } \mathcal{U}(t, \eta) &= U(t, l(t, \eta)). \end{aligned}$$

These identities follow from equality (3.9) since  $\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta))$  and  $P(t, \xi) = p(t, y(t, \xi))$ , and analogously  $\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta))$  and  $U(t, \xi) = u(t, y(t, \xi))$ .

Following the same argument as in the smooth case (see (2.11)), we end up with

$$0 \leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1, \quad 2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq 1, \quad \text{and} \quad 0 \leq \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq 1. \quad (3.22)$$

*Differentiability with respect to  $t$ :* As far as the differentiability of  $l(t, \eta)$  with respect to time is concerned, the argument is a bit more involved. First of all, note that for any time  $t$  we have  $J(t, l(t, \eta)) = \eta$  and hence

$$J(t, l(t, \eta)) = J(\tilde{t}, l(\tilde{t}, \eta)) \quad \text{for all } \eta \text{ and } \tilde{t}.$$

In particular, we can conclude that

$$\begin{aligned} J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta)) &= J(\tilde{t}, l(\tilde{t}, \eta)) - J(t, l(\tilde{t}, \eta)) \\ &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) ds \\ &= 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) ds. \end{aligned}$$

We already showed that  $Q_t(t, \xi)$  exists and is given by (3.16). However, a close look reveals that  $Q_t(t, \xi)$  is uniformly bounded both with respect to space and time and  $Q_t(t, \xi)$  is continuous with respect to space for fixed  $t$ . The same holds for

$$H_t(t, \xi) = U^3(t, \xi) - 2PU(t, \xi).$$

Thus we have that

$$\lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}} = \lim_{\tilde{t} \rightarrow t} 2 \int_t^{\tilde{t}} \frac{Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))}{t - \tilde{t}} ds \quad (3.23)$$

from a formal point of view. Thus it is left to establish that the limit exists and is finite.

Since  $J_t(t, \xi) = 2Q_t(t, \xi) + 2H_t(t, \xi)$  and both  $Q_t(t, \xi)$  and  $H_t(t, \xi)$  can be bounded uniformly both in space and time by a constant only dependent on the total energy  $C$ , it follows that there exists a constant  $\mathcal{O}(1)$  such that

$$|J_t(t, \xi)| \leq \mathcal{O}(1) \quad \text{for all } t \text{ and } \xi.$$

Thus it follows that for all  $s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|]$ , we have

$$J(s, \xi) \in [J(t, \xi) - \mathcal{O}(1)|\tilde{t} - t|, J(t, \xi) + \mathcal{O}(1)|\tilde{t} - t|].$$

Accordingly we can conclude, since  $J(t, l(t, \eta)) = \eta$  for all  $\eta$ , that

$$J(s, l(t, \eta)) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|],$$

and hence

$$l(t, \eta) = l(s, \eta(s)) \quad \text{for some } \eta(s) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|], \quad (3.24)$$

and, in particular,

$$|\eta - \eta(s)| \leq \mathcal{O}(1)|\tilde{t} - t| \quad \text{for all } s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|].$$

Similar considerations yield that for all  $s \in [\tilde{t} - |t - \tilde{t}|, \tilde{t} + |t - \tilde{t}|]$ , we can find  $\tilde{\eta}(s)$  such that

$$l(\tilde{t}, \eta) = l(s, \tilde{\eta}(s)) \quad \text{and} \quad |\eta - \tilde{\eta}(s)| \leq \mathcal{O}(1)|t - \tilde{t}|.$$

Let us return to the integral we are actually interested in. Namely,

$$\begin{aligned} \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) ds &= 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) ds \\ &= 2 \int_t^{\tilde{t}} (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))) ds \\ &\quad + 2 \int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) \\ &\quad - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta)))) ds \\ &= \tilde{I}_1 + \tilde{I}_2. \end{aligned}$$

What we are hoping for is that the second term  $\tilde{I}_2$  is of order  $o(|t - \tilde{t}|)$ , and hence plays no role when computing (3.23). Therefore observe that

$$\begin{aligned} H_{t\xi}(t, \xi) &= 3U^2U_\xi(t, \xi) - 2QUy_\xi(t, \xi) - 2PU_\xi(t, \xi), \\ Q_{t\xi}(t, \xi) &= -2U^2U_\xi(t, \xi) + 2QUy_\xi(t, \xi) + 2PU_\xi(t, \xi) \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} \\ &\quad \times \left( \frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (t, \sigma) d\sigma y_\xi(t, \xi). \end{aligned}$$

Notice that the second derivative  $Q_{t\xi}$  is well defined by a dominated convergence argument as before for  $Q_t$ . Recalling (3.24), we have

$$\begin{aligned} \tilde{I}_2 &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) - J_t(s, l(t, \eta)) ds \\ &= 2 \int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta)))) ds \\ &= 2 \int_t^{\tilde{t}} \int_{l(t, \eta)}^{l(\tilde{t}, \eta)} (Q_{t, \xi} + H_{t, \xi})(s, z) dz ds \\ &= 2 \int_t^{\tilde{t}} \int_{l(t, \eta)}^{l(\tilde{t}, \eta)} \left[ U^2U_\xi(s, z) \right. \\ &\quad \left. - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(z - \sigma) e^{-|y(s, z) - y(s, \sigma)|} \right. \end{aligned}$$

$$\begin{aligned}
& \times \left( \frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (s, \sigma) d\sigma y_\xi(s, z) \Big] dz ds \\
& = 2 \int_t^{\tilde{t}} \int_{l(s, \eta(s))}^{l(s, \tilde{\eta}(s))} \left[ U^2 U_\xi(s, z) \right. \\
& \quad \left. - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(z - \sigma) e^{-|y(s, z) - y(s, \sigma)|} \right. \\
& \quad \times \left. \left( \frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (s, \sigma) d\sigma y_\xi(s, z) \right] dz ds \\
& = 2 \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \mathcal{U}^2 \mathcal{U}_\eta(s, m) dm ds \\
& \quad - \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \\
& \quad \times \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm ds,
\end{aligned}$$

where  $\eta$ ,  $\eta(s)$ , and  $\tilde{\eta}(s)$  satisfy

$$l(t, \eta) = l(s, \eta(s)) \quad \text{and} \quad l(\tilde{t}, \eta) = l(s, \tilde{\eta}(s)),$$

and

$$|\eta - \eta(s)| \leq \mathcal{O}(1)|t - \tilde{t}| \quad \text{and} \quad |\eta - \tilde{\eta}(s)| \leq \mathcal{O}(1)|t - \tilde{t}|. \quad (3.25)$$

As far as the first term is concerned, we can apply (3.10), (3.22) and (3.25) as follows:

$$\begin{aligned}
& \left| 2 \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \mathcal{U}^2 \mathcal{U}_\eta(s, m) dm ds \right| \\
& \leq \int_t^{\tilde{t}} \|\mathcal{U}(s, \cdot)\|_{L^\infty([0, 2C])} |\eta(s) - \tilde{\eta}(s)| ds \leq \mathcal{O}(1)|t - \tilde{t}|^2. \quad (3.26)
\end{aligned}$$

As far as the second and last term is concerned, we want to apply the Cauchy–Schwarz inequality to the integral term at first. Indeed,

$$\begin{aligned}
& \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \right. \\
& \quad \times \left. \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm \right| \\
& \leq \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \left[ \frac{2}{3} \mathcal{U}^2 \sqrt{\mathcal{Y}_\eta} (\mathcal{U}^2 \mathcal{Y}_\eta)^{1/2} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + 2\mathcal{P}\sqrt{\mathcal{Y}_\eta}(\mathcal{U}^2\mathcal{Y}_\eta)^{1/2} + \mathcal{P}\sqrt{\mathcal{Y}_\eta}\sqrt{\mathcal{H}_\eta} \Big] (s, n) dn \mathcal{Y}_\eta(s, m) dm \Big| \\
& \leq \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left( \frac{2}{3}\mathcal{U}^2\sqrt{\mathcal{Y}_\eta} + 3\mathcal{P}\sqrt{\mathcal{Y}_\eta} \right) \sqrt{\mathcal{H}_\eta}(s, n) dn \mathcal{Y}_\eta(s, m) dm \right| \\
& \leq \int_{\eta(s)}^{\tilde{\eta}(s)} \left( \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left( \frac{2}{3}\mathcal{U}^2 + 3\mathcal{P} \right)^2 \mathcal{Y}_\eta(s, n) dn \right)^{1/2} \\
& \quad \times \left( \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) dn \right)^{1/2} \mathcal{Y}_\eta(s, m) dm \\
& \leq \int_{\eta(s)}^{\tilde{\eta}(s)} \left( \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left( \frac{8}{9}\mathcal{U}^4 + 18\mathcal{P}^2 \right) \mathcal{Y}_\eta(s, n) dn \right)^{1/2} \\
& \quad \times \left( \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) dn \right)^{1/2} \mathcal{Y}_\eta(s, m) dm \\
& \leq \mathcal{O}(1) \left( \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
& \quad \times \left( \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
& \leq \mathcal{O}(1) \left( \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
& \quad \times \left( \int_{\eta(s)}^{\tilde{\eta}(s)} 4\mathcal{P}\mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
& \leq \mathcal{O}(1) \left( \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
& \quad \times \sqrt{|\eta(s) - \tilde{\eta}(s)|} \\
& \leq \mathcal{O}(1) \left( |t - \tilde{t}| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2}.
\end{aligned}$$

Thus it is left to show that

$$\left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right|$$

is bounded uniformly with respect to both space and time. Therefore observe that, since all the involved terms are positive, we have

$$\begin{aligned}
& \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right| \\
& \leq \int_0^{2C} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm
\end{aligned}$$

$$\begin{aligned}
 &= \int_0^{2C} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{Y}_\eta(s, m) dm (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s, n) dn \\
 &\leq 2 \int_0^{2C} (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s, n) dn \leq 6C,
 \end{aligned}$$

where we used (3.22) in the last step. Thus we obtain

$$\begin{aligned}
 &\left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \right. \\
 &\quad \times \left. \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm \right| \\
 &\leq \mathcal{O}(1) \left( |t - \tilde{t}| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
 &\leq \mathcal{O}(1) |t - \tilde{t}|^{1/2},
 \end{aligned}$$

and

$$\begin{aligned}
 &\left| \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \right. \\
 &\quad \times \left. \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm ds \right| \\
 &\leq \mathcal{O}(1) |t - \tilde{t}|^{3/2}.
 \end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we end up with

$$\begin{aligned}
 |\tilde{I}_2| &= \left| 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))) ds \right| \\
 &\leq \mathcal{O}(1) (|t - \tilde{t}|^{3/2} + |t - \tilde{t}|^2).
 \end{aligned}$$

As far as  $\tilde{I}_1$  is concerned, we would like to apply the mean-value theorem. Note therefore that we showed before that the function  $(Q_t + H_t)(t, \xi)$  is continuous with respect to time, and hence the function  $(Q_t + H_t)(s, l(t, \eta))$ , considered as a function of  $s$ , is continuous with respect to  $s$ . Thus we end up with

$$\tilde{I}_1 = 2 \int_t^{\tilde{t}} (Q_t + H_t)(s, l(t, \eta)) ds = 2(\tilde{t} - t)(Q_t + H_t)(\tilde{s}, l(t, \eta))$$

for some  $\tilde{s}$  between  $t$  and  $\tilde{t}$ .

Last but not least, we therefore have

$$\begin{aligned} 2 \int_t^{\tilde{t}} (Q_t + H_t)(s, l(\tilde{t}, \eta)) ds &= \tilde{I}_1 + \tilde{I}_2 \\ &= 2(\tilde{t} - t)(Q_t + H_t)(\tilde{s}, l(t, \eta)) + \mathcal{O}(1)|\tilde{t} - t|^{3/2} \\ &\quad + \mathcal{O}(1)|\tilde{t} - t|^2, \end{aligned}$$

for some  $\tilde{s}$  between  $t$  and  $\tilde{t}$ , which implies (cf. (3.23)) that

$$\begin{aligned} \lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}} &= \lim_{\tilde{t} \rightarrow t} 2 \int_t^{\tilde{t}} \frac{(Q_t + H_t)(s, l(\tilde{t}, \eta))}{t - \tilde{t}} ds \\ &= 2(Q_t + H_t)(t, l(t, \eta)). \end{aligned}$$

Recalling that  $J(t, l(t, \eta)) = \eta$  for all  $t$  and  $\eta$ , we have that

$$J^{-1}(t, J(t, l(t, \eta))) = l(t, \eta),$$

if we, as before, denote the inverse to  $J(t, \cdot)$  by  $J^{-1}(t, \cdot)$ . Thus we have

$$\begin{aligned} l_t(t, \eta) &= \lim_{\tilde{t} \rightarrow t} \frac{l(t, \eta) - l(\tilde{t}, \eta)}{t - \tilde{t}} \\ &= \lim_{\tilde{t} \rightarrow t} \frac{J^{-1}(t, J(t, l(t, \eta))) - J^{-1}(t, J(t, l(\tilde{t}, \eta)))}{t - \tilde{t}} \\ &= \lim_{\tilde{t} \rightarrow t} \frac{J^{-1}(t, J(t, l(t, \eta))) - J^{-1}(t, J(t, l(\tilde{t}, \eta)))}{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))} \\ &\quad \times \lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}}. \end{aligned}$$

We have established that the last limit on the right-hand side exists and the existence of the first one follows from the fact that  $J^{-1}(t, \cdot)$  is differentiable almost everywhere.

**REMARK 3.2.** The above argument relies heavily on the fact that the function  $l(t, \cdot): [0, 2C] \rightarrow \mathbb{R}$  is continuous and strictly increasing, the reason being that  $J(t, \cdot)$  is continuous and strictly increasing. This is in contrast to the HS equation in [16], where the function  $H(t, \cdot)$  is increasing, but not necessarily strictly increasing and its inverse may have jumps. In the HS equations, if  $H(t, \cdot)$  is constant on some interval  $I$ , then  $H(s, \cdot)$  will be constant on  $I$  for any  $s$ , since  $H$  is independent of time. Furthermore, this means that the jumps in the new coordinates might change in height with respect to time, but never change their position. This is the essential difference to the CH equation where the jumps would turn up and disappear again immediately, which motivates the choice of  $G(t, x)$  and  $J(t, \xi)$ .

**3.3. New system: the right quantities.** Since we have shown that  $l(t, \eta)$  is differentiable almost everywhere with respect to both space and time, we can now establish rigorously the system of differential equations in our new coordinates. Recall that  $J(t, l(t, \eta)) = \eta$  and hence direct calculations yield

$$\begin{aligned} J_t(t, l(t, \eta)) + J_\xi(t, l(t, \eta))l_t(t, \eta) &= 0, \\ J_\xi(t, l(t, \eta))l_\eta(t, \eta) &= 1. \end{aligned}$$

According to the time evolution in Lagrangian coordinates, we thus end up with

$$\begin{aligned} l_\eta(t, \eta) &= \frac{1}{J_\xi(t, l(t, \eta))}, \\ l_t(t, \eta) &= -\frac{J_t(t, l(t, \eta))}{J_\xi(t, l(t, \eta))} = -J_t(t, l(t, \eta))l_\eta(t, \eta). \end{aligned}$$

As far as the new coordinates are concerned, we recall that

$$\begin{aligned} \mathcal{Y}(t, \eta) &= y(t, l(t, \eta)), \quad \mathcal{U}(t, \eta) = U(t, l(t, \eta)), \\ \text{and } \mathcal{P}(t, \eta) &= P(t, l(t, \eta)), \end{aligned}$$

and direct calculations using the differentiability proved for  $\mathcal{Q}$ ,  $\mathcal{P}$ , and  $l$  in Subsections 3.1 and 3.2 yield the following differential equations for the unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$ .

**THEOREM 3.3.** *Let  $(u, \mu)$  denote a weak conservative solution of (1.1). Define  $\mathcal{Y}$  by (2.3),  $\mathcal{U}$  by (2.6), and  $\mathcal{P}$  by (2.8). Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (3.28a)$$

$$\mathcal{U}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (3.28b)$$

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (3.28c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (3.29a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta, \quad (3.29b)$$



$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta (t, \theta) d\theta. \end{aligned} \tag{3.29c}$$

REMARK 3.4. This system is identical to the system derived in the smooth case (cf. (2.9)) as expected. We also remind the reader that  $\mathcal{Q}(t, \eta) = \mathcal{Q}(t, l(t, \eta))$ ,  $\mathcal{S}(t, \eta) = J_t(t, l(t, \eta)) - \frac{2}{3} \mathcal{U}^3(t, \eta)$ , and  $\mathcal{R}(t, \eta) = P_t(t, l(t, \eta)) - \mathcal{Q}\mathcal{U}(t, \eta)$ .

**3.4. Functional setting: consistency of the new coordinates.** The next main question is which functional space we should work in such that the right-hand side of system (3.28) can be regarded as a Lipschitz function of the chosen unknowns. For instance, it is difficult or rather impossible to establish the Lipschitz continuity of  $\mathcal{Q}$  with respect to  $\mathcal{P}$ ,  $\mathcal{U}$ , and  $\mathcal{Y}$ . However, we will see that one can establish that  $\mathcal{Q}$  is Lipschitz continuous with respect to  $\mathcal{P}^{1/2}$ ,  $\mathcal{U}$ , and  $\mathcal{Y}$ . At first sight this seems to be surprising, but on the other hand we established in Eulerian coordinates that  $2p(t, x) \geq u^2(t, x)$ , which rewrites in our new coordinates as

$$2\mathcal{P}(t, \eta) \geq \mathcal{U}^2(t, \eta). \tag{3.30}$$

Thus it seems somehow natural that  $\mathcal{P}(t, \eta)^{1/2}$  and  $|\mathcal{U}(t, \eta)|$  will behave similarly.

The aim of this subsection is to derive the system of differential equations for  $\mathcal{Y}$ ,  $\mathcal{U}$ , and  $\mathcal{P}^{1/2}$  and subsequently to establish that all terms turning up in this system are well defined by assuming that each of the new variables ( $\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2}$ ) is in  $L^2([0, 2C])$ .

Replacing the equation for  $\mathcal{P}$  with the corresponding one for  $\mathcal{P}^{1/2}$ , we find that system (3.28) reads as

$$\mathcal{Y}_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S}\right) \mathcal{Y}_\eta = \mathcal{U}, \tag{3.31a}$$

$$\mathcal{U}_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S}\right) \mathcal{U}_\eta = -\mathcal{Q}, \tag{3.31b}$$

$$(\mathcal{P}^{1/2})_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S}\right) (\mathcal{P}^{1/2})_\eta = \frac{\mathcal{Q}\mathcal{U}}{2\mathcal{P}^{1/2}} + \frac{\mathcal{R}}{2\mathcal{P}^{1/2}}, \tag{3.31c}$$

where  $\mathcal{Q}$ ,  $\mathcal{S}$ , and  $\mathcal{R}$  are given by (3.29).

Assume for the moment that  $\mathcal{P}^{1/2}$ ,  $\mathcal{U}$ , and  $\mathcal{Y}$  belong to  $L^2([0, 2C])$ . Then we want to show that all terms appearing in the above system also belong to  $L^2([0, 2C])$ . Therefore it is important to keep in mind, in addition to (3.30), that

$$2\mathcal{P}\mathcal{Y}_\eta(t, \eta) - \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) = 1 \tag{3.32}$$

and, in particular (cf. (3.12), (3.21), and (3.22)),

$$2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq \mathcal{H}_\eta(t, \eta) \leq 1, \tag{3.33a}$$

$$\mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq \mathcal{H}_\eta(t, \eta) \leq 1, \tag{3.33b}$$

$$2|\mathcal{Q}|\mathcal{Y}_\eta(t, \eta) \leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1. \tag{3.33c}$$

As an immediate consequence we obtain that

$$\mathcal{U}^3\mathcal{Y}_\eta(t, \eta) \quad \text{and} \quad \mathcal{U}^3\mathcal{U}_\eta(t, \eta)$$

are uniformly bounded. The last term of this form can be estimated as follows:

$$|\mathcal{U}^3(\mathcal{P}^{1/2})_\eta(t, \eta)| = \left| \frac{\mathcal{U}^3\mathcal{Q}\mathcal{Y}_\eta}{2\mathcal{P}^{1/2}}(t, \eta) \right| \leq \left| \frac{\mathcal{U}^3}{2\mathcal{P}^{1/2}}(t, \eta) \right| \leq \mathcal{U}^2(t, \eta),$$

and is therefore uniformly bounded. Here we used (3.30).

As far as  $\mathcal{S}$  is concerned, we want to establish that

$$|\mathcal{S}(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}(t, \eta),$$

implying that

$$|\mathcal{S}\mathcal{Y}_\eta(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq \mathcal{O}(1),$$

$$|\mathcal{S}\mathcal{U}_\eta(t, \eta)| \leq \mathcal{S}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) \leq \mathcal{O}(1),$$

$$|\mathcal{S}(\mathcal{P}^{1/2})_\eta(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}^{3/2}\mathcal{Y}_\eta(t, \eta) \leq \mathcal{O}(1).$$

Indeed, by the definition of  $\mathcal{S}$  and since  $|\mathcal{Q}| \leq \mathcal{P}$  and (cf. (3.12a))

$$\mathcal{U}_\eta^2(t, \eta) \leq \mathcal{H}_\eta\mathcal{Y}_\eta(t, \eta),$$

we have

$$\begin{aligned} |\mathcal{S}(t, \eta)| &= \left| \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left( \frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right)(t, \theta) d\theta \right| \\ &\leq \|\mathcal{U}(t, \cdot)\|_{L^\infty([0, 2C])} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}^2\mathcal{Y}_\eta(t, \theta) d\theta \\ &\quad + 2 \left( \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{P}^2\mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}^2\mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{H}_\eta(t, \theta) d\theta \right)^{1/2} \\
 & \leq 4\sqrt{C} \mathcal{P}(t, \eta) + 6 \left( \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \mathcal{P}^{1/2}(t, \eta).
 \end{aligned}$$

Thus it is left to show that

$$\int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \leq \mathcal{O}(1) \mathcal{P}(t, \eta). \quad (3.34)$$

As several times before, the main tool will be integration by parts together with

$$\mathcal{P}(t, \eta) = \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (2\mathcal{U}^2 - \mathcal{P}) \mathcal{Y}_\eta(t, \theta) + 1 d\theta.$$

Direct computations yield

$$\begin{aligned}
 & \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \\
 & = \mathcal{P}^2(t, \eta) - \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} 2\mathcal{P}\mathcal{Q}\mathcal{Y}_\eta(t, \theta) d\theta \\
 & = \mathcal{P}^2(t, \eta) - 2\mathcal{P}\mathcal{Q}(t, \eta) \\
 & \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2\mathcal{Q}^2 \mathcal{Y}_\eta + 2\mathcal{P}\mathcal{Q}_\eta)(t, \theta) d\theta \\
 & = \mathcal{P}^2(t, \eta) - 2\mathcal{P}\mathcal{Q}(t, \eta) \\
 & \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} 2(\mathcal{P}^2 + \mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
 & \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2(\mathcal{P} - \mathcal{U}^2) \mathcal{Y}_\eta(t, \theta) - 1) \mathcal{P}(t, \theta) d\theta. \quad (3.35)
 \end{aligned}$$

Here we have used (3.4b) and (3.32). Thus, rearranging the terms, we end up with

$$\begin{aligned}
 & \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \\
 & \leq \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (\mathcal{P}^2 + 2\mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
 & = 2\mathcal{P}\mathcal{Q}(t, \eta) - \mathcal{P}^2(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1)\mathcal{P}(t, \theta) d\theta \\
 & \leq 2\mathcal{P}^2(t, \eta) + 4\|\mathcal{P}(t, \cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t, \eta) \\
 & \leq 6\|\mathcal{P}(t, \cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t, \eta).
 \end{aligned} \tag{3.36}$$

Similar computations yield that

$$\begin{aligned}
 \int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta & \leq \int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} (\mathcal{P}^2 + 2\mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
 & = -\mathcal{P}^2(t, \eta) - 2\mathcal{P}\mathcal{Q}(t, \eta) \\
 & \quad + \int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} \\
 & \quad \times (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1)\mathcal{P}(t, \theta) d\theta \\
 & \leq 2\mathcal{P}^2(t, \eta) + 4\|\mathcal{P}(t, \cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t, \eta) \\
 & \leq 6\|\mathcal{P}(t, \cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t, \eta).
 \end{aligned}$$

This proves (3.34) due to (3.13).

There are two terms left to investigate. Namely, the term  $\mathcal{Q}\mathcal{U}/\mathcal{P}^{1/2}$ , and we easily find that

$$\left| \frac{\mathcal{Q}\mathcal{U}}{\sqrt{2}\mathcal{P}^{1/2}}(t, \eta) \right| \leq \mathcal{P}(t, \eta),$$

and hence it is uniformly bounded. The last term  $\mathcal{R}/\mathcal{P}^{1/2}$  is a bit more involved. We have

$$\begin{aligned}
 & |\mathcal{R}(t, \eta)| \\
 & \leq \left| \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \left( \frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \right| \\
 & \quad + \left| \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta \right| \\
 & \leq \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) |\mathcal{U}|(t, \theta) d\theta \\
 & \quad + \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} 2\mathcal{P} |\mathcal{U}| \mathcal{Y}_\eta(t, \theta) d\theta \\
 & \quad + \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} |\mathcal{U} \mathcal{Q}| \mathcal{Y}_\eta(t, \theta) d\theta \\
 & \leq \|\mathcal{U}(t, \cdot)\|_{L^\infty([0,2C])} \mathcal{P}(t, \eta) + \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (\mathcal{P}^2 + \mathcal{U}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
 & \leq \mathcal{O}(1) \mathcal{P}(t, \eta).
 \end{aligned} \tag{3.37}$$

Thus

$$\left| \frac{\mathcal{R}}{\mathcal{P}^{1/2}(t, \eta)} \right| \leq \mathcal{O}(1)\mathcal{P}^{1/2}(t, \eta),$$

and hence belongs to  $L^2([0, 2C])$ .

Later on, we will need in some of our estimates (cf. (3.3a)) that

$$\mathcal{P}(t, \eta) = \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (\mathcal{U}^2 \mathcal{Y}_\eta + \mathcal{H}_\eta)(t, \theta) d\theta,$$

which implies the estimate

$$\int_0^\eta e^{-\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)} (\mathcal{U}^2 \mathcal{Y}_\eta + \mathcal{H}_\eta)(t, \theta) d\theta \leq 4\mathcal{P}(t, \eta). \quad (3.38)$$

**3.5. The choice of the distance.** In order to motivate the choice of the distance between two solutions with the same energy and to outline the strategy for proving the Lipschitz continuity, let us come back to system (3.31). We observe that the unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  are advected by the same velocity field in the  $\eta$  variable. Moreover, since all terms make sense under the assumption that the unknowns belong to  $L^2([0, 2C])$ , as checked in the previous subsection, we consider a toy problem of the form

$$f_t + \mathcal{A}(f)f_\eta = \mathcal{B}(f),$$

where  $f$  might be a vector valued function in  $L^2([0, 2C])$ . This toy problem resembles our situation with the complicated operator dependencies on the unknowns. Within this functional framework, it is natural to look at the evolution of the  $L^2$ -norm of the unknown  $f$ . Given two solutions  $f_1$  and  $f_2$  to  $f_t + \mathcal{A}(f)f_\eta = \mathcal{B}(f)$ , direct computations yield

$$\begin{aligned} \frac{d}{dt} \|f_1 - f_2\|_{L^2([0, 2C])}^2 &= 2 \int_0^{2C} (f_1 - f_2)(f_{1,t} - f_{2,t})(t, \theta) d\theta \\ &= -2 \int_0^{2C} (\mathcal{A}(f_1)f_{1,\eta} - \mathcal{A}(f_2)f_{2,\eta})(f_1 - f_2)(t, \theta) d\theta \\ &\quad + 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta \\ &= -2 \int_0^{2C} \mathcal{A}(f_1)(f_{1,\eta} - f_{2,\eta})(f_1 - f_2)(t, \theta) d\theta \\ &\quad - 2 \int_0^{2C} f_{2,\eta}(\mathcal{A}(f_1) - \mathcal{A}(f_2))(f_1 - f_2)(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
 &+ 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta \\
 = &\mathcal{A}(f_1)(f_1 - f_2)^2(t, 0) - \mathcal{A}(f_1)(f_1 - f_2)^2(t, 2C) \\
 &+ \int_0^{2C} \mathcal{A}'(f_1)f_{1,\eta}(f_1 - f_2)^2(t, \theta) d\theta \\
 &- 2 \int_0^{2C} f_{2,\eta}(\mathcal{A}(f_1) - \mathcal{A}(f_2))(f_1 - f_2)(t, \theta) d\theta \\
 &+ 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta,
 \end{aligned}$$

where the first two terms in the last line are going to vanish if we impose the correct boundary conditions or rather if we have a good behavior of the solutions at both boundaries. Under this assumption, we can then use norm estimates if we know that  $\mathcal{A}'(f_1)f_{1,\eta}$  and  $f_{2,\eta}$  are uniformly bounded by a constant  $\mathcal{O}(1)$  and that  $\mathcal{A}$  and  $\mathcal{B}$  are Lipschitz continuous with Lipschitz constant  $\mathcal{O}(1)$  with respect to  $f$  to conclude that

$$\frac{d}{dt} \|f_1 - f_2\|_{L^2([0,2C])}^2 \leq \mathcal{O}(1) \|f_1 - f_2\|_{L^2([0,2C])}^2,$$

and Gronwall’s lemma then implies

$$\| (f_1 - f_2)(t) \|_{L^2([0,2C])} \leq \| (f_1 - f_2)(0) \|_{L^2([0,2C])} e^{\mathcal{O}(1)t}.$$

Hence the strategy consists in proving first a propagation in time for the  $L^2$ -norm of  $f$  in order to get bounds on the unknowns and check the validity of the approach. Then the second, and the most important and technical point, is to establish the Lipschitz estimates. We will come back to this point in Section 4, where we will also fix the following problem in the definition of our metric: the domain of definition of our unknowns depends on the total energy. This is unsatisfactory, if we want to compare solutions with different total energies.

**3.6. Propagation of  $L^2$  bounds: moment conditions.** Consider  $(u, \mu)$  a weak conservative solution of (1.1) and  $f = (\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ , defined by (2.3), (2.6), and (2.8), the solution to system (3.31). We want to show that the  $L^2$ -norm of  $f$  is propagated in time. Since both  $\mathcal{U}$  and  $\mathcal{P}$  are bounded functions due to (3.10) and (3.13), we are reduced to show the propagation of the  $L^2$ -norm of  $\mathcal{Y}$ . First of all, note that we have

$$\mathcal{Y}(t, G(t, x)) = x \quad \text{for all } x \in \mathbb{R},$$

where  $G(t, x)$  is given by

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x).$$

Notice that the  $x$ -distributional derivative of  $G$  is a measure that we denoted by  $\nu(t, \cdot) \in \mathcal{M}_+(\mathbb{R})$  given by

$$d\nu(t, x) = (2p - u^2)(t, x) + d\mu(t, x).$$

Since  $\mathcal{Y}(t, \eta)$  is the pseudoinverse to  $G(t, x)$ , then  $\mathcal{Y}$  pushes forward the uniform distribution on  $[0, 2C]$  to  $\nu$  (see [67]) and then

$$\int_{\mathbb{R}} x^2 d\nu = \int_0^{2C} \mathcal{Y}^2(t, \eta) d\eta.$$

Thus  $\mathcal{Y} \in L^2([0, 2C])$  is equivalent to  $\nu(t, \cdot)$  having a finite second moment.

**PROPOSITION 3.5.** *Let  $(u, \mu)$  denote a weak conservative solution of (1.1), and by  $f = (\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  the solution to system (3.31); then*

$$\int_0^{2C} \mathcal{Y}^2(t, \eta) d\eta \leq e^{\mathcal{O}(1)t} \left( 1 + \int_0^{2C} \mathcal{Y}^2(0, \eta) d\eta \right).$$

*Proof.* We show the propagation of the second moment in time using the Lagrangian coordinates. Note first that

$$\begin{aligned} \int_{\mathbb{R}} x^2 d\mu(t, x) &= \int_{\mathbb{R}} y^2 H_{\xi}(t, \xi) d\xi, \\ \int_{\mathbb{R}} x^2 u^2(t, x) dx &= \int_{\mathbb{R}} y^2 U^2 y_{\xi}(t, \xi) d\xi, \\ \int_{\mathbb{R}} x^2 p(t, x) dx &= \int_{\mathbb{R}} y^2 P y_{\xi}(t, \xi) d\xi. \end{aligned}$$

We start by computing the derivatives of  $y^2 H_{\xi}$ ,  $y^2 U^2 y_{\xi}$ , and  $y^2 P y_{\xi}$  with respect to time. Direct computations using the formulas in Section 3.1 then yield

$$\begin{aligned} |(y^2 H_{\xi})_t| &= |2yU H_{\xi} + y^2(3U^2 U_{\xi} - 2PU_{\xi} - 2QU y_{\xi})| \\ &\leq y^2 H_{\xi} + U^2 H_{\xi} + 3y^2 |U| H_{\xi} + 2y^2 P^2 y_{\xi} + 2y^2 H_{\xi} \\ &\leq \mathcal{O}(1)(y^2 H_{\xi} + y^2 P y_{\xi} + H_{\xi}) \end{aligned}$$

and

$$\begin{aligned} |(y^2 P y_\xi)_t| &= |2y U P y_\xi + y^2 P_t y_\xi + y^2 P U_\xi| \\ &\leq y^2 P^2 y_\xi + U^2 y_\xi + y^2 |P_t| y_\xi + y^2 P (y_\xi + H_\xi) \\ &\leq \mathcal{O}(1)(y^2 H_\xi + y^2 P y_\xi + H_\xi), \end{aligned}$$

where we used (3.20). Thus

$$\frac{d}{dt} (\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1}) \leq \mathcal{O}(1) (\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1} + C)$$

and

$$\int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(t, \xi) d\xi \leq e^{\mathcal{O}(1)t} \left( \int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(0, \xi) d\xi + C \right).$$

Since  $y^2 U^2 y_\xi(t, \xi) \leq y^2 H_\xi(t, \xi)$ , it follows that also

$$\int_{\mathbb{R}} y^2 U^2 y_\xi(t, \xi) d\xi$$

remains finite for all times, which proves the desired estimate since

$$\int_{\mathbb{R}} x^2 dv(t, x) = \int_{\mathbb{R}} x^2 (2p - u^2)(t, x) dx + \int_{\mathbb{R}} x^2 d\mu(t, x). \quad \square$$

The propagation of moments not only implies the feasibility of the strategy illustrated in Section 3.5 but also gives us a control on the ‘boundary terms’. In fact, since  $(u^2 + u_x^2)(t, x) \leq d\mu(t, x)$  and  $(2p - u^2)(t, x) \geq 0$ , the previous result implies that

$$\int_{\mathbb{R}} x^2 p(t, x) dx, \quad \int_{\mathbb{R}} x^2 u^2(t, x) dx, \quad \int_{\mathbb{R}} x^2 u_x^2(t, x) dx, \quad \text{and} \quad \int_{\mathbb{R}} x^2 d\mu(t, x) \quad (3.39)$$

are all finite. We show next that (3.39) implies that

$$xu(t, x) \rightarrow 0 \quad \text{and} \quad x^2 p(t, x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (3.40)$$

Indeed, (3.39) and  $u \in H^1(\mathbb{R})$  imply that

$$xu(t, x) \in L^2(\mathbb{R}) \quad \text{and} \quad (xu(t, x))_x = u(t, x) + xu_x(t, x) \in L^2(\mathbb{R}).$$

Hence  $xu(t, x)$  belongs to  $H^1(\mathbb{R})$  for any fixed  $t$  and in particular  $xu(t, x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  for any fixed  $t$ .



The argument for  $xp(t, x)$  is a bit more involved, but follows the same lines. To be more precise,  $p(t, x) \in L^2(\mathbb{R})$ ,  $|p_x(t, x)| \leq p(t, x)$ , and

$$\int_{\mathbb{R}} x^2 p^2(t, x) dx \leq \mathcal{O}(1) \int_{\mathbb{R}} x^2 p(t, x) dx < \infty,$$

imply that

$$xp(t, x) \in L^2(\mathbb{R}) \quad \text{and} \quad (xp(t, x))_x = p(t, x) + xp_x(t, x) \in L^2(\mathbb{R}).$$

Hence  $xp(t, x)$  belongs to  $H^1(\mathbb{R})$  for any fixed  $t$  and in particular  $xp(t, x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ .

Finally, it is left to show that  $x\sqrt{p(t, x)} \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Therefore note that  $p(t, x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and

$$|(x\sqrt{p(t, x)})_x| = \left| \sqrt{p(t, x)} + x \frac{p_x(t, x)}{2\sqrt{p(t, x)}} \right| \leq \sqrt{p(t, x)} + |x|\sqrt{p(t, x)},$$

which together with (3.39) implies that

$$x\sqrt{p(t, x)} \in L^2(\mathbb{R}) \quad \text{and} \quad (x\sqrt{p(t, x)})_x \in L^2(\mathbb{R}).$$

Hence  $x\sqrt{p(t, x)}$  belongs to  $H^1(\mathbb{R})$  for any fixed  $t$  and in particular  $x^2 p(t, x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . These properties will allow us to carry out several integrations by parts when deriving the anticipated Lipschitz estimate.

#### 4. The right metric: solutions with different energies

For the remaining part of this paper, we will consider two distinct solutions  $(u_j, \mu_j)$  for  $j = 1, 2$  of the CH equation (1.1) and estimate their difference using a carefully selected norm based on our new variables  $(\mathcal{Y}_j, \mathcal{U}_j, \mathcal{P}_j^{1/2})$  for  $j = 1, 2$  that yields a Lipschitz metric. The main idea remains to a large extent the one from the HS equation in [16], where a Lipschitz metric for measuring the distance between solutions with nonzero energy has been constructed.

We want to define a Lipschitz metric, which can measure the distance between solutions with different total energy based on our new coordinates. At first sight this seems to be impossible for two reasons:

1. The support of the new variables depends on the total energy since

$$\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2} : [0, 2C] \rightarrow \mathbb{R}.$$

2. In the case of the zero solution, that is,  $(u, \mu) = (0, 0)$ , the function  $\mathcal{Y}$  is not defined and the same applies to the new variables  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ .

The key idea is to use a rescaling; more precisely, given  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ , where  $\mathcal{Y} : [0, 2C] \rightarrow \mathbb{R}$  with  $C \neq 0$ . Then we can introduce

$$\begin{aligned}\tilde{\mathcal{Y}}(t, \eta) &= A\mathcal{Y}(t, A^2\eta), & \tilde{\mathcal{U}}(t, \eta) &= A\mathcal{U}(t, A^2\eta), \\ \tilde{\mathcal{P}}^{1/2}(t, \eta) &= A\mathcal{P}^{1/2}(t, A^2\eta), & \tilde{\mathcal{H}}(t, \eta) &= A^3\mathcal{H}(t, A^2\eta),\end{aligned}\quad (4.1)$$

with

$$A = \sqrt{2C} \quad (4.2)$$

for notational purposes.

This rescaling has three main properties that motivate our choice:

1. The support of the variables  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$  is independent of the energy, which allows us to compare arbitrary solutions with nonzero energy.
2. In the last section, we discussed that the natural solution space for the unknowns  $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$  is  $L^2([0, 2C])$ . Since our rescaling preserves the  $L^2$ -norm, the natural solution space for the unknowns  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$  is  $L^2([0, 1])$ .
3. The most important property is that this rescaling allows us to set

$$(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2}) = (0, 0, 0) \quad \text{for } (u, \mu) = (0, 0).$$

A close look at the definition of  $\tilde{\mathcal{Y}}(t, \eta)$  reveals that  $\tilde{\mathcal{Y}}(t, \eta)$  is the pseudoinverse to  $\tilde{G}(t, x)$ , which is given by

$$\tilde{G}(t, x) = \frac{1}{A^2} G\left(t, \frac{x}{A}\right) \quad \text{for } A \neq 0.$$

If one compares  $\tilde{G}(t, x)$  with  $G(t, x)$  for small  $A$ , then the graph of  $G(t, x)$  is squeezed in the  $x$  direction, but stretched in the  $y$  direction (cf. Figure 1). Taking a sequence of functions  $G_n(t, x)$  with  $A_n$  tending to zero, then the corresponding sequence of functions  $\tilde{G}_n(t, x)$  tends to the Heaviside function, which has as pseudoinverse  $\tilde{\mathcal{Y}}(t, \eta) = 0$  (cf. Figure 2).

Direct calculation yields the following theorem.

**THEOREM 4.1.** *Let  $(u, \mu)$  denote a weak conservative solution of (1.1). Define  $\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}$  and  $\tilde{\mathcal{P}}^{1/2}$  by (4.1). Then the following system of differential equations holds:*

$$\tilde{\mathcal{Y}}_t + \left( \frac{2}{3} \frac{1}{A^5} \tilde{\mathcal{U}}^3 + \frac{1}{A^6} \tilde{\mathcal{S}} \right) \tilde{\mathcal{Y}}_\eta = \tilde{\mathcal{U}}, \quad (4.3a)$$

$$\tilde{\mathcal{U}}_t + \left( \frac{2}{3} \frac{1}{A^5} \tilde{\mathcal{U}}^3 + \frac{1}{A^6} \tilde{\mathcal{S}} \right) \tilde{\mathcal{U}}_\eta = -\frac{1}{A^2} \tilde{\mathcal{Q}}, \quad (4.3b)$$

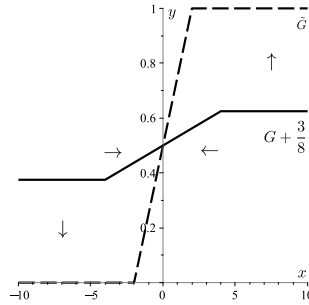


Figure 1. An example of how the rescaling changes the graph of a function  $G : \mathbb{R} \rightarrow \frac{1}{4}$ . The picture shows the function  $G + \frac{3}{8}$  (solid) and its rescaled version  $\tilde{G}$  (dash).

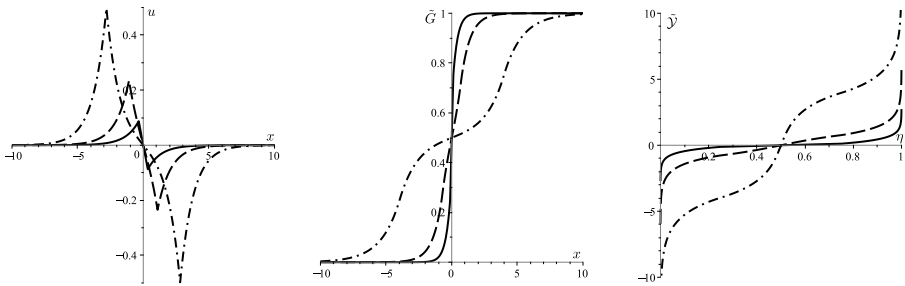


Figure 2. Three different peakon–antipeakon solutions  $u$  (left) with total energy 1 (dash–dot), 0.5 (dash), and 0.25 (solid) and the corresponding functions  $G$  (middle) and  $\tilde{Y}$  (right).

$$(\tilde{\mathcal{P}}^{1/2})_t + \left( \frac{2}{3} \frac{1}{A^5} \tilde{U}^3 + \frac{1}{A^6} \tilde{S} \right) (\tilde{\mathcal{P}}^{1/2})_\eta = \frac{1}{2A^2} \frac{\tilde{Q}\tilde{U}}{\tilde{\mathcal{P}}^{1/2}} + \frac{1}{2A^3} \frac{\tilde{R}}{\tilde{\mathcal{P}}^{1/2}}, \quad (4.3c)$$

where

$$\begin{aligned} \tilde{Q}(t, \eta) &= A^3 Q(t, A^2 \eta) \\ &= -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A} |\tilde{Y}(t, \eta) - \tilde{Y}(t, \theta)|} (2(\tilde{U}^2 - \tilde{\mathcal{P}}) \tilde{Y}_\eta(t, \theta) + A^5) d\theta, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \tilde{S}(t, \eta) &= A^4 S(t, A^2 \eta) \\ &= \int_0^1 e^{-\frac{1}{A} |\tilde{Y}(t, \eta) - \tilde{Y}(t, \theta)|} \left( \frac{2}{3} \tilde{U}^3 \tilde{Y}_\eta - \tilde{U}_\eta \tilde{Q} - 2\tilde{\mathcal{P}} \tilde{U} \tilde{Y}_\eta \right) (t, \theta) d\theta, \end{aligned} \quad (4.4b)$$

$$\begin{aligned}
\tilde{\mathcal{R}}(t, \eta) &= A^5 \mathcal{R}(t, A^2 \eta) \\
&= \frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{\lambda} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} \left( \frac{2}{3} A \tilde{\mathcal{U}}^3 \tilde{\mathcal{Y}}_\eta + A^6 \tilde{\mathcal{U}} \right) (t, \theta) \\
&\quad - \frac{1}{2} \int_0^1 e^{-\frac{1}{\lambda} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} \tilde{\mathcal{U}} \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta (t, \theta) d\theta.
\end{aligned} \tag{4.4c}$$

Now, we can define the metric between two general conservative solutions to (1.1). To simplify the notation we will from now on assume that all norms are on  $L^2([0, 1])$ , unless otherwise indicated, and write

$$\|\Psi\| = \|\Psi\|_{L^2([0,1])},$$

for any function  $\Psi$ . We will not always explicitly indicate the time dependence in all quantities and write  $\|\Psi\|$  rather than  $\|\Psi(t)\|$ .

**DEFINITION 4.2.** Let  $(u_i, \mu_i)$  for  $i = 1, 2$  denote two conservative solutions of (1.1). We define the distance between them as

$$\begin{aligned}
d((u_1, \mu_1), (u_2, \mu_2)) &= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
&\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|,
\end{aligned}$$

where  $A_i = \sqrt{2C_i}$  and  $C_i = \mu_i(\mathbb{R})$ ,  $i = 1, 2$ , and  $(\tilde{\mathcal{Y}}_i, \tilde{\mathcal{U}}_i, \tilde{\mathcal{P}}_i^{1/2})$  are given by (4.1).

**REMARK 4.3.** Notice that in the case of one of the two solutions being the trivial solution, the distance reduces to the  $L^2$ -norm of the solution in the right functional space, that is,

$$d((u, \mu), (0, 0)) = \|\tilde{\mathcal{Y}}\| + \|\tilde{\mathcal{U}}\| + \|\tilde{\mathcal{P}}^{1/2}\| + \sqrt{2C},$$

where  $C = \mu(\mathbb{R})$ , that we already estimated in Section 3.6. It remains to show that the right-hand side of system (4.3) is Lipschitz continuous with respect to the new unknowns  $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ .

The main result of this work can finally be stated. The proof consists in showing the corresponding Lipschitz estimates in each of the components of the distance in Definition 4.2. This is done in the next section in Lemmas 5.3–5.5. Collecting these results leads to our main theorem due to Gronwall's lemma.

**THEOREM 4.4.** Consider initial data  $u_{i,0} \in H^1(\mathbb{R})$ ,  $\mu_{i,0} \in \mathcal{M}_+(\mathbb{R})$  such that  $d(\mu_{ac})_{i,0} = (u_i^2 + u_{i,x}^2) dx$  and  $C_i = \mu_i(\mathbb{R})$ , and let  $(u_i, \mu_i)$  for  $i = 1, 2$  denote

the corresponding conservative solutions of the CH equation (1.1). Then we have that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq e^{\mathcal{O}(1)t} d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

where  $\mathcal{O}(1)$  denotes a constant depending only on  $\max_j(C_j)$  remaining bounded as  $\max_j(C_j) \rightarrow 0$ .

Now, let us start to do some preparatory work for the next section by collecting the main estimates we need in the new variables. Introduce

$$\begin{aligned} \tilde{\mathcal{D}}(t, \eta) &= \int_0^\eta e^{\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\theta) - \tilde{\mathcal{Y}}(t,\eta))} \left( (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \\ &= \frac{1}{2} \int_0^\eta e^{\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\theta) - \tilde{\mathcal{Y}}(t,\eta))} (\tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{H}}_\eta)(t, \theta) d\theta, \end{aligned} \quad (4.5a)$$

$$\begin{aligned} \tilde{\mathcal{E}}(t, \eta) &= \int_\eta^1 e^{\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \left( (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \\ &= \frac{1}{2} \int_\eta^1 e^{\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} (\tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{H}}_\eta)(t, \theta) d\theta, \end{aligned} \quad (4.5b)$$

so that we can write

$$\tilde{\mathcal{P}}(t, \eta) = \frac{1}{2A} (\tilde{\mathcal{D}}(t, \eta) + \tilde{\mathcal{E}}(t, \eta)) \quad \text{and} \quad \tilde{\mathcal{Q}}(t, \eta) = \frac{1}{2} (-\tilde{\mathcal{D}}(t, \eta) + \tilde{\mathcal{E}}(t, \eta)) \quad (4.6)$$

and

$$\tilde{\mathcal{Q}}(t, \eta) = -\tilde{\mathcal{D}}(t, \eta) + A\tilde{\mathcal{P}}(t, \eta). \quad (4.7)$$

Note that both  $\tilde{\mathcal{D}}(t, \eta)$  and  $\tilde{\mathcal{E}}(t, \eta)$  have some very nice properties. Namely,

$$0 \leq \tilde{\mathcal{D}}(t, \eta) \leq 2A\tilde{\mathcal{P}}(t, \eta) \quad \text{and} \quad 0 \leq \tilde{\mathcal{E}}(t, \eta) \leq 2A\tilde{\mathcal{P}}(t, \eta), \quad (4.8)$$

and

$$\left| \frac{d}{d\eta} \tilde{\mathcal{D}}(t, \eta) \right| = \left| (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \eta) + \frac{1}{2} A^5 - \frac{1}{A} \tilde{\mathcal{D}} \tilde{\mathcal{Y}}_\eta(t, \eta) \right| \leq \mathcal{O}(1) A^5, \quad (4.9)$$

$$\left| \frac{d}{d\eta} \tilde{\mathcal{E}}(t, \eta) \right| = \left| -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \eta) - \frac{1}{2} A^5 + \frac{1}{A} \tilde{\mathcal{E}} \tilde{\mathcal{Y}}_\eta(t, \eta) \right| \leq \mathcal{O}(1) A^5. \quad (4.10)$$

The scaling leads to some changes in the standard estimates. Some of the problematic terms depending on  $\frac{1}{\lambda}$  in system (4.3) can be controlled since

$$\begin{aligned} 2\tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \eta) &\leq A^5, \quad 2|\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta(t, \eta)| \leq A^4, \quad \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta(t, \eta) \leq A^5, \\ &\text{and} \quad \tilde{\mathcal{H}}_\eta(t, \eta) \leq A^5, \end{aligned} \quad (4.11)$$

from (3.33) and (4.1), as well as

$$A^2 \tilde{\mathcal{U}}_\eta^2(t, \eta) \leq \tilde{\mathcal{H}}_\eta \tilde{\mathcal{Y}}_\eta(t, \eta) \leq A^5 \tilde{\mathcal{Y}}_\eta(t, \eta). \quad (4.12)$$

In addition, equation (3.32) becomes

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t, \eta) - \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta(t, \eta) + \tilde{\mathcal{H}}_\eta(t, \eta) = A^5. \quad (4.13)$$

Moreover, we can show that

$$\begin{aligned} 4\|\tilde{\mathcal{P}}(t, \cdot)\|_{L^\infty} &= 4\|A^2\mathcal{P}(t, \cdot)\|_{L^\infty} \leq A^4 \\ 2\|\tilde{\mathcal{U}}^2(t, \cdot)\|_{L^\infty} &= 2\|A^2\mathcal{U}^2(t, \cdot)\|_{L^\infty} \leq A^4. \end{aligned} \quad (4.14)$$

We now collect all the estimates we will need on our solutions in the new variables with the explicit dependence on  $A$ . This will be repeatedly used in the next section.

$$0 \leq 4\tilde{\mathcal{P}} \leq A^4, \quad [\text{from (4.14) and (3.13)}], \quad (4.15a)$$

$$\sqrt{2}|\tilde{\mathcal{U}}| \leq A^2, \quad [\text{from (4.14) and (3.10)}], \quad (4.15b)$$

$$|\tilde{\mathcal{Q}}| \leq A\tilde{\mathcal{P}}, \quad [\text{from common sense}], \quad (4.15c)$$

$$\tilde{\mathcal{U}}^2 \leq 2\tilde{\mathcal{P}}, \quad [\text{from (3.30)}], \quad (4.15d)$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15e)$$

$$2|\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta| \leq A^4, \quad [\text{from (4.11)}], \quad (4.15f)$$

$$0 \leq \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15g)$$

$$\tilde{\mathcal{U}}_\eta^2 \leq A^3\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.12) and (3.12a)}], \quad (4.15h)$$

$$\sqrt{2}|\tilde{\mathcal{U}}|\tilde{\mathcal{P}}^{1/2}\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.15e) and (4.15g)}], \quad (4.15i)$$

$$|\tilde{\mathcal{R}}| \leq \mathcal{O}(1)A^3\tilde{\mathcal{P}}, \quad [\text{from (3.37)}], \quad (4.15j)$$

$$0 \leq \tilde{\mathcal{H}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15k)$$

$$0 \leq \tilde{\mathcal{Y}}_\eta, \quad [\text{from the definition}], \quad (4.15l)$$

$$A^2\tilde{\mathcal{U}}_\eta^2 \leq \tilde{\mathcal{H}}_\eta\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.12) and (3.12a)}], \quad (4.15m)$$

$$0 \leq \tilde{\mathcal{D}} \leq 2A\tilde{\mathcal{P}}, \quad [\text{from (4.8)}], \quad (4.15n)$$

$$0 \leq \tilde{\mathcal{E}} \leq 2A\tilde{\mathcal{P}}, \quad [\text{from (4.8)}], \quad (4.15o)$$

$$2\sqrt{2}\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta \leq A^6, \quad [\text{from (4.15a), (4.15e), and (4.15h)}], \quad (4.15p)$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leq A^8, \quad [\text{from (4.15e) and (4.15h)}], \quad (4.15q)$$

$$4\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leq A^7\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.15a) and (4.15h)}]. \quad (4.15r)$$

In addition, we have several integrals that appear frequently:

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \leq 6\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (4.16g)}], \quad (4.16a)$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq \frac{3}{2} A^5 \tilde{\mathcal{P}}(t, \eta),$$

[from (3.36) and (4.15a)], (4.16b)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (3.38)}], \quad (4.16c)$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (3.38)}], \quad (4.16d)$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 2A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16e)

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16f)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \leq 6\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16g)

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16h)

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) d\theta \leq 7\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16i)

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16j)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 3\frac{1+\beta}{\beta} \frac{A^{1+4\beta}}{4^\beta} \tilde{\mathcal{P}}(t, \eta), \quad \beta > 0,$$

[from Lemma A.10]. (4.16k)

As for derivatives of the quantity  $\tilde{\mathcal{P}}^{1/2}$  we have

$$|(\tilde{\mathcal{P}}^{1/2})_\eta| \leq \frac{1}{2A} \tilde{\mathcal{P}}^{1/2} \tilde{\mathcal{Y}}_\eta, \quad (4.17a)$$

$$|\tilde{\mathcal{U}}(\tilde{\mathcal{P}}^{1/2})_\eta| \leq \frac{3}{8} A^4, \quad (4.17b)$$

$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leq \frac{1}{8} A^3 \tilde{\mathcal{Y}}_\eta, \quad (4.17c)$$

$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leq \frac{1}{16} A^2 \tilde{\mathcal{Y}}_\eta^2. \quad (4.17d)$$

## 5. Lipschitz estimates in the new metric

We will repeatedly use the following elementary identity and estimate.

LEMMA 5.1. *Let  $a_j, b_j \in \mathbb{R}$  for  $j = 1, 2$ . Then we have*

$$a_2 b_2 - a_1 b_1 = (b_2 - b_1)(a_2 \mathbb{1}_{b_1 < b_2} + a_1 \mathbb{1}_{b_1 \geq b_2}) + \min(b_1, b_2)(a_2 - a_1).$$

Here

$$\mathbb{1}_{\mathcal{K}} = \begin{cases} 1, & \mathcal{K} \text{ is true,} \\ 0, & \mathcal{K} \text{ is false.} \end{cases}$$

We also use  $\mathbb{1}_{\mathcal{K}}$  to denote the characteristic (indicator) function of a set  $\mathcal{K}$ . Furthermore, we have the estimate

$$|\min(a_1, b_1) - \min(a_2, b_2)| \leq \max(|a_1 - a_2|, |b_1 - b_2|).$$

LEMMA 5.2. *Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be two Lipschitz continuous functions with Lipschitz constants  $c_f$  and  $c_g$ , respectively. Then the function  $h: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $h = \min(f, g)$  is a Lipschitz continuous function with Lipschitz constant bounded by  $\max(c_f, c_g)$ .*

*Proof.* Use the previous lemma. □

Introduce for any function  $\Phi$  the functions

$$\Phi^- = \min(0, \Phi), \quad \Phi^+ = \max(0, \Phi). \quad (5.1)$$

We then have

$$\Phi^- \leq 0 \leq \Phi^+, \quad \Phi^- \Phi^+ = 0, \quad \Phi = \Phi^+ + \Phi^-, \quad |\Phi| = \Phi^+ - \Phi^-. \quad (5.2)$$

For two functions  $\Phi, \Psi$  we have

$$|\Phi^\pm - \Psi^\pm| \leq |\Phi - \Psi|. \quad (5.3)$$

Frequently, we will have to estimate quantities like

$$\int_0^1 e^{-\frac{1}{\lambda} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} (\dots)(t, \theta) d\theta.$$



We will rewrite it as follows:

$$\begin{aligned}
 & \int_0^1 e^{-\frac{1}{\lambda}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\dots)(t,\theta) d\theta \\
 &= \left( \int_0^\eta + \int_\eta^1 \right) e^{-\frac{1}{\lambda}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\dots)(t,\theta) d\theta \\
 &= \int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\dots)(t,\theta) d\theta \\
 &\quad + \int_\eta^1 e^{\frac{1}{\lambda}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\dots)(t,\theta) d\theta.
 \end{aligned} \tag{5.4}$$

Both integrals can be estimated in the same manner, using that the argument in the exponential is negative in both cases. Eliminating the absolute value will allow us to perform integration by parts.

**5.1. Lipschitz estimates for  $\tilde{\mathcal{Y}}$ .** From the system of differential equations, we have

$$\tilde{\mathcal{Y}}_{i,t} + \left( \frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) \tilde{\mathcal{Y}}_{i,\eta} = \tilde{\mathcal{U}}_i, \tag{5.5}$$

where

$$\begin{aligned}
 \tilde{\mathcal{P}}_i(t,\eta) &= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{\lambda_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{Q}}_i(t,\eta) &= -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{\lambda_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{S}}_i(t,\eta) &= \int_0^1 e^{-\frac{1}{\lambda_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} \left( \frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta} \right) (t,\theta) d\theta.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \frac{d}{dt} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) d\eta \\
 &= 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,t} - \tilde{\mathcal{Y}}_{2,t})(t,\eta) d\eta \\
 &= 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) d\eta \\
 &\quad + \frac{4}{3} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left( \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right) (t,\eta) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &+ 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left( \frac{1}{A_2^6} \tilde{\mathcal{S}}_2 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1 \tilde{\mathcal{Y}}_{1,\eta} \right) (t, \eta) d\eta \\
 &= I_1 + I_2 + I_3.
 \end{aligned} \tag{5.6}$$

The strategy is to use integration by parts for the last two integrals  $I_2$  and  $I_3$ , while we want to use straightforward estimates for  $I_1$ , which will finally yield that

$$\begin{aligned}
 &\frac{d}{dt} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which depends on  $A = \max_j(A_j)$  and which remains bounded as  $A \rightarrow 0$ .

*The first integral  $I_1$ :* Note that

$$\begin{aligned}
 |I_1| &= \left| 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (t, \eta) d\eta \right| \\
 &\leq \int_0^1 ((\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 + (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2) (t, \eta) d\eta \\
 &= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
 \end{aligned}$$

*The second integral  $I_2$ :* Note that

$$\begin{aligned}
 \frac{3}{4} |I_2| &= \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left( \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right) (t, \eta) d\eta \right| \\
 &\leq \frac{1}{(\max_j(A_j))^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}) (t, \eta) d\eta \right| \\
 &\quad + \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \right| \\
 &\leq \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (t, \eta) d\eta \right| \\
 &\quad + \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) [\tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}] \right. \\
 &\quad \times (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) (t, \eta) d\eta \left. \right| \\
 &\quad + \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta}) (t, \eta) d\eta \right|
 \end{aligned}$$

$$\begin{aligned}
& + \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta \right| \\
& = J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We will write

$$A = \max_j(A_j), \quad a = \min_j(A_j). \quad (5.7)$$

We commence with  $J_1$ : Since  $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A_i^5 \leq A^5$  for all  $t$  and  $\eta$ , we have

$$\begin{aligned}
J_1 & = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right| \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
\end{aligned}$$

Next term is  $J_2$ : Using once more that  $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A^5$  for all  $t$  and  $\eta$ , we get

$$\begin{aligned}
J_2 & = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) [\tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}] \right. \\
& \quad \left. \times (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)(t, \eta) d\eta \right| \\
& \leq \frac{1}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| (2\tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + 2\tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2})(t, \eta) d\eta \\
& \leq 2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).
\end{aligned}$$

Next, it is  $J_3$ : Here integration by parts plays an essential role. Thus we have to determine first whether or not the function  $\eta \mapsto \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta)$  is differentiable, and, if so, if its derivative is bounded. Recall from Lemma A.2 (ii) that the function  $\eta \mapsto \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta)$  is Lipschitz continuous with Lipschitz constant at most  $A^4$ . Thus

$$\left| \frac{d}{d\eta} \left( \tilde{\mathcal{U}}_1(t, \eta) \min_j(\tilde{\mathcal{U}}_j^2(t, \eta)) \right) \right| \leq \frac{3}{2} A^4 \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \mathcal{O}(1) A^6, \quad (5.8)$$

and integration by parts together with (3.40) yields

$$\begin{aligned}
J_3 & = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) d\eta \right| \\
& = \left| -\frac{1}{2A^5} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta) \right|_{\eta=0}^1
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2A^5} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t, \eta) d\eta \Big| \\
 & = \frac{1}{2A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t, \eta) d\eta \right| \\
 & \leq \frac{3}{4A} \|\tilde{\mathcal{U}}_1(t, \cdot)\|_{L^\infty} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned}$$

Finally, we consider  $J_4$ : Direct calculations yield

$$\begin{aligned}
 J_4 & \leq \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left( \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \right| \right. \\
 & \quad \left. + \mathbb{1}_{A_2 < A_1} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right| \right) \\
 & \leq \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} (\mathbb{1}_{A_1 \leq A_2} A_1^7 + \mathbb{1}_{A_2 < A_1} A_2^7) \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\
 & \leq \frac{|A_1^5 - A_2^5|}{A^3} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \leq 5A |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where we again used that  $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} \leq A_i^5$  and  $\|\tilde{\mathcal{U}}_i\|_{L^\infty} \leq A_i^2$ .

The third integral  $I_3$ : We will consider several smaller parts of  $I_3$  by inserting the definition of  $\tilde{\mathcal{S}}_i$ , and combine them in the end. We write

$$I_3 = \frac{4}{3} I_{31} - 4 I_{32} - 2 I_{33}, \tag{5.9}$$

where

$$\begin{aligned}
 I_{31} & = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 & \quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta,
 \end{aligned}$$

$$\begin{aligned}
 I_{32} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta, \\
 I_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

Recall definitions (5.1). Then we have

$$\begin{aligned}
 I_{31} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{K}^- + \tilde{K}^+)(t, \eta) d\eta.
 \end{aligned}$$

Since both integrals have the same structure, it suffices to consider the second integral. Furthermore, applying the device described in (5.4), it suffices to study the terms, denoted by  $K^\pm$ , with the upper limit of the inner integral replaced by  $\eta$ .

Therefore observe that we can write

$$\begin{aligned}
& K^+(t, \eta) \\
&= \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
&\quad - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
&= \frac{1}{(\max_j(A_j))^6} \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
&\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
&\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
&= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+(t, \theta)} d\theta \right) \\
&\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+(t, \theta)} d\theta \right) \\
&\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
&\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
&\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
&= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
\end{aligned}$$

$$\begin{aligned}
 & \times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 & + \frac{1}{A_6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \right. \\
 & \times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \left. \right) \\
 & + \frac{1}{A_6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & + \frac{1}{A_6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 & + \frac{1}{A_6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) \\
 & - \frac{1}{A_6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \left. \right) \\
 & + \frac{1}{A_6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \left. \right) \\
 & - \frac{1}{A_6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \left. \right) \\
 & + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & = (J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10})(t, \eta). \tag{5.10}
 \end{aligned}$$

Here

$$B(\eta) = \{(t, \theta) \mid e^{\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \eta)} \leq e^{\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta)}\} \tag{5.11}$$

which means especially that  $B(\eta)$  depends heavily on  $\eta$ ! Observe that the set  $B(\eta)$  is scale invariant in the sense that  $(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$  and  $(c\tilde{\mathcal{Y}}_1, c\tilde{\mathcal{Y}}_2)$  define the same set  $B(\eta)$  for any constant  $c$ .

As far as the first two terms  $J_1$  and  $J_2$  are concerned, they have the same structure, and hence we only consider the term  $J_1$ , that is,

$$\begin{aligned} & \int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \, d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) \, d\theta \right) \, d\eta. \end{aligned}$$

The main ingredients are the following observations:

$$|(\tilde{\mathcal{U}}_1^+)^3(t, \eta) - (\tilde{\mathcal{U}}_2^+)^3(t, \eta)| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+} \leq 3\tilde{\mathcal{U}}_2^2(t, \eta) |\tilde{\mathcal{U}}_1(t, \eta) - \tilde{\mathcal{U}}_2(t, \eta)|,$$

and (4.16c).

Thus we have

$$\begin{aligned} & \left| \int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \, d\eta \right| \\ &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\ & \quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) \, d\theta \right) \, d\eta \right| \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) \, d\theta \right)^2 \, d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & \quad + \frac{9}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) \, d\theta \right) \, d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \end{aligned}$$



$$\begin{aligned}
 & + \frac{36}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
 \leq & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{18}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta d\eta \\
 = & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & - \frac{18A_2}{A^6} \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
 & + \frac{18A_2}{A^6} \int_0^1 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \\
 \leq & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + 18\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
 \end{aligned}$$

where in the last step we used (4.15b) and (4.16c), which imply that

$$\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \leq 8A^5 \tilde{\mathcal{P}}_2(t, \eta), \tag{5.12}$$

and hence the left-hand side of (5.12) tends to 0 as  $\eta$  to 0 or 1 according to (3.40). Thus the second term above vanishes.

As far as the third and fourth terms  $J_3$  and  $J_4$  are concerned, they again have the same structure, and hence we only consider the integral corresponding to  $J_3$ , that is,

$$\begin{aligned}
 & \int_0^1 J_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \\
 = & \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

Recall Lemma A.1(ii). Direct computations yield

$$\begin{aligned}
 & \left| \int_0^1 J_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\
 = & \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
 & \times \left. \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\
 \leq & \frac{4|A_1 - A_2|}{aA^6 e} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 + \frac{16|A_1 - A_2|^2}{a^2 A^{12} e^2} \mathbb{1}_{A_1 \leq A_2} \\
 & \quad \times \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{U}_1| |\tilde{U}_2|^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 \\
 & \quad + \frac{16a^4 |A_1 - A_2|^2}{a^2 A^{12} e^2} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 \\
 & \quad + \frac{64|A_1 - A_2|^2}{A^9 e^2} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 \\
 & \quad + \frac{32|A_1 - A_2|^2}{A^4 e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta d\eta \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 + \frac{64A_2 |A_1 - A_2|^2}{A^4 e^2} \\
 & \quad \times \left( - \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0} + \int_0^1 \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \eta) d\eta \right) \\
 & \leq \| \tilde{Y}_1 - \tilde{Y}_2 \|^2 + \frac{128A^2}{e^2} |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1) (\| \tilde{Y}_1 - \tilde{Y}_2 \|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

As far as the terms  $J_5$  and  $J_6$  are concerned, they again have the same structure, and hence we only consider the integral corresponding to  $J_5$ , that is,

$$\begin{aligned}
 \int_0^1 J_5(\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) d\eta &= \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\
 & \quad \left. \times \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

The main ingredient is the estimate

$$\begin{aligned}
 & |e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}| \\
 & \leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (|\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)| + |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)|) \\
 & \quad \text{for all } (t, \theta) \in B(\eta),
 \end{aligned} \tag{5.13}$$

which follows from Lemma A.1 (i).

Direct computations yield

$$\begin{aligned}
 & \left| \int_0^1 J_5(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\
 & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \right. \\
 & \quad \left. \left. \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\
 & \quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \frac{1}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \frac{4}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \quad + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq (2A + 1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{a^2 A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{a^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{a^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)} \\
 &\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
 &= \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad - \frac{2}{aA^6} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \\
 &\quad + \frac{2}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ . We used (cf. (4.16c)) that

$$\begin{aligned}
 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta &\leq \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &\leq 4A_2 \tilde{\mathcal{P}}_2(t, \eta).
 \end{aligned}$$

In the last step we used (4.15b), (4.15g), and finally that

$$\begin{aligned}
 &\left| \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
 &\leq (\|\tilde{\mathcal{Y}}_1 \tilde{\mathcal{U}}_1\|_{L^\infty}^2 + \|\tilde{\mathcal{Y}}_2 \tilde{\mathcal{U}}_2\|_{L^\infty}^2) 8A \tilde{\mathcal{P}}_2(t, \eta).
 \end{aligned}$$

Since  $\tilde{\mathcal{P}}_i(t, \eta)$  tends to 0 as  $\eta$  tends to 0 and 1, the term on the left-hand side tends to zero as  $\eta$  tends to 0 and 1, respectively (cf. (3.40)).

Consider next the terms  $J_7$  and  $J_8$ , that is,

$$\begin{aligned}
 (J_7 + J_8)(t, \eta) &= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right).
 \end{aligned}$$

Here we have to be a bit more careful. Introducing

$$E = \left\{ (t, \eta) \mid \int_0^\eta \Omega(t, \eta, \theta) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \int_0^\eta \Omega(t, \eta, \theta) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}, \tag{5.14}$$

with

$$\Omega(t, \eta, \theta) = \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta),$$

we can write

$$\begin{aligned}
 &(J_7 + J_8)(t, \eta) \\
 &= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 &= \frac{1}{A^6} (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
 &\quad \times \min_k \left[ \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \\
 &\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\
 &\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})
 \end{aligned}$$

$$\begin{aligned} & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\ & = (L_1 + L_2 + L_3)(t, \eta). \end{aligned}$$

As far as the first term  $L_1$  is concerned, the corresponding integral can be estimated as follows (we use (3.40)):

$$\begin{aligned} & \left| \int_0^1 L_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\ & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \right. \\ & \quad \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \left. \right| \\ & = \left| -\frac{1}{2A^6} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\ & \quad \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \Big|_{\eta=0}^1 \\ & \quad + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\ & \quad \times \frac{d}{d\eta} \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \left. \right| \\ & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2, \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant only depending on  $A$ , which remains bounded as  $A \rightarrow 0$ , since the derivative

$$\frac{d}{d\eta} \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right]$$

exists and is uniformly bounded; see Lemma A.8.

As far as the last term  $L_3$  (a similar argument works for  $L_2$ ) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned} & \int_0^1 L_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \\ & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
 &\quad \times \left[ (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right]_{\theta=0}^{\eta} \\
 &\quad - \int_0^{\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \\
 &\quad \times \left[ \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
 &\quad \left. + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right) \right] d\theta \Big] d\eta \\
 &= -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
 &\quad \times \int_0^{\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \\
 &\quad \times \left[ \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
 &\quad \left. + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right) \right] d\theta d\eta \\
 &= L_{31} + L_{32}.
 \end{aligned}$$

As far as the first term  $L_{31}$  is concerned, we have, since

$$\min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A^5 \min_j(\tilde{\mathcal{U}}_j^+) \leq \frac{1}{\sqrt{2}} A^7,$$

that

$$\begin{aligned}
 |L_{31}| &\leq \left| \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \right| \\
 &\leq \frac{A}{\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes a constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

The second term  $L_{32}$ , on the other hand, is a bit more demanding. We start by considering the first part of  $L_{32}$ . From Lemma A.2 we have that

$$\left| \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}(t, \theta)).$$

This implies that

$$\begin{aligned}
& \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left. \left( \frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min(\tilde{\mathcal{U}}_j^+)^3(t, \theta) d\theta d\eta \right| \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left. \left( \frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min(\tilde{\mathcal{U}}_j^+)^3(t, \theta) d\theta \right)^2 d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \times \left. \min(\tilde{\mathcal{U}}_j^+)^3 \max(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta) d\theta \right)^2 d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{a^2}{2A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left. \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) d\theta \right)^2 d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
& \quad \times \left( \int_0^\eta e^{-\frac{3}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{1}{2A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
& \quad \times \left( \int_0^\eta e^{-\frac{3}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta
\end{aligned}$$



$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{2A^5} \int_0^1 8A\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 2A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + 4AA_2 \left( - \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Here we used (4.13) and (4.16e). Furthermore, we use (4.16f).

Next, we turn to the second half of  $L_{32}$ . Recall first (4.16g).

From Lemma A.2 we have that

$$\left| \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta).$$

Thus we can conclude as before

$$\begin{aligned}
&\frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \left. \times \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) d\theta d\eta \right| \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \left. \times \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{a_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{24}{A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + 12A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \\
 &= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + 12aA \left( - \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\
 &\quad \left. + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \right) \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned}$$

We conclude that

$$|L_{32}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Finally, we have a look at  $J_9$  (the argument for  $J_{10}$  follows the same lines). We have

$$\begin{aligned}
 &\left| \int_0^1 J_9(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\
 &= \left| \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^2 \mathbb{1}_{A_1 \leq A_2} \\
 &\quad \times \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq 4 \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^3 \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\leq 2 \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^8 \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\
 &\leq 6\sqrt{2}A(A_2 - A_1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq 6A(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

We now turn our attention to  $I_{32}$ , that is,

$$I_{32} = \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{Y}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta)|} \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{Y}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta)|} \tilde{P}_1 \tilde{U}_1 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta.$$

As before, we are only going to establish the estimates for one part of it, since the other parts can be treated similarly. Let

$$\tilde{I}_{32} = \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{Y}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{Y}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ = \frac{1}{A_6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \tilde{Y}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \tilde{Y}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ + \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{1,\eta}(t, \eta) \\ \times \left( \int_0^\eta e^{-\frac{1}{A_1} (\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ + \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \\ \times \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3. \tag{5.15}$$

Direct calculations yield for  $\tilde{K}_2$  (and similarly for  $\tilde{K}_3$ ) that

$$|\tilde{K}_2| \leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \\ \times \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{1,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1} (\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ \leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{Y}_1 - \tilde{Y}_2\| \\ \times \left( \int_0^1 \tilde{Y}_{1,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1} (\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2}$$

$$\begin{aligned}
 &\leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\
 &\quad \times \left. \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta d\eta \right)^{1/2} \\
 &\leq \sqrt{6} \frac{A_2^6 - A_1^6}{A_1^3 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left( \int_0^1 \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) d\eta \right)^{1/2} \\
 &\leq \frac{\sqrt{6}}{2} \frac{A_2^6 - A_1^6}{A^4} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2),
 \end{aligned}$$

where we used (4.16b), (4.16c), and  $A_1 \leq A_2$ .

On the other hand, the term  $\tilde{K}_1$  needs to be rewritten a bit more. Namely,

$$\begin{aligned}
 \tilde{K}_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5.
 \end{aligned}$$

We start by having a close look at  $\tilde{J}_1$  ( $\tilde{J}_2$  can be handled similarly). One has

$$\begin{aligned}
 |\tilde{J}_1| & \leq \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
 & \quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \quad \times \left. (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})(\tilde{\mathcal{P}}_2^{1/2} + \tilde{\mathcal{P}}_1^{1/2}) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left( 2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \\
 & \quad \times \left( \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{8A_2}{A^6} \left( - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
& + \int_0^1 (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}$$

where we used that

$$\begin{aligned}
0 & \leq \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
& \leq A^4 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq A^4 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \leq A^5 \mathcal{O}(1) \tilde{\mathcal{P}}_2^{1/2}(t, \eta),
\end{aligned}$$

where the very last term tends to 0 as  $\eta \rightarrow 0, 1$ . In the last step we used (4.16b).

Next, we investigate  $\tilde{\mathcal{J}}_3$  ( $\tilde{\mathcal{J}}_4$  can be handled in a similar way). The argument is a bit more involved and hence we start with some preliminary estimates. One has

$$\begin{aligned}
& \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq 2A^4 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq 2A^4 \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \leq A^5 \mathcal{O}(1) \tilde{\mathcal{P}}_2^{1/4}(t, \eta),
\end{aligned}$$

where the term on the right-hand side tends to 0 for  $\eta \rightarrow 0, 1$ . In the last step we used (4.16j); cf. (3.35) (and the derivative  $\tilde{\mathcal{Q}}_{2,\eta}$  can be found from (4.7) and (4.9)).

With these estimates in mind, we end up with, recalling (4.16e),

$$\begin{aligned}
 |\tilde{J}_3| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{P}_j) (\tilde{U}_2^+ - \tilde{U}_1^+) \tilde{Y}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{U}_1^+ \leq \tilde{U}_2^+}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\quad + \frac{1}{A^{12}} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{P}_j) (\tilde{U}_2^+ - \tilde{U}_1^+) \tilde{Y}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{U}_1^+ \leq \tilde{U}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\quad + \frac{1}{A^{12}} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{U}_2^+ - \tilde{U}_1^+)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{U}_2^+ - \tilde{U}_1^+)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\quad + \frac{1}{A^6} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} (\tilde{U}_2^+ - \tilde{U}_1^+)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\quad + \frac{1}{A^6} \left( -2A_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{U}_2^+ - \tilde{U}_1^+)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0} \right. \\
 &\quad \left. + 2A_2 \int_0^1 (\tilde{U}_2^+ - \tilde{U}_1^+)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \eta) d\eta \right) \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{U}_2 - \tilde{U}_1\|^2).
 \end{aligned}$$

As far as  $\tilde{J}_5$  is concerned, we have to rewrite it a bit more. Namely,

$$\begin{aligned}
 \tilde{J}_5 &= \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \\
 &\quad \times \left( \tilde{Y}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{P}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 = & \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 & \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right.
 \end{aligned}$$



$$\begin{aligned} & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\ & = \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5, \end{aligned}$$

where  $B(\eta)$  is given by (5.11).

Both  $\tilde{L}_1$  and  $\tilde{L}_2$  can be handled in much the same way, and therefore we only consider  $\tilde{L}_1$ . One has

$$\begin{aligned} |\tilde{L}_1| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ & \quad \left. \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\ & \leq \frac{4}{aA^6 e} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{a^2 A^{12} e^2} \mathbb{1}_{A_1 \leq A_2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{5}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{32}{A^9 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A^4 e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{4A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \\ & \quad \times \int_0^\eta e^{\frac{1}{4A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta |A_1 - A_2|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{64}{A^3 e^2} \left( - \int_0^\eta e^{-\frac{1}{4A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\ &\quad \left. + \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) |A_1 - A_2|^2 \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where we used (4.16h).

Next we turn our attention to  $\tilde{L}_3$  and  $\tilde{L}_4$ , which can be handled in much the same way, and therefore we only consider  $\tilde{L}_3$ . One has

$$\begin{aligned} |\tilde{L}_3| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \right. \\ &\quad \left. \times \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \Big| \\ &\leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \right. \\ &\quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| \right. \\ &\quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{1}{\sqrt{2}A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\sqrt{2}A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \frac{3}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^5} \left( - \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\
 &\quad \left. + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where we used (4.16k) for  $\beta = \frac{1}{2}$ .

Finally, we can turn our attention to  $\tilde{L}_5$ , which we need to split into several parts. We write

$$\begin{aligned}
 \tilde{L}_5 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
& \quad \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \left. \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
& \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \left. \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
& \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \left. \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
& = \tilde{L}_{51} + \tilde{L}_{52} + \tilde{L}_{53},
\end{aligned}$$

where  $\tilde{E}$  is given by

$$\begin{aligned}
\tilde{E} & = \left\{ (t, \eta) \mid \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. \leq \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}. \quad (5.16)
\end{aligned}$$

As far as the first term  $\tilde{L}_{51}$  is concerned, it can be estimated as follows (we use (3.40)):

$$\begin{aligned}
|\tilde{L}_{51}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \right. \\
& \quad \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \left. \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \Big|
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2A^6} \left| -(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
 &\quad \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 &\quad \times \left. \left. \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
 &\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\
 &\quad \times \frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 &\quad \times \left. \left. \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2, \tag{5.17}
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant only depending on  $A$ , which remains bounded as  $A \rightarrow 0$ , provided we can show that the derivative in the latter integral exists and is uniformly bounded. In fact, from Lemma A.6 we have that

$$\frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right)$$

exists and is uniformly bounded.

As far as the term  $\tilde{L}_{52}$  (a similar argument works for  $\tilde{L}_{53}$ ) is concerned, the integral can be estimated as follows:

$$\begin{aligned}
 \tilde{L}_{52} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 &\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left[ (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
 &\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) (t, \theta) \right]_{\theta=0}^\eta
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \\
 & \times \left[ \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
 & \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right] d\eta \\
 = & -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 & \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \\
 & \times \left[ \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
 & \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right] d\eta \\
 = & M_1 + M_2.
 \end{aligned}$$

As far as the first term  $M_1$  is concerned, we have since

$$2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \leq \frac{A^7}{\sqrt{2}},$$

that

$$\begin{aligned}
 |M_1| & \leq \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 & \leq \frac{A}{2\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

The second term  $M_2$ , on the other hand, is a bit more demanding. (i): First of all, recall (A.4), that is,

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta),$$

which implies that

$$\begin{aligned}
& \frac{1}{A_6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left. \left( \frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \right. \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{A_{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left. \left( \frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{4a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{a^2}{8A^2 A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{8A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \\
& \quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{A_2^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{A}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \\
& = \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 - AA_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
& \quad + AA_2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
& \leq \mathcal{O}(1) \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2,
\end{aligned}$$

where we used (4.16e) and (4.16f).

(ii): First of all, we have to establish that  $\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)$  is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.3 we show that

$$\left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \theta) \right| \leq 2A^4(\min_j(\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta).$$

We are now ready to establish a Lipschitz estimate for the second part of  $M_2$ . Indeed,

$$\begin{aligned} & \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\ & \quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ & \quad \times \left. \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta d\eta \right| \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\ & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\ & \quad \times \left. \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\ & \quad \times \left. |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\min_j(\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4(1 + \sqrt{2})^2}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & \quad + \frac{36}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \end{aligned}$$



$$\begin{aligned}
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{252}{A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + 126A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + 126A \left( -2A_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
 &\quad + \int_0^1 2A_2 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where we used (4.16i). Thus we find that

$$|\tilde{L}_{52}| + |\tilde{L}_{53}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

We now turn our attention to  $I_{33}$ , that is,

$$\begin{aligned}
 I_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

As before, we apply (5.4). Thus it suffices to study the following term.

$$\begin{aligned}
 \bar{I}_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & = \bar{K}_1 + \bar{K}_2 + \bar{K}_3.
 \end{aligned}$$

Direct calculations yield for  $\bar{K}_2$  (and similarly for  $\bar{K}_3$ ) that

$$\begin{aligned}
 |\bar{K}_2| & \leq \frac{A_2^6 - A_1^6}{A_1^5 A_2^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \theta) d\theta \right) d\eta \\
 & \leq \frac{A_2^6 - A_1^6}{A_1^5 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \quad \times \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
 & \leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \right. \\
 & \quad \left. \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{H}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 & \leq \sqrt{6} \frac{A_2^6 - A_1^6}{A_1^3 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left( \int_0^1 \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) d\eta \right)^{1/2} \\
 & \leq \frac{\sqrt{6}}{2} \frac{A_2^6 - A_1^6}{A^4} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Here we used (4.15m), (4.16b), and (4.16d).

$\bar{K}_1$ , on the other hand, needs to be rewritten a bit more. Recall (4.7). Then we can write

$$\begin{aligned}
 \bar{K}_1 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( A_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - A_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right. \\
 &\left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right) d\eta. \tag{5.18}
 \end{aligned}$$

In the next step we are applying integration by parts, so that we get rid of  $\tilde{\mathcal{U}}_{i,\eta}(t, \theta)$  in the integrands and hence can therefore have a splitting into positive and negative parts. Therefore note that for  $i = 1, 2$ , we have

$$\begin{aligned}
 &\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
 &= e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \theta) \Big|_{\theta=0}^\eta \\
 &\quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left( \frac{1}{A_i} \tilde{\mathcal{P}}_i + \frac{1}{A_i^2} \tilde{\mathcal{Q}}_i \right) \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
 &= \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \eta) \\
 &\quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left( \frac{2}{A_i} \tilde{\mathcal{P}}_i - \frac{1}{A_i^2} \tilde{\mathcal{D}}_i \right) \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
 &= \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \eta) \\
 &\quad - \frac{2}{A_i} \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
 &\quad + \frac{1}{A_i^2} \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta,
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
 &= \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \\
 &\quad \times \left( \int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\theta) - \tilde{\mathcal{Y}}_i(t,l))} \left( \tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
 &= \int_0^\eta \left( \int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,l))} \left( \tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
 &= \left( \int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,l))} \left( \tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \tilde{\mathcal{U}}_i(t, \theta) \Big|_{\theta=0}^\eta \\
 &\quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left( \tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + \frac{1}{2} A_i^5 \tilde{\mathcal{U}}_i(t, \theta) d\theta
 \end{aligned}$$

$$= \tilde{D}_i \tilde{U}_i(t, \eta) - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{Y}_i(t, \eta) - \tilde{Y}_i(t, \theta))} \left( \tilde{U}_i^2 - \tilde{P}_i \right) \tilde{Y}_{i, \eta}(t, \theta) + \frac{1}{2} A_i^5 \tilde{U}_i(t, \theta) d\theta.$$

Here we used that for  $\theta \leq \eta$ ,

$$\begin{aligned} 0 &\leq \int_0^\theta e^{-\frac{1}{A_i}(\tilde{Y}_i(t, \eta) - \tilde{Y}_i(t, l))} \left( \tilde{U}_i^2 - \tilde{P}_i \right) \tilde{Y}_{i, \eta}(t, l) + \frac{1}{2} A_i^5 dl \\ &\leq \int_0^\eta e^{-\frac{1}{A_i}(\tilde{Y}_i(t, \eta) - \tilde{Y}_i(t, l))} \left( \tilde{U}_i^2 - \tilde{P}_i \right) \tilde{Y}_{i, \eta}(t, l) + \frac{1}{2} A_i^5 dl \\ &\leq \tilde{D}_i(t, \eta) \leq 2A_i \tilde{P}_i(t, \eta). \end{aligned}$$

We finally end up with

$$\begin{aligned} \bar{K}_1 &= \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(A_2 \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2, \eta} - A_1 \tilde{P}_1 \tilde{U}_1 \tilde{Y}_{1, \eta})(t, \eta) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(\tilde{D}_1 \tilde{U}_1 \tilde{Y}_{1, \eta} - \tilde{D}_2 \tilde{U}_2 \tilde{Y}_{2, \eta})(t, \eta) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \\ &\times \left( \tilde{Y}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t, \eta) - \tilde{Y}_2(t, \theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2 \tilde{Y}_{2, \eta}(t, \theta) \right. \\ &- \tilde{Y}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t, \eta) - \tilde{Y}_1(t, \theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1 \tilde{Y}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{3}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \tilde{Y}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t, \eta) - \tilde{Y}_1(t, \theta))} \tilde{P}_1 \tilde{U}_1 \tilde{Y}_{1, \eta}(t, \theta) d\theta \right. \\ &- \tilde{Y}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t, \eta) - \tilde{Y}_2(t, \theta))} \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \tilde{Y}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t, \eta) - \tilde{Y}_2(t, \theta))} \tilde{U}_2^3 \tilde{Y}_{2, \eta}(t, \theta) d\theta \right. \\ &- \tilde{Y}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t, \eta) - \tilde{Y}_1(t, \theta))} \tilde{U}_1^3 \tilde{Y}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{1}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)(t, \eta) \left( \tilde{Y}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t, \eta) - \tilde{Y}_2(t, \theta))} A_2^5 \tilde{U}_2(t, \theta) d\theta \right. \\ &- \tilde{Y}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t, \eta) - \tilde{Y}_1(t, \theta))} A_1^5 \tilde{U}_1(t, \theta) d\theta \Big) d\eta \\ &= \bar{K}_{11} + \bar{K}_{12} + \bar{K}_{13} + \bar{K}_{14} + \bar{K}_{15} + \bar{K}_{16}. \end{aligned} \tag{5.19}$$

We start by considering  $\bar{K}_{11}$ , which can be further split into

$$\begin{aligned}
 \bar{K}_{11} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) d\eta \\
 &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
 &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
 &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
 &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &= \bar{B}_{11} + \bar{B}_{12} + \bar{B}_{13} + \bar{B}_{14} + \bar{B}_{15} + \bar{B}_{16}.
 \end{aligned}$$

For  $\bar{B}_{11}$  we have (and similarly for  $B_{12}$ ) that

$$\begin{aligned}
 |\bar{B}_{11}| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right| \\
 &\leq \frac{A}{2\sqrt{2}} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|).
 \end{aligned}$$

Recalling (4.15i), we have for  $\bar{B}_{13}$  (and similarly for  $\bar{B}_{14}$ ) that

$$\begin{aligned}
 |\bar{B}_{13}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \right| \\
 &\leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 &\leq \sqrt{2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2).
 \end{aligned}$$

As far as  $\bar{B}_{15}$  is concerned, we want to use integration by parts. Therefore it is important to recall (3.40) and Lemma A.3 (ii), which imply that

$$\begin{aligned}
 |\bar{B}_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \, d\eta \right| \\
 &= \frac{a}{2A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 &\quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \Big| \\
 &= \frac{a}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \right| \\
 &\leq \frac{A}{2\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned} \tag{5.20}$$

$\bar{B}_{16}$  is straightforward. Indeed, one has using (4.15e) that

$$\begin{aligned}
 |\bar{B}_{16}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \, d\eta \right| \\
 &\leq \frac{1}{2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \, d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|.
 \end{aligned}$$

We continue with  $\bar{K}_{12}$ . We can split  $\bar{K}_{12}$  as follows:

$$\begin{aligned}
 \bar{K}_{12} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) \, d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2) \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) \, d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \eta) \, d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \, d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \, d\eta \\
 &= \bar{B}_{21} + \bar{B}_{22} + \bar{B}_{23} + \bar{B}_{24}.
 \end{aligned}$$

As the first step we need to establish an estimate for  $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t, \eta)$ . Note that  $|\tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \eta)| \leq A^{5/2} \sqrt{\tilde{\mathcal{Y}}_{i,\eta}(t, \eta)}$ , and hence it is in general unbounded. Thus our estimate for  $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t, \eta)$  must take care of this problem.

Applying Lemma A.9 finally yields

$$\begin{aligned}
|\bar{B}_{21}| &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2| |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta)| \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) d\eta \\
&\leq \frac{2}{A^{9/2}} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{\alpha}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{3}{4\lambda}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{\alpha}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) |\tilde{\mathcal{U}}_1| |\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta)
\end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A} \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{1}{4} \tilde{\mathcal{Y}}_1(t,\eta)} \\
 & \times \left( \int_0^\eta e^{\frac{1}{A} \tilde{\mathcal{Y}}_1(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 + (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 + A(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + 12A^{1/2} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \times \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\eta)} \left( \int_0^\eta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
 & + \frac{3A^{1/2}}{2} \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{2}{A} \tilde{\mathcal{Y}}_1(t,\eta)} \left( \int_0^\eta e^{\frac{2}{A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + 6A^{1/2} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \times \left( \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\eta)} \left( \int_0^\eta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Following the same lines, one obtains

$$|\bar{B}_{22}| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

The estimate for  $\bar{B}_{23}$  is straightforward. Namely,

$$\begin{aligned}
 |\bar{B}_{23}| & \leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
 \end{aligned}$$

As far as  $\bar{B}_{24}$  is concerned, recall from Lemma A.4 (ii) that

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta)) \right| \leq \mathcal{O}(1) A^7,$$

which yields, together with (3.40), that

$$\begin{aligned}
 |\bar{B}_{24}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
 & = \frac{1}{2A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 & \quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big|
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} (\min(\tilde{\mathcal{D}}_j \tilde{\mathcal{U}}_2)(t, \eta)) d\eta \right| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned}$$

Next, we have a look at  $\bar{K}_{13}$ , which can be rewritten as follows:

$$\begin{aligned}
 \bar{K}_{13} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \bar{K}_{13}^+ + \bar{K}_{13}^-.
 \end{aligned}$$

Note that both  $\bar{K}_{13}^+$  and  $\bar{K}_{13}^-$  have the same structure and hence we are only going to present the details for  $\bar{K}_{13}^+$ , which needs to be rewritten a bit more. Namely,

$$\begin{aligned}
 \bar{K}_{13}^+ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta \left( \min_j \left( e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) - \min_j \left( e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta \left( \min_j \left( e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) - \min_j \left( e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
 & \times \min_k \left( \int_0^\eta \min_j \left( e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \\
 & \times \left( \int_0^\eta \min_j \left( e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_D(t, \eta) \\
 & \times \left( \int_0^\eta \min_j \left( e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right. \\
 & \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 & = \hat{B}_{31} + \hat{B}_{32} + \bar{B}_{31} + \bar{B}_{32} + \bar{B}_{33} + \bar{B}_{34} \\
 & + \bar{B}_{35} + \bar{B}_{36} + \hat{B}_{35} + \hat{B}_{36} + \bar{B}_{37} + \bar{B}_{38} + \bar{B}_{39},
 \end{aligned}$$

where  $B(\eta)$  by (5.11) and

$$D = \left\{ (t, \eta) \mid \int_0^\eta \gamma(t, \eta, \theta) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \int_0^\eta \gamma(t, \eta, \theta) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}, \tag{5.21}$$

where

$$\Upsilon(t, \eta, \theta) = \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+).$$

We start by estimating  $\hat{B}_{31}$  (a similar argument works for  $\hat{B}_{32}$ ). Direct calculations yield

$$\begin{aligned} |\hat{B}_{31}| &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{aA^7} |A_2 - A_1| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\ &\leq \frac{2\sqrt{6}}{A^4} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) d\eta \\ &\leq \sqrt{6}A |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

For  $\bar{B}_{31}$  (a similar argument works for  $\bar{B}_{32}$ ) we would like to apply some of the estimates established when investigating  $\bar{K}_1$ . Thus we split  $\bar{B}_{31}$  into even smaller parts, that is,

$$\begin{aligned} |\bar{B}_{31}| &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\bar{d}_{11} + \bar{d}_{12}) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\bar{\mathcal{D}}_1 \leq \bar{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_1 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_3 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 & = B_{311} + B_{312} + B_{313} + B_{314}.
 \end{aligned}$$

By (A.12) and (A.13), we have that

$$(\bar{d}_{11} + \bar{d}_{12})(t, \theta) \leq 2A^{3/2} \tilde{\mathcal{D}}_2^{1/2} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) + 2\sqrt{2}A^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \theta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|,$$

and hence

$$\begin{aligned}
 B_{311} & \leq \frac{2}{A^{11/2}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{2\sqrt{2}}{A^{11/2}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{4}{A^{11}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{2\sqrt{2}}{A^{11/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \times \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{4}{A^{11}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{2\sqrt{2}}{A^{11/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
 & \times \left. \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{8}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{4}{A^{5/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \times \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{8A_2}{A^5} \left[ - \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\
 & \left. + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right] \\
 & + 4A^{1/2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned}$$

Recalling the estimate for  $\bar{T}_1$ , we have that

$$\begin{aligned}
 B_{312} & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{4}{A^4} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^3} \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} |\tilde{Y}_1 - \tilde{Y}_2|(t, l) dl \right) \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{8\sqrt{2}}{\sqrt{3}eA^3} \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
 \leq & 5\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{Y}_1 - \tilde{Y}_2| |\tilde{U}_2|^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{16}{A^8} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{U}_1 - \tilde{U}_2)^2(t, l) dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{2}{A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{1}{A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} |\tilde{Y}_1 - \tilde{Y}_2|(t, l) dl \right) \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{128}{3e^2A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
 \leq & 5\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^4 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{Y}_1 - \tilde{Y}_2)^2 \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{16}{A^8} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{U}_1 - \tilde{U}_2)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{2}{A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} |\tilde{Y}_1 - \tilde{Y}_2|(t, l) dl \right)^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{128}{3e^2 A^6} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq 5 \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 & + \frac{2}{A^4} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + \frac{64}{A^7} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right.
 \end{aligned}$$



$$\begin{aligned}
 & \times \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{U}_1 - \tilde{U}_2)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{8}{A^5} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{4}{A^5} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{512}{3e^2 A^5} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 \leq & 5 \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 & + A \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{A_2} \tilde{Y}_2(t,\theta)} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + \frac{32}{A^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{Y}_{2,\eta}(t, \theta) e^{-\frac{1}{2A} \tilde{Y}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-(\frac{1}{2A} \tilde{Y}_2(t,\eta) - \frac{1}{A} \tilde{Y}_2(t,l))} (\tilde{U}_1 - \tilde{U}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
 & + 4 \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{Y}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-(\frac{1}{2A_2} \tilde{Y}_2(t,\eta) - \frac{1}{A_2} \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
 & + 2 \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{Y}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-(\frac{1}{2A_2} \tilde{Y}_2(t,\eta) - \frac{1}{A_2} \tilde{Y}_2(t,l))} dl \right) d\theta \right) d\eta \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 & + \frac{256}{3e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{Y}_{2,\eta}(t, \theta) e^{-\frac{1}{4A} \tilde{Y}_2(t,\theta)} \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\theta e^{-\left(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l)\right)} dl \right) d\theta \Big) d\eta |A_1 - A_2|^2 \\
 \leq & 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + A \left[ -A_2 \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\
 & + \left. \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \right] \\
 & + \frac{32}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left[ -2 \int_0^\theta e^{\frac{1}{A} \tilde{\mathcal{Y}}_2(t,l) - \frac{1}{2A} (\tilde{\mathcal{Y}}_2(t,\eta) + \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \Big|_{\theta=0}^\eta \right. \\
 & + \left. 2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right] d\eta \\
 & + 4A_2 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left[ -2 \int_0^\theta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l) - \frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\eta) + \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \Big|_{\theta=0}^\eta \right. \\
 & + \left. 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right] d\eta \\
 & + 2A_2 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left[ -2 \int_0^\theta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l) - \frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\theta) + \tilde{\mathcal{Y}}_2(t,\eta))} dl \Big|_{\theta=0}^\eta \right. \\
 & + \left. 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right] d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{256A}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left[ -4 \int_0^\theta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l) - \left(\frac{1}{4A} \tilde{\mathcal{Y}}_2(t,\theta) + \frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta)\right)} \Big|_{\theta=0}^\eta \right. \\
 & + \left. 4 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right] d\eta |A_1 - A_2|^2 \\
 \leq & \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{64}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + 8A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + 4A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1024A}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
 \leq & \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + 64 \left[ -2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
 & + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \Big] \\
 & + 8AA_2 \left[ -2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
 & + 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
 & + 4AA_2 \left[ -2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{1024A^2}{3e^2} \left[ -2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] |A_1 - A_2|^2 \\
 \leq & \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Similar calculations yield for  $B_{313}$ ,

$$\begin{aligned}
 B_{313} \leq & \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{2\sqrt{2}}{A^4} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{\sqrt{2}A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{2A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{4\sqrt{2}}{\sqrt{3}eA^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{8}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{1}{2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{1}{4A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{32}{3e^2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{8}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{4A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{32}{3e^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{1}{2A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + \frac{32}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{2}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{128}{3e^2 A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{A}{4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + \frac{16}{A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A} \tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) d\theta \right) d\eta \\
 & + \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
 & + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l)\right)} dl \right) d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{64}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{4A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l)\right)} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{A}{4} \left[ -A_2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
 & + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
 & + \frac{16}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left[ -2 \int_0^\theta e^{\frac{1}{A} \tilde{Y}_2(t,l) - \frac{1}{2A} (\tilde{Y}_2(t,\eta) + \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right]_{\theta=0}^\eta \\
 & + 2 \int_0^\eta e^{\frac{1}{A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \Big] d\eta \\
 & + A_2 \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \\
 & \times \left[ -2 \int_0^\theta e^{\frac{1}{A_2} \tilde{Y}_2(t,l) - \frac{1}{2A_2} (\tilde{Y}_2(t,\eta) + \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right]_{\theta=0}^\eta \\
 & + 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big] d\eta \\
 & + \frac{A_2}{2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left[ -2 \int_0^\theta e^{\frac{1}{A_2} \tilde{Y}_2(t,l) - \frac{1}{2A_2} (\tilde{Y}_2(t,\theta) + \tilde{Y}_2(t,\eta))} dl \right]_{\theta=0}^\eta \\
 & + 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \Big] d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{64A}{3e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left[ -4 \int_0^\theta e^{\frac{3}{4A} \tilde{Y}_2(t,l) - (\frac{1}{4A} \tilde{Y}_2(t,\theta) + \frac{1}{2A} \tilde{Y}_2(t,\eta))} \right]_{\theta=0}^\eta \\
 & + 4 \int_0^\eta e^{\frac{1}{2A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \Big] d\eta |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{32}{A} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A} \tilde{Y}_2(t,\theta)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right) d\eta \\
 & + 2A \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 & + A \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{256A}{3e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{Y}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A} \tilde{Y}_2(t,\theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + 32 \left[ -2 \int_0^\eta e^{\frac{1}{2A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
 & + 2 \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \eta) d\eta \Big]
 \end{aligned}$$

$$\begin{aligned}
 &+ 2AA_2 \left[ -2 \int_0^\eta e^{\frac{1}{2A_2}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
 &+ 2 \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \eta) d\eta \Big] \\
 &+ AA_2 \left[ -2 \int_0^\eta e^{\frac{1}{2A_2}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &+ \frac{256A^2}{3e^2} \left[ -2 \int_0^\eta e^{\frac{1}{2A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] |A_1 - A_2|^2 \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Last but not least, direct calculations for  $B_{314}$  yield

$$\begin{aligned}
 B_{314} &\leq \frac{6}{A^3} |A_1 - A_2| \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{6}{A^3} |A_1 - A_2| \|\tilde{Y}_1 - \tilde{Y}_2\| \\
 &\quad \times \left( \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
 &\leq \frac{6}{A^3} |A_1 - A_2| \|\tilde{Y}_1 - \tilde{Y}_2\| \\
 &\quad \times \left( \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right)^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 &\leq \frac{12}{A^{5/2}} |A_1 - A_2| \|\tilde{Y}_1 - \tilde{Y}_2\| \\
 &\quad \times \left( \int_0^1 \tilde{\mathcal{P}}_2 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 &\leq 6\sqrt{2} |A_1 - A_2| \|\tilde{Y}_1 - \tilde{Y}_2\|
 \end{aligned}$$



$$\begin{aligned} & \times \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{4\kappa} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\ & \times \left. \left. \left( \int_0^\theta e^{\frac{3}{4\kappa} \tilde{\mathcal{Y}}_2(t,l) - \frac{1}{2\kappa} \tilde{\mathcal{Y}}_2(t,\eta)} dl \right) d\theta \right) d\eta \right)^{1/2} \\ & \leq \mathcal{O}(1) |A_1 - A_2| \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|. \end{aligned}$$

Collecting the estimates we have obtained, we conclude

$$| \bar{B}_{31} | \leq \mathcal{O}(1) (\| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + |A_1 - A_2|^2).$$

Recalling (4.16c), a direct computation yields for  $\bar{B}_{33}$  (and in much the same way for  $\bar{B}_{34}$ ) that

$$\begin{aligned} | \bar{B}_{33} | & \leq \frac{1}{A^7} \int_0^1 | \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 | \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \times \left( \int_0^\eta e^{-\frac{1}{2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{D}}_j) | \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 | \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \times \left( \int_0^\eta e^{-\frac{1}{2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} | \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 | (t, \theta) d\theta \right)^2 d\eta \\ & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\ & + \frac{4}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \times \left( \int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} ( \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 )^2(t, \theta) d\theta \right) d\eta \\ & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\ & + \frac{4}{A^6} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ( \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 )^2(t, \theta) d\theta \right) d\eta \\ & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\ & + \frac{2}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} ( \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 )^2(t, \theta) d\theta \right) d\eta \\ & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\ & + \frac{2A_2}{A} \left[ -2 \int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ( \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 )^2(t, \theta) d\theta \right]_{\eta=0}^1 \end{aligned}$$

$$\begin{aligned}
 & + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \, d\eta \Big] \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).
 \end{aligned}$$

Direct calculations yield for  $\bar{B}_{35}$  (and in much the same way for  $\bar{B}_{36}$ ) that

$$\begin{aligned}
 |\bar{B}_{35}| & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 & \quad \times \min_j(\tilde{D}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) \, d\theta \Big) \, d\eta \\
 & \leq \frac{1}{A^7 A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta (|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) + |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta)) \right. \\
 & \quad \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{D}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \Big) \, d\eta \\
 & \leq \frac{1}{A^7 A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{D}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \, d\eta \\
 & \quad + \frac{1}{A^7 A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{D}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) \, d\theta \right) \, d\eta \\
 & \leq \frac{2}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{P}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} \, d\eta \\
 & \quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \quad + \frac{4}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{P}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) \, d\theta \right) \, d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2\sqrt{6}}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \, d\eta \\
 &\quad + \frac{16}{A^{13}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) \, d\theta \right) d\eta \\
 &\leq (1 + \sqrt{6}A) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) \, d\theta \right) d\eta \\
 &= (1 + \sqrt{6}A) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + AA_2 \left[ - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) \, d\theta \right]_{\eta=0}^1 \\
 &\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \, d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2. \tag{5.22}
 \end{aligned}$$

Direct calculations yield for  $\hat{B}_{35}$  (and in much the same way for  $\hat{B}_{36}$ ) that

$$\begin{aligned}
 |\hat{B}_{35}| &\leq \mathbb{1}_{A_1 \leq A_2} \frac{4|A_1 - A_2|}{aA^7 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) d\eta \\
 &\leq \frac{8}{aA^6 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \frac{16}{aA^{11/2} e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{256}{a^2 A^{11} e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
 & + \frac{16A}{e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1) (\| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

As far as the term  $\bar{B}_{37}$  is concerned, it can be estimated as follows (we use (3.40)):

$$\begin{aligned}
 |\bar{B}_{37}| & = \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) \right. \\
 & \quad \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \left. \right) d\eta \left. \right| \\
 & = \frac{1}{2A^7} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 & \quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \left. \right) \Big|_{\eta=0}^1 \\
 & \quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \left. \right) d\eta \left. \right| \\
 & = \frac{1}{2A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 & \quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \left. \right) d\eta \left. \right| \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2, \tag{5.23}
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant only depending on  $A$ , which remains bounded as  $A \rightarrow 0$ , provided we can show that the derivative in the latter integral exists and is uniformly bounded; see Lemma A.7.

As far as the term  $\bar{B}_{38}$  (a similar argument works for  $\bar{B}_{39}$ ) is concerned, the integral can be estimated as follows:

$$\begin{aligned}
 \bar{B}_{38} &= \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
 &= \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \left[ (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \Big|_{\theta=0}^\eta \\
 &\quad - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \left[ \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \\
 &\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right] d\eta \right] \\
 &= -\frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{D^c}(t, \eta) \\
 &\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \left[ \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \\
 &\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right] d\eta \\
 &= \bar{M}_1 + \bar{M}_2.
 \end{aligned}$$

As far as the first term  $\bar{M}_1$  is concerned, we have since

$$\begin{aligned}
 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) &\leq 2 \min_j(A_j \tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \\
 &\leq A^6 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \leq \frac{A^8}{\sqrt{2}},
 \end{aligned}$$

that

$$\begin{aligned}
 |\bar{M}_1| &\leq \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \, d\eta \\
 &\leq \frac{A}{\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

The second term  $\bar{M}_2$ , on the other hand, is a bit more demanding.

(i): First of all, recall (A.4), that is,

$$\left| \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta),$$

which implies that

$$\begin{aligned}
 &\frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{D^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
 &\quad \times \left. \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \, d\theta \, d\eta \right| \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{D^c}(t, \eta) \\
 &\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\
 &\quad \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \, d\theta \right)^2 \, d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{2a^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \, d\theta \right)^2 \, d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{a^2}{2A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{2a^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \, d\theta \right)^2 \, d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2a^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \, d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
 \leq & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{1}{A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
 \leq & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + \frac{6}{A_2^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
 \leq & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & + 3A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \\
 = & \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 - 6AA_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
 & + 6AA_2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
 \leq & \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where we used (4.13), (4.16e), and (4.16f). Again,  $\mathcal{O}(1)$  denotes some constant only depending on  $A$ , which remains bounded as  $A \rightarrow 0$ .

(ii): First of all, we have to establish that  $\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)$  is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.4 we show that

$$\left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \theta) \right| \leq \mathcal{O}(1) \sqrt{AA^4} (\min_j(\tilde{\mathcal{D}}_j))^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta).$$

We are now ready to establish a Lipschitz estimate for the second part of  $\bar{M}_2$ . Indeed,

$$\begin{aligned}
 & \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \left. \times \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta d\eta \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2 \mathbb{1}_{D^c}(t, \eta) \\
&\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \left. \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \right. \\
&\quad \times \left. (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) A_2 \left( -2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 2(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$



where we used (4.16i). Thus we conclude that

$$|\bar{B}_{38}| + |\bar{B}_{39}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Next we have a look at  $\bar{K}_{14}$ , which can be rewritten as follows:

$$\begin{aligned} \bar{K}_{14} &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \bar{K}_{14}^+ + \bar{K}_{14}^-. \end{aligned}$$

Note that both  $\bar{K}_{14}^+$  and  $\bar{K}_{14}^-$  have the same structure. Moreover, having a close look at  $\bar{K}_{14}^+$  one has

$$\bar{K}_{14}^+ = -3\tilde{K}_1,$$

where  $\tilde{K}_1$  is defined in (5.15). Thus we can immediately conclude that

$$|\bar{K}_{14}^+| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|).$$

Next, we have a look at  $\bar{K}_{15}$ , which can be rewritten as follows:

$$\begin{aligned} \bar{K}_{15} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \end{aligned}$$

$$\begin{aligned}
 & - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & = \bar{K}_{15}^+ + \bar{K}_{15}^-.
 \end{aligned}$$

Note that both  $\bar{K}_{15}^+$  and  $\bar{K}_{15}^-$  have the same structure. Having a close look at  $\bar{K}_{15}^+$  one has

$$\bar{K}_{15}^+ = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t, \eta) d\eta,$$

where  $J_1, \dots, J_8$  are defined in (5.10). Thus we can conclude immediately that

$$\begin{aligned}
 |\bar{K}_{15}^+| & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Finally, we have a look at  $\bar{K}_{16}$ , which can be rewritten as follows:

$$\begin{aligned}
 \bar{K}_{16} & = \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left( \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
 & = \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta \right) d\eta \\
 & \quad + \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
 & \quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
 & \quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta (\min_j (e^{-\frac{1}{A_2}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j (e^{-\frac{1}{A_1}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \tilde{U}_2(t, \theta) d\theta \Big) d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{1,\eta}(t, \eta) \left( \int_0^\eta (\min_j (e^{-\frac{1}{A_2}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j (e^{-\frac{1}{A_1}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \tilde{U}_1(t, \theta) d\theta \Big) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) (\tilde{U}_2^+ - \tilde{U}_1^+) \mathbb{1}_{\tilde{U}_1^+ \leq \tilde{U}_2^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) (\tilde{U}_2^- - \tilde{U}_1^-) \mathbb{1}_{\tilde{U}_2^- \leq \tilde{U}_1^-}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{1,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) (\tilde{U}_2^+ - \tilde{U}_1^+) \mathbb{1}_{\tilde{U}_2^+ \leq \tilde{U}_1^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) \tilde{Y}_{1,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) (\tilde{U}_2^- - \tilde{U}_1^-) \mathbb{1}_{\tilde{U}_1^- \leq \tilde{U}_2^-}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) (\tilde{Y}_{2,\eta} - \tilde{Y}_{1,\eta})(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j (\tilde{U}_j^+)(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2) (\tilde{Y}_{2,\eta} - \tilde{Y}_{1,\eta})(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \max_j (\tilde{U}_j^-)(t, \theta) d\theta \right) d\eta \\
 & = \bar{B}_{61} + \bar{B}_{62} + \bar{B}_{63} + \bar{B}_{64} + \bar{B}_{65} + \bar{B}_{66} \\
 & \quad + \bar{B}_{67}^+ + \bar{B}_{67}^- + \bar{B}_{68}^+ + \bar{B}_{68}^- + \bar{B}_{69}^+ + \bar{B}_{69}^-.
 \end{aligned}$$

The key observation, which rescues the whole paper, is again

$$A_i^5 \leq 2(\tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})(t, \eta),$$

which yields for  $\bar{B}_{61}$  (and similarly for  $\bar{B}_{62}$ ) that

$$\begin{aligned} |\bar{B}_{61}| &\leq \frac{A_2^5 - A_1^5}{2A^{11}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{5}{2A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t, \theta) \right)^2 d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left[ \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 \right. \right. \\ &\quad \left. \left. + \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^2 \right] d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left( \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \right. \\ &\quad \times \left[ \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad \left. \left. + \|\tilde{\mathcal{U}}_2(t, \cdot)\|_{L^\infty}^2 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^2 \right] d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left( \int_0^1 (6A^6 + 16A^2 \|\tilde{\mathcal{U}}_2(t, \cdot)\|_{L^\infty}^2) \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) d\eta \right)^{1/2} \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2). \end{aligned}$$

Next, we have a close look at  $\bar{B}_{63}$  (and similarly for  $\bar{B}_{64}$ ). Recalling the definition of  $B(\eta)$  (5.11), we have

$$\begin{aligned}
 |\bar{B}_{63}| &\leq \frac{a^5}{2A^6A_2} \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta (|\tilde{Y}_1 - \tilde{Y}_2|(t, \eta) + |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta)) \right. \\
 &\quad \left. \times e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{U}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{a^5}{2A^6A_2} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)^2 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{U}_2|(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{a^5}{2A^6A_2} \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{U}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{a^5}{A^6A_2^6} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)^2 \tilde{Y}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{P}_2 \tilde{Y}_{2,\eta} + \tilde{H}_{2,\eta}) |\tilde{U}_2|(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{a^5}{A^6A_2^6} \int_0^1 |\tilde{Y}_1 - \tilde{Y}_2| \tilde{Y}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{P}_2 \tilde{Y}_{2,\eta} + \tilde{H}_{2,\eta}) |\tilde{U}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^6A_2} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)^2 \tilde{Y}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 &\quad \times \left( \frac{1}{A_2} \tilde{P}_2^2 \tilde{Y}_{2,\eta} + A_2 \tilde{U}_2^2 \tilde{Y}_{2,\eta} + \|\tilde{U}_2(t, \cdot)\|_{L^\infty} \tilde{H}_{2,\eta} \right) (t, \theta) d\theta \Big) d\eta \\
 &\quad + \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\quad + \frac{1}{A^{12}A_2^2} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 &\quad \times \left. (\tilde{P}_2 \tilde{Y}_{2,\eta} + \tilde{H}_{2,\eta}) |\tilde{U}_2|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 (\tilde{Y}_1 - \tilde{Y}_2)^2 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \eta) d\eta \\
 &\quad + \frac{2}{A^{12}A_2^2} \int_0^1 \tilde{Y}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left[ \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 |\tilde{U}_2| \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{A_1^2 A_2^2} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \\
 & \quad \times \left[ \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad + \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad \left. \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) \right] d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \quad + \frac{8}{A_1^2 A_2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \\
 & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
 & = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4A_2}{A^8} \left( - \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 & \quad \times (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \Big|_{\eta=0}^1 \\
 & \quad \left. + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \eta) d\eta \right) \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned}$$

Next, we have a look at  $\bar{B}_{65}$  (a similar argument works for  $\bar{B}_{66}$ ). Direct calculations yield

$$\begin{aligned}
 |\bar{B}_{65}| & \leq \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\
 & \quad \times \left( \frac{4}{ae} \int_0^\eta \min_j (e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2|
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{A^2 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{3}{4\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq \frac{4}{A^2 A_2^2 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{3}{4\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq |A_1 - A_2|^2 + \frac{32}{A^4 A_2^{10} e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left[ \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad + \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \left. \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \right] d\eta \\
&\leq |A_1 - A_2|^2 + \frac{128}{A^4 A_2^9 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{64}{A^4 A_2^4 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad \times \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2\lambda_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2\lambda_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{128}{A^4 A_2^3 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad \times \left[ - \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right]_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \eta) d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Next, we have a look at  $\bar{B}_{67}^+$  (a similar argument works for  $\bar{B}_{67}^-$  and  $\bar{B}_{68}^\pm$ ). Direct calculations yield

$$\begin{aligned}
 |\bar{B}_{67}^+| &\leq \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+(t, \theta)| d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{A_2^5}{4A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{2A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left[ \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 \right. \\
 &\quad \left. + \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 \right] d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{4}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{2A_2}{A^6} \left( -2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Bigg|_{\eta=0}^1 \\
 &\quad + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{4A_2}{A^6} \left( - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
 & + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \eta) d\eta \Big) \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).
 \end{aligned} \tag{5.24}$$

Finally, we consider  $\bar{B}_{69}^+$  (a similar argument works for  $\bar{B}_{69}^-$ ). Here integration by parts will play the main role. Indeed, we have

$$\begin{aligned}
 |\bar{B}_{69}^+| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) \right. \\
 & \quad \times \left. \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \right| \\
 &= \frac{a^5}{4A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
 & \quad \times \left. \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
 & \quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\
 & \quad \times \frac{d}{d\eta} \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \Big| \\
 & \leq \frac{a^5}{4A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
 & \quad \times \left. \frac{d}{d\eta} \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|,
 \end{aligned} \tag{5.25}$$

where  $\mathcal{O}(1)$  denotes some constant depending on  $A$ , which remains bounded as  $A \rightarrow 0$ , provided that we can show that both

$$a^5 \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \leq \mathcal{O}(1)A^2 \min(\tilde{\mathcal{P}}_j)(t, \eta)$$

and the derivative

$$a \frac{d}{d\eta} \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \tag{5.26}$$

exist and are uniformly bounded.

Direct computations yield

$$\begin{aligned}
 & a^5 \int_0^\eta \min_j (e^{-\frac{1}{\lambda_i}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
 & \leq \int_0^\eta e^{-\frac{1}{\lambda_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} A_i^5 |\tilde{\mathcal{U}}_i|(t, \theta) d\theta \\
 & \leq 2 \int_0^\eta e^{-\frac{1}{\lambda_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} (\tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta}) |\tilde{\mathcal{U}}_i|(t, \theta) d\theta \\
 & \leq 2 \int_0^\eta e^{-\frac{1}{\lambda_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left( \frac{1}{A_i} \tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta} + A_i \tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \|\tilde{\mathcal{U}}_i(t, \cdot)\|_{L^\infty} \tilde{\mathcal{H}}_{i,\eta} \right) (t, \theta) d\theta \\
 & \leq \mathcal{O}(1) A_i^2 \tilde{\mathcal{P}}_i(t, \eta).
 \end{aligned}$$

The result for (5.26) is contained in Lemma A.5.

LEMMA 5.3. Let  $\tilde{\mathcal{Y}}_i$  denote two solutions of (5.5). Then we have

$$\begin{aligned}
 \frac{d}{dt} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \quad (5.27)
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which depends on  $A = \max_j(A_j)$ , which remains bounded as  $A \rightarrow 0$ .

**5.2. Lipschitz estimates for  $\tilde{\mathcal{U}}$ .** From the system of differential equations, we have

$$\tilde{\mathcal{U}}_{i,t} + \left( \frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) \tilde{\mathcal{U}}_{i,\eta} = -\frac{1}{A_i^2} \tilde{\mathcal{Q}}_i, \quad (5.28)$$

where

$$\begin{aligned}
 \tilde{\mathcal{Q}}_i(t, \eta) & = -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{\lambda_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{S}}_i(t, \eta) & = \int_0^1 e^{-\frac{1}{\lambda_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} \left( \frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta} \right) (t, \theta) d\theta.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \frac{d}{dt} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \\
 & = 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{1,t} - \tilde{\mathcal{U}}_{2,t})(t, \eta) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left( \frac{1}{A_2^2} \tilde{\mathcal{Q}}_2 - \frac{1}{A_1^2} \tilde{\mathcal{Q}}_1 \right) (t, \eta) d\eta \\
 &\quad + \frac{4}{3} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left( \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \\
 &\quad + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left( \frac{1}{A_2^6} \tilde{\mathcal{S}}_2 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$

The strategy is to use integration by parts for the last two integrals  $I_2$  and  $I_3$ , while we want to use straightforward estimates for  $I_1$ , which will finally yield that

$$\begin{aligned}
 &\frac{d}{dt} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which only depends on  $A = \max_j(A_j)$  and which remains bounded as  $A \rightarrow 0$ .

*The first integral  $I_1$ :* Note that we can split  $I_1$  as follows:

$$\begin{aligned}
 I_1 &= \mathbb{1}_{A_1 \leq A_2} 2 \frac{A_1^2 - A_2^2}{A_1^2 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{Q}}_1(t, \eta) d\eta \\
 &\quad + \mathbb{1}_{A_2 < A_1} 2 \frac{A_1^2 - A_2^2}{A_1^2 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{Q}}_2(t, \eta) d\eta \\
 &\quad + 2 \frac{1}{A^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{Q}}_2 - \tilde{\mathcal{Q}}_1)(t, \eta) d\eta \\
 &= I_{11} + I_{12} + I_{13}.
 \end{aligned}$$

As far as  $I_{11}$  is concerned (a similar argument works for  $I_{12}$ ), we have

$$\begin{aligned}
 |I_{11}| &\leq \mathbb{1}_{A_1 \leq A_2} 4 \frac{|A_1 - A_2|}{A_1 A_2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1(t, \eta) d\eta \\
 &\leq A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Note that

$$\begin{aligned} |I_{13}| &= \left| 2 \frac{1}{A^2} \int_0^1 (\tilde{u}_1 - \tilde{u}_2)(\tilde{Q}_1 - \tilde{Q}_2)(t, \eta) d\eta \right| \\ &\leq \int_0^1 \left( (\tilde{u}_1 - \tilde{u}_2)^2 + \frac{1}{A^4} (\tilde{Q}_1 - \tilde{Q}_2)^2 \right)(t, \eta) d\eta, \end{aligned}$$

and hence it suffices to show that

$$\begin{aligned} \|\tilde{Q}_1 - \tilde{Q}_2\| &\leq \mathcal{O}(1)A^2(\|\tilde{Y}_1 - \tilde{Y}_2\| + \|\tilde{u}_1 - \tilde{u}_2\| \\ &\quad + \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\| + |A_1 - A_2|), \end{aligned} \quad (5.29)$$

which is equivalent to

$$\begin{aligned} \|\tilde{Q}_1 - \tilde{Q}_2\|^2 &\leq \mathcal{O}(1)A^4(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{u}_1 - \tilde{u}_2\|^2 \\ &\quad + \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

To begin with, we observe that we can write

$$\begin{aligned} (\tilde{Q}_1 - \tilde{Q}_2)(t, \eta) &= (A_1\tilde{P}_1 - A_2\tilde{P}_2)(t, \eta) \\ &\quad + (\tilde{D}_2 - \tilde{D}_1)(t, \eta) \\ &= (A_1 - A_2)\tilde{P}_1 \\ &\quad + A_2(\tilde{P}_1^{1/2} + \tilde{P}_2^{1/2})(\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2})(t, \eta) \\ &\quad + (\tilde{D}_2 - \tilde{D}_1)(t, \eta) \\ &= K_1(t, \eta) + K_2(t, \eta) + K_3(t, \eta). \end{aligned}$$

As far as  $K_1(t, \eta)$  is concerned, we have

$$|A_1 - A_2| \|\tilde{P}_1\|_{L^\infty} \leq \frac{A^4}{4} |A_1 - A_2|.$$

As far as  $K_2(t, \eta)$  is concerned, we have

$$\|\tilde{P}_1 - \tilde{P}_2\| \leq A^2 \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|,$$

since  $\|\tilde{P}_i^{1/2}\|_{L^\infty}$  can be bounded by a constant, which only depends on  $A$ .

As far as  $K_3(t, \eta)$  is concerned, Lemma A.9 implies immediately that

$$\begin{aligned} \|\tilde{D}_1 - \tilde{D}_2\| &\leq \mathcal{O}(1)A^2(\|\tilde{Y}_1 - \tilde{Y}_2\| + \|\tilde{u}_1 - \tilde{u}_2\| \\ &\quad + \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\| + |A_1 - A_2|). \end{aligned}$$

This finishes the proof of (5.29).

The second integral  $I_2$ : Note that we can write

$$\begin{aligned}
 \frac{3}{4}I_2 &= \int_0^1 (\tilde{u}_1 - \tilde{u}_2) \left( \frac{1}{A_2^5} \tilde{u}_2^3 \tilde{u}_{2,\eta} - \frac{1}{A_1^5} \tilde{u}_1^3 \tilde{u}_{1,\eta} \right) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^3 \tilde{u}_{2,\eta} - \tilde{u}_1^3 \tilde{u}_{1,\eta}) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_1^3 \tilde{u}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{u}_2^3 \tilde{u}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2 - \tilde{u}_1) \tilde{u}_2^2 \tilde{u}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^2 \tilde{u}_{2,\eta} - \tilde{u}_1^2 \tilde{u}_{1,\eta}) \tilde{u}_1 (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_1^3 \tilde{u}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{u}_2^3 \tilde{u}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2 - \tilde{u}_1) \tilde{u}_2^2 \tilde{u}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^2 - \tilde{u}_1^2) \tilde{u}_{2,\eta} \tilde{u}_1 (t, \eta) \mathbb{1}_{\tilde{u}_2^2 \leq \tilde{u}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^2 - \tilde{u}_1^2) \tilde{u}_1 \tilde{u}_{1,\eta} (t, \eta) \mathbb{1}_{\tilde{u}_2^2 < \tilde{u}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_{2,\eta} - \tilde{u}_{1,\eta}) \tilde{u}_1 \min(\tilde{u}_j^2) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_1^3 \tilde{u}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{u}_2^3 \tilde{u}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2 - \tilde{u}_1) \tilde{u}_2^2 \tilde{u}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^2 - \tilde{u}_1^2) \tilde{u}_{2,\eta} \tilde{u}_1 (t, \eta) \mathbb{1}_{\tilde{u}_2^2 \leq \tilde{u}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_2^2 - \tilde{u}_1^2) \tilde{u}_1 \tilde{u}_{1,\eta} (t, \eta) \mathbb{1}_{\tilde{u}_2^2 < \tilde{u}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{2A^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2)^2 \frac{d}{d\eta} (\tilde{u}_1 \min(\tilde{u}_j^2)) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{u}_1 - \tilde{u}_2) (\tilde{u}_1^3 \tilde{u}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{u}_2^3 \tilde{u}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta,
 \end{aligned}$$

where we used integration by parts in the last step together with  $\tilde{\mathcal{U}}_i(t, \eta) \rightarrow 0$  as  $\eta \rightarrow 0, 1$ . As far as the derivative in the last integral is concerned, observe that

$$|\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta)| \leq |\tilde{\mathcal{U}}_1^2 \tilde{\mathcal{U}}_{1,\eta}(t, \eta)| \leq \frac{A^4}{2} \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \frac{A^6}{2\sqrt{2}}, \tag{5.30}$$

since  $2|\tilde{\mathcal{U}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \eta)| \leq A_i^4 \leq A^4$  for all  $t$  and  $\eta$ . Furthermore, we established before that the function  $\theta \mapsto \min_j(\tilde{\mathcal{U}}_j^2(t, \eta))$  is Lipschitz continuous with Lipschitz constant at most  $A^4$ ; see (5.8). Thus

$$\left| \frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta) \right| \leq \frac{3}{2} A^4 \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \frac{3}{2\sqrt{2}} A^6 \tag{5.31}$$

and

$$\left| \frac{1}{A_1^5 A_2^5} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{U}}_{i,n} \mathbb{1}_{A_i=a} \right| \leq \frac{1}{A_1^5 A_2^5} \frac{1}{4} A_i^8 \mathbb{1}_{A_i=a} \leq \frac{1}{A^2}.$$

Finally, we get that

$$\begin{aligned} \frac{3}{4} |I_2| &= \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left( \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^2} |A_1^5 - A_2^5| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

*The third integral  $I_3$ :* We will split  $I_3$  into several terms that we treat separately, and combine them in the end.

Recall that we introduced the functions  $\tilde{\mathcal{U}}_i^- = \min(0, \tilde{\mathcal{U}}_i)$  and  $\tilde{\mathcal{U}}_i^+ = \max(0, \tilde{\mathcal{U}}_i)$  with properties (5.2) and (5.3). We write

$$I_3 = \frac{4}{3} I_{31} - 4 I_{32} - 2 I_{33},$$

where

$$\begin{aligned} I_{31} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta, \tag{5.32a} \\ I_{32} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \end{aligned}$$

$$- \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta, \tag{5.32b}$$

$$I_{33} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta. \tag{5.32c}$$

Thus we have

$$I_{31} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.$$

Since both inner integrals have the same structure, it suffices to consider the second integral. Recall rewrite (5.4). Thus we need to estimate the following term:

$$\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\ - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ = \frac{1}{A_6} \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.$$

$$\begin{aligned}
 & -\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 = & \frac{1}{A_6^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & - \frac{1}{A_6^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 = & \frac{1}{A_6^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \frac{1}{A_6^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
 & - \frac{1}{A_6^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{A^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & - \frac{1}{A^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \hat{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & = (J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10})(t, \eta), \tag{5.33}
 \end{aligned}$$

where  $B(\eta)$  is given by (5.11).

As far as the integral that contains  $J_1$  (a similar argument works for  $J_2$ ) is concerned, we have

$$\begin{aligned}
 & \left| \int_0^1 J_1(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
 & \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \int_0^1 J_1^2(t, \eta) d\eta \\
 & \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|_{L^2}^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 & \quad + \frac{1}{A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} 3(\tilde{\mathcal{U}}_2^+)^2 | \tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+ | \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 & \quad + \frac{9}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 & \leq \left( 1 + \frac{36}{A^3} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2, \tag{5.34}
 \end{aligned}$$

where we used (4.15e), (4.15g), and (4.15h).

As far as the third and fourth terms  $J_3$  and  $J_4$  are concerned, they again have the same structure, and hence we only consider the integral corresponding to  $J_3$ .

$$\begin{aligned}
 & \left| \int_0^1 J_3(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
 & = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right|
 \end{aligned}$$

$$\begin{aligned}
& \times \int_0^\eta \left( e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \, d\eta \Big| \\
& \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \, d\eta |A_1 - A_2| \\
& \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \\
& \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{32}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \, d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

As far as the third and fourth terms  $J_5$  and  $J_6$  are concerned, they again have the same structure, and hence we only consider the integral corresponding to  $J_5$ . Thus, using (5.13),

$$\begin{aligned}
& \left| \int_0^1 J_5(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \right| \\
& = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \left( \int_0^\eta \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \right) \right. \right. \\
& \quad \left. \left. \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) \, d\theta \right) \, d\eta \right| \\
& \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \left( \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\
& \quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \, d\eta \\
& \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \, d\eta \\
& \quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta)
\end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \frac{4a}{A^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2, \eta}|(t, \eta) d\eta + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \frac{1}{a^2 A^9} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq (\sqrt{2}A^2 + 1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2). \tag{5.35}
 \end{aligned}$$

Next are the terms  $J_7$  and  $J_8$ . Therefore recall (5.14), which implies

$$\begin{aligned}
 J_7 + J_8 &= \frac{1}{A^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad - \frac{1}{A^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
 &= \frac{1}{A^6} (\tilde{\mathcal{U}}_{2, \eta} - \tilde{\mathcal{U}}_{1, \eta})(t, \eta) \\
 & \quad \times \min_k \left[ \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right] \\
 & \quad + \frac{1}{A^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\
 & \quad + \frac{1}{A^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\
 &= L_1 + L_2 + L_3.
 \end{aligned}$$

As far as the first term  $L_1$  is concerned, the corresponding integral can be estimated as follows, using (A.11),

$$\begin{aligned}
 & \left| \int_0^1 L_1(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \right| \\
 &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta})(t, \eta) \right. \\
 &\quad \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) \, d\theta \right] \left. d\eta \right| \\
 &= \left| -\frac{1}{2A^6} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \right. \\
 &\quad \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) \, d\theta \right] \Big|_{\eta=0}^1 \\
 &\quad + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \\
 &\quad \times \frac{d}{d\eta} \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) \, d\theta \right] \left. d\eta \right| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \tag{5.36}
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant only depending on  $A$ , which remains bounded as  $A \rightarrow 0$ .

As far as the last term  $L_3$  (a similar argument works for  $L_2$ ) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned}
 & \left| \int_0^1 L_3(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \right| \\
 &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) \, d\theta \mathbb{1}_{E^c}(t, \eta) \, d\eta \left. \right| \\
 &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \right. \\
 &\quad \times \left[ (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right]_{\theta=0}^\eta \\
 &\quad - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \left[ \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
 &\quad \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) \right] \mathbb{1}_{E^c}(t, \eta) \, d\eta \left. \right|
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{A^6} \left| - \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(\tilde{Y}_1 - \tilde{Y}_2) \min_j(\tilde{U}_j^+)^3 \tilde{U}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \right. \\
 &\quad + \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
 &\quad \times \int_0^\eta (\tilde{Y}_1 - \tilde{Y}_2)(t, \theta) \left[ \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \min_j(\tilde{U}_j^+)^3(t, \theta) \right. \\
 &\quad \left. + \min_j(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{U}_j^+)^3 \right)(t, \theta) \right] d\theta d\eta \Big| \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2) + \frac{1}{A^{12}} \int_0^1 \tilde{U}_{2,\eta}^2(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) \right. \\
 &\quad \times \left[ \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j(\tilde{U}_j^+)^3 \max_j(\tilde{Y}_{j,\eta})(t, \theta) \right. \\
 &\quad \left. \left. + 2A^4 \min_j(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j(\tilde{U}_j^+)(t, \theta) \right] d\theta \right)^2 d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2) + \frac{54}{A} \int_0^1 \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \eta) d\eta \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2), \tag{5.37}
 \end{aligned}$$

where we used (A.4), (4.16a), and (A.6).

Finally, we have a look at the integral, which contains  $J_9$  (a similar argument works for  $J_{10}$ ), where we can assume  $A_1 \leq A_2$ . Thus

$$\begin{aligned}
 &\left| \int_0^1 J_9(\tilde{U}_1 - \tilde{U}_2)(t, \eta) d\eta \right| \\
 &= \left( \frac{1}{A_1^6} - \frac{1}{A_2^6} \right) \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} (\tilde{U}_1^+)^3 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &= \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} (\tilde{U}_1^+)^3 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq 3 \frac{A_2 - A_1}{A_1^2 A_2} \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big|
 \end{aligned}$$

$$\begin{aligned}
 &\leq |A_1 - A_2|^2 + \frac{9}{A_1^4 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{U}}_{1,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq |A_1 - A_2|^2 + \frac{9}{A_1 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq |A_1 - A_2|^2 + \frac{9}{A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq |A_1 - A_2|^2 + \frac{36}{A} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\leq |A_1 - A_2|^2 + 18A^4 \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2. \tag{5.38}
 \end{aligned}$$

We now turn to the integral  $I_{32}$ . Recall equation (5.32b), namely,

$$\begin{aligned}
 I_{32} = &\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

By first using the decomposition  $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ , and then involving (5.4), we need to estimate terms of the type

$$\begin{aligned}
 \tilde{I}_{32} = &\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

We invoke Lemma 5.1 and find

$$\begin{aligned}
 \tilde{I}_{32} = &\frac{1}{A_6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leq A_1} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & = N_1 + N_2 + N_3.
 \end{aligned}$$

We consider first  $N_1$ , where we get

$$\begin{aligned}
 N_1 & = \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left[ \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
 & = \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\
 & \quad \times \left[ \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{U}_2^+ \tilde{Y}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right. \\
 & \quad + \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{U}_1^+ \tilde{Y}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \\
 & \quad + \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{U}_2^+ - \tilde{U}_1^+) \tilde{Y}_{2,\eta} \mathbb{1}_{\tilde{U}_1^+ \leq \tilde{U}_2^+}(t, \theta) d\theta \\
 & \quad + \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{U}_2^+ - \tilde{U}_1^+) \tilde{Y}_{1,\eta} \mathbb{1}_{\tilde{U}_2^+ < \tilde{U}_1^+}(t, \theta) d\theta \\
 & \quad + \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{2,\eta}(t, \theta) d\theta \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
 & = N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}. \tag{5.39}
 \end{aligned}$$

The terms  $N_{11}$  and  $N_{12}$  can be treated similarly. To that end, we find

$$\begin{aligned}
 |N_{11}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \right. \\
 & \quad \times \left[ \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{U}_2^+ \tilde{Y}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right] d\eta \Big|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 &\quad + \frac{4}{A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \frac{16}{A^8} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1)(\| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2),
 \end{aligned}$$

where we have used (4.15e), (4.15h), and (4.16c).

The terms  $N_{13}$  and  $N_{14}$  follow the same estimates. More precisely,

$$\begin{aligned}
 |N_{13}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
 &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+| \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
 &\leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+| \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
 &\quad + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$



$$\begin{aligned} &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2, \end{aligned}$$

by applying (4.15e), (4.15q), and (4.16e).

The term  $N_{15} + N_{16}$  can be estimated as follows:

$$\begin{aligned} N_{15} + N_{16} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left[ \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\ &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left[ \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
 & \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
 & = N_{151} + N_{152} + N_{153} + N_{154} + N_{155}.
 \end{aligned}$$

Unfortunately, each of these terms needs a special treatment.

The terms  $N_{151}$  and  $N_{152}$  can be handled as follows:

$$\begin{aligned}
 |N_{151}| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
 & \left. \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \frac{4}{A^6 A_1 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
 & \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{16}{A^{12} A_1^2 e^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
 & \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{8}{A^4 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) d\eta |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where we used (4.16e).

The terms  $N_{153}$  and  $N_{154}$  can be handled as follows:

$$\begin{aligned}
 |N_{153}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 & \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))})
 \end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \Big| \\
 & \leq \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \\
 & \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
 & = \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
 & \quad + \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
 & \leq \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 & \quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{12} a^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2 \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \frac{\sqrt{3}}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \frac{9}{4A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

where we used (4.16k). We consider now  $N_{155}$ :

$$\begin{aligned}
 N_{155} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\
 &\quad \times \left[ \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)_\eta(t, \eta) \\
 &\quad \times \min \left[ \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta d\theta \right) d\eta \\
 &= N_{1551} + N_{1552} + N_{1553},
 \end{aligned}$$

which, yet again, requires separate treatment. Here  $\tilde{E}$  is defined in (5.16).

The term  $N_{1551}$  can be handled as the term  $\tilde{L}_{31}$  (cf. (5.17)),

$$|N_{1551}| \leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.$$

The terms  $N_{1552}$  and  $N_{1553}$  can be treated in the same manner:

$$\begin{aligned}
 |N_{1552}| &\leq \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) |\tilde{\mathcal{U}}_{2,\eta}| \mathbb{1}_{\tilde{E}^c}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left[ \frac{d}{d\theta} \left( \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \right) \right] d\theta \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \frac{d}{d\theta} \left( \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \Big] (t, \theta) d\theta d\eta | \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).
 \end{aligned}$$

The term  $N_2$  (and also  $N_3$ ) can be treated as follows, keeping in mind that  $A_2 \leq A_1$ .

$$\begin{aligned}
 |N_2| & \leq \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}| (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 & \leq \sqrt{6} \frac{A_1^6 - A_2^6}{A^6 A_2^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}| (t, \eta) d\eta \\
 & \leq 3\sqrt{3} A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| (t, \eta) d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Finally, we now turn to the integral  $I_{33}$ . Recall equation (5.32c), namely,

$$\begin{aligned}
 I_{33} & = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

By first involving (5.4) and then Lemma 5.1, we see that it suffices to estimate

$$\begin{aligned}
 \tilde{I}_{33} & = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad + \mathbb{1}_{A_2 \leq A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{1}_{A_1 < A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta d\eta \\
 & = M_1 + M_2 + M_3.
 \end{aligned}$$

To our dismay, the estimate for  $M_1$  is rather involved. We estimate, using first that  $\tilde{Q}_i = A_i \tilde{P}_i - \tilde{D}_i$  (cf. (4.7)):

$$\begin{aligned}
 M_1 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{Q}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} A_2 \tilde{P}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} A_1 \tilde{P}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad - \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{D}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{D}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (A_2 \tilde{P}_2 \tilde{U}_2 \tilde{U}_{2,\eta} - A_1 \tilde{P}_1 \tilde{U}_1 \tilde{U}_{1,\eta})(t, \eta) d\eta \\
 & \quad - \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{D}_2 \tilde{U}_2 \tilde{U}_{2,\eta} - \tilde{D}_1 \tilde{U}_1 \tilde{U}_{1,\eta})(t, \eta) d\eta \\
 & \quad + \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\
 & \quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad - \frac{3}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\
 & \quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left( \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
 & - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \Big) d\eta \\
 & = W_1 + W_2 + W_3 + W_4 + W_5 + W_6.
 \end{aligned}$$

Here we have used the rewrite employed when manipulating the term  $\bar{K}_1$  from the expression (5.18) to (5.19). We start by considering the term  $W_1$ :

$$\begin{aligned}
 W_1 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta})(t, \eta) d\eta \\
 & = \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) d\eta \\
 & \quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
 & \quad - \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) d\eta \\
 & = W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16}.
 \end{aligned}$$

For  $W_{11}$  we have (and similarly for  $W_{12}$ ) that

$$\begin{aligned}
 |W_{11}| & \leq \frac{A^2}{8} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

The terms  $W_{13}$  and  $W_{14}$  are similar:

$$\begin{aligned}
 |W_{11}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{U}_2 \tilde{U}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) \, d\eta \right| \\
 &\leq \frac{2a}{A^6} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |\tilde{U}_2 \tilde{U}_{2,\eta}|(t, \eta) \, d\eta \\
 &\leq \frac{A}{2} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| \, d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{U}_1 - \tilde{U}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2),
 \end{aligned}$$

using (4.15a) and (4.15f).

The term  $W_{15}$  goes as follows:

$$\begin{aligned}
 |W_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{U}_{2,\eta} - \tilde{U}_{1,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{U}_2(t, \eta) \, d\eta \right| \\
 &= \left| \frac{a}{2A^6} (\tilde{U}_1 - \tilde{U}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{U}_2(t, \eta) \right|_{\eta=0}^1 \\
 &\quad - \frac{a}{2A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{U}_2)(t, \eta) \, d\eta \Big| \\
 &\leq \frac{a}{2A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)^2 \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{U}_2) \right|(t, \eta) \, d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{U}_1 - \tilde{U}_2\|^2;
 \end{aligned}$$

see the estimates for  $\bar{B}_{15}$  (cf. (5.20)). As for the term  $W_{16}$ , we get

$$\begin{aligned}
 |W_{16}| &= \frac{a}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)^2 \min_j(\tilde{\mathcal{P}}_j) |\tilde{U}_{1,\eta}|(t, \eta) \, d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{U}_1 - \tilde{U}_2\|^2,
 \end{aligned}$$

using (4.15p).

As for the term  $W_2$ , we find

$$\begin{aligned}
 -W_2 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{\mathcal{D}}_2 \tilde{U}_2 \tilde{U}_{2,\eta} - \tilde{\mathcal{D}}_1 \tilde{U}_1 \tilde{U}_{1,\eta})(t, \eta) \, d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{U}_1 \tilde{U}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) \, d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{U}_2 \tilde{U}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \eta) \, d\eta
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \, d\eta \\
 & - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \, d\eta \\
 & = W_{21} + W_{22} + W_{23} + W_{24}.
 \end{aligned}$$

The terms  $W_{21}$  and  $W_{22}$  can be treated similarly. We need to estimate  $\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1$ . Applying Lemma A.9, we have

$$\begin{aligned}
 |W_{21}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) \, d\eta \right| \\
 &\leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
 &\times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) \, d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
 &\times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) \, d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
 &\times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) \, d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\
 &+ \frac{12\sqrt{2}}{\sqrt{3}eA^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
 &\times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \, d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta |A_1 - A_2| \\
 &+ \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
 &\times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) \, d\theta \right) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
 \leq & \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{A^{13/2}}{2\sqrt{2}} d\eta \\
 & + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{3A^8}{8} d\eta \\
 & + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \frac{A^{13/2}}{2\sqrt{2}} d\eta \\
 & + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
 & + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
 & + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
 & + \frac{12\sqrt{2}}{\sqrt{3}eA^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \frac{A^4}{2} d\eta |A_1 - A_2| \\
 & + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) \frac{A^4}{2} d\eta \\
 & + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \frac{A^4}{2} d\eta \\
 \leq & \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where we have used estimates (4.15a), (4.15b), (4.15f), and (4.15n). The term  $W_{23}$  goes as follows:

$$\begin{aligned}
 |W_{23}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
 & \leq \left| \frac{1}{2A^6} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 & \quad - \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\
 & \leq \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2) \right|(t, \eta) d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
 \end{aligned}$$

by applying Lemma A.4 (ii).

The term  $W_{24}$  goes as follows:

$$\begin{aligned} |W_{24}| &\leq \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)^2 \min_j (\tilde{D}_j) |\tilde{U}_{1,\eta}|(t, \eta) \, d\eta \\ &\leq \mathcal{O}(1) \|\tilde{U}_1 - \tilde{U}_2\|^2, \end{aligned}$$

using

$$\min_j (\tilde{D}_j) |\tilde{U}_{1,\eta}| \leq 2A_1 \tilde{\mathcal{P}}_1 |\tilde{U}_{1,\eta}| \leq \frac{A^7}{\sqrt{2}},$$

from (4.15n) and (4.15p).

Next, we turn to the term  $W_3$ :

$$\begin{aligned} W_3 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\ &\quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) \, d\theta \right. \\ &\quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1 \tilde{Y}_{1,\eta}(t, \theta) \, d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\ &\quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) \, d\theta \right. \\ &\quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1^+ \tilde{Y}_{1,\eta}(t, \theta) \, d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\ &\quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2^- \tilde{Y}_{2,\eta}(t, \theta) \, d\theta \right. \\ &\quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1^- \tilde{Y}_{1,\eta}(t, \theta) \, d\theta \right) d\eta \\ &= W_{31} + W_{32}. \end{aligned}$$

These terms can be treated similarly:

$$\begin{aligned} W_{31} &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \\ &\quad \times \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) \, d\theta \right. \end{aligned}$$

$$\begin{aligned}
& - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
= & \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\
 & \times \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\
 & \times \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{1,\eta} \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & - \frac{1}{A^7} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) (\tilde{U}_1 - \tilde{U}_2)_\eta(t, \eta) \\
 & \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) \tilde{Y}_{k,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) (\tilde{Y}_{2,\eta} - \tilde{Y}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta} \mathbb{1}_D(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \min_j(\tilde{D}_j) \min_j(\tilde{U}_j^+) (\tilde{Y}_{2,\eta} - \tilde{Y}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 & = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9 + Z_{10} + Z_{11} + Z_{12} + Z_{13},
 \end{aligned}$$

where the set  $D$  is defined by (5.21). At this point, it cannot come as a surprise that we have to treat these terms separately.

Let us start with  $Z_1$  ( $Z_2$  is similar):

$$\begin{aligned}
 |Z_1| & \leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{1,\eta}|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{U}_1^2 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1^2 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right)^{1/2} d\eta
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{2\sqrt{6}}{A^4} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms  $Z_3$  and  $Z_4$  go as follows:

$$\begin{aligned} |Z_3| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) \, d\theta \right) d\eta \right| \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

by Lemma A.9. The terms  $Z_5$  and  $Z_6$  can be treated similarly:

$$\begin{aligned} |Z_5| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) \, d\theta \right) d\eta \right| \\ &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) \, d\theta \right)^2 d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \end{aligned}$$

by applying (4.16e), estimating  $\tilde{\mathcal{D}}_2 \leq 2A_2\tilde{\mathcal{P}}_2$  (cf. (4.15n)), and subsequently  $2\sqrt{2}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2 \leq A_2^6$  (cf. (4.15q)). The terms  $Z_7$  and  $Z_8$  follow this pattern:

$$\begin{aligned} |Z_8| &\leq \frac{4}{A^7 a e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} 2A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} |\tilde{\mathcal{U}}_2|(t, \theta) \, d\theta \right) d\eta |A_1 - A_2| \\ &\leq \frac{4\sqrt{2}}{A^5 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \, d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \, d\theta \right)^{1/2} d\eta |A_1 - A_2| \end{aligned}$$

$$\begin{aligned} &\leq \frac{4\sqrt{6}}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1^{1/2} |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \, d\eta |A_1 - A_2| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms  $Z_9$  and  $Z_{10}$  follow this pattern:

$$\begin{aligned} |Z_9| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \right. \\ &\quad \left. \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) \, d\theta \right) \, d\eta \right| \\ &\leq \frac{a}{\sqrt{2}A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left( \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\ &\quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \, d\eta \\ &\leq \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \, d\eta \\ &\quad + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) \, d\theta \right) \, d\eta \\ &\leq \frac{\sqrt{2}}{A^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right)^{1/2} \, d\eta \\ &\quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^{10}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2(t, \theta) \, d\theta \right) \, d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2). \end{aligned}$$

The term  $Z_{11}$ :

$$\begin{aligned}
 |Z_{11}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \right. \\
 &\quad \times \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|_2^2,
 \end{aligned}$$

following the estimates employed for the term  $\bar{B}_{37}$ ; see (5.23).

The terms  $Z_{12}$  and  $Z_{13}$  may be treated as follows:

$$\begin{aligned}
 |Z_{12}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \Big| \\
 &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left[ \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
 &\quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \right) d\theta \right] d\eta \Big| \\
 &\leq \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{D^c} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right| \\
 &\quad + \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \right) d\theta \right) d\eta \Big| \\
 &\leq \frac{A^2}{4} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \right. \\
 &\quad \times \left. \left. \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \right| (t, \theta) d\theta \right) d\eta \\
 &= \tilde{M}_1 + \tilde{M}_2.
 \end{aligned}$$

Here we find

$$\tilde{M}_1 \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),$$



while  $\tilde{M}_2$  requires more care:

$$\begin{aligned}
 \tilde{M}_2 &= \frac{1}{A^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \left| \frac{d}{d\theta} \left( \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \right. \right. \\
 &\quad \times \left. \left. \min(\tilde{D}_j) \min(\tilde{U}_j^+)(t, \theta) \right| d\theta \right) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \right. \\
 &\quad \times \left| \frac{d}{d\theta} \left( \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \min(\tilde{D}_j) \min(\tilde{U}_j^+) \right. \\
 &\quad \left. \left. + \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \frac{d}{d\theta} \left( \min(\tilde{D}_j) \min(\tilde{U}_j^+) \right) \right| (t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \left| \frac{d}{d\theta} \left( \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right) \right. \right. \\
 &\quad \times \left. \left. \min(\tilde{D}_j) \min(\tilde{U}_j^+)(t, \theta) \right| d\theta \right) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \left| \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right. \right. \\
 &\quad \times \left. \left. \frac{d}{d\theta} \left( \min(\tilde{D}_j) \min(\tilde{U}_j^+) \right) (t, \theta) \right| d\theta \right) d\eta \\
 &\leq \frac{1}{aA^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right. \\
 &\quad \times \left. \max_j(\tilde{Y}_{j,\eta}) \min(\tilde{D}_j) \min(\tilde{U}_j^+)(t, \theta) d\theta \right) d\eta \\
 &\quad + \mathcal{O}(1) \frac{1}{A^{5/2}} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \left( \min(\tilde{D}_j) \right)^{1/2} + |\tilde{U}_2| \right) (t, \theta) d\theta \right) d\eta \\
 &= \tilde{M}_{21} + \tilde{M}_{22},
 \end{aligned}$$

where estimates for the derivatives come from Lemmas A.2 and A.4. We find for the term  $\tilde{M}_{21}$  that

$$\begin{aligned}
 \tilde{M}_{21} &= \frac{1}{aA^7} \int_0^1 |\tilde{U}_1 - \tilde{U}_2| |\tilde{U}_{2,\eta}|(t, \eta) \left( \int_0^\eta |\tilde{Y}_1 - \tilde{Y}_2| \min(e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right. \\
 &\quad \times \left. \max_j(\tilde{Y}_{j,\eta}) \min(\tilde{D}_j) \min(\tilde{U}_j^+)(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{a^2 A^{14}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big)^2 d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{A^3}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{A^3}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

where we used the estimate

$$\max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \leq 2 \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (A_j \tilde{\mathcal{P}}_j) \leq 2A \max_j (\tilde{\mathcal{P}}_j \tilde{\mathcal{Y}}_{j,\eta}) \leq A^6,$$

using (4.15e) and (4.15n), as well as (4.15q) and (4.16a). The term  $\tilde{M}_{22}$  reads as

$$\begin{aligned}
 \tilde{M}_{22} &= \mathcal{O}(1) \frac{1}{A^{5/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j (\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta)
 \end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{U}_1 - \tilde{U}_2\|^2 + \|\tilde{Y}_1 - \tilde{Y}_2\|^2). \end{aligned}$$

Here we employed

$$\min_j (\tilde{D}_j)^{1/2} + |\tilde{U}_2| \leq \sqrt{2A_2} \tilde{P}_2^{1/2} + \sqrt{2} \tilde{P}_2^{1/2} \leq \sqrt{2}(1 + \sqrt{A_2}) \tilde{P}_2^{1/2},$$

as well as (4.15q) and (4.16i).

Next, we turn to the term  $W_4$ :

$$\begin{aligned} -W_4 &= \frac{3}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= W_4^+ + W_4^-, \end{aligned}$$

with

$$\begin{aligned} W_4^\pm &= \frac{3}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2^\pm \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_1^\pm \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{U}_1 - \tilde{U}_2\|^2 + \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|^2 + \|\tilde{Y}_1 - \tilde{Y}_2\|^2 + |A_1 - A_2|^2), \end{aligned}$$

using that  $W_4^+ = 3N_1$  (see (5.39)) and similarly for  $W_4^-$ .

Next, we turn to the term  $W_5$ :

$$\begin{aligned} W_5 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{U}_1^3 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

We take positive and negative parts of the term  $\tilde{U}_j^3$ , and thus it suffices to study the term

$$\begin{aligned} W_5^+ &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \left[ \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{U}_2^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} (\tilde{U}_1^+)^3 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right] d\eta. \end{aligned}$$

Having a close look at  $W_5^+$  one has

$$W_5^+ = \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t, \eta) d\eta,$$

where  $J_1, \dots, J_6$  are defined in (5.33). Thus we can conclude immediately that

$$\begin{aligned} |W_5^+| &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2 \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

Next, we turn to the term  $W_6$ :

$$\begin{aligned} 2W_6 &= \frac{1}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2)(t, \eta) \left( \tilde{U}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{U}_2(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{U}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{U}_1(t, \theta) d\theta \right) d\eta \\ &= \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{U}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{U}_1(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{U}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^5} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{U}_1(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{U}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{U}_1 - \tilde{U}_2) \tilde{U}_{1,\eta}(t, \eta) \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta d\eta \\
 & + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
 & + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_2^- \leq \tilde{\mathcal{U}}_1^-}(t, \theta) d\theta d\eta \\
 & + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta d\eta \\
 & + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_1^- \leq \tilde{\mathcal{U}}_2^-}(t, \theta) d\theta d\eta \\
 & - \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \\
 & - \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \\
 & \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{U}}_j^-)(t, \theta) d\theta d\eta \\
 & = W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} + W_{67}^\pm + W_{68}^\pm + W_{69}^\pm.
 \end{aligned}$$

Here we go again! The terms  $W_{61}$  and  $W_{62}$ :

$$\begin{aligned}
 |W_{61}| &= \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 & \quad \left. \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \right| \\
 & \leq \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta \\
 & \leq \frac{\sqrt{65}}{A^2} |A_2 - A_1| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \\
 & \leq \sqrt{35} A^2 |A_2 - A_1| \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
 & \leq \mathcal{O}(1) (|A_1 - A_2|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),
 \end{aligned}$$

using (4.15q) and (4.16a).

The terms  $W_{63}$  and  $W_{64}$ :

$$\begin{aligned}
 |W_{63}| & \leq \frac{4a^4}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta |A_1 - A_2| \\
 & \leq \frac{4}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 & \leq \frac{4\sqrt{6}}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta |A_1 - A_2| \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Next comes the terms  $W_{65}$  and  $W_{66}$ :

$$\begin{aligned}
 |W_{65}| & = \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 & \quad \times \left. \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \right| \\
 & \leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 & \quad \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\
 & \quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
 &\quad + \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
 &\leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\quad + \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \frac{\sqrt{6}}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \\
 &\quad + \frac{\sqrt{6}}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

using (4.15q), (4.16a), and (5.13).

The terms  $W_{67}^\pm$  and  $W_{68}^\pm$  have a similar structure:

$$\begin{aligned}
 |W_{67}^+| &= \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)(t, \theta) d\theta \right)^2 d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 &\quad + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \left( \int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
 \end{aligned}$$

using (4.15h).

Finally, the terms  $W_{69}^\pm$ :

$$\begin{aligned}
 |W_{69}^+| &= \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \right| \\
 &\leq \frac{1}{2A} \left| \left( (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
 &\quad - \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \frac{d}{d\eta} \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \Big| \\
 &\leq \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2;
 \end{aligned}$$

see estimates for  $\tilde{B}_{67}^\pm$  (cf. (5.25)) and Lemma A.5.

The terms  $M_2$  and  $M_3$  can be treated similarly. More precisely,

$$\begin{aligned}
 |M_2| &\leq \mathbb{1}_{A_2 \leq A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta d\eta \right| \\
 &\leq \mathbb{1}_{A_2 \leq A_1} \frac{|A_1^6 - A_2^6|}{A^6 A_2^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t, \theta) d\theta d\eta \\
 &\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{A A_2^5} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
 &\quad \times \left( \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t, \theta) d\theta \right)^2 d\eta \right)^{1/2}
 \end{aligned}$$



$$\begin{aligned}
 &\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{AA_2^6} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
 &\quad \times \left( \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
 &\quad \left. \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 &\leq \mathbb{1}_{A_2 \leq A_1} 6\sqrt{6} \frac{|A_1 - A_2|}{AA_2^3} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \left( \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2 \tilde{\mathcal{P}}_2^2(t, \eta) d\eta \right)^{1/2} \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Here we used

$$\tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}| \leq \frac{1}{A_2} \tilde{\mathcal{P}}_2 \sqrt{\tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{H}}_{2,\eta}}$$

(cf. (4.15m)) as well as (4.15a) and (4.15q). In addition, we applied (4.16b) and (4.16d).

We have shown the anticipated result.

LEMMA 5.4. Let  $\tilde{\mathcal{U}}_i$  be two solutions of (5.28). Then we have

$$\begin{aligned}
 &\frac{d}{dt} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which only depends on  $A = \max_j(A_j)$  and which remains bounded as  $A \rightarrow 0$ .

**5.3. Lipschitz estimates for  $\tilde{\mathcal{P}}$  (or  $\tilde{\mathcal{P}}^{1/2}$ ).** From the system of differential equations we recall

$$(\tilde{\mathcal{P}}_i^{1/2})_t + \left( \frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) (\tilde{\mathcal{P}}_i^{1/2})_\eta = \frac{1}{2A_i^2} \frac{\tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} + \frac{1}{2A_i^3} \frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}}, \tag{5.40}$$

where

$$\tilde{\mathcal{Q}}_i(t, \eta) = -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i} |\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta, \tag{5.41a}$$

$$\tilde{\mathcal{S}}_i(t, \eta) = \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} \left( \frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta} \right) (t, \theta) d\theta, \tag{5.41b}$$

$$\begin{aligned} \tilde{\mathcal{R}}_i(t, \eta) &= \frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} \left( \frac{2}{3} A_i \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} + A_i^6 \tilde{\mathcal{U}}_i \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} \tilde{\mathcal{U}}_i \tilde{\mathcal{Q}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta. \end{aligned} \tag{5.41c}$$

Thus we have

$$\begin{aligned} &\frac{d}{dt} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \eta) d\eta \\ &= 2 \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_1^{1/2})_t - (\tilde{\mathcal{P}}_2^{1/2})_t)(t, \eta) d\eta \\ &= -\frac{2}{3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_1^3} \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \right) (t, \eta) d\eta \\ &\quad - \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_1^6} \tilde{\mathcal{S}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^6} \tilde{\mathcal{S}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta \right) (t, \eta) d\eta \\ &\quad + \frac{1}{2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &\quad + \frac{1}{2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_1^3} \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^3} \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &= \frac{2}{3} I_1 + I_2 + \frac{1}{2} I_3 + \frac{1}{2} I_4. \end{aligned}$$

We will estimate each of these terms, yielding that

$$\begin{aligned} &\frac{d}{dt} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which only depends on  $A = \max_j(A_j)$  and which remains bounded as  $A \rightarrow 0$ .

The term  $I_1$ : Here we do as follows:

$$\begin{aligned} I_1 &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \right) (t, \eta) d\eta \\ &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta) (t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2})(t, \eta) d\eta \\
= & \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^2 (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1(t, \eta) d\eta \\
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2})(t, \eta) d\eta \\
= & \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) (\tilde{\mathcal{P}}_2^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2}(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) (\tilde{\mathcal{P}}_1^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_2^2 < \tilde{\mathcal{U}}_1^2}(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta) d\eta \\
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta \\
= & I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned}$$

We first estimate

$$|(\tilde{\mathcal{P}}_i^{1/2})_\eta| = \frac{|\tilde{\mathcal{P}}_{i,\eta}|}{2\tilde{\mathcal{P}}_i^{1/2}} = \frac{|\tilde{\mathcal{Q}}_i \tilde{\mathcal{Y}}_{i,\eta}|}{2A_i^2 \tilde{\mathcal{P}}_i^{1/2}} \leq \frac{1}{2A_i} \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta},$$

and thus

$$|\tilde{\mathcal{U}}_i \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta}| \leq \frac{1}{2} (\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta}) \leq \frac{3}{4} A_i^5.$$

We start with the term  $I_{11}$ :

$$\begin{aligned}
|I_{11}| & \leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \eta) d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2).
\end{aligned}$$

Subsequently (and similarly for  $I_{13}$ ),

$$\begin{aligned} |I_{12}| &\leq \frac{1}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| |\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta \tilde{\mathcal{U}}_1| \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2}(t, \eta) \, d\eta \\ &\leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| \tilde{\mathcal{U}}_2^2 |(\tilde{\mathcal{P}}_2^{1/2})_\eta| (t, \eta) \, d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2). \end{aligned}$$

The term  $I_{14}$  requires more estimates (see (5.31)):

$$\begin{aligned} |I_{14}| &= \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1 \min(\tilde{\mathcal{U}}_j^2)(t, \eta) \, d\eta \right| \\ &= \left| \frac{1}{2A^5} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \tilde{\mathcal{U}}_1 \min(\tilde{\mathcal{U}}_j^2)(t, \eta) \right|_{\eta=0}^1 \\ &\quad - \frac{1}{2A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \left( \frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min(\tilde{\mathcal{U}}_j^2) \right) (t, \eta) \, d\eta \Big| \\ &= \frac{1}{2A^5} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \left( \frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min(\tilde{\mathcal{U}}_j^2) \right) (t, \eta) \, d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2. \end{aligned}$$

The last term can be estimated as follows:

$$\begin{aligned} |I_{15}| &\leq \left| \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \right. \\ &\quad \times (\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1})(t, \eta) \, d\eta \Big| \\ &\leq 5 \frac{|A_1 - A_2| A^4}{A_1^5 A_2^5} \\ &\quad \times \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (|\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta| \mathbb{1}_{A_1 \leq A_2} + |\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{A_2 < A_1})(t, \eta) \, d\eta. \end{aligned}$$

We consider, in particular, the term

$$\frac{A^4}{A_1^5 A_2^5} |\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta| \mathbb{1}_{A_1 \leq A_2} \leq \frac{A^4}{A_1^5 A_2^5} \frac{3}{16} A_1^8 \mathbb{1}_{A_1 \leq A_2} \leq \frac{3}{16} A^2.$$

Thus

$$\begin{aligned}
 |I_{15}| &\leq \frac{30}{16} A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).
 \end{aligned}$$

The term  $I_2$ : Recall that the term reads as follows:

$$I_2 = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_2^6} \tilde{\mathcal{S}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta \right) (t, \eta) d\eta.$$

Here we can follow the estimates of the term  $I_3$ ; cf. (5.6) and (5.9). Here we go. Define

$$\begin{aligned}
 I_{21} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta,
 \end{aligned}$$

$$\begin{aligned}
 I_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta,
 \end{aligned}$$

$$\begin{aligned}
 I_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{\lambda_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

Using the definition (5.41b) of  $\tilde{\mathcal{S}}$ , we can write

$$I_2 = \frac{2}{3} I_{21} - 2I_{22} - I_{23}.$$

We commence with the estimates for  $I_{21}$ :

$$I_{21} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta)$$

$$\begin{aligned}
 & \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 & = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 & + \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

The two terms are similar, and we only discuss the second one. Next, we use trick (5.4). Thus we have to consider the term

$$\begin{aligned}
 & \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 & - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 & = \frac{1}{A^6} \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 & = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 & + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 & - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 = & \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 & + \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta d\eta \\
 & + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) \\
 & - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \left. \right) \\
 & + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left( \int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
 & + \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 & + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 & = (I_{211} + I_{212} + I_{213} + I_{214} + I_{215} + I_{216} + I_{217} + I_{218} + I_{219} + I_{220})(t, \eta).
 \end{aligned} \tag{5.42}$$

The same estimate works for  $I_{211}$  and  $I_{212}$  (cf. (5.34)):

$$\left| \int_0^1 I_{211}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \int_0^1 I_{211}^2(t, \eta) d\eta,$$

which shows that it suffices to estimate

$$\begin{aligned}
 \int_0^1 I_{211}^2(t, \eta) d\eta & \leq \frac{1}{A^{12}} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 & \quad \left. \times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2,
 \end{aligned}$$

where we used that  $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leq \frac{1}{8} A_2^3 \tilde{\mathcal{Y}}_{2, \eta}$ .

Regarding the term  $I_{213}$  (and  $I_{214}$ ), we find

$$\begin{aligned}
 & \left| \int_0^1 I_{213}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\
 & = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \quad \left. \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \right| \\
 & \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta | A_1 - A_2 | \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 \\
 & \quad + \frac{8}{A^{10} e^2} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)
 \end{aligned}$$



$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{2A^2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{4}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{Y}_{2,\eta}(t, \eta) d\eta |A_1 - A_2|^2 \\ & \leq \mathcal{O}(1) (\| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + |A_1 - A_2|^2). \end{aligned}$$

Regarding the term  $I_{215}$  (and  $I_{216}$ ), we find

$$\begin{aligned} & \left| \int_0^1 I_{215} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\ & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\ & \quad \times \left. \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \left. \right| \\ & \leq \frac{1}{aA^6} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\ & \quad \times \left( \int_0^\eta (| \tilde{Y}_2(t, \eta) - \tilde{Y}_1(t, \eta) | + | \tilde{Y}_2(t, \theta) - \tilde{Y}_1(t, \theta) |) \right. \\ & \quad \times \left. e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \frac{1}{aA^6} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | \tilde{Y}_1 - \tilde{Y}_2 | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\ & \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta d\eta \\ & \quad + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{a^2 A^{12}} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \\ & \quad \times \left( \int_0^\eta | \tilde{Y}_1(t, \theta) - \tilde{Y}_2(t, \theta) | e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \min_j (\tilde{U}_j^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \mathcal{O}(1) (\| \tilde{Y}_1 - \tilde{Y}_2 \|^2 + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2), \end{aligned}$$

using the estimates in (5.35) and  $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leq \frac{1}{8} A^3 \tilde{Y}_{2,\eta}$ .

The next terms are  $I_{217} + I_{218}$ :

$$I_{217} + I_{218} = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta)$$

$$\begin{aligned}
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & - \frac{1}{A_6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 & = \frac{1}{A_6} ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
 & \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \\
 & + \frac{1}{A_6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\
 & + \frac{1}{A_6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
 & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\
 & = L_{211} + L_{212} + L_{213}.
 \end{aligned}$$

The term  $L_{211}$  can be estimated as in (5.36), yielding the result

$$\left| \int_0^1 L_{211} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2.$$

The terms  $L_{212}$  and  $L_{213}$  can be treated similarly; we list  $L_{213}$ :

$$\left| \int_0^1 L_{213} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2);$$

see (5.37). The terms  $I_{219}$  and  $I_{220}$  share the same behavior. We find, when  $A_1 \leq A_2$  (otherwise  $I_{219}$  vanishes; see (5.38)), that

$$\begin{aligned}
 & \left| \int_0^1 I_{219} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\
 & = \left( \frac{1}{A_1^6} - \frac{1}{A_2^6} \right) \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \right. \\
 & \quad \left. \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right|
 \end{aligned}$$

$$\leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).$$

Next, we consider the term  $I_{22}$ :

$$\begin{aligned} I_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

We start by replacing the integrals  $\int_0^1(\dots) d\theta$  with  $\int_0^\eta(\dots) d\theta$ ; cf. (5.4). Furthermore, we write  $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ , and hence it suffices to consider

$$\begin{aligned} \tilde{I}_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leq A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &= N_1 + N_2 + N_3. \end{aligned}$$

We consider first  $N_1$ , where we get

$$\begin{aligned}
 N_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left[ (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right] d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left[ (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right. \\
 &\quad + (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right] d\eta \\
 &= N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}. \tag{5.43}
 \end{aligned}$$

The terms  $N_{11}$  and  $N_{12}$  can be treated similarly. To that end, we find

$$\begin{aligned}
 |N_{11}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 \\
 & \quad + \frac{1}{2A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} | \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} |^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{2}{A^8} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} | \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} |^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2,
 \end{aligned}$$

where we have used (4.15e) and (4.17c). The terms  $N_{13}$  and  $N_{14}$  follow the same lines. More precisely,

$$\begin{aligned}
 |N_{13}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \min_j (\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
 & \leq \frac{1}{A^6} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \min_j (\tilde{\mathcal{P}}_j) | \tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+ | \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \min_j (\tilde{\mathcal{P}}_j) | \tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+ | \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned} &\leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1)(\| \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} \|^2 + \| \tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1 \|^2), \end{aligned}$$

by applying (4.15e), (4.16e), and (4.17c). The term  $N_{15} + N_{16}$  can be estimated as follows:

$$\begin{aligned} N_{15} + N_{16} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left[ (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \\ &\quad \times \left. \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \times \left. \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \times \left. \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left[ (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 & - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \left. \right] d\eta \\
 & = N_{151} + N_{152} + N_{153} + N_{154} + N_{155},
 \end{aligned}$$

which all, unfortunately, need a special treatment.

The terms  $N_{151}$  and  $N_{152}$  can be handled as follows:

$$\begin{aligned}
 |N_{151}| & \leq \mathbb{1}_{A_1 \leq A_2} \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
 & \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 & \leq \frac{2\sqrt{2}a}{A^3 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta |A_1 - A_2| \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

The terms  $N_{153}$  and  $N_{154}$  can be handled as follows:

$$\begin{aligned}
 |N_{153}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \left. \right| \\
 & \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \, d\eta \\
 = & \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \, d\eta \\
 & + \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \, d\eta \\
 \leq & \frac{1}{\sqrt{2}A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right)^{1/2} \, d\eta \\
 & + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right)^2 \, d\eta \\
 \leq & \frac{\sqrt{3}}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \, d\eta + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & + \frac{1}{2A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{2a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right) \, d\eta \\
 \leq & \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

using (4.15e), (4.16b), (4.16e), and (4.17a). We consider now  $N_{155}$ :

$$\begin{aligned}
 N_{155} = & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left[ (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \, d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \right] \, d\eta
 \end{aligned}$$



$$\begin{aligned}
 &= -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 &\quad \times \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
 &= N_{1551} + N_{1552} + N_{1553},
 \end{aligned}$$

which, yet again, requires separate treatment. Here  $\tilde{E}$  is defined in (5.16). The term  $N_{1551}$  can be handled as the term  $\tilde{L}_{31}$  (cf. (5.17)),

$$|N_{1551}| \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2.$$

The terms  $N_{1552}$  and  $N_{1553}$  can be treated in the same manner:

$$\begin{aligned}
 |N_{1552}| &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left| \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right| d\eta \\
 &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \left| \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left[ \frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \right. \\
 &\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \frac{d}{d\theta} (\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)) \right] (t, \theta) d\theta \right| d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).
 \end{aligned}$$

The term  $N_2$  (and also  $N_3$ ) can be treated as follows:

$$\begin{aligned}
 |N_2| &\leq \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \sqrt{6} \frac{A_1^6 - A_2^6}{A^6 A_2^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2 |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1) |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

This concludes the discussion of the term  $I_{22}$ .

The term  $I_{23}$  is handled as follows:

$$\begin{aligned}
 I_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

We start by replacing the integrals  $\int_0^1 (\dots) d\theta$  with  $\int_0^\eta (\dots) d\theta$ ; cf. (5.4). Thus

$$\begin{aligned}
 \tilde{I}_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \mathbb{1}_{A_2 \leq A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{Q}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta d\eta \\
 & + \mathbb{1}_{A_1 < A_2} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2})(\tilde{P}_1^{1/2})_\eta(t, \eta) \\
 & \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta d\eta \\
 & = M_1 + M_2 + M_3.
 \end{aligned}$$

We estimate, using first that  $\tilde{Q}_i = A_i \tilde{P}_i - \tilde{D}_i$  (cf. (4.7)):

$$\begin{aligned}
 M_1 &= \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) \left( (\tilde{P}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{Q}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - (\tilde{P}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) \left( A_2 (\tilde{P}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - A_1 (\tilde{P}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{P}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad - \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) \left( (\tilde{P}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{D}_2 \tilde{U}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - (\tilde{P}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \tilde{D}_1 \tilde{U}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) (A_2 \tilde{P}_2 \tilde{U}_2 (\tilde{P}_2^{1/2})_\eta - A_1 \tilde{P}_1 \tilde{U}_1 (\tilde{P}_1^{1/2})_\eta)(t, \eta) d\eta \\
 & \quad - \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) (\tilde{D}_2 \tilde{U}_2 (\tilde{P}_2^{1/2})_\eta - \tilde{D}_1 \tilde{U}_1 (\tilde{P}_1^{1/2})_\eta)(t, \eta) d\eta \\
 & \quad + \frac{1}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) \\
 & \quad \times \left( (\tilde{P}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{D}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\
 & \quad \left. - (\tilde{P}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{D}_1 \tilde{U}_1 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & \quad - \frac{3}{A^6} \int_0^1 (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}) \\
 & \quad \times \left( (\tilde{P}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{P}_2 \tilde{U}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right.
 \end{aligned}$$

$$\begin{aligned}
 & - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
 & = W_1 + W_2 + W_3 + W_4 + W_5 + W_6.
 \end{aligned}$$

Here we have used the rewrite employed when manipulating the term  $\bar{K}_1$  from the expression (5.18) to (5.19). We start by considering the term  $W_1$ :

$$\begin{aligned}
 W_1 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) d\eta \\
 & = \mathbb{1}_{A_1 \leq A_2} \frac{A_2 - A_1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
 & \quad + \mathbb{1}_{A_2 < A_1} \frac{A_2 - A_1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
 & \quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \\
 & = W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16}.
 \end{aligned}$$

Using (4.15a), (4.15b), (4.15e), (4.17a), and (4.17b), we find that

$$\begin{aligned}
 & W_{11} + W_{12} + W_{13} + W_{14} + W_{16} \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Regarding the term  $W_{15}$ :

$$\begin{aligned} |W_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \, d\eta \right| \\ &= \left| \frac{a}{2A^6} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\ &\quad - \frac{a}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \left| \right. \\ &= \frac{a}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) \, d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using the same estimates as for  $\tilde{B}_{13}$  (cf. (5.20)) and Lemma A.3 (ii).

As for the term  $W_2$ , we find

$$\begin{aligned} -W_2 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \, d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) \, d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \eta) \, d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \, d\eta \\ &\quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \, d\eta \\ &= W_{21} + W_{22} + W_{23} + W_{24}. \end{aligned}$$

The terms  $W_{21}$  and  $W_{22}$  can be treated similarly. We need to estimate  $\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1$ . Applying Lemma A.9, we have

$$\begin{aligned} |W_{21}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) \, d\eta \right| \\ &\leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) \, d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) \, d\eta \\ &\quad + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) \, d\eta \end{aligned}$$

$$\begin{aligned}
 & + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & + \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
 & \leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{A^{13/2}}{4} d\eta \\
 & + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{3A^8}{8\sqrt{2}} d\eta \\
 & + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^{13/2}}{4} d\eta \\
 & + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta \\
 & + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta \\
 &+ \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) \frac{A^4}{2\sqrt{2}} d\eta \\
 &+ \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^4}{2\sqrt{2}} d\eta \\
 &+ \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^4}{2\sqrt{2}} d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where we have used estimates (4.15a), (4.15b), (4.15n), and (4.17b). Furthermore,

$$\begin{aligned}
 |W_{23}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
 &\leq \left| \frac{1}{2A^6} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 &\quad - \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\
 &= \frac{1}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2,
 \end{aligned}$$

by applying Lemma A.4 (ii). The term  $W_{24}$  goes as follows:

$$\begin{aligned}
 |W_{24}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \right| \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2)
 \end{aligned}$$

using

$$\min_j(\tilde{\mathcal{D}}_j) |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leq 2A_1 \tilde{\mathcal{P}}_1 |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leq \tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} \leq \frac{A^7}{4}$$

from (4.15a), (4.15e), (4.15n), and (4.17a).

Next comes  $W_3$ , namely,

$$\begin{aligned}
 W_3 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 &\quad \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \left( \tilde{\mathcal{P}}_1^{1/2} \right)_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
 = & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 & \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 & \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 = & W_{31} + W_{32},
 \end{aligned}$$

and the two terms can be treated in the same manner. Thus

$$\begin{aligned}
 W_{31} = & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 & \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 = & \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left( \frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \Big) d\eta \\
 & + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
 & + \mathbb{1}_{A_2 < A_1} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \Big) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
 &\times \min \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \\
 &+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \\
 &\times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 &+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_e(t, \eta) \\
 &\times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
 &= Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9 + Z_{10} + Z_{11} + Z_{12} + Z_{13},
 \end{aligned}$$

where the set  $e$  is given by (5.21). Here there is no alternative but to treat these terms more or less separately. For the terms  $Z_1$  and  $Z_2$ , we find

$$\begin{aligned}
 |Z_1| &\leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \\
 &\times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

For the terms  $Z_3$  and  $Z_4$ , we find

$$\begin{aligned}
 |Z_3| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 &\times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2 + |A_2 - A_1|^2)
 \end{aligned}$$

by Lemma A.9.

For the terms  $Z_5$  and  $Z_6$ , we find

$$\begin{aligned}
 |Z_5| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 &\times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^{14}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{42}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2), \end{aligned}$$

by applying (4.16e), estimating  $\tilde{\mathcal{D}}_2 \leq 2A_2\tilde{\mathcal{P}}_1$  (cf. (4.15n)), and subsequently (4.17a).

For the terms  $Z_7$  and  $Z_8$ , we find

$$\begin{aligned} |Z_7| &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{42}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{41}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \left. \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\ &\leq \frac{4}{A_1 A^7 e} \left| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \right. \\ &\quad \left. \times \left( \int_0^\eta e^{-\frac{3}{42}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| |A_1 - A_2| \\ &\leq \frac{4\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

For the terms  $Z_9$  and  $Z_{10}$ , we find

$$\begin{aligned} |Z_9| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \left. \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \right| \\ &\leq \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta (|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) + |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta)) \right. \\
 & \times \left. e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \frac{\sqrt{2}}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 & + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & + \frac{2}{A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

applying the same set of estimates applied when studying  $\bar{B}_{35}$ ; see (5.22). For the term  $Z_{11}$ , we find

$$\begin{aligned}
 |Z_{11}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \right. \\
 & \quad \times \left. \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2;
 \end{aligned}$$

see estimates for  $\bar{B}_{37}$  (cf. (5.23)).

For the terms  $Z_{12}$  and  $Z_{13}$ , we find

$$\begin{aligned}
 |Z_{12}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \right| \\
 &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left[ \left( \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
 &\quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left( \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right] d\eta \Big| \\
 &= \left| \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right. \\
 &\quad - \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \\
 &\quad \times \left. \frac{d}{d\theta} \left( \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \frac{A^2}{4\sqrt{2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
 &\quad \times \left. \left| \frac{d}{d\theta} \left( \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \right| (t, \theta) d\theta \right) d\eta \\
 &= \tilde{M}_1 + \tilde{M}_2.
 \end{aligned}$$

We find directly

$$|\tilde{M}_1| \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).$$

For the other term  $\tilde{M}_2$ , we proceed as follows:

$$\begin{aligned}
 |\tilde{M}_2| &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
 &\quad \times \left. \left| \frac{d}{d\theta} \left( \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \right| (t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left[ \left| \left( \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right| \left| \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right| \right. \right. \\
 & \left. \left. + \left| \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)) \right| \right] (t, \theta) d\theta \right) d\eta \\
 & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) \right. \\
 & \quad \times \left[ \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
 & \quad \left. \left. + \mathcal{O}(1)A^{9/2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) \right] d\theta \right) d\eta \\
 & = \tilde{M}_{21} + \tilde{M}_{22},
 \end{aligned}$$

by using Lemmas A.2 and A.4. For  $\tilde{M}_{21}$ , we find

$$\begin{aligned}
 \tilde{M}_{21} &= \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. \times |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left| \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right| (t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{a^2A^{14}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. \times |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & \quad + \frac{1}{2} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & \quad + \frac{1}{8A^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \mathcal{O}(1) \int_0^1 \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
 \end{aligned}$$

using

$$\max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \leq 2 \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(A_i \tilde{\mathcal{P}}_j) \leq 2 \max_j(A_i \tilde{\mathcal{P}}_j \tilde{\mathcal{Y}}_{j,\eta}) \leq A^6,$$

and (4.15e), (4.16a), and (4.17c). For  $\tilde{M}_{22}$ , we find

$$\begin{aligned} \tilde{M}_{22} &= \frac{\mathcal{O}(1)}{A^{5/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \\ &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{\mathcal{O}(1)}{A^5} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{\mathcal{O}(1)}{A^5} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\quad + \frac{\mathcal{O}(1)}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right) d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\quad + \frac{\mathcal{O}(1)}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2), \end{aligned}$$

using that (cf. (4.15d) and (4.15n))

$$\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \leq \sqrt{2A_2} \tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2| \leq \sqrt{2}(1 + \sqrt{A_2}) \tilde{\mathcal{P}}_2^{1/2},$$

and (4.15e), (4.16i), and (4.17a).

We now consider the term  $W_4$ :

$$\begin{aligned} W_4 &= -\frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\ &\quad \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \end{aligned}$$

$$\begin{aligned}
 & - \left( \tilde{\mathcal{P}}_1^{1/2} \right)_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 = & -\frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 & \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & - \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
 & \times \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 = & W_4^+ + W_4^-.
 \end{aligned}$$

Since  $W_4^+ = -3N_1$  (see (5.43)), we find that

$$|W_4| \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2).$$

The next term  $W_5$  would be laborious:

$$\begin{aligned}
 W_5 = & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 = & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 = & W_5^+ + W_5^-.
 \end{aligned}$$

Unfortunately, having a close look at  $W_5^+$  one has

$$W_5^+ = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(I_{211} + I_{212} + I_{213} + I_{214} + I_{215} + I_{216} + I_{217} + I_{218})(t, \eta) d\eta,$$



where  $I_{211}, \dots, I_{218}$  are defined in (5.42). Thus we can conclude immediately that

$$|W_5^+| \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

For the term  $W_6$ , we find

$$\begin{aligned} W_6 &= \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\ &= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta d\eta \\ &\quad + \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \tilde{\mathcal{U}}_2(t, \theta) d\theta \right) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_2^- \leq \tilde{\mathcal{U}}_1^-}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_1^- < \tilde{\mathcal{U}}_2^-}(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
 & \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
 & + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
 & \times \left( \int_0^\eta \max_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^-)(t, \theta) d\theta \right) d\eta \\
 & = W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} \\
 & + W_{67}^+ + W_{67}^- + W_{68}^+ + W_{68}^- + W_{69}^+ + W_{69}^-.
 \end{aligned}$$

The terms  $W_{61}$  and  $W_{62}$ :

$$\begin{aligned}
 |W_{61}| &= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \right| \\
 & \leq 5 \frac{A_2 - A_1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta \\ & \leq 5\sqrt{6} \frac{A_2 - A_1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\ & \leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2), \end{aligned}$$

using (4.15e), (4.16a), and (4.17a).

The terms  $W_{63}$  and  $W_{64}$ :

$$\begin{aligned} |W_{63}| &= \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \left| \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{U}}_2(t, \theta) d\theta \right| d\eta \\ & \leq \frac{4a^5}{2aA^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \frac{2}{A^2 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\ & \leq \frac{2\sqrt{6}}{A^2 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta |A_1 - A_2| \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms  $W_{65}$  and  $W_{66}$ :

$$\begin{aligned} |W_{65}| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ & \quad \times \left. \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \right| \\ & \leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\ & \quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
 &\quad + \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
 &\leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\quad + \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \frac{\sqrt{3}}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\
 &\quad + \frac{\sqrt{3}}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

using (4.15e), (4.16a), (4.17a), and (5.13).

The terms  $W_{67}^\pm$  and  $W_{68}^\pm$ :

$$\begin{aligned}
 |W_{67}^+| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^2} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 &\quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{16A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_2^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 \\
 & + \frac{A}{32} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) (\| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2)
 \end{aligned}$$

following the estimates used for  $\bar{B}_{65}^+$ ; see (5.24). The term  $W_{69}^{\pm}$ :

$$\begin{aligned}
 |W_{69}^+| & = \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \right. \\
 & \quad \times \left. \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \right| \\
 & = \frac{a^5}{4A^6} \left| \left( (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
 & \quad - \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \Big| \\
 & \leq \frac{a^5}{4A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \\
 & \quad \times \left| \frac{d}{d\eta} \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| d\eta \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2;
 \end{aligned}$$

see estimates employed for  $\bar{B}_{67}$  (cf. (5.25)) and Lemma A.5.

The terms  $M_2$  and  $M_3$  can be treated similarly. More precisely,

$$\begin{aligned}
 |M_2| & \leq \mathbb{1}_{A_2 \leq A_1} \left( \frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta d\eta \right| \\
 & \leq \mathbb{1}_{A_2 \leq A_1} \frac{|A_1^6 - A_2^6|}{A^6 A_2^5} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 | \tilde{\mathcal{U}}_{2, \eta} | (t, \theta) d\theta d\eta \\
 & \leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{AA_2^5} \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \| \\
 & \quad \times \left( \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 | \tilde{\mathcal{U}}_{2, \eta} | (t, \theta) d\theta \right)^2 d\eta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{AA_2^6} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
 &\quad \times \left( \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
 &\leq \mathbb{1}_{A_2 \leq A_1} 6\sqrt{6} \frac{|A_1 - A_2|}{AA_2^3} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left( \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \tilde{\mathcal{P}}_2^2(t, \eta) d\eta \right)^{1/2} \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Here we used

$$\tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2, \eta}| \leq \frac{1}{A_2} \tilde{\mathcal{P}}_2 \sqrt{\tilde{\mathcal{Y}}_{2, \eta} \tilde{\mathcal{H}}_{2, \eta}},$$

cf. (4.15m), as well as (4.15c), (4.15e), and (4.17d). In addition, we applied (4.16b) and (4.16d).

The term  $I_3$ : We have the following estimates:

$$\begin{aligned}
 |I_3| &\leq \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \left| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right| (t, \eta) d\eta \\
 &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \left\| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2.
 \end{aligned}$$

We consider the latter term and find

$$\begin{aligned}
 &\left\| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
 &\leq 3 \mathbb{1}_{A_1 \leq A_2} \left| \frac{1}{A_1^2} - \frac{1}{A_2^2} \right|^2 \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} \right\|^2 + 3 \mathbb{1}_{A_2 < A_1} \left| \frac{1}{A_1^2} - \frac{1}{A_2} \right|^2 \left\| \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
 &\quad + 3 \frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
 &\leq \mathcal{O}(1) |A_1 - A_2|^2 + 3 \frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2,
 \end{aligned}$$

where we have used that

$$\left| \frac{\tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} \right| \leq \frac{A_i \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} = A_i \tilde{\mathcal{P}}_i^{1/2} |\tilde{\mathcal{U}}_i| \leq \mathcal{O}(1) A_i^5.$$

We consider the latter term and find

$$\begin{aligned}
 & \frac{1}{A^4} \left\| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
 &= \frac{1}{A^4} \int_0^1 \left| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right|^2 (\mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} + \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1})(t, \eta) \, d\eta \\
 &\leq \frac{2}{A^4} \int_0^1 \left| \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} (\tilde{Q}_1 \tilde{U}_1 - \tilde{Q}_2 \tilde{U}_2) + \left( \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} \right) \tilde{Q}_1 \tilde{U}_1 \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) \, d\eta \\
 &\quad + \frac{2}{A^4} \int_0^1 \left| \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} (\tilde{Q}_1 \tilde{U}_1 - \tilde{Q}_2 \tilde{U}_2) + \left( \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} \right) \tilde{Q}_2 \tilde{U}_2 \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) \, d\eta \\
 &\leq \frac{4}{A^4} \int_0^1 \left| \frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} (\tilde{Q}_1(\tilde{U}_1 - \tilde{U}_2) + \tilde{U}_2(\tilde{Q}_1 - \tilde{Q}_2)) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) \, d\eta \\
 &\quad + \frac{4}{A^4} \int_0^1 \left| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) \, d\eta \\
 &\quad + \frac{4}{A^4} \int_0^1 \left| \frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} (\tilde{Q}_1(\tilde{U}_1 - \tilde{U}_2) + \tilde{U}_2(\tilde{Q}_1 - \tilde{Q}_2)) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) \, d\eta \\
 &\quad + \frac{4}{A^4} \int_0^1 \left| \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) \, d\eta \\
 &= 4(I_{31} + I_{32} + I_{33} + I_{34}).
 \end{aligned}$$

We treat terms  $I_{31}$  and  $I_{32}$ ; the others are similar.

$$\begin{aligned}
 I_{31} &\leq \frac{2}{A^4} \int_0^1 \left| \left( \frac{\tilde{Q}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right)^2 |\tilde{U}_1 - \tilde{U}_2|^2 \right. \\
 &\quad \left. + \left( \frac{\tilde{U}_2}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right)^2 |\tilde{Q}_1 - \tilde{Q}_2|^2 \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) \, d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{Y}_1 - \tilde{Y}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

where we have used (5.29), that

$$\begin{aligned}
 \left| \frac{\tilde{Q}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} &\leq \frac{A_1 \tilde{\mathcal{P}}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \\
 &\leq A \frac{\max_i(\tilde{\mathcal{P}}_i)}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq A \max_i(\tilde{\mathcal{P}}_i^{1/2}) \leq \frac{A^3}{2},
 \end{aligned}$$

and

$$\left| \frac{\tilde{\mathcal{U}}_2}{\max_i (\tilde{\mathcal{P}}_i^{1/2})} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \frac{|\tilde{\mathcal{U}}_2|}{\tilde{\mathcal{P}}_2^{1/2}} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \sqrt{2}.$$

Next follows

$$\begin{aligned} I_{32} &\leq \frac{1}{A^4} \int_0^1 \left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2, \end{aligned}$$

using that

$$\left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1} \right| \leq |A_1 \tilde{\mathcal{U}}_1| \leq \frac{A^3}{\sqrt{2}}.$$

This proves that

$$\begin{aligned} &\frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

and thus  $I_3$  has the right form.

*The term  $I_4$ :*

$$\begin{aligned} I_4 &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{A_1^3} \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^3} \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} (t, \eta) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} (t, \eta) d\eta \\ &\quad + \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left( \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2) \right. \\ &\quad \left. - \left( \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} \right) (\tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1} + \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}) \right) (t, \eta) d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} (t, \eta) d\eta \end{aligned}$$



$$\begin{aligned}
& + \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t, \eta) d\eta \\
& + \frac{1}{A^3} \int_0^1 \left[ \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2) - (\tilde{\mathcal{P}}_1^{1/2} \right. \\
& \quad \left. - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1} + \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}) \right] (t, \eta) d\eta \\
& = J_1 + J_2 + J_3 - J_4 - J_5,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \mathbb{1}_{A_1 \leq A_2} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}(t, \eta) d\eta, \\
J_2 &= \mathbb{1}_{A_2 < A_1} \left( \frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t, \eta) d\eta, \\
J_3 &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)(t, \eta) d\eta, \\
J_4 &= \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta, \\
J_5 &= \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta.
\end{aligned}$$

For  $J_1$  (and similarly for  $J_2$ ), we find

$$\begin{aligned}
|J_1| &\leq \frac{A_2^3 - A_1^3}{a^3 A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \mathcal{O}(1) a^5 d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we used that

$$\left| \frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}} \right| \leq \mathcal{O}(1) A_i^3 \tilde{\mathcal{P}}_i^{1/2} \leq \mathcal{O}(1) A_i^5.$$

For  $J_3$ , we find

$$\begin{aligned}
J_3 &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)(t, \eta) d\eta \\
&= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^1 \text{sign}(\eta - \theta) \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \left( e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} A_1 \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} - e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} A_2 \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \right)(t, \theta) d\theta \Big) d\eta \\
 & + \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left( \int_0^1 \text{sign}(\eta - \theta) (e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} A_1^6 \tilde{\mathcal{U}}_1 \right. \\
 & \left. - e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} A_2^6 \tilde{\mathcal{U}}_2) (t, \theta) d\theta \right) d\eta \\
 & + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left( \int_0^1 (e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
 & \left. - e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1,\eta} \right)(t, \theta) d\theta \Big) d\eta \\
 & = J_{11} + J_{12} + J_{13}.
 \end{aligned}$$

Next we replace all the inner integrals  $\int_0^1 \dots d\theta$  by  $(\int_0^\eta + \int_\eta^1) \dots d\theta$  (cf. (5.4)), consider only the terms with  $\int_0^\eta \dots d\theta$ , and call the corresponding quantities  $\tilde{J}_{11}$ ,  $\tilde{J}_{12}$ ,  $\tilde{J}_{13}$ . Thus

$$\begin{aligned}
 \tilde{J}_{11} = & \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} A_1 \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right. \\
 & \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} A_2 \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \right)(t, \theta) d\theta \Big) d\eta, \tag{5.44}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{J}_{12} = & \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} A_1^6 \tilde{\mathcal{U}}_1 \right. \\
 & \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} A_2^6 \tilde{\mathcal{U}}_2) (t, \theta) d\theta \right) d\eta, \tag{5.45}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{J}_{13} = & \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
 & \left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1,\eta} \right)(t, \theta) d\theta \Big) d\eta. \tag{5.46}
 \end{aligned}$$

For the term  $\tilde{J}_{11}$  we write  $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ , collect terms, and study the positive part and the negative part separately.

With a slight abuse of notation, we need to consider the term

$$\begin{aligned}
\tilde{J}_{11} &= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1 (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta} \right. \\
&\quad \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2 (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left( \mathbb{1}_{A_2 < A_1} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right. \\
&\quad + \mathbb{1}_{A_1 \leq A_2} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad - a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad - a \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
&\quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad - a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
&\quad - a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& - a \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& + a \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10}.
\end{aligned}$$

It never stops, but we need to study the terms in groups.

The term  $K_1$  can be estimated as follows ( $K_2$  is similar):

$$\begin{aligned}
|K_1| &= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left| \mathbb{1}_{A_2 < A_1} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| d\eta \\
&\leq \frac{\sqrt{2}}{3} \int_0^1 \frac{\tilde{\mathcal{P}}_1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta |A_1 - A_2| \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The term  $K_3$  can be estimated as follows ( $K_4$  is similar):

$$\begin{aligned}
|K_3| &= \frac{a}{6A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right| \\
&\leq \frac{1}{2A^2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{2} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})^2} \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \frac{1}{2} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&\quad + \frac{1}{2A^4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})^2} \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} |\tilde{U}_1^+ - \tilde{U}_2^+|^2 (\tilde{U}_1^+)^2 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2), \end{aligned}$$

using (4.15g), (4.16c), and (5.3).

As for  $K_5$  ( $K_6$  is similar), we traverse the following path:

$$\begin{aligned} |K_5| &= \frac{1}{6A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \times \left( -a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\ & \quad \left. \left. \times \min_j ((\tilde{U}_j^+)^3) \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \right| \\ & \leq \frac{2}{3A^3 e} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \min_j ((\tilde{U}_j^+)^3) \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \frac{2\sqrt{2}A^2}{3} \int_0^1 \frac{\tilde{\mathcal{P}}_1^{1/2}}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta |A_1 - A_2| \\ & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

As for  $K_7$  ( $K_8$  is similar), we traverse the following path:

$$\begin{aligned} |K_7| &= \frac{a}{6A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\ & \quad \left. \left. \times \min_j ((\tilde{U}_j^+)^3) \tilde{Y}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \right| \\ & \leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\ & \quad \left. \times (|\tilde{Y}_2(t, \eta) - \tilde{Y}_1(t, \eta)| + |\tilde{Y}_2(t, \theta) - \tilde{Y}_1(t, \theta)|) (\tilde{U}_2^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{Y}_2 - \tilde{Y}_1|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{U}_2^+)^3 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 \leq & \frac{\sqrt{2}}{3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2(t, \eta) d\eta \\
 & + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & + \frac{1}{36A^6} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 \leq & \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
 & + \frac{1}{36A^6} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 (\tilde{\mathcal{U}}_2^+)^4 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 \leq & \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
 & + \frac{1}{9A^5} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2} \tilde{\mathcal{P}}_2(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 (\tilde{\mathcal{U}}_2^+)^4 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 \leq & \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2).
 \end{aligned}$$

Here we have applied (4.15b), (4.15g), (4.16c), and (5.13). Lo and behold, we can do  $K_9$  and  $K_{10}$  in one sweep:

$$\begin{aligned}
 K_9 + K_{10} & = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 & \times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 & \left. - \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 & = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
& \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
& = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left[ \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
& \quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \right) (t, \theta) d\theta \right] d\eta \\
& = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta \\
& \quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (t, \theta) d\theta \right) d\eta \\
& \quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \theta) d\theta \right) d\eta \\
& = L_1 + L_2 + L_3,
\end{aligned}$$

where

$$\begin{aligned}
L_1 & = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta, \\
L_2 & = \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (t, \theta) d\theta \right) d\eta, \\
L_3 & = \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

We easily find

$$|L_1| \leq \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta$$

$$\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),$$

using (4.15b) and (4.15d). Furthermore, applying Lemma A.2,

$$|L_2| \leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)$$

$$\times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right.$$

$$\times \left. \min_j((\tilde{\mathcal{U}}_j^+)^3) \max_j(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta) d\theta \right) d\eta$$

$$\leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)$$

$$\times \left( \int_0^\eta \min_j(e^{-\frac{1}{A_j}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{U}}_j^2(t, \theta) d\theta \right)^{\frac{1}{2}}$$

$$\times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 \max_j(\tilde{\mathcal{U}}_j^+ \tilde{\mathcal{Y}}_{j,\eta})^2(t, \theta) d\theta \right)^{\frac{1}{2}} d\eta$$

$$\leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|$$

$$\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),$$

using (4.15g), (4.16a), and

$$\min_j((\tilde{\mathcal{U}}_j^+)^3) \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \leq \min_j(\tilde{\mathcal{U}}_j^+) \max_j(\tilde{\mathcal{U}}_j^+ \tilde{\mathcal{Y}}_{j,\eta}) \leq \frac{A^7}{\sqrt{2}}.$$

Another application of Lemma A.2 yields

$$|L_3| \leq \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)$$

$$\times \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left| \frac{d}{d\theta} \min_j((\tilde{\mathcal{U}}_j^+)^3) \right| (t, \theta) d\theta \right) d\eta$$

$$\leq \frac{A^2}{3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)$$



$$\begin{aligned}
& \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
& \leq \frac{A^2}{3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2(t, \theta) d\theta \right)^{1/2} d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.16a). We find

$$\begin{aligned}
|K_9 + K_{10}| & \leq |L_1| + |L_2| + |L_3| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2).
\end{aligned}$$

Thus we can sum up the estimates for  $\tilde{J}_{11}$ , and we find

$$|\tilde{J}_{11}| \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2).$$

We now proceed to the next term from (5.45):

$$\begin{aligned}
\tilde{J}_{12} & = \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^6 \tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^6 \tilde{\mathcal{U}}_2)(t, \theta) d\theta \right) d\eta \\
& = \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left( (A_1^6 - A_2^6) \mathbb{1}_{A_1 < A_2} \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
& \quad + (A_1^6 - A_2^6) \mathbb{1}_{A_2 \leq A_1} \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta \\
& \quad \left. + a^6 \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2)(t, \theta) d\theta \right) d\eta \\
& = M_1 + M_2 + M_3.
\end{aligned}$$

The term  $M_1$  ( $M_2$  is similar) can be estimated as follows:

$$\begin{aligned} |M_1| &\leq \frac{1}{4A^3} |A_1^6 - A_2^6| \mathbb{1}_{A_1 < A_2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{3A^2}{2} |A_1 - A_2| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right)^{1/2} d\eta \\ &\leq \frac{3\sqrt{3}A^2}{\sqrt{2}} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

using (4.16a). As for the next term  $M_3$ , we first write  $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ , collect terms, and study the positive part and the negative part separately. In the interest of the reader, we do not change the notation. Thus

$$\begin{aligned} M_3 &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+)(t, \theta) d\theta d\eta \\ &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \right. \\ &\quad + \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \\ &\quad \left. + \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ &= M_{31} + M_{32} + M_{33}. \end{aligned}$$

The terms  $M_{31}$  and  $M_{32}$  can be treated similarly. Thus

$$\begin{aligned} |M_{31}| &= \frac{a^6}{4A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ &\quad \left. \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \right) d\eta \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{4A^2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| d\eta \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),
 \end{aligned}$$

using

$$\begin{aligned}
 &a^5 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|(t, \theta) d\theta \\
 &\leq \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &= \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((2\tilde{\mathcal{P}}_2 - \tilde{\mathcal{U}}_2^2)\tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{H}}_{2, \eta}) |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &= 2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &\quad - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &\leq 2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 &\leq 2 \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad + \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\ & \leq \mathcal{O}(1) A^3 \tilde{\mathcal{P}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|, \end{aligned}$$

using (4.13), (4.15e), (4.15k), (4.16d), and (4.16e). The term  $M_{33}$  goes as follows:

$$\begin{aligned} M_{33} &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ & \quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \mathbb{1}_{A_1 \leq A_2} \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ & \quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \mathbb{1}_{A_2 < A_1} \left( \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ & \quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)} \right. \\ & \quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)^c} \right. \end{aligned}$$

$$\begin{aligned} & \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & = M_{331} + M_{332} + M_{333} + M_{334}, \end{aligned}$$

where  $B(\eta)$  is defined by (5.11). The terms  $M_{331}$  and  $M_{332}$  can be treated in the same manner. More specifically,

$$\begin{aligned} |M_{331}| & \leq \frac{A^2}{e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \frac{A^2}{e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms  $M_{333}$  and  $M_{334}$  can be treated in a similar manner. More specifically,

$$\begin{aligned} |M_{333}| & \leq \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta |e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}| \mathbb{1}_{B(\eta)} \right. \\ & \quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\ & \quad \left. \times (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \mathbb{1}_{B(\eta)} \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \quad + \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta \\
 &\quad + \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta \\
 &\quad + \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
 \end{aligned}$$

using (4.16a) and (5.13).

And the final term from  $J_3$  (cf. (5.46)) can be estimated as follows:

$$\begin{aligned}
 \tilde{J}_{13} = &\frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
 &\left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

We introduce the positive and negative parts of  $\tilde{\mathcal{U}}_j$ , that is,  $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$  (see (5.1)) and introduce  $\tilde{\mathcal{Q}}_j = A_j \tilde{\mathcal{P}}_j - \tilde{\mathcal{D}}_j$ . We study the term with  $\tilde{\mathcal{P}}_j$  first:

$$\begin{aligned}
 \tilde{J}_{131} = &\frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\times \left( \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
 &\left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left[ \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad + \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \mathbb{1}_{\tilde{\mathcal{P}}_2 \leq \tilde{\mathcal{P}}_1} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 &\quad + a \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad + a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \mathbb{1}_{B(\eta)} \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \mathbb{1}_{B(\eta)^c} \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
 &\quad + a \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \Big] d\eta \\
 &= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10} + K_{11}.
 \end{aligned}$$

The terms  $K_1$  and  $K_2$  can be treated similarly. Thus

$$\begin{aligned}
 |K_1| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left. \left( \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

The terms  $K_3$  and  $K_4$  can be treated similarly. Thus

$$\begin{aligned}
 |K_3| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \frac{1}{A^2} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \right| \\
 &\leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2}(t, \eta)
 \end{aligned}$$



$$\begin{aligned} & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} d\eta \\ & \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using (4.15e) and (4.16c). As for  $K_5$  and  $K_6$ , we find

$$\begin{aligned} |K_5| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \left. + \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\ & \leq \frac{1}{2A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad + \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ & \quad + \frac{1}{4A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+)^2(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^3} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} \tilde{\mathcal{P}}_2(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+)^2(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2), \end{aligned}$$

using (4.15e) and (4.16e).

The terms  $K_7$  and  $K_8$  allow for the following estimates:

$$\begin{aligned}
|K_7| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad \times \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \left. \right| \\
&\leq \frac{2}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta |A_1 - A_2| \\
&\leq \frac{2}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left( \int_0^1 e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left( \int_0^1 e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The terms  $K_9$  and  $K_{10}$  allow for the following estimates:

$$\begin{aligned}
|K_9| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad + \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \mathbb{1}_{B(\eta)} \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \left. \right| \\
&\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + \frac{1}{4A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
 \end{aligned}$$

using (4.15a), (4.15e), and (4.16c). The last term  $K_{11}$  receives special treatment

$$\begin{aligned}
 |K_{11}| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \right. \\
 &\quad \left. \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{1, \eta} - \tilde{\mathcal{Y}}_{2, \eta})(t, \theta) d\theta \right) d\eta \right| \\
 &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \left[ \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right) \Big|_{\theta=0}^1 \right. \\
& - \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \\
& \times \left. \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \right) d\eta \right] \\
\leq & \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
& + \frac{a}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \\
& \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \right| (t, \theta) d\theta d\eta \\
\leq & \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \\
& \left. + 2aA^4 \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j(\tilde{\mathcal{P}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|) (t, \theta) d\theta d\eta \right) \\
\leq & \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) d\eta \\
& + A^2 \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)
\end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) (\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ & = \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) + K_{71} + K_{72} + K_{73}, \end{aligned}$$

using Lemmas A.2 and A.3, and

$$\begin{aligned} & \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) \\ & \leq e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) + e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+(t, \theta). \end{aligned}$$

Here  $K_{71}$  and  $K_{72}$  allow for the same treatment:

$$\begin{aligned} 2K_{71} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) d\eta \\ & \leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}^2 \tilde{\mathcal{P}}_1^2 (\tilde{\mathcal{U}}_1^+)^2(t, \theta) d\theta \right)^{1/2} d\eta \\ & \leq A^2 \sqrt{2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_1^{1/2}(t, \eta)) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2), \end{aligned}$$

using (4.15g) and (4.16b). Furthermore,

$$\begin{aligned} K_{73} &= A^2 \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) (\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
&\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j(e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&\quad + \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
&\quad \times \left( \int_0^\eta e^{-\frac{3}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \mathcal{O}(1) \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} \tilde{\mathcal{P}}_2(t, \eta) \\
&\quad \times \left( \int_0^\eta e^{-\frac{1}{2\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15d) and (4.16i). This proves that

$$\tilde{J}_{131} \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).$$

It remains to consider  $\tilde{J}_{132}$ :

$$\begin{aligned}
-\tilde{J}_{132} &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left( \int_0^\eta (e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta} \right. \\
&\quad \left. - e^{-\frac{1}{\lambda_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{D}}_1 \tilde{\mathcal{Y}}_{1, \eta} \right)(t, \theta) d\theta \Big) d\eta \\
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left[ \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right. \\
&\quad + \int_0^\eta e^{-\frac{1}{\lambda_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \theta) d\theta \\
&\quad \left. + \int_0^\eta (e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} - e^{-\frac{1}{\lambda_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}) \right)
\end{aligned}$$

$$\begin{aligned} & \times \min_j(\tilde{\mathcal{D}}_j)(t, \theta) d\theta \Big] d\eta \\ & = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3. \end{aligned}$$

The terms  $\tilde{A}_1$  and  $\tilde{A}_2$  allow for the same treatment. Specifically, (see Lemma A.9)

$$\begin{aligned} |\tilde{A}_1| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \times \left. \left[ \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \right| \\ & \leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left[ \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\bar{d}_{11} + \bar{d}_{12}) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right. \\ & \quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_1 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \\ & \quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_2 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \\ & \quad \left. + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_3 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \\ & = \frac{1}{2}(A_{11} + A_{12} + A_{13} + A_{14}). \end{aligned}$$

Again we are forced to consider individual terms,

$$\begin{aligned} A_{11} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left[ \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\bar{d}_{11} + \bar{d}_{12}) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \\ & \leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left[ \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (2A^{3/2} \tilde{\mathcal{D}}_2^{1/2} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\ & \quad \left. + 2\sqrt{2}A^{3/2} \tilde{\mathcal{D}}_2^{1/2} \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|)(t, \theta) d\theta \right] d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{4}{A^3} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^+ \tilde{Y}_{2,\eta} \tilde{D}_2^{1/2} |\tilde{Y}_2 - \tilde{Y}_1|(t, \theta) d\theta \right)^2 d\eta \\
 & + \frac{2\sqrt{2}}{A^{3/2}} \|\tilde{Y}_2 - \tilde{Y}_1\| \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\| \left( \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \right. \\
 & \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^+ \tilde{Y}_{2,\eta} \tilde{D}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
 & \leq \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|^2 \\
 & + \frac{4}{A^3} \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{D}_2 \tilde{Y}_{2,\eta} |\tilde{Y}_2 - \tilde{Y}_1|^2(t, \theta) d\theta \right) d\eta \\
 & + \frac{2\sqrt{2}}{A^3} \|\tilde{Y}_2 - \tilde{Y}_1\| \|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\| \left[ \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \right. \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} 2A \tilde{P}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \right]^{1/2} \\
 & \leq \mathcal{O}(1)(\|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|^2 + \|\tilde{Y}_2 - \tilde{Y}_1\|^2),
 \end{aligned}$$

using (4.15e), (4.15n), and (4.16c). Next we find (see (A.14))

$$\begin{aligned}
 A_{12} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{P}_j^{1/2})} |\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^+ \tilde{Y}_{2,\eta} T_1 \mathbb{1}_{\tilde{D}_1 < \tilde{D}_2}(t, \theta) d\theta \right) d\eta \\
 & \leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{P}_j^{1/2})} |\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}|(t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} |\tilde{U}_2|^3 \tilde{Y}_{2,\eta} |\tilde{Y}_2 - \tilde{Y}_1|(t, \theta) d\theta \right) d\eta \\
 & + 4 \int_0^1 \frac{1}{\max_j(\tilde{P}_j^{1/2})} |\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{U}_2 - \tilde{U}_1)^2(t, l) dl \right)^{1/2} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$



$$\begin{aligned}
& + \sqrt{2}A \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + A \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{8\sqrt{2}A}{\sqrt{3}e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4\lambda}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^{1/2} |A_1 - A_2| \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
\leq & 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
& \times \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
& + 16 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& + 2A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& + A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{128A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} \left( \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4\lambda}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
\leq & 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^4 \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\
& + 16 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + 2A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right)^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{128A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
 & + \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + B_1 + B_2 + B_3,
 \end{aligned}$$

using (4.15b), (4.15g), and (4.16c). As for  $B_1$ , we find

$$\begin{aligned}
 B_1 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & = \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) d\theta \right) d\eta \\
 & = \mathcal{O}(1) \int_0^1 \left[ \left( -2Ae^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\
 & \times \left. \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \right) \Big|_{\theta=0}^\eta \right. \\
 & \left. + 2A \int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,\theta)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right] d\eta \\
 & = -2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta d\eta \\
 & \quad + 2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2,
 \end{aligned}$$

while for  $B_2$ , we estimate as follows:

$$\begin{aligned}
 B_2 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) d\theta \right) d\eta \\
 &= \mathcal{O}(1) \int_0^1 \left[ \left( -2A_2 e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\left(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \right) \Big|_{\theta=0}^\eta \right. \\
 &\quad \left. + 2A_2 \int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-\left(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)\right)} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right] d\eta \\
 &= -2A_2 \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta d\eta \\
 &\quad + 2A_2 \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2,
 \end{aligned}$$

while for  $B_3$ , we estimate as follows:

$$\begin{aligned}
 B_3 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 &\leq \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2
 \end{aligned}$$

$$\begin{aligned}
 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 &\quad \times \left. \left( \int_0^\theta e^{-\left(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l)\right)} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
 &= \mathcal{O}(1) \int_0^1 \left[ \left( -\frac{8}{3} A e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\left(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l)\right)} dl \right) \right) \Big|_{\theta=0}^\eta \right. \\
 &\quad \left. + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\theta)} e^{-\left(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,\theta)\right)} d\theta \right] d\eta |A_1 - A_2|^2 \\
 &= -\frac{8}{3} A \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta d\eta |A_1 - A_2|^2 \\
 &\quad + \frac{8}{3} A \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{3}{8A} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta d\eta |A_1 - A_2|^2 \\
 &\leq \mathcal{O}(1) |A_1 - A_2|^2.
 \end{aligned}$$

Thus we find that

$$A_{12} \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

Next we find (see (A.15))

$$\begin{aligned}
 A_{13} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} T_2 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \left( \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) \right. \right. \\
 &\quad \left. \left. + 2\sqrt{2} A^3 \left( \int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \right. \right. \\
 &\quad \left. \left. + \frac{A^4}{\sqrt{2}} \left( \int_0^\theta e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \right. \right. \\
 &\quad \left. \left. + \frac{A^4}{2} \left( \int_0^\theta e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \right) \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{4\sqrt{2}A^4}{\sqrt{3}e} \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} |A_1 - A_2| \Big) d\theta \Big) d\eta \\
 \leq & 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_j | \tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1 | (t, \theta) d\theta \right)^2 d\eta \\
 & + 8 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} d\theta \right)^2 d\eta \\
 & + \frac{A^2}{2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} d\theta \right)^2 d\eta \\
 & + \frac{A^2}{4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} | \tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1 | (t, l) dl \right) d\theta \right)^2 d\eta \\
 & + \frac{32A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
 & \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
 \leq & 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & + 8 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2})^2(t, l) dl \right) d\theta \Big) d\eta \\
& + \frac{A^2}{2} \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{Y}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
& + \frac{A^2}{4} \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{Y}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} |\tilde{Y}_2 - \tilde{Y}_1|(t, l) dl \right)^2 d\theta \right) d\eta \\
& + \frac{32A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{P}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \tilde{U}_2^+ \tilde{Y}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2}\|^2 + \|\tilde{Y}_2 - \tilde{Y}_1\|^2) \\
& + \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
& \quad \times \left. \left( \int_0^\theta e^{-\frac{1}{A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{P}_1^{1/2} - \tilde{P}_2^{1/2})^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
& \quad \times \left. \left( \int_0^\theta e^{-\frac{1}{a}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} (\tilde{Y}_2 - \tilde{Y}_1)^2(t, l) dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\
& \quad \times \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,l))} dl \right) \tilde{Y}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2
\end{aligned}$$

$$\begin{aligned} &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + B_4 + B_2 + B_3 \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2), \end{aligned}$$

following the approach used for  $A_{12}$ . As for  $B_4$ , we find

$$\begin{aligned} B_4 &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2\lambda}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\ &\quad \times \left. \left( \int_0^\theta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2\lambda}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\ &\quad \times \left. \left( \int_0^\theta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \mathcal{O}(1) \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\ &\quad \times \left. \left( \int_0^\theta e^{-\left(\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{\lambda}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) d\theta \right) d\eta \\ &= \mathcal{O}(1) \int_0^1 \left[ \left( -2A e^{-\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\ &\quad \times \left. \left. \left( \int_0^\theta e^{-\left(\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{\lambda}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \right) \right]_{\theta=0}^\eta \\ &\quad + 2A \int_0^\eta e^{-\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-\left(\frac{1}{2\lambda}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{\lambda}\tilde{\mathcal{Y}}_2(t,\theta)\right)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \Big] d\eta \\ &= -2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta d\eta \\ &\quad + 2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2\lambda}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta d\eta \\ &\leq \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2. \end{aligned}$$

Thus we find that

$$A_{12} \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$



Next we find (see (A.16))

$$\begin{aligned}
 A_{14} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_3 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
 &\leq 12A|A_1 - A_2| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \\
 &\leq 12A|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left( \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right)^2 d\eta \right)^{1/2} \\
 &\leq 12A|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
 &\quad \times \left( \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^2 d\theta \right) d\eta \right)^{1/2} \\
 &\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left( \int_0^1 \left( \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^2 d\theta \right) d\eta \right)^{1/2} \\
 &\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
 &\quad \times \left( \int_0^1 \left( \int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \right)^{1/2} \\
 &\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
 &\quad \times \left( \int_0^1 \left( \int_0^\eta e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t, \theta)} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
 &\quad \times \left. \left. \left( \int_0^\theta e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A}\tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left( \int_0^1 \left[ \left( -\frac{8}{3} A e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\theta)} \int_0^\theta e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l))} dl \right) \Big|_{\theta=0}^\eta \right. \right. \\
&\quad \left. \left. + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\theta)} e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right] d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left( \int_0^1 \left[ -\frac{8}{3} A \int_0^\eta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right. \right. \\
&\quad \left. \left. + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right] d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).
\end{aligned}$$

Thus we have shown that

$$\tilde{A}_1 + \tilde{A}_2 \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

We next consider the term  $\tilde{A}_3$ :

$$\begin{aligned}
\tilde{A}_3 &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \\
&\quad - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j)(t, \theta) d\theta d\eta \\
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left[ \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right. \\
&\quad \left. + \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right. \\
&\quad \left. + \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. + \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\
&\quad \left. + \int_0^\eta (e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
 & + \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
 & \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \\
 & + \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big] d\eta \\
 & = A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36} + A_{37}. \tag{5.47}
 \end{aligned}$$

Terms  $A_{31}$  and  $A_{32}$  allow for the same treatment:

$$\begin{aligned}
 |A_{31}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 & \quad \times \left. \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+|(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+|(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
 & \quad + \frac{2}{A^3} \int_0^1 \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2),
 \end{aligned}$$

using (4.15e), (4.15n), and (4.16e).

The terms  $A_{33}$  and  $A_{34}$  can be treated as follows:

$$\begin{aligned}
 |A_{33}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \right. \\
 &\quad \times \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \left. \right| \\
 &\leq \frac{\sqrt{2}a}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
 &\leq \frac{2\sqrt{2}}{A^2 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Furthermore, the terms  $A_{35}$  and  $A_{36}$  can be estimated like this:

$$\begin{aligned}
 |A_{35}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left( \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) d\eta \left. \right| \\
 &\leq \frac{1}{2aA^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{a}{2\sqrt{2}A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{a}{2\sqrt{2}A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{\sqrt{2}A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\quad + \frac{1}{\sqrt{2}A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
 \end{aligned}$$

using (4.15e), (4.15n), and (4.16e). As for  $A_{37}$ , we follow this path:

$$\begin{aligned}
 |A_{37}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 &\quad \left. \left. \times \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \right| \\
 &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \left. \times \left[ \left( \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \right]_{\theta=0}^\eta \right. \\
 &\quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right|
 \end{aligned}$$

$$\begin{aligned}
 & \times \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \Big] d\eta \Big| \\
 & \leq \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right| \\
 & + \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \right. \\
 & \times \left. \left. \frac{d}{d\theta} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right) d\eta \right| \\
 & \leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) d\eta \\
 & + \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left( \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \right. \\
 & \times \left( \left( \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right. \\
 & \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( \frac{d}{d\theta} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) \right) \right) (t, \theta) d\theta \Big) d\eta \Big| \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
 & + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 & \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \frac{1}{a} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 & \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \\
 & \left. \left. + \mathcal{O}(1) A^5 \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j (\tilde{\mathcal{D}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|) \right) (t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) + \frac{1}{2} (A_{371} + A_{372}),
 \end{aligned}$$

using (4.15a), (4.15b), (4.15n), and Lemmas A.2 and A.4. Here

$$\begin{aligned}
 A_{371} &= \frac{1}{aA^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 & \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
 & \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \Big) d\eta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\sqrt{2}a}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
 &\quad \times \left. \left( e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} + e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right)(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{\sqrt{2}}{A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left[ \left( \int_0^\eta e^{-\frac{3}{2A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \right. \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad + \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left. \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} \right] d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

using  $\max_j(a_j) \leq a_1 + a_2$  and  $\min_j(b_j) \leq b_k$ . Then

$$\begin{aligned}
 A_{372} &= \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) (\min_j(\tilde{\mathcal{D}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left( \int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
 \end{aligned}$$

using (4.15d), (4.15n), and (4.16i). This completes the estimate for  $\tilde{A}_3$  (see (5.47)):

$$\tilde{A}_3 \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).$$

For  $J_4$  (and similarly for  $J_5$ ) the estimates read as

$$\begin{aligned} |J_4| &\leq \frac{1}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|^2 \frac{|\tilde{\mathcal{R}}_2|}{\tilde{\mathcal{P}}_2}(t, \eta) d\eta \\ &\leq \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using (4.15j).

We have shown the anticipated result.

LEMMA 5.5. *Let  $\tilde{\mathcal{P}}_i^{1/2}$  be two solutions of (5.40) for  $i = 1, 2$ . Then we have*

$$\begin{aligned} \frac{d}{dt} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant which only depends on  $A = \max_j(A_j)$  and which remains bounded as  $A \rightarrow 0$ .

### Acknowledgements

KG and HH were supported by the grants *Waves and Nonlinear Phenomena (WaNP)* and *Wave Phenomena and Stability — a Shocking Combination (WaPheS)* from the Research Council of Norway.

JAC was partially supported by the EPSRC grant number EP/P031587/1 and the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 883363).

JAC, KG, and HH are very grateful to the Institut Mittag-Leffler, Stockholm for generous support and hospitality during the program Interactions between Partial Differential Equations and Functional Inequalities. HH would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program *Nonlinear Water Waves* when work on this paper was undertaken.



## Appendix A. Lipschitz continuity and uniformly bounded Lipschitz constants

We need to establish that a number of complicated functions are Lipschitz continuous with uniformly bounded Lipschitz constants. In a desperate attempt to ease the readability of the main estimates, we collect these results here together with some other estimates that are essential in Section 5.

LEMMA A.1. (i) *We have*

$$|e^{a_1-a_2} - e^{b_1-b_2}| \leq \max(e^{a_1-a_2}, e^{b_1-b_2})(|b_1 - a_1| + |b_2 - a_2|) \quad (\text{A.1})$$

$$\leq |b_1 - a_1| + |b_2 - a_2|, \quad (\text{A.2})$$

$$a_1 < a_2, \quad b_1 < b_2.$$

(ii) *Let  $0 < a \leq A_j \leq A$ ,  $j = 1, 2$ . Then we have*

$$|e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| \leq \frac{4}{ae} e^{-\frac{3}{4A}x} |A_2 - A_1|, \quad x \in [0, \infty). \quad (\text{A.3})$$

*Proof.* (i) The result follows from the elementary inequality

$$|e^a - e^b| = \left| \int_b^a e^x dx \right| \leq e^b \left| \int_b^a dx \right| \leq e^b |b - a|, \quad a < b.$$

(ii) For  $x \in [0, \infty)$  one can write

$$\begin{aligned} |e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| &= \left| \int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{s}x} ds \right| \\ &\leq e^{-\frac{3}{4A}x} \left| \int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{4s}x} ds \right| \\ &\leq \frac{4}{ae} e^{-\frac{3}{4A}x} |A_2 - A_1|. \end{aligned}$$

Here we used in the last step that for  $s \in [a, A]$ , the function  $f: [0, \infty) \rightarrow [0, \infty)$  with  $f(x) = \frac{1}{s^2} x e^{-\frac{1}{4s}x}$  attains its maximum at  $x = 4s$  and

$$0 \leq f(4s) = \frac{4}{se} \leq \frac{4}{ae}. \quad \square$$

LEMMA A.2. (i) *The function  $\theta \mapsto \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))})$  is nondecreasing for almost every  $\eta$  and thus differentiable almost everywhere. We have that*

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))}) \max_j (\tilde{Y}_{j,\eta}(t, \theta)). \quad (\text{A.4})$$

(ii) The function  $\theta \mapsto \min_j (\tilde{\mathcal{U}}_j^2)(t, \theta)$  is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^2)(t, \theta) \right| \leq A^4. \quad (\text{A.5})$$

(iii) The function  $\theta \mapsto \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta)$  is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta). \quad (\text{A.6})$$

*Proof.* (i): First of all, note that the function  $\theta \mapsto \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))})$  is nondecreasing and hence differentiable almost everywhere. Consider in the following  $\theta < \eta$ . Assume that for fixed  $\theta$ , such that the given function is differentiable, we have

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \quad (\text{A.7})$$

and that there exists a sequence  $\theta_n \uparrow \theta$  such that

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned} & \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) \right| \\ &= e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \\ &\leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta_n)), \end{aligned}$$

since  $\tilde{\mathcal{Y}}_i(t, \cdot)$  is nondecreasing.

$$\begin{aligned} \left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| &\leq \lim_{\theta_n \uparrow \theta} \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta_n)}{\theta - \theta_n} \\ &= \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \\ &= \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta). \end{aligned}$$

Assume, on the other hand, that for fixed  $\theta$  we have

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}$$

and that there exists a sequence  $\theta_n \downarrow \theta$  such that

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned} & \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta_n))}) \right| \\ &= e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta_n))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \\ &\leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta_n))} (\tilde{\mathcal{Y}}_1(t, \theta_n) - \tilde{\mathcal{Y}}_1(t, \theta)), \end{aligned}$$

since  $\tilde{\mathcal{Y}}_i(t, \cdot)$  is nondecreasing. Thus

$$\begin{aligned} \left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| &\leq \lim_{\theta_n \downarrow \theta} \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta_n))} \frac{\tilde{\mathcal{Y}}_1(t, \theta_n) - \tilde{\mathcal{Y}}_1(t, \theta)}{\theta_n - \theta} \\ &= \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \\ &= \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta). \end{aligned}$$

Thus in case (A.7) we find

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta).$$

If we instead of (A.7) assume

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))},$$

a similar argument yields

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta).$$

Thus we conclude that in general we have

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{Y}}_{j,\eta})(t, \theta).$$

(ii): We have (cf. Lemma 5.1) that

$$\begin{aligned} & \left| \min_j (\tilde{\mathcal{U}}_j^2(t, \eta)) - \min_j (\tilde{\mathcal{U}}_j^2(t, \theta)) \right| \\ &\leq \max(|\tilde{\mathcal{U}}_1^2(t, \eta) - \tilde{\mathcal{U}}_1^2(t, \theta)|, |\tilde{\mathcal{U}}_2^2(t, \eta) - \tilde{\mathcal{U}}_2^2(t, \theta)|) \\ &\leq 2 \max_j \|\tilde{\mathcal{U}}_j \tilde{\mathcal{U}}_{j,\eta}\|_\infty |\eta - \theta| \leq A^4 |\eta - \theta|, \end{aligned} \tag{A.8}$$

using (4.15f).

(iii): Note that one has that for any positive function  $m(x)$ ,

$$\begin{aligned} |m^3(x) - m^3(y)| &= (m^2(x) + m(x)m(y) + m^2(y))|m(x) - m(y)| \\ &\leq (m^2(x) + 2m(x)m(y) + m^2(y))|m(x) - m(y)| \\ &= (m(x) + m(y))^2|m(x) - m(y)| \\ &= (m(x) + m(y))|m^2(x) - m^2(y)|. \end{aligned}$$

If we replace  $m(x)$  by  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$ , we have

$$\begin{aligned} &|\min_j(\tilde{\mathcal{U}}_j^+)^3(t, \eta) - \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta)| \\ &\leq (\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta))|\min_j(\tilde{\mathcal{U}}_j^+)^2(t, \eta) - \min_j(\tilde{\mathcal{U}}_j^+)^2(t, \theta)| \\ &\leq (\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)) \\ &\quad \times \max(|(\tilde{\mathcal{U}}_1^+)^2(t, \eta) - (\tilde{\mathcal{U}}_1^+)^2(t, \theta)|, |(\tilde{\mathcal{U}}_2^+)^2(t, \eta) - (\tilde{\mathcal{U}}_2^+)^2(t, \theta)|) \\ &\leq A^4(\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta))|\eta - \theta| \end{aligned}$$

(see (A.8)), and hence  $\min_j(\tilde{\mathcal{U}}_j^+)^3(t, \eta)$  is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta). \quad \square$$

LEMMA A.3. (i) *The function  $\eta \mapsto \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$  is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \eta) \right| \leq 2A^4 (\min_j(\tilde{\mathcal{P}}_j))^{1/2} + |\tilde{\mathcal{U}}_k|)(t, \eta), \quad k = 1, 2.$$

(ii) *The function  $\eta \mapsto \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_k(t, \eta)$ ,  $k = 1, 2$  is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_k)(t, \eta) \right| \leq \frac{1}{\sqrt{2}} A^6.$$

*Proof.* (i) We only present the proof for the case  $k = 2$ , since the case  $k = 1$  is similar.

Given  $0 \leq \eta_1 < \eta_2 \leq 1$ , we assume without loss of generality that

$$0 \leq \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1).$$

To ease the notation, we introduce the function

$$d(t, \eta) = \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta).$$

We will distinguish several cases:

(1): If  $d(t, \eta_2) = 0$ , then  $d(t, \eta_1) = 0$  and one has

$$\begin{aligned} 0 &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \left| \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \right| \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \left( |\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)| \right) \\ &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2): If  $d(t, \eta_2) > 0$ , then  $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) > 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) > 0$  or equivalently

$$\tilde{\mathcal{P}}_1(t, \eta_2) > 0, \quad \tilde{\mathcal{P}}_2(t, \eta_2) > 0, \quad \tilde{\mathcal{U}}_1(t, \eta_2) > 0, \quad \text{and} \quad \tilde{\mathcal{U}}_2(t, \eta_2) > 0.$$

(2a): Assume that  $d(t, \eta_1) = 0$  and  $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) = 0$ . Then

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \left( \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \right) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \left( |\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)| \right) \\ &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2b): Assume that  $d(t, \eta_1) = 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = 0$ . Then one either has that

(i):  $\tilde{\mathcal{U}}_2(t, \eta_1) \leq \tilde{\mathcal{U}}_2^+(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1)$ , and we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \left( \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \end{aligned}$$

$$\begin{aligned} &\leq \int_{\eta_1}^{\eta_2} \left( |\tilde{\mathcal{U}}_{2,\eta}| \min_j(\tilde{\mathcal{P}}_j) + |\tilde{\mathcal{U}}_2| \left( \frac{1}{A_1} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right) \right) (t, s) ds \\ &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|, \end{aligned}$$

or

(ii):  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \tilde{\mathcal{U}}_1^+(t, \eta_1)$ . Then there exists a maximal interval  $[\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, a)$  and  $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$ . Moreover, there exists a maximal interval  $[b, a] \subset [\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, a)$  and  $\tilde{\mathcal{U}}_1^+(t, b) = 0$ . Hence we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq d(t, \eta_2) - d(t, a) + d(t, a) - d(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left( \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \\ &\quad + \int_b^a \left( \tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \\ &\leq \int_a^{\eta_2} \left( \frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2| \right) (t, s) ds \\ &\quad + \int_b^a \left( \frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1| \right) (t, s) ds \\ &\leq \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, \eta_2]$ , the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b),$$

where  $(b, \eta_2]$  denotes the maximal interval such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, \eta_2]$ .

(2c): Assume that  $d(t, \eta_1) > 0$ ; then  $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) > 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) > 0$ . Then one either has that

(i):  $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_2(t, \eta_1)$ , and we have (as before)

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \\ &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|, \end{aligned}$$

or

(ii):  $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_1(t, \eta_1)$ . Then there exists a maximal interval  $[\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, a)$  and  $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$ . Moreover, there exists a maximal interval  $[b, a] \subset [\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, a]$  and  $\tilde{\mathcal{U}}_1^+(t, b) = 0$ . Hence we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq d(t, \eta_2) - d(t, a) + d(t, a) - d(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left( \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t, s) ds \\ &\quad + \int_b^a \left( \tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t, s) ds \\ &\leq \int_a^{\eta_2} \left( \frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2| \right)(t, s) ds \\ &\quad + \int_b^a \left( \frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1| \right)(t, s) ds \\ &\leq \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, \eta_2]$ , the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b),$$

where  $(b, \eta_2]$  denotes the maximal interval such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, \eta_2]$ .

Thus we showed that

$$|d(t, \eta_2) - d(t, \eta_1)| \leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,$$

or, in other words,  $d(t, \cdot) = \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \cdot)$  is Lipschitz continuous with Lipschitz constant  $\frac{3}{2\sqrt{2}} A^6$ , which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$\begin{aligned} |d(t, \eta_2) - d(t, \eta_1)| &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1| \\ &\quad + \frac{A^4}{\sqrt{2}} \left| \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds \right| + A^4 \left| \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \right|. \end{aligned}$$

Since both  $|\tilde{\mathcal{U}}_2|(t, \cdot)$  and  $\min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, \cdot)$  are continuous, the fundamental theorem of calculus implies that

$$\begin{aligned} \left| \frac{d(t, \eta_2) - d(t, \eta_1)}{\eta_2 - \eta_1} \right| &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\quad + \frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, \tilde{\eta}) + A^4 |\tilde{\mathcal{U}}_2|(t, \tilde{\eta}) \end{aligned}$$

for some  $\tilde{\eta}$  between  $\eta_1$  and  $\eta_2$ . Letting  $\eta_2 \rightarrow \eta_1$  we thus obtain for almost every  $\eta$  that

$$\begin{aligned} \left| \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \right| &= \left| \frac{d}{d\eta} d(t, \eta) \right| \\ &\leq 2A^4 \left( \min_j(\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \right)(t, \eta). \end{aligned}$$



(ii) We have that

$$\begin{aligned} & \left| \min_j(\tilde{\mathcal{P}}_j)(t, \eta) - \min_j(\tilde{\mathcal{P}}_j)(t, \tilde{\eta}) \right| \\ & \leq \max(|\tilde{\mathcal{P}}_1(t, \eta) - \tilde{\mathcal{P}}_1(t, \tilde{\eta})|, |\tilde{\mathcal{P}}_2(t, \eta) - \tilde{\mathcal{P}}_2(t, \tilde{\eta})|) \\ & \leq \frac{A^4}{2} |\eta - \tilde{\eta}|. \end{aligned}$$

Thus, for almost every  $\eta$ ,

$$\begin{aligned} \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_k)(t, \eta) \right| &= \left| \left( \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) \tilde{\mathcal{U}}_k(t, \eta) + \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_{k,\eta}(t, \eta) \right| \\ &\leq \frac{A^4}{2} \|\tilde{\mathcal{U}}_k\|_\infty + \|\tilde{\mathcal{P}}_k\tilde{\mathcal{U}}_{k,\eta}\|_\infty \\ &\leq \frac{1}{\sqrt{2}} A^6. \end{aligned} \quad \square$$

LEMMA A.4. (i) *The function  $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$  is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\begin{aligned} & \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \eta) \right| \\ & \leq \mathcal{O}(1)\sqrt{AA^4}(\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_k|)(t, \eta), \quad k = 1, 2. \end{aligned}$$

(ii) *The function  $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k(t, \eta)$ ,  $k = 1, 2$ , is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k)(t, \eta) \right| \leq \mathcal{O}(1)A^7.$$

*Proof.* (i) We only present the proof for the case  $k = 2$ , since the case  $k = 1$  is similar.

Given  $0 \leq \eta_1 < \eta_2 \leq 1$ , we assume without loss of generality that

$$0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1).$$

To ease the notation, we introduce the function

$$\bar{d}(t, \eta) = \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta).$$

We will distinguish several cases:

(1): If  $\bar{d}(t, \eta_2) = 0$ , then  $\bar{d}(t, \eta_1) = 0$  and one has

$$\begin{aligned} 0 &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \left| \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) \right| \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{D}}_1(t, \eta_2) - \tilde{\mathcal{D}}_1(t, \eta_1)| + |\tilde{\mathcal{D}}_2(t, \eta_2) - \tilde{\mathcal{D}}_2(t, \eta_1)|) \\ &\leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2): If  $\bar{d}(t, \eta_2) > 0$ , then  $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) > 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) > 0$  or equivalently

$$\tilde{\mathcal{D}}_1(t, \eta_2) > 0, \quad \tilde{\mathcal{D}}_2(t, \eta_2) > 0, \quad \tilde{\mathcal{U}}_1(t, \eta_2) > 0, \quad \text{and} \quad \tilde{\mathcal{U}}_2(t, \eta_2) > 0.$$

(2a): Assume that  $\bar{d}(t, \eta_1) = 0$  and  $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) = 0$ . Then

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq (\min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1)) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{D}}_1(t, \eta_2) - \tilde{\mathcal{D}}_1(t, \eta_1)| + |\tilde{\mathcal{D}}_2(t, \eta_2) - \tilde{\mathcal{D}}_2(t, \eta_1)|) \\ &\leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2b): Assume that  $\bar{d}(t, \eta_1) = 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = 0$ . Then one either has that

(i):  $\tilde{\mathcal{U}}_2(t, \eta_1) \leq \tilde{\mathcal{U}}_2^+(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1)$  and we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j)(t, s) + \tilde{\mathcal{U}}_2 \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j)(t, s) ds \\ &\leq \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_{2,\eta}| \min_j(\tilde{\mathcal{D}}_j)(t, s) ds \\ &\quad + \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2| \left( \left| (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta} - \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{Y}}_{1,\eta} + \frac{1}{2} A_1^5 \right| \right. \\ &\quad \left. + \left| (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \frac{1}{2} A_2^5 \right| \right) (t, s) ds \end{aligned}$$

$$\begin{aligned} &\leq \sqrt{AA^4} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1| \end{aligned}$$

or

(ii):  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \tilde{\mathcal{U}}_1^+(t, \eta_1)$ . Then there exists a maximal interval  $[\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, a)$  and  $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$ . Moreover, there exists a maximal interval  $[b, a] \subset [\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, a]$  and  $\tilde{\mathcal{U}}_1^+(t, b) = 0$ . Hence we can write

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \bar{d}(t, \eta_2) - \bar{d}(t, a) + \bar{d}(t, a) - \bar{d}(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left( \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\quad + \int_b^a \left( \tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\leq \int_a^{\eta_2} (\sqrt{AA^4} \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_2|)(t, s) ds \\ &\quad + \int_b^a (\sqrt{AA^4} \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_1|)(t, s) ds \\ &\leq \sqrt{AA^4} \int_b^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \sqrt{AA^4} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, \eta_2]$ , the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b),$$

where  $(b, \eta_2]$  denotes the maximal interval such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, \eta_2]$ .

(2c): Assume that  $\bar{d}(t, \eta_1) > 0$ ; then  $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) > 0$  and  $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) > 0$ . Then one either has that

(i):  $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_2(t, \eta_1)$ , and we have (as before)

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \left( \frac{1}{2} \sqrt{AA^6} \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2| \right)(t, s) ds \\ &\leq \mathcal{O}(1)A^7|\eta_2 - \eta_1|, \end{aligned}$$

or

(ii):  $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_1(t, \eta_1)$ . Then there exists a maximal interval  $[\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, a)$  and  $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$ . Moreover, there exists a maximal interval  $[b, a] \subset [\eta_1, a]$  such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, a)$  and  $\tilde{\mathcal{U}}_1^+(t, b) = 0$ . Hence we can write

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \bar{d}(t, \eta_2) - \bar{d}(t, a) + \bar{d}(t, a) - \bar{d}(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left( \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\quad + \int_b^a \left( \tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\leq \int_a^{\eta_2} \left( \frac{1}{2} \sqrt{AA^6} \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2| \right)(t, s) ds \\ &\quad + \int_b^a \left( \frac{1}{2} \sqrt{AA^6} \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_1| \right)(t, s) ds \\ &\leq \sqrt{AA^4} \int_b^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \sqrt{AA^4} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1)A^7|\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that  $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$  for all  $s \in [\eta_1, \eta_2]$ , the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b),$$

where  $(b, \eta_2]$  denotes the maximal interval such that  $\tilde{\mathcal{U}}_1^+(t, s) > 0$  for all  $s \in (b, \eta_2]$ .

Thus we showed that

$$|\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)| \leq \mathcal{O}(1)A^7|\eta_2 - \eta_1|,$$

or, in other words,  $\bar{d}(t, \cdot) = \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \cdot)$  is Lipschitz continuous with Lipschitz constant  $\mathcal{O}(1)A^7$ , which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$\begin{aligned} & |\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)| \\ & \leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2)|\eta_2 - \eta_1| \\ & \quad + \sqrt{A}A^4 \left| \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds \right| + \mathcal{O}(1)A^5 \left| \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \right|. \end{aligned}$$

Since both  $|\tilde{\mathcal{U}}_2|(t, \cdot)$  and  $\min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, \cdot)$  are continuous, the fundamental theorem of calculus implies that

$$\begin{aligned} \left| \frac{\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)}{\eta_2 - \eta_1} \right| & \leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ & \quad + \sqrt{A}A^4 \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, \tilde{\eta}) + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_2|(t, \tilde{\eta}) \end{aligned}$$

for some  $\tilde{\eta}$  between  $\eta_1$  and  $\eta_2$ . Letting  $\eta_2 \rightarrow \eta_1$  we thus obtain for almost every  $\eta$  that

$$\begin{aligned} \left| \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \right| & = \left| \frac{d}{d\eta} \bar{d}(t, \eta) \right| \\ & \leq \mathcal{O}(1)\sqrt{A}A^4 (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \eta). \end{aligned}$$

(ii) We have that

$$\begin{aligned} & |\min_j(\tilde{\mathcal{D}}_j)(t, \eta) - \min_j(\tilde{\mathcal{D}}_j)(t, \tilde{\eta})| \\ & \leq \max(|\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_1(t, \tilde{\eta})|, |\tilde{\mathcal{D}}_2(t, \eta) - \tilde{\mathcal{D}}_2(t, \tilde{\eta})|) \\ & \leq \mathcal{O}(1)A^5 |\eta - \tilde{\eta}|, \end{aligned}$$

and hence, for almost every  $\eta$ ,

$$\left| \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j)(t, \eta) \right| \leq \mathcal{O}(1)A^5.$$

This implies, for almost every  $\eta$ ,

$$\begin{aligned} \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k)(t, \eta) \right| &= \left| \left( \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right) \tilde{\mathcal{U}}_k(t, \eta) + \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_{k,\eta}(t, \eta) \right| \\ &\leq \mathcal{O}(1)A^7 + 2A\|\tilde{\mathcal{P}}_k\|\tilde{\mathcal{U}}_{k,\eta} \\ &\leq \mathcal{O}(1)A^7. \end{aligned} \quad \square$$

LEMMA A.5. *The function*

$$\eta \mapsto a \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with

$$\left| \frac{d}{d\eta} \left( a \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1)A^2.$$

*Proof.* To prove the existence and boundedness of the derivative, we will prove Lipschitz continuity. Let  $0 \leq \eta_1 < \eta_2 \leq 1$ ; then

$$\begin{aligned} &a \left| \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right. \\ &\quad \left. - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right| \\ &\leq a \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\ &\quad + a \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ &\quad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))})) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\ &\leq \frac{aA^2}{\sqrt{2}}|\eta_2 - \eta_1| \\ &\quad + a \int_0^{\eta_1} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_1) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_2) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
 &+ a \int_0^{\eta_1} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta_1) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta_2) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
 \leq &\frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
 &+ \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
 &+ \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
 \leq &\frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
 &+ \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
 &+ \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
 \leq &\frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| + \frac{1}{A_1^5} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \right. \\
 &\times \left. \left( \frac{1}{A_1} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1, \eta} + A_1 \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta} + \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{H}}_{1, \eta} \right) (t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
 &+ \frac{1}{A_2^5} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 &\times \left. \left( \frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{H}}_{2, \eta} \right) (t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
 \leq &\frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
 &+ \mathcal{O}(1) \frac{1}{A_1^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
 &+ \mathcal{O}(1) \frac{1}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
 \leq &\mathcal{O}(1) A^2 |\eta_2 - \eta_1|. \quad \square
 \end{aligned}$$

LEMMA A.6. *The function*

$$\eta \mapsto \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right)$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left| \frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1)A^7. \quad (\text{A.9})$$

*Proof.* Introduce

$$\begin{aligned} \tilde{a}(t, \eta) &= \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \\ \tilde{b}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ \tilde{c}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta. \end{aligned}$$

Thus

$$\tilde{a}(t, \eta) = \min(\tilde{b}, \tilde{c})(t, \eta).$$

Then we have to show that  $\tilde{a}(t, \cdot)$  is Lipschitz continuous with a Lipschitz constant, which only depends on  $A$ . Clearly, we have that

$$|\tilde{a}(t, \eta_1) - \tilde{a}(t, \eta_2)| \leq \max(|\tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2)|, |\tilde{c}(t, \eta_1) - \tilde{c}(t, \eta_2)|),$$

and it suffices to show that both  $\tilde{b}(t, \cdot)$  and  $\tilde{c}(t, \cdot)$  are Lipschitz continuous with a Lipschitz constant, which only depends on  $A$ . We are only going to establish the Lipschitz continuity for  $\tilde{b}$ , since the argument for  $\tilde{c}$  follows the same lines.

Let  $0 \leq \eta_1 < \eta_2 \leq 1$ . Then we have to consider two cases:

$$0 \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \quad \text{and} \quad 0 \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1).$$

(i):  $0 \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1)$ : By definition, we have

$$\begin{aligned} \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1) &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\ &\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\ &\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \end{aligned}$$



$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & + \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \\
 & - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))})) \\
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \leq \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\
 & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta,
 \end{aligned}$$

where we used in the last step that  $0 \leq \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta)$  and that  $\tilde{\mathcal{Y}}_1(t, \eta)$  is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) \geq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}).$$

Moreover, note that  $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \leq 1$ , for  $0 \leq \eta_1 \leq \theta \leq \eta_2$ , and that

$$0 \leq 2\sqrt{2} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \leq A_1^7 \leq A^7,$$

by (4.15b) and (4.15e). Thus

$$\begin{aligned}
 0 & \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1) \\
 & \leq \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \leq \frac{A^7}{2\sqrt{2}} |\eta_2 - \eta_1|.
 \end{aligned}$$

(ii):  $0 \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2)$ : By definition, we have

$$\begin{aligned}
 0 & \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \\
 & = \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \quad - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & = \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}))
 \end{aligned}$$

$$\begin{aligned}
 & \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))})) \\
 & \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta.
 \end{aligned}$$

Now we have to be much more careful than before. Namely, we have (as before)

$$\begin{aligned}
 0 & \leq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))}) \\
 & \leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{\frac{1}{a}(\tilde{\mathcal{V}}_1(t,\theta) - \tilde{\mathcal{V}}_1(t,s))} \tilde{\mathcal{Y}}_{1,\eta}(t, s) + e^{\frac{1}{a}(\tilde{\mathcal{V}}_2(t,\theta) - \tilde{\mathcal{V}}_2(t,s))} \tilde{\mathcal{Y}}_{2,\eta}(t, s)) ds.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \\
 & \leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_1) - \tilde{\mathcal{V}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t,\eta_2) - \tilde{\mathcal{V}}_j(t,\theta))})) \\
 & \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{V}}_1(t,s) - \tilde{\mathcal{V}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & \quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{V}}_2(t,s) - \tilde{\mathcal{V}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 & = \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{V}}_1(t,s) - \tilde{\mathcal{V}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 & \quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{V}}_2(t,s) - \tilde{\mathcal{V}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 & \leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{V}}_1(t,s) - \tilde{\mathcal{V}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 & \quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{V}}_2(t,s) - \tilde{\mathcal{V}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 & = \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds.
 \end{aligned}$$

As far as  $\tilde{B}_1(t, s)$  is concerned, we have

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{1}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/4} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{1}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left( \int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq 3A^2 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{3}{2} A^7 |\eta_2 - \eta_1|, \end{aligned}$$

using (4.16k).

As far as  $\tilde{B}_2(t, s)$  is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) - \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\begin{aligned} & \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \frac{\sqrt{2}}{a} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \left( 2\tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta} + \frac{A_2}{\sqrt{2}} \tilde{\mathcal{H}}_{2,\eta} \right) (t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \left( 2\sqrt{2} A_2^2 \tilde{\mathcal{P}}_2(t, s) + 2 \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right). \end{aligned}$$

Note that the integral term can be bounded by  $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$  since

$$\int_0^s e^{-\frac{1}{2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \frac{15}{\sqrt{2}} A_2^5 \tilde{\mathcal{P}}_2(t, s)$$

by (4.16k). We end up with

$$\begin{aligned} \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \\ &\quad \times \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{17}{\sqrt{2}} \frac{A_1^5}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \\ &\leq \frac{17}{2\sqrt{2}} A^7 |\eta_2 - \eta_1|. \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds \\ &\leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1|, \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

Finally combining both cases yields that there exists a constant  $\mathcal{O}(1)$ , which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ , such that

$$|\tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1)| \leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1|$$

and subsequently

$$|\tilde{a}(t, \eta_2) - \tilde{a}(t, \eta_1)| \leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1|.$$

This proves that the derivative exists for almost every  $\eta$  and is bounded by (A.9). □

LEMMA A.7. *The function*

$$\eta \mapsto \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right)$$

is Lipschitz continuous with uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left| \frac{d}{d\eta} \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1)A^8. \tag{A.10}$$

*Proof.* We present the following argument. Introduce

$$\begin{aligned}\bar{a}(t, \eta) &= \min_k \left( \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right), \\ \bar{b}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta, \\ \bar{c}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta.\end{aligned}$$

Thus

$$\bar{a}(t, \eta) = \min(\bar{b}, \bar{c})(t, \eta).$$

We have to show that  $\bar{a}(t, \cdot)$  is Lipschitz continuous with a Lipschitz constant, which only depends on  $A$ . Clearly, we have that

$$|\bar{a}(t, \eta_1) - \bar{a}(t, \eta_2)| \leq \max(|\bar{b}(t, \eta_1) - \bar{b}(t, \eta_2)|, |\bar{c}(t, \eta_1) - \bar{c}(t, \eta_2)|),$$

and it suffices to show that both  $\bar{b}(t, \cdot)$  and  $\bar{c}(t, \cdot)$  are Lipschitz continuous with a Lipschitz constant, which only depends on  $A$ . We are only going to establish the Lipschitz continuity for  $\bar{b}$  since the argument for  $\bar{c}$  follows the same lines. Let  $0 \leq \eta_1 < \eta_2 \leq 1$ . Then we have to consider two cases:

$$0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \quad \text{and} \quad 0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1).$$

(i):  $0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1)$ : By definition, we have

$$\begin{aligned}& \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1) \\ &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad + \int_0^{\eta_1} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\quad - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta\end{aligned}$$

$$\begin{aligned} &\leq \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))}) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta, \end{aligned}$$

where we used in the last step that  $0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \eta)$  and that  $\tilde{\mathcal{Y}}_i(t, \eta)$  is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_1) - \tilde{\mathcal{V}}_j(t, \theta))}) \geq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))}).$$

Moreover, note that  $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))}) \leq 1$  for  $0 \leq \eta_1 \leq \theta \leq \eta_2$  and that

$$0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \leq \frac{1}{\sqrt{2}} A_1^8 \leq \frac{1}{\sqrt{2}} A^8$$

by (4.15b), (4.15e), and (4.15n). Thus

$$0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1) \leq \frac{1}{\sqrt{2}} A^8 |\eta_2 - \eta_1|.$$

(ii)  $0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2)$ : By definition, we have

$$\begin{aligned} &0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \\ &= \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_1) - \tilde{\mathcal{V}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_1) - \tilde{\mathcal{V}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))})) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_1) - \tilde{\mathcal{V}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{V}}_j(t, \eta_2) - \tilde{\mathcal{V}}_j(t, \theta))})) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta. \end{aligned}$$

Now we have to be much more careful than before. Namely, we have (as before)

$$\begin{aligned} 0 &\leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) + e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s)) ds. \end{aligned}$$

Hence

$$\begin{aligned} &\bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \\ &\leq \int_0^{\eta_1} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1, \eta}(t, s) \\ &\quad \times \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2, \eta}(t, s) \\ &\quad \times \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1, \eta}(t, s) \\ &\quad \times \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2, \eta}(t, s) \\ &\quad \times \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &= \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds. \end{aligned}$$

As far as  $\bar{B}_1(t, s)$  is concerned, we have

$$\begin{aligned}
 & \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds \\
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{a}(\bar{Y}_1(t,s) - \bar{Y}_1(t,\theta))} \min_j(\bar{D}_j) \min_j(\bar{U}_j^+) \bar{Y}_{1,\eta}(t, \theta) d\theta \right) \bar{Y}_{1,\eta}(t, s) ds \\
 &\leq \sqrt{2}A \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\bar{Y}_1(t,s) - \bar{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/4} \bar{U}_1^+ \bar{Y}_{1,\eta}(t, \theta) d\theta \right) \bar{Y}_{1,\eta}(t, s) ds \\
 &\leq \sqrt{2}A \int_{\eta_1}^{\eta_2} \left( \int_0^s e^{-\frac{1}{A_1}(\bar{Y}_1(t,s) - \bar{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/2} \bar{Y}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left( \int_0^s e^{-\frac{1}{A_1}(\bar{Y}_1(t,s) - \bar{Y}_1(t,\theta))} \bar{U}_1^2 \bar{Y}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \bar{Y}_{1,\eta}(t, s) ds \\
 &\leq 6A^3 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \bar{Y}_{1,\eta}(t, s) ds \\
 &\leq 3A^8 |\eta_2 - \eta_1|,
 \end{aligned}$$

where we used (4.15n).

As far as  $\bar{B}_2(t, s)$  is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{Y}_{2,\eta}(t, \eta) - \tilde{U}_2^2 \tilde{Y}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{Y}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\begin{aligned}
 & \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \min_j(\bar{D}_j) \min_j(\bar{U}_j^+) \bar{Y}_{1,\eta}(t, \theta) d\theta \\
 &\leq \frac{2\sqrt{2}A}{a} \int_0^s e^{-\frac{1}{A_2}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)^{3/2} \bar{Y}_{1,\eta}(t, \theta) d\theta \\
 &\leq 2A \int_0^s e^{-\frac{1}{A_2}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} \tilde{\mathcal{P}}_1 \bar{Y}_{1,\eta}(t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \int_0^s e^{-\frac{1}{A_2}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} (2\tilde{\mathcal{P}}_2 \tilde{Y}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \int_0^s e^{-\frac{1}{A_2}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \left( 2\tilde{\mathcal{P}}_2^{5/4} \tilde{Y}_{2,\eta} + \frac{A_2}{\sqrt{2}} \tilde{\mathcal{H}}_{2,\eta} \right) (t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \left( 2\sqrt{2}A_2^2 \tilde{\mathcal{P}}_2(t, s) + 2 \int_0^s e^{-\frac{1}{A_2}(\bar{Y}_2(t,s) - \bar{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{Y}_{2,\eta}(t, \theta) d\theta \right).
 \end{aligned}$$



Note that the integral term can be bounded by  $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$  since

$$\int_0^s e^{-\frac{1}{a}(\hat{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \frac{15}{\sqrt{2}} A_2^2 \tilde{\mathcal{P}}_2(t, s)$$

by (4.16k). We end up with

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{17\sqrt{2}A^6}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \\ &\leq \frac{17A^8}{\sqrt{2}} |\eta_2 - \eta_1|. \end{aligned}$$

Moreover,

$$\begin{aligned} \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds \\ &\leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|, \end{aligned}$$

where  $\mathcal{O}(1)$  denotes some constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

Finally, combining both cases yields that there exists a constant  $\mathcal{O}(1)$ , which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ , such that

$$|\bar{b}(t, \eta_2) - \bar{b}(t, \eta_1)| \leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|$$

and subsequently

$$|\bar{a}(t, \eta_2) - \bar{a}(t, \eta_1)| \leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|.$$

This proves that the derivative exists for almost every  $\eta$  and is bounded by (A.10). □

LEMMA A.8. *The function*

$$\eta \mapsto \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right]$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies

$$\left| \frac{d}{d\eta} \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \right| \leq \mathcal{O}(1)A^6, \quad (\text{A.11})$$

where  $\mathcal{O}(1)$  denotes a constant, which only depends on  $A$  and which remains bounded as  $A \rightarrow 0$ .

*Proof.* To that end, we present the following argument. Introduce

$$\begin{aligned} a(t, \eta) &= \min_k \left[ \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right], \\ b(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta, \\ c(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta. \end{aligned}$$

Thus

$$a(t, \eta) = \min(b, c)(t, \eta).$$

Then we have to show that  $a(t, \cdot)$  is Lipschitz continuous with a Lipschitz constant, which only depends on  $A$  and hence on  $C$ . Clearly, we have that

$$|a(t, \eta_1) - a(t, \eta_2)| \leq \max(|b(t, \eta_1) - b(t, \eta_2)|, |c(t, \eta_1) - c(t, \eta_2)|),$$

and it suffices to show that both  $b(t, \cdot)$  and  $c(t, \cdot)$  are Lipschitz continuous with a Lipschitz constant, which only depends on  $A$ . We are only going to establish the Lipschitz continuity for  $b$ , since the argument for  $c$  follows the same lines. Let  $0 \leq \eta_1 < \eta_2 \leq 1$ . Then we have to consider two cases:

$$0 \leq b(t, \eta_1) - b(t, \eta_2) \quad \text{and} \quad 0 \leq b(t, \eta_2) - b(t, \eta_1).$$

(i):  $0 \leq b(t, \eta_2) - b(t, \eta_1)$ : By definition, we have

$$\begin{aligned} &b(t, \eta_2) - b(t, \eta_1) \\ &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^{\eta_1} \left( \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right. \\
 &- \left. \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right) \\
 &\times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &\leq \int_{\eta_1}^{\eta_2} \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta,
 \end{aligned}$$

where we used in the last step that  $0 \leq \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta)$  and that  $\tilde{\mathcal{Y}}_1(t, \cdot)$  is increasing, which implies

$$\begin{aligned}
 &\min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \\
 &= \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \eta_2) + \tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \\
 &\geq \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \eta_2))} \right) \\
 &\geq \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right).
 \end{aligned}$$

Moreover, note that  $\min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \leq 1$  since  $\eta_1 \leq \theta \leq \eta_2$ , and that

$$0 \leq \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \leq |\tilde{\mathcal{U}}_1^+|^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \leq \frac{1}{\sqrt{2}} A_1^7 \leq \frac{1}{\sqrt{2}} A^7$$

by (4.15b) and (4.15g). Thus

$$\begin{aligned}
 &0 \leq b(t, \eta_2) - b(t, \eta_1) \\
 &\leq \int_{\eta_1}^{\eta_2} \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &\leq \frac{1}{\sqrt{2}} A^7 |\eta_2 - \eta_1|.
 \end{aligned}$$

(ii):  $0 \leq b(t, \eta_1) - b(t, \eta_2)$ : By definition, we have

$$\begin{aligned}
 &b(t, \eta_1) - b(t, \eta_2) \\
 &= \int_0^{\eta_1} \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &- \int_0^{\eta_2} \min_j \left( e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\eta_1} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
 &\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
 &\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &\quad - \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &\leq \int_0^{\eta_1} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
 &\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
 &\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta.
 \end{aligned}$$

Now we have to be much more careful than before. Namely, if  $\min_j(\tilde{\mathcal{Y}}_j(t, \theta) - \tilde{\mathcal{Y}}_j(t, \eta_2)) = \tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2)$ , we have

$$\begin{aligned}
 0 &\leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 &= \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2))} \\
 &\leq e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_1))} - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2))} \\
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, s))} \tilde{\mathcal{Y}}_{i, \eta}(t, s) \, ds \\
 &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, s))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) + e^{\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, s))} \tilde{\mathcal{Y}}_{2, \eta}(t, s)) \, ds.
 \end{aligned}$$

Hence

$$\begin{aligned}
 &b(t, \eta_1) - b(t, \eta_2) \\
 &\leq \int_0^{\eta_1} \left( \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
 &\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
 &\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \\
 &\leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \, ds \\
 &\quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \, d\theta \, ds
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &= \int_{\eta_1}^{\eta_2} B_1(t, s) ds + \int_{\eta_1}^{\eta_2} B_2(t, s) ds.
 \end{aligned}$$

As far as  $B_1(t, s)$  is concerned, recall that

$$\begin{aligned}
 \tilde{\mathcal{P}}_i(t, s) &= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s) - \tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta \\
 &= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s) - \tilde{\mathcal{Y}}_i(t,\theta)|} (\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})(t, \theta) d\theta,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 &\int_{\eta_1}^{\eta_2} B_1(t, s) ds \\
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\leq \frac{a}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
 &\leq \frac{4}{\sqrt{2}} A^2 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
 &\leq \sqrt{2} A^7 |\eta_2 - \eta_1|.
 \end{aligned}$$

As far as  $B_2(t, s)$  is concerned, we have to be more careful. Therefore recall that we have

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) - \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta$$

$$\begin{aligned}
 &\leq \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \, d\theta \\
 &\leq \frac{A_1^5}{aA_2^5} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) \, d\theta \\
 &\leq \frac{A_1^5}{aA_2^5} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \left( A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + A_2^2 \tilde{\mathcal{H}}_{2,\eta} \right) (t, \theta) \, d\theta \\
 &\leq \frac{A_1^5}{aA_2^5} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \left( A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + A_2^2 \tilde{\mathcal{H}}_{2,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} \right) (t, \theta) \, d\theta \\
 &\leq \frac{A_1^5}{aA_2^5} \left( 4A_2^2 \tilde{\mathcal{P}}_2(t, s) + 4A_2^3 \tilde{\mathcal{P}}_2(t, s) \right. \\
 &\quad \left. + \frac{1}{A_2} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right).
 \end{aligned}$$

Note that the integral term can be bounded by  $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$  since

$$\begin{aligned}
 &\int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \\
 &= A_2^7 \int_0^1 e^{-|\mathcal{Y}_2(t, A_2^2 s) - \mathcal{Y}_2(t, A_2^2 \theta)|} \mathcal{P}_2^2 \mathcal{Y}_{2,\eta}(t, A_2^2 \theta) \, d\theta \\
 &= A_2^5 \int_0^{A_2^2} e^{-|\mathcal{Y}_2(t, A_2^2 s) - \mathcal{Y}_2(t, \theta)|} \mathcal{P}_2^2 \mathcal{Y}_{2,\eta}(t, \theta) \, d\theta \\
 &\leq \mathcal{O}(1)A_2^5 \mathcal{P}_2(t, A_2^2 s) = \mathcal{O}(1)A_2^3 \tilde{\mathcal{P}}_2(t, s)
 \end{aligned}$$

by (3.34). We end up with

$$\begin{aligned}
 &\int_{\eta_1}^{\eta_2} B_2(t, s) \, ds \\
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left( \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \, d\theta \right) ds \\
 &\leq \frac{A_1^5}{aA_2^5} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \\
 &\quad \times \left( A_2^2 \mathcal{O}(A) \tilde{\mathcal{P}}_2(t, s) + \frac{1}{A_2} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \, d\theta \right) ds \\
 &\leq \frac{\mathcal{O}(1)A_1^5 A_2^2}{aA_2^5} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) \, ds \\
 &\leq \mathcal{O}(1)A^6 |\eta_2 - \eta_1|.
 \end{aligned}$$

Moreover,

$$\begin{aligned} b(t, \eta_1) - b(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} B_1(t, s) ds + \int_{\eta_1}^{\eta_2} B_2(t, s) ds \\ &\leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|, \end{aligned}$$

where  $\mathcal{O}(1)$  only depends on  $A$  and hence on  $C = \frac{A^2}{2}$ , and remains bounded as  $A \rightarrow 0$ .

Finally, combining both cases yields that

$$|b(t, \eta_2) - b(t, \eta_1)| \leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|$$

and subsequently

$$|a(t, \eta_2) - a(t, \eta_1)| \leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|,$$

where  $\mathcal{O}(1)$  remains bounded as  $A \rightarrow 0$ . This proves that the derivative exists for almost every  $\eta$  and is bounded by (A.11).  $\square$

We need to estimate the pointwise difference between two functions  $\tilde{\mathcal{D}}_j$ ,  $j = 1, 2$ . This is the content of the following lemma.

LEMMA A.9. *We have that*

$$\begin{aligned} |\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| &\leq 2A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| \\ &\quad + 2\sqrt{2}A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad + 4A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + 2\sqrt{2}A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + \frac{3A^4}{\sqrt{2}} \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + \frac{12\sqrt{2}A^4}{\sqrt{3}e} \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} d\theta \right)^{1/2} |A_1 - A_2| \\ &\quad + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\ &\quad + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\ &\quad + \frac{3A^4}{2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \end{aligned}$$

$$+ 6A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2|,$$

for any value of  $j = 1, 2$ .

*Proof.* Direct calculations yield

$$\begin{aligned} & |\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| \\ &= \left| \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \left( (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \\ &\quad \left. - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left( (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \\ &\leq \left| \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \times \left( (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \left| \right. \\ &\quad \left. + \left| \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \right. \\ &\quad \times \left( (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left| \right. \\ &\quad \left. + \left| \int_0^\eta \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \right. \\ &\quad \left. \left. - \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left( (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \right. \\ &= \bar{d}_{11}(t, \eta) + \bar{d}_{12}(t, \eta) + \bar{d}_{13}(t, \eta), \end{aligned}$$

where  $B(\eta)$  is defined in (5.11).

For  $\bar{d}_{11}(t, \eta)$  we immediately obtain

$$\begin{aligned} \bar{d}_{11}(t, \eta) &\leq \frac{1}{A_1} \int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \\ &\quad \times e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \left( (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \\ &\leq \frac{\tilde{\mathcal{D}}_1(t, \eta)}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\leq A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|. \end{aligned} \tag{A.12}$$



Following the same lines, we end up with

$$\begin{aligned} \bar{d}_{12}(t, \eta) &\leq \frac{1}{A_2} \int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \\ &\quad \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \left| (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) + \frac{1}{2} A_2^5 \right| d\theta \\ &\leq A_2^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_2^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|. \end{aligned} \tag{A.13}$$

The last term  $\bar{d}_{13}(t, \eta)$  needs a bit more work. Indeed,

$$\begin{aligned} \bar{d}_{13}(t, \eta) &= \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left( (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \\ &\quad \left. - \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left( (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \\ &\leq \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\quad + \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\quad + \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{1}{2} A_1^5 \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{1}{2} A_2^5) d\theta \right| \\ &= \bar{T}_1(t, \eta) + \bar{T}_2(t, \eta) + \bar{T}_3(t, \eta). \end{aligned}$$

To estimate  $\bar{T}_1(t, \eta)$ , recall (3.40), (A.4), (A.5), and Lemma A.1 (ii), which imply

$$\begin{aligned} \bar{T}_1(t, \eta) &= \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\leq \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{U}}_2^2) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2}(t, \theta) d\theta \right| \end{aligned}$$

$$\begin{aligned}
 & + \left| \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{U}}_2^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}(t, \theta) d\theta \right| \\
 & + \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta (\min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
 & + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta (\min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| \\
 & + \left| \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \theta) d\theta \right| \\
 \leq & 2 \int_0^\eta \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 & + 2 \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
 & + \mathbb{1}_{A_1 \leq A_2} \frac{4}{ae} \left| \int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
 & + \mathbb{1}_{A_2 < A_1} \frac{4}{ae} \left| \int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
 & + \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \right|_{\theta=0}^\eta \\
 & - \int_0^\eta \frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2)) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) d\theta \Big| \\
 \leq & 2 \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 & + 2 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 & + \frac{2\sqrt{2}A}{e} \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{2\sqrt{2}A}{e} \left( \int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
 & \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
 & + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 & + \frac{1}{a} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{1,\eta} + \tilde{\mathcal{Y}}_{2,\eta}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
 & + A^4 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
 & \leq 4A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 & + \frac{8\sqrt{2}A^4}{\sqrt{3}e} \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
 & + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 & + \sqrt{2}A^4 \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 & + A^4 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta. \tag{A.14}
 \end{aligned}$$

Following the same lines, one has

$$\begin{aligned}
 \bar{T}_2(t, \eta) & = \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \right. \\
 & \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta)) d\theta \right| \\
 & \leq \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_2 \leq \tilde{\mathcal{P}}_1}(t, \theta) d\theta \right| \\
 & \quad + \left| \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right| \\
 & \quad + \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
 & \quad + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 & \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right|
 \end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \theta) d\theta \right| \\
\leq & 2 \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{P}}_1^{1/2} |\tilde{\mathcal{Y}}_{1,\eta}| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \theta) d\theta \\
& + 2 \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{Y}}_{2,\eta}| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \frac{4}{ae} \left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \mathbb{1}_{A_2 < A_1} \frac{4}{ae} \left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \left| \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \right|_{\theta=0}^\eta \\
& - \int_0^\eta \frac{d}{d\theta} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) d\theta \left| \right. \\
\leq & 2 \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
& + 2 \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left( \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
& + \frac{2A}{e} \left( \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \frac{2A}{e} \left( \int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \frac{1}{a} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{Y}}_{1,\eta} + \tilde{\mathcal{Y}}_{2,\eta}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
& + \frac{A^4}{2} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
 &\leq 2\sqrt{2}A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad + \frac{4\sqrt{2}A^4}{\sqrt{3}e} \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
 &\quad + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad + \frac{A^4}{\sqrt{2}} \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad + \frac{A^4}{2} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta. \tag{A.15}
 \end{aligned}$$

For the last term  $\bar{T}_3(t, \eta)$ , we have

$$\begin{aligned}
 2\bar{T}_3(t, \eta) &= \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) A_1^5 - (\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) A_2^5) d\theta \right| \\
 &\leq \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (A_1^5 - A_2^5) d\theta \right| \\
 &\quad + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (A_1^5 - A_2^5) d\theta \right| \\
 &\quad + a^5 \int_0^\eta \left| \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| d\theta \\
 &\leq 10A^4 \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2| \\
 &\quad + \frac{4a^4}{e} \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2| \\
 &\leq 12A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2|. \tag{A.16}
 \end{aligned}$$

Thus we end up with

$$\begin{aligned}
 |\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| &\leq 2A^{3/2} \max_j(\tilde{\mathcal{D}}_j^{1/2})(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| \\
 &\quad + 2\sqrt{2}A^{3/2} \max_j(\tilde{\mathcal{D}}_j^{1/2})(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 &\quad + 4A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad + 2\sqrt{2}A^3 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad + \frac{3A^4}{\sqrt{2}} \left( \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{12\sqrt{2}A^4}{\sqrt{3}e} \left( \int_0^\eta e^{-\frac{3}{4A}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
 &+ \tilde{U}_j^2 |\tilde{Y}_1 - \tilde{Y}_2|(t, \eta) \\
 &+ \tilde{P}_j |\tilde{Y}_1 - \tilde{Y}_2|(t, \eta) \\
 &+ \frac{3A^4}{2} \int_0^\eta e^{-\frac{1}{A}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))} |\tilde{Y}_1 - \tilde{Y}_2|(t, \theta) d\theta \\
 &+ 6A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{Y}_j(t,\eta) - \tilde{Y}_j(t,\theta))} d\theta |A_1 - A_2|,
 \end{aligned}$$

which proves the lemma. □

LEMMA A.10. *We have the following estimates:*

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P} \tilde{Y}_\eta(t, \theta) d\theta \leq 2A \tilde{P}(t, \eta), \tag{A.17}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{H}_\eta(t, \theta) d\theta \leq 4A \tilde{P}(t, \eta), \tag{A.18}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{U}^2(t, \theta) d\theta \leq 6\tilde{P}(t, \eta), \tag{A.19}$$

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P} \tilde{Y}_\eta(t, \theta) d\theta \leq 4A \tilde{P}(t, \eta), \tag{A.20}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P}(t, \theta) d\theta \leq 7\tilde{P}(t, \eta), \tag{A.21}$$

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P}^2 \tilde{Y}_\eta(t, \theta) d\theta \leq A\mathcal{O}(1) \tilde{P}^{1/2}(t, \eta), \tag{A.22}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P}^{1+\beta} \tilde{Y}_\eta(t, \theta) d\theta \leq 3 \frac{1 + \beta}{\beta} \frac{A^{1+4\beta}}{4^\beta} \tilde{P}(t, \eta), \quad \beta > 0. \tag{A.23}$$

*Proof.* The proof of (A.17) goes as follows:

$$\begin{aligned}
 &\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P} \tilde{Y}_\eta(t, \theta) d\theta \\
 &= \frac{2}{3} A \tilde{P}(t, \eta) - \frac{2}{3A} \int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{Q} \tilde{Y}_\eta(t, \theta) d\theta \\
 &\leq \frac{2}{3} A \tilde{P}(t, \eta) + \frac{2}{3} \int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta) - \tilde{Y}(t,\theta))} \tilde{P} \tilde{Y}_\eta(t, \theta) d\theta,
 \end{aligned}$$

which implies

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 2A \tilde{\mathcal{P}}(t, \eta).$$

Next, we use that

$$\begin{aligned} \int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta &\leq \int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \\ &\leq 4A \tilde{\mathcal{P}}(t, \eta) \end{aligned}$$

(see (4.16d)), showing (A.18).

For (A.19) we find

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \\ &= \theta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \left( \frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta \right) (t, \theta) d\theta \\ &= \eta \tilde{\mathcal{U}}^2(t, \eta) - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \left( \frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta \right) (t, \theta) d\theta \\ &\leq \tilde{\mathcal{U}}^2(t, \eta) + \int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \left( \frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2|\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta| \right) (t, \theta) d\theta \\ &\leq \tilde{\mathcal{U}}^2(t, \eta) + 4\tilde{\mathcal{P}}(t, \eta) \\ &\leq 6\tilde{\mathcal{P}}(t, \eta). \end{aligned}$$

The proof of (A.20),

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A \tilde{\mathcal{P}}(t, \eta),$$

follows in the same manner as (A.17).

Furthermore, for (A.21) we find

$$\begin{aligned} &\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) d\theta \\ &= \theta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - \int_0^\eta \theta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \left( \frac{3}{2A} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta + \frac{1}{A^2} \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta \right) (t, \theta) d\theta \end{aligned}$$

$$\begin{aligned} &\leq \eta \tilde{\mathcal{P}}(t, \eta) + \frac{3}{A} \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\leq 7\tilde{\mathcal{P}}(t, \eta), \end{aligned}$$

which follows from (A.17).

In order to prove (A.22), we do as follows:

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= 2A\tilde{\mathcal{P}}^2(t, \eta) - 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t, \eta) \\ &\quad + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \left( \tilde{\mathcal{P}}^2 + \frac{1}{A^2} \tilde{\mathcal{Q}}^2 \right) \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\quad + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \left( (\tilde{\mathcal{P}} - \tilde{\mathcal{U}}^2) \tilde{\mathcal{Y}}_\eta(t, \theta) - \frac{1}{2} A^5 \right) \tilde{\mathcal{P}}(t, \theta) d\theta, \end{aligned}$$

and hence

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\leq 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t, \eta) + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \left( \tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}} \right) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \Big) \tilde{\mathcal{P}}(t, \theta) d\theta \\ &\leq 8A\tilde{\mathcal{P}}^2(t, \eta) + 8 \left( \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \left( \tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}} \right) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \Big)^{1/2} \\ &\quad \times \left( \int_0^\eta \left( \tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}} \right) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) \tilde{\mathcal{P}}^2(t, \theta) d\theta \Big)^{1/2} \\ &\leq A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t, \eta). \end{aligned}$$

As for the proof of (A.23), we proceed as follows:

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= A e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^{1+\beta}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - A(1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{P}}_\eta(t, \theta) d\theta \\ &= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - \frac{1}{A}(1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - (1 + \beta) e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} (\beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{P}}_\eta \tilde{\mathcal{Q}} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta)(t, \theta) d\theta \end{aligned}$$



$$\begin{aligned}
&= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - (1 + \beta)\tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \eta) \\
&\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left( \beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta \frac{1}{A^2} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta \right) (t, \theta) d\theta.
\end{aligned} \tag{A.24}$$

Recall (4.6), (4.9), and (4.10), which together imply

$$\tilde{\mathcal{Q}}_\eta(t, \eta) = \frac{1}{2}(\tilde{\mathcal{E}}_\eta - \tilde{\mathcal{D}}_\eta)(t, \eta) = -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t, \eta) - \frac{1}{2}A^5 + \tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t, \eta).$$

Thus

$$\begin{aligned}
&\beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta \frac{1}{A^2} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta \\
&= \frac{\beta}{A^2} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^{\beta+1} \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^\beta \left( -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta - \frac{1}{2}A^5 \right).
\end{aligned}$$

Inserting this expression in (A.24) and re-ordering the terms, we find

$$\begin{aligned}
&\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
&\quad + \frac{1 + \beta}{A^2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
&= (1 + \beta)\tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \eta) - A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) \\
&\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \left( (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta.
\end{aligned}$$

Estimating this, using (4.5a), we find

$$\begin{aligned}
&\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
&\leq (1 + \beta)\tilde{\mathcal{P}}^\beta |\tilde{\mathcal{Q}}|(t, \eta) \\
&\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \left( (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta \\
&\leq (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta A\tilde{\mathcal{P}}(t, \eta) \\
&\quad + (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left( (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta \\
&\leq (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta A\tilde{\mathcal{P}}(t, \eta) + (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta \tilde{\mathcal{D}}(t, \eta) \\
&\leq 3(1 + \beta) \left( \frac{A^4}{4} \right)^\beta A\tilde{\mathcal{P}}(t, \eta),
\end{aligned}$$

where we used (4.8), (4.15a), and (4.15c).  $\square$

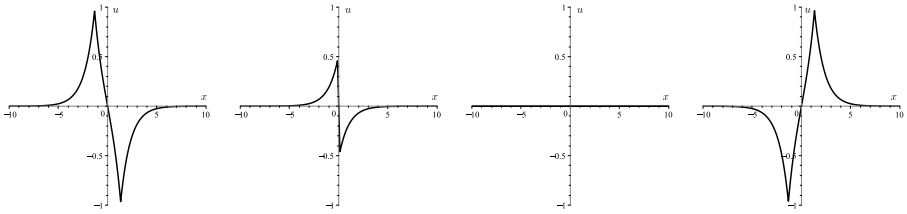


Figure B.3. Time evolution of  $u(t, x)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

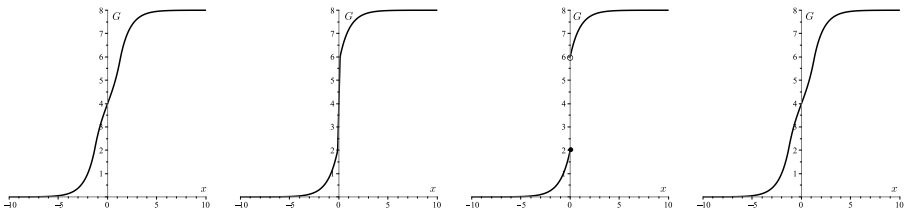


Figure B.4. Time evolution of  $G(t, x)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

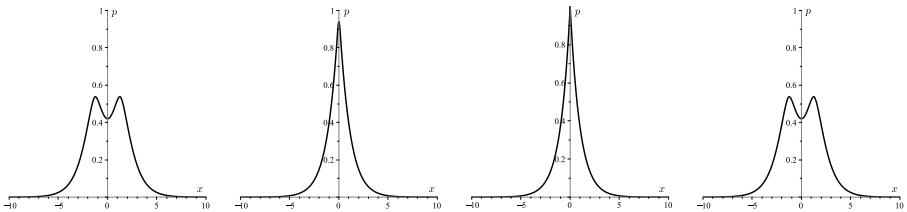


Figure B.5. Time evolution of  $p(t, x)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

## Appendix B. The antisymmetric peak–antipeakon example

Fortunately, one can compute explicitly the quantities described in this paper in the important case of an antisymmetric peak–antipeakon solution. The various functions are depicted on Figures B.3–B.14.

More precisely, consider the function [41]

$$u(t, x) = \begin{cases} \beta(t) \sinh(x), & |x| \leq \gamma(t), \\ \text{sign}(x)\alpha(t)e^{-|x|}, & |x| \geq \gamma(t), \end{cases}$$

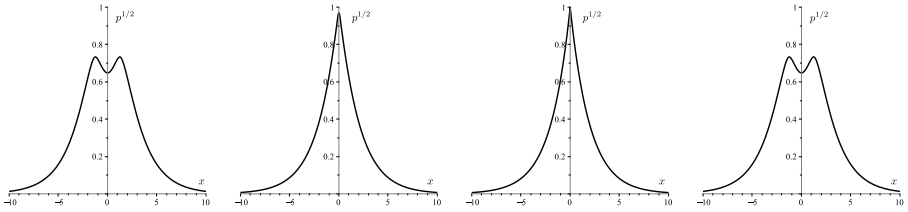


Figure B.6. Time evolution of  $p^{1/2}(t, x)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

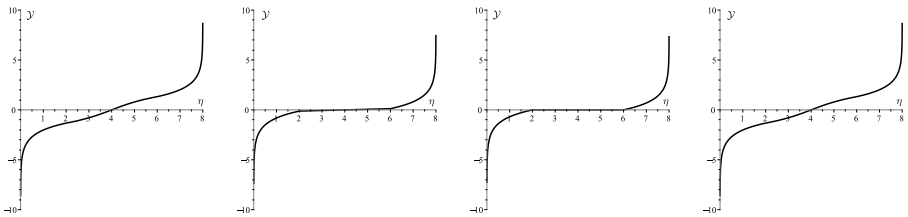


Figure B.7. Time evolution of  $\mathcal{Y}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

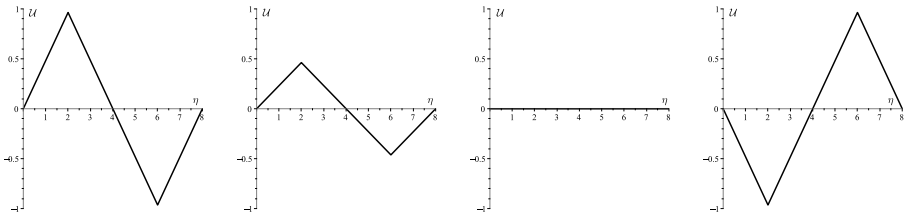


Figure B.8. Time evolution of  $\mathcal{U}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

$$= \begin{cases} -\alpha(t)e^x, & x \leq -\gamma(t), \\ \beta(t) \sinh(x), & -\gamma(t) \leq x \leq \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leq x, \end{cases}$$

where

$$\alpha(t) = \frac{E}{2} \sinh\left(\frac{E}{2}(t - t_0)\right), \quad \beta(t) = E \frac{1}{\sinh\left(\frac{E}{2}(t - t_0)\right)},$$

$$\gamma(t) = \ln\left(\cosh\left(\frac{E}{2}(t - t_0)\right)\right).$$

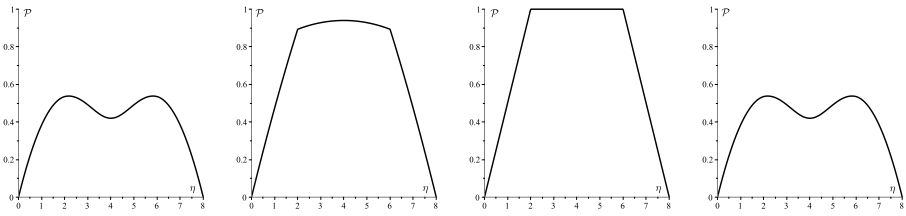


Figure B.9. Time evolution of  $\mathcal{P}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

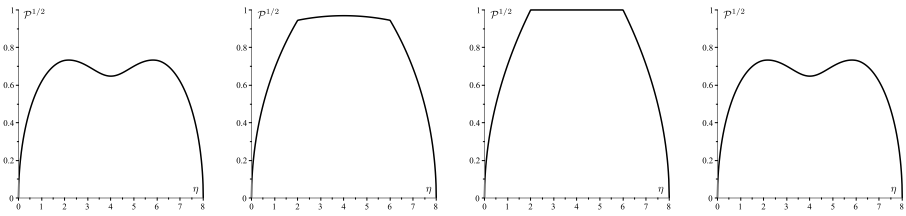


Figure B.10. Time evolution of  $\mathcal{P}^{1/2}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

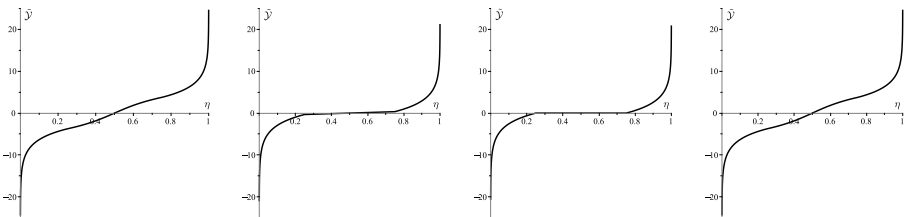


Figure B.11. Time evolution of  $\tilde{\mathcal{Y}}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

Here  $E = \|u(t)\|_{H^1}$ ,  $t \neq t_0$ , denotes the total energy. The corresponding energy density is given by

$$\mu(t, x) = (u^2 + u_x^2)(t, x) = \begin{cases} 2\alpha^2(t)e^{2x}, & x \leq -\gamma(t), \\ \beta^2(t) \cosh(2x), & -\gamma(t) \leq x \leq \gamma(t), \\ 2\alpha^2(t)e^{-2x}, & \gamma(t) \leq x, \end{cases} \quad t \neq t_0,$$

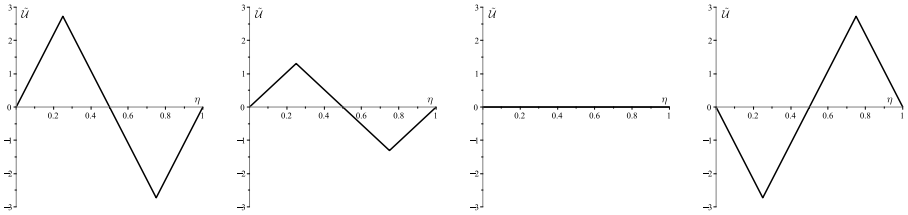


Figure B.12. Time evolution of  $\tilde{U}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

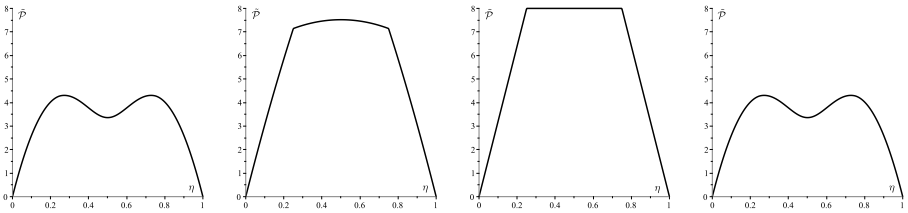


Figure B.13. Time evolution of  $\tilde{P}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

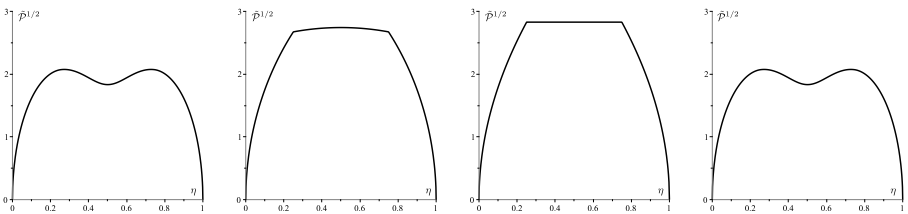


Figure B.14. Time evolution of  $\tilde{P}^{1/2}(t, \eta)$  with  $C = E^2 = 4$  and  $t_0 = 2$  at  $t = 0, 1.5, 2, 4$ .

with  $\mu(t_0, x) = E^2 \delta_0(x)$  for  $t = t_0$ . Hence  $C = \mu(t, \mathbb{R}) = E^2$ , and

$$F(t, x) = \begin{cases} \alpha(t)^2 e^{2x}, & x < -\gamma(t), \\ \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = -\gamma(t), \\ \frac{1}{2}E^2 + \frac{1}{2}\beta(t)^2 \sinh(2x), & -\gamma(t) < x < \gamma(t), \\ E^2 - \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = \gamma(t), \\ E^2 - \alpha(t)^2 e^{-2x}, & \gamma(t) < x. \end{cases}$$

In particular, this solution experiences wave breaking at time  $t = t_0$ , that is,  $u_x(t, 0)$  tends to  $-\infty$  as  $t \rightarrow t_0^-$  and

$$F(t_0^-, x) = \begin{cases} 0, & x \leq 0, \\ E^2, & 0 < x. \end{cases}$$

The corresponding function  $p_x(t, x)$ , which can be computed using  $p_x(t, x) = -u_t(t, x) - uu_x(t, x)$ , is given by

$$p_x(t, x) = \begin{cases} \alpha'(t)e^x - \alpha(t)^2e^{2x}, & x < -\gamma(t), \\ \frac{E^2}{4} - \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = -\gamma(t), \\ -\beta'(t) \sinh(x) - \frac{1}{2}\beta(t)^2 \sinh(2x), & -\gamma(t) < x < \gamma(t), \\ -\frac{E^2}{4} + \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & \gamma(t) = x, \\ -\alpha'(t)e^{-x} + \alpha(t)^2e^{-2x}, & \gamma(t) < x. \end{cases}$$

In particular, one obtains at wave breaking time  $t = t_0$  that

$$p_x(t_0^-, x) = \begin{cases} \frac{E^2}{4}e^x, & x < 0, \\ -\frac{E^2}{4}e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that  $p_x(t_0^-, x)$  has a (negative) jump of height  $-\frac{E^2}{2}$  at  $x = 0$  at time  $t = t_0$ . For all other points  $x \in \mathbb{R}$ , the function  $p_x(t_0^-, x)$  is continuous.

Thus the function  $G(t, x) = 2p_x(t, x) + 2F(t, x)$  is given by

$$G(t, x) = \begin{cases} 2\alpha'(t)e^x, & x < -\gamma(t), \\ \frac{E^2}{2}, & x = -\gamma(t), \\ E^2 - 2\beta'(t) \sinh(x), & -\gamma(t) < x < \gamma(t), \\ \frac{3E^2}{2}, & x = \gamma(t), \\ 2E^2 - 2\alpha'(t)e^{-x}, & \gamma(t) < x. \end{cases}$$

In particular, one observes that  $G_x(t, x) > 0$  for all  $x \in \mathbb{R}$ ,

$$\lim_{x \rightarrow -\infty} G(t, x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} G(t, x) = 2E^2 = 2C.$$

Thus the limits at  $\pm\infty$  are independent of time. Moreover, one has

$$G(t_0-, x) = \begin{cases} \frac{E^2}{2} e^x, & x \leq 0, \\ 2E^2 - \frac{E^2}{2} e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that  $G(t_0-, x)$  has a jump of size  $E^2$  due to the wave breaking at  $t = t_0$ , that is,  $\mu(t_0, x) = E^2 \delta_0(x)$ .

Direct computations using  $p = p_{xx} + \frac{1}{2}u^2 + \frac{1}{2}d\mu$  yield

$$p(t, x) = \begin{cases} \alpha'(t)e^x - \frac{1}{2}\alpha(t)^2 e^{2x}, & x \leq -\gamma(t), \\ -\beta'(t) \cosh(x) - \frac{1}{2}\beta(t)^2 \cosh^2(x), & -\gamma(t) \leq x \leq \gamma(t), \\ \alpha'(t)e^{-x} - \frac{1}{2}\alpha(t)^2 e^{-2x}, & \gamma(t) \leq x. \end{cases}$$

In the new coordinates, using  $G(t, \mathcal{Y}(t, \eta)) = \eta$ , the solution reads as

$$\mathcal{Y}(t, \eta) = \begin{cases} \ln\left(\frac{\eta}{2\alpha'(t)}\right), & 0 < \eta \leq \frac{E^2}{2}, \\ \sinh^{-1}\left(\frac{E^2 - \eta}{2\beta'(t)}\right), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \ln\left(\frac{2\alpha'(t)}{2E^2 - \eta}\right), & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

and, applying  $\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta))$ ,

$$\mathcal{U}(t, \eta) = \begin{cases} -\frac{\alpha(t)}{2\alpha'(t)}\eta, & 0 < \eta \leq \frac{E^2}{2}, \\ \frac{\beta(t)}{2\beta'(t)}(E^2 - \eta), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \frac{\alpha(t)}{2\alpha'(t)}(2E^2 - \eta), & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

and, by invoking  $\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta))$ ,

$$\mathcal{P}(t, \eta) = \begin{cases} \frac{1}{2}\eta - \frac{\alpha(t)^2}{8\alpha'(t)^2}\eta^2, & 0 < \eta \leq \frac{E^2}{2}, \\ -\beta'(t)\sqrt{1 + \frac{(E^2 - \eta)^2}{4\beta'(t)^2}} \\ - \frac{1}{2}\beta(t)^2\left(1 + \frac{(E^2 - \eta)^2}{4\beta'(t)^2}\right), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \frac{1}{2}(2E^2 - \eta) - \frac{\alpha(t)^2}{8\alpha'(t)^2}(2E^2 - \eta)^2, & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

where we used the convention that  $\sinh^{-1}(x)$  denotes the inverse of  $\sinh(x)$ .

For the scaled quantities we find, when we introduce  $A = \sqrt{2C} = \sqrt{2}E$ , that

$$\tilde{\mathcal{Y}}(t, \eta) = A\mathcal{Y}(t, A^2\eta) = \sqrt{2C} \begin{cases} \ln\left(\frac{C\eta}{\alpha'(t)}\right), & 0 < \eta \leq \frac{1}{4}, \\ \sinh^{-1}\left(\frac{C(1 - 2\eta)}{2\beta'(t)}\right), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ \ln\left(\frac{\alpha'(t)}{C(1 - \eta)}\right), & \frac{3}{4} \leq \eta < 1, \end{cases}$$

$$\tilde{\mathcal{U}}(t, \eta) = A\mathcal{U}(t, A^2\eta) = \sqrt{2C} \begin{cases} -\frac{\alpha(t)}{\alpha'(t)}C\eta, & 0 < \eta \leq \frac{1}{4}, \\ \frac{\beta(t)}{2\beta'(t)}C(1 - 2\eta), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ \frac{\alpha(t)}{\alpha'(t)}C(1 - \eta), & \frac{3}{4} \leq \eta < 1, \end{cases}$$

$$\tilde{\mathcal{P}}(t, \eta) = A^2\mathcal{P}(t, A^2\eta)$$

$$= 2C \begin{cases} C\eta - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2\eta^2, & 0 < \eta \leq \frac{1}{4}, \\ -\beta'(t)\sqrt{1 + \frac{C^2(1 - 2\eta)^2}{4\beta'(t)^2}} \\ - \frac{1}{2}\beta(t)^2\left(1 + \frac{C^2(1 - 2\eta)^2}{4\beta'(t)^2}\right), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ C(1 - \eta) - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2(1 - \eta)^2, & \frac{3}{4} \leq \eta < 1. \end{cases}$$



**Conflict of Interest:** None.

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