Joint Default Probabilities: A Model with Time-varying and Correlated Sharpe Ratios and Volatilities

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Abstract

The probabilities of joint default among companies are one of the major concerns in credit risk management, mainly because it affects the distribution of loan portfolio losses and is therefore critical when allocating capital for solvency purposes. This paper proposes a multivariate model with time-varying and correlated Sharpe ratios and volatilities for the value of the firms, calibrated to fit sample averages between and within the rating categories A and Ba. We found that, in the standard Merton framework, the model performs well with one average A-rated firm and one average Ba-rated firm and with two average Ba-rated firms when the joint default probabilities are compared with similar empirical probabilities.

Preface

This paper is submitted to the Norwegian University of Science and Technology (NTNU) as a completion of the two-year program Master of Science in Financial Economics at the Department of Economics.

I would like to thank Prof. Dr. Cristina Danciulescu at Trondheim Business School for her guidance, encouragement and help throughout the work with this paper. Many thanks are also owed to my fellow students for rewarding discussions. Finally, I will give my gratitude to my wife, Marita, for her support and love.

The responsibility for any errors in this paper rests on my shoulders alone.

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1 Introduction

Modeling dependence between multiple default events are one of the biggest challenges in credit risk modeling. The most obvious reason is that it affects the distribution of loan portfolio losses and is therefore critical in determining quantiles or other risk measures used for allocating capital for solvency purposes. For long term investments, default correlation can be quite a significant factor if the underlying firm values are highly correlated. Also, a bank with large loan portfolios can have significant exposures to a group of related counterparties, i.e. firms in the same economic sector or geographic region. Thus, they need good performing models for capturing and forecasting the risks associated with its large portfolios. Empirical results indicate that conventional methodologies for portfolio default losses that are "typically estimated for meeting bank capital requirements (...) are downward biased by a full order of magnitude on typical test portfolios" (Duffie, Eckner, Horel and Saita, 2009).

Also, dependence modeling is necessary in trying to understand the risk of simultaneous defaults by, for example, financial institutions. Standard credit risk models cannot explain the observed clustering of defaults, which is sometimes described as "credit contagion". A breakdown in the financial system could affect the entire economy. Avoiding such breakdowns are a major motivation behind regulation.

Conventional portfolio loss risk models and default correlations are mainly estimated in two ways. One method uses historical data and "assumes that borrower-level conditional default probabilities depend on measured firm-specific or marketwide factors. Portfolio loss distributions are typically based on the correlating influence of such observable factors" (Duffie, Eckner, Horel and Saita, 2009). The second approach "utilizes a particular theoretical structure of the default process. The most popular structure in practice is based on Merton's (1974) framework" (Zhou, 2001). Moody's KMV approach to measure the default probabilities for different firms, which is based on the Merton's framework, is considered as the best approach in the industry.

Among others, Chen, Collin-Dufresne and Goldstein (2009) found that a univariate model for the standard Merton model predicted spreads that falls below historical market spreads when the model was calibrated to match historical default rates, recovery rates, means and volatilities. They resolved this "credit spread puzzle" by the fact that default rates and Sharpe ratios are strongly correlated; both are high during recessions and low during booms. Further, they investigated credit spread implications of the Campbell and Cochrane (1999) pricing kernel calibrated to equity returns and aggregate consumption data. By identifying the historical surplus consumption ratio from aggregate consumption data, they found that

the implied level and time variation of spreads match historical levels well.

Within the lines of Chen, Collin-Dufresne and Goldstein (2009), we propose a multivariate model with correlated and time-varying means and volatilities, and investigate in a standard Merton framework its performance with respect to the joint probabilities of default. Using Monte Carlo simulations, we generate firm value distributions and investigate how this affects the joint probabilities of default. Further, we compare these probabilities with empirically default data from Moody's Investors Service. We also compare the performance of a standard multivariate model which is a model with constant and independent means and volatilities.

The paper proceeds as follows. The next section explains the standard Merton model and the Moody's KMV approach to measure the default probabilities for different firms. Section (3) derives the probabilities of joint default and default correlations. Section (4) outlines the proposed model and Section (5) explains the Monte Carlo analysis, including the calibration strategy, the Monte Carlo algorithm and the simulation results. In Section (6), we investigate the risk management implications and compare the model with empirical data. Finally, we conclude in Section (7).

2 The Merton model

2.1 Model overview

The famous framework from Merton (1974) is widely used in credit risk modeling today. The model employs the general equilibrium theory of option pricing developed by Black & Scholes in 1973. According to Merton (1974), the value of the corporate debt depends on three items: the rate of return on the riskless debt (e.g. government bonds that is assumed to be risk free), the specification of the particular issue, i.e. the maturity date, coupon rate, seniority in the event of default etc., and the probability that the firm will be unable to satisfy some or all of the issued claims on the company. This is defined as the probability of default. The risk free interest rate is thought of as an exogenous variable. Thus, changes in the value of corporate debt are solely caused by changes in the firm's probability of default.

The following Black-Scholes assumptions are made to develop the pricing model:

1. There are no transactions costs, taxes, or problems with indivisibilities of assets.

2. There are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.

3. There exists an exchange market for borrowing and lending at the same rate of interest.

4. Short-sales of all assets, with full use of the proceeds, is allowed.

5. Trading in assets takes place continuously in time.

6. The Modigliani-Miller theorem that the value of the firm is invariant to its capital structure obtains.

7. The term-structure is "flat" and known with certainty, i.e. the price, P, of a riskless discount bond which promises a payment of one dollar at time, τ, in the future is $P(\tau) = exp(-r\tau)$ where r is the instantaneous annual riskless rate of interest, the same for all time and exp is the exponential function.

8. The dynamics for the value of the firm, V, through time can be described by a diffusion-type stochastic process with the stochastic differential equation

$$dV = [\alpha V - C] \, dt + \sigma V \, dz, \tag{1}$$

where α is the instantaneous expected rate of return on the firm per unit of time, C is the total dollar payouts by the firm per unit of time to either the shareholders or the liabilities-holders, σ^2 is the instantaneous variance of the return on the firm per unit of time and dz is a standard Brownian motion.

Suppose there exists a security whose market value, Y, at any point in time can be written as a function of the value of the firm and time, i.e. $Y = F(V, t)$. The dynamics of this security's value is

$$dY = [\alpha_Y Y - C_Y] \, dt + \sigma_Y Y \, dz_Y, \tag{2}$$

where α_Y, C_Y, σ_Y^2 and z_Y have similar interpretations as for the firm dynamics. From Itô's lemma, it follows that the process followed by Y is

$$dY = \left[\frac{\partial F}{\partial V}(\alpha V - C) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial V^2} \sigma^2 V^2 \right] dt + \frac{\partial F}{\partial V} \sigma V \, dz. \tag{3}$$

The terms in the equations (2) and (3) can be compared. Then the following will be true:

$$\alpha_Y Y = \frac{\partial F}{\partial V}(\alpha V - C) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial V^2} \sigma^2 V^2 + C_Y \tag{4}$$

$$\sigma_Y Y = \frac{\partial F}{\partial V} \sigma V \tag{5}$$

$$dz_Y = dz \tag{6}$$

The instantaneous returns on Y and V are perfectly correlated. Following the Merton derivation of the Black & Scholes model, consider forming a three-security portfolio containing the firm, the particular security and the riskless debt. The number of dollar invested in the firm, the security and the riskless debt, are W_1, W_2 and W_3, respectively. The portfolio is constructed such that it requires zero net investments, so $W_3 \equiv -(W_1 + W_2)$. Thus, the instantaneous return to the portfolio, dx, would be[1]

$$dx = [W_1(\alpha - r) + W_2(\alpha_Y - r)] \, dt + [W_1 \sigma + W_2 \sigma_Y] \, dz. \tag{7}$$

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[1]See Merton (1973)

If W_1 and W_2 are chosen such that dz always will be zero, dx would be non-stochastic. To avoid arbitrage profits, the expected return on the portfolio must be zero. I.e. the following two equations must be satisfied:

$$W_1\sigma + W_2\sigma_Y = 0 \tag{8}$$

$$W_1(\alpha - r) + W_2(\alpha_Y - r) = 0. \tag{9}$$

A solution to (8) and (9) exists if and only if

$$(\frac{\alpha - r}{\sigma}) = (\frac{\alpha_Y - r}{\sigma_Y}). \tag{10}$$

Substituting from equations (4) and (5) into equation (10) we find

$$(\frac{\alpha - r}{\sigma}) = \frac{\frac{\partial F}{\partial V}(\alpha V - C) + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial V^2}\sigma^2 V^2 + C_Y - rF}{\frac{\partial F}{\partial V}\sigma V}. \tag{11}$$

Rearranging and simplifying this equation gives

$$0 = \frac{1}{2}\frac{\partial^2 F}{\partial V^2}\sigma^2 V^2 + (rV - C)\frac{\partial F}{\partial V} - rF + \frac{\partial F}{\partial t} + C_Y \tag{12}$$

Equation (12) is the Black-Scholes-Merton differential equation. This must be satisfied by any security whose value can be written as a function of the value of the firm and the time. The particular derivative that is obtained when the equation is solved depends on the boundary conditions and initial conditions that are used. It is precisely these boundary condition specifications which distinguish one security from another. From equation (12), there is possible to see that there is only the interest rate, the volatility, the firm value, time and the payout policy of the firm that affect the value of the security. However, it does not depend on the expected rate of return from the firm or the risk-preferences of the investor. Two investors with different utility functions would therefore agree on the value of the security.

In the following, we assume that the company has issued two types of claims: a single, homogenous class of debt and the residual claim, equity. The bond issue promises to pay D dollars on a specified calendar date T. If this payment is not met, the bondholders immediately take over the company. Two further assumptions are made: the firm cannot issue any

new claims on the company nor can it pay cash dividends, so $C = 0$. With those assumptions, the payoffs to debt, $B(T)$, and equity, $S(T)$, at date T are given as

$$B(T) = min(D, V(T)) = D - max(D - V(T), 0), \qquad (13)$$

$$S(T) = max(V(T) - D, 0). \qquad (14)$$

As illustrated in Figure (1), in the Merton model the company will default if the firm value is less than its liabilities at maturity. If $V(T) < D$, then the equity holders receive nothing and the debt holders get the "recovery" of $V(T)$ instead of the promised payment D. But if the firm value exceeds the debt, the debt holders receive their promised payment and the equity holders receive the remaining part of the firm value. One could see from equation (13) that the debt can be viewed as the difference between a riskless bond and a put option on the firm's assets and from equation (14) that the equity can be viewed as a call option on the firm's assets. Note that in this framework there exist no tax advantages by issuing debt and no bankruptcy costs. A consequence of this is that at any time t, the relationship $V(t) = B(t) + S(t)$ should hold.
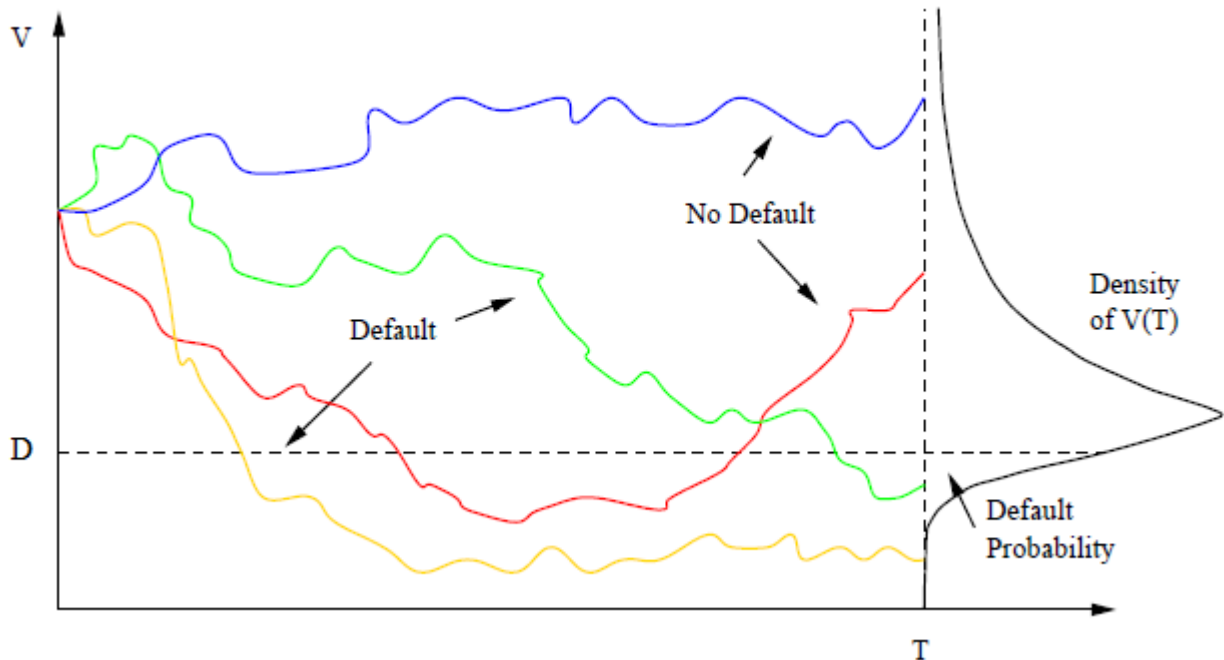
Figure 1: Default in the Merton model. Source: Giesecke (2004).

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The question is then how the debt and equity are valued prior to the maturity date T. From the Black-Scholes-Merton differential equation (12) with the boundary conditions given by the equations (13) and (14) it can be shown that the following is true:[2] Given the current firm level V, the volatility σ of the firm and the riskless rate r, we let $C^{BS}(V, D, \sigma, r, T - t)$ denote the Black & Scholes price of an European call option with strike price D and time to maturity $T - t$. Then, at time t

$$C^{BS}(V(t), D, \sigma, r, T - t) = V\Phi(d_1) - Dexp\left[-r(T - t)\right]\Phi(d_2) \tag{15}$$

where $\Phi(x; \mu, \sigma) \equiv \frac{1}{\sigma\sqrt{2\pi}}exp\left[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\right]$ is the normal distribution function and

$$d_1 = \frac{log(V/D) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \tag{16}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}. \tag{17}$$

Similarly, the formula for an European put is

$$P^{BS}(V(t), D, \sigma, r, T - t) = Dexp\left[-r(T - t)\right]\Phi(-d_2) - Vexp\left[-r(T - t)\right]\Phi(-d_1), \tag{18}$$

where d_1 and d_2 are given by the equations (16) and (17), respectively. Thus, the value of equity and debt at time t are

$$S(t) = C^{BS}(V(t), D, \sigma, r, T - t), \tag{19}$$

$$B(t) = Dexp\left[-r(T - t)\right] - P^{BS}(V(t), D, \sigma, r, T - t), \tag{20}$$

respectively. Note that since the sum of debt and equity values is the asset value, we can also write $B(t) = V(t) - C^{BS}(V(t), D, \sigma, r, T - t)$.

This basic version of the Merton model was extended in a number of ways. One way was to assume that the firm will default whenever the asset value falls below a barrier level. This is called the "Black-Cox Setup". Default can now occur prior to the maturity of the bond and it will happen when the level of the asset value hits a lower boundary, modeled as a

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[2]See Black and Scholes (1973).

deterministic function of time. In the original approach of Black and Cox, the boundary represents the point at which bond safety covenants cause a default. (Black and Cox, 1976). Another extension of the Merton model was to allow for stochastic default-free interest rates. This extension was considered due to the empirical evidence which found that the interest rates on treasury bonds are stochastic. The model was also extended to allow for jumps in the asset value (Lando, 2004).

2.2 The probability of default in the Merton model

In the basic version of the Merton model, default will occur if the firm value at time T, $V(T)$, is bellow the face value of the debt, D. The model can therefore produce a probability of default for each firm in a sample at any given point in time. Bharath and Shumway (2008) argues that the distance to default can be calculated as

$$DD = \frac{log(V/D) + (\mu - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \tag{21}$$

where r is replaced with the expected continuously compounded return on the firm, μ. This is due to the fact that this solution is obtained under the true probability measure and can be compared with empirically observed default probabilities. DD measures the size (in standard deviations) of the random shock required to induce bankruptcy for a firm. Then, the corresponding implied probability, P, of default is

$$P(V(T) < D|V(t)) = \Phi\left[-\frac{log(V/D) + (\mu - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right] = \Phi\left[-DD\right]. \tag{22}$$

If the Merton model holds, this probability of default should be a sufficient statistic for the default forecasts. This requires efficient markets (Lando, 2004). For a given time to maturity, the probability of default is a function of the firm's value to debt ratio, its volatility and its asset return. *Therefore, this measure can be both increasing and decreasing with time to maturity and the evolution of the return and the volatility through time.*

For a firm with asset to debt ratio of 1.5, asset mean of 10% and a volatility of 20%, Figure (2) shows the probability of default in the Merton model for 10 years to maturity. The probability of default is increasing with time until a certain point where it peak. This is because the uncertainty is low for a short horizon when the firm to debt ratio exceeds 1. For different calibrations, one may obtain different shapes. If the default probability is 2.5%, the DD is 1.96. The financial interpretation is that the debt value is lower than the asset value by 1.96 standard deviations.
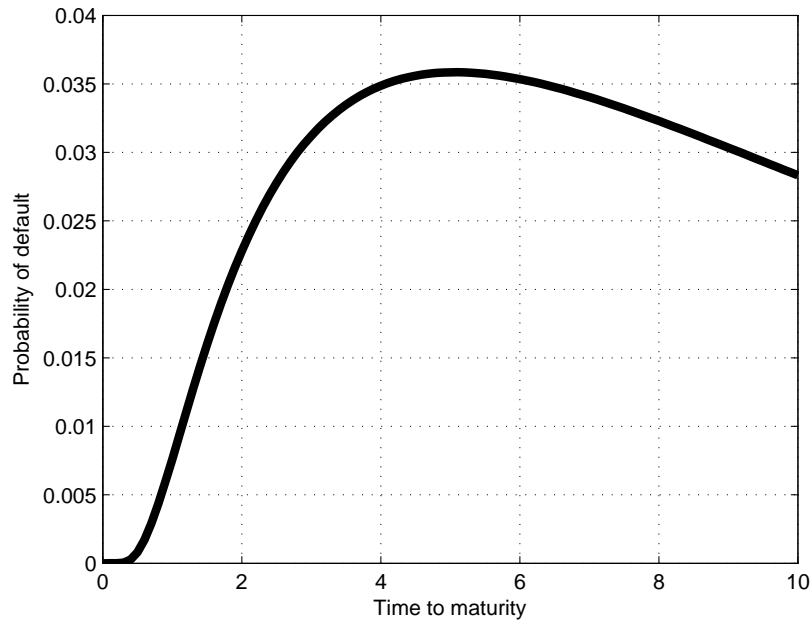
Figure 2: Probability of default as a function of time to maturity in the Merton model. Calibration: $\mu = 0.1$; $\sigma = 0.2$; $\frac{V}{D} = 1.5$; $T - t = 10$

Focusing on asset values implies valuing all future cash flows of the firm, rather than focusing on a single period. The classical corporate finance approach to default is to project future free cash flows to see whether they allow the timely repayment of debt, interest and principal. The relevant cash flows are then the cash flows from operations left for the lenders and equity holders after all economic outflows required to maintain the operating ability of the firm are paid. These future free cash flows should be high enough to face the future debt obligations. Using the discounted value of the free cash flows as the firm's asset value is an elegant way of summarizing the information. A short term view of the cash flows is not a relevant criterion for solvency. A temporary shortage of cash does not trigger default as long as there are chances of improvement in the future. Persistent cash flow deficiencies make a default highly likely. Temporary deficiencies do not, but the market view synthesizes this information.

2.3   The Moody's KMV (MKMV) approach

Bharath and Shumway (2008) states that the first step to empirically implement the Merton model is to estimate the standard deviation of the equity, σ_E, either from historical stock return data or from option-implied volatility data. Further, it can be shown that the relationship between equity and asset volatility under the Black & Scholes assumptions is

$$\sigma_E = (\frac{V}{S})\Phi(d_1)\sigma, \qquad (23)$$

where σ is the standard deviation of the firm's asset value and d_1 is given by equation (16). The second step is to measure the firm's liabilities. Crosbie and Bohn (2003) defined the default point as

$$0.5 \times long - term\ debt + short - term\ debt, \qquad (24)$$

because MKMV has found that firms generally do not default when their asset values reaches the book value of their total liabilities. The debt level triggering a default is unclear since debt amortizes by fractions according to some schedule. Refinancing can pull down the default point and the long-term nature of some of the liabilities provides some breathing space.

The third step is to collect values of the risk free rate and the market value of the firm's equity. After performing these three steps we have values for all variables in the Merton model except for V. The fourth step in estimating the model is to solve the equations (15) and (23) numerically for the value of V and σ.

"An actual test of whether this is a good model for default would then look at how well DD has historically predicted defaults. A simple test would be to group the distances to default into small intervals, small enough to consider the default probability as a constant over the interval but large enough to include enough firms, then the default frequency within each bucket would be a reasonably accurate estimate of the default probability. This would produce an empirical curve." (Lando, 2004). Those measures are found to differ from the model. Therefore, the measure of distance to default utilized by Moody's KMV is slightly different and is reported in Crosbie and Bohn (2003) as

$$\frac{Distance}{to\ default} = \frac{\begin{array}{c} Market\ value \\ of\ assets \end{array} - \begin{array}{c} Default \\ point \end{array}}{\begin{array}{c} Market\ value \\ of\ assets \end{array} \times \begin{array}{c} Asset \\ volatility \end{array}}. \qquad (25)$$

The relevant net worth of the company is the market value of the firm's assets minus the firm's default point. The market measure of net worth must be considered in the context of the firm's business risk. The asset risk is measured by the asset volatility and is the standard deviation of the annual percentage change in the asset value. This measure for the risk of

default compares the market net worth to the size of a one standard deviation move in asset value. The default probability can be computed directly from the distance-to-default if the probability distribution of the asset is known.
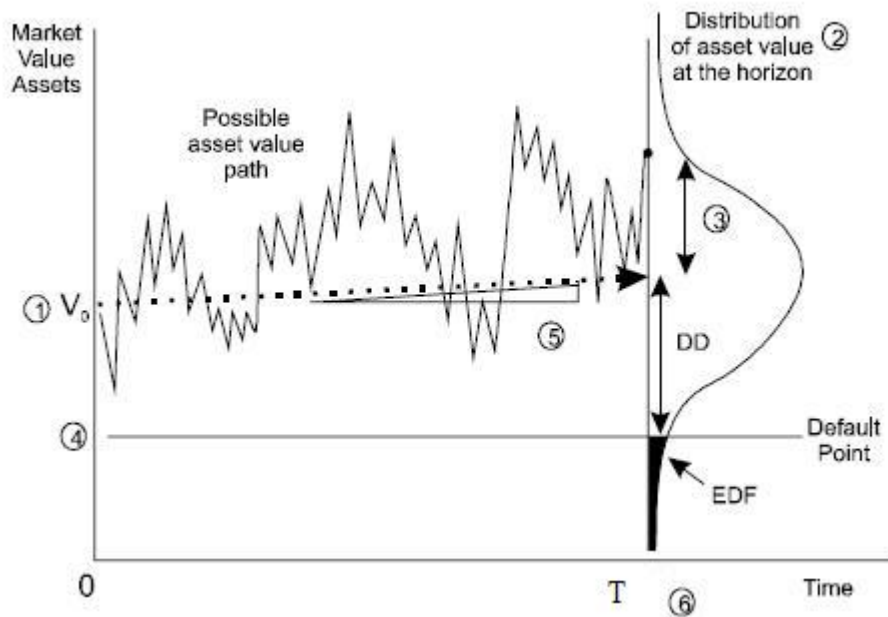
Figure 3: Determination of the default probability in the MKMV approach. Source: Crosbie and Bohn (2003).

Figure (3) illustrates graphically the MKMV approach. There are six variables that determine the default probability of a firm over some horizon, from now until time T:

1. The current asset value

2. The distribution of the asset value at time T.

3. The volatility of the future asset value.

4. The level of the default point.

5. The expected rate of growth in the asset value over the horizon.

6. The length of the horizon T.

Once the values for these variables are obtained, equation (25) is used to find the distance to default. MKMV obtain the relationship between distance to default and the probability of default from data on historical defaults and bankruptcy frequencies. This distribution

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has wider tails than the Normal distribution mainly because of uncertainty in calculating the default point. The shaded area in Figure (3) is the Expected Default Frequency (EDF) which is defined as the probability of default during a given horizon of for example 1 year.

Default is a relatively rare event and there is considerable variation in default probabilities across firms. For example, the probability of default for an AAA-rated company is around 0.02% per annum, but the same probability is around 4% for a CCC-rated firm. Moody's KMV has implemented an empirical version of the Black-Scholes-Merton model which actually performs well. Their one year accuracy ratio was on average over 80% between 1983 and 2002 (Cantor and Mann, 2003).

It is important to point out that there are a number of things which differentiate the Merton DD model and the Moody's KMV (Bharath and Shumway, 2008). First of all, the MKMV allows for various classes and maturities of debt. Second, they use their own large historical database to estimate the empirical distribution of changes in distance to default and calculate default probabilities based on that distribution. Finally, they also make proprietary adjustments to the accounting information that they use to calculate the face value of the debt.

In this paper, we use an extended Merton model to investigate the probabilities of joint default. Due to the advent of innovative corporate debt products and credit derivatives, academics and practitioners have shown renewed interest in models that forecast corporate defaults. Different applications of the Merton model are widely applied in this research and it is a clever application of classic finance theory. But it is a unusual forecasting model. Most forecasting models constructed by econometricians involve posing a model and then estimating the model with method of moments or maximum-likelihood techniques. The Merton model replaces estimation with something more like calibration.

How well a model performs in forecasting depends on how realistic its assumptions are. Among other things, the standard Merton model assumes that the underlying value of each firm follows a geometric Brownian motion. If the models strong assumptions are violated, it should be possible to construct a reduced-form model with more accuracy. The standard Merton model is known for systematically underestimate default probability when compared with historical default rates. One explanation of this is that it uses the normal distribution which not includes the possibility of fat tails. Default correlation can influence the asset value distribution and should be included in a structural model of default.

3 Default correlation and joint default probability

3.1 Definitions and applications

Default probabilities are influenced by common background variables which can be observable or unobservable. Firms share a common dependence on the economic environment, which result in cyclical correlation. Therefore, default events of companies are often correlated.

A risk manager should be concerned about three quantities while measuring the credit risk: the probability of default for each individual position over various investment horizons, the joint probability of default between every pair of counterparties over various investment horizons and the loss given default for each firm. The most crucial and the most difficult part is estimating the joint probability of default.

Zhou (2001) argues that the joint probability of default is important in credit risk analysis for several reasons. First, there is an increasing market for credit derivatives. For example, one type of credit derivatives, letter of credit-backed debt, is issued by a financial institution to the buyer of the debt and promises that the seller will receive payment on time and in the correct amount. Thus, two types of credit events have to occur before the issuer experience a loss; both the financial institution and the buyer have to default. Furthermore, a simple credit default swap (CDS) can be viewed as default insurance on loans or bonds. The buyer pays a premium in the form of an annuity until the time of the credit event or until the maturity, whichever is first (Duffie and Singleton, 2003). This insurance will only be valid if the issuer has not defaulted and loss will happen if both the issuer of the CDS and the debt default.

Second, "credit clustering" generates greater dispersion in the distribution of credit losses. This implies greater likelihood of large losses. Constructing a portfolio implies combining assets/securities of multiple firms. An increase in the joint probabilities of default could lead to losses that exceed most of the worst estimates. As an example, collateralized debt obligations backed by subprime debt have been at the heart of the financial crisis that started in 2007 and lead to a great number of bank failures. Hence, quantifying correctly the probabilities of joint default in a portfolio of fixed income securities is crucial.

To understand default correlation, it is convenient to define two random variables $D_1(t)$ and $D_2(t)$ that describe the default status of firm 1 and firm 2 at time T, as in Zhou (2001):

$$D_i(t) = \begin{cases} 1 & if\ firm\ i\ defaults\ by\ T \\ 0 & otherwise \end{cases}, \quad i = 1, 2. \tag{26}$$

It is reasonable to assume that when one entity defaults, the other entity have a higher likelihood of defaulting either because of the pressure from the general economy or because both firms belong to the same industry. From the definition of the correlation between two variables and because $D_1(t)$ and $D_2(t)$ are Bernoulli binomial random variables, we have that the probability, P, of joint default is

$$P(D_1(t) = 1 \ and \ D_2(t) = 1) = E\left[D_1(t)\right] E\left[D_2(t)\right] + Corr\left[D_1(t), D_2(t)\right] \sqrt{Var\left[D_1(t)\right] Var\left[D_2(t)\right]}, \tag{27}$$

where E is the expectation operator, $Corr\left[D_1(t), D_2(t)\right]$ is the correlation between the default status of the two firms and $Var\left[D_i(t)\right]$ is the variance to the default status of firm i. The expected default probability and the variance for firm i are given by

$$E\left[D_i(t)\right] = P(D_i(t) = 1) \tag{28}$$

and

$$Var\left[D_i(t)\right] = P(D_i(t) = 1) \times \left[1 - P(D_i(t) = 1)\right], \tag{29}$$
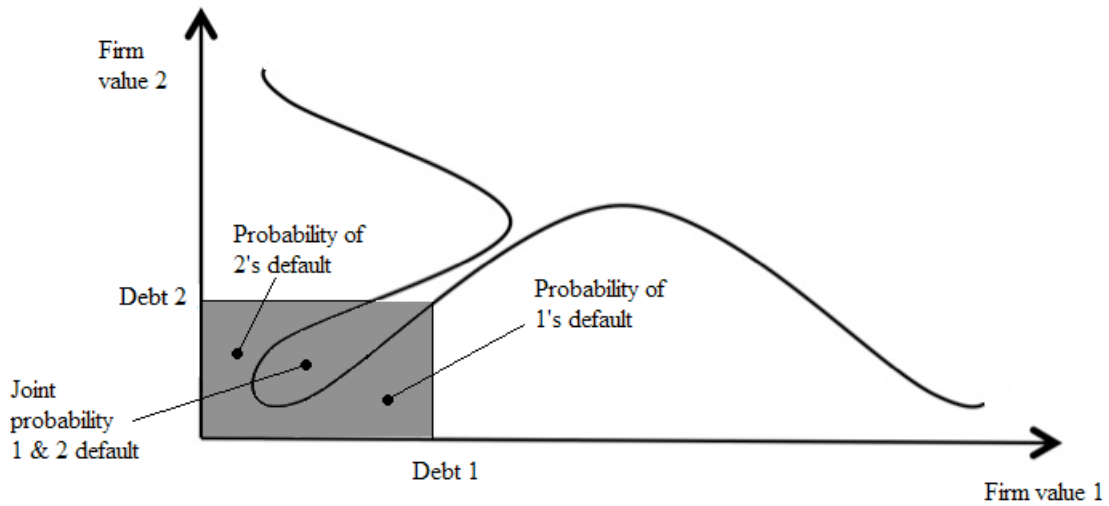
respectively.

Figure 4: The joint default probability of two firms.

Figure (4) shows possible firm value distributions for two companies for a given horizon. As we see from these distributions, Firm 1 and 2 have different debt levels, volatilities and expected asset values. In this example, we assume that the default event occur whenever the asset value gets lower than the debt and that the joint default of a pair of obligors occurs whenever both of them have asset values bellow the debt values. The joint probability of default is represented by the area under the left and down rectangle between the origin of the axes in Figure (4) and the two lines representing the debt levels of two firms. The joint probability of default embeds the correlation of default events. If the annual probabilities of default for firm 1 and 2 are 1% and 3%, then the annual joint probability of default is 1%·3%=0.03% if they are uncorrelated. But if the correlation equals 0.3, the annual joint probability of default would increase by $0.3 \cdot \sqrt{0.009 \cdot 0.0291}$=0.51% to 0.54%. This is a huge increase in the joint probability of default.

Asset value simulations are usually employed when generating correlation between credit events. Within the option theoretic framework, there is common to use a Monte Carlo simulation methodology when generating correlated stochastic processes for the asset values. The random asset values at the horizon depend on common factors and firm specific risk. The first can be generated through a correlation structure in the asset dynamics, the latter can be generated through the firm specific volatility. A model of two firms that should capture the probability of joint default have to generate simulated asset values for both firms and assigning default values whenever both the simulated asset values falls below the default points.

3.2 Default correlation in the standard Merton model

3.2.1 The case of two firms

In the simplest case of joint default, we consider firm 1 and firm 2. With $t = 0$, the asset dynamics of firm i is

$$\frac{dV^i(t_i)}{V^i(t_i)} = \mu_i dt + \sigma_i dW^i(t_i), \quad V_0^i > 0; \quad i = 1, 2. \tag{30}$$

$(W^1(t_1), W^2(t_2))$ is a two dimensional Brownian motion with correlation ρ, i.e. $(W^1(t_1), W^2(t_2)) \sim \Phi[0, \Sigma_W]$ with variance-covariance matrix

$$\Sigma_W = \begin{pmatrix} t_1 & \rho\sqrt{t_1 t_2} \\ \rho\sqrt{t_1 t_2} & t_2 \end{pmatrix}, \tag{31}$$

where t_i is the change in time.

Using Itô's lemma we get

$$V^i(t) = V_0^i exp((\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W^i(t_i)). \tag{32}$$

In the classical Merton framework, the joint probability of firm 1 to default at T_1 and firm 2 to default at T_2 is

$$P(T_1, T_2) = P(V^1(T_1) < D_1, V^2(T_2) < D_2)$$

$$= \Phi_2 \left[\rho, -\frac{log(V^1/D_1)+(\mu_1-\frac{1}{2}\sigma_1^2)T_1}{\sigma_1\sqrt{T_1}}, -\frac{log(V^2/D_2)+(\mu_2-\frac{1}{2}\sigma_2^2)T_2}{\sigma_2\sqrt{T_2}} \right] \tag{33}$$

where $\Phi_2[\rho, x, y]$ is the bivariate standard normal distribution function, with linear correlation parameter $\rho < 1$, given by

$$\Phi_2(\rho, x, y) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi\sqrt{1-\rho^2}} exp(\frac{2\rho xy - x^2 - y^2}{2(1-\rho^2)})dxdy. \tag{34}$$

By using results from probability theory, a host of other useful probabilities can be derived. For example, the probability of default of firm 1 at time T_1 conditional on the default of firm 2 at time T_2 is

$$P(T_1 \mid T_2) = \frac{\Phi_2 \left[\rho, -\frac{log(V^1/D_1)+(\mu_1-\frac{1}{2}\sigma_1^2)T_1}{\sigma_1\sqrt{T_1}}, -\frac{log(V^2/D_2)+(\mu_2-\frac{1}{2}\sigma_2^2)T_2}{\sigma_2\sqrt{T_2}} \right]}{\Phi \left[-\frac{log(V^2/D_2)+(\mu_2-\frac{1}{2}\sigma_2^2)T_2}{\sigma_2\sqrt{T_2}} \right]}. \tag{35}$$

Also, the probability of at least one default or survival probabilities can be calculated.

3.2.2 The multiple firms' case

In the case of many issuers, the computation is a bit more challenging because it involves the use of the multivariate normal distribution. If the probability that firm i will default at time t is defined as

$$P(V_i(T) < D_i|V_i(t)) = \Phi \left[-\frac{log(V_i/D_i) + (\mu_i - \frac{1}{2}\sigma_i^2)(T-t)}{\sigma_i\sqrt{T-t}} \right] = \Phi[-DD_i], \tag{36}$$

then for N firms, the probability of joint default is given by

16

$$P(V_1(T) < D_1, V_2(T) < D_2, ..., V_N(T) < D_N) = \Phi_N \left[-DD_1, -DD_2, ..., -DD_N, \rho_{12}, \rho_{13}, ... \right],$$
(37)

where DD_i is defined as in equation (36) and Φ_N is the Nth dimensional multivariate standard normal distribution.

3.3 Moody's KMV approach to account for correlation

To create a comprehensive credit portfolio model, Moody's KMV uses, in addition to the EDF values for each firm, the joint default frequency (JDF) to determine asset value correlations (Kealhofer and Bohn, 1993). This measure can be calculated by focusing on the EFD_i measure for the individual firm i and the correlation between each part of the firm's market asset value. Mathematically, this is given by

$$JDF = \Phi_2(\Phi(-EDF_1), \Phi(-EDF_2), \rho_A),$$
(38)

where ρ_A is the correlation between firm 1 asset return and firm 2 asset return and Φ is the standard normal distribution function.

4 A model of joint default probabilities

4.1 Time-varying and correlated Sharpe ratios and volatilities.

The credit spread puzzle is defined as the inability of structural models, when calibrated to default probabilities, loss rates and Sharpe ratios, to predict spread levels across rating categories consistent with historical market spreads (Amato and Remolona, 2003). Chen, Collin-Dufresne and Goldstein (2008) found that the standard Merton model predicted a four-year Baa-Aaa spread that falls well below historical spreads when they calibrated the model to match historical default rates, recovery rates, means and volatilities. They investigated channels through which the predicted spreads could be increased. One of these channels was modeling time-variation in the Sharpe ratio, which is defined as the ratio of the expected equity premium to the expected standard deviation,

$$S = \frac{\mu - r}{\sigma}. \tag{39}$$

Time variation in Sharpe ratios has long been recognized as an important channel for explaining the high risk premium on stocks. It seems therefore natural to investigate how a model consisting of this time-variation, calibrated to match empirical probabilities of default, can explain the joint probabilities of default between two companies. In this paper, we propose a multivariate model to investigate the impact of time varying and correlated means and volatilities on the probabilities of joint default. The analysis is conducted in the standard Merton model framework.

It is an empirical fact that the default rates are highest during recessions. "When times are bad, the default probability is high, (...) and when times are good, the default probability is low" (Bruche and Gonzalez-Aguado, 2006). A company's ability to pay its liabilities depends on the ability to generate profits, which may be sharply impaired in a recession. Helwege and Kleiman (1996) for example investigated the relationship between credit quality and macroeconomic variables on default rates and found that the model improved significantly when the GDP growth was included in the regression. High default rates during recessions are generated in the Merton model through firm value dropping toward the default point.

Also, changes in equity markets are correlated. Longin and Solnik (1995), among others, studied the correlation of monthly excess return for seven major countries over 30 years and found that the correlation matrices where high and unstable over time. Many other papers have also found similar results. Changes in equity values are tightly linked to changes in firm values.

Time-variation in Sharpe ratio could be understood within the framework developed by Cochrane (2001). He states that asset prices should equal expected discounted cash flows. But empirical evidence found that discount rates vary over time. A significant part of the variation in prices is due to news. Following Cochrane (2001), we define the price of an asset, p_t, as

$$p_t = E(m_{t+1}x_{t+1}), \tag{40}$$

where x_{t+1} is the asset payoff and m_{t+1} is the stochastic discount factor. If we divide the payoff, x_{t+1}, by the price, p_t, we get the gross return, R_{t+1}. Thus, equation (40) becomes

$$1 = E(m_{t+1}R_{t+1}). \tag{41}$$

This basic pricing equation should hold for any asset, stock, bond, option, etc. In a standard asset pricing model, a representative investor chooses how much to save and how much to consume subject to his budget constraint. Further, we assume that the investor's utility function, U, is defined over current and future values of consumption,

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t\left[u(c_{t+1})\right]. \tag{42}$$

Here, c_t denotes the consumption at date t and β is the subjective discount factor which capture investor's impatience and aversion to risk. The representative investor is maximizing utility subject to his budget constraints. The maximization problem delivers the first order conditions for an optimal consumption and portfolio choice. Solving these first order conditions for the prices, p_t, gives[3]

$$p_t = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}\right], \tag{43}$$

where $u'(c_t)$ is the marginal utility at time t. We can now define the stochastic discount factor as

$$m_{t+1} \equiv \beta\frac{u'(c_{t+1})}{u'(c_t)} \tag{44}$$

which measures the rate at which the investor is willing to substitute consumption at time t+1 for consumption at time t. By the use of the definition of covariance between two

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[3]See Cochrane (2001) for detailed derivations.

variables and the result that the risk free rate is given by $r = \frac{1}{E[m]}$, the expected excess return of asset i is

$$E(R^i) - r = -\frac{cov\left[u'(c_{t+1}), R^i_{t+1}\right]}{E\left[u'(c_t)\right]}. \tag{45}$$

If an asset has positive correlation with consumption, it will have a negative correlation with marginal utility if $u''(\cdot) < 0$. This is a standard assumption of a utility function. Assets whose returns covary positively with consumption make consumption more volatile and must earn return in excess of the risk free rate to induce investors to hold them.

Campbell and Cochrane (1999) explain why a firm's value drops far more than the drop in expected dividends through recessions. They explain the drop through a variation in investors' attitude toward risk. When the economy goes into recessions, investors have fewer resources to maintain their accustomed living standards and thus are less willing to bear financial risk. To induce them to hold stocks instead of risk free short-term Treasury bills, for a given level of stock market risk, the expected equity premium must increase. Therefore, stock prices fall during recessions because dividends are discounted by a higher rate as a result of the increase in the equity premium. This time-variation in the expected equity premium can be generated in our model by allowing the gross return of the firms to change over time, denoted as the mean, μ.

The expected excess returns are correlated through time and between firms on the basis of macro-conditions. Ferson and Harvey (1991) found that the average risk premium of size and industry-grouped common stock portfolios is associated with their sensitivity to common economic variables. Among a group of economic variables, the risk premium associated with a stock market index captures the largest component of the predictable variation in the stock market return. Other papers, like Lewellen (1997), found that factors like book-to-market predict economically and statistically significant time-variation in expected stock returns. In our multivariate model, we therefore also allow the returns (means) to be correlated across firms and across time.

The standard Merton model assumes constant asset volatility. However, there is strong empirical evidence that the volatility of the stock returns are time-varying. For instance, Schwert (1989) found that asset return are more volatile during a recession. A number of other features of volatility are observed:[4]

- Volatility clustering - the tendency for volatility to appear in bunches. Thus large returns of either signs are expected to follow large returns and vice versa.

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[4]See for example Brooks (2002).

· Leverage effect - the tendency for volatility to rise more following a large price fall than following a price rise of the same magnitude, i.e. stock volatility is negatively correlated with stock returns.

This gives good reasons to allow for time-variation in the volatility. There is also good evidence that volatility is mean-reverting. If standard deviations on equities, T-bills, etc. had been constant, all lines in Figure (5) have been flat. But the lines indicate that the uncertainty is lower for the long run than the short run. The annualized standard deviation of the return on equities is 18% for one year horizon, whereas it is 14% with a horizon of 25 years. This means that the standard deviation on equities has historically been mean reverting toward a long run mean.
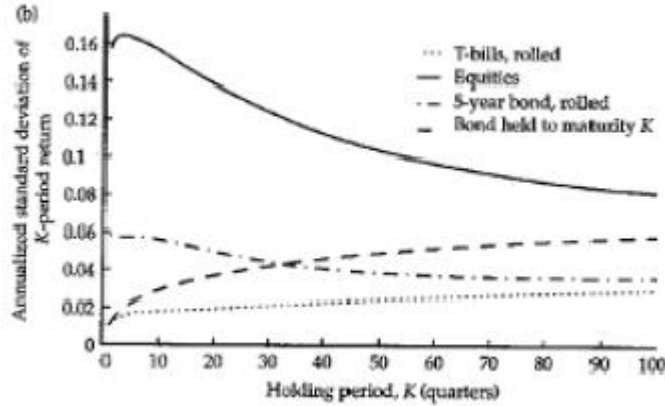

Figure 5: Variability of multi-period asset returns. Quarterly data, 1952(1)-1999(4). Source: Campbell and Viceira (2002)

The characteristics of the volatility outlined above have been for the univariate time series, relating the volatility of the series to only information contained in that series' history. But financial asset prices do not evolve independently of the market around them. Other assets may therefore contain relevant information for the volatility of a series. Such evidence has been found by for example Engle, Ng and Rothschild (1990). In addition, it is possible that deterministic events also have an impact on the volatility series. In our model, we therefore also allow for time-varying volatilities which may be correlated between firms.

4.2 The proposed model

We consider two firms and specify the dynamics of firm i as

$$\frac{dV^i(t)}{V^i(t)} = \mu_i(t)dt + \sigma_i(t)dB^i(t), \quad V_0^i > 0; \quad for \ i = 1, 2. \tag{46}$$

$(B^1(t), B^2(t))$ is a two-dimensional correlated Brownian motion with variance-covariance matrix given by

$$\Sigma_B = \begin{pmatrix} 1 & \rho_V \\ \rho_V & 1 \end{pmatrix}, \tag{47}$$

where ρ_V is the correlation between the two Brownian motions. The difference from the previous literature approach is that we allow the drift and volatility terms to be time-varying and correlated. The means in the firms' dynamics are assumed to follow a bivariate geometric Brownian motion (GBM)

$$\begin{bmatrix} d\mu_1(t) \\ d\mu_2(t) \end{bmatrix} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \begin{pmatrix} \mu_1(t) \\ \mu_2(t) \end{pmatrix} dt + \begin{pmatrix} \mu_1(t) & 0 \\ 0 & \mu_2(t) \end{pmatrix} \begin{pmatrix} \sigma_{11}^\mu & \sigma_{12}^\mu \\ \sigma_{21}^\mu & \sigma_{22}^\mu \end{pmatrix} \begin{pmatrix} dZ^1(t) \\ dZ^2(t) \end{pmatrix}. \tag{48}$$

$(Z^1(t), Z^2(t))$ is a two-dimensional correlated Brownian motion with variance-covariance matrix given by

$$\Sigma_Z = \begin{pmatrix} 1 & \rho_\mu \\ \rho_\mu & 1 \end{pmatrix}, \tag{49}$$

where ρ_μ is the correlation between the two Brownian motions. β_1 and β_2 are the expected drift rates per unit of time for μ_1 and μ_2. $\Sigma^\mu = \begin{pmatrix} \sigma_{11}^\mu & \sigma_{12}^\mu \\ \sigma_{21}^\mu & \sigma_{22}^\mu \end{pmatrix}$ is the volatility matrix of the bivariate process.

The volatilities are assumed to follow a bivariate CIR process

$$\begin{bmatrix} d\sigma_1(t) \\ d\sigma_2(t) \end{bmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \end{pmatrix} \end{bmatrix} + \begin{pmatrix} \sqrt{\sigma_1(t)} & 0 \\ 0 & \sqrt{\sigma_2(t)} \end{pmatrix} \begin{pmatrix} \sigma_{11}^\sigma & \sigma_{12}^\sigma \\ \sigma_{21}^\sigma & \sigma_{22}^\sigma \end{pmatrix} \begin{pmatrix} dW^1(t) \\ dW^2(t) \end{pmatrix}. \tag{50}$$

$(W^1(t), W^2(t))$ is a two-dimensional correlated Brownian motion with variance-covariance matrix given by

$$\Sigma_W = \begin{pmatrix} 1 & \rho_\sigma \\ \rho_\sigma & 1 \end{pmatrix}, \tag{51}$$

where ρ_σ is the correlation between the two Brownian motions. $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ is the long run mean of the volatility vector. Therefore, the matrix $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ can be interpreted as the mean rate of reversion to the long run means of the volatility. The size of the parameters controls how fast the drift coefficients change. $\Sigma^\sigma = \begin{pmatrix} \sigma_{11}^\sigma & \sigma_{12}^\sigma \\ \sigma_{21}^\sigma & \sigma_{22}^\sigma \end{pmatrix}$ is the volatility matrix of the bivariate process.

Since we consider the Merton framework, we assume that default happens only at maturity of the debt if the firm value is below the default point. The probability of bankruptcy at time T for firm i, conditional on the value of assets at time t is

$$P(V^i(T) < D^i | V^i(t)) = \Phi\left[-\frac{log(V^i(t)/D^i) + (\mu_i(t) - \frac{1}{2}\sigma_i^2(t))(T - t)}{\sigma_i(t)\sqrt{T - t}} \right] = \Phi\left[-DD_i(t) \right]. \tag{52}$$

The mean, μ_i, and the volatility, σ_i, are now assumed to be functions of time and are given by the equations (48) and (50).

The joint probability of firm 1 to default at T_1 and firm 2 to default at T_2 is

$$P(T_1, T_2) = p(V^1(T_1) < D_1, V^2(T_2) < D_2)$$

$$= \Phi_2\left[r, -\frac{log(V^1(t)/D_1) + (\mu_1(t) - \frac{1}{2}\sigma_1^2(t))T_1}{\sigma_1(t)\sqrt{T_1}}, -\frac{log(V^2(t)/D_2) + (\mu_2(t) - \frac{1}{2}\sigma_2^2(t))T_2}{\sigma_2(t)\sqrt{T_2}} \right], \tag{53}$$

where r is the correlation coefficient.

The model can be extended to account for multiple firms, but in this paper we will only consider the bivariate case.

5 Monte Carlo analysis

5.1 Calibration strategy

The benchmark calibration in this paper follows mostly the work in Zhang, Zhou and Zhu (2009). We calibrate the parameters to fit the sample averages of one firm with rating category A (Firm 1) and one firm with rating category Ba (Firm 2). Both firm values, V_1 and V_2, are initially set to 100.

We set the starting value for the volatility process of Firm 1, σ_{10}, to 24.65%, the long run asset volatility, θ_1, to 4.24% and the mean reversion coefficient, α_1, to 0.74. Historical leverage ratio is 43.29% for A-rated companies. In the original Merton framework, the default point is set to the face value of debt. However, Leland (2004) argues that the default boundary is close to 75% of the face value of the debt. This agrees with Moody's KMV. They found that the default point generally lies somewhere between total liabilities and short term liabilities (Crosbie and Bohn, 2003). The default point, D^1, is therefore set to 32.47. The starting value for the mean process, μ_{10}, is set to 9%. This reflects a risk-free rate, r, of 5%, a risk premium, $E(R^1) - r$, of 6% and a payout ratio, δ_1, of 2%.

We set the starting value for the volatility process of Firm 2, σ_{20}, to 30.27%, the long run asset volatility, θ_2, to 4.9% and the mean reversion coefficient, α_2, to 0.8. The historical leverage ratio is 58.63% for Ba-rated companies. The default point, D^2, is therefore set to 75% of the leverage ratio, which is 43.97. The starting value for the mean process, μ_{20}, is set to 11.5%. This reflects a risk-free rate, r, of 5%, a risk premium, $E(R^2) - r$, of 8.5% and a payout ratio, δ_2, of 2%.

The basic strategy for the remaining parameters in the model are to calibrate them such that the Monte Carlo simulations gives approximately 5-year and 10-year default probabilities that correspond to the empirical default probabilities, as reported by Zhou (2001). The 5-year default probabilities are 0.62% for an average A-firm and 11.85% for an average Ba-firm. The 10-year default probabilities are 1.96% for an average A-firm and 19.48% for an average Ba-firm. By trial and error, the correlations between the firm values, ρ_V, means, ρ_μ, and volatilities, ρ_σ, are set to 0.1, 0.1 and 0.2, respectively. The volatility matrices of the bivariate processes driving the means and the volatilities are set to $\Sigma^\mu = \begin{pmatrix} 0.2 & 0.25 \\ 0.5 & 0.2 \end{pmatrix}$ and $\Sigma^\sigma = \begin{pmatrix} 0.025 & 0.03 \\ 0.03 & 0.05 \end{pmatrix}$. The drifts of the mean processes, β_1 and β_2, are set to 0.77 and 0.3.

Table (1) from the Appendix displays a summary of the benchmark calibration.

5.2 Monte Carlo algorithm

We used Matlab R2011a for simulation of the model. For an initial calibration, we set the time to maturity, T, to 10 years and the total number of steps within these 10 years, N, to 100. The number of simulations for each j steps where j=1:N, Nsim, are set to 1,000,000. The main steps of the Monte Carlo algorithm are as follows:

1. Generate a 2xN matrix for the two-dimensional correlated Brownian motion driving the firm values,

2. Generate a 2xN matrix for the two-dimensional correlated Brownian motion driving the means,

3. Generate a 2xN matrix for the two-dimensional correlated Brownian motion driving the volatilities,

4. For each j step, generate Nsim means, volatilities and values for each firm. The processes for those are given by (48), (50) and (46) respectively,

5. For each j step, we count for each firm how many times, out of Nsim simulations, the firm value is bellow the default point,

6. For each j step, we count how many times, out of Nsim simulations, both firm values are bellow their default points,

7. For each j step, we calculate the probability of default for each firm and the joint probability of default by dividing the numbers obtained in steps 5 and 6 by Nsim.

The probabilities are plotted as a function of time to maturity. See the Appendix for the Matlab code.

5.3 Simulation results

5.3.1 Benchmark calibration

The benchmark calibration matches the sample averages of one firm with rating category A and one firm with rating category Ba. Figure (6) plots the probability of joint default from the proposed model for a horizon of 10 years to maturity. The joint probability of default varies significantly over different horizons. One can see that the probability of joint default is generally very small over the short horizon because quick defaults are rare and nearly

idiosyncratic. Then it increases and finally it slowly decreases as it approaches the maturity date. The decrease in the joint probability of default over longer horizons may be due to the relationship of the time period being studied to the average business cycle. If the time period studied covers the entire ebb of the business cycle, the defaults caused by the general economic conditions average out over the period, thus lowering the default correlation.
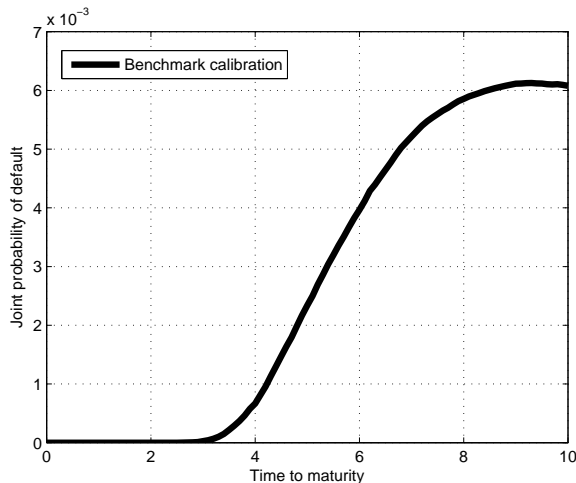

Figure 6: The joint probability of default as a function of time to maturity for the benchmark calibration.

5.3.2 Two A-rated firms and two Ba-rated firms

We can also calibrate the proposed model such that it matches the sample averages of two A-rated firms and two Ba-rated firms. We set both firm values, V_1 and V_2, to 100 as before.

For the calibration "Two A-rated firms", we set the starting values for the volatilities, σ_{10} and σ_{20}, to 23% and 26% with both long run asset volatilities, θ_1 and θ_2, to 4.24% and both mean reversion coefficients, α_1 and α_2, to 0.74. The default points, D^1 and D^2, are set to 30 and 35. Both starting values for the means, μ_{10} and μ_{20}, are set to 9% and the drifts of the asset means, β_1 and β_2, are set to 0.77. The remaining parameters are similar to the benchmark calibration.

For the calibration "Two Ba-rated firms", we set the starting values for the volatilities, σ_{10} and σ_{20}, to 29% and 32% with both long run asset volatilities, θ_1 and θ_2, to 4.9% and both mean reversion coefficients, α_1 and α_2, to 0.8. The default points, D^1 and D^2, are set to 42 and 45. Both starting values for the means, μ_{10} and μ_{20}, are set to 11.5% and the drifts of the asset means, β_1 and β_2, are set to 0.3. The remaining parameters are similar to the benchmark calibration.

Figure (7) shows the probability of joint default from the proposed model for two A-rated firms and two Ba-rated firms. One could see that the probability of joint default is generally low for A-rated companies. This is mainly due to the fact that when the credit quality increases, the probability of default decrease. Furthermore, companies with high rating are more susceptible to company-specific problems (Lucas, 1995). Thus, defaults are typically isolated to the individual company and do not produce so much default correlation.

The probability of joint default for two Ba-rated companies is significantly larger compared to the benchmark calibration and the case of two A-rated companies. Lucas (1995) argues that lower rated firms have asset values closer to their default points and are relatively more susceptible to problems in the general economy. They are therefore more likely to be pushed into default because of an economic downturn. As economic conditions affect all low-rated credits simultaneously, defaults among these firms are more likely to be correlated.
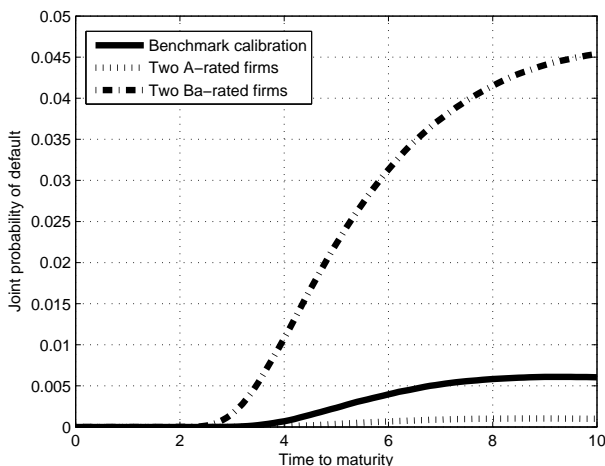

Figure 7: The joint probability of default as a function of time to maturity for two A-rated firms and two Ba-rated firms.

5.4 Sensitivity analysis

To analyze the dynamics of the proposed model, we investigate how increases and decreases in the values for the parameters of the proposed model impact the probability of joint default.

5.4.1 Changes in D^1 and D^2

Figures (8) and (9) show the joint probability of default as a function of time to maturity for different default points for Firm 1 and Firm 2, respectively.

When the default point of one of the firms increases (decreases), the joint probability of default increases (decreases). This is because the firm's asset value is closer to the default point and the probability of default increases. Thus the probability of joint default also increases.

With the same intuition, if one of the firms asset value increases (decreases), the joint probability of default decreases (increases).
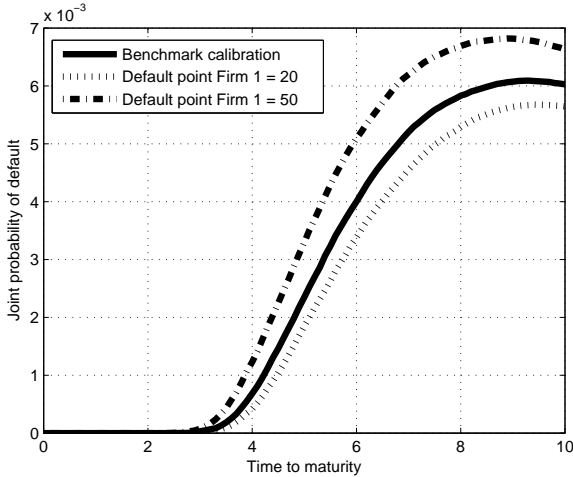

Figure 8: The joint probability of default as a function of time to maturity for different D^1.
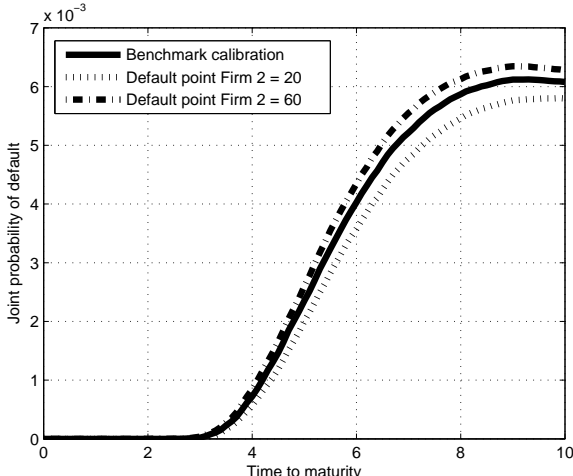
Figure 9: The joint probability of default as a function of time to maturity for different D^2.

5.4.2 Changes in ρ_V, ρ_μ and ρ_σ

Figures (10)-(12) display the joint probability of default as a function of time to maturity for different correlations between the firm values, means and volatilities, respectively.

In general, one could see that higher (lower) correlation between the firm values, the means or the volatilities implies higher (lower) joint probability of default. If one firm defaults, it is more likely that the value of the other firm also have declined and moved closer to its default point given that the firm values, the means or the volatilities are highly correlated.

If the firm values, means or volatilities are negatively correlated, the probability of joint default generally decreases. This is because the default correlation and asset level correlation have the same sign.
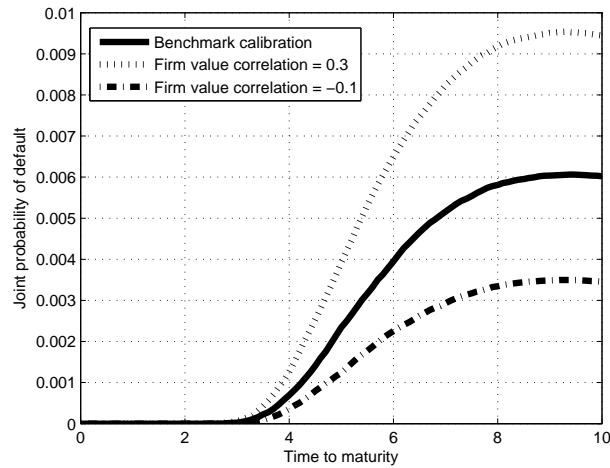

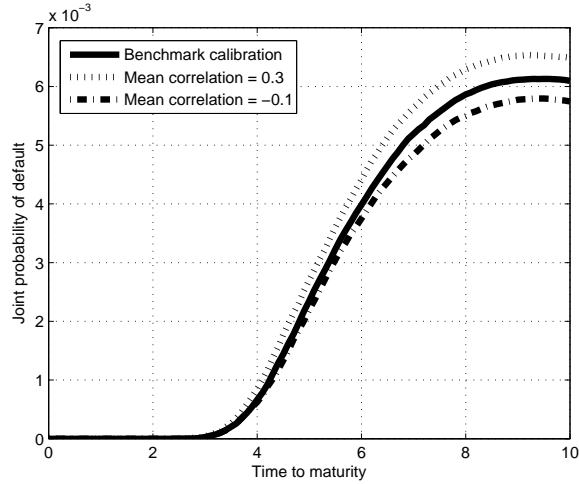Figure 10: The joint probability of default as a function of time to maturity for different ρ_V.

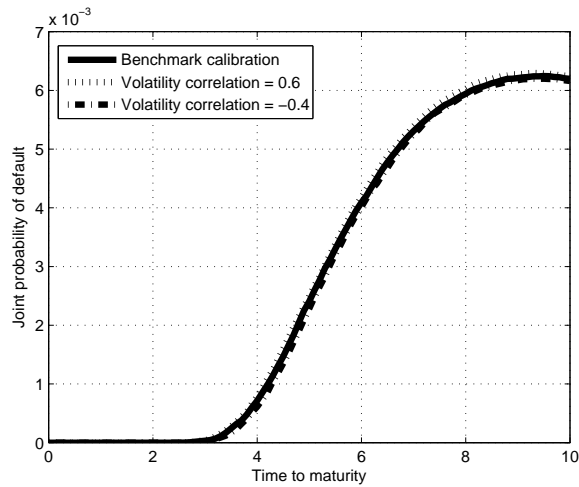Figure 11: The joint probability of default as a function of time to maturity for different ρ_μ.

Figure 12: The joint probability of default as a function of time to maturity for different ρ_σ.

5.4.3 Changes in σ_{10} and σ_{20}

Figures (13) and (14) show the joint probability of default as a function of time to maturity when different starting values for the volatilities are considered for Firm 1 and Firm 2, respectively.

One could notice that higher (lower) initial volatility generally implies higher (lower) joint probability of default. Higher uncertainty for one of the firms increases the probability that the firm will default. This increases the joint probability of default, given the correlation structure.
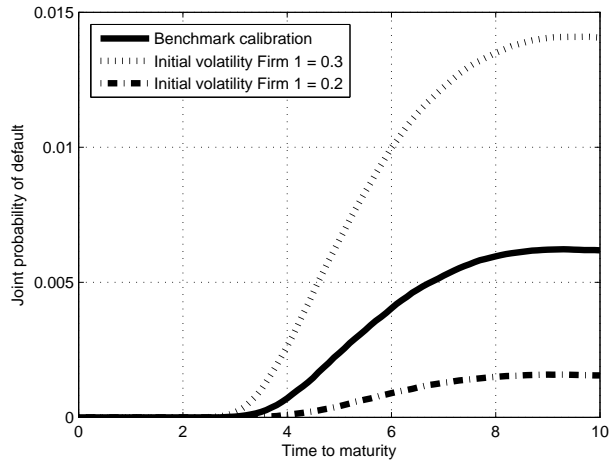
Figure 13: The joint probability of default as a function of time to maturity for different σ_{10}.
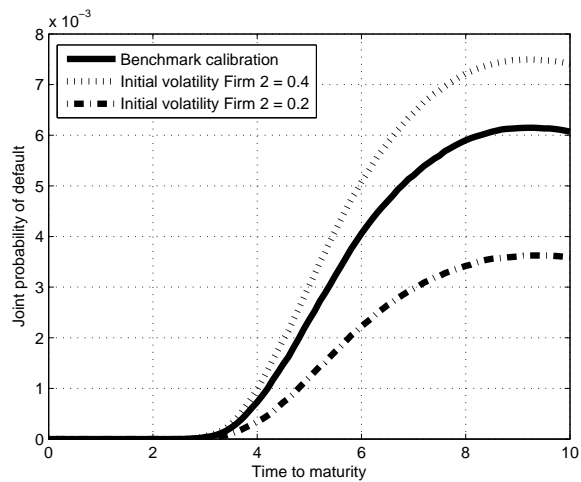

Figure 14: The joint probability of default as a function of time to maturity for different σ_{20}.

5.4.4 Changes in θ_1 and θ_2

Figures (15) and (16) show the joint probability of default as a function of time to maturity for different long run means of the volatilities of the two firms.

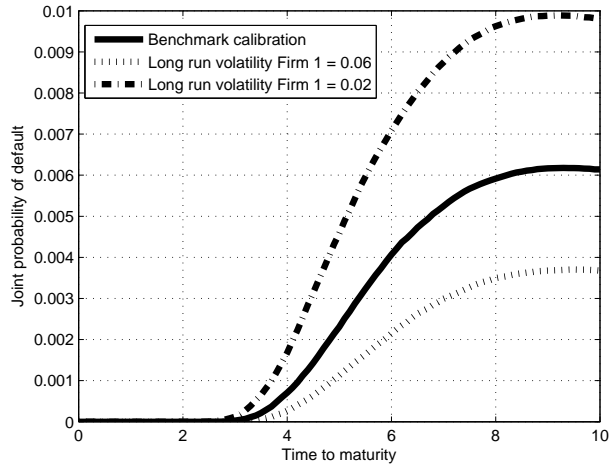Higher (lower) long run volatility implies lower (higher) joint probability of default.

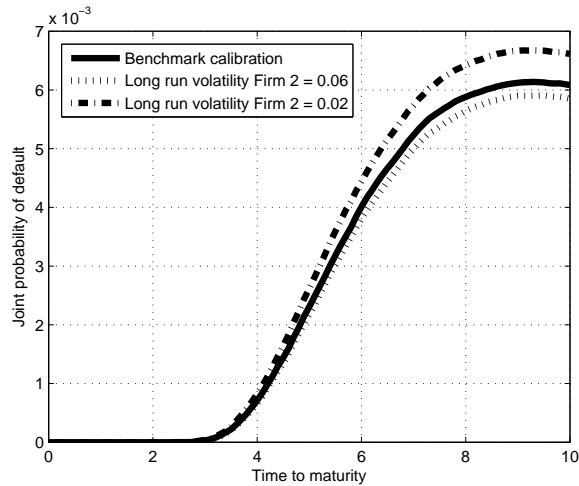Figure 15: The joint probability of default as a function of time to maturity for different θ_1.

Figure 16: The joint probability of default as a function of time to maturity for different θ_2.

5.4.5   Changes in α_1 and α_2

Figures (17) and (18) show the joint probability of default as a function of time to maturity for different mean rate of reversion of the two firms.

Higher (lower) mean rate of reversion implies lower (higher) joint probability of default. Higher mean rate of reversion implies that the volatility stabilizes around the long run volatility faster and thus lowers the joint probability of default.
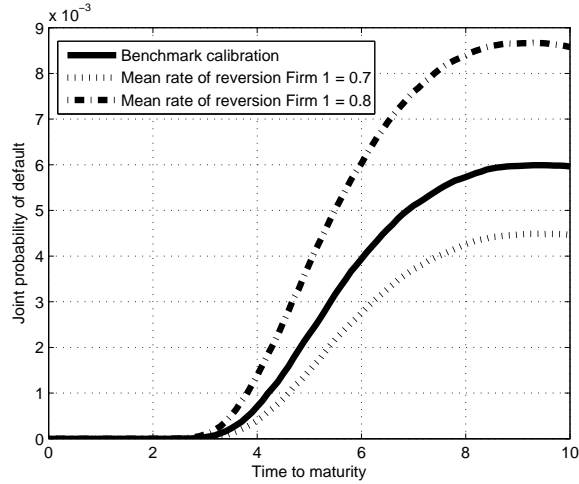
Figure 17: The joint probability of default as a function of time to maturity for different α_1.

Figure 18: The joint probability of default as a function of time to maturity for different α_2.

5.4.6 Changes in σ_{11}^{σ} and σ_{12}^{σ}

Figures (19) and (20) display the joint probability of default as a function of time to maturity for different σ_{11}^{σ} and σ_{12}^{σ}.

One could see that higher (lower) σ_{11}^{σ} and σ_{12}^{σ} implies higher (lower) joint probability of default. Higher volatility of the volatility process increases the uncertainty and thus increases the joint probability of default.

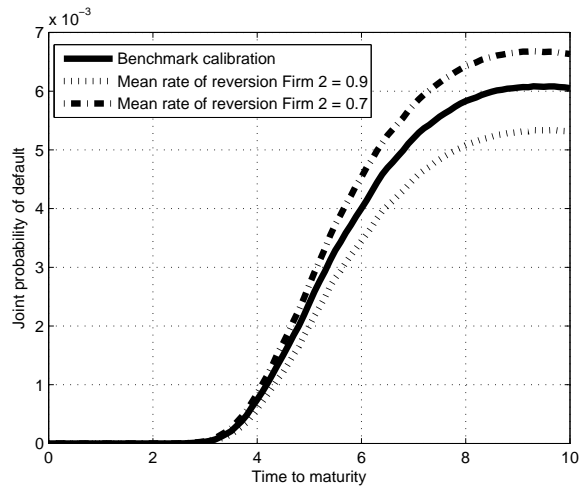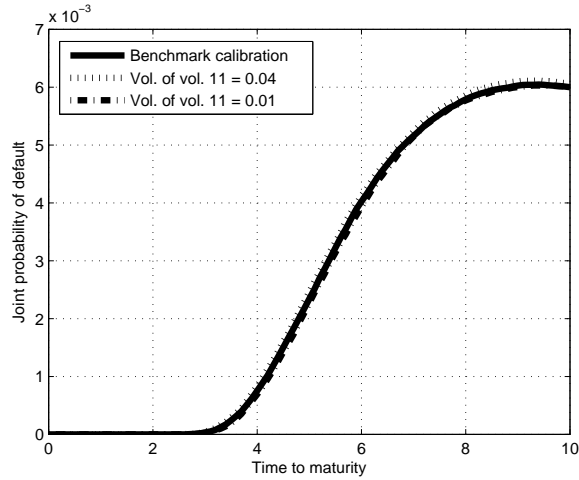Figure 19: The joint probability of default as a function of time to maturity for different σ_{11}^{σ}.

Figure 20: The joint probability of default as a function of time to maturity for different σ_{12}^{σ}.

5.4.7 Changes in σ_{21}^{σ} and σ_{22}^{σ}

Figures (21) and (22) show the joint probability of default as a function of time to maturity for different σ_{21}^{σ} and σ_{22}^{σ}.

One can notice that higher (lower) σ_{21}^{σ} and σ_{22}^{σ} implies higher (lower) joint probability of default. Same as above, higher volatility of the volatility process increases the uncertainty and thus increases the joint probability of default.

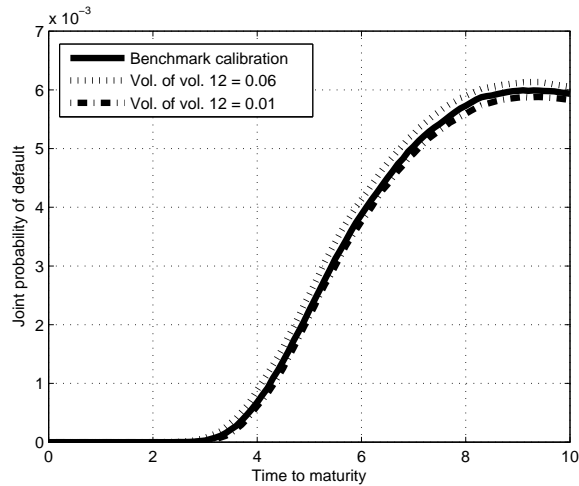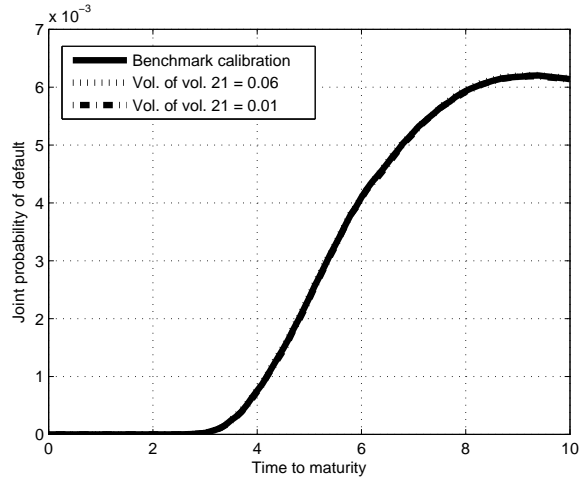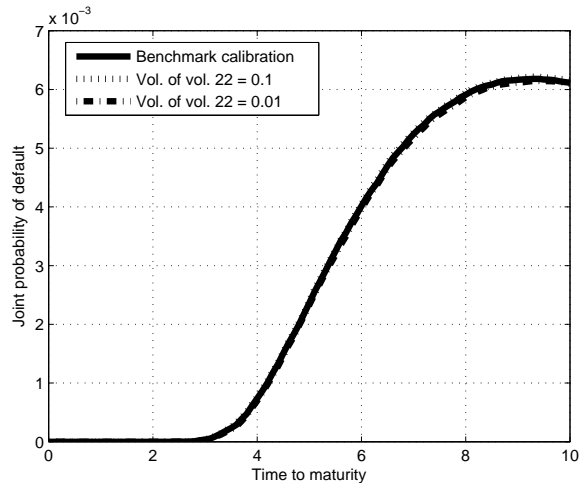Figure 21: The joint probability of default as a function of time to maturity for different σ_{21}^{σ}.

Figure 22: The joint probability of default as a function of time to maturity for different σ_{22}^{σ}.

5.4.8 Changes in μ_{10} and μ_{20}

Figures (23) and (24) show the joint probability of default as a function of time to maturity when different starting values for the means of Firm 1 and Firm 2 are considered.

Higher (lower) mean implies lower (higher) joint default probability. When the expected return of one of the firms increases, there is less likely that the asset value of the firm will move toward the default boundary. This implies a lower joint probability of default.
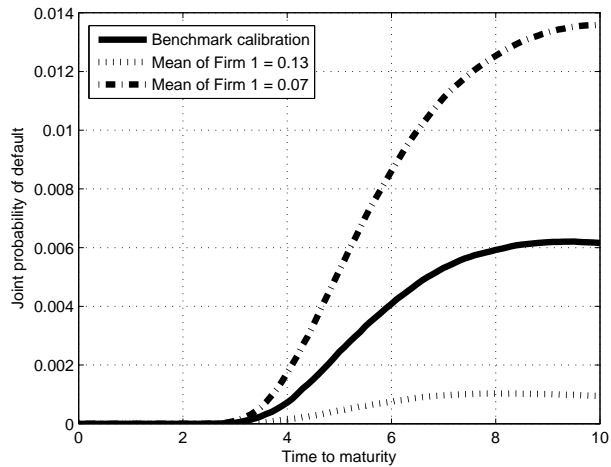
Figure 23: The joint probability of default as a function of time to maturity for different μ_{10}.

Figure 24: The joint probability of default as a function of time to maturity for different μ_{20}.

5.4.9 Changes in β_1 and β_2

Figures (25) and (26) show the joint probability of default as a function of time to maturity for different expected drift rates of the mean processes of the two firms.

One could see that higher (lower) expected drift rates for the mean processes imply lower (higher) joint probability of default. This is because higher expected drift rate for the mean process of one of the firms implies a lower probability of default. This decreases the probability of joint default.

Figure 25: The joint probability of default as a function of time to maturity for different β_1.

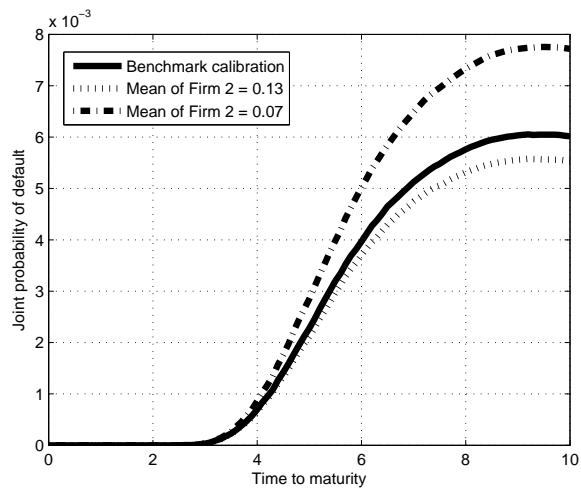Figure 26: The joint probability of default as a function of time to maturity for different β_2.

5.4.10 Changes in σ_{11}^{μ} and σ_{12}^{μ}

Figures (27) and (28) show the joint probability of default as a function of time to maturity for different σ_{11}^{μ} and σ_{12}^{μ}.

Higher (lower) σ_{11}^{μ} and σ_{12}^{μ} implies higher (lower) joint probability of default. Higher volatility of the mean process increases the uncertainty and thus increases the joint probability of default.
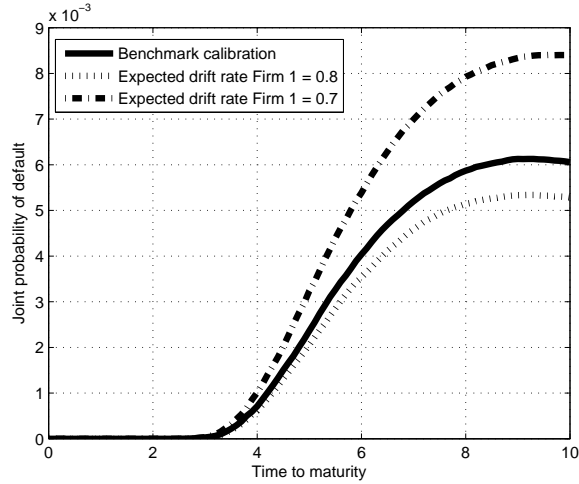
Figure 27: The joint probability of default as a function of time to maturity for different σ_{11}^{μ}.

Figure 28: The joint probability of default as a function of time to maturity for different σ_{12}^{μ}.

5.4.11 Changes in σ_{21}^{μ} and σ_{22}^{μ}

Figures (29) and (30) show the joint probability of default as a function of time to maturity for different σ_{21}^{μ} and σ_{22}^{μ}.

Higher (lower) σ_{21}^{μ} and σ_{22}^{μ} implies higher (lower) joint probability of default. As before, higher volatility of the mean process increases the uncertainty and thus increases the joint probability of default.
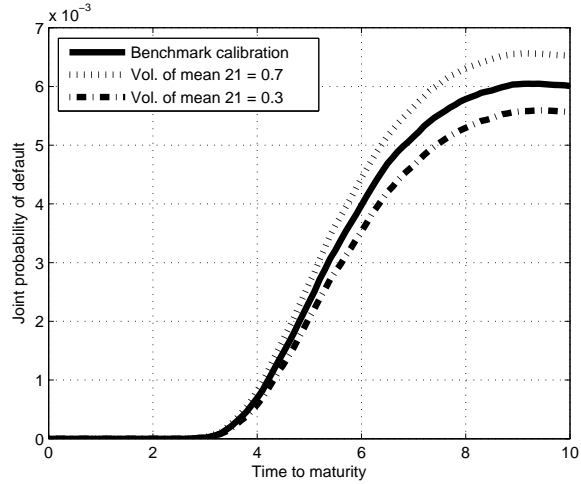
Figure 29: The joint probability of default as a function of time to maturity for different σ_{21}^{μ}.
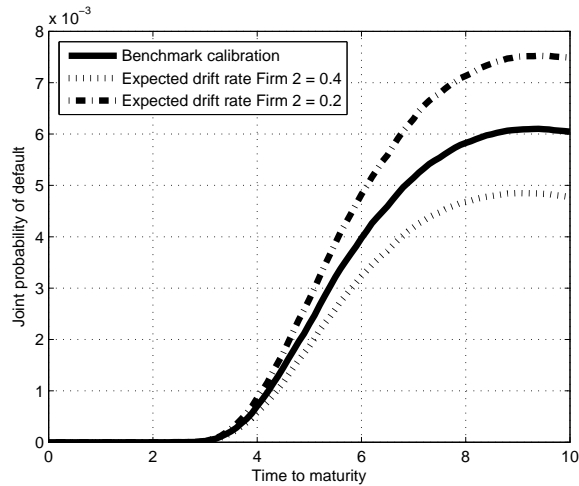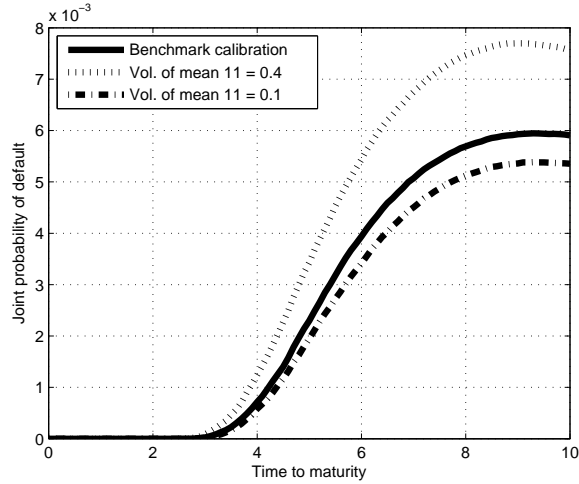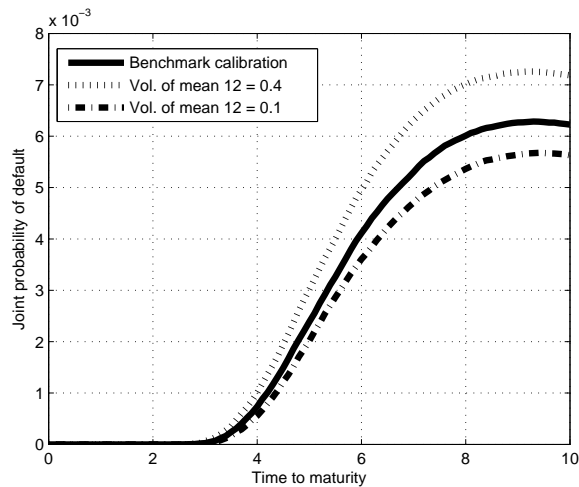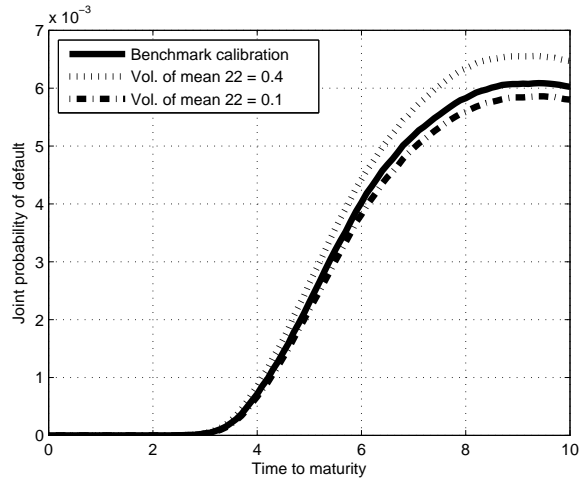

Figure 30: The joint probability of default as a function of time to maturity for different σ_{22}^{μ}.

5.4.12   Changes in T

Figure (31) shows the joint probability of default for different T.

In the benchmark calibration, the time to maturity is set to 10 years. One could see that there is no change in the probability of joint default within these 10 years when the time to maturity is changed to 5 years or 20 years. Further, the probability of joint default continues to decrease when the time to maturity exceed 10 years.
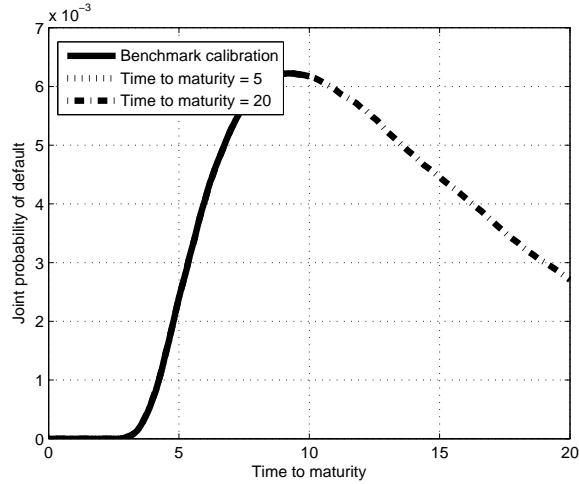
Figure 31: The joint probability of default for different T.

5.4.13 Changes in N and Nsim

Figures (32) and (33) show the joint probability of default for different N and Nsim.

A higher number of time steps, N, will decrease the length of each step. One could see that the number of steps does not impact significantly the shape of the probability of joint default. Further, increasing the number of simulations for each step, Nsim, will give more accurate approximations and smoother looking curves because of the central limit theorem.

Figure 32: The joint probability of default for different N.

Figure 33: The joint probability of default for different Nsim.

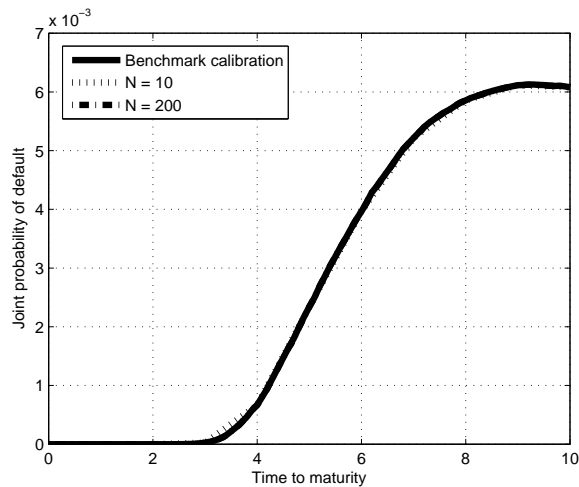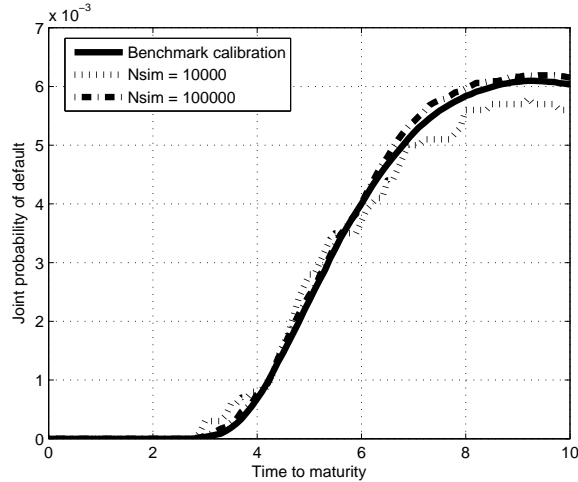## 5.5 Comparison of the proposed model and the standard model

The proposed model incorporates time-varying and correlated processes for the means and volatilities driving firm values. To see how this affects the probability of joint default, we compare the proposed model with a standard model which only capture time-varying and correlated firm values. Thus, the standard model has constant and independent means and volatilities. The joint probability of default from the standard model is simulated within the same Monte Carlo algorithm as in the proposed model.

The standard model is calibrated within the lines of the benchmark calibration. Both firm values, V_1 and V_2, are initially set to 100. The default points, D^1 and D^2, are set to 32.47 and 43.97. The volatility of the firms, σ_1 and σ_2, are set to 24.65% and 30.27%. Finally, the means, μ_1 and μ_2, are set to 9% and 11.5%. The correlation between the firm values, ρ_V, is set to 0.1.

In Figure (34), the joint probability of default from the proposed model and the standard model are plotted. One could see that the joint probability paths from the two models are significantly different. The standard model delivers a larger joint probability of default at short horizons, but a lower joint probability of default for longer time horizons. It also has an earlier peak.

The lower values for the joint probability of default generated by the standard model is mainly due to lower correlation between the firms. Because means and volatilities now are constant, the standard model does not produce so high correlation between the firm values and thus lower the probability that both firms will default simultaneously.

Further, the proposed model produces means that on average slowly increase over the horizon because of the positive drift parameters. The standard model has constant means set to the starting values in the proposed model (μ_{10} and μ_{20}). Due to this feature, the probabilities of default are driven down for longer horizons in the standard model as opposed to the proposed model.

Finally, as one could see in Figure (35) and (36), the proposed model simulate firm values that have greater dispersion than the standard model for a horizon of 5 years. This effect leads to a sharply increase in the probability of joint default for the proposed model.
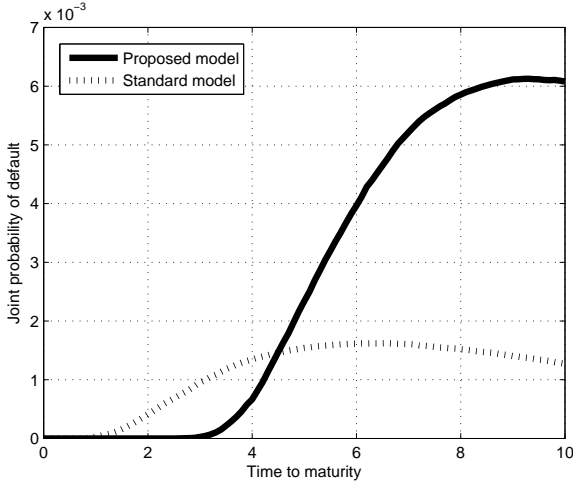

Figure 34: The joint probability of default as a function of time to maturity with the proposed model and the standard model.
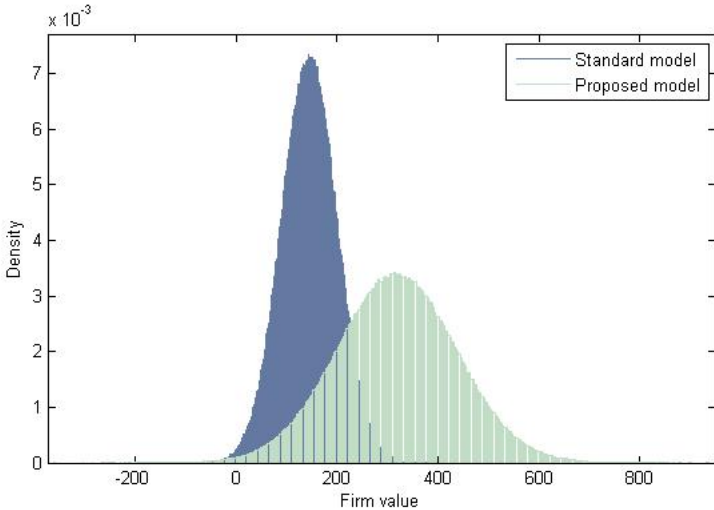

Figure 35: Firm value distribution for Firm 1 with the proposed model and the standard model for a horizon of 5 years.
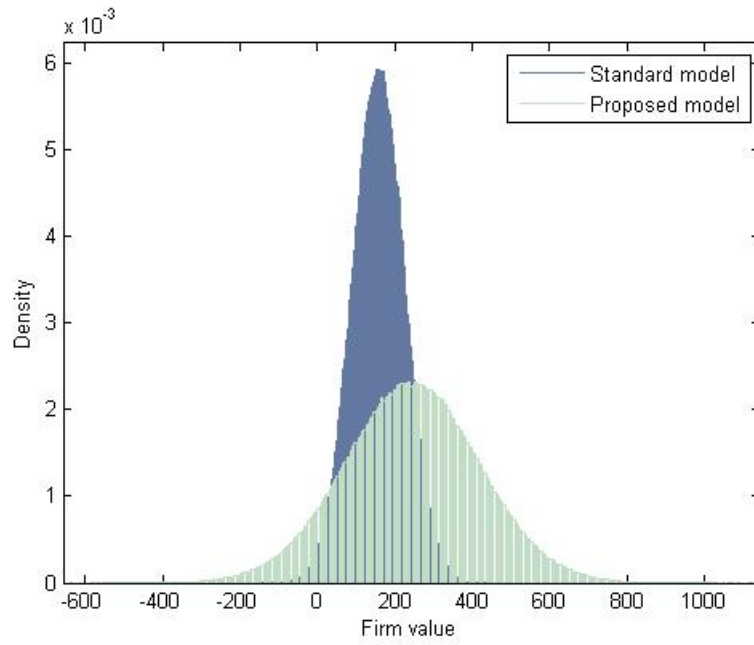
Figure 36: Firm value distribution for Firm 2 with the proposed model and the standard model for a horizon of 5 years.

6 The implications and applications of the proposed model

6.1 Risk management implications of the proposed model

The proposed model implies that the joint probability of default is small for short horizons. Thus, for loan portfolios consisting of short term credit (0-4 years), risk management should mainly be focused upon default probabilities. But for loan portfolios consisting of long-term credit (i.e. 5 to 10 years), the joint probability of default can be a significant factor. Also, the proposed model implies that the joint probability of default is generally large for portfolios consisting of low-rated companies. Risk management requires models that are able to capture these effects because changes in the joint probability of default will affect the loss distribution for the portfolio.

Since lower correlation between asset values, means or volatilities generally implies a lower joint probability of default, diversification between regions and industries should lower the probability of joint default for loan portfolios.

If the default probability of two loans doubles, the probability of joint default can be significantly more than double. Because of this dynamic nature of the probability of joint default, active management is required. Changes in the parameters of the proposed model (volatility, correlation, default point etc.) can significantly influence the joint probability of default.

6.2 The proposed model compared with empirical data

In order to compare the proposed model with empirical data, we need firm specific information for two firms. Due to the lack of readily statistics on firm-specific default correlations, we use pooled data to see if the proposed model can match average empirical joint default probabilities between and within the rating categories A and Ba. These are computed using historical cumulative default rates provided by Moody's Investors Service covering twenty-four years of annual data from 1970 through 1993 as reported in Zhou (2001). The empirical joint default probabilities are computed and reported by Lucas (1995) from the same data set. The annual values are reported in Table (2) in the Appendix.

The cross-sectional diversity of the pool represent over 4 000 issuers, but the time series of the data is short. Lucas (1995) states that twenty-four time periods are studied for the one-year default correlation statistic, but only fifteen overlapping time periods for the ten-year statistic (1970 through 1979, 1971 through 1980,...,1984 through 1993). Thus, this pooled data could give biased estimates of joint default probabilities.

Furthermore, this comparison with empirical data depends on the true underlying default probabilities for each rating category and that the default correlations remain the same over time. However, this is not a pattern observed in time series.

Also, it is hard to say anything about default correlation among and between specific industries. The statistics used here are based on diversified portfolios, i.e. the whole universe of Moody's-rated companies. Thus, it can be considered as an estimate of the base default correlation caused by issuer reaction to the general economy (Lucas, 1995).

For this exercise, the calibration of the model follows mostly Zhang, Zhou and Zhu (2004). However, some of the parameters are chosen such that the proposed model generates approximately 5-year and 10-year default probabilities that correspond to the cumulative default rates provided by Moody's Investors Service. Figures (37), (38) and (39) compare the model simulated joint probability of default with one A-rated firm and one Ba-rated firm, two Ba-rated firms and two A-rated firms with the empirical ones.

For the case with one A-rated firm and one Ba-rated firm, shown in Figure (37), the model simulated joint probability of default matches the empirical data well. Also for the case with two Ba-rated firms, shown in Figure (38), the model matches the empirical data well.

For the case with two A-rated firms, shown in Figure (39), the model simulated joint probability of default falls below the empirical data. There are several potential explanations for this. In the first place, the values for the parameters of the model could be misspecified. Most of the parameters are tied down to historical values, but some of them are chosen such that the model generates 5-year and 10-year default probabilities which correspond to the cumulative default rates. Further, the model itself could be misspecified. We propose a new way to model the stochastic processes for the assets, but it is possible that the model does not capture all patterns in the asset dynamics. Finally, default rates are in general very small for A-rated companies. Thus, the short time period considered in this data set can give wrong estimates of default probabilities and default correlation.

Figure 37: Model simulated joint probability of default compared with empirical joint probability of default for an average A-rated firm and an average Ba-rated firm. Source: Lucas (1995), Zhou (2001).

Figure 38: Model simulated joint probability of default compared with empirical joint probability of default for two average Ba-rated firms. Source: Lucas (1995), Zhou (2001).
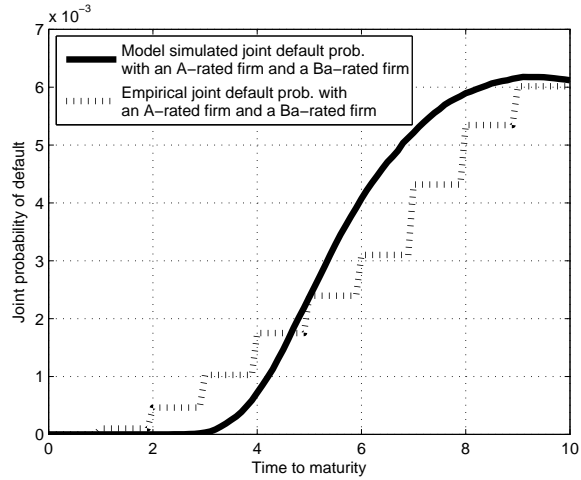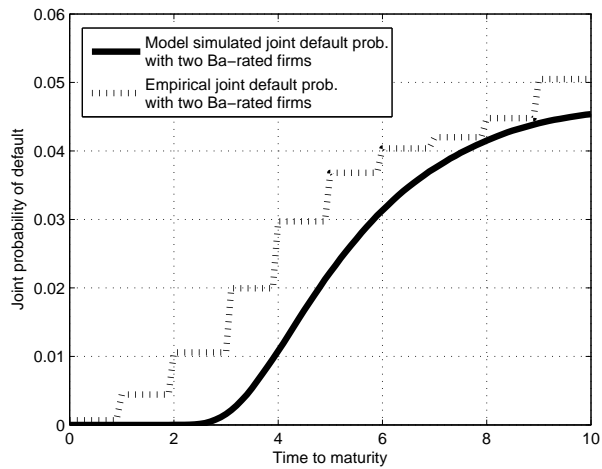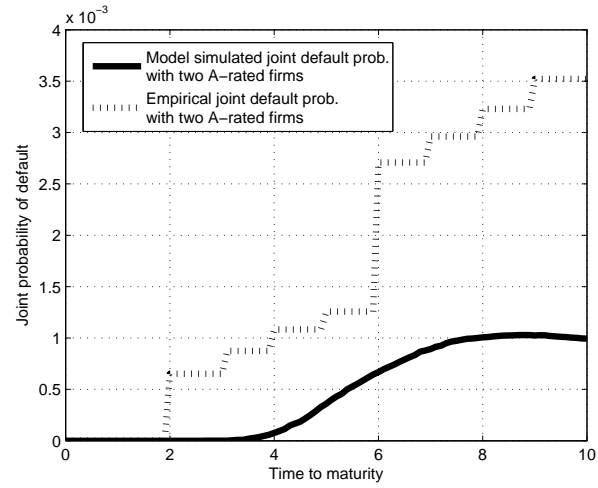
Figure 39: Model simulated joint probability of default compared with empirical joint probability of default for two average A-rated firms. Source: Lucas (1995), Zhou (2001).

7 Conclusion

In this paper, we proposed a multivariate stochastic process for firm values that fits better the observed features of the financial data than the diffusion-type stochastic processes assumed in the standard Merton model. The proposed model incorporates two important features: the fact that the Sharpe ratios and the volatilities of firms are time-varying and correlated over time. Within the standard Merton framework, we calibrated the proposed model to fit sample averages between and within the rating categories A and Ba. By Monte Carlo simulations, we investigated the dynamics of the proposed model with respect to the joint probability of default.

The joint probability of default is important in understanding and predicting the behavior of credit portfolios. If default events are correlated among firms, the joint probability of default can be a significant source of risk for a loan portfolio. Defaults can be more heavily clustered than envisioned in the default correlation models currently used by financial institutions. The financial crisis led to bank failures, mainly because of losses that exceeded the worst estimates. Consequently, significantly more capital might be required in order to survive default losses, especially at high confidence levels as the ones required by regulation. An understanding of the joint probability of default is also crucial for the rating and risk analysis of structured credit products, such as letter of credit-backed debt or credit default swaps.

There are also major concerns about the calibrations of portfolio and credit risk models. "In particular, estimation of default correlations is difficult because they cannot be directly measured for specific obligors" (Jorion and Zhang, 2009). Current models seem to be unable to reproduce the actual pattern of correlation among default events. Unlike the standard model, we found that our proposed model delivers probabilities of the joint default closer to what is observed in the data.

In this paper we considered only the case of two firms, but the model could be extended to the case of multiple firms. Furthermore, an analysis of how time-varying and correlated means and volatilities for firms' values affect the loss distributions for typically bank portfolios consisting of risky debt, would be of high interest. Future work will investigate in these directions.

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Appendix

Table 1: Benchmark calibration.

Calibrated variables	Firm 1	Firm 2	Calibration details/Literature source
Initial asset value V_1, V_2	100	100	Zhang, Zhou and Zhu (2004)
Initial asset mean μ_{10}, μ_{20} (%)	9	11.5	Zhang, Zhou and Zhu (2004)
Initial asset volatility σ_{10}, σ_{20} (%)	24.65	30.27	Zhang, Zhou and Zhu (2004)
Default point D^1, D^2 (% of firm value)	32.47	43.97	Zhang, Zhou and Zhu (2004), Leland (2004)
Asset mean: drift β_1, β_2	0.77	0.3	Calibrated to match historical default probabilities
Vol. of mean Firm 1 σ_{11}^{μ}, σ_{12}^{μ}	0.2	0.25	Calibrated to match historical default probabilities
Vol. of mean Firm 2 σ_{21}^{μ}, σ_{22}^{μ}	0.5	0.2	Calibrated to match historical default probabilities
Asset vol.: mean reversion α_1, α_2	0.74	0.8	Zhang, Zhou and Zhu (2004)
Asset vol.: long-run mean θ_1, θ_2 (%)	4.24	4.9	Zhang, Zhou and Zhu (2004)
Vol. of vol. Firm 1 σ_{11}^{σ}, σ_{12}^{σ}	0.025	0.03	Calibrated to match historical default probabilities
Vol. of vol. Firm 2 σ_{21}^{σ}, σ_{22}^{σ}	0.03	0.05	Calibrated to match historical default probabilities
Risk-free rate (%)	5	5	Zhang, Zhou and Zhu (2004)
Initial equity premium (%)	6	8.5	Zhang, Zhou and Zhu (2004)
Payout ratio (%)	2	2	Zhang, Zhou and Zhu (2004)
Firm values: Correlation ρ_V	0.1		Calibrated to match historical default probabilities
Means: Correlation ρ_μ	0.1		Calibrated to match historical default probabilities
Volatilities: Correlation ρ_σ	0.2		Calibrated to match historical default probabilities

Table 2: Empirical default probabilities, default correlations and probabilities of joint default between an average A-rated firm and an average Ba-rated firm, between two average A-rated firms and between two average Ba-rated firms. Source: Moody's Investors Service, Lucas (1995).

Year	Historical cumulative default rates $P(D_i(t))$		Variance $Var[D_i]$		Default correlation between A and Ba $Corr[D_A(t), D_{Ba}(t)]$	Default correlation between A and A $Corr[D_{A1}(t), D_{A2}(t)]$	Default correlation between Ba and Ba $Corr[D_{Ba1}(t), D_{Ba2}(t)]$	Joint default prob. between A and Ba $P(D_A(t)=1$ and $D_{Ba}(t)=1)$	Joint default prob. between A and A $P(D_{A1}(t)=1$ and $D_{A2}(t)=1)$	Joint default prob. between Ba and Ba $P(D_{Ba1}(t)=1$ and $D_{Ba2}(t)=1)$
	A-rating	Ba-rating	A-rating	Ba-rating						
1	0,0001	0,0179	0,00009999	0,01757959	0	0	0,02	0,0000179	0,00000001	0,000672002
2	0,0009	0,0438	0,00089919	0,04188156	0,01	0	0,06	0,000100787	0,00000081	0,004431334
3	0,0028	0,0692	0,00279216	0,06441136	0,02	0,01	0,09	0,000461974	0,000651954	0,010585662
4	0,0046	0,0941	0,00457884	0,08524519	0,03	0,01	0,13	0,001025559	0,000873612	0,019936685
5	0,0062	0,1185	0,00616156	0,10445775	0,04	0,01	0,15	0,001749489	0,00183018	0,029710913
6	0,0083	0,1378	0,00823111	0,11881116	0,04	0,01	0,15	0,002394626	0,001257002	0,036810514
7	0,0106	0,1533	0,01048764	0,12979911	0,04	0,02	0,13	0,003100805	0,002708342	0,040374774
8	0,0131	0,1675	0,01292839	0,13944375	0,05	0,02	0,1	0,00431721	0,002960485	0,042000625
9	0,0161	0,1814	0,01584079	0,14849404	0,05	0,02	0,08	0,005345547	0,003229091	0,044785483
10	0,0196	0,1948	0,01921584	0,15685296	0,04	0,02	0,08	0,006014099	0,003521219	0,050495277

Matlab code for the Monte Carlo simulation

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%% BENCHMARK CALIBRATION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Parameters for the Monte Carlo simulation:

T=10;            % the time to maturity
Nsim=1000000;    % the number of simulations for each step considered
N=100;           % the number of steps considered
h=T/N;           % the step size (dt)
l=(0:h:T);       % the vector [0 1h 2h 3h ... Nh] to be used to rescale the x-axis for↙
plotting

% The two brownian motions driving the means of the firms' values are
% correlated and consequently we make draws from a multivariate normal:

mumu=[0 0];
varmu=[1 0.1; 0.1 1];
r2=mvnrnd(mumu,varmu,Nsim);
epsZ1=r2(:,1)';
epsZ2=r2(:,2)';

% The two brownian motions driving the volatilities of the firms' values are
% correlated and consequently we make draws from a multivariate normal:

musigma=[0 0];
varsigma=[1 0.2; 0.2 1];
r1=mvnrnd(musigma,varsigma,Nsim);
epsW1=r1(:,1)';
epsW2=r1(:,2)';

% The two brownian motions driving the values of the firms are correlated
% and consequently we make draws from a multivariate normal:

muV=[0 0];
varV=[1 0.1; 0.1 1];
r3=mvnrnd(muV,varV,Nsim);
epsB1=r3(:,1)';
epsB2=r3(:,2)';

% Parameters from Zhang, Zhou and Zhu (2004) and Leland (2004):

V1=100;          % the initial value of firm 1
V2=100;          % the initial value of firm 2

D1=43.29*0.75;   % the initial value of debt in firm 1 times the default boundary
D2=58.63*0.75;   % the initial value of debt in firm 2 times the default boundary

mu1=0.09;        % the initial value for the drift of firm 1
mu2=0.115;       % the initial value for the drift of firm 2

vol1=0.2465;     % the initial value for the volatility of firm 1
vol2=0.3027;     % the initial value for the volatility of firm 2

mvol1=0.0424;    % the long run volatility of firm 1
mvol2=0.049;     % the long run volatility of firm 2

kvol1=0.74;      % the mean reversion coefficient of firm 1
kvol2=0.8;       % the mean reversion coefficient of firm 2
```

```matlab
% Calibrated parameters to fit historical default probabilities:

kmu11=0.2;      % the volatility of the mean 11
kmu12=0.25;     % the volatility of the mean 12
kmu21=0.5;      % the volatility of the mean 21
kmu22=0.2;      % the volatility of the mean 22

kvol11=0.025;   % the volatility of the volatility 11
kvol12=0.03;    % the volatility of the volatility 12
kvol21=0.03;    % the volatility of the volatility 21
kvol22=0.05;    % the volatility of the volatility 22

b1=0.77;        % the drift of the mean for firm 1
b2=0.3;         % the drift of the mean for firm 2

% The Monte Carlo simulation:

jointPD=zeros(N,1);     % make a vector to store the joint probability of default
probdef1=zeros(N,1);    % make a vector to store the probability of default Firm 1
probdef2=zeros(N,1);    % make a vector to store the probability of default Firm 2

for j=1:N
    t=h*j;
    sumjointprobab=0;
    sumprob1=0;
    sumprob2=0;
    for i=1:Nsim

% Generate the time-varying and correlated means:

 mu11=mu1+b1*mu1*t+mu1*(kmu11*sqrt(t)*epsZ1(i)+kmu12*sqrt(t)*epsZ2(i));
 mu21=mu2+b2*mu2*t+mu2*(kmu21*sqrt(t)*epsZ1(i)+kmu22*sqrt(t)*epsZ2(i));

% Generate the time-varying and correlated volatilities:

 vol11=vol1+(kvol1*(mvol1-vol1))*t+sqrt(vol1)*(kvol11*sqrt(t)*epsW1(i)+kvol12*sqrt(t) ↙
*epsW2(i));
 vol21=vol2+(kvol2*(mvol2-vol2))*t+sqrt(vol2)*(kvol21*sqrt(t)*epsW1(i)+kvol22*sqrt(t) ↙
*epsW2(i));

% Generate the time-varying and correlated firm values:

 V11=V1+mu11*V1*t+V1*vol11*sqrt(t)*epsB1(i);
 V21=V2+mu21*V2*t+V2*vol21*sqrt(t)*epsB2(i);

% Count each time firm 1 default:

 if V11<D1
    sumprob1=sumprob1+1;
 end

% Count each time firm 2 default:

 if V21<D2
    sumprob2=sumprob2+1;
 end

% Count each time both firms defaults:

 if V11<D1
```

```matlab
    if V21<D2
        sumjointprobab=sumjointprobab+1;
    end
    end
end
% the probability of default for firm 1:

    probdef1(j,1)=sumprob1/Nsim;

% the probability of default for firm 2:

    probdef2(j,1)=sumprob2/Nsim;

% the probability of joint default:

    jointPD(j,1)=sumjointprobab/Nsim;

end

% Adjust the length of the vectors to be consistent with the x-axis:

jointPD;
jointPD=[zeros(1,1);jointPD];
probdef1;
probdef1=[zeros(1,1);probdef1];
probdef2;
probdef2=[zeros(1,1);probdef2];

% Make the figure of joint default probability as a function of time to maturity:

figure(1)
plot(l,jointPD,'-k','LineWidth',4);
xlabel('Time to maturity');
ylabel('Joint probability of default');
hleg1 = legend('Benchmark calibration');
grid on
```