



Continuous Optimization

Solving oligopolistic equilibrium problems with convex optimization<sup>☆</sup>Ruud Egging-Bratseth<sup>a,\*</sup>, Tobias Baltensperger<sup>a,b</sup>, Asgeir Tomasgard<sup>a</sup><sup>a</sup> Department of Industrial Economics and Technology Management, NTNU, Trondheim, Norway<sup>b</sup> Department of Mechanical Engineering, ETH Zurich, Switzerland

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## ABSTRACT

The approach of choice to analyze markets with oligopolistic competition has traditionally been complementarity modeling. In this paper we show that the majority of partial equilibrium models under imperfect competition in the (energy-)economic literature can in fact be cast as optimization models, not requiring the derivation and implementation of Karush–Kuhn–Tucker conditions. This is achieved by adding appropriate terms accounting for market power exertion to the well-known social welfare maximization objective. The method is applicable to both spatial Cournot oligopoly models and hybrid competition forms often implemented using conjectural variation approaches. We show how optimization and complementarity problems are equivalent, and provide a rationale for the terms accounting for market power exertion. Resulting models are solved orders of magnitude faster using off-the-shelf optimization software, compared to solving complementarity problems. Large problem instances take minutes rather than hours, and one instance solves 640 times faster. The drastically reduced solution times greatly enhance modeling capabilities as they allow increased geographical scope and represent economic, technical and other characteristics in much more detail in equilibrium problems with imperfect competition. We present practical implications for the partial and multi-level equilibrium modeling community.

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## 1. Introduction.

Equilibria in perfectly-competitive markets and monopoly markets – under mild convexity conditions – can be found using maximization of a single objective: social welfare or supplier profit, respectively. However, many resource and energy markets are characterized by an imperfectly competitive or oligopolistic market structure, wherein each agent maximizes its own objective function while market clearing conditions govern the interaction between agents. For such problems, researchers do not rely on optimization but commonly formulate partial equilibrium models cast as complementarity problems, e.g., Devine, Gabriel, and Moryadee (2016), Egging and Gabriel (2006), Egging, Gabriel, Holz, and Zhuang (2008), Egging, Holz, Von Hirschhausen, and

Gabriel (2009), Egging and Huppmann (2012), Gabriel, Zhuang, and Kiet (2005b), Gabriel, Kiet, and Zhuang (2005a), Gabriel and Smeers (2006), Gabriel, Zhuang, and Egging (2009), Hobbs (2001), Hobbs, Rijkers, and Boots (2005), Holz, Richter, and Egging (2016), Huppmann and Holz (2009), Huppmann and Holz (2012), Huppmann and Egging (2014), Lise, Hobbs, and Van Oostvoorn (2008), Lise and Kruseman (2008), Metzler, Hobbs, and Pang (2003), Moryadee, Gabriel, and Avetisyan (2014a), Moryadee, Gabriel, and Rehulka (2014b), Zhuang and Gabriel (2008), Zwart and Mulder (2006), Zwart (2008).

To solve the implemented complementarity problems, specialized solvers are used, such as PATH (Dirkse and Ferris (1993); Ferris and Munson (2000)). In this paper, we show that the partial equilibrium models in all the aforementioned references can, in fact, be cast and solved as optimization problems.

As early as 1952, Samuelson (1952) formulated a linear program to find equilibria in spatial markets. Spence (1976b) presented a maximization formulation for a rather general non-spatial monopolistic competition problem with affine demand. Spence (1976a) discussed cases wherein monopolistic competition implicitly maximizes some function and indicates that it is not the social welfare that is implicitly maximized. Murphy, Sherali, and Soyster (1982) formulated a family of convex programs to deter-

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mine Nash equilibria for oligopolies. They used a line-search approach to find solutions for these. Hashimoto (1985) showed how a static spatial oligopoly with fixed transportation cost, and without capacity restrictions, can be formulated as a maximization problem. Bergstrom and Varian (1985) stated conditions so that a Cournot equilibrium implicitly maximizes an objective function. Slade (1994) showed the existence of a fictitious objective function for quasi Cournot equilibria. Monderer and Shapley (1996) proved that there is at most one fictitious objective function (refer to Slade (1994)) and coin a new term: the potential function, for quasi-Cournot settings with linear demand and arbitrary differentiable cost functions. Based on Hashimoto (1985), Hobbs (2001) introduced a term that, when added to a social welfare maximization objective, accurately accounts for the competitive behavior of Nash–Cournot suppliers in a power market. He applied a quadratic problem to solve a spatial Nash equilibrium model for a power market wherein the network is governed by Kirchhoffs laws and line flow limits.

The Symmetry Principle (e.g., Facchinei & Pang, 2003; Gabriel, Rosendahl, Egging, Avetisyan, & Siddiqui, 2012) provides conditions such that a complementarity system  $0 \leq F(q) \perp q \geq 0$  has a corresponding optimization problem  $\min_{q \in M^Z(q)} F(q)$ ,  $M \in \mathbb{R}^n$ ,  $M$  convex. For  $F$  continuously differentiable:

$$\exists y : F(q) = \nabla y(q) \Leftrightarrow \text{Jacobian } JF(q) \text{ is symmetric} \\ \Leftrightarrow F \text{ is integrable.}$$

Cournot oligopoly problems with affine inverse demand and (piece-wise) convex costs fulfil the conditions. The list of references on the previous page meet these conditions.<sup>1</sup>

At present, the research community has not realized the (full) potential of this. With this paper we aim to remind researchers of these results. We show that the term introduced by Hashimoto (1985) has much wider applicability, e.g., in multi-period settings, conjectural variation approaches, stochastic models, multiple energy carriers and certain types of regulation, including emission ceilings and tradeable certificates (cf., Ansari, Holz, and Basri Tosun (2019) for an example of the latter two). Therefore, under mild convexity conditions which are fulfilled in the references in the long list on the previous page and many others, equilibria of imperfect market models can be found using convex optimization.<sup>2</sup>

The generalization of the applicability of the results by Spence (1976b), Hashimoto (1985), and Hobbs (2001) is the first of four main contributions of this paper. Second, we provide an economic-theoretical rationale for the term and explain how it can be derived from basic economic principles. Third, as a consequence of being able to use convex optimization, the solution times for a wide range of partial equilibrium problems will be drastically reduced. The convex formulation allows use of a broad range of off-the-shelf optimization software rather than specialized algorithms to solve complementarity problems. By drastically reducing solution times, larger and more detailed data instances of

(market) equilibrium models can be solved using this method. Fourth, an additional benefit is that imperfect markets can be analyzed without the – often tedious – derivation and implementation of Karush–Kuhn–Tucker (KKT) conditions.

To illustrate the benefits, we present solution times for several instances of a stochastic stylized resource market model (Baltensperger & Egging, 2017) and a large-scale deterministic gas market model with convex – but not quadratic – supply cost (cf., Egging, 2013; Egging & Holz, 2016; Egging, Holz, & Gabriel, 2010; Holz, Richter, & Egging, 2015; Holz et al., 2016). The results demonstrate superior scalability of the convex optimization formulation compared to the equivalent MCP formulations.

The remainder of this paper is organized as follows. In Section 2 we introduce a general resource markets model and present the equivalent convex formulation. Section 3 provides an economic rationale for the *market power adjustment term*. Section 4 discusses the application for conjectural variation approaches. Section 5 shows the applicability for modeling under uncertainty. Section 6 presents solution times for different formulations for a stylized multi-stage stochastic and a large-scale deterministic gas market model. Section 7 concludes and discusses some broader implications of the drastic improvements in numerical tractability.

## 2. Mathematical representation of stylized commodity markets

This section provides stepping stones that facilitate showing that the convex optimization problem we construct, based on adding extra terms to the social welfare maximization objective, does provide the same solution as the partial equilibrium problem we claim it solves. The intuition behind this is that if two different problems for which KKT conditions are necessary and sufficient (cf., Bazaraa, Sherali, & Shetty, 2004) have the same KKT conditions, they must have the same solution set. It does not matter if one of the problems is a pure optimization problem and the other an equilibrium problem. Hence, we design an optimization problem that has the same KKT conditions as the imperfect equilibrium problem we aim to solve.

After introducing notation, the first step is to derive the KKT conditions for a social welfare (SW) maximizing problem. (Social welfare is the net market surplus, the sum of surpluses of all actors in the market including the end-users.) Second, we derive the KKT conditions for a perfectly competitive (PC) market with price-taking agents, and show that these are exactly the same as for the SW maximizing problem. Third, we derive the KKT conditions for an oligopoly market with traders competing à la Cournot (CO). We observe that the only KKT condition that is different from the ones in the PC and SW problems, is the KKT condition for sold quantities to final consumers. All other KKT conditions are the same. Fourth, we introduce a term that when added to the SW-objective leads to KKT conditions accurately reflecting Cournot behavior. In the sections after this, we provide a theoretical rationale for the *market power adjustment-term*, and present variants for conjectural variation approaches and stochastic equilibrium modeling.

### 2.1. Notation

Table 1 presents sets and indices, Table 2 parameters, and Table 3 variables and (dual) prices. Example units of measurement: quantities or volumes in kilogram or cubic meter in a period. Costs and prices in \$ per kilogram or € per cubic meter.

### 2.2. Social welfare maximization

We set up an SW maximization problem for a general resource or commodity market, wherein traders make use of different types

<sup>1</sup> Notably, Boots, Rijkers, and Hobbs (2004) solve a successive oligopoly version of model COMPETES using optimization. Other references have implemented MCP models for social welfare maximizing problems that could have been solved as optimization problems, e.g., Abrell and Weigt (2012), Haftendorn and Holz (2010), Haftendorn, Holz, and Von Hirschhausen (2012), Neumann, Viehrig, and Weigt (2009). In contrast, Holz, Von Hirschhausen, and Kemfert (2008) employ iso-elastic demand curves. In that case, the approach in this paper can only work for symmetric suppliers, which is hardly ever the case for realistic data instances. Metzler et al. (2003) includes a Stackelberg, hence bi-level, model, which reduces to a single level model when substituting the optimal followers' responses into the leaders' objectives. This allows for applying a variant of the MPA-term discussed in this paper.

<sup>2</sup> The Symmetry Principle is crucial. For instance, symmetry will not be fulfilled when inverse demand functions are not affine. In some such cases a trick can be used, e.g., Murphy et al. (1982) present an iterative linearization and optimization scheme to solve a constant elasticity of demand Cournot problem.

**Table 1**

Sets and indices.

Index set	Description
$f \in \mathcal{F}$	Traders; trading firms
$n \in \mathcal{N}$	Spatial nodes
$n \in \mathcal{N}_z^+$	Node receiving commodity from infrastructure, e.g., node at end of a pipeline
$n \in \mathcal{N}_z^-$	Node delivering commodity to infrastructure, e.g., storage addition
$t \in \mathcal{T}$	Time periods (e.g., years)
$d \in \mathcal{D}$	Sub periods (e.g., seasons)
$z \in \mathcal{Z}$	Infrastructure service (e.g. production, transport, storage withdrawal)
$z \in \mathcal{Z}_n^+$	Infrastructure services delivering commodity to node, e.g. production, or inward pipeline
$z \in \mathcal{Z}_n^-$	Infrastructure services receiving commodity from node, e.g. outward pipeline, or storage injection

We write  $n_z^+$ ,  $n_z^-$ , etc. as shorthand when the index scope is clear in the context.

**Table 2**

Parameters.

Parameter	Description
$a_{ntd}$	Intercept of the inverse demand curve – strictly positive
$b_{ntd}$	Slope of the inverse demand curve – strictly positive
$c_{ztd}(\cdot)$	Cost of infrastructure usage – convex, increasing
$\bar{q}_{ztd}$	Capacity of infrastructure service
$\theta_{fntd}$	Conjectural variation parameter – value in [0,1]

**Table 3**

Variables.

Variable	Description
$q_{fntd} \geq 0$	Sales to consumers by specific trader
$q_{ztd} \geq 0$	Infrastructure service usage
$q_{fztd} \geq 0$	Infrastructure service usage by specific trader
$p_{ntd}$ f.i.s.	(Dual) market price for commodity
$p_{ztd}$ f.i.s.	(Dual) market price for infrastructure service
$\lambda_{ztd}$ f.i.s.	Dual to infrastructure capacity limit
$\phi_{fntd}$ f.i.s.	Dual to nodal mass balance of specific trader
$\phi_{fztd}$ f.i.s.	Dual to infrastructure mass balance of specific trader

Abbreviation *f.i.s.* stands for free in sign. The model formulations in this paper ensure that all *f.i.s.* variables will have non-negative values in the solutions. For two-directional flows, the formulation defines flows and capacities in both directions.

of infrastructure to sell commodity to end-users. We assume an underlying network of nodes connected by arcs. Infrastructure services can be used to deliver or extract commodities at specific nodes. Generally, feasible regions are not affected by market power assumptions. To not overly complicate the derivations, we ignore losses, discounting, and capacity restrictions spanning multiple periods (such as aggregate storage capacity) in the problem set up. We also assume twice differentiable, increasing convex infrastructure service cost functions and affine decreasing inverse demand functions.

### 2.2.1. Social welfare maximization problem

The social welfare maximizing objective function (1a) includes three sets of terms: consumer surplus plus trader revenues minus infrastructure services costs. The nodal mass balance (1b) represents that for each trader, in each sub period, total delivered commodity at a node must equal total received commodity plus consumption. The storage cycle constraint (1c) models that for each trader, aggregate withdrawals in a time period cannot exceed aggregate additions. The capacity constraint (1d) reflects that aggregate production and transport volumes, storage additions and storage withdrawals in any sub period cannot exceed their respective

capacity limits.

$$\max_{\substack{q_{fntd}, \\ q_{fztd}}} \sum_{t,d} \left[ \sum_n \left( \frac{1}{2} b_{ntd} \left( \sum_f q_{fntd} \right)^2 + \left( a_{ntd} - b_{ntd} \sum_f q_{fntd} \right) \sum_f q_{fntd} \right) - \sum_z c_{ztd}(q_{ztd}) \right] \quad (1a)$$

$$\text{s.t. } \forall f, n, t, d : \quad \sum_{z_n^+} q_{fztd} = \sum_{z_n^-} q_{fztd} + q_{fntd} \quad (1b)$$

$$\forall f, z, t : \quad \sum_d q_{fz^+td} = \sum_d q_{fz^-td} \quad (1c)$$

$$\forall z, t, d : \quad \sum_f q_{fztd} \leq \bar{q}_{ztd} \quad (1d)$$

### 2.2.2. Social welfare problem in standard form

To simplify derivation of the KKT conditions we present the problem above in so-called *standard form*: a minimization problem with only  $\leq$  inequality constraints and the equations ordered such that: *sinks* – *sources* = 0. We also assign the dual variables used in the KKT conditions below. Abbreviation *f.i.s.* stands for *free in sign* (or unrestricted in sign).

$$\min_{\substack{q_{fntd}, \\ q_{fztd}}} \sum_{t,d} \left[ \sum_n \left( \frac{1}{2} b_{ntd} \left( \sum_f q_{fntd} \right)^2 - a_{ntd} \sum_f q_{fntd} \right) + \sum_z c_{ztd}(q_{ztd}) \right] \quad (2a)$$

$$\text{s.t. } \forall f, n, t, d : \quad \sum_{z_n^-} q_{fztd} + q_{fntd} - \sum_{z_n^+} q_{fztd} = 0 \quad (\phi_{fntd} \text{ f.i.s.}) \quad (2b)$$

$$\forall f, z, t : \quad \sum_d q_{z^+td} - \sum_d q_{z^-td} = 0 \quad (\phi_{fzt} \text{ f.i.s.}) \quad (2c)$$

$$\forall z, t, d : \quad \sum_f q_{fztd} - \bar{q}_{ztd} \leq 0 \quad (\lambda_{ztd} \geq 0) \quad (2d)$$

### 2.2.3. Social welfare maximization – KKT conditions

The following are the KKT conditions that solve the SW optimization problem in the previous subsection. The first two conditions are stationarity conditions for sold quantities and used infrastructure services respectively. The last three reflect the feasible region.

$$\forall f, n, t, d : \quad 0 \leq q_{fntd} \perp - \left( a_{ntd} - b_{ntd} \sum_f q_{fntd} \right) + \phi_{fntd} \geq 0 \quad (3a)$$

$$\forall f, z, t, d : \quad 0 \leq q_{fztd} \perp \frac{\partial c_{ztd}(q_{fztd})}{\partial q_{fztd}} - \phi_{fz^+td} + \phi_{fz^-td} + \phi_{fz^+t} - \phi_{fz^-t} + \lambda_{ztd} \geq 0 \quad (3b)$$

$$\forall f, n, t, d : \quad \phi_{fntd} \text{ f.i.s.}, \quad \sum_{z_n^-} q_{fztd} + q_{fntd} - \sum_{z_n^+} q_{fztd} = 0 \quad (3c)$$

$$\forall f, z, t : \quad \phi_{fzt} \text{ f.i.s., } \sum_d q_{fz^+td} - \sum_d q_{fz^-td} = 0 \quad (3d)$$

$$\forall z, t, d : \quad 0 \leq \lambda_{ztd} \perp \bar{q}_{ztd} - \sum_f q_{fztd} \geq 0 \quad (3e)$$

A different way to determine an SW maximizing market outcome is to assume a perfectly competitive (PC) market wherein all agents are price-takers. We introduce price-taking infrastructure service operators who provide services to traders at marginal costs, including a scarcity, or congestion, rent, if applicable. The following shows a model with price-taking agents for the same market as in the previous subsection. At the end of this subsection we show that the two models have the exact same KKT conditions.

### 2.3. Perfect competition

We develop two types of profit maximization problems, one for traders, and one for service providers. Market clearing conditions link the agent problems together.

#### 2.3.1. Trader profit maximization

For traders, the market prices for commodities  $p_{ntd}$  as well as services  $p_{ztd}$  are exogenous. Traders maximize revenues from commodity sales minus purchase costs for infrastructure services (4a). Nodal mass balance (4b) is identical to Eq. (1b) in the SW problem. The storage cycle constraint (4c) is also equal to the Eq. (1c) above.

Trader  $f$ :

$$\max_{q_{fntd}, q_{fztd}} \sum_{t,d} \left[ \sum_n p_{ntd} q_{fntd} - \sum_z p_{ztd} q_{fztd} \right] \quad (4a)$$

$$\text{s.t. } \forall n, t, d : \quad \sum_{z_n^+} q_{fztd} = \sum_{z_n^-} q_{fztd} + q_{fntd} \quad (\phi_{fntd}) \quad (4b)$$

$$\forall z, t : \quad \sum_d q_{fz^+td} = \sum_d q_{fz^-td} \quad (\phi_{fztd}) \quad (4c)$$

#### 2.3.2. Infrastructure service provider profit maximization

For the price-taking infrastructure service providers the market prices for services  $p_{ztd}$  are exogenous. The service providers maximize profit (5a) which consists of revenues from providing services to traders minus the infrastructure operational costs. Here, we ignore capacity investment, depreciation and losses. Eq. (5b) imposes infrastructure capacity limits.

Infrastructure service provider  $z$ :

$$\max_{q_{ztd}} \sum_{z,t,d} \left[ p_{ztd} q_{ztd} - c_{ztd}(q_{ztd}) \right] \quad (5a)$$

$$\text{s.t. } \forall z, t, d : \quad q_{ztd} \leq \bar{q}_{ztd} \quad (\lambda_{ztd}) \quad (5b)$$

#### 2.3.3. Perfect competition – market clearing conditions

Market clearing conditions (m.c.c.) link the problems of different agents together. Market prices for commodities  $p_{ntd}$  and services  $p_{ztd}$  are determined outside of the agent problems as the dual variables to the m.c.c. Total supply of infrastructure services must equal demand by traders: Eq. (6a). The inverse demand curve reflects the market price as a function of total sales in a consumption node: Eq. (6b).

$$\forall z, t, d : \quad q_{ztd} = \sum_f q_{fztd} \quad (p_{ztd}) \quad (6a)$$

$$\forall n, t, d : \quad p_{ntd} = a_{ntd} - b_{ntd} \sum_f q_{fntd} \quad (p_{ntd}) \quad (6b)$$

#### 2.3.4. Perfect competition – KKT conditions

Eqs. (7a)–(7d) are the trader KKT conditions. Eqs. (7e)–(7f) are the service provider KKT conditions. Eqs. (7g)–(7h) are the market clearing conditions. Together, these constitute a mixed complementarity problem (MCP).

$$\forall f, n, t, d : \quad 0 \leq q_{fntd} \perp -p_{ntd} + \phi_{fntd} \geq 0 \quad (7a)$$

$$\forall f, z, t, d : \quad 0 \leq q_{fztd} \perp p_{ztd} - \phi_{fztd} + \phi_{fz^+td} + \phi_{fz^-td} - \phi_{fz^-t} \geq 0 \quad (7b)$$

$$\forall f, n, t, d : \quad \phi_{fntd} \text{ f.i.s., } \sum_{z_n^-} q_{fztd} + q_{fntd} - \sum_{z_n^+} q_{fztd} = 0 \quad (7c)$$

$$\forall f, n, t : \quad \phi_{fzt} \text{ f.i.s., } \sum_d q_{fz^+td} - \sum_d q_{fz^-td} = 0 \quad (7d)$$

$$\forall z, t, d : \quad 0 \leq q_{ztd} \perp -p_{ztd} + \frac{\partial c_{ztd}(q_{fztd})}{\partial q_{fztd}} + \lambda_{ztd} \geq 0 \quad (7e)$$

$$\forall z, t, d : \quad 0 \leq \lambda_{ztd} \perp \bar{q}_{ztd} - q_{ztd} \geq 0 \quad (7f)$$

$$\forall z, t, d : \quad p_{ztd} \text{ f.i.s., } q_{ztd} - \sum_f q_{fztd} = 0 \quad (7g)$$

$$\forall n, t, d : \quad p_{ntd} \text{ f.i.s., } p_{ntd} - a_{ntd} + b_{ntd} \sum_f q_{fntd} = 0 \quad (7h)$$

This system has eight conditions, three more than the system presented above for SW. We show next that the two systems are identical.

#### 2.3.5. Perfect competition – KKT conditions – reduced model

Following Baltensperger, Fuchsli, Krütli, and Lygeros (2016), we modify the system of equations by making several substitutions which reduce the number of equations but do not change the solution set. We substitute Eq. (7h) into Eq. (7a), Eq. (7g) into Eq. (7f) and Eq. (7e) into Eq. (7b). After these steps  $p_{ntd}$ ,  $p_{ztd}$ , and  $q_{ztd}$  are no longer part of the model, although they are completely determined by the remaining model and can be calculated *ex post*. The following five conditions remain:

$$\forall f, n, t, d : \quad 0 \leq q_{fntd} \perp - \left( a_{ntd} - b_{ntd} \sum_f q_{fntd} \right) + \phi_{fntd} \geq 0 \quad (8a)$$

$$\forall f, z, t, d : \quad 0 \leq q_{fztd} \perp \frac{\partial c_{ztd}(q_{fztd})}{\partial q_{fztd}} + \lambda_{ztd} - \phi_{fz^+td} + \phi_{fz^-td} + \phi_{fz^+t} - \phi_{fz^-t} \geq 0 \quad (8b)$$

$$\forall f, n, t, d : \quad \phi_{fntd} \text{ f.i.s., } \sum_{z_n^-} q_{fztd} + q_{fntd} - \sum_{z_n^+} q_{fztd} = 0 \quad (8c)$$

$$\forall f, n, t : \quad \phi_{fzt} \text{ f.i.s., } \sum_d q_{fz^+td} - \sum_d q_{fz^-td} = 0 \quad (8d)$$

$$\forall z, t, d : \quad 0 \leq \lambda_{ztd} \perp \bar{q}_{ztd} - \sum_f q_{fztd} \geq 0 \quad (8e)$$

Indeed – after minor reordering of terms – the systems Eqs. (3a)–(3e) and (8a)–(8e) are identical. This provides the basis for comparison when deriving the KKT conditions for the Cournot oligopoly in the next subsection.



## 2.4. Cournot oligopoly

Here we assume trading firms competing à la Cournot. This means that traders take competitors' sales as given, and exert monopoly power facing the residual demand curve  $p_{fntd}() = a_{ntd} - b_{ntd} \sum_{f'} q_{f'ntd} = [a_{ntd} - b_{ntd} \sum_{f' \neq f} q_{f'ntd}] - b_{ntd} q_{fntd}$ .

### 2.4.1. Trader profit maximization

The objective reflects that traders maximize revenues minus purchase costs for infrastructure services (9). Nodal mass balance and the storage cycle constraint are the same as for perfectly competitive traders.

Trader  $f$ :

$$\begin{aligned} \max_{q_{fntd}, q_{fztd}} \quad & \sum_{t,d} \left[ \sum_n \left( a_{ntd} - b_{ntd} \sum_{f'} q_{f'ntd} \right) q_{fntd} - \sum_z p_{ztd} q_{fztd} \right] \quad (9) \\ \text{s.t.} \quad & \text{Eqs. (4b) – (4c)} \end{aligned}$$

The optimization problems for service providers as well as the market clearing conditions for services are the same as for the PC market. The inverse demand curve is part of the objective function and does not need to be stated separately. Hence, market power behavior by the traders only affects the objective function of the traders. The rest of the model is the same as the PC problem above.

## 2.5. Cournot oligopoly – KKT conditions

Compared to the PC system (7) the Cournot (CO) KKT conditions differ only in the stationarity condition for sales Eq. (10), reflecting the exertion of market power. (Eq. (7h) would be redundant in this system.)

$$\begin{aligned} \forall f, n, t, d : \quad & 0 \leq q_{fntd} \perp - \left( a_{ntd} - b_{ntd} \left( \sum_{f'} q_{f'ntd} + q_{fntd} \right) \right) \\ & + \phi_{fntd} \geq 0 \quad (10) \end{aligned}$$

Eqs. (7b) – (7g)

## 2.6. Cournot oligopoly – KKT conditions – reduced model

The system does not reflect the inverse demand curve separately (cf., Eq. (7h)) so it does not have to be substituted out. We apply the other two substitutions implemented above for the PC model, to arrive at the reduced CO system:

Eqs. (10), (8b) – (8e)

This concludes the groundwork to introduce and verify the market power adjustment term. This is the topic of the following subsection.

## 2.7. A term for market power representation in convex optimization models

Two different problems for which KKT conditions are necessary and sufficient with the same KKT conditions have the same solution set, independent of whether the problems from which the KKT conditions are derived are (pure) optimization problems or (imperfect) equilibrium problems. Hence, we design a convex optimization problem that has the same KKT conditions as the imperfect equilibrium problem reflecting the Cournot oligopoly in Section 2.6.

Above, the only difference between the KKT conditions of the CO system and of the PC system is in the stationarity conditions for

sales: Eq. (10) vs. Eq. (8a). Therefore, we add a term to the SW objective such that the KKT conditions for the adjusted model are the same as for the CO model. Note that this adjusted objective function does not calculate an economic surplus measure; it is merely a construct to find solutions using an optimization approach rather than an equilibrium approach (cf., Spence (1976b) calls it *the wrong function*, Slade (1994) the *fictitious objective function*, and Monderer and Shapley (1996) the *potential function*). However, the adjusted model will determine all quantities correctly and any economic rents and prices (for commodity as well as services) can be calculated *ex post*.

The difference between Eqs. (10) and (8a) is term  $b_{ntd} q_{fntd}$ . Integrating this term gives:  $\frac{1}{2} b_{ntd} (q_{fntd})^2$ . This should be present in the objective function  $\forall f, n, t, d$ . Hence, to reflect market power exertion à la Cournot the following term should be added to the SW maximization objective function (*maximization*, hence the minus sign):

$$-\frac{1}{2} \sum_{f,n,t,d} b_{ntd} (q_{fntd})^2 \quad (11)$$

This term is equivalent to what is proposed by Hobbs (2001). Below, we show that the stationarity conditions for sales for the adjusted problem are indeed the same as Eq. (10). To do so, we first create objective function (12) by adding Term (11) to SW objective function (2a).

$$\begin{aligned} \max_{q_{fntd}, q_{fztd}} \quad & \sum_{t,d,n} \left( \frac{1}{2} b_{ntd} \left( \sum_f q_{fntd} \right)^2 \right) \quad (12) \\ & + \sum_{t,d,n} \left( \left( a_{ntd} - b_{ntd} \sum_f q_{fntd} \right) \sum_f q_{fntd} \right) \\ & - \sum_{t,d,z} c_{ztd} \left( \sum_f q_{fztd} \right) - \sum_{f,n,t,d} \frac{1}{2} b_{ntd} (q_{fntd})^2 \\ \text{s.t.} \quad & \text{Eqs. (2b) – (2d)} \end{aligned}$$

We derive the KKT stationarity condition for  $q_{fntd}$  to find:

$$\begin{aligned} \forall f, n, t, d : \quad & 0 \leq q_{fntd} \perp - \left( b_{ntd} \sum_{f'} q_{f'ntd} \right) \\ & - \left( a_{ntd} - 2b_{ntd} \sum_{f'} q_{f'ntd} \right) + b_{ntd} q_{fntd} + \phi_{fntd} \geq 0 \quad (13) \end{aligned}$$

This can be rewritten to Eq. (14), which is identical to Eq. (10).

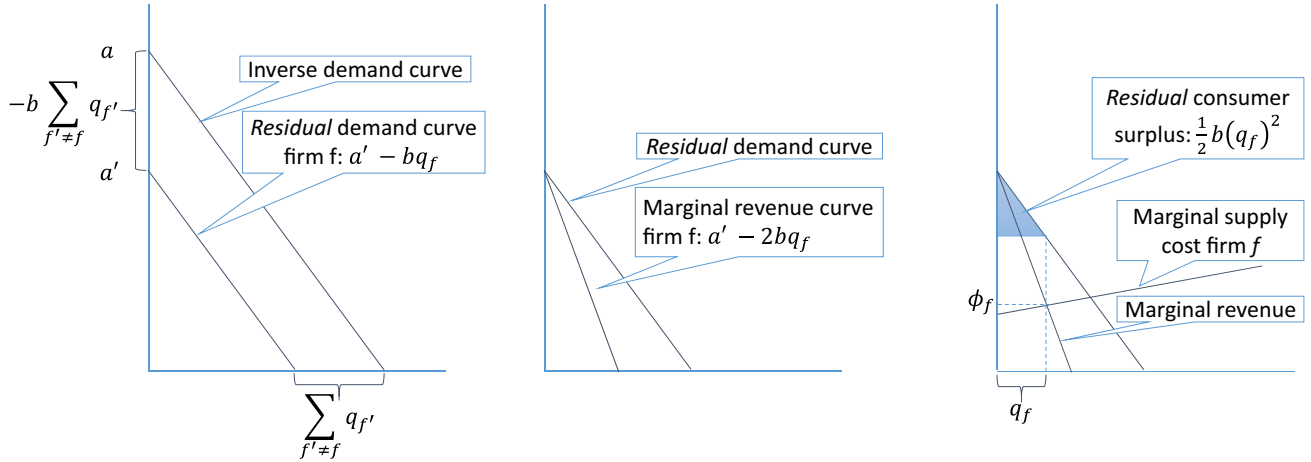
$$\begin{aligned} \forall f, n, t, d : \quad & 0 \leq q_{fntd} \perp - \left( a_{ntd} - b_{ntd} \left( \sum_{f'} q_{f'ntd} + q_{fntd} \right) \right) \\ & + \phi_{fntd} \geq 0 \quad (14) \end{aligned}$$

All other KKT conditions are not affected by the addition of the market power adjustment term. This verifies that a convex optimization problem with an objective consisting of the traditional SW maximization objective and added Term (11) will find the equilibrium for the Cournot oligopoly market.

Next, we provide a rationale why this somewhat pragmatic approach is in line with micro economic theory.

## 3. Theoretical rationale

Here, we provide a theoretic rationale for Term (11), which was used by several references (see Section 1) but without providing a rationale.



**Fig. 1.** Illustration of residual demand and residual consumer surplus. To the left, the original and residual demand curves, c.f., Eq. (15). In the middle, the marginal revenue curve for a monopoly supplier:  $a' - 2bq_f$ , c.f., Eq. (14). To the right, the resulting equilibrium and residual consumer surplus from the perspective of supplier  $f$ . We have omitted indices  $n, t, d$  for readability.

Consider the most extreme situation: when there is just one market power exerting trader. One can easily see that if there is only one trading firm, Term (11) is the negative of the modelwide consumer surplus. Subtracting this from the SW objective will give a model that ignores consumer surplus and only accounts for agent profits, indeed a trader monopoly.

Next, consider a Cournot oligopoly market with multiple market-power exerting agents. According to Cournot theory, each supplier assumes the supplies by the competitors as fixed, and acts as a monopolist on the residual demand curve, Eq. (15).

Original demand curve:

$$p_{ntd}() = a_{ntd} - b_{ntd} \sum_f q_{fntd}$$

Residual demand curve for supplier  $f$  :

$$\begin{aligned}
 p_{fntd}() &= a_{ntd} - b_{ntd} \sum_{f'} q_{f'ntd} \\
 &= \left( a_{ntd} - b_{ntd} \sum_{f' \neq f} q_{f'ntd} \right) - b_{ntd} q_{fntd} \\
 &= a'_{ntd} - b_{ntd} q_{fntd}
 \end{aligned}
 \tag{15}$$

Considering the residual demand curve, we loosely introduce a concept residual consumer surplus, which only considers the supply by trader  $f$ :  $\frac{1}{2} b_{ntd} (q_{fntd})^2$  (see Fig. 1). The market power adjustment term (11) exactly cancels residual consumer surplus from the perspective of the specific trader. Effectively, the supplier ignores the consumer surplus, and acts as a monopolist on the residual demand curve, in line with the theory.

#### 4. Conjectural variation

Conjectural variation approaches allow different levels of market power exertion by different suppliers in the same model or levels of market power somewhere in between perfectly competitive and purely Cournot (or even collusion, see, e.g., Day, Hobbs, & Pang (2002)). The approach has been subject to critique and is perceived by some to lack a proper theoretical foundation or rationale for parameter value choices (e.g., Fischer & Kamerschen, 2003; Fudenberg & Jean, 1991; Liu, Ni, Wu, & Cai, 2007; Mulligan & Fik, 1989; Pfouts & Ferguson, 1960).<sup>3</sup>

<sup>3</sup> Note: conjectured supply function (CSF) approach equilibria – wherein firms adjust their supplies in response to price changes – is a very different method. In CSF the agents do not act as Cournot agents considering a residual demand curve tak-

This section does not take a stance regarding reasonableness or defendability of a conjectural variation approach or specific value choices. It merely illustrates that, with a slight adjustment to Term (11), imperfect equilibrium models with conjectural variation can also be formulated and solved by convex optimization.

Let  $\theta_{fntd}$  with values in the range  $[0,1]$  reflect the conjectural variation parameter. Higher values reflect more market power. A value  $\theta_{fntd} = 0$  reflects no market power,  $\theta_{fntd} = 1$  pure Cournot behavior, and other values in between. Term (16) is the alternative for Term (11) to adjust an SW objective to reflect market power using conjectural variations.

$$-\frac{1}{2} \sum_{f,n,t,d} \theta_{fntd} b_{ntd} (q_{fntd})^2 \tag{16}$$

One can easily verify that the stationarity condition for sales after this adjustment, KKT condition (17), accurately represents CV market power exertion. E.g., substituting values  $\theta_{fntd} = 0$  and  $\theta_{fntd} = 1$  clearly give the KKT conditions for PC and CO, respectively.

$$\begin{aligned}
 \forall f, n, t, d : 0 \leq q_{fntd} \perp & - \left( a_{ntd} - b_{ntd} \left( \sum_{f'} q_{f'ntd} + \theta_{fntd} q_{fntd} \right) \right) \\
 & + \phi_{fntd} \geq 0
 \end{aligned}
 \tag{17}$$

#### 5. Stochastic partial equilibrium problems

Here, we consider explicitly uncertainty in the objective functions of model agents, but not in feasible regions. Uncertainty in feasible regions is not dependent on the market power behavior of agents, has the same impact on price-taking and market power exerting agents, and affects the KKT conditions for both agent types in the same way.

Examples of uncertainty in objective functions are uncertainty in costs, prices, or inverse demand curves. We impose a multi-stage scenario tree as the information structure (ref, e.g., Kall & Wallace (1994)).

We introduce notation index  $r$  for scenario tree nodes. Probability  $\rho_r \in [0, 1]$  is the likelihood that the events represented by scenario tree node  $r$  occur. In the notation of parameters and decision variables, replace index  $t$  by  $r$ . E.g.,  $q_{fnrd}$  reflects the quantity

ing supply of competitors as given, but rather have conjectures how competitors' supplies respond to price changes. E.g., Day et al. (2002); García-Alcalde, Ventosa, Rivier, Ramos, and Relación (2002).

sold in scenario tree node  $r$ . The trader decision problem under uncertainty is the following:

Cournot trader  $f$ :

$$\begin{aligned} \max_{q_{fnd}, q_{fzd}} \quad & \sum_r \rho_r \sum_d \left[ \sum_n \left( a_{nrd} - b_{nrd} \sum_{f'} q_{f'nrd} \right) q_{fnd} - \sum_z p_{zrd} q_{fzrd} \right] \\ \text{s.t.} \quad & \end{aligned} \quad (18)$$

Eqs. (4b) – (4c), with indices appropriately adjusted ( $r$  for  $t$ ).

For brevity we do not show the formulation for infrastructure service operators under uncertainty. Market clearing conditions are the same as for the deterministic problem, with indices appropriately adjusted ( $r$  for  $t$ ). KKT condition (19) reflects the stationarity condition for sales under uncertainty.

$$\begin{aligned} \forall f, n, r, d: \quad & 0 \leq q_{fnd} \perp -\rho_r \left( a_{nrd} - b_{nrd} \left( \sum_{f'} q_{f'nrd} + q_{fnd} \right) \right) \\ & + \phi_{fnd} \geq 0 \end{aligned} \quad (19)$$

One can easily verify that Term (20) is the alternative for Term (11) to adjust a stochastic SW objective with market power.

$$-\frac{1}{2} \sum_{f,n,r,d} \rho_r b_{nrd} (q_{fnd})^2 \quad (20)$$

The stochastic SW maximizing objective function is as follows:

$$\begin{aligned} \max_{q_{fnd}, q_{fzd}} \quad & \frac{1}{2} \sum_{r,d,n} \rho_r \left( b_{nrd} \left( \sum_f q_{fnd} \right)^2 \right) \\ & + \sum_{r,d,n} \rho_r \left( \left( a_{nrd} - b_{nrd} \sum_f q_{fnd} \right) \sum_f q_{fnd} \right) \\ & - \sum_{r,d,z} \rho_r c_{zrd} \left( \sum_f q_{fzrd} \right) - \sum_{f,n,r,d} \rho_r \frac{1}{2} b_{nrd} (q_{fnd})^2 \end{aligned} \quad (21)$$

It is easily verified that the KKT condition of this objective for the sales variable  $q_{fnd}$  is equal to Eq. (19).

The following section demonstrates how the ability to find equilibria for imperfect market models using convex optimization has dramatic impact on solution times.

## 6. Computational results

This section demonstrates the solution time reduction that can be obtained by solving equilibrium problems with convex optimization using the approach in this paper. We solve three different implementations of two resource market models. Both market models employ affine decreasing demand curves and convex increasing costs, and, thus, meet the necessary conditions. Model SRM is a stylized multi-stage stochastic resource market model consisting of a network with nine interlinked nodes (Baltensperger & Egging, 2017). SRM has three seasons per stage, inter-seasonal storage, infrastructure expansions, quadratic production costs, and stochastic demand.<sup>4</sup> For this stochastic model, the number of scenarios increases exponentially with the number of stages. The GGM

<sup>4</sup> In the current paper we do not show the validity of the approach when a model includes infrastructure capacity limits spanning multiple periods, which is needed to adequately reflect inter-seasonal storage with more than two seasons, nor infrastructure expansions. (Eq. (1c) does, however, cover the storage volume balance.) These features, that only affect feasible regions, will not hamper the validity of the convex approach.

**Table 4**

Knitro solver setting non-default values.

Parameter	Value	Default value
algorithm	1	0
bar_murule	5	4
bar_switchrule	1	2
pivot	10 <sup>-7a</sup>	10 <sup>-8</sup>

<sup>a</sup> only for GGM SW with 11 stages.

is the deterministic Global Gas Model (Egging, 2013; 2010; Holz et al., 2015; 2016), which consists of more than 90 geographical nodes and several hundred transport arcs. GGM features the non-quadratic Golombek production cost function (Golombek, Gjelsvik, & Rosendahl, 1994) and is calibrated on real-world data.

For both models we have three implementations: an equilibrium problem cast as both (1) an MCP and (2) a convex optimization problem *convex NLP*; and (3) a SW maximizing problem variant SW. These will serve to show the computation time impacts of using an optimization solver rather than a complementarity solver, and evaluate the impact of the market power adjustment term on computation time vs. an SW optimization.

All model versions are implemented in GAMS v24.7.1 (Brooke, Kendrick, Meeraus, & Raman, 1998), and solved using a personal computer (3.40 gigahertz CPU, 16 gigabyte RAM). The MCP formulations were solved with PATH v4.7.04 (Dirkse & Ferris, 1993; Ferris & Munson, 2000) and default solver settings. For SRM, the *convex NLP* is a quadratic program, and to solve the instances, we use IBM ILOG CPLEX v12.6.3.0 IBM (2016) with default solver settings. For GGM, the *convex NLP* is not quadratic. We solve the instances using Artelys Knitro v10.0.1 Artelys (2016) with slightly adapted solver settings (Table 4).<sup>5</sup>

To illustrate scalability, we vary the number of time periods (stages) for which the models are solved. As a result, we have 30 test instances. Of these, the two largest MCP instances for SRM did not finish within a preset time limit of 24 hours (86,400 seconds).

Fig. 2 shows the problem sizes and solution times for 30 instances tried.<sup>6</sup> On the left, results for instances of SRM, and on the right, results for GGM.

The figures show that the optimization models *convex NLP* and SW solve all stochastic and the deterministic model instances orders of magnitude faster than the MCP. All optimization problems solve within minutes even for the largest problem sizes. Fig. 2(a) shows the largest reduction in computation time. This is for the stochastic instances with six stages, where the MCP takes 640 times as long to solve as the *convex NLP* (five hours and 43 minutes vs. 32 seconds). For seven stages, the *convex NLP* solves in 89 seconds, whereas the MCP does not solve within 24 hours, which is almost 1,000 times as long. The smallest reduction in solution time is a factor of 27 for GGM with nine stages (one hour and 13 minutes vs. two minutes 43 seconds). Overall, the results show clearly that the *convex NLP* solves orders of magnitude faster than MCP in all simulations.

<sup>5</sup> With default settings Knitro only solved small problem instances.

<sup>6</sup> The reported variable numbers are indicative for the relative model sizes. As expected, the MCP instances are larger than the *convex NLP* and SW instances. However, in contrast to what the Figures show, based on the model formulations the number of variables of *convex NLP* and SW should always be the same. We observe this for GGM, but not for SRM. This is a consequence of the less lean implementation of SRM compared to GGM. For GGM, most variables that will be zero in a solution are fixed to zero before the model is pre-solved, thus the reported number of variables depends less on the pre-solver. In contrast, none of the variables were fixed to zero in the SRM variants, and the pre-solver seems to miss some model reduction options in some cases. Computation time differences between SW and CP are likely to be due to parallel activities on the same machine and should not be given too much meaning. The message is in the order of magnitude difference between CP and MCP.

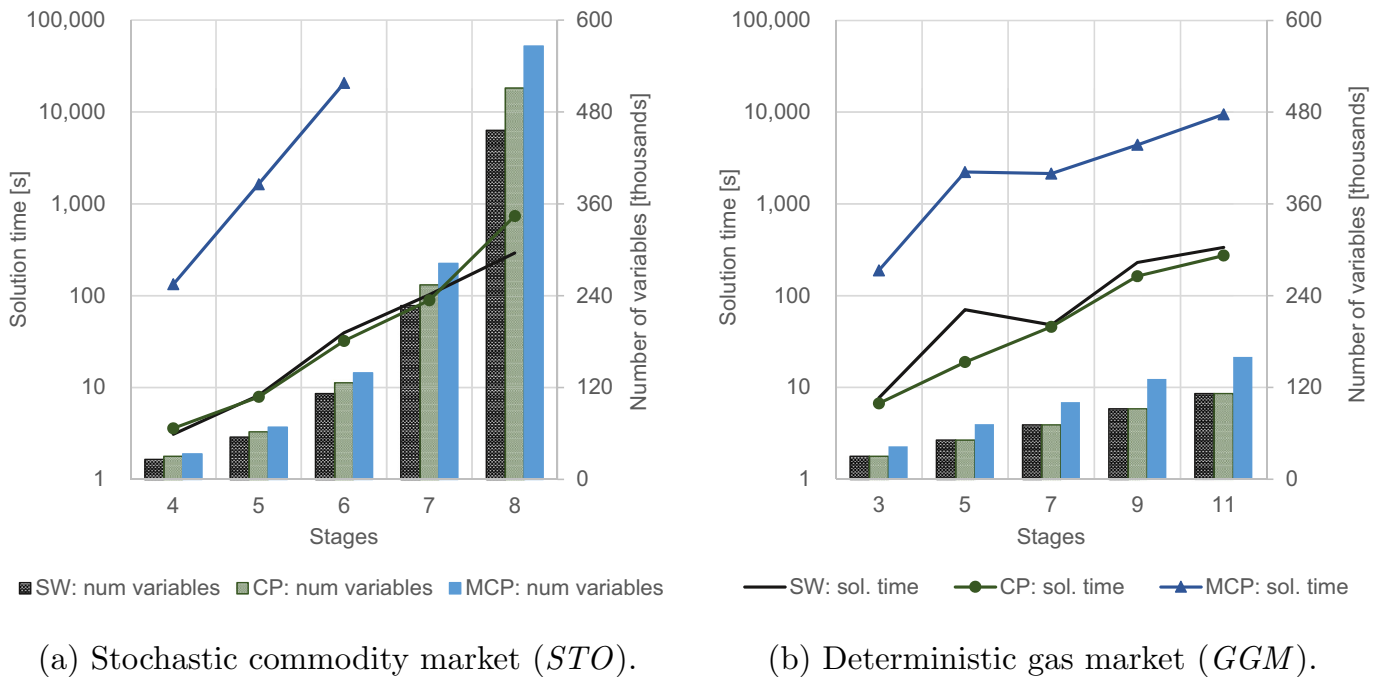


Fig. 2. Number of variables and solution times for different model versions and problem sizes. Note that solution times are presented on a logarithmic scale.

For reference, in an online Appendix we have provided the Cournot oligopoly example from Egging and Ansari (2019) implemented in GAMS as an MCP and as an optimization problem. The full convex optimization version of the Global Gas Model and its documentation is available at: <https://www.ntnu.edu/web/iot/ggm>. This version employs quadratic production costs and is solved using quadratic optimization.

## 7. Conclusion and implications

This paper demonstrates the equivalence of a convex problem formulation and an MCP for a general resource market model with imperfectly competing agents under mild convexity conditions. We demonstrate this by showing that the KKT conditions are the same for both problems. Thus, many of the equilibrium models for energy and resource markets in the literature can be cast as optimization problems. The convex problem is based off the standard social welfare maximization problem, modified by appropriate terms reflecting market power exertion. We show for several instances of a deterministic and of a stochastic model, that the convex problems solves orders of magnitudes faster using off-the-shelf optimization software compared to solving the equivalent MCP.

Our findings imply that oligopolistic and CV-based resource and energy market models can be scaled up in size significantly while maintaining computational tractability. This allows for increased geographical scope and detail, and more granularly represent economic, technical, and other problem characteristics, or include many more scenarios in stochastic problems than is currently the case in state-of-the-art equilibrium model formulations. For instance, we foresee that a new version of Multimod (Huppmann & Egging (2014)) can represent all main energy consuming and producing countries, as well as more detail in the number of energy technologies and carriers. Stochastic versions of Multimod can be developed on representative rather than toy data instances.

Furthermore, existing large-scale continuous optimization problems can easily be extended to address market power exertion with minimal impact on computational tractability. This is of par-

ticular relevance for electricity and energy market analysis where market power still plays a role in some markets, but is often ignored in modeling and analysis.

Moreover, decomposition techniques can be applied without special modifications towards MCP formulations. For instance, the standard Benders Decomposition approach can be used to solve large-scale stochastic market equilibrium problems, instead of the advanced variational inequality-based variants developed and applied by Fuller and Chung (2008), Gabriel and Fuller (2010), and Egging (2013).

Another implication is that several multi-level problems in the classes MPEC and EPEC (cf., Gabriel, Conejo, Fuller, Hobbs, & Ruiz (2013))) are, in fact, multi-level optimization problems (c.f., Metzler et al. (2003)). This will not resolve issues of non-existence or non-uniqueness of solutions (e.g., Gabriel et al. (2013)), but may help to more rapidly explore the set of solutions (cf., Huppmann & Egerer (2015)). The extent of possible solution time reductions in multi-level equilibrium problems can be considered in future research.

## Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2020.01.025.

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