CHANGE OF RINGS AND SINGULARITY CATEGORIES

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Dedicated to the memory of Ragnar-Olaf Buchweitz

ABSTRACT. We investigate the behavior of singularity categories and stable categories of Gorenstein projective modules along a morphism of rings. The natural context to approach the problem is via change of rings, that is, the classical adjoint triple between the module categories. In particular, we identify conditions on the change of rings to induce functors between the two singularity categories or the two stable categories of Gorenstein projective modules. Moreover, we study this problem at the level of 'big singularity categories' in the sense of Krause [30]. Along the way we establish an explicit construction of a right adjoint functor between certain homotopy categories. This is achieved by introducing the notion of 0-cocompact objects in triangulated categories and proving a dual version of Bousfield's localization lemma. We provide applications and examples illustrating our main results.

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1. INTRODUCTION

Singularity categories via morphisms of rings. The singularity category of a noetherian ring R, introduced by Buchweitz in his unpublished manuscript [11] as the Verdier quotient $D_{sg}(R) = D^b (\text{mod } R) / \text{perf } R$, is by now a celebrated invariant. This category vanishes precisely when R has finite global dimension and, in a sense, describes how far R is from being regular. Indeed, if R is commutative then $D_{sg}(R)$ is a categorical measure for the complexity of the singularities of the spectrum of

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R. It should be remarked that Orlov [37] later considered $D^{b}(\operatorname{coh} X)/\operatorname{perf} X$ for an algebraic variety X not only in order to understand the singularities of X, but also to provide new insight into Kontsevich's homological mirror symmetry conjecture [29]. Denoting by <u>Gproj</u> R the stable category of finitely generated Gorenstein projective R-modules, there is a natural triangle functor <u>Gproj</u> $R \hookrightarrow D_{sg}(R)$ which is always fully faithful. By a famous theorem due to Buchweitz [11], obtained independently by Happel in [21, Theorem 4.6], this is even a triangle equivalence provided R is Gorenstein. Notice that this is a more general version of the well known equivalence between the singularity category and the stable module category of a selfinjective algebra, due to Rickard [40, Theorem 2.1].

The notion of singular equivalence of finite dimensional algebras has recently attracted much attention. In particular, Chen [12, 13, 14, 16] has investigated when certain extensions of rings have equivalent singularity categories. In [39] this topic was studied from the point of view of recollements. Explicitly, each idempotent element e in R gives rise to a recollement of module categories

$\operatorname{mod} R/\langle e \rangle \hookrightarrow \operatorname{mod} R \xrightarrow{\pi} \operatorname{mod} eRe,$

and the authors of that paper gave necessary and sufficient conditions for the quotient functor π to induce a singular equivalence. Moreover, Chen [15] investigated what is happening in the left hand part of the diagram.

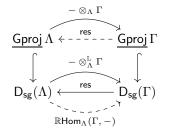
It is thus natural to explore singularity categories and stable categories of Gorenstein projective modules along a more general morphism of rings. Such an $f: \Lambda \to \Gamma$ gives rise to change of rings via the classical adjoint triple

$$\operatorname{\mathsf{Mod}}\nolimits \Lambda \underbrace{\xleftarrow{}_{\mathsf{res}}}_{\mathsf{Hom}_\Lambda(\Gamma,-)} \operatorname{\mathsf{Mod}}\nolimits \Gamma.$$

If moreover the two rings are module finite over a commutative noetherian ring i.e. they are 'noetherian algebras' — then these functors restrict to an adjoint triple on the level of finitely generated modules. It is our aim to understand when these functors induce functors on the levels of singularity categories and stable categories of Gorenstein projective modules, and to investigate what kind of properties these induced functors have. Our main results in this direction are summarized below.

Theorem I. Let $f: \Lambda \to \Gamma$ be a morphism of noetherian algebras such that the projective dimension of Γ is finite both as left and as right Λ -module. Then the following hold.

(i) We have the solid part of the commutative diagram



where the two functors on the bottom level form an adjoint pair.

- (ii) If moreover RHom_Λ(Γ, Λ) belongs to perf Γ then res restricts to a functor between stable categories of finitely generated Gorenstein projective modules, and it has a right adjoint RHom_Λ(Γ, -) on the level of singularity categories, as indicated by the dashed arrows.
- (iii) If moreover Λ is module finite over some regular ring and Cone(f) belongs to perf Λ^e, then the functor ⊗^L_Λ Γ, hence also ⊗_Λ Γ, is fully faithful. In particular, in this case the pair (D_{sg}(Λ), Ker(res_{D_{sg}})) is a stable t-structure in D_{sg}(Γ) which restricts to a stable t-structure (Gproj Λ, Ker(res)) in Gproj Γ when the condition in (ii) is met.

The above theorem is a summary of results of Section 3. We remark that (iii) immediately gives a generalization of the main result in [15], by choosing Γ to be the quotient of Λ by a homological ideal of finite projective dimension as bimodule.

Homotopy categories of big modules. Often it is necessary to consider all modules, not just finitely generated ones. Also in the realm of singularity categories, 'big' categories — i.e. triangulated categories which admit small coproducts — give a different perspective.

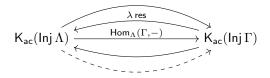
In this case, the category one considers is the homotopy category of acyclic complexes of injective modules, $K_{ac}(\ln j \Lambda)$, which should be considered the natural 'big singularity category' since it is compactly generated by $D_{sg}(\Lambda)$ — see Krause [30] and Theorem 2.20 below. This big singularity category plays a key role in recent developments in the theory of support varieties. In particular, if R is locally a hypersurface ring, then by [41] the lattice of (compactly generated) localizing subcategories of $K_{ac}(\ln j R)$ is isomorphic to the lattice of (specialization closed) subsets of the singular locus of R.

As a big version of the stable category of Gorenstein projective modules, it seems natural to consider the homotopy category of totally acyclic complexes of projective (or injective) modules. However, unfortunately, the connection here is not as good as one might have hoped: While the Gorenstein projective modules are always compact in $K_{tac}(Proj \Lambda)$, they only generate this category if the algebra is virtually Gorenstein — a class of algebras introduced by Beligiannis–Reiten [6].

For these big versions of the categories in Theorem I we obtain the following result, which is an excerpt of Section 4 — see in particular the Summary at the end of the section where we also provide a version in terms of the homotopy categories of coacyclic and totally acyclic complexes of projective modules.

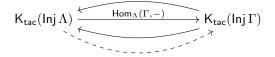
Theorem II. Let $f \colon \Lambda \to \Gamma$ be a morphism of noetherian algebras such that the projective dimension of Γ is finite both as left and as right Λ -module.

 (i) There are adjunctions as indicated by the solid arrows in the following diagram.



If moreover $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda)$ belongs to perf Γ , then there is an additional adjoint functor, as indicated by the dashed arrow.

 (ii) If Λ and Γ admit dualizing complexes D_Λ and D_Γ, respectively, such that there is an isomorphism Hom_Λ(Γ, D_Λ) ≃ D_Γ of complexes of Λ-Γ-bimodules, then there are adjunctions as indicated by the solid arrows in the following diagram.



If moreover $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda)$ belongs to perf Γ , then there is an additional adjoint functor, as indicated by the dashed arrow.

(iii) If moreover the conditions of Theorem I (iii) are met, then both occurences of $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ are fully faithful. In this case the pair $(\operatorname{K}_{\operatorname{ac}}(\operatorname{Inj}\Lambda), \operatorname{Ker}(\lambda \operatorname{res}))$ is a stable t-structure in $\operatorname{K}_{\operatorname{ac}}(\operatorname{Inj}\Gamma)$ which also restricts to the level of totally acyclic complexes.

Here λ denotes injective resolutions, which become necessary since the restriction of a complex of injectives need not consist of injectives any more. While we do not unveil the precise constructions of all the functors above in this introduction, it is worth mentioning that they are all given completely constructively, at least under the hypotheses of (ii).

Adjoint functors. In some cases the mere existence of an adjoint functor has important consequences. For instance, the right adjoint of the inclusion functor $K_{tac}(\operatorname{Proj} R) \hookrightarrow K(\operatorname{Proj} R)$, which exists under rather weak restrictions on R, was used by Jørgensen [26] in order to establish the contravariant finiteness of the Gorenstein projective modules in Mod R. In Corollary 5.4 we continue in this direction and add to the class of rings over which such right approximations exist.

On the other hand, for the purpose of doing actual computations, learning from formal arguments that a functor exists is often not satisfactory. In the age of Brown representability, due to Neeman [33], such scenarios seem to arise quite often. In certain cases, however, Bousfield's 'localization lemma' [9, 10] — as presented by Neeman [31, Lemma 1.7] — tells us precisely what a wealth of left adjoint functors actually look like. To apply his result it suffices to assume that T is a triangulated category admitting coproducts. The lemma asserts that if X is a set of compact objects in T, then the inclusion $X^{\perp} \hookrightarrow T$ admits a left adjoint and, most remarkably, that the latter is explicitly given by assigning to each object T of T the homotopy colimit of the cones of iterated right Add X-approximations of T. Such a good grasp on this functor has been proven most useful, e.g. in [31] where massive generalizations of results from algebraic geometry were obtained.

We provide the following dual of Bousfield's lemma. Since non-zero cocompact objects rarely exist in the categories we typically consider, we relax the assumption on the set of objects X to a notion which we call 0-cocompactness in order to get an interesting result.

Theorem III (Theorem 6.6). Let T be a triangulated category which admits products. Let X be a set of 0-cocompact objects in T Then

$$\left({}^{\perp}\mathsf{X},({}^{\perp}\mathsf{X}){}^{\perp}\right)$$

is a stable t-structure in T.

For the sake of brevity the reader is referred to Section 6 for the precise definition of 0-cocompactness and a detailed account of the construction itself. Here " \perp " refers to all extensions vanishing.

It should be noted that the above stable *t*-structure provides us with a functorial triangle $T_{\perp X} \to T \to T_X \to T_{\perp X}[1]$, where the objects $T_{\perp X}$ in ${}^{\perp}X$ and T_X in $({}^{\perp}X)^{\perp}$, as well as the maps, are explicitly given by a construction dual to the one of Bous-field. We emphasize that a set of 0-cocompact objects in a triangulated category cogenerates a stable *t*-structure, in contrast to Bousfield's case.

We moreover show that the hypotheses of Theorem III are satisfied in several cases of interest. In particular, we provide explicit descriptions of the right adjoint functors to the following inclusions — see Corollary 6.12.

Outline. Section 2 recalls and to some extent introduces notions and results that will be employed at later stages. This section consists of seven short subsections. In the first subsection we recall well known concepts and results from the theory of triangle functors and adjoints, as well as homotopy limits and colimits. Our first contributions appear in Subsection 2.2, where we provide a description of the right adjoint of the inclusion functor $\mathsf{K}(\mathsf{Proj}\,R) \hookrightarrow \mathsf{K}(\mathsf{Mod}\,R)$ and a description of the left adjoint of the embedding $\mathsf{K}(\mathsf{Inj}\,R) \hookrightarrow \mathsf{K}(\mathsf{Mod}\,R)$. In Subsection 2.3 we discuss dualizing complexes and recall a result of Iyengar–Krause stating that a dualizing complex for a noetherian ring R induces an equivalence between $K(\operatorname{Proj} R)$ and $K(\ln R)$. In Proposition 2.12 we show that this equivalence interacts nicely with change of rings. Subsection 2.4 is devoted to acyclic, coacyclic and totally acyclic complexes: For an Artin algebra Λ we show in Proposition 2.17 how the subcategories $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda)$ and $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda)$ of $\mathsf{K}(\mathsf{Proj}\,\Lambda)$ can be described as certain left perpendicular classes, and obtain similar descriptions of the subcategories $K_{ac}(\ln j \Lambda)$ and $K_{coac}(Inj \Lambda)$ in $K(Inj \Lambda)$. In Subsection 2.5 we recall several aspects of Gorenstein homological algebra that are used throughout the paper. In Subsection 2.6 we recall in Theorem 2.20 that if R is a noetherian ring, then the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,R)$ and the singularity category $\mathsf{D}_{\mathsf{sg}}(R)$ can be realized as the subcategories of compact objects in certain compactly generated triangulated categories — a result due to Krause [30]. Under additional assumptions on R, we show a similar statement for the stable category of Gorenstein projective modules $\operatorname{Gproj} R$. Finally, in Subsection 2.7 we discuss contravariantly finite subcategories and torsion pairs. Moreover, we recall in Theorem 2.23 a result of Jørgensen [24, 26]providing sufficient conditions on a ring R so that the subcategory of Gorenstein projective modules $\operatorname{\mathsf{GProj}} R$ is contravariantly finite in $\operatorname{\mathsf{Mod}} R$.

In Section 3 we show Theorem I above by breaking the proof down into several smaller and managable steps, and provide examples that illustrate the result. The claims (i) and (ii) are Theorem 3.1, while (iii) is Proposition 3.7 and Corollary 3.8 where we view f as a morphism of bimodules over Λ . As a byproduct of Theorem I, we obtain the main result of Chen [15] as the case $f: \Lambda \to \Lambda/I$ for Λ a finite dimensional algebra with a homological ideal I of finite projective dimension as bimodule — see Corollary 3.9. As a further application, we show for certain trivial extensions $\Gamma = \Lambda \ltimes X$ that there are fully faithful functors $-\bigotimes_{\Lambda}^{\mathbb{L}} \Gamma: \mathsf{D}_{sg}(\Lambda) \to \mathsf{D}_{sg}(\Gamma)$

and $-\otimes_{\Lambda} \Gamma$: <u>Gproj</u> $\Lambda \to \underline{\text{Gproj}} \Gamma$ — see Corollary 3.13. The results of this section are illustrated with two examples — see Example 3.12 and Example 3.14.

In Section 4 we extend the functors of Theorem I to certain homotopy categories of projective and of injective modules. Adjunctions turn out to be plentiful and Theorem II above reflects only a part of the full picture. In particular, we show that Theorem I is in fact the restriction of a picture — see Diagram 4.1 in the Summary of Section 4 — that exists on the level of certain homotopy categories, where the singularity category as well as the stable category of Gorenstein projective modules serve as subcategories of compact objects. The claim (i) in Theorem II is Proposition 4.2 and Proposition 4.4, (ii) is Proposition 4.6, while (iii) is Corollary 4.5 and 4.7. Moreover, we obtain 'big' versions of some of the applications of Section 3 — see Corollary 4.5 and Corollary 4.7.

Section 5 deals with right approximations and shows how, for any ring R, a contravariantly finite subcategory of $\mathsf{K}(\mathsf{Proj}\,R)$ induces one in $\mathsf{Mod}\,R$ — see Proposition 5.2. As a consequence, we slightly expand Jørgensen's result [26] by showing that the category of Gorenstein projective modules is contravariantly finite in $\mathsf{Mod}\,R$ as long as R is a noetherian ring with a dualizing complex — see Corollary 5.4.

The final Section 6 contains a proof of Theorem III and demonstrates how this result provides explicit descriptions of several functors that appear at earlier stages of the paper. In particular, Theorem III completes the explicit descriptions of the remaining functors of Diagram 4.1 — see Corollary 6.12. We also discuss our notion of a 0-cocompact object in a triangulated category and, in particular, show in Corollary 6.10 that if Λ is an Artin algebra, then each finite complex of finitely generated Λ -modules is 0-cocompact in the homotopy category K(Mod Λ).

Notation. By a *noetherian algebra* we mean a ring which is module finite over some commutative noetherian ring. By a module we mean a right module. When A is an additive category and X is a class of objects in A, the left and right *orthogonal classes* of X are the subcategories

$$^{\perp}\mathsf{X} = \{A \in \mathsf{A} \,|\, \mathsf{A}(A,\mathsf{X}) = 0\} \text{ and } \mathsf{X}^{\perp} = \{A \in \mathsf{A} \,|\, \mathsf{A}(\mathsf{X},A) = 0\},\$$

respectively. When T is a triangulated category and X is a class of objects in $\mathsf{T},$ we write

$$^{\perp} \mathsf{X} = \{ T \in \mathsf{T} \mid \mathsf{T}(T,\mathsf{X}[i]) = 0 \ \forall \ i \} \text{ and } \mathsf{X}^{\perp} = \{ T \in \mathsf{T} \mid \mathsf{T}(\mathsf{X}[i], T) = 0 \ \forall \ i \}.$$

If A has coproducts, then Add X is the subcategory consisting of summands of coproducts of objects in X. Dually, if A has products, then Prod X denotes the subcategory of summands of products of objects in X.

2. Preliminaries

2.1. Triangle functors and adjoints. Let us collect a few facts about functors between triangulated categories that will be employed in the sequel.

Lemma 2.1 (Orlov [37, Lemma 1.2]). Let $F: \mathsf{T} \to \mathsf{T}'$ be an exact functor between triangulated categories with right adjoint G. Let X and X' be triangulated subcategories of T and T' , respectively, satisfying $F(\mathsf{X}) \subseteq \mathsf{X}'$ and $G(\mathsf{X}') \subseteq \mathsf{X}$. Then the induced functors $\overline{F}: \mathsf{T} / \mathsf{X} \to \mathsf{T}' / \mathsf{X}'$ and $\overline{G}: \mathsf{T}' / \mathsf{X}' \to \mathsf{T} / \mathsf{X}$ form an adjoint pair $(\overline{F}, \overline{G})$. Moreover, if F is fully faithful, then so is \overline{F} . We now need to recall a few key notions. Suppose T is a triangulated category admitting coproducts. A triangulated subcategory of T is called *thick* if it is closed under direct summands, and *localizing* if it is closed under coproducts. It is not hard to show that a localizing subcategory is automatically thick. An object C in T is called *compact* if the functor T(C, -) preserves coproducts, i.e. if any morphism $C \to \prod T_i$ factors through a finite subcoproduct. T is *compactly generated* if it admits a set C of compact objects that generate T in the sense that T(C, X) = 0implies X = 0. In this case T coincides with the smallest localizing subcategory containing C. Moreover, the subcategory T^c of all compact objects in T coincides with the smallest thick subcategory containing C.

The following theorem is a consequence of Brown representability for compactly generated triangulated categories [33, Theorem 8.6.1].

Theorem 2.2 (Neeman [32, Theorems 4.1 and 5.1] and [33, Theorem 8.6.1]). Suppose $F: \mathsf{T} \to \mathsf{T}'$ is an exact functor between triangulated categories with T compactly generated.

- (i) F admits a right adjoint if and only if it preserves coproducts.
- (ii) F admits a left adjoint if and only if it preserves products.
- (iii) If F admits a right adjoint G, then F preserves compact objects if and only if G preserves coproducts.

The proof of the following observation is a standard dévissage argument and can be found in the Appendix.

Lemma 2.3. Let T and T' be triangulated categories with coproducts and suppose T is compactly generated. Let $F: \mathsf{T} \to \mathsf{T}'$ be exact and coproduct preserving. If F restricts to a fully faithful functor $\mathsf{T}^{\mathsf{c}} \to \mathsf{T}'^{\mathsf{c}}$, then F is fully faithful.

As the only monomorphisms and epimorphisms in a triangulated category are the ones that are split, limits and colimits rarely exist. In certain cases the following machinery is still useful. Suppose we are given a sequence

$$T_0 \xrightarrow{t_0} T_1 \xrightarrow{t_1} T_2 \to \cdots$$

in a triangulated category admitting coproducts, and denote by $t: \coprod T_i \to \coprod T_i$ the morphism induced by the t_i . The homotopy colimit of such a sequence, a notion which originates from algebraic topology and was introduced in algebra through [8], is defined by the triangle

$$\coprod T_i \xrightarrow{1-t} \coprod T_i \to \operatorname{hocolim} T_i \to \left(\coprod T_i\right)[1].$$

In particular, taking hocolim is not in general a functorial procedure, but it does commute with left adjoint functors. For each T_j there is a morphism $\mu_j: T_j \rightarrow$ hocolim T_i which is compatible with the t_i . It is easy to check that if t_i is invertible for each i, then each μ_j is an isomorphism. Moreover, if each $t_i = 0$, then hocolim T_i vanishes. Using homotopy colimits, it is not difficult to show that a triangulated category with coproducts is automatically idempotent complete. Homotopy colimits also (often) enable us to construct the totalization of a complex in a triangulated category. This should be thought of as an analog of the fundamental notion of the total complex with respect to coproducts of a double complex.

Dually, let T be a triangulated category with products and consider a sequence

$$\cdots \to S_2 \xrightarrow{s_2} S_1 \xrightarrow{s_1} S_0.$$

The *homotopy limit* of this sequence is given by the triangle

$$\left(\prod S_i\right)[-1] \to \operatorname{holim} S_i \to \prod S_i \xrightarrow{1-s} \prod S_i,$$

where the *i*-th component of s is the map $S_i \xrightarrow{s_i} S_{i-1} \to \prod S_i$. Note again that the holim is determined up to a non-unique isomorphism.

2.2. **Projective and injective resolutions.** Let R be a ring. The homotopy category $\mathsf{K}(\mathsf{Proj}\,R)$ is not in general compactly generated, but by [34] it is always 'well generated' which means that it still satisfies Brown representability. Hence the coproduct preserving inclusion $\mathsf{K}(\mathsf{Proj}\,R) \hookrightarrow \mathsf{K}(\mathsf{Mod}\,R)$ has a right adjoint, which we will denote by ρ . Similarly, by [36] the inclusion $\mathsf{K}(\mathsf{Inj}\,R) \hookrightarrow \mathsf{K}(\mathsf{Mod}\,R)$ has a left adjoint which we will denote by λ .

The aim of this subsection is twofold: First we give explicit descriptions of ρ and λ for complexes whose terms have bounded homological dimension — see Proposition 2.5. Then, in Proposition 2.6, we provide a somewhat surprising syzygy-cosyzygy adjunction which will prove to be useful later.

Both these results hinge on the following slightly technical lemma.

Lemma 2.4. Let $0 \to X \to C \to Y \to 0$ be a short exact sequence in C(Mod R), and assume C is contractible. (Note that such a sequence typically does not give rise to a triangle in K(Mod R).)

(i) If $M \in \mathsf{C}(\mathsf{Mod}\,R)$ is such that $\mathsf{Ext}^1_R(M^i, X^j) = 0$ for all *i* and *j*, then

 $\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(M, X) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(M, Y[-1]).$

(ii) If $N \in \mathsf{C}(\mathsf{Mod}\,R)$ is such that $\mathsf{Ext}^1_R(Y^i, N^j) = 0$ for all *i* and *j*, then

 $\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(X, N) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(Y[-1], N).$

Proof. We only prove the first statement, the second one is dual.

The vanishing of Ext^1 means that $0 \to X \to C \to Y \to 0$ induces a short exact sequence

 $0 \to \mathcal{H}om(M, X) \to \mathcal{H}om(M, C) \to \mathcal{H}om(M, Y) \to 0.$

Since the middle term is still contractible, this short exact sequence reveals a quasiisomorphism $\mathcal{H}om(M, Y)[-1] \to \mathcal{H}om(M, X)$. The claim follows, as the morphisms in the homotopy category appear as the homologies of these $\mathcal{H}om$ -complexes. \Box

Recall that the (abelian) category of complexes C(Mod R) has enough projectives. Indeed, the *i*-th term X^i of X admits an epimorphism from a projective *R*-module P_{X^i} , and hence X itself admits an epimorphism from the projective object given by

$$P_X = \dots \to P_{X^i} \oplus P_{X^{i-1}} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} P_{X^{i+1}} \oplus P_{X^i} \to \dots$$

In particular P_X is a contractible complex. As usual, we denote by $\Omega(X)$ the syzygy of X, i.e. the kernel of the natural projection $P_X \to X$.

Dually, X embeds in a contractible complex I_X consisting of injective modules, and we let $\Im X$ denote the cokernel of $X \hookrightarrow I_X$.

Proposition 2.5. Let X be a complex of R-modules. If each term of X has projective dimension no larger than some fixed number d, then

$$\rho(X) = \Omega^d(X)[d].$$

Dually, if each term of X has injective dimension no larger than d, then

$$\lambda(X) = \mho^d(X)[-d].$$

Proof. Consider the short exact sequence of complexes

$$0 \to \Omega(X) \to P_X \to X \to 0$$

with contractible middle term. For any complex Q of projective modules we have

$$\mathsf{Ext}^1_R(Q^i, \Omega(X)^j) = 0,$$

and hence Lemma 2.4 asserts that

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(Q, X) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(Q, \Omega(X)[1]).$$

Iterating we obtain

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(Q, X) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(Q, \Omega^d(X)[d]).$$

But $\Omega^d(X)[d]$ is a complex of projective modules by the assumption on the projective dimensions of the terms of X. Thus $\Omega^d(X)[d]$ satisfies the defining isomorphism for the right adjoint, so $\rho(X) = \Omega^d(X)[d]$.

The proof of the second claim is dual.

Proposition 2.6. Let X and Y be complexes of R-modules such that

$$\mathsf{Ext}_{R}^{d+1}(X^{m},Y^{n}) = 0$$

for all terms X^m of X and Y^n of Y. Then

$$\operatorname{Hom}_{\mathsf{K}}(\Omega^d X, Y) \cong \operatorname{Hom}_{\mathsf{K}}(X, \mathbb{O}^d Y),$$

and this isomorphism is functorial in X and Y satisfying the term-wise vanishing condition (*).

Proof. Note that, by dimension shift, for any $0 \le i \le d$ we have

$$\mathsf{Ext}^1_R((\Omega^i X)^m, (\mho^{d-i} Y)^n) = 0.$$

Thus, by Lemma 2.4 the short exact sequences

$$0 \to \mathcal{O}^{d-i}(Y) \to I_{\mathcal{O}^{d-i}(Y)} \to \mathcal{O}^{d-i+1}(Y) \to 0$$

and

$$0 \to \Omega^{i+1} X \to P_{\Omega^i X} \to \Omega^i X \to 0$$

give rise to isomorphisms

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i}X, \mathbb{C}^{d-i}Y) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i}X, \mathbb{C}^{d-i+1}Y[-1])$$

and

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i}X, \mho^{d-i}Y) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i+1}X[1], \mho^{d-i}Y),$$

respectively. Combining the latter we find

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i}X, \mathbb{O}^{d-i}Y) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{\mathsf{Mod}} R)}(\Omega^{i-1}X, \mathbb{O}^{d-i+1}Y),$$

and the claim follows by composing isomorphisms of this form.

2.3. **Dualizing complexes.** We now introduce Theorem 2.11, due to Iyengar–Krause, which will serve as the foundation for much of the sequel.

Definition 2.7 (Iyengar–Krause [23]). A dualizing complex for a noetherian ring R is a bounded complex D_R of R–R-bimodules such that

- (i) the terms of D_R are injective both as left and as right R-modules;
- (ii) the homology of D_R is finitely generated both as left and as right R-module;
- (iii) the canonical maps $R \to \operatorname{Hom}_R(D_R, D_R)$ and $R^{\operatorname{op}} \to \operatorname{Hom}_{R^{\operatorname{op}}}(D_R, D_R)$ are quasi-isomorphisms.

This terminology is justified by the following result, due to [22] for commutative rings. A translation of the proof to the non-commutative setting can be found in [23].

Theorem 2.8 (Hartshorne [22]). A dualizing complex D_R induces a duality

 $\mathcal{H}om_R(-, D_R)$: $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,R) \leftrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,R^{\mathsf{op}})^{\mathsf{op}}$: $\mathcal{H}om_{R^{\mathsf{op}}}(-, D_R)$.

Example 2.9. If Λ is an Artin algebra, i.e. module finite over a commutative artinian ring k, then $D_{\Lambda} = \text{Hom}_k(\Lambda, E)$ is a dualizing complex for Λ , where $\text{Hom}_k(-, E)$ is the standard duality.

Recall that a ring is *Gorenstein* if it has finite injective dimension as left and as right module over itself — see e.g. [21].

Example 2.10. If k is a commutative noetherian Gorenstein ring and Λ is a module finite (not necessarily commutative) k-algebra, then $D_{\Lambda} = \text{Hom}_k(\Lambda, i_k)$ is a dualizing complex for Λ , where i_k is a finite injective resolution of k over itself.

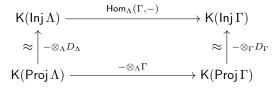
Theorem 2.11 (Iyengar–Krause [23]). A dualizing complex D_{Λ} induces an equivalence

$$-\otimes_{\Lambda} D_{\Lambda} \colon \mathsf{K}(\mathsf{Proj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$$

with quasi-inverse $\rho \operatorname{Hom}_{\Lambda}(D_{\Lambda}, -)$.

In the setting of Example 2.10, this equivalence interacts nicely with change of rings:

Proposition 2.12. Let $f: \Lambda \to \Gamma$ be a morphism of noetherian algebras admitting dualizing complexes D_{Λ} and D_{Γ} , respectively, such that $\operatorname{Hom}_{\Lambda}(\Gamma, D_{\Lambda}) \cong D_{\Gamma}$ as complexes of Λ - Γ -bimodules. Then the equivalences of Theorem 2.11 make the following square commutative.



In particular, the required isomorphism $\operatorname{Hom}_{\Lambda}(\Gamma, D_{\Lambda}) \cong D_{\Gamma}$ is satisfied by the dualizing complexes D_{Λ} and D_{Γ} of Example 2.10 above.

Proof. By assumption, for the first claim it suffices to show that the natural transformation $- \otimes_{\Lambda} \operatorname{Hom}_{\Lambda}(\Gamma, D_{\Lambda}) \to \operatorname{Hom}_{\Lambda}(\Gamma, - \otimes_{\Lambda} D_{\Lambda})$ is an isomorphism on complexes over $\operatorname{Proj} \Lambda$. This follows readily since the functors coincide on Λ and moreover commute with coproducts. For the last claim, it suffices to observe that

$$\operatorname{Hom}_{\Lambda}(\Gamma, D_{\Lambda}) = \operatorname{Hom}_{\Lambda}(\Gamma, \operatorname{Hom}_{k}(\Lambda, i_{k})) = \operatorname{Hom}_{k}(\Gamma, i_{k}) = D_{\Gamma}.$$

2.4. Acyclic, coacyclic and totally acyclic complexes. Let R be a ring. We denote by $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,R)$ the subcategory of $\mathsf{K}(\mathsf{Proj}\,R)$ consisting of acyclic complexes. Clearly, $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,R) = (\mathsf{Proj}\,R)^{\perp} = (R)^{\perp}$ as subcategories of $\mathsf{K}(\mathsf{Proj}\,R)$. Dually, we consider $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,R) = {}^{\perp}(\mathsf{Proj}\,R) \subset \mathsf{K}(\mathsf{Proj}\,R)$ and call an object in this subcategory a *coacyclic complex of projectives*. Note that $X \in \mathsf{K}(\mathsf{Proj}\,R)$ is coacyclic if and only if the complex $\mathsf{Hom}_R(X, P)$ is exact for each $P \in \mathsf{Proj}\,R$. Finally, a complex over $\mathsf{Proj}\,R$ is *totally acyclic* if it belongs to

$$\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,R) = \mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,R) \cap \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,R).$$

On the other hand, $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,R)$ is the subcategory of $\mathsf{K}(\mathsf{Inj}\,R)$ whose objects are the acyclic complexes. One easily sees that $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,R) = {}^{\perp}(\mathsf{Inj}\,R) = (\lambda R)^{\perp}$ as subcategories of $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,R)$ (recall that λ denotes an injective resolution). We let $\mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,R) = (\mathsf{Inj}\,R)^{\perp} \subset \mathsf{K}(\mathsf{Inj}\,R)$ and call an object of this subcategory a *coacyclic complex of injectives*. Dual to the projective case, $Y \in \mathsf{K}(\mathsf{Inj}\,R)$ is coacyclic if and only if the complex $\mathsf{Hom}_R(I,Y)$ is exact for each $I \in \mathsf{Inj}\,R$. Finally, a complex over $\mathsf{Inj}\,R$ is *totally acyclic* if it belongs to the category

$$\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,R) = \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,R) \cap \mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,R).$$

Right orthogonal classes. It would be convenient if (co)acyclicity was detected by a single object, rather than by all of $\operatorname{Proj} R$ or $\operatorname{Inj} R$. We have already seen how $\operatorname{K}_{ac}(\operatorname{Proj} R)$ and $\operatorname{K}_{ac}(\operatorname{Inj} R)$ are obtained as the right orthogonal classes of R and λR , respectively. We now show how the presence of a dualizing complex enables us to also write $\operatorname{K}_{coac}(\operatorname{Proj} R)$ and $\operatorname{K}_{coac}(\operatorname{Inj} R)$ as $(X)^{\perp}$ for some object X. Lemma 2.14 below is the key ingredient, but to prove this we need the following fact from homological algebra.

Lemma 2.13. Let Λ be a noetherian ring. Suppose $J \in \text{Inj } \Lambda$ and that I is a Λ - Λ -bimodule such that ${}_{\Lambda}I \in \text{Inj } \Lambda^{\text{op}}$. Then $\text{Hom}_{\Lambda}(I, J)$ is a flat Λ -module.

Proof. The natural transformation $\operatorname{Hom}_{\Lambda}(I, J) \otimes_{\Lambda} - \to \operatorname{Hom}_{\Lambda}(\operatorname{Hom}_{\Lambda^{\operatorname{op}}}(-, I), J)$ of right exact functors is an isomorphism on Λ and thus on all of $\operatorname{mod} \Lambda^{\operatorname{op}}$. Hence $\operatorname{Hom}_{\Lambda}(I, J) \otimes_{\Lambda} -$ is left exact on $\operatorname{mod} \Lambda^{\operatorname{op}}$. This suffices, as flatness of $\operatorname{Hom}_{\Lambda}(I, J)$ is even implied by the vanishing of $\operatorname{Tor}_{1}^{\Lambda}(\operatorname{Hom}_{\Lambda}(I, J), \Lambda/L)$ for each left ideal L. \Box

In the current subsection we denote by T the equivalence $-\otimes_{\Lambda} D_{\Lambda}$: $\mathsf{K}(\mathsf{Proj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$ of Theorem 2.11 and by S its quasi-inverse $\rho \operatorname{Hom}_{\Lambda}(D_{\Lambda}, -)$. Claim (i) below is essentially [23, Proposition 4.7] in the non-commutative setting.

Lemma 2.14. Let Λ be a noetherian ring with a dualizing complex D_{Λ} . Then

- (i) T restricts to an equivalence $\mathsf{K}^{\mathsf{b}}(\mathsf{Proj}\,\Lambda) \cong \mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\,\Lambda)$;
- (ii) T restricts to an equivalence $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) \cong \mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda)$;
- (iii) T restricts to an equivalence $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) \cong \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$;
- (iv) T restricts to an equivalence $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda) \cong \mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda)$.

Proof. Let us start with (i). The functor T does restrict as desired, and we only need to show that so does S. For $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\Lambda)$, the bounded complex $\mathsf{Hom}_{\Lambda}(D_{\Lambda}, Y)$

consists of flat modules by Lemma 2.13. As flat Λ -modules have finite projective dimension by [17], the description of ρ in Proposition 2.5 settles the claim. Statement (ii) now follows from

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) = (\mathsf{Proj}\,\Lambda)^{\perp} = \mathsf{K}^{\mathsf{b}}(\mathsf{Proj}\,\Lambda)^{\perp} \stackrel{^{T}}{\cong} \mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\,\Lambda)^{\perp} = (\mathsf{Inj}\,\Lambda)^{\perp} = \mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda)$$

and a similar argument shows (iii), whence (iv) is immediate.

Proposition 2.15. Let Λ be a noetherian ring with a dualizing complex D_{Λ} . Then

- (i) $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) = (\rho \,\mathbb{R}\mathsf{Hom}_{\Lambda}(D_{\Lambda},\Lambda))^{\perp}$ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$;
- (ii) $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) = (\lambda\Lambda)^{\perp} in \,\mathsf{K}(\mathsf{Inj}\,\Lambda);$
- (iii) $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) = (\Lambda)^{\perp} in \mathsf{K}(\mathsf{Proj}\,\Lambda);$
- (iv) $\mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda) = (D_\Lambda)^{\perp}$ in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$.

Moreover, in (i)-(iv) the objects defining the right orthogonal classes are all compact in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$ and $\mathsf{K}(\mathsf{Inj}\,\Lambda)$, respectively.

Proof. Claims (ii) and (iii) were observed above. We obtain (i) from the following identifications

$$\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) \stackrel{^{T}}{\cong} \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) = (\lambda\Lambda)^{\perp} \stackrel{^{S}}{\cong} (\rho \operatorname{\mathsf{Hom}}_{\Lambda}(D_{\Lambda},\lambda\Lambda))^{\perp} = (\rho \operatorname{\mathbb{R}Hom}_{\Lambda}(D_{\Lambda},\Lambda))^{\perp}.$$

For (iv), Lemma 2.14 implies $\mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\,\Lambda) = \mathsf{thick}(\mathsf{Add}\,D_{\Lambda})$ from which we infer

$$\mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda) = (\mathsf{Inj}\,\Lambda)^{\perp} = \mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\,\Lambda)^{\perp} = \mathsf{thick}(\mathsf{Add}\,D_{\Lambda})^{\perp} = (D_{\Lambda})^{\perp}.$$

Finally, since $\lambda\Lambda$ is compact in $\mathsf{K}(\mathsf{Inj}\Lambda)$ it follows that $S(\lambda\Lambda) = \rho \mathbb{R}\mathsf{Hom}_{\Lambda}(D_{\Lambda},\Lambda)$ is compact in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$. Similarly, the compactness of Λ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$ implies the compactness of $T(\Lambda) = D_{\Lambda}$ in $\mathsf{K}(\mathsf{Inj}\Lambda)$. \square

Left orthogonal classes. Contrary to what one might expect, $K_{coac}(Proj \Lambda)$ need not coincide with $^{\perp}(\Lambda)$ even for a noetherian ring Λ . In fact, exactness of Hom_{Λ} (X, Λ) fails to imply coacyclicity of a complex X over $\operatorname{Proj} \Lambda$ already if Λ is a complete local domain [23, Remark 5.11]. Knowing this, the below Proposition 2.17 is more or less what one could hope for in the pursuit of describing (co)acyclicity as $^{\perp}(X)$ for a single object X. We will need the following observation, the proof of which uses the machinery of pure-injective modules and can be found in the Appendix.

Lemma 2.16. If Λ is an Artin algebra, then the natural monomorphism $\Lambda^{(I)} \to \Lambda^{I}$ is split for any index set I.

Below, D_{Λ} refers to the dualizing complex from Example 2.9.

Proposition 2.17. Let Λ be an Artin algebra. Then

- (i) $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) = {}^{\perp}(\Lambda)$ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$;
- (ii) $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) = {}^{\perp}(D_{\Lambda})$ in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$;
- (iii) $\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) = {}^{\perp}(\rho D_{\Lambda}) \text{ in } \mathsf{K}(\mathsf{Proj}\,\Lambda);$ (iv) $\mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda) = {}^{\perp}(D_{\Lambda} \otimes_{\Lambda}^{\mathbb{L}} D_{\Lambda}) \text{ in } \mathsf{K}(\mathsf{Inj}\,\Lambda).$

Proof. To prove (i) observe that Lemma 2.16 implies $\operatorname{Proj} \Lambda = \operatorname{Prod} \Lambda$, which yields $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) = {}^{\perp}(\mathsf{Proj}\,\Lambda) = {}^{\perp}(\mathsf{Prod}\,\Lambda) = {}^{\perp}(\Lambda).$ Now (ii) follows once we identify

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \stackrel{S}{\cong} \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) = {}^{\bot}(\Lambda) \stackrel{T}{\cong} {}^{\bot}(D_{\Lambda}).$$

For (iii), notice that we also have $\ln \Lambda = \operatorname{Prod} D_{\Lambda}$ for our dualizing complex. Moreover, a complex X is acyclic if and only if $\operatorname{Hom}_{\Lambda}(X, I)$ is exact for each $I \in \operatorname{Inj} \Lambda$. Hence in $\mathsf{K}(\operatorname{\mathsf{Mod}} \Lambda)$ we have

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) = {}^{\bot}(\mathsf{Inj}\,\Lambda) \cap \mathsf{K}(\mathsf{Proj}\,\Lambda) = {}^{\bot}(\mathsf{Prod}\,D_{\Lambda}) \cap \mathsf{K}(\mathsf{Proj}\,\Lambda) = {}^{\bot}(D_{\Lambda}) \cap \mathsf{K}(\mathsf{Proj}\,\Lambda).$$

Observe next that by adjunction we have $\operatorname{Hom}_{\mathsf{K}}(P,\rho D_{\Lambda}) = \operatorname{Hom}_{\mathsf{K}}(P,D_{\Lambda})$ for each complex P over $\operatorname{Proj} \Lambda$, which reveals that $\operatorname{K}_{\mathsf{ac}}(\operatorname{Proj} \Lambda) = {}^{\perp}(\rho D_{\Lambda})$ in $\operatorname{K}(\operatorname{Proj} \Lambda)$. Now (iv) is implied by identifying

$$\mathsf{K}_{\mathsf{coac}}(\mathsf{Inj}\,\Lambda) \stackrel{S}{\cong} \mathsf{K}_{\mathsf{ac}}(\mathsf{Proj}\,\Lambda) = {}^{\bot}(\rho D_{\Lambda}) \stackrel{T}{\cong} {}^{\bot}(\rho D_{\Lambda} \otimes_{\Lambda} D_{\Lambda}) = {}^{\bot}(D_{\Lambda} \otimes_{\Lambda}^{\mathbb{L}} D_{\Lambda}). \quad \Box$$

Remark 2.18. For an additive category A one lets $K_{tac}(A) = {}^{\perp}(A) \cap (A)^{\perp}$ upon viewing A as the stalk complexes of K(A). In this tradition, if R is a ring, then $K_{coac}(\operatorname{proj} R)$ should be defined as $K(\operatorname{proj} R) \cap {}^{\perp}(\operatorname{proj} R)$. On the other hand, it would also be natural to define $K_{coac}(\operatorname{proj} R)$ as $K(\operatorname{proj} R) \cap K_{coac}(\operatorname{Proj} R)$. Thankfully, this seeming conflict solves itself. Indeed, the fact that each finitely generated module is compact in Mod R implies that ${}^{\perp}(\operatorname{proj} R) = {}^{\perp}(\operatorname{Proj} R)$ as subcategories of $K(\operatorname{proj} R)$, so the competing definitions agree. In particular, and useful in Section 3 below, this means that a complex P over $\operatorname{proj} R$ does belong to $K_{coac}(\operatorname{proj} R)$ if $\operatorname{Hom}_{K}(P, R) = 0$.

2.5. Gorenstein homological algebra. Let R be a ring. An R-module is *Gorenstein projective* if it appears as the 0-boundaries of a totally acyclic complex over Proj R. The Gorenstein projective R-modules form a Frobenius exact subcategory GProj R of Mod R, and assigning $X \mapsto B^0(X)$ gives a triangle equivalence

$$\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,R) \cong \mathsf{GProj}\,R.$$

Provided R is right noetherian, $\mathsf{Gproj}\,R=\mathsf{GProj}\,R\cap\mathsf{mod}\,R$ is again Frobenius exact in $\mathsf{mod}\,R,$ and

$$\mathsf{K}_{\mathsf{tac}}(\mathsf{proj}\,R) \cong \mathsf{Gproj}\,R$$

by restriction of the above equivalence. Dually, an *R*-module is called *Gorenstein* injective if it is isomorphic to the 0-cycles of some totally acyclic complex over lnj R, and assigning $Y \mapsto B^0(Y)$ gives triangle equivalences

$$\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,R) \cong \overline{\mathsf{GInj}}\,R \text{ and } \mathsf{K}_{\mathsf{tac}}(\mathsf{inj}\,R) \cong \overline{\mathsf{Ginj}}\,R.$$

Notice that if R happens to be noetherian with a dualizing complex, then

$$\operatorname{\mathsf{GProj}} R \cong \overline{\operatorname{\mathsf{GInj}}} R \text{ and } \operatorname{\mathsf{Gproj}} R \cong \overline{\operatorname{\mathsf{Ginj}}} R$$

by Lemma 2.14.

If Γ is an Artin algebra, then the duality between $\mathsf{Gproj}\,\Gamma$ and $\mathsf{Ginj}\,\Gamma$ is pleasant enough to ensure $(\mathsf{Gproj}\,\Gamma)^{\perp} = {}^{\perp}(\mathsf{Ginj}\,\Gamma)$ in $\mathsf{mod}\,\Gamma$ (see [3]). However, this does not necessarily hold true for big modules. To amend this oddity, in [6] the class of *virtually Gorenstein* algebras was introduced as the algebras Γ for which $(\mathsf{GProj}\,\Gamma)^{\perp} = {}^{\perp}(\mathsf{Glnj}\,\Gamma)$. We remark that the class of virtually Gorenstein algebras is rather large. Indeed, it contains the algebras of finite representation type and is closed under derived equivalence. In fact it seems that the first example of an Artin algebra which is not virtually Gorenstein appeared as recently as [5, Example 4.3].

Denote now by Λ a noetherian ring. The *singularity category* of Λ is the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(\Lambda) = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)/\mathsf{perf}\,\Lambda$$

introduced in [11]. Notice that there is a natural triangle functor

$$\iota \colon \operatorname{\mathsf{Gproj}} \Lambda \to \mathsf{D}_{\mathsf{sg}}(\Lambda).$$

Indeed, upon identifying <u>Gproj</u> $\Lambda = \mathsf{K}_{\mathsf{tac}}(\mathsf{proj}\,\Lambda)$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda) = \mathsf{K}^{-,\mathsf{b}}(\mathsf{proj}\,\Lambda)$, ι is given by assigning to a totally acyclic complex X over $\mathsf{proj}\,\Lambda$ the object in $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ represented by its brutal truncation $\tau^0 X$.

Below, (i) is the fundamental theorem of Buchweitz [11, Theorem 4.4.1]. The partial converses in (ii) are due to [2,7] (see also [13] for a relative version).

Theorem 2.19. The functor ι is fully faithful. Moreover,

- (i) if Λ is Gorenstein, then ι gives an equivalence $\mathsf{Gproj} \Lambda \cong \mathsf{D}_{\mathsf{sg}}(\Lambda)$;
- (ii) if Λ is commutative local or an Artin algebra such that ι is an equivalence, then Λ is Gorenstein.

2.6. Compactly generated completions. When Λ is a noetherian ring, the triangulated categories $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ and $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ may be realized as the subcategories of compact objects in familiar compactly generated triangulated categories. If Λ is moreover virtually Gorenstein, then the same goes for <u>Gproj</u> Λ . We now explain how these embeddings come about.

Theorem 2.20. If Λ is a noetherian ring, then we have the following.

- (i) $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is compactly generated with $\mathsf{K}(\mathsf{Inj}\,\Lambda)^{\mathsf{c}} \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$;
- (ii) $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$ is compactly generated with $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)^{\mathsf{c}} \cong \mathsf{D}_{\mathsf{sg}}(\Lambda)$;
- (iii) $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda)$ is compactly generated if Λ admits a dualizing complex. If moreover Λ is an Artin algebra, then $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda)^{\mathsf{c}} \cong \underline{\mathsf{Gproj}}\,\Lambda$ if and only if Λ is virtually Gorenstein.

Idea of proof. Claims (i) and (ii) are due to Krause [30] — let us give a brief account of his argument. It is straightforward to verify that the injective resolution of a finitely generated module is compact in $K(\ln j \Lambda)$, from which it follows that $K(\ln j \Lambda)$ is compactly generated. In other words, taking injective resolutions embeds $D^{b}(mod \Lambda)$ as the compact objects of $K(\ln j \Lambda)$.

Recall that $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$ is the subcategory $(\lambda\Lambda)^{\perp}$ of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ by Proposition 2.15, so in particular it is closed under coproducts. As the compact generation of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ has already been established, we may invoke [31] which constructs a compact preserving left adjoint I_{λ} to the inclusion $I \colon \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \to \mathsf{K}(\mathsf{Inj}\,\Lambda)$. As such an I_{λ} automatically takes a set of generating objects to a set of generating objects, the compact generation of $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$ follows. Now Theorem 2.2 yields the existence of a recollement

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(\mathsf{Mod}\,\Lambda)$$

which induces an equivalence up to direct summands

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)^{\mathsf{c}} \cong \mathsf{K}(\mathsf{Inj}\,\Lambda)^{\mathsf{c}} / \mathsf{D}(\mathsf{Mod}\,\Lambda)^{\mathsf{c}},$$

and the fact that $\mathsf{D}(\mathsf{Mod}\,\Lambda)^{\mathsf{c}} = \mathsf{perf}\,\Lambda$ is well known.

We now turn to (iii). As in the previous paragraph, the compact generation of $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\Lambda)$ follows from the compact generation of $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda)$ since $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\Lambda)$ is the subcategory $(D_{\Lambda})^{\perp}$ of $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda)$ by Proposition 2.15. For the last claim, <u>Gproj</u> $\Lambda \subset (\underline{\mathsf{GProj}}\Lambda)^{\mathsf{c}}$ is always true. By [4], the reversed inclusion holds precisely when Λ is virtually Gorenstein, which is not entirely surprising once one learns that Λ is virtually Gorenstein if and only if each Gorenstein projective module is a filtered colimit of finitely generated Gorenstein projective modules [5, Theorem 5]. Hence the proof is complete since $\mathsf{GProj} \Lambda \cong \mathsf{K}_{\mathsf{tac}}(\mathsf{Inj} \Lambda)$.

In the above, the homotopy categories appearing are over $\ln \Lambda$, even though Lemma 2.14 suggests that we could equally well work over $\operatorname{Proj} \Lambda$. Indeed, each relevant property will formally carry over by transport of structure, but homotopy categories over $\ln \Lambda$ seem to be intrinsically better behaved than their counterparts over $\operatorname{Proj} \Lambda$. For instance, and crucial above, if M is a finitely generated Λ -module, then λM is compact in $\operatorname{K}(\operatorname{Inj} \Lambda)$ while ρM need not be compact in $\operatorname{K}(\operatorname{Proj} \Lambda)$. Indeed, ρM is compact in $\operatorname{K}(\operatorname{Proj} \Lambda)$ if and only if M is compact in $\operatorname{D}(\operatorname{Mod} \Lambda)$, i.e. precisely when M has finite projective dimension.

A further reason for preferring injectives is that generalizations are then more often within reach. Indeed, in the above (i) and (ii) one may replace $\mathsf{Mod} \Lambda$ by any locally noetherian Grothendieck category \mathcal{A} such that $\mathsf{D}(\mathcal{A})$ is compactly generated, which happens for instance when \mathcal{A} has finite global dimension. On the other hand, there is no reason why such an \mathcal{A} should even have enough projectives. Nevertheless, Jørgensen [25] showed directly, i.e. with no allusion to the above, that $\mathsf{K}(\mathsf{Proj} \Lambda)$ is compactly generated if Λ is noetherian. Funnily enough, the compact objects of $\mathsf{K}(\mathsf{Proj} \Lambda)$ arise as $\mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\rho M, \Lambda)$ for $M \in \mathsf{mod} \Lambda^{\mathsf{op}}$, and it turns out that $\mathsf{K}(\mathsf{Proj} \Lambda)^{\mathsf{c}}$ is naturally equivalent to $\mathsf{D}^{\mathsf{b}}(\mathsf{mod} \Lambda^{\mathsf{op}})^{\mathsf{op}}$. Note that the latter should be expected in light of the Grothendieck-type duality of Theorem 2.8.

For later reference we end with a porism of Theorem 2.20. Below, the statements involving $\operatorname{Proj} \Lambda$ follow from those involving $\operatorname{Inj} \Lambda$ by restricting the equivalence $\mathsf{K}(\operatorname{Inj} \Lambda) \cong \mathsf{K}(\operatorname{Proj} \Lambda)$.

Observation 2.21. Let Λ be a noetherian ring admitting a dualizing complex. Then

- (i) the inclusions K_{ac}(Inj Λ) → K(Inj Λ) and K_{coac}(Proj Λ) → K(Proj Λ) both admit a left and a right adjoint;
- (ii) the inclusions $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$ and $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda) \hookrightarrow \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda)$ both admit a left and a right adjoint.

2.7. Contravariant finiteness and torsion pairs. Let A be a category with a subcategory B. A right B-approximation of $A \in A$ is a morphism $B \to A$ with $B \in B$ through which each morphism $B' \to A$ with $B' \in B$ factors. B is called contravariantly finite in A if each $A \in A$ admits a right B-approximation. The dual notions are those of a left approximation and a covariantly finite subcategory, respectively. In the following sense, it is easy to generate such categories.

Lemma 2.22. Let A be a category and B a skeletally small subcategory. If A has coproducts, then Add B is contravariantly finite in A. Dually, if A has products, then Prod B is covariantly finite in A.

Proof. Assume that A has coproducts. For $A \in A$, let I denote the collection of all morphisms $B_i \to A$ with $B_i \in B$. Then I is a set and the canonical morphism

$$\prod_{i\in I} B_i \to A$$

is a right Add B-approximation. The remaining claim is dual.

A basic problem in Gorenstein homological algebra is determining when $\mathsf{GProj} R$ is contravariantly finite in $\mathsf{Mod} R$. For some time, an affirmative answer could

only be given under rather strong restrictions — see [18, Theorem 2.9] and [19, Theorem 3.4]. Jørgensen vastly improved on these when he showed in [24] that any module over an Artin algebra admits a right approximation by Gorenstein projective modules, and later used similar techniques in order to extend his theorem to the one below. We remark that one could hope to go even further, as there seems to be no known example of a ring R such that GProj R fails to be contravariantly finite in Mod R.

Theorem 2.23 (Jørgensen [26]). Consider either of the following two situations.

- (i) Λ is a commutative noetherian ring admitting a dualizing complex.
- (ii) Λ is a left coherent, right noetherian algebra over a field k for which there exists a left noetherian k-algebra Γ and a dualizing complex $_{\Gamma}D_{\Lambda}$ in the sense of [42].

Then $\operatorname{\mathsf{GProj}} \Lambda$ is contravariantly finite in $\operatorname{\mathsf{Mod}} \Lambda$.

A pair of subcategories (X, Y) of a triangulated catgory T is a *torsion pair* if T(X, Y) = 0, and each object $T \in T$ appears in a triangle

$$X_T \to T \to Y_T \to X_T[1]$$

with $X_T \in \mathsf{X}$ and $Y_T \in \mathsf{Y}$. If in addition we have $\mathsf{X}[1] \subseteq \mathsf{X}$, the torsion pair (X, Y) is called a *t-structure*. In this case X is an *aisle* and Y a *coaisle* in T . We remark that torsion pairs according to [6, Definition I.2.1] are in fact precisely t-structures. An aisle is always contravariantly finite since the above $X_T \to T$ is a right X-approximation of T. Moreover, approximations coming from aisles are functorial, since assigning $T \mapsto X_T$ gives a right adjoint R to the inclusion $\mathsf{X} \hookrightarrow \mathsf{T}$ —originally due to Keller–Vossieck [28]. As the dual claims hold for coaisles, the following picture always accompanies a t-structure.

$$X \underset{R}{\longleftrightarrow} T \xleftarrow{L} Y$$

This 'decomposition' of the ambient category T means that an aisle is a much stronger tool than a contravariantly finite subcategory. It is correspondingly more difficult to get a hold of and, indeed, the matter of generating aisles has become a central issue in modern algebra. Let us mention one remarkable result.

Theorem 2.24 (Neeman [35]). Let T be an idempotent complete triangulated category, and S a thick contravariantly finite subcategory. Then S is an aisle in T.

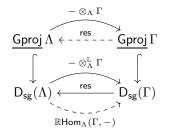
A stronger notion still is that of a *stable t-structure* on T, i.e. a t-structure (X, Y) in which X and Y are both closed under all shifts. In this case X and Y become thick subcategories of T, and the above adjoints R and L induce triangle equivalences $T / Y \cong X$ and $T / X \cong Y$, respectively.

3. SINGULARITY CATEGORIES AND GORENSTEIN PROJECTIVES

In this section we are concerned with how change of rings affects 'small' categories, that is categories derived in some way from categories of finitely generated modules. Therefore, throughout this section, we assume $f: \Lambda \to \Gamma$ to be a morphism of noetherian algebras such that $\mathsf{pdim}_{\Lambda}\Gamma$ and $\mathsf{pdim}\Gamma_{\Lambda}$ are finite.

The following theorem sums up the primary result of this section.

Theorem 3.1. Under the standing assumptions above we have the solid part of the following commutative diagram

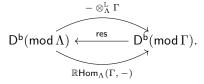


where the two functors on the bottom level form an adjoint pair.

If moreover $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda) \in \text{perf }\Gamma$, then res restricts to a functor between stable categories of finitely generated Gorenstein projective modules, and it has a right adjoint $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, -)$ on the level of singularity categories, as indicated by the dashed arrows above.

Singularity categories. Our strategy here is to observe first that the desired functors exist between bounded derived categories, and then transfer them to singularity categories.

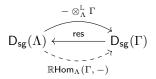
Lemma 3.2. Under the assumptions at the beginning of this section, there is an adjoint triple



Proof. Deriving the initial adjoint triple of functors between the big module categories gives an adjoint triple between $D(Mod \Lambda)$ and $D(Mod \Gamma)$. It is straightforward to verify that the latter restricts to bounded derived categories in our setup.

This diagram immediately gives us the lower part of Theorem 3.1:

Proposition 3.3. There is an adjoint pair of functors between singularity categories as in the following diagram.



If moreover $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda) \in \text{perf }\Gamma$, then there is an adjoint triple as indicated by the dashed arrow.

Proof. By Lemma 2.1 it suffices to check that the respective functors between bounded derived categories preserve perfect complexes. For $-\otimes_{\Lambda}^{\mathbb{L}} \Gamma$ this is automatic, and for res it follows since Γ is quasi-isomorphic to a perfect complex of Λ -modules by assumption. Finally, $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, -)$ preserves perfect complexes if and only if $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda) \in \text{perf }\Gamma$. Stable categories of Gorenstein projective modules. Recall that in the current section, the coacyclicity of a complex of projective modules is detected by the ring itself — see Remark 2.18.

Proposition 3.4. If $M \in \text{Gproj } \Lambda$, then $M \otimes_{\Lambda} \Gamma$ is a Gorenstein projective Γ -module and $\text{Tor}_n^{\Lambda}(M, \Gamma) = 0$ for n > 0.

Proof. By definition, there is a totally acyclic complex P over $\operatorname{proj} \Lambda$ such that $M = \mathsf{B}^0(P)$. As P is exact we see that $\mathsf{H}^i(P \otimes_{\Lambda} \Gamma) = \operatorname{Tor}_{j-i}(\mathsf{B}^{j+1}(P), \Gamma)$ for any j > i. Since the projective dimension of Γ as left Λ -module is assumed to be finite, we can always choose j sufficiently big so that this Tor vanishes. Thus $P \otimes_{\Lambda} \Gamma$ is exact.

It follows that $\operatorname{Tor}_n^{\Lambda}(M, \Gamma) = 0$, and that $M \otimes_{\Lambda} \Gamma = \mathsf{B}^0(P \otimes_{\Lambda} \Gamma)$. Thus the proof is complete provided we manage to show that $P \otimes_{\Lambda} \Gamma$ is totally acyclic. Acyclicity is already established. For coacyclicity note that

$$\operatorname{Hom}_{\Lambda}(P \otimes_{\Lambda} \Gamma, \Gamma) = \operatorname{Hom}_{\Lambda}(P, \Gamma),$$

and that the latter is exact because $\Gamma \in \mathsf{perf} \Lambda$ as right Λ -modules, and $\mathsf{Hom}_{\Lambda}(P, \Lambda)$ is exact by assumption.

Proposition 3.5. Assume that $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda) \in \text{perf }\Gamma$. Then the restriction functor between the singularity categories restricts to a functor between the stable categories of finitely generated Gorenstein projective modules.

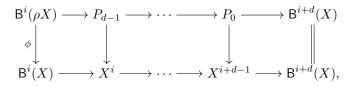
For the proof, we prepare the following technical observation.

Lemma 3.6. Let X be an acyclic complex of finitely generated Λ -modules, and suppose the projective dimensions of the terms of X are bounded by d. Then, for each i, there is a morphism $\phi: \mathsf{B}^i(\rho X) \to \mathsf{B}^i(X)$ such that the cone of ϕ is quasiisomorphic to a complex of finitely generated projectives concentrated in degrees -dto 0.

Proof. By Proposition 2.5 we have $\rho X = \Omega^d X[d]$. It follows directly from the construction of syzygies for complexes that syzygies commute with taking boundaries of exact complexes, thus

$$\mathsf{B}^{i}(\rho X) = \mathsf{B}^{i}(\Omega^{d} X[d]) = \Omega^{d} \mathsf{B}^{i+d}(X).$$

We hence obtain a commutative diagram



from which it follows that there is a quasi-isomorphism

$$\mathsf{Cone}(\phi) \to \left[P_{d-1} \to P_{d-2} \oplus X^i \to \dots \to \mathsf{B}^{i+d}(X) \oplus X^{i+d-1} \to \mathsf{B}^{i+d}(X) \right].$$

Note that the rightmost map here is a split epimorphism. After splitting off the term $\mathsf{B}^{i+d}(X)$, the complex

$$P_{d-1} \to P_{d-2} \oplus X^i \to \dots \to P_0 \oplus X^{i+d-2} \to X^{i+d-1}$$

extends from degrees -1 to d-1. Moreover, this complex admits non-zero homology only in degrees -1 and 0. In particular it is quasi-isomorphic to the complex

$$P_{d-1} \to K,$$

where K is the kernel of the map $P_{d-2} \oplus X^i \to P_{d-3} \oplus X^{i+1}$. Since K appears as the kernel of the exact sequence

$$0 \to K \to P_{d-2} \oplus X^i \to \dots \to P_0 \oplus X^{i+d-2} \to X^{i+d-1} \to 0$$

in which all other terms have projective dimension at most d, it follows that also the projective dimension of K is bounded above by d. Thus, replacing K by a minimal projective resolution we obtain the claim.

Proof of Proposition 3.5. Let $M \in \mathsf{Gproj}\,\Gamma$. What we need to show is that $\mathsf{res}\,M$ is, up to isomorphism in $\mathsf{D}_{\mathsf{sg}}(\Lambda)$, a Gorenstein projective Λ -module.

Let P be a complete resolution of M, that is a totally acyclic complex over $\operatorname{proj} \Gamma$ such that $M = B^{0}(P)$. Since finitely generated projective Γ -modules have bounded projective dimension as Λ -modules we are in the situation of Lemma 3.6 above. In particular

res
$$M = \mathsf{B}^0(\mathsf{res}\,P) \cong \mathsf{B}^0(\rho\,\mathsf{res}\,P)$$
,

since $B^0(\operatorname{res} P)$ and $B^0(\rho \operatorname{res} P)$ only differ by a finite complex of projectives.

It only remains to show that $B^0(\rho \operatorname{res} P)$ is Gorenstein projective over Λ . For this it clearly suffices to check that $\rho \operatorname{res} P$ is totally acyclic. The fact that $\rho \operatorname{res} P$ is acyclic follows directly from the construction. Thus it only remains for us to show that $\operatorname{Hom}_{\Lambda}(\rho \operatorname{res} P, \Lambda)$ is also acyclic. One may observe that the homologies of this complex are, for sufficiently large j, given by

$$\begin{split} \mathsf{H}^{i}(\mathsf{Hom}_{\Lambda}(\rho \operatorname{res} P, \Lambda)) &= \mathsf{Ext}_{\Lambda}^{i+j} \left(\mathsf{B}^{j+1}(\rho \operatorname{res} P), \Lambda\right) \\ &= \mathsf{Ext}_{\Lambda}^{i+j} \left(\mathsf{B}^{j+1}(\operatorname{res} P), \Lambda\right) \\ &= \mathsf{Ext}_{\Lambda}^{i+j} \left(\operatorname{res} \mathsf{B}^{j+1}(P), \Lambda\right) \\ &= \mathsf{Ext}_{\Gamma}^{i+j} \left(\mathsf{B}^{j+1}(P), \mathbb{R}\mathsf{Hom}_{\Lambda}(\Gamma, \Lambda)\right) \end{split}$$

which vanishes provided $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda)$ is perfect, since P is totally acyclic.

This furnishes the proof of Theorem 3.1.

Applications. For the remainder of the current section, we assume that Λ and Γ are module finite over some regular commutative noetherian ring k, and that f is k-linear. Our next result can be interpreted as saying that Λ is 'less singular' than Γ under the following condition on the cone of f.

Proposition 3.7. Suppose Cone $(f) \in perf \Lambda^{e}$. Then, in the notation of Theorem 3.1.

- (i) the functor ⊗[⊥]_Λ Γ: D_{sg}(Λ) → D_{sg}(Γ) is fully faithful;
 (ii) the functor ⊗_Λ Γ: Gproj Λ → Gproj Γ is fully faithful.

Proof. For (i) it suffices to show that the unit η of the adjunction $\left(-\otimes_{\Lambda}^{\mathbb{L}}\Gamma, \mathsf{res}\right)$ is an isomorphism on the level of singularity categories. For this purpose, notice that the triangle

$$\Lambda \xrightarrow{f} \Gamma \to \mathsf{Cone}(f) \to \Lambda[1]$$

in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda^{\mathsf{e}})$ will induce, for each $X \in \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$, the triangle

$$X \otimes^{\mathbb{L}}_{\Lambda} \mathsf{Cone}(f)[-1] \to X \xrightarrow{\eta_X} X \otimes^{\mathbb{L}}_{\Lambda} \Gamma \to X \otimes^{\mathbb{L}}_{\Lambda} \mathsf{Cone}(f)$$

in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$. By assumption the outer terms belong to $\mathsf{perf}\,\Lambda$, which means that η becomes an isomorphism in the quotient category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ as desired. Claim (ii) is now immediate, since the stable categories of Gorenstein projective modules are full subcategories of the respective singularity categories.

We collect an immediate consequence, using Proposition 3.7 in order to view $\mathsf{Gproj}\Lambda$ and $\mathsf{D}_{sg}(\Lambda)$ as subcategories of $\mathsf{Gproj}\Gamma$ and $\mathsf{D}_{sg}(\Gamma)$, respectively.

Corollary 3.8. Suppose $Cone(f) \in perf \Lambda^{e}$. Then we have the following.

(i) The pair of subcategories

$$(\mathsf{D}_{\mathsf{sg}}(\Lambda), \mathsf{Ker}(\mathsf{res}_{\mathsf{D}_{\mathsf{sg}}}))$$

is a stable t-structure in $\mathsf{D}_{\mathsf{sg}}(\Gamma)$. In particular, the fully faithful functor $-\otimes^{\mathbb{L}}_{\Lambda}\Gamma$ induces a triangle equivalence

$$\mathsf{D}_{\mathsf{sg}}(\Lambda) \cong \mathsf{D}_{\mathsf{sg}}(\Gamma) / \operatorname{\mathsf{Ker}}(\operatorname{\mathsf{res}}_{\mathsf{D}_{\mathsf{sg}}})$$

(ii) If moreover $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda)$ lies in perf Γ , then the pair of subcategories

 $(\mathsf{Gproj}\,\Lambda,\mathsf{Ker}(\mathsf{res}))$

is a stable t-structure in Gproj Γ . In this case the fully faithful functor $-\otimes_{\Lambda}\Gamma$ induces a triangle equivalence

$$\operatorname{\mathsf{Gproj}} \Lambda \cong \operatorname{\mathsf{Gproj}} \Gamma / \operatorname{\mathsf{Ker}}(\operatorname{\mathsf{res}}).$$

Let us include an easy application. Recall from [20] that f is a homological epimorphism if res: $D^{b}(mod \Gamma) \rightarrow D^{b}(mod \Lambda)$ is fully faithful. In this case also the functor res: $D_{sg}(mod \Gamma) \rightarrow D_{sg}(mod \Lambda)$ is fully faithful by Lemma 2.1. In particular, the kernel of the latter then vanishes, so Corollary 3.8 reveals the following.

Corollary 3.9. Suppose f is a homological epimorphism and that $Cone(f) \in perf \Lambda^e$. Then Λ and Γ are singularly equivalent.

Observe that the main result of [15] is recovered as the case $f: \Lambda \to \Lambda/I$ for Λ a finite dimensional algebra with a homological ideal I of finite projective dimension as bimodule.

Examples. We illustrate Proposition 3.7 and Corollary 3.8 with two examples — both of which will first be presented in a generic form and then illustrated completely explicitly for certain Nakayama algebras.

Corollary 3.10. Let Λ be a module finite algebra over some regular commutative noetherian ring k. Suppose I is an ideal of Λ which, as a bimodule, is isomorphic to a tensor product

$$I \cong M \otimes_k N$$

where M and N are left and right Λ -modules, respectively, both of finite projective dimension. Then the functors

$$-\otimes_{\Lambda}^{\mathbb{L}} \Lambda/I \colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(\Lambda/I) \ and \ -\otimes_{\Lambda} \Lambda/I \colon \operatorname{\underline{\mathsf{Gproj}}} \Lambda \to \operatorname{\underline{\mathsf{Gproj}}} \Lambda/I$$

are both fully faithful.

Proof. The cone of $\Lambda \to \Lambda/I$ is I[1]. Thus, by Proposition 3.7 it suffices for I to be a perfect bimodule. This is guaranteed if I is a tensor product of two perfect modules.

Corollary 3.11. Let Λ be a finite dimensional algebra over field k, and let I be a 1-dimensional ideal. Assume that the projective dimension of I is finite both as left and as right module. Then the functors

 $-\otimes^{\mathbb{L}}_{\Lambda} \Lambda/I \colon \operatorname{\mathsf{D}_{sg}}(\Lambda) \to \operatorname{\mathsf{D}_{sg}}(\Lambda/I) \ and \ -\otimes_{\Lambda} \Lambda/I \colon \operatorname{\underline{\mathsf{Gproj}}} \Lambda \to \operatorname{\underline{\mathsf{Gproj}}} \Lambda/I$

are both fully faithful.

Proof. Since I is 1-dimensional, we have $I \cong I \otimes_k I$ as bimodules. Thus the claim follows from Corollary 3.10 above.

Example 3.12. Let k be a field and $\Lambda = k [1 \underbrace{\smile}_{a}^{b} 2]/(ab)^{n}$ for some n > 0.

Consider the ideal $I = (ba)^n$. Then I is isomorphic to the simple at vertex 1 both as left and as right module, and moreover these simples have projective dimension 1. Thus Corollary 3.11 applies, and we obtain fully faithful functors

$$-\otimes_{\Lambda}^{\mathbb{L}} \Lambda/I \colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(\Lambda/I) \text{ and } -\otimes_{\Lambda} \Lambda/I \colon \underline{\mathsf{Gproj}} \Lambda \to \underline{\mathsf{Gproj}} \Lambda/I.$$

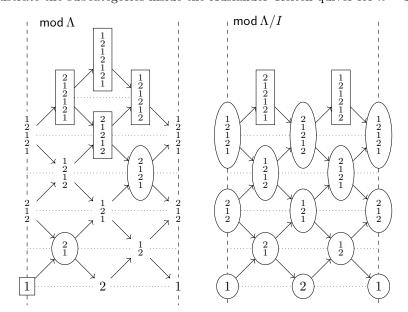
In fact, in this example both Λ and Λ/I are Gorenstein, so the singularity categories coincide with the respective stable categories of Gorenstein projective modules. Moreover, Λ/I is even self-injective, so $\mathsf{D}_{\mathsf{sg}}(\Lambda/I) = \operatorname{mod} \Lambda/I$. One may observe that

$$\mathsf{Ker}(\mathsf{res}_{\mathsf{D}_{\mathsf{sg}}}) = \{ M \in \underline{\mathsf{mod}}(\Lambda/I) \mid \mathsf{pdim}\, M_{\Lambda} < \infty \} = \mathsf{add}\{S_1, P_2/S_1\}.$$

It now follows from Corollary 3.8 that

$$\mathsf{D}_{\mathsf{sg}}(\Lambda) = {}^{\perp}\mathsf{Ker}(\mathsf{res}_{\mathsf{D}_{\mathsf{sg}}}) = \mathsf{add}\{P_2/\mathsf{rad}^2 P_2, P_2/\mathsf{rad}^4 P_2, \dots, P_2/\mathsf{rad}^{2n-2} P_2\}.$$

We illustrate the subcategories inside the Auslander-Reiten quiver for $n = 3$:



In both cases, the dashed lines are identified. The rectangles mark modules of finite projective dimension (including the projective modules at the top), and the ellipses mark non-projective Gorenstein projective modules. In the picture clearly the ellipses on the left are a subset of those on the right.

Corollary 3.13. Let Λ be module finite over a regular commutative noetherian ring k. Assume M and N are left and right Λ -modules of finite projective dimension, respectively. Set

$$\Gamma = \Lambda \ltimes (M \otimes_k N).$$

Then there are fully faithful functors

$$-\otimes_{\Lambda}^{\mathbb{L}} \Gamma \colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(\Gamma) \ and \ -\otimes_{\Lambda} \Gamma \colon \operatorname{\underline{\mathsf{Gproj}}} \Lambda \to \operatorname{\underline{\mathsf{Gproj}}} \Gamma.$$

Proof. This is an immediate application of Proposition 3.7 above.

Example 3.14. Let k be a field and $\Lambda = k \left[\begin{array}{c} 1 \xrightarrow{k} \\ c \\ s \end{array} \right] / (cb, bac).$

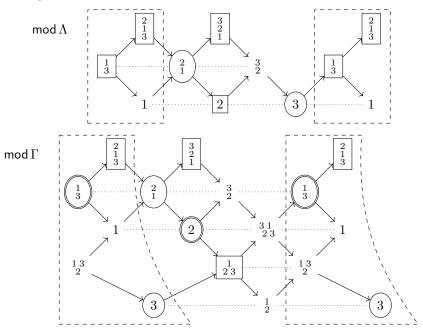
We see that the simple at vertex 2 has projective dimension 1 as right Λ -module, and so does the simple at vertex 1 as left Λ -module. Therefore, for

$$\Gamma = \Lambda \ltimes (S_1^{\mathsf{left}} \otimes_k S_2) = k \left[\underbrace{ 1_{\overbrace{c}{\overset{\vee}{3}}}^{x} 2}_{\underset{c}{\overset{\vee}{3}}} \right] / (cb, bac, ax, xa),$$

Corollary 3.13 applies and we obtain fully faithful functors

$$-\otimes^{\mathbb{L}}_{\Lambda} \Gamma \colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(\Gamma) \text{ and } -\otimes_{\Lambda} \Gamma \colon \underline{\mathsf{Gproj}}\,\Lambda \to \underline{\mathsf{Gproj}}\,\Gamma.$$

As before, we illustrate the relevant categories inside the Auslander–Reiten quivers in order to get an immediate visual confirmation of the result.



The dashed boxes indicate which parts of the quivers are identified. The rectangles indicate modules of finite projective dimension — for Γ these are only the projective modules. In particular, one sees that not all injective modules are inside rectangles, that is the algebras are not Gorenstein. The modules inside ellipses are representatives of the indecomposable objects in the singularity category. (For both algebras, all projective resolutions are eventually periodic, so the singularity categories are Krull–Schmidt with indecomposable objects being represented by the infinite syzygies. However, the singularity category of Γ is not a subcategory of the stable module category: Contrary to what the picture indicates, there is no non-zero morphism from $\frac{2}{1}$ to 2 in the singularity category of Γ , and this singularity category is in fact semisimple.) The doubly circled modules are the non-projective Gorenstein projective modules. One immediately verifies that the circles in the upper picture are a subcollection of the circles in the lower one, and similarly — albeit vacuously — for double circles.

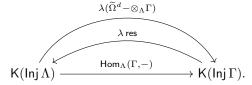
4. BIG SINGULARITY CATEGORIES

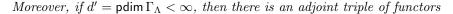
We now investigate to what extent change of rings gives rise to functors between the corresponding homotopy categories of projective or injective modules, and prominent subcategories thereof. In particular, the results of this section show that the diagram in Theorem 3.1 is in fact the restriction of a picture that even exists on the level of the compactly generated completions from Subsection 2.6.

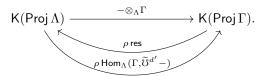
The functors ρ and λ will occur frequently, and we write $\Omega = \Omega \circ [1]$ and $\mho = \mho \circ [-1]$ for the sake of brevity — confer Proposition 2.5.

Full homotopy categories. We start by demonstrating how from a morphism of rings subject only to rather lenient assumptions, an abundance of adjoints are induced on the level of full homotopy categories.

Proposition 4.1. Let $f: \Lambda \to \Gamma$ be a morphism of rings. If $d = pdim_{\Lambda}\Gamma < \infty$, then there is an adjoint triple of functors







Proof. We only discuss the first claim. The second one is dual, and in fact easier since it avoids the slight extra problem of translating left projective to right injective dimension.

First observe that $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ maps injective Λ -modules to injective Γ -modules. Indeed $\operatorname{Hom}_{\Gamma}(-, \operatorname{Hom}_{\Lambda}(\Gamma, I)) \cong \operatorname{Hom}_{\Lambda}(\operatorname{res} -, I)$ is exact for any injective Λ -module I. Therefore the bottom functor is well-defined.

Since res is left adjoint to $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ on the level of homotopy categories of all modules, and λ is left adjoint to the inclusion of the homotopy category of injectives into that category, λ res is left adjoint to $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ on the level of homotopy categories of injectives.

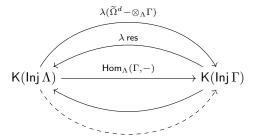
Next we observe that the right injective dimension of res I for injective Γ -modules I is bounded by d. Indeed, since $-\otimes_{\Lambda}\Gamma$ is left adjoint to res we have an isomorphism $\operatorname{Hom}_{\Lambda}(-, \operatorname{res} I) \cong \operatorname{Hom}_{\Gamma}(-\otimes_{\Lambda}\Gamma, I)$, and the latter has at most d derived functors.

We can thus apply Propositions 2.5 and 2.6 in the first two lines below, and obtain the following isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(X,\lambda\,\mathsf{res}\,Y) &= \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(X,\mho^{d}\,\mathsf{res}\,Y) \\ &= \mathsf{Hom}_{\mathsf{K}(\Lambda)}(\widetilde{\Omega}^{d}X,\mathsf{res}\,Y) \\ &= \mathsf{Hom}_{\mathsf{K}(\Gamma)}(\widetilde{\Omega}^{d}X\otimes_{\Lambda}\Gamma,Y) \\ &= \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Gamma)}(\lambda(\widetilde{\Omega}^{d}X\otimes_{\Lambda}\Gamma),Y). \end{aligned}$$

We infer that $\lambda(\widetilde{\Omega}^d - \otimes_{\Lambda} \Gamma)$ is left adjoint to λ res.

Proposition 4.2. Let $f: \Lambda \to \Gamma$ be a morphism of noetherian algebras with $d = \mathsf{pdim}_{\Lambda}\Gamma < \infty$. Then there is an adjoint quadruple as indicated by the solid arrows in the following diagram. If also $\mathsf{pdim}\,\Gamma_{\Lambda} < \infty$, then we even have a five-tuple of adjoint functors.



Proof. Consider first the functor $\operatorname{Hom}_{\Lambda}(\Gamma, -)$. We observe that its left adjoint λ respectives to the respective subcategories of compacts. Therefore $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ has a right adjoint by Theorem 2.2.

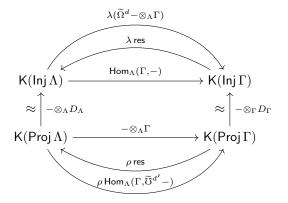
Similarly, for the bottom solid functor, its left adjoint $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ preserves compact objects if and only if Γ_{Λ} has finite projective dimension. Thus, in this case we obtain the additional dashed adjoint.

Unfortunately, the two new functors in the above proposition are not explicit. However, in the case that our rings are module finite over some commutative Gorenstein ring, we get a much more explicit picture.

Proposition 4.3. Let $f: \Lambda \to \Gamma$ be a morphism of module finite algebras over a commutative noetherian Gorenstein ring. Assume that $d = pdim_{\Lambda}\Gamma < \infty$ and that $d' = pdim \Gamma_{\Lambda} < \infty$. Then we have the following diagram of adjoints, with a

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commutative square in the middle.



Proof. This is a combination of Proposition 4.1 (for the adjunctions) and Proposition 2.12 (for the commutative square). \Box

Acyclic and coacyclic complexes. We now consider the subcategories of acyclic and coacyclic complexes. Our strategy is to investigate which of the above established functors restrict to this level under which conditions.

Proposition 4.4. Let $f: \Lambda \to \Gamma$ be a morphism of rings. Assume that $d = p\dim \Gamma_{\Lambda} < \infty$. Then we have the following.

- (i) The functors Hom_Λ(Γ, -) and λ res restrict to functors between homotopy categories of acyclic complexes of injectives.
- (ii) The functor − ⊗_Λ Γ restricts to a functor between homotopy categories of coacyclic complexes of projectives.
- (iii) If additionally for any projective Λ-module Q the complex RHom_Λ(Γ, Q) is quasi-isomorphic to a bounded complex of projective Γ-modules, then the functor ρ res restricts to a functor between categories of coacyclic complexes of projectives.

If Γ is perfect as Λ -module, the condition is satisfied when $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda)$ lies in perf Γ .

Proof. For the first part of (i) observe that if $I \in \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda)$, then

$$\mathsf{H}^{i}(\mathsf{Hom}_{\Lambda}(\Gamma, I)) = \mathsf{Ext}_{\Lambda}^{i+j}(\Gamma, \mathsf{B}^{-j}(I)) = 0 \text{ for } j \gg 0$$

since $\operatorname{pd}\Gamma_{\Lambda} < \infty$. For the second part of (i) note that res preserves acyclicity, and so does λ by the construction in Proposition 2.5.

For (ii), let $P \in \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda)$. For $Q \in \mathsf{Proj}\,\Gamma$ we have

$$\operatorname{Hom}_{\mathsf{K}(\Gamma)}(P \otimes_{\Lambda} \Gamma, Q) = \operatorname{Hom}_{\mathsf{K}(\Lambda)}(P, Q),$$

which vanishes since $\operatorname{Hom}_{\mathsf{K}(\Lambda)}(P, \operatorname{Proj} \Lambda[i]) = 0$, and Q has a finite projective resolution over Λ .

For (iii), we let $P \in \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Gamma)$ and look at $\mathsf{Hom}_{\mathsf{K}(\Lambda)}(\rho \operatorname{res} P, Q)$. By adjunction on the level of full homotopy categories of projectives we have

$$\operatorname{Hom}_{\mathsf{K}(\Lambda)}(\rho\operatorname{res} P,Q) = \operatorname{Hom}_{\mathsf{K}(\Gamma)}(P,\rho\operatorname{Hom}_{\Lambda}(\Gamma,\mho^{d}Q)).$$

Note that the second argument in the $\mathsf{Hom}_{\mathsf{K}(\Gamma)}$ is just a projective resolution of $\mathbb{R}\mathsf{Hom}_{\Lambda}(\Gamma, Q)$. As this is finite by assumption, the Hom-space vanishes since $\mathsf{Hom}_{\mathsf{K}(\Gamma)}(P, -)$ vanishes on all projectives.

Finally, if Γ is perfect as a Λ -module then $\mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda^{(I)}) \cong \mathbb{R}\text{Hom}_{\Lambda}(\Gamma, \Lambda)^{(I)}$, whence it suffices to consider the projective Λ -module Λ .

Observe that there is a certain asymmetry in the above, in the sense that λ res automatically restricts, while for ρ res we resort to the peculiar extra condition. We collect this as yet another argument for prefering injective to projective objects in this context.

'Big' versions of Proposition 3.7 and its consequences now come for free.

Corollary 4.5. Let Λ and Γ be module finite over some regular commutative noetherian ring k, and the ring homomorphism f be k-linear. Assume that $Cone(f) \in$ perf Λ^{e} . Then the functor

$$- \otimes_{\Lambda} \Gamma \colon \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) \to \mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Gamma)$$

is fully faithful. In particular, if additionally $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda) \in perf \Gamma$ then the pair

 $(\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda),\mathsf{Ker}(\rho\,\mathsf{res}))$

is a stable t-structure in $K_{coac}(\operatorname{Proj} \Gamma)$.

Proof. Under these assumptions, $-\otimes_{\Lambda} \Gamma$ does give a functor between the categories in question by Proposition 4.4. Moreover, $-\otimes_{\Lambda}^{\mathbb{L}} \Gamma$ is fully faithful on the level of singularity categories by Proposition 3.7. By Lemma 2.3 this suffices since the singularity category embeds as the compact objects in the homotopy category of coacyclic complexes of projective modules.

Totally acyclic complexes. We now descend further to the homotopy categories of totally acyclic complexes. As before, we investigate which of the above functors exist on this level under which conditions.

Proposition 4.6. Let $f: \Lambda \to \Gamma$ be a morphism of rings. Assume that $d = \mathsf{pdim}_{\Lambda}\Gamma < \infty$ and that $d' = \mathsf{pdim}_{\Gamma} \Lambda < \infty$ Then we have the following.

- (i) The functor $\operatorname{Hom}_{\Lambda}(\Gamma, -)$ restricts to a functor between homotopy categories of totally acyclic complexes of injectives.
- (ii) The functor ⊗_Λ Γ restricts to a functor between homotopy categories of totally acyclic complexes of projectives.
- (iii) If additionally $\mathbb{R}Hom_{\Lambda}(\Gamma, Q) \in \mathsf{K}^{\mathsf{b}}(\mathsf{Proj}\,\Gamma)$ for any $Q \in \mathsf{Proj}\,\Lambda$, then the functor ρ res restricts to a functor between categories of totally acyclic complexes of projectives.

If Γ is perfect as Λ -module, the condition is satisfied when $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda)$ lies in perf Γ .

(iv) If additionally $J \otimes_{\Lambda}^{\mathbb{L}} \Gamma \in \mathsf{K}^{\mathsf{b}}(\mathsf{Inj}\,\Gamma)$ for any $J \in \mathsf{Inj}\,\Lambda$, then the functor λ respectives to a functor between categories of totally acyclic complexes of injectives.

Proof. The first three points can be proved very similarly to Proposition 4.4 above. We focus on (iv). Also as above, one sees that for a totally acyclic complex of injectives I, $\lambda \operatorname{res} I$ is acyclic again. For it to be totally acyclic, it remains to check that $\operatorname{Hom}_{\mathsf{K}(\Gamma)}(J, \lambda \operatorname{res} I) = 0$ for all injective Λ -modules J. By adjunction we have

$$\operatorname{Hom}_{\mathsf{K}(\Lambda)}(J,\lambda\operatorname{res} I) = \operatorname{Hom}_{\mathsf{K}(\Gamma)}(\lambda(\Omega^{d}J\otimes_{\Lambda}\Gamma),I).$$

 \sim ,

We observe that the left argument of the final Hom-above is an injective resolution of $J \otimes_{\Lambda}^{\mathbb{L}} \Gamma$, so it is a bounded complex of injectives by assumption. Thus the total acyclicity of I implies that these Hom-groups are zero.

By restriction of Corollary 4.5 we immediately get the following. Notice that it is in fact not necessary to invoke Proposition 3.7 and Lemma 2.3 directly by assuming virtual Gorensteinness.

Corollary 4.7. Let Λ and Γ be module finite over some regular commutative noetherian ring k, and the ring homomorphism f be k-linear. Assume that $Cone(f) \in$ perf Λ^{e} . Then there is a fully faithful functor

$$- \otimes_{\Lambda} \Gamma \colon \mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda) \to \mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Gamma).$$

In particular, if additionally $\mathbb{R}Hom_{\Lambda}(\Gamma, \Lambda) \in perf \Gamma$ then the pair

$$(\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda), \mathsf{Ker}(\rho\,\mathsf{res}))$$

is a stable t-structure in $K_{tac}(\operatorname{Proj} \Gamma)$.

Summary. Restricted to the case where $f: \Lambda \to \Gamma$ is a morphism of module finite algebras over some commutative noetherian Gorenstein ring such that $d = \max\{\mathsf{pdim}_{\Lambda}\Gamma,\mathsf{pdim}\Gamma_{\Lambda}\} < \infty$, the results of the current section can be brought together in Diagram 4.1 below. A few comments are in order.

The top level is just a restatement of Proposition 4.3.

Evidently, the two lower levels are comprised of subcategories of the top level, so we have the vertical inclusions. Moreover, by Observation 2.21 these inclusions have both left and right adjoints, as indicated by the downward pointing arrows labeled L and R, respectively. It should be noted that these functors are given explicitly in restricted setups: The left adjoints are constructed in Bousfield's localization lemma — see Corollary 6.2. For the right adjoints, we give an explicit description in the case that the two rings are Artin algebras in Corollary 6.12.

Most of the arrows, i.e. those not involving any R or L, in the two lower levels of Diagram 4.1 arise by observing that the corresponding arrows on the top level restrict to these subcategories. In Propositions 4.4 (middle level) and 4.6 (lower level) we collected which functors automatically restrict. The same propositions also contain information on more functors restricting under certain additional hypothesis, which gives a stronger version of Diagram 4.1 in a restricted setup.

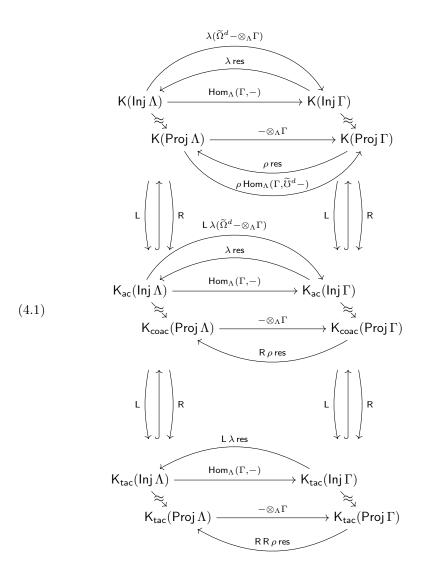
Finally, on each of the two lower levels, we obtain one 'extra adjoint' by composing adjoints around the square above the functor in question. For instance, the functor $\lambda \operatorname{res}$: $\operatorname{K}_{\operatorname{ac}}(\operatorname{Inj} \Gamma) \to \operatorname{K}_{\operatorname{ac}}(\operatorname{Inj} \Lambda)$ may be written as the composition

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Gamma) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Gamma) \xrightarrow{\lambda \operatorname{res}} \mathsf{K}(\mathsf{Inj}\,\Lambda) \xrightarrow{\mathsf{R}} \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda),$$

whence it has a left adjoint given by the composition

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda) \xrightarrow{\lambda(\widetilde{\Omega}^d - \otimes_\Lambda \Gamma)} \mathsf{K}(\mathsf{Inj}\,\Gamma) \xrightarrow{\mathsf{L}} \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Gamma).$$

Similarly we obtain all the extra adjoints involving L or R.



5. Approximations of modules

In [24] and the subsequent [26], Jørgensen plays the following game in order to obtain right approximations by Gorenstein projective modules. Under rather forgiving hypotheses on Λ , he establishes the contravariant finiteness of $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda)$ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$ before passing to $\mathsf{Mod}\,\Lambda$ by taking boundaries, a procedure which turns out to take a right $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda)$ -approximation of a projective resolution ρM of M to a right $\mathsf{GProj}\,\Lambda$ -approximation of M. We will now follow these ideas in two directions: We first observe that over an arbitrary ring R, any contravariantly finite subcategory of $\mathsf{K}(\mathsf{Proj}\,R)$ gives rise to a contravariantly finite subcategory of $\mathsf{Mod}\,R$ by taking cokernels. Next, an ever so slight expansion of the above Theorem 2.23 is obtained. **Lemma 5.1.** Let A be a Frobenius exact category. If \underline{B} is a contravariantly finite subcategory in \underline{A} , then its preimage B is contravariantly finite in A.

Proof. Each $X \in A$ admits a right B-approximation

$$B \oplus P_X \xrightarrow{\left(\pi \ p \right)} X$$

where p is a projective cover and π is any lift of a right B-approximation.

Proposition 5.2. Let R be a ring, and suppose S is a contravariantly finite subcategory of K(Proj R). Denote by T the preimage of S in C(Proj R) and by Cok(T) the subcategory of Mod R consisting of all the cokernels of the form $Cok(T^{-1} \rightarrow T^0)$ for T a complex belonging to T. Then Cok(T) is contravariantly finite in Mod R.

Proof. For each $X \in \mathsf{C}(\mathsf{Proj} R)$ we denote by $\mathsf{Cok}(X) = \mathsf{Cok}(X^{-1} \to X^0) \in \mathsf{Mod} R$ and let $\pi_X : X \to \mathsf{Cok}(X)$ be the natural projection. Pick $M \in \mathsf{Mod} R$ and choose a projective resolution ρM . In particular $M = \mathsf{Cok}(\rho M)$. Since T is contravariantly finite in $\mathsf{C}(\mathsf{Proj} R)$ by Lemma 5.1, ρM admits a right T-approximation $\alpha : t(\rho M) \to \rho M$. We claim that the induced morphism $a : \mathsf{Cok}(t(\rho M)) \to M$ satisfying

$$a \circ \pi_{t(\rho M)} = \pi_{\rho M} \circ \alpha$$

is a right $\mathsf{Cok}(\mathsf{T})$ -approximation in $\mathsf{Mod} R$. What we need to show is that each morphism $b: \mathsf{Cok}(T) \to M$ in $\mathsf{Mod} R$ with $T \in \mathsf{T}$ factors through a. By 'comparison', such a b lifts to a chain map $\beta: T \to \rho M$ such that

$$b \circ \pi_T = \pi_{\rho M} \circ \beta,$$

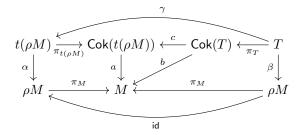
and since α is a right T-approximation, there is a chain map $\gamma: T \to t(\rho M)$ making

$$\beta = \alpha \circ \gamma$$

Letting c: $\mathsf{Cok}(T) \to \mathsf{Cok}(t(\rho M))$ be the induced morphism in $\mathsf{Mod} R$ ensuring

$$c \circ \pi_T = \pi_{t(\rho M)} \circ \gamma,$$

we have by now established the diagram



with each square commutative. It follows that $b \circ \pi_T = a \circ c \circ \pi_T$ and hence the desired factorization $b = a \circ c$ since π_T is an epimorphism.

The alluded to existence of right approximations by Gorenstein projective modules is now immediate once one makes the following standard observation.

Lemma 5.3. Let A be a category and suppose B is a subcategory such that the inclusion μ : B \hookrightarrow A has a right adjoint μ_* . Then B is contravariantly finite in A.

Proof. It suffices to show that, for each $A \in A$, the counit $\varepsilon_A \colon \mu \circ \mu_*(A) \to A$ is a right B-approximation. But the isomorphism of the adjunction is given, for each $B \in B$, by

$$\mathsf{B}(B,\mu_*(A)) \xrightarrow{\mu} \mathsf{A}(\mu(B),\mu \circ \mu_*(A)) \xrightarrow{\mathsf{A}(\mu(B),\varepsilon_A)} \mathsf{A}(\mu(B),A).$$

In particular this means that $A(\mu(B), \varepsilon_A)$ is an epimorphism.

Corollary 5.4. Let Λ be a noetherian ring with a dualizing complex. Then $\mathsf{GProj} \Lambda$ is contravariantly finite in $\mathsf{Mod} \Lambda$.

Proof. A combination of Observation 2.21 and Lemma 5.3 reveals the contravariant finiteness of $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda)$ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$, so Proposition 5.2 settles the claim.

6. An explicit right adjoint

The aim of this section is to provide a somewhat strengthened dual of Bousfield's localization lemma [9,10]. We start by recalling the construction as presented in [31], see also [6, Chapter III, Theorem 2.3] and [1, Theorem 4.3].

Here, for the sake of generality, we consider perpendicular categories with respect to only certain shifts, rather than all shifts. More precisely, we write

$$\mathsf{X}^{\perp_{0}} = \{ T \in \mathsf{T} \mid \mathsf{T}(\mathsf{X}, T) = 0 \}, \qquad \mathsf{X}^{\perp_{\leq 0}} = \{ T \in \mathsf{T} \mid \forall i \leq 0 \colon \mathsf{T}(\mathsf{X}, T[i]) = 0 \},$$

$${}^{\perp_{0}}\mathsf{X} = \{T \in \mathsf{T} \mid \mathsf{T}(T,\mathsf{X}) = 0\}, \qquad {}^{\perp_{\leq 0}}\mathsf{X} = \{T \in \mathsf{T} \mid \forall i \leq 0 \colon \mathsf{T}(T,\mathsf{X}[i]) = 0\},$$

and similar for $\perp_{>0}$. With this convention we obviously have

(6.1)
$$\mathsf{X}^{\perp} = \{X[i] \mid X \in \mathsf{X}, i \in \mathbb{Z}\}^{\perp_0}$$

and similar for left perpendicular subcategories.

Theorem 6.1. Let T be a triangulated category which has coproducts, and let X be a set of compact objects.

(i) The pair

$$\left({}^{\perp_0}(\,\mathsf{X}^{\perp_0}),\mathsf{X}^{\perp_0}\right)$$

is a torsion pair on T.

(ii) The pair

$$\left(\stackrel{\perp_{\leq 0}}{\overset{(}} \mathsf{X}^{\perp_{\leq 0}}), \mathsf{X}^{\perp_{\leq 0}} \right)$$

is a t-structure, and moreover $^{\perp_{\leq 0}}(X^{\perp_{\leq 0}})$ is the smallest subcategory of T containing X and closed under [1], coproducts, and extensions.

From the proof of this theorem, we only recall the following explicit construction of the triangle

$$T_{\mathsf{X}} \to T \to T_{\mathsf{X}^{\perp_0}} \to T_{\mathsf{X}}[1]$$

with $T_{\mathsf{X}} \in {}^{\perp_0}(\mathsf{X}^{\perp_0})$ and $T_{\mathsf{X}^{\perp_0}} \in \mathsf{X}^{\perp_0}$:

Let $T_0 = T$. Pick a right Add X-approximation $X_0 \to T_0$ of T_0 — this approximation exists since X is a set and T has coproducts — and complete it to a triangle $X_0 \to T_0 \to T_1 \to X_0[1]$. Pick a right Add X-approximation $X_1 \to T_1$ of T_1 and complete it to a triangle $X_1 \to T_1 \to T_2 \to X_1[1]$. Then $T_{\mathsf{X}^{\perp_0}} = \mathsf{hocolim} T_i$, and T_X is obtained by completing the natural map $T \to T_{\mathsf{X}^{\perp_0}}$ to a triangle.

This construction covers each functor L appearing in Diagram 4.1 in the Summary of Section 4. In that setup we consider subcategories X^{\perp} orthogonal to a

given object with respect to all extensions as specified in Proposition 2.15. By (6.1) this is a special case of the setup of Theorem 6.1, and in this case we obtain a stable t-structure $(^{\perp}(X^{\perp}), X^{\perp})$. Thus we obtain the following descriptions of the left adjoints.

Corollary 6.2. Let Λ be a noetherian ring with a dualizing complex D_{Λ} . Then the following hold.

(i) The left adjoint of the inclusion $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Proj}\,\Lambda)$ is given by

 $T \mapsto T_{(\rho \, \mathbb{R} \mathsf{Hom}_{\Lambda}(D_{\Lambda}, \Lambda))^{\perp}};$

(ii) The left adjoint of the inclusion $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda)$ is given by

 $T \mapsto T_{(\lambda \Lambda)^{\perp}};$

(iii) The left adjoint of the inclusion $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Proj}\,\Lambda)$ is given by

 $T \mapsto T_{((\rho \mathbb{R} Hom_{\Lambda}(D_{\Lambda}, \Lambda)) \oplus \Lambda)^{\perp}};$

(iv) The left adjoint of the inclusion $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda)$ is given by

 $T \mapsto T_{((\lambda\Lambda) \oplus D_{\Lambda})^{\perp}}.$

Unfortunately, the naive dual of Theorem 6.1 is not as useful as the original in the situations we are typically interested in. Indeed, triangulated categories which somehow derive from module categories rarely have any non-zero cocompact objects. However, as we will point out in the sequel, we can get away with the weaker notion of 0-cocompactness which is not as elusive in concrete situations.

This section consists of two essentially independent parts. First, in Theorem 6.6 we provide the desired dual. Then, in Theorem 6.8 and in particular Corollary 6.10 we show that the prerequisites of our dual are met in reasonable situations.

We start by introducing our notion of 0-cocompactness.

Definition 6.3. A sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$$

of objects and morphisms in an abelian category is dual Mittag-Leffler if for each i the increasing sequence

$$0 \subset \operatorname{Ker} f_i \subset \operatorname{Ker} f_{i+1} f_i \subset \cdots$$

stabilizes.

An object X in a triangulated category T is 0-cocompact if $T(\operatorname{holim} Y_i, X) = 0$ for each sequence

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

in ${\sf T}$ with the property that the induced sequence

$$\mathsf{T}(Y_0, X[1]) \to \mathsf{T}(Y_1, X[1]) \to \mathsf{T}(Y_2, X[1]) \to \cdots$$

is dual Mittag-Leffler and colim $T(Y_i, X) = 0$.

Remark 6.4. Let us motivate the name "0-cocompact" and explain in what way it is a weak version of cocompact.

Recall that in a triangulated category T with set-indexed coproducts, an object X is *compact* if the functor $T(X, -): T \to Ab$ commutes with coproducts. If T has set-indexed products, then the dual notion should be formulated as follows. An

object $X \in \mathsf{T}$ is 'cocompact' if the functor $\mathsf{T}(-, X)$: $\mathsf{T}^{\mathsf{op}} \to \mathsf{Ab}$ turns products to coproducts, i.e. for any set of objects $(Y_i \mid i \in I)$ in T we have an isomorphism

(6.2)
$$\operatorname{Hom}_{\mathsf{T}}(\prod_{i\in I}Y_i, X)\cong \coprod_{i\in I}\operatorname{Hom}_{\mathsf{T}}(Y_i, X).$$

Consider now a finite dimensional algebra Λ . Clearly, Λ is a compact object in the derived category $D(Mod \Lambda)$. On the other hand, the dual $D\Lambda$ awkwardly fails to be cocompact in $D(Mod \Lambda)$. However, we show in Theorem 6.8 that $D\Lambda$ is in fact 0-cocompact in the homotopy category $K(Mod \Lambda)$.

Replacing products by homotopy limits and coproducts by colimits, X being 0cocompact simply means that if the right hand side of (6.2) is zero and some extra conditions hold, then the left hand side of (6.2) is also zero. As it turns out, this property suffices for our purposes. Let us revisit the cocomplete case and consider a sequence $X_0 \to X_1 \to X_2 \to \cdots$ in T. Then by Neeman [31], we know that there is the following isomorphism as functors on T^c .

(6.3)
$$\operatorname{colim} \mathsf{T}(-, X_i) \cong \mathsf{T}(-, \operatorname{hocolim} X_i)$$

From this point of view, the 0-cocompactness of X means that the vanishing of the dual left side of (6.3) implies the vanishing of the dual right side of (6.3).

Remark 6.5. The above definition might seem quite unnatural: Why would one want to require the dual Mittag-Leffler condition at all? And why for shifts?

Consider the case that T = D(A), where A is an abelian category with exact products. For a sequence

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

of complexes, we see that the defining triangle

$$\operatorname{holim} Y_i \to \prod Y_i \to \prod Y_i \to$$

gives rise to the exact sequence

$$\begin{split} \prod \mathsf{H}^{0}(Y_{i}[-1]) & \to \prod \mathsf{H}^{0}(Y_{i}[-1]) \longrightarrow \mathsf{H}^{0}(\operatorname{holim} Y_{i}) \longrightarrow \prod \mathsf{H}^{0}(Y_{i}) \longrightarrow \prod \mathsf{H}^{0}(Y_{i}) \\ & \swarrow & \swarrow & \swarrow & \swarrow \\ & \varprojlim^{1} \mathsf{H}^{0}(Y_{i}[-1]) & \varprojlim \mathsf{H}^{0}(Y_{i}) \end{split}$$

Thus we see that, even if a Hom-functor behaves as nicely as homology, we can only get a result on holim Y_i provided we have a condition on the derived projective limit of the shifted sequence.

To make this more explicit, take \mathcal{A} to be the category of vector spaces over a field k, and choose X = k. Then $\operatorname{Hom}_{D(k)}(-, k) = D \operatorname{H}^{0}$. Thus we have the short exact sequence

$$0 \to \operatorname{Hom}_{\mathsf{D}(k)}(\varprojlim Y_i, k) \to \operatorname{Hom}_{\mathsf{D}(k)}(\operatorname{holim} Y_i, k) \to \operatorname{Hom}_{\mathsf{D}(k)}(\varprojlim^1 Y_i, k[1]) \to 0$$

showing that we need to make both end terms vanish in order to conclude that the middle also vanishes.

We are now ready to state and prove the main result of this section.

Theorem 6.6. Let T be a triangulated category which has products. Let X be a set of 0-cocompact objects.

(i) The pair

$$\left({}^{\perp_{\geq 0}}\mathsf{X},({}^{\perp_{\geq 0}}\mathsf{X})^{\perp_{\geq 0}}\right)$$

is a co-t-structure in T.

(ii) The pair

$$\left({}^{\perp}\mathsf{X},({}^{\perp}\mathsf{X}){}^{\perp}\right)$$

is a stable t-structure, and moreover $({}^{\perp}X)^{\perp}$ is the smallest triangulated subcategory of T which contains X and is closed under products.

Proof. In (i), the pair is clearly Hom-orthogonal. It is also immediate that $^{\perp \geq 0}X$ is closed under [-1] and that $(^{\perp \geq 0}X)^{\perp \geq 0}$ is closed under [1]. So it remains to show that each $T \in \mathsf{T}$ appears in a triangle

(6.4)
$$T_{\perp_{\geq 0}\chi} \to T \to T_{\chi} \to T_{\perp_{\geq 0}\chi}[1]$$

with $T_{\perp_{\geq 0}\mathsf{X}} \in {}^{\perp_{\geq 0}}\mathsf{X}$ and $T_\mathsf{X} \in ({}^{\perp_{\geq 0}}\mathsf{X})^{\perp_{\geq 0}}$.

Consider the following construction: Write $T_0 = T$, and $X[\ge 0] = \{X[i] \mid X \in X \text{ and } i \ge 0\}$. Pick a left $\operatorname{Prod} X[\ge 0]$ -approximation $T_0 \to X_0$ of T_0 — which exists by Lemma 2.22 — and complete it to a triangle

$$(6.5) T_1 \to T_0 \to X_0 \to T_1[1]$$

Pick a left $\operatorname{\mathsf{Prod}} X[\geq 0]$ -approximation $T_1 \to X_1$ of T_1 , complete it to a triangle

$$(6.6) T_2 \to T_1 \to X_1 \to T_2[1]$$

and iterate in this fashion. Then put

$$T_{\perp \geq 0}_{\mathsf{X}} = \operatorname{holim} T_i \text{ and } T_{\mathsf{X}} = \operatorname{Cone}[T_{\perp \geq 0}_{\mathsf{X}} \to T].$$

It remains for us to show that these objects $T_{\perp \geq 0X}$ and T_X belong to the subcategories ${}^{\perp \geq 0}X$ and $({}^{\perp \geq 0}X)^{\perp \geq 0}$, respectively.

It follows by construction that each map in the sequence

$$\mathsf{T}(T_0,\mathsf{X}[i]) \to \mathsf{T}(T_1,\mathsf{X}[i]) \to \mathsf{T}(T_2,\mathsf{X}[i]) \to \cdots$$

vanishes for all positive *i*. Indeed, for any map $T_j \to X'[i]$ with X' in X, the composition $T_{j+1} \to T_j \to X'[i]$ is zero since the map $T_j \to X'[i]$ factors through X_j and by the triangle (6.5) the composition $T_{j+1} \to T_j \to X_j$ is zero. This implies that the above sequences are dual Mittag-Leffler and have vanishing colimit. Since the objects in X are 0-cocompact, it follows that $T_{\perp>0_X} = \operatorname{holim} T_i$ lies in $^{\perp\geq_0}X$.

The proof that $T_{\mathsf{X}} \in ({}^{\perp \geq 0}\mathsf{X})^{\perp \geq 0}$ is almost verbatim the same as steps (ii) to (v) in the proof of [1, Proposition 4.5]. This completes the proof of (i).

The fact that the pair in (ii) is a stable t-structure follows immediately from (i): Either apply (i) to the set of 0-cocompacts $X[\leq 0] = \{X[i] \mid X \in X \text{ and } i \leq 0\}$, or replace any occurrence of $X[\geq 0]$ in the proof of (i) by $X[\mathbb{Z}] = \{X[i] \mid X \in X \text{ and } i \in \mathbb{Z}\}$.

For the proof of the final claim let us denote by \widehat{X} the smallest triangulated subcategory of T containing X and closed under products. It is immediate that $\widehat{X} \subseteq ({}^{\perp}X)^{\perp}$, since the latter category it triangulated, contains X, and is closed under products.

To see the converse inclusion, if suffices to show that $T_X \in \widehat{X}$ for any $T \in T$. We write Y_i for the cone of the composed map $T_i \to T$. In particular $Y_0 = 0$ and $Y_1 = X_0$. By the octahedral axiom the Y_i also appear (iteratively) in triangles

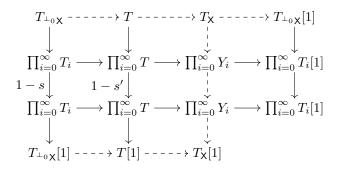
$$X_i \to Y_{i+1} \to Y_i \to X_i[1],$$

and in particular Y_i is in the smallest triangulated subcategory containing X.

Now we take the product of all the triangles defining the Y_i , that is the triangle

$$\prod_{i=0}^{\infty} T_i \to \prod_{i=0}^{\infty} T \to \prod_{i=0}^{\infty} Y_i \to \prod_{i=0}^{\infty} T_i[1]$$

Observe that the solid square in the following diagram commutes, where the vertical maps between products are the ones defining the homotopy limits. (That is the left vertical map is 1 - s, where s is given by the morphisms connecting the T_i , and the middle vertical map is 1 - s', where s' just sends each factor T identically to the previous one.). Hence, by the triangulated 3×3 -lemma the diagram may be completed to a commutative one with triangles in all rows and columns.



Note that the triangle $T \to \prod_{i=0}^{\infty} T \stackrel{1-s'}{\to} \prod_{i=0}^{\infty} T \to T[1]$ splits. Therefore the first dashed horizontal map is the natural map holim $T_i \to T$, and the top triangle is the one defining T_X . Now the vertical dashed triangle shows that T_X is an extension of $\prod_{i=0}^{\infty} Y_i[-1]$ and $\prod_{i=0}^{\infty} Y_i$, hence in \widehat{X} .

Remark 6.7. There is an arguably more conceptual proof of (ii) above in case T has an enhancement: Assume T sits at the base of a stable derivator. Then one may consider the triangles

$$T_i \to T \to Y_i \to T_i$$

and obtain a new triangle taking the degree-wise homotopy limit

holim
$$T_i \to T \to \text{holim } Y_i \to \text{holim } T_i[1]$$
.

This is true by dual of [27, Proposition A.5, Corollary A.6] — note that one may have to adjust the maps connecting the Y_i . Now holim Y_i is easily seen to lie in \hat{X} .

We now provide examples of 0-cocompact objects.

Theorem 6.8. Let Λ be an algebra over a commutative artinian ring k and denote by $D = \operatorname{Hom}_k(-, E)$ the standard duality. For any finite complex of finitely generated left Λ -modules X, the complex DX of right Λ -modules is 0-cocompact in K(Mod Λ).

Proof. We denote by F the functor given by taking zeroth homology of the total tensor product with X, i.e. $F = \mathsf{H}^0(-\bigotimes_{\Lambda}^{\mathsf{tot}} X)$, and observe that

$$\begin{aligned} \mathsf{Hom}_{\mathsf{K}}(-,DX) &= \mathsf{H}^{0}(\mathcal{H}\mathsf{om}_{\Lambda}(-,DX)) = \mathsf{H}^{0}(D(-\otimes^{\mathsf{tot}}_{\Lambda}X)) \\ &= D\,\mathsf{H}^{0}(-\otimes^{\mathsf{tot}}_{\Lambda}X) = DF. \end{aligned}$$

Moreover, since X is finitely generated, the tensor product and hence also F commute with products.

After these preparations we are ready to verify the 0-cocompactness of DX. Let

$$\cdots \rightarrow T_2 \rightarrow T_1 \rightarrow T_0$$

be a sequence of morphisms and objects in $\mathsf{K}(\mathsf{Mod}\,\Lambda)$ such that

$$\operatorname{Hom}_{\mathsf{K}}(T_0, DX[1]) \to \operatorname{Hom}_{\mathsf{K}}(T_1, DX[1]) \to \operatorname{Hom}_{\mathsf{K}}(T_2, DX[1]) \to \cdots$$

is dual Mittag-Leffler and that $\operatorname{colim} \operatorname{Hom}_{\mathsf{K}}(T_i, DX) = 0$.

Dualizing this second assumption, and using the fact that D turns colimits into limits, we obtain that

$$\cdots \rightarrow D \operatorname{Hom}_{\mathsf{K}}(T_2, DX) \rightarrow D \operatorname{Hom}_{\mathsf{K}}(T_1, DX) \rightarrow D \operatorname{Hom}_{\mathsf{K}}(T_0, DX)$$

has vanishing limit, i.e. $\lim D^2 F(T_i) = 0$. Since limits are left exact it follows that also $\lim F(T_i) = 0$. In particular the morphism

$$\prod F(T_i) \xrightarrow{1-(f_i)} \prod F(T_i)$$

is monic.

On the other hand, note that the sequence in the first assumption is the dual of

$$\cdots \to F(T_2[-1]) \to F(T_1[-1]) \to F(T_0[-1]).$$

It follows that the latter sequence satisfies the Mittag-Leffler condition, and in particular that $\lim^{1} F(T_{i}[-1]) = 0$. Thus the map

$$\prod F(T_i[-1]) \xrightarrow{1-(f_i)} \prod F(T_i[-1])$$

is epic.

Since F commutes with products, the exact sequence

$$F(\prod T_i[-1]) \to F(\prod T_i[-1]) \to F(\mathsf{holim}\,T_i) \to F(\prod T_i) \to F(\prod T_i)$$

starts with an epimorphism and ends with a monomorphism, showing that the middle term vanishes. Thus

$$\operatorname{Hom}_{\mathsf{K}}(\operatorname{holim} T_i, DX) = DF(\operatorname{holim} T_i) = 0.$$

Remark 6.9. In the above theorem, it is not relevant that Λ is concentrated in degree 0, i.e. the theorem still holds for Λ a dg k-algebra.

However, here we are actually most interested in an even more special case:

Corollary 6.10. Let Λ be an Artin algebra. Then any finite complex of finitely generated Λ -modules is 0-cocompact in $\mathsf{K}(\mathsf{Mod}\,\Lambda)$.

Proof. This follows directly from Theorem 6.8, using only the observation that the duals of finitely generated left Λ -modules are precisely the finitely generated right Λ -modules.

What is more, for Artin algebras, Corollary 6.12 below says that each functor R appearing in Diagram 4.1 in the Summary of Section 4 is covered by the explicit construction in (the proof of) Theorem 6.6. As usual, D_{Λ} denotes the dualizing complex for Λ — see Example 2.9.

By Proposition 2.17, the relevant categories are given as left orthogonal to certain objects. Our first task is to make sure that these objects are 0-cocompact.

Lemma 6.11. Let Λ be an Artin algebra. Then

- (i) the complex Λ is 0-cocompact in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$;
- (ii) the complex D_{Λ} is 0-cocompact in $\mathsf{K}(\mathsf{Inj}\,\Lambda)$;
- (iii) the complex ρD_{Λ} is 0-cocompact in K(Proj Λ);
- (iv) the complex $D_{\Lambda} \otimes_{\Lambda}^{\mathbb{L}} D_{\Lambda}$ is 0-cocompact in K(Inj Λ).

Proof. (i) and (ii) are immediate from Corollary 6.10. In order to show (iii), let

 $\cdots \to X_2 \to X_1 \to X_0$

be a sequence in $\mathsf{K}(\mathsf{Proj}\Lambda)$ such that the induced sequence

 $\operatorname{Hom}_{\mathsf{K}(\operatorname{Proj}\Lambda)}(X_0,\rho D_{\Lambda}[1]) \to \operatorname{Hom}_{\mathsf{K}(\operatorname{Proj}\Lambda)}(X_1,\rho D_{\Lambda}[1]) \to \cdots$

is dual Mittag-Leffler and colim $\operatorname{Hom}_{\mathsf{K}(\operatorname{Proj}\Lambda)}(X_i, \rho D_\Lambda) = 0$. Since ρ is right adjoint to the inclusion $\mathsf{K}(\operatorname{Proj}\Lambda) \hookrightarrow \mathsf{K}(\operatorname{Mod}\Lambda)$, the sequence

 $\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(X_0, D_{\Lambda}[1]) \to \operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(X_1, D_{\Lambda}[1]) \to \cdots$

is again dual Mittag-Leffler and the colimit of $\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,\Lambda)}(X_i, D_\Lambda)$ vanishes. The 0-cocompactness of D_Λ in $\mathsf{K}(\mathsf{Mod}\,\Lambda)$ thus implies the 0-cocompactness of ρD_Λ in $\mathsf{K}(\mathsf{Proj}\,\Lambda)$. Indeed,

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{Proj}\Lambda)}(\operatorname{holim} X_i, \rho D_\Lambda) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{Mod}\Lambda)}(\operatorname{holim} X_i, D_\Lambda) = 0$$

since $\operatorname{Proj} \Lambda$ is closed under products. Claim (iv) now follows from the fact that the equivalence $- \otimes_{\Lambda} D_{\Lambda}$: $\mathsf{K}(\operatorname{Proj} \Lambda) \to \mathsf{K}(\operatorname{Inj} \Lambda)$ preserves 0-cocompactness. \Box

As in Corollary 6.2, we consider the perpendiculars to these objects with respect to all extensions, that is we consider the stable t-structures $({}^{\perp}X, ({}^{\perp}X)^{\perp})$. We obtain the following descriptions of the right adjoints.

Corollary 6.12. Let Λ be an Artin algebra. Then

(i) the right adjoint of the inclusion $\mathsf{K}_{\mathsf{coac}}(\mathsf{Proj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Proj}\,\Lambda)$ is given by

$$T \mapsto T_{\perp(\Lambda)}$$

(ii) the right adjoint of the inclusion $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda)$ is given by

$$T \mapsto T_{\perp(D_{\Lambda})};$$

(iii) the right adjoint of the inclusion $\mathsf{K}_{\mathsf{tac}}(\mathsf{Proj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Proj}\,\Lambda)$ is given by

 $T \mapsto T_{\perp(\Lambda \oplus (\rho D_{\Lambda}))};$

(iv) the right adjoint of the inclusion $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}\,\Lambda) \hookrightarrow \mathsf{K}(\mathsf{Inj}\,\Lambda)$ is given by

$$T \mapsto T_{\perp}(D_{\Lambda} \oplus (D_{\Lambda} \otimes_{\Lambda}^{\mathbb{L}} D_{\Lambda}))$$

Observe that if Λ is an Artin algebra, then the right GProj Λ -approximations in Mod Λ of Corollary 5.4 are also explicitly described by Proposition 6.12. Indeed, a right GProj Λ -approximation of M appears as $\operatorname{Cok}(X^{-1} \to X^0) \to M$ for a right $\mathsf{K}_{\mathsf{tac}}(\operatorname{Proj} \Lambda)$ -approximation $X \to \rho M$ of ρM .

Appendix

We provide the two proofs that are missing from Section 2.

Lemma 2.3. Let T and T' be triangulated categories with coproducts and suppose T is compactly generated. Let $F \colon \mathsf{T} \to \mathsf{T}'$ be exact and coproduct preserving. If F restricts to a fully faithful functor $\mathsf{T}^{\mathsf{c}} \to \mathsf{T}'^{\mathsf{c}}$, then F is fully faithful.

Proof. Suppose F restricts to a fully faithful functor $\mathsf{T}^{\mathsf{c}} \to \mathsf{T}'^{\mathsf{c}}$. Fix $X \in \mathsf{T}^{\mathsf{c}}$ and let $X_* = \{B \in \mathsf{T} \mid F \colon \mathsf{T}(X, \Sigma^i B) \to \mathsf{T}'(FX, F\Sigma^i B) \text{ is bijective } \forall i \in \mathbb{Z}\}.$

First, observe that X_* is a triangulated subcategory of T. Indeed, given a triangle

$$Y \to Y' \to Y'' \to \Sigma Y$$

in T with $Y, Y'' \in X_*$, we get the triangle

$$FY \to FY' \to FY'' \to \Sigma FY$$

in T' and hence the following diagram with exact rows, revealing $Y' \in X_*$.

Second, X_* is closed under coproducts. Namely, for each family $(B_i | i \in I)$ of objects in X_* we have

$$\mathsf{T}(X, \coprod_{i\in I} B_i) \cong \coprod_{i\in I} \mathsf{T}(X, B_i)$$

since X is compact, while

$$\mathsf{T}'(FX, F \coprod_{i \in I} B_i) \cong \mathsf{T}'(FX, \coprod_{i \in I} FB_i) \cong \coprod_{i \in I} \mathsf{T}'(FX, FB_i)$$

as F preserves coproducts and FX is compact, and hence $\coprod_{i \in I} B_i \in X_*$. Since $\mathsf{T}^{\mathsf{c}} \subset X_*$ by assumption, this implies $X_* = \mathsf{T}$.

Fix now an arbitrary $B \in \mathsf{T}$ and consider the subcategory

$$B^* = \{ A \in \mathsf{T} \mid F \colon \mathsf{T}(\Sigma^i A, B) \to \mathsf{T}'(F\Sigma^i A, FB) \text{ is bijective } \forall i \in \mathbb{Z} \}.$$

We claim that $B^* = \mathsf{T}$, which would clearly suffice. As above, it suffices to show that B^* contains T^{c} and is closed under extensions and coproducts. First of all, the above obtained $X_* = \mathsf{T}$ for each compact X means that $\mathsf{T}^{\mathsf{c}} \subset B^*$. Further, B^* is closed under extensions by an argument similar to the above. Finally, B^* is closed under coproducts, since for each family $(A_i | i \in I)$ of objects in B^* , we have

$$\mathsf{T}(\prod_{i\in I}A_i,B)\cong\prod_{i\in I}\mathsf{T}(A_i,B)$$

and on the other hand

$$\mathsf{T}'(F \coprod_{i \in I} A_i, FB) \cong \mathsf{T}'(\coprod_{i \in I} FA_i, FB) \cong \prod_{i \in I} \mathsf{T}'(FA_i, FB).$$

Lemma 2.16. If Λ is an Artin algebra, then the natural monomorphism $\Lambda^{(I)} \to \Lambda^{I}$ is split for any index set I.

Proof. We first recall a few notions. Let R be a ring. A monomorphism $X \to Y$ in Mod R is called *pure* if $X \otimes_R N \to Y \otimes_R N$ remains a monomorphism for each finitely presented left Λ -module N. An R-module M is called *pure-injective* if it is injective with respect to pure monomorphisms, i.e. given a pure monomorphism $f: X \to Y$ in Mod R, each morphism $g: X \to M$ factors through f:



Observe now that the natural monomorphism $\mu_{\Lambda} \colon \Lambda^{(I)} \to \Lambda^{I}$ is in fact pure for any indexing set I. Indeed, let N be a finitely presented left R-module. Then $\Lambda^{I} \otimes_{\Lambda} N = N^{I}$, and we always have $\Lambda^{(I)} \otimes_{\Lambda} N = N^{(I)}$. Hence the induced $\mu_{\Lambda} \otimes$ $N \colon \Lambda^{(I)} \otimes_{\Lambda} N \to \Lambda^{I} \otimes_{\Lambda} N$ is nothing but the monomorphism $\mu_{N} \colon N^{(I)} \to N^{I}$. Since Λ is an Artin algebra it has finite length when viewed as a module over its endomorphism ring, which means that $\Lambda^{(I)}$ is pure-injective — see e.g. [38]. In particular this means that we have



i.e. a splitting of μ_{Λ} .

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