On the Pricing of Sovereign Debt and the Option to Default

Sondre Solvoll Bakketun

Department of Economics, Norwegian University of Science and Technology (NTNU)

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This thesis marks the end of my two-year MSc. program in Financial Economics at the Department of Economics, NTNU. It has provided me with challenging, educational and interesting work. I believe it will be a valuable experience to draw on later in life.

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1 Introduction

It is, or at least has been, a widespread belief that sovereign default is a problem concerning emerging markets and that the developed economies of the world do not default (Feenstra & Taylor, 2006). However the financial crisis that struck the world in 2008 has led to solvency problems for many European countries regarded as developed economies. Foremost for the so called PIGS countries (Portugal, Ireland, Greece and Spain) as they have been, and are, the ones with the most severe problems.

The matter of sovereign default is very complex and parts itself on a number of areas from the matter of corporate default. When a company borrows money it obligates itself to repay the loan at a future agreed upon date and if it cannot meet this obligation the debt holders gain control over the company. The debt holders have a claim on the company's assets and this claim is enforced by the legal system. In a way debt holders and equity holders own the company together, but the debt holders' investment has a limited upside and downside while the equity holders' investment has *un*limited upside and downside (debt holders may of course also lose all of their investment, but in most cases they do get some of their investment back). If a country borrows money it also obligates itself to pay back the loan at an agreed upon date, but if it cannot meet this obligation the debt holders will not gain control over the assets of the country. In fact there is very little a creditor can do if a country decides to default on its debt. So why would anybody lend money to a country if the country is free to default on its debt? There is broad agreement that the existence of government debt is possible due to the fact that default is costly. If a country defaults on its debt it may face punishments such as exclusion from capital markets and trade embargos that directly affect the economy. It may also face higher interest rates when it gains access to capital markets again. A default may also have consequences that are not as easily measured economically, for instance could a default prove to be very costly for political leaders.

The idea to this thesis came from the fact that a country can default on its debt without creditors having the same claim on their assets as in a corporate case, and that the government therefore has a valuable option to default. The nature of sovereign debt and credit default swaps on sovereign debt will hence be different than the nature of corporate debt and credit default swaps on corporate debt. Using a simple framework I adopt a procedure similar to the one Robert C. Merton (1974) used in his famous paper on corporate debt for the purpose of analyzing sovereign debt.

In the following I first present some technical issues that I use in my analysis of sovereign debt, and then I present the central theory about sovereign debt and the costs of default. Next, I develop a simple model for the very complex matter of sovereign debt and the option to default. The model can be used to price sovereign debt, credit default swaps on sovereign debt and the option to default. In the last chapter of the thesis I confront the model with data on CDS prices and try to calculate the same prices in the model to see whether it can produce sensible results or not.

2 Theoretical Background

In this chapter I present the matter of sovereign debt and default and some theoretical concepts used, in order to ease the reader's experience.

Black & Scholes (1973) and Merton (1974) derived partial differential equations (called pricing PDE's) based on hedging arguments in their option analysis, but their arguments assumed that the underlying asset was tradable or could be spanned by tradable securities. When dealing with an untradeable variable, such as the output of a country, it makes more sense to use a different approach where the underlying asset need not be tradable. Chapter 2.1 - 2.3 develop the theoretical background for the pricing method used in chapter three. It is based on Hull (2009) and McDonald (2006).

2.1 Geometric Brownian Motion (GBM)

Throughout the thesis I assume that the output Y of a country follows a *continuous time stochastic process*. Such a process is, as the name suggests, continuous in the sense that the variable can take on any value within a given range. In this case it is natural to assume that output Y can only take on non-negative values. I will assume that Y follows a continuous process known as *Geometric Brownian Motion* (GBM) which means that the marginal change dY is given by

$$(2.1.1)\ dY = \mu Y dt + \sigma Y dz$$

where μ is the expected growth rate in Y, σ is the volatility of the growth rate of Y and the parameter dz is a Wiener process. A Wiener process has the following two properties:

1. The change in z during a small period of time is given by

$$(2.1.2)\ \Delta z = \epsilon\sqrt{\Delta t}$$

   where ϵ is a standard normal distributed variable.

2. For two different intervals of time, Δt, the values of Δz are independent.

It follows that Δz also is normally distributed and has an expected value of zero and a variance of Δt.

Using Itô's lemma I derive the process followed by $\ln\left(\frac{Y_t}{Y_0}\right)$. Y_0 is the level of output today, which is observable. Define $G_t = \ln\left(\frac{Y_t}{Y_0}\right)$, then

$$dG_t = \frac{\partial G_t}{\partial Y_t} dY_t + \frac{\partial G_t}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G_t}{\partial Y_t^2} (dY_t)^2$$

$$\frac{\partial G_t}{\partial Y_t} = \frac{1}{Y_t}, \frac{\partial G_t}{\partial t} = 0, \frac{\partial^2 G_t}{\partial {Y_t}^2} = -\frac{1}{Y_t^2}$$

$$\Rightarrow (2.1.3)\ dG_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz$$

Solving equation (2.1.3) gives

$$\ln Y_t = \ln Y_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dv + \int_0^t \sigma dz_v$$

$$\ln Y_t = \ln Y_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma z_t$$

$$\Leftrightarrow\ (2.1.4)\ Y_t = Y_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma z_t}$$

This expression for *Y* will prove useful later in the analysis.

2.2 The market price of risk

In chapter 3 of the thesis I will develop pricing formulas for derivatives dependent on the output of a country. The first step towards a method of pricing such derivatives is to examine the properties of derivatives dependent on the value of a single variable, which is done here. Assume that θ follows the process

$$(2.2.1)\ \frac{d\theta}{\theta} = a dt + b dz_\theta$$

Where dz_θ is a Wiener process as defined earlier, and a and b are the expected growth rate in θ and the volatility of θ respectively. In this context there is no restriction on what θ can be. It can be an asset price, but it can also be a non-financial variable such as the temperature of a given place.

Assume that f_1 and f_2 are the prices of two derivatives that depend only on θ and time *t*. The derivatives provide no income during the time period under consideration. The derivatives both follow GBM given by

$$\frac{df_1}{f_1} = \mu_1 dt + \sigma_1 dz_\theta$$

and

$$\frac{df_2}{f_2} = \mu_2 dt + \sigma_2 dz_\theta$$

μ_1, μ_2, σ_1 and σ_2 are functions of θ and *t*, and the dz_θ parts are the same as in equation (2.2.1) since the only source of uncertainty in f_1 and f_2 are the uncertainty of θ. The derivatives f_1 and f_2 are however not necessarily perfectly correlated with θ, since both their drifts and diffusion parts are functions of θ and t.

Now consider a portfolio consisting of f_1 and f_2. The value of this portfolio is given by

$$(2.2.2)\ \pi = af_1 + bf_2$$

where *a* and *b* are the amounts invested in f_1 and f_2 respectively. The process for this portfolio is then given by

$$(2.2.3)\ d\pi = adf_1 + bdf_2 = (a\mu_1 f_1 + b\mu_2 f_2)dt + (a\sigma_1 f_1 + b\sigma_2 f_2)dz_\theta$$

Assume we want the portfolio to be instantaneously risk-free. This means that we want the stochastic part to disappear. To achieve this we must have

$$a\sigma_1 f_1 + b\sigma_2 f_2 = 0$$

Or equivalently

$$a = -b\frac{\sigma_2 f_2}{\sigma_1 f_1}$$

A possible solution to this is $a = \sigma_2 f_2$ and $b = -\sigma_1 f_1$. Inserting this in equation (2.2.3) gives

$$(2.2.4)\ d\pi = (\sigma_2 f_2 \mu_1 f_1 - \sigma_1 f_1 \mu_2 f_2)dt$$

which is an instantaneously risk-free portfolio. Because it is risk-free, in the absence of arbitrage, it must earn the risk-free rate. That is

$$d\pi = r\pi dt$$

Substituting equation (2.2.2) and (2.2.4) into this gives

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

or equivalently

$$(2.2.5)\ \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

Define λ as the value of each side of equation (2.2.5), so that

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$$

So if f is the price of a derivative that depends only on θ and *t*, and follows the process

$$\frac{df}{f} = \mu dt + \sigma dz_\theta$$

then

$$(2.2.6)\ \frac{\mu - r}{\sigma} = \lambda$$

The parameter λ is known as the *market price of risk* of θ. It is a price in the sense that it gives the excess return over the risk-free rate that the market requires per unit of risk. Equation (2.2.6) can be written as

$$(2.2.7)\ \mu = r + \lambda\sigma$$

Note the analogy to the Capital Asset Pricing Model (CAPM) which relates the expected return on a stock to its risk. In CAPM the expected return on a stock is given by the risk-free rate plus the beta of the stock times the excess return of the market portfolio over the risk-free rate. Substituting equation (2.2.7) into the process followed by f gives

$$(2.2.8)\ \frac{df}{f} = (r + \lambda\sigma)dt + \sigma dz_\theta$$

Dependent on what "world" you assume the parameter λ has different values. Different "worlds" could be the real world and the risk-neutral world. In the risk-neutral world, the market price of risk equals zero, so that $\mu = r$. Different "worlds" means different assumptions about risk preferences, and an assumption about what "world" to use is also referred to as defining the probability measure. In the risk-neutral world the assumption is that all investors are risk neutral and thus the expected return on securities is the risk-free rate. The market price of risk is not necessarily known (it can be estimated by the use of CAPM), but, as chapter 2.3 will show, the market price of risk can be *chosen* so as to enable the development of a pricing method for derivatives.

2.3 Martingales and the Equivalent Martingale Measure Result

In addition to the concepts already developed, knowledge of martingales is needed to develop the pricing method adopted in this thesis.

2.3.1 Martingales

A sequence of random variables $X_0, X_1, \ldots$ is a martingale if for all $i > 0$,

$$E[X_i | X_{i-1}, X_{i-2}, \ldots X_0] = X_{i-1}$$

If a variable φ is to follow a martingale it has the process

$$d\varphi = \sigma_\varphi dz_\varphi$$

where dz_φ is a Wiener process and the variable σ_φ may be stochastic. The reason we care about martingales is that it has the appealing property that its expected value at any future time equals its value today. Mathematically written this gives

$$E[\varphi_T] = \varphi_0$$

This property arises from the fact that the expected change over any very small time interval is given by

$$E[d\varphi] = E\left[\sigma_\varphi dz_\varphi\right] = 0$$

because of the fact that a Wiener process is normally distributed with mean zero.

2.3.2 The Equivalent Martingale Measure Result

Using the concepts developed up until now this section derives a method for the pricing of derivatives dependent on a single source of uncertainty.

Consider two derivatives dependent on a single source of uncertainty with prices *f* and g that follow GBMs. The securities provide no income during the period considered. Define the relative price of f with respect to g as $\varphi = \frac{f}{g}$. The price g is here referred to as the *numeraire*. Assume a world where the market price of risk is given by the volatility of g, σ_g. From equation (2.2.8) it follows that the processes followed by *f* and g are

$$df = (r + \sigma_g \sigma_f) f dt + \sigma_f f dz_\varphi$$

and

$$dg = (r + \sigma_g^2) g dt + \sigma_g g dz_\varphi$$

Using Itô's lemma I find the process followed by φ

$$d\varphi = \frac{\partial \varphi}{\partial f} df + \frac{1}{2}\frac{\partial^2 \varphi}{\partial f^2}(df)^2 + \frac{\partial \varphi}{\partial g} dg + \frac{1}{2}\frac{\partial^2 \varphi}{\partial g^2}(dg)^2 + \frac{1}{2}\frac{\partial^2 \varphi}{\partial f \partial g} df dg + \frac{1}{2}\frac{\partial^2 \varphi}{\partial g \partial f} dg df$$

$$\Rightarrow (2.3.1)\ d\varphi = (\sigma_f - \sigma_g)\varphi dz_\varphi$$

This is known as the *equivalent martingale measure* result and shows that φ is a martingale. When dealing with a world where the market price of risk is the volatility of g, you say that the world is *forward risk neutral* with respect to g. Because of the appealing property of martingales discussed earlier it follows that

$$\varphi_0 = \frac{f_0}{g_0} = E_g\left[\frac{f_T}{g_T}\right]$$

or equivalently

$$(2.3.2)\ f_0 = g_0 E_g\left[\frac{f_T}{g_T}\right]$$

E_g is the expected value in a world that is forward risk neutral with respect to g. This result can then be used to value derivatives.

One example of a numeraire that can be used could be a risk-free bond that pays off 1 at maturity. Suppose $g_T = 1$, then $g_0 = e^{-rT}$ (r is the risk-free rate). Inserting this into equation (2.3.2) gives

$$(2.3.3)\ f_0 = e^{-rT} E_Q[f_T]$$

Equation (2.3.3) is consistent with the risk-neutral valuation principle that arose from Black, Scholes and Merton's options analysis. In short, risk-neutral valuation says that the price of a security is given by its expected future value, assuming it earns the risk-free rate, discounted back by the risk-free rate. This is exactly what equation (2.3.3) says.

2.4 Merton's model for the pricing of Corporate Debt

In a famous paper Merton (1974) develops a theory for the pricing of risky corporate liabilities. In this thesis I give an attempt to do the same thing for sovereign liabilities. In order to compare some results with Merton I briefly present his findings for corporate liabilities here.

Assuming the value of the firm follows GBM, Merton develops the following PDE which the pricing formula for a zero-coupon bond has to satisfy.

$$(2.4.1)\ \frac{1}{2}\sigma^2 V^2 F_{VV} + rVF_V - rF - F_{T-t} = 0,$$

where V is the value of the firm, $F(V, T-t)$ is the value of the bond (subscripts denotes partial derivatives and $T-t$ is time to maturity), r is the risk-free rate and σ is the instantaneous standard deviation of the total return on the firm (that is the return on V). To solve this PDE one need three boundary conditions. They are given by

$$F(0, T-t) = 0$$

$$F(V, T-t) \leq V$$

$$F(V, 0) = \min(V, D)$$

where *D* is the amount the firm promises to pay the creditors at maturity. The first boundary condition tell us that if the firm becomes worthless so does the debt, the second condition shows that the debt cannot be worth more than the company, while the third condition says that at maturity the debt holders has the senior claim on the company. The last condition implies that if the company cannot repay its debt, the creditors take control over whatever values are left in the company.

Using the three boundary conditions it is possible to solve equation (2.4.1) directly using some advanced mathematics beyond my knowledge. Instead of using these complex techniques Merton show that the problem of finding the value of equity is identical to that of finding the value of a European call option on a non-dividend-paying stock where the firm value corresponds to the stock price and *D* corresponds to the strike price. Using the already developed Black-Scholes equation and the fact that the value of equity $f(V, T-t)$ is given by

$$f(V, T-t) = V - F(V, T-t)$$

Merton show that the value of debt is given by

$$(2.4.1)\ F(V, T-t) = V_0 N(-d_1) + e^{-r(T-t)} D N(d_2)$$

where

$$d_1 = \frac{\ln \frac{V}{B} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma(T-t)}, \qquad d_2 = d_1 - \sigma(T-t)$$

and *N(x)* gives the cumulative standard normal distribution. Merton arrives at a pricing formula which shares the appealing properties of the Black & Scholes formula that all the parameters are observable (except the volatility which can be estimated).

2.5 Sovereign debt and default

This chapter presents the concept of sovereign debt and default, and discusses some of the theory and empirical results on the subject.

2.5.1 Default

A sovereign default can be defined as the failure of a sovereign to meet the obligations given in a debt contract. The extreme case is of course when a sovereign defaults completely on its

debt and leaves the creditors with nothing. This is however a very rare outcome. Russia's default in 1918 is a famous total default, but even they had to suffer a token payment on its defaulted debt before reentering the capital markets sixty-nine years later. It turns out that almost every default end up in partial repayment, although in some cases this may be small and occur many years later. In addition to regular defaults, debt rescheduling is regarded by rating agencies as negotiated partial defaults. A typical debt rescheduling involves longer repayment schedules and often lower interest rates. This will in effect be a partial default (Reinhart & Rogoff, 2009).

In the literature it is common to distinguish between default on external debt and default on domestic debt. External debt is defined as the total debt obligations of a country with foreign creditors, both public and private. Normally this kind of debt is subject to the jurisdiction of the foreign creditors or international law. Domestic debt (more precisely government domestic debt) is defined as all the liabilities that are issued under and are subject to national jurisdiction (Reinhart & Rogoff, 2009).

Historically default on external debt has been given the most attention both in the media and in the academic literature, but, although given little attention, default on domestic debt has also occurred frequently throughout history. Well known episodes of domestic default are Mexico's defaults in 1982, 1994-1995 and 2001, but looking closer these defaults either coincided with external defaults (82 and 01) or a vast part of the creditors were foreigners (94-95) (Reinhart & Rogoff, 2009). The defaults that have the most impact on the global economy are the ones that affect foreign citizens, i.e. external defaults.

The problem of default and serial default is, rightfully, mainly regarded as an emerging market problem. The vast majority of developed economies, when they still were in a developing stage, had periods where they struggled with the problem of serial default, but as they graduated into developed economies they also graduated from the problem of serial default (Reinhart & Rogoff, 2009). Now however, in the aftermath of the financial crisis of 2008, the ghost of "default past" has come to visit some of the European economies thought to have graduated from defaulting on debt. Countries such as Greece, Ireland and Portugal all face the risk of having to default on their debt.

2.5.2 The costs of default

Because a sovereign debtor has the option to default without the same consequences as a corporate debtor, the academic research on sovereign debt has mainly dealt with the task of explaining the existence of sovereign debt. The literature is in agreement that there has to be some sort of costs associated with a default to make a market for sovereign debt possible, but there are great differences as to what the costs actually are.

The costs of default are complex and not easily measured. This makes the choice of defaulting or not a difficult one, which has to be based on an extensive cost-benefit analysis. In the earlier parts of the history of sovereign default, when it was normal to use force when collecting debt, the costs of default were a bit more "hands on". During the colonial times of the nineteenth century, superpowers were known to use military force in their "negotiations" with borrowers. Britain invaded Istanbul in 1872 and Egypt in1882 with concerns about repayments of debt as their justifying reasons. The U.S. also used concerns about debt repayments as part of the reasons for its conflict with Venezuela that began in the 1890s and its occupation of Haiti in 1915 (Reinhart & Rogoff, 2009). The use of force has however not been reserved only for creditors. Borrowers have also been known to use force, or at least their ability to use force, as a tool when defaulting. Throughout the sixteenth, seventeenth and eighteenth centuries, major borrowers like France and Spain, which also had great armies, were hardly afraid of being invaded if they were to repudiate their debt. Also domestic lenders faced an uncertain repayment scheme when lending to these countries. French monarchs were at the time known for their use of executions when dealing with major domestic creditors. (Reinhart & Rogoff, 2009). As one considers the more recent parts of sovereign default history it is evident that the use of force or military power has become more and more out of style. Hopefully this is because the world has become a more civilized place, but perhaps just as likely because the costs of using force in debt negotiations now greatly exceeds the costs of other solutions.

There are two types of costs associated with default that has been given much attention in the literature. These are reputational costs and costs arising from direct sanctions. An important paper by Eaton & Gersovitz (1981) argues that what makes a sovereign debtor repay is the threat of receiving a reputation as a defaulter. If a sovereign gets a reputation as a defaulter it typically faces lower credit ratings (and as a consequence higher borrowing costs) and in extreme cases exclusion from capital markets. Eaton & Gersovitz assume that the punishment

of defaulting is *permanent* exclusion from capital markets, an assumption they themselves anticipated would be challenged.

Bulow & Rogoff (1989) argue that reputational costs alone are not sufficient to make countries repay their debt. They consider a country that can borrow money by entering into what they call reputation-for-repayment contracts. In these contracts the creditors have no other sanctions to impose on a defaulting country than excluding it from future borrowing. In their analysis they show that given fairly general assumptions it can be optimal for a country to default on these contracts and use the funds that was meant for repayments to invest in the international capital markets instead. If they do this they can be better off when receiving interest on their investments than if they honor their obligations. On the basis of this analysis they conclude that creditors need to be able to impose direct sanctions on sovereign borrowers, like trade embargos or freezing of assets, to make sovereign borrowing possible.

English (1996) conduct a thorough analysis of the defaults of the U.S. States in the 1840s to test different models of sovereign debt. He points out that the U.S. Constitution prevented foreign creditors from obtaining payment in the federal courts and, as the U.S. states were part of a powerful union, they were insulated from direct sanctions from their creditors. Following Bulow and Rogoff this should make all the states default on their debt, but in reality most of the states repaid their debt. English concludes therefore that in practice the threat of being excluded from international borrowing is sufficient to make sovereigns repay debt. Direct sanctions seem unnecessary to provoke repayment. He also observes that in the years that lead up to the Civil War the states that repaid their debt were able to borrow internationally, while those that did not repay were unable to do so.

Empirically there are some support to the existence of both reputational costs and direct sanctions. Sandleris, Gaston Gelos & Ratna Sahay (2004) find that countries, on average, were excluded from capital markets for about four years after a default in the 1980s. After the 1980s the exclusion period was shorter (0-2 years). Other papers show similar results (Panizza, Sturzenegger, & Zettelmeyer, 2009). This evidence of course rejects the assumption of Eaton & Gersovitz (1981) of the exclusion from capital markets being permanent. Recent research also conclude that given the high levels of debt observed, the threat of exclusion from capital markets cannot alone encourage sovereign debtors to repay debt (Arellano & Jonathan, 2008).

As mentioned earlier the use of direct sanctions in debt negotiations occurred in the earlier parts of history when "gunboat diplomacy" was commonly adopted. In more recent history this has however (thankfully) not been observed. There is evidence that defaults has been accompanied by declines in trade, but whether this is an effect of direct sanctions is not clear. These effects are also, like reputational costs, short-lived. There has also been taken measures to enhance the legal rights of sovereign creditors and this has led to the seizing of sovereign assets in some cases, but the legal system is still too weak to fully protect a sovereign creditor against sovereign default. Assets held abroad are also generally not large enough to service substantial amounts of debt (Panizza, Sturzenegger, & Zettelmeyer, 2009).

A third cost that may arise from defaults on sovereign debt is domestic costs. Sturzenegger (2004) finds that defaults are accompanied by a reduction in growth of approximately 0.6 percentage points. If accompanied by a banking crisis the drop is 2.2 percentage points. This may imply that a default will lead to losses in output, but it may also be the other way around. A decrease in output may lead to debt problems. The causal direction of these mechanisms does not seem to be settled in the literature.

A fourth and final possible cost is the political costs of a default. These costs differ from the costs discussed above in that they do not affect the entire economy directly. Instead they affect the political leaders that take the decision of whether to default or not. The worst case scenario for a government is of course if it is forced to resign as a consequence of its actions. This makes the decision of defaulting or not a decision composed of two elements. One element that concerns the economy and what is best for it and another that concerns only the political leaders currently in charge. The latter can then be thought of possibly creating a distortion in the decision making because the government may not only do what is best for the country, but also what is best for the government in a political sense. Studies have found that the proximity of elections, increases in indicators of political instability, changes in the central bank governors or the finance minister and the presence of a presidential democratic regime instead of a parliamentary democratic regime are all associated with higher probabilities of default (see for instance Citron & Nickelsburg (1987), Block & Vaaler (2004) and Kohlscheen (2009)). Political indicators may be significant only because of their correlation with economic variables (such as the accumulation of debt), but when controlled for such variables the results still hold (Hatchondo & Martinez, 2010).

The discussion above demonstrates that the costs of default are complex. The research conducted provide no clear answer as to what the costs of default actually are, but at least it agrees that a default has to be costly or else there would be no market for sovereign debt.

2.5.3 The Pricing of Sovereign Debt

The literature on sovereign debt has mainly focused on the justification of the very existence of sovereign debt. The pricing of it has been given much less attention. I will adopt an option approach similar to the one Merton (1974) used, but I am not the first to use such an approach.

Cohen (1991) assumes that if the debtor country in question chooses to default, it has to pay default costs equal to a fraction γ of its income Q. He also assumes that these costs are paid to the creditors. In effect the creditors' payoff at maturity,$V(D)$, is given by

$$V(D) = \min(D, \gamma Q)$$

where D is the face value of the debt. This is identical to the value of debt at maturity in Merton (1974) with γQ replacing the total value of the firm. Cohen then assumes that the income of the country follows GBM and derives a formula for today's value of the debt along the lines of Black & Scholes and arrives at the same formula as Merton.

Claessens & van Wijnbergen (1990) also adopt an option pricing approach to the pricing of sovereign debt. They assume that the value of debt at maturity $R(D)$ is given by

$$R(D) = \min(D, FX)$$

where D is the face value of debt and FX is the value of the foreign exchange reserves available to service debt. This is also equivalent to the condition in Merton (1974), with FX replacing the total value of the firm. Assuming that the value of the foreign exchange reserves follows GBM the derived formula for the value of debt is the same as the one Merton finds.

Both Cohen and Claessen & van Wijnbergen implicitly assume that sovereign debt is equal to corporate debt in that creditors *have* a claim on assets or payments in the event of default. I would argue that when assuming this they sidestep the greatest difference between corporate and sovereign debt, namely the fact that sovereign creditors have *no* claim on sovereign

assets/payments in the event of a default. Although I use a similar approach I do not sidestep this fact and therefore believe that my model is more realistic.

3 A Pricing Model for Sovereign Debt

In this chapter of the thesis I develop a model for the pricing of sovereign debt. The model can also be used to value credit default swaps on the debt and it identifies the option to default and enables valuation of it.

3.1 The model under the assumption of GBM

In this section I adopt the following assumptions

- Markets are complete (no transaction costs, taxes or other frictions and all relevant information is available for investors).
- There exists a risk-free security which promises a return of *r*, the risk free-rate.
- Assets are traded continuously in time.

3.1.1 The Sovereign

Consider a country with output Y_t at time *t*. Assume the country borrows money at time *t* and has to repay D at the maturity date T. The debt is assumed coupon free. Given that the country repays its debt it is left with $Y_T - D$ at time T. The repayment of debt is assumed to be taken directly from the country's output. One can for instance think of government expenditure as being reduced with the amount *D*. The country can choose to default on its debt in which case it repays nothing. A default is costly and the costs are here defined as a fraction *c* of *Y*, so that if it defaults the country is left with $Y_T - cY_T$.

As discussed in chapter 2 the costs of default are complex and not easily measured or defined. I have chosen a simple way of accounting for the costs by putting them all in one constant fraction of output. When a country defaults it seems to have an observable negative effect on the country's output, but a default may also yield other costs that are not easily observed. The costs included in the fraction *c* of output can be thought to also include costs that does not have an observable effect on output, but still are relevant for the decision of whether to default or not. This could include political costs and in addition it could be thought to include costs of defaulting that will incur in the years following the default year. These costs must also be considered when deciding whether or not to default. This means that in the event of a default one would not necessarily see that the output declined with the whole fraction *c*, because

some of the costs included in it may account for non-observable costs and costs that will incur in the years yet to come. The fraction *c* may in other words be artificially high.

I assume that *c* is an exogenously given constant. Assuming the country wishes to maximize its output it will choose the following output/payoff at time *T*:

$$\pi_T^Y = \max(Y_T - D, Y_T - cY_T)$$

In words, the country will choose to default if the costs of doing so are less than the costs of repaying[1]. When modeled like this it is implicitly assumed that it is only willingness to repay that matters and not ability. The country's attitude is: "We will repay our debt, but only if it is the best solution for us". By adding and subtracting $Y_T - D$ this can be written as

$$(3.1.1)\ \pi_T^Y = Y_T - D + \max(0, D - cY_T)$$

The value of the country's output in period *T* can be viewed as the output it will be left with if it pays back its debt $(Y_T - D)$ plus a put option on cY_T with strike price *D*. This put option represents the country's option to default. An interesting feature about this option is that in a sense the country is both the writer and the buyer of the option. Figure 1 presents equation (3.1.1) graphically for given values of *c* and *D*.

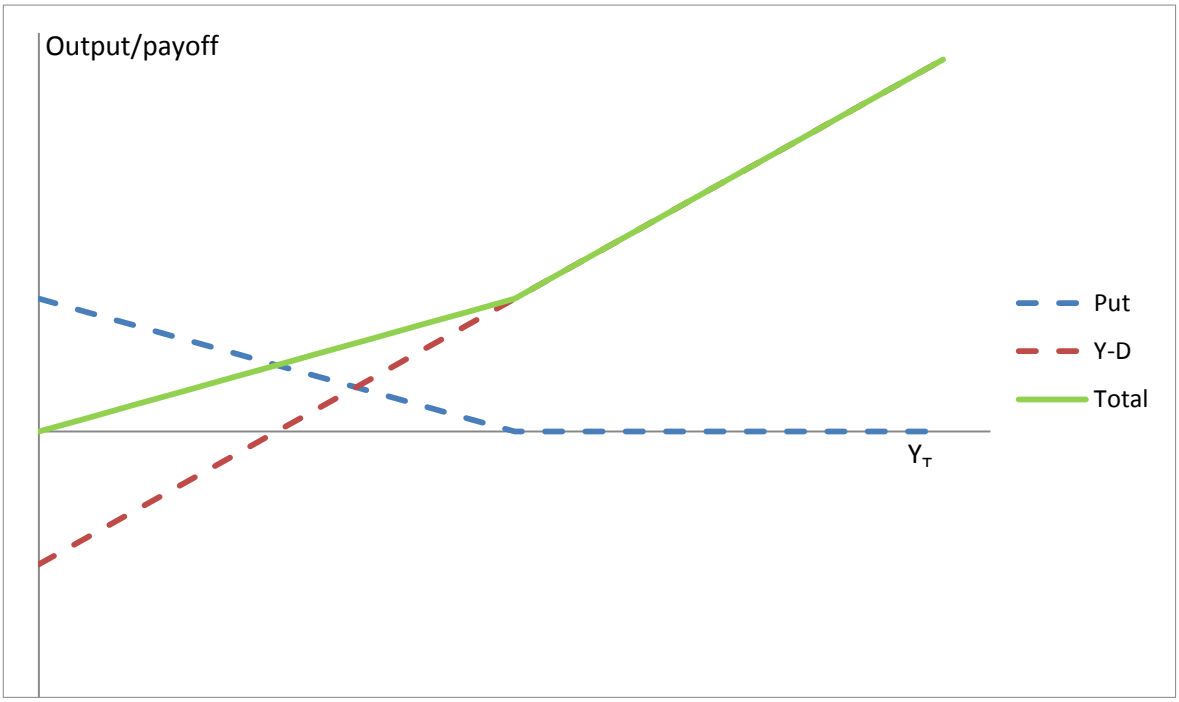

Figure 1 Output/payoff at maturity

[1] The setup of this simple model is inspired by a model in Feenstra and Taylor (2006).

The value of equation (3.1.1) at time *t* can be found using the equivalent martingale measure result developed in chapter 2. The value at time *t*, denoted by π_t^Y, is given by[2]

$$(3.1.2)\ \pi_t^Y = Y_t - e^{-r(T-t)}D + e^{-r(T-t)}DN(-h_2) - cY_tN(-h_1)$$

where $h_1 = \frac{\ln\frac{cY_t}{D} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, $h_2 = h_1 - \sigma\sqrt{T-t}$ and $N(x)$ gives the cumulative standard normal distribution.

Denote the value of the option to default as π_t^{def}, so that

$$\pi_t^{def} = e^{-r(T-t)}DN(-h_2) - cY_tN(-h_1)$$

The Greeks for this put are analogous to those for an ordinary put (except from the presence of *c*), but in addition it has a derivative for the constant *c*. This derivative is given by

$$\frac{\partial \pi_t^{def}}{\partial c} = -Y_tN(-h_1)$$

As expected this derivative is negative, meaning that the option to default gets less valuable as the costs associated with default increase.

3.1.2 Sovereign Debt

Assume the debt holders lend money to the country at time *t* and, given that the country chooses to repay, receive *D* at the maturity date *T*. If the country chooses to default the debt holders get nothing. Given these assumptions the payoff of the debt holders at maturity is given by

$$(3.1.3)\ \pi_T^D = D1_A, \qquad 1_A = \begin{cases} 1 \text{ if } D < cY_T \\ 0 \text{ otherwise} \end{cases}$$

The value at time *t* of equation (3.1.3) is then given as[3]

$$(3.1.4)\ \pi_t^D = e^{-r(T-t)}DN(h_2)$$

When comparing this result to Merton's result for a corporate liability the difference is obvious: In the case of a corporate liability the value of the company enters the pricing

[2] See appendix A for complete derivation.
[3] See appendix A for complete derivation.

formula for the debt, while the sovereign liability formula above does *not* contain the GDP for the country (it enters in the probabilities, but not as an element on its own). The formula given in equation (3.1.4) is equivalent to the last part of equation (2.4.1). This illustrates the fact that sovereign creditors have no legal claim on any values the sovereign possess and has to rely on the willingness of the sovereign to repay debt.

3.1.3 Credit Default Swaps (CDS)

The fact that a sovereign can default on its debt gives rise to a market for insurance against such a default. Securities that provide such insurance are called Credit Default Swaps (CDS). In the event of default a CDS pays the amount lost on the debt. In the case discussed here the payoff of a CDS equals the amount *D*. This payoff is contingent on the event of default, which gives a payoff at maturity of

$$\pi_T^{CDS} = D1_A, \qquad 1_A = \begin{cases} 1 \text{ if } D > cY_T \\ 0 \text{ otherwise} \end{cases}$$

the value at time *t* of the CDS is then given by

$$(3.1.4)\ \pi_t^{CDS} = e^{-r(T-t)}DN(-h_2)$$

It is obvious that a portfolio consisting of a long position in both the CDS and the underlying sovereign bond is a risk-free portfolio.

3.2 Extensions of the model

The model presented in the previous section is very simple, but catches the basic properties of sovereign debt and can be built further upon. In this section I discuss possible extensions of the model to make it more realistic.

3.2.1 Recovery Rate

3.2.1.1 The Sovereign

A complete repudiation of sovereign debt is, as noted in chapter 2, a very rare event. In almost every case of sovereign default there has been some sort of repayment to creditors. This fact suggests that the model in section 3.1 is unrealistic when it assumes that the country either repays its debt or not. To incorporate a more realistic scenario I now assume that, in the case

of a default, the country repays a fraction R of its debt. The constant R can thus be viewed as the recovery rate. The choice of the debtor country at maturity is then given by

$$\pi_T^Y = \max(Y_T - D, Y_T - RD - cY_T)$$

By adding and subtracting $Y_T - D$ this can be written as

$$(3.2.1)\ \pi_T^Y = Y_T - D + \max(0, (1-R)D - cY_T)$$

which is the equivalent of equation (3.1.1), but now the strike price of the put has changed to $(1-R)D$. Equation (3.2.1) is depicted in figure 2 below.

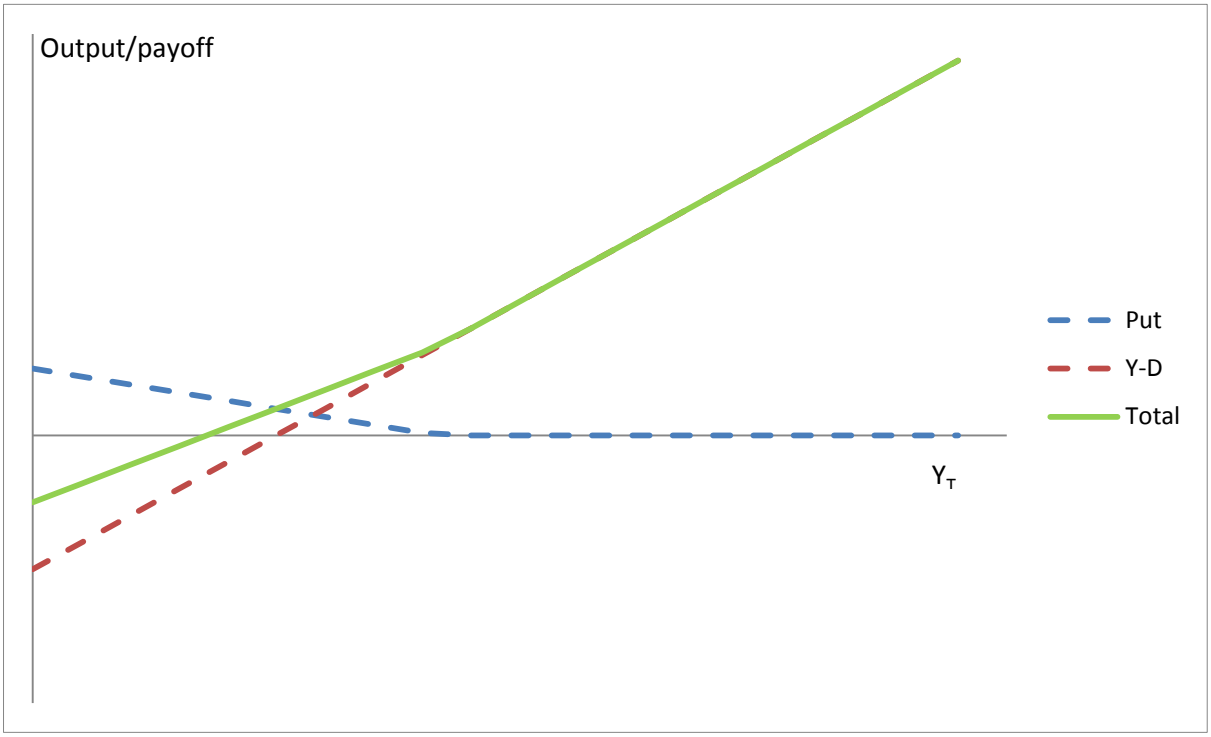

Figure 2 Output/payoff at maturity

Figure 2 reveals an immediate problem with this representation. For sufficiently low levels of output, the output the country is left with after defaulting is negative, which is an unfeasible feature. This feature arises because of the assumption that the country repays an exogenously given amount of the debt in the case of a default. I will return to this later and discuss how it can be resolved.

The value of equation (3.2.1) at time t is given by

$$(3.2.2)\ \pi_t^Y = Y_t - e^{-r(T-t)}D + e^{-r(T-t)}(1-R)DN(-b_2) - cY_tN(-b_1)$$

where $b_1 = \frac{\ln\frac{cY_t}{(1-R)D} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, b_2 = b_1 - \sigma\sqrt{T-t}.$

This model yields another derivative of the option price $\pi_t^{def} = e^{-r(T-t)}(1-R)DN(-b_2) - cY_tN(-b_1)$. The derivative with respect to *R*, which is given by

$$\frac{\partial \pi_t^{def}}{\partial R} = -DN(-b_2)$$

As expected it is negative, meaning that an increase in the recovery rate leads to a decline in the value of the option to default.

3.2.1.2 Creditors and CDS

With this new assumption the creditors receive *D* if the country honors its obligation and *RD* if the country chooses to default. The payoff at maturity for the creditors is thus given by

$$(3.2.3)\ \pi_T^D = D1_A + (1-1_A)RD = RD + (1-R)D1_A, \qquad 1_A = \begin{cases} 1 \text{ if } (1-R)D < cY_T \\ 0 \text{ otherwise} \end{cases}$$

The price of the debt at time *t* is then given by

$$(3.2.4)\ \pi_t^D = e^{-r(T-t)}D\left(R + (1-R)N(b_2)\right)$$

The CDS pays off the amount lost on the debt in case of a default. In this framework the payoff on the CDS becomes

$$\pi_T^{CDS} = (1-R)D1_A, \quad 1_A = \begin{cases} 1 \text{ if } (1-R)D > cY_T \\ 0 \text{ otherwise} \end{cases}$$

The price of the CDS at time *t* is then given by

$$(3.2.5)\ \pi_t^{CDS} = e^{-r(T-t)}(1-R)DN(-b_2)$$

3.2.1.3 The Costs of Default and the Recovery Rate

Until now I have assumed that the costs of default and the recovery rate are exogenously given constants. However appealing the simplicity this assumption results in, it does not change the fact that it is highly unrealistic.

First the costs of default are likely to change with the severity of the default. I have no empirical support for this assumption, but I am quite confident that the costs related to a default will tend to increase as the amount the defaulter repays decreases. A simple way of incorporating such a feature in the model is to model the costs of default as a decreasing linear function of the recovery rate. A simple example of a relationship between the recovery rate and the costs of default could be

$$c = 1 - aR$$

where *a* is a given constant. Assuming $a = 1$ substantially simplifies the pricing relationships developed earlier. Looking at the sovereign, the value of its position becomes

$$(3.2.6)\ \pi_t^Y = Y_t - e^{-r(T-t)}D + (1-R)\big(DN(-d_2) - Y_t N(-d_1)\big)$$

where $d_1 = \frac{\ln\frac{Y_t}{D} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}$. Given the assumption about the relationship between the costs of default and the recovery rate, the option to default now consists of $(1-R)$ number of put options on *Y* with strike price *D*.

The value of the debt holders' claim now becomes

$$\pi_t^D = e^{-r(T-t)}D[R + (1-R)N(d_2)]$$

And the CDS pricing formula becomes

$$\pi_t^{CDS} = e^{-r(T-t)}(1-R)DN(-d_2)$$

A critique of such a simple relationship between the costs of default and the recovery rate will be that there are other things which also affect the costs of default. The political costs of defaulting depend on the attitudes of the citizens of the country, which can be challenging to forecast. During the latest debt problems in Europe it has become obvious that it is also politically costly to perform budget cuts that are necessary to honor debt obligations (more so in some countries than other). This could imply that the political costs of defaulting need not be that vast when a country has severe problems.

Perhaps there is a non-linear relationship between political costs of default and the severity of the debt problems of the country. The political costs may be large if the problems of the country are small and increase with the magnitude of the crisis up to some point, but when the

problems of the country becomes large enough the political costs of defaulting actually decrease again. An example could be Iceland where there is a complete national agreement that the people will not pay for the problems their enormous banks caused during the financial crisis. Politicians that refuse to make such payments actually gain political support.

Second, the recovery rate is not constant and not known in advance. The problem noted with equation (3.2.1) that the output of the country in the case of a default may be negative arises because of the assumption of a constant recovery rate. It is however not realistic that a debtor country will repay the same fraction of debt regardless of how severe their problems become. If a country suffers a large negative shock to its output it is likely that it will repay less than if it suffered only a minor shock. In other words the recovery rate is likely to be negatively related with the ratio of debt to GDP. The higher the ratio of debt to GDP at the time of default, the lower is the recovery rate.

3.2.2 Coupon Bonds

I have assumed that there are no coupon payments on the bonds. It is however simple to expand the model to include coupon payments. This can be done by regarding the coupon payments as continuous payments which then enters the pricing formulas in the same way as continuous dividends does in a formula for a standard European option on a stock. To further ease the restrictions on the model one can assume that the sovereign can exercise their option to default at any time up until maturity meaning that their option becomes an American option. A closed form solution for American options is yet to be developed, but the value can be estimated using numerical techniques such as Monte Carlo simulation.

3.2.3 Different processes

3.2.3.1 Merton's Jump-Diffusion Model

The assumption that the output of a country follows GBM is appealing because of its statistical properties, but there may be other processes that describe actual output processes in a better way. One possibility would be to model output so as to include the possibility of jumps. Merton (1976) uses a model that includes such a feature. Today the model is commonly known as Merton's mixed jump-diffusion model. The model is presented in the following.

The process for the output of a country in Merton's mixed jump-diffusion model is given by

$$(3.2.7)\ dY = (\alpha - Jk)Ydt + \sigma dz_Y + dq$$

where dq equals 0 if no jump occurs and $(X-1)$ if there is a jump, J is the expected number of jumps per unit time and $k = E[X-1]$. Accordingly $(X-1)$ is the actual percentage change in Y in the case of a jump, so that in the case of a jump Y changes to XY. The percentage change X can both be a random variable and a constant. The inclusion of jumps in the process means that the process no longer is continuous in time and that the derivation of a closed form solution to an option pricing problem is restricted to special cases. One such special case is that of $X = 0$. Under this assumption the price of an option equals the price of an option under the regular assumption of GBM, but with the risk-free rate replaced by $r + J$.

Another special case is the one where the jump magnitude is lognormally distributed. Given lognormally distributed X with standard deviation s the price of an option is given by

$$\sum_{n=0}^{\infty} \frac{e^{-J'(T-t)}\left(J'(T-t)\right)^n}{n!} f_n$$

where $J' = J(1+k)$ and f_n is the Black-Scholes option price with variance

$$\sigma^2 + \frac{ns^2}{T-t}$$

and risk free rate given by

$$r - Jk + \frac{n\ln(1+k)}{T-t}$$

If one cannot derive a closed form solution for the option price one can apply Monte Carlo simulation to estimate the price. The problem then is that one has to make assumptions about the discount rate since it may not be correct to use the risk-free rate.

3.2.3.2 Expanding the Macroeconomic Environment of the Model

In the world of option pricing where options depend on underlying variables such as stocks, which tend to be difficult to model, the use of GBM and other random-walk-models have proven to be useful tools. The output of a country tends to be far less volatile than stocks and exhibit much clearer relationships with fundamental variables such as capital stock. Based on

this it would be natural to expand the process of output so as to include these fundamental variables. Such a process could be

$$(3.2.8)\ dY = \alpha K dt + \alpha K \sigma_Y dz_Y$$

where K is the capital stock, α is the marginal product of capital, σ_Y is the volatility of the rate of change in Y and dz is a Wiener process (Turnovsky, 1995). This will of course complicate the valuation of the securities and perhaps make closed form solutions impossible to derive.

4 Calibrating the Model

In this section I attempt to confront the model with data and thereby get an impression of how it performs. In the analysis I estimate CDS prices using my model and compare them to the actual CDS prices observed in the market. I have collected data for Portugal, Ireland, Greece and Spain (PIGS).

4.1 The Data

The dataset consists of daily observations on one-year CDS prices, quarterly observations on GDP, and daily data on the 12 month LIBOR rate. I have not been able to find data on the residual maturity structure of debt for each individual nation, but I have found a time series for the Euro 16 area in which all four countries enter. This time series gives the ratio of general government debt with residual maturity of up to one year to GDP. In lack of country specific data I use this as a proxy for each country. This is of course an inaccurate proxy on the amount of short term debt each country has, but it may give a relevant picture of how the amount has developed over time. The use of this proxy means that the short term debt to GDP ratio will be equal for all four countries, and so the only variables that will create differences in the CDS prices (presented in basis points) between countries is the volatility and the costs of default (which I deduct from the model).

Because of this inaccuracy in the dataset the following analysis will not be suitable for assessing the exact prices the model produce. However, given that we believe the short term debt level of the Euro 16 area is a good proxy for the development of short term debt in the countries in question, it may shed some light on how well the model depicts the *development* in the prices. The data show that the short term debt level of the Euro 16 area has increased from almost 15% of GDP in 2005 to over 22% in 2010[4]. The data is collected from Thomson DataStream, Reuters EcoWin Pro and ECB Statistics.

4.2 The Parameters of the model

Most of the parameters in the model are observable or can be estimated from the data, but a couple of parameters present some challenges.

[4] See Appendix B.

One challenge is how to handle the debt parameter. In the model debt simply enters as a constant D, whereas in real life the debt structure of a sovereign is very complex. Ideally I should have had data on the residual maturity structure of each country's debt, but as already noted above this data was not obtainable so I use the already discussed proxy.

The recovery rate enters the model as a constant and has to be estimated. For this estimation I rely on an annual analysis conducted by Moody's Investors Service on sovereign default and recovery rates. They estimate a 50 % issuer-weighted mean recovery rate[5] based on a dataset from 1983 to 2009 (Moody's Credit Research, 2010). I therefore use $R = 0.5$ in my model.

It is natural to assume that the correct benchmarks for risk-free rates are the rates on Treasury bills and Treasury bonds, but traders in the market usually use LIBOR rates as proxies for short-term risk-free rates. They argue that the rates on Treasury securities are too low and therefore the LIBOR rates serve as better proxies. There are three arguments behind this reasoning: First, the demand for T-bills and T-bonds is artificially high because financial institutions must purchase them to fulfill a variety of regulatory requirements. Second, the amount of capital an institution is required to hold to support investments in T-bills and T-bonds is substantially smaller compared to the capital required to support similar investments in instruments with very low risk (thus boosting demand). And third, treasury securities are not taxed at the state level and are hence given a favorable tax treatment compared to most other fixed-income investments (Hull, 2009). I will not try to conclude whether these arguments are correct or not, but if the traders in the market use LIBOR rates I believe the correct rates to use when trying to explain market prices will be LIBOR rates. I therefore use the 12-month LIBOR rate as a proxy for the 1 year risk-free rate.

The cost of default parameter is the most challenging to estimate in the model. Since not all costs can be measured (or even identified) it is hard to give *one* estimate to what the constant,c, should be. As mentioned in chapter 3 the costs of default may differ between countries and also depend on the severity of the crisis. Instead of trying to estimate the cost of default parameter, I find the value that minimizes the sum of the squared residuals between the model and the observed prices. This sum is given by

[5] Issuer-weighted mean recovery rates are derived by estimating the mean recovery rates for each issuer, and then averaging them across issuer (Moody's Credit Research, 2010).

$$\sum_{i=1}^{n}\left(P_i - \hat{P}_i\right)^2$$

Where P_i and $\hat{P}_i$ is the observed price and the estimated price respectively. The value of the cost of default parameter which minimizes this sum is thus the value of the parameter that results in the best fit for the model given all other variables.

4.3 Estimating CDS Prices

In chapter 3 I presented three different formulas for the pricing of a CDS. I consider the formula given by equation (3.2.5) the most realistic of these formulas, and hence use that one to calculate the CDS prices. Below I display a table with the results.

Table 1 CDS prices

	Portugal		Ireland		Greece		Spain	
Date	**Real**	**Estimates**	**Real**	**Estimates**	**Real**	**Estimates**	**Real**	**Estimates**
Q1 2005	NA	0.00	1.12	0.00	1.60	0.00	NA	0.00
Q2 2005	NA	0.00	4.84	0.00	2.05	0.00	2.50	0.00
Q3 2005	2.79	0.00	8.19	0.00	2.03	0.00	1.28	0.00
Q4 2005	3.10	0.00	3.77	0.00	1.92	0.00	6.05	0.00
Q1 2006	4.16	0.00	0.99	0.00	3.32	0.00	1.17	0.00
Q2 2006	4.50	0.00	1.06	0.00	3.32	0.00	1.01	0.00
Q3 2006	3.85	0.00	1.05	0.00	3.29	0.00	1.10	0.00
Q4 2006	2.80	0.00	2.06	0.00	2.22	0.00	1.33	0.00
Q1 2007	2.74	0.00	4.07	0.00	1.68	0.00	1.18	0.00
Q2 2007	2.37	0.00	2.91	0.00	1.24	0.00	1.15	0.00
Q3 2007	4.70	0.00	3.50	0.00	1.54	0.00	2.06	0.00
Q4 2007	5.59	0.00	4.55	0.00	7.15	0.00	5.88	0.00
Q1 2008	19.16	0.00	17.65	0.00	22.60	0.00	20.55	0.00
Q2 2008	16.62	0.00	16.62	0.00	23.52	0.00	17.43	0.00
Q3 2008	14.89	0.00	20.86	0.00	24.06	0.00	15.51	0.00
Q4 2008	47.96	0.00	81.07	0.00	99.02	0.00	50.07	0.00
Q1 2009	80.91	0.00	210.49	0.81	190.31	0.00	90.29	0.00
Q2 2009	53.02	0.00	163.75	2.04	123.47	0.00	66.97	0.00
Q3 2009	39.67	0.00	122.60	5.67	96.75	0.02	51.10	0.00
Q4 2009	38.86	0.00	105.28	7.83	96.83	0.03	39.03	0.00
Q1 2010	140.27	294.71	109.68	233.36	402.21	722.22	95.95	178.92
Q2 2010	306.85	234.75	201.49	217.53	867.20	701.54	187.26	166.05
Q3 2010	335.95	220.99	345.16	210.66	945.89	766.18	237.38	168.80

The real prices are averages of the daily quotes over the quarter. All prices are quoted in basis points. One hundred basis points is the same as 1%, hence a one-year CDS price of 50, means that the CDS premium is 0.5% of the face value of the debt. Clearly the model produces different prices than those observed in reality. The estimated prices are zero for all dates up until Q1 2010 for Portugal, Greece and Spain, while they for Ireland are zero up until Q1 2009. Because of the lack of nation specific debt data there is no point in assessing the individual prices produced by the model. What is interesting is if the model is able to trace the *development* in the prices. To see the development clearer the data from table 1 are displayed in a figure for each country below.

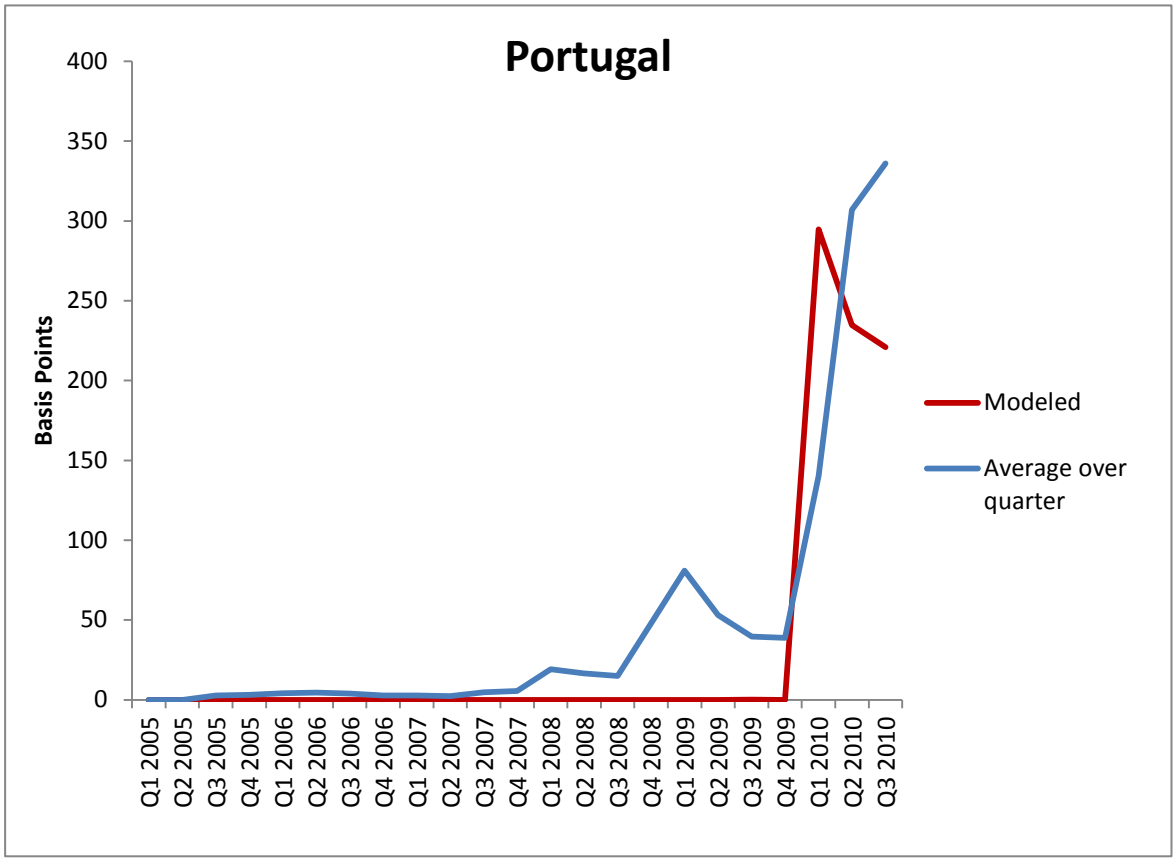

Figure 3 One-year CDS prices Portugal

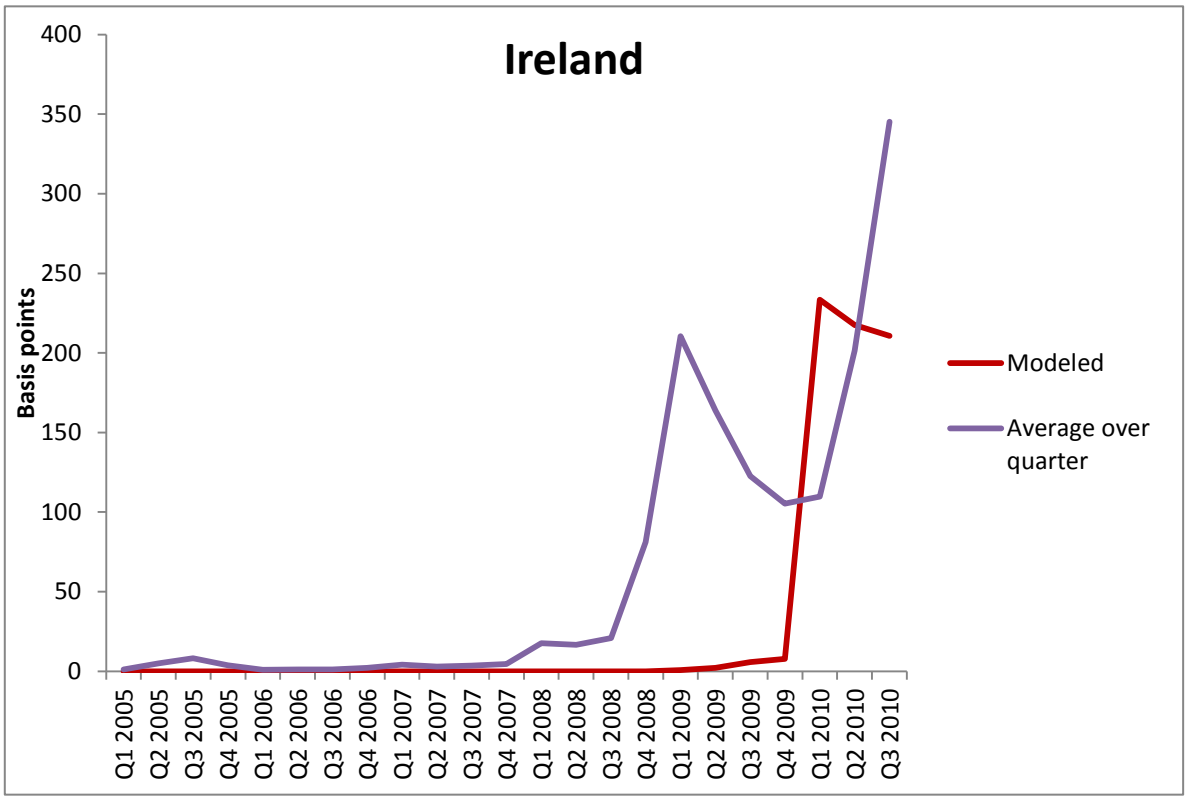

Figure 4 One-year CDS prices Ireland

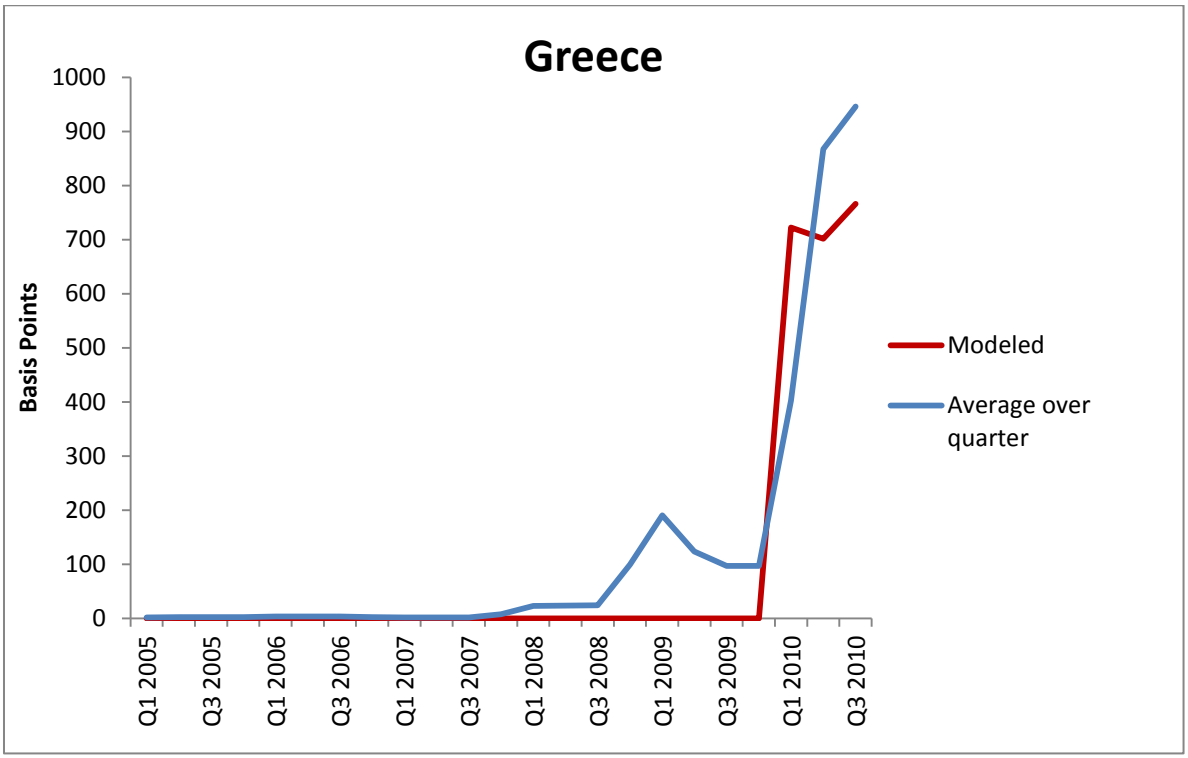

Figure 5 One-year CDS prices Greece

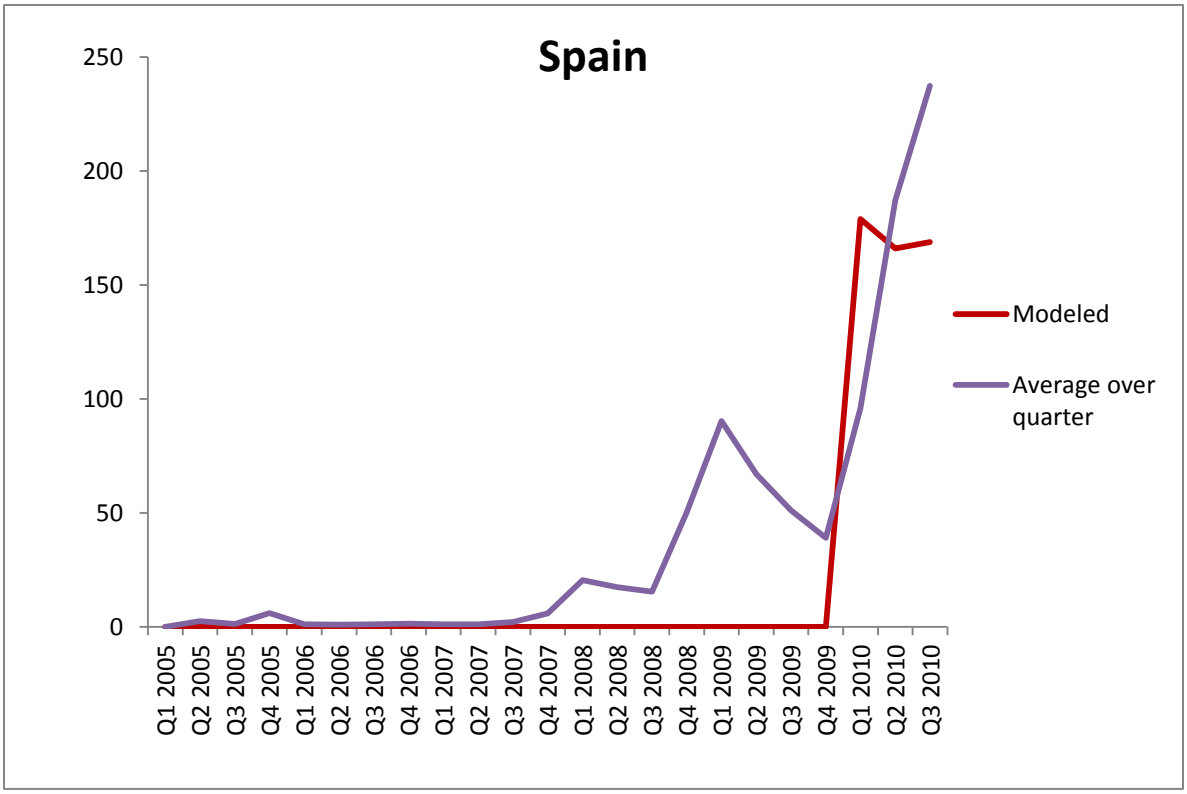

Figure 6 One-year CDS prices Spain

There are several noteworthy features of the above figures. The observed CDS prices have very similar paths for all four countries. They are very low up until the end of 2007 (at the wake of the financial crisis) and then they all start to rise. First they rise to a top around Q1 2009 then they show a decline before they all have a sharp rise in 2010. The similarity between these price paths indicates that the CDS prices were driven by the same events for all countries. To see if this pattern of similar development is present in other variables I consider the development in the GDP for the countries. Figure 7 shows the year-to-year change in GDP for each country[6].

[6] See appendix B for the data used

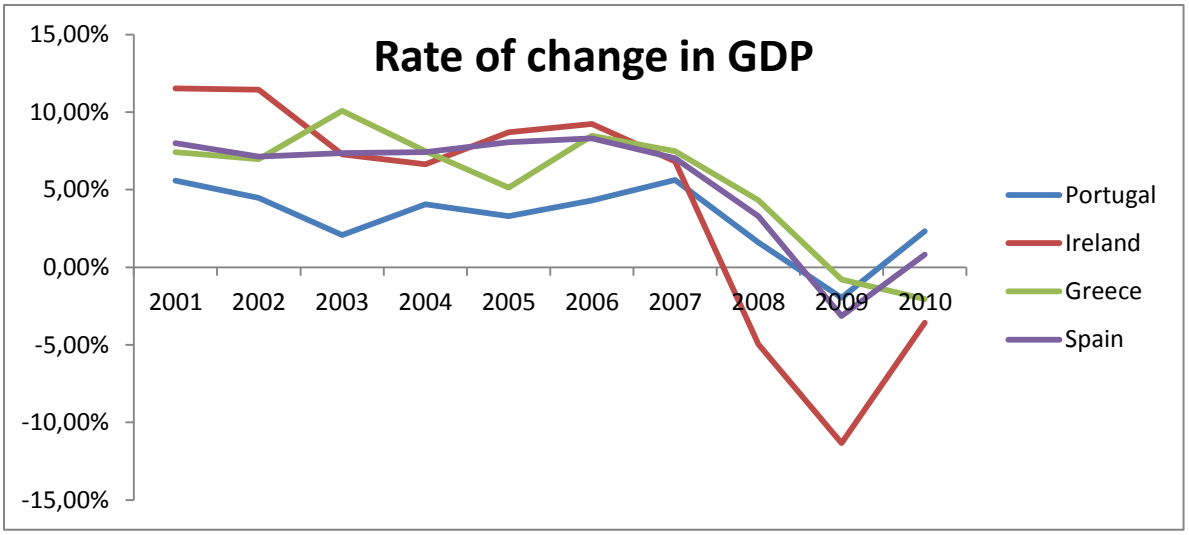

Figure 7 Rate of change in GDP

Clearly the developments in the countries' GDP have been quite similar over the last years. Assuming the model is correct; the similar one-year CDS price development then implies that the evolvement of debt with short term residual maturity also is similar. Although the use of the same proxy for all four countries' residual maturity is inaccurate it will at least capture this feature.

Since I do not have nation specific data on the residual maturity of debt, I am not able to assess the value of the cost of default parameter extracted from the model. An interesting observation though (and a weakness of the model) is the fact that the CDS price is highly sensitive to changes in the costs of default as shown from the sensitivity table below.

Table 2 CDS prices and costs of default

c	0.105	0.110	0.115	0.120
CDS price (in basis points)	4815.650	2048.459	48.031	0.031

As seen from table 2, the CDS price is very sensitive to small variations in the costs of default. Remember that the costs of default are defined as a fraction of output. A change in the cost of default of 0.005 then equals 0.5 percentage points. For a country with GDP of 200 billion, 0.5 percent equals 1 billion. A large number, but hardly large enough to defend the large price movements seen in table 1. To see why the model has this feature, I look at the derivative of the risk-neutral probability of a default with respect to the costs of default. The risk-neutral probability of default is given by

$$N(-b_2) = N\left(-\frac{\ln\left(\frac{cY}{(1-R)D}\right) + (r - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

The derivative is then given by

$$\frac{\partial N(-b_2)}{\partial c} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-b_2)^2} \cdot \frac{1}{c\sigma\sqrt{T-t}}$$

When both c and σ is low (like they are in this case: the cost of default parameter is about 0.11 for all countries and the volatility is below 5% for all) this derivative will be large which explains the large price changes accompanying small changes in the costs of default.

As expected the model predicts quite different prices than what is observed in reality and does not seem to explain the development very well, but it does exhibit the same spike in the last quarters as the real series perhaps indicating that it has some explaining power.

4.4 Higher Volatility

At the end of 2007 all the real CDS prices start to rise. This is possibly explained by the growing amount of short term debt, but the prices also respond to changes in risk. If the perceived risk of default increases due to e.g., the financial crisis, CDS prices will be affected.

When calibrating my model above I have calculated the volatility of the rate of change in output, like always, from historical data, but seeing the large incline in prices around 07/08 (which is not depicted by the model) perhaps the uncertainty incorporated in the observed CDS prices is larger than the historical volatility and this is part of the reason for the rise in prices. The estimated volatilities are all very low, between 1.5% and 2% for Portugal, Spain and Greece, and between 3.5% and 4.5% for Ireland. I have conducted an experiment where I incorporate a somewhat higher volatility of 10% in the model to see if this will improve the fit. Below I display the resulting figure for Greece[7].

[7] The data used is given in appendix B, and the corresponding figures for the other countries are given in appendix C.

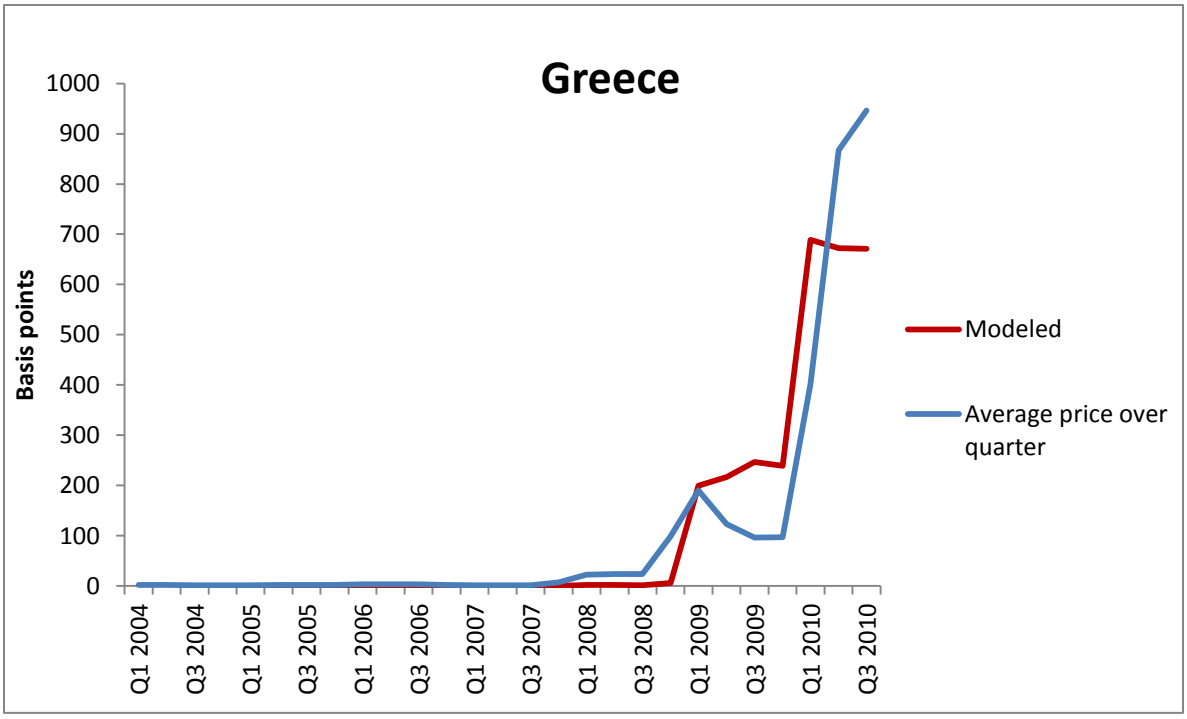

Figure 8 One-year CDS prices (higher volatility)

Clearly the incorporation of higher volatility enables a better fit for the model, possibly implying that expected volatility is higher than the historical. To see how the volatility works in the model consider the derivative of the relative CDS price with respect to volatility given by

$$\frac{\partial \frac{\pi_t^{CDS}}{D}}{\partial \sigma} = e^{-r(T-t)}(1-R)n(-b_2)\left(b_2\frac{1}{\sigma}+\sqrt{T-t}\right)$$

where $n(-b_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(-b2)^2}$. The sign of this derivative will depend on the sign of the last parenthesis$\left(b_2\frac{1}{\sigma}+\sqrt{T-t}\right)$. This will be negative if $b_2\frac{1}{\sigma} < -\sqrt{T-t}$, which is equivalent to

$$\frac{1}{2}\sigma^2 < -\left(\frac{\ln\left(\frac{cY}{(1-R)D}\right)}{T-t}+r\right)$$

If $\frac{\ln\left(\frac{cY}{(1-R)D}\right)}{T-t}+r$ is positive (which is the case as long as$(1-R)D < cY$) this obviously doesn't hold and hence the derivative is positive. An increase in volatility leads to an increase in the price. If however $\frac{\ln\left(\frac{cY}{(1-R)D}\right)}{T-t}+r$ is negative, the derivative may be negative and an

increase in volatility will then lead to a decrease in the price. In my analysis the debt level is not large enough to change the sign of the derivative from positive to negative; hence there is a direct positive effect of an increase in the volatility on the price level.

The volatility also has an indirect effect on the price in that it affects how sensitive the price is to changes in the debt level. The derivative of the relative CDS price with respect to debt is given by

$$\frac{\partial \frac{\pi_t^{CDS}}{D}}{\partial D} = e^{-r(T-t)}(1-R)n(-b_2)\frac{1}{D\sigma\sqrt{T-t}}$$

which is obviously positive. The derivative of this with respect to volatility is given by

$$\frac{\partial^2 \frac{\pi_t^{CDS}}{D}}{\partial D \partial \sigma} = e^{-r(T-t)}(1-R)n(-b_2)\frac{1}{D\sigma\sqrt{T-t}}\left(b_2^2\frac{1}{\sigma} + b_2\sqrt{T-t} - \frac{1}{\sigma}\right)$$

The sign of this derivative is dependent on the size of b_2. If it is larger than 1 the derivative is positive and an increase in the volatility increases the sensitivity of the price to changes in debt. Since time to maturity in my analysis is 1, $b_2\sqrt{T-t}$ will never be larger in absolute value than $b_2^2\frac{1}{\sigma}$ and hence even if b_2 is negative the sum of the two parts will always be positive (given that the absolute value is larger than 1). The derivative will only be negative if b_2 is less than 1 and $\frac{1}{\sigma}$ is larger in absolute value than $b_2^2\frac{1}{\sigma} + b_2\sqrt{T-t}$. In my analysis b_2 is always positive and before the increase in volatility it is below 1 only for high levels of debt for Ireland, but when I increase the volatility to 10% it gets below 1 for the largest levels of debt in all countries and thus the sign of the derivative changes from positive to negative for these debt levels. The increase in volatility improves the fit because it makes the prices at high levels of debt relatively less sensitive to debt changes than prices at low levels of debt. Thus where there was no movement in prices when volatility was low there is now a movement similar to the one observed in reality.

The improvement of the fit due to the incorporation of higher volatility is also present for the other countries. The analysis conducted leads to the conclusion that although the model is highly stylized, it is able to trace some of the development in actual CDS prices.

5 Final Remarks

In this thesis I have developed a model for the pricing of sovereign debt and the option to default. I have adopted an option analysis approach similar to the one developed by Merton (1974). The formulas derived relies on the assumption that it is the willingness of a country to repay debt that is significant and not its ability, since most countries would be able to repay debt if sufficient budget cuts could be made. It also assumes that the country wishes to maximize output and chooses to default or not based on that. The model is very simple and in its current state probably mostly interesting on a theoretical level and possibly as a starting point for a more thorough option analysis of sovereign liabilities, and some possible extensions have been suggested in this thesis. Compared to other similar models it stands out as more realistic in its assumptions regarding the creditors' payoff structure. This makes it possible to isolate the option to default and thereby enable the valuation of this option. To my knowledge this has not been done previously.

The value added of the analysis in chapter four is limited because of its shortcomings when it comes to data quality. It should therefore not be seen as a careful empirical analysis, but rather as an illustration of the mechanisms at hand. The illustration indicates that the model has potential to explain at least some of the development in the CDS prices. As discussed in chapter 2 the costs of default are subject to much research and discussion, and it is yet to be established with certainty what the main costs of a default really are. To conduct a fully-fledged empirical analysis, it would be of critical importance to model in more detail the costs of default and the recovery rate.

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Appendix

A Derivation of The Model

A.1 Equation (3.1.2)

The starting point for the derivation is the following equation

$$(3.1.1)\ \pi_T^Y = Y_T - D + \max(0, D - cY_T)$$

Using the equivalent martingale measure result the value at time *t* can be found by using a risk-free asset with value 1 at time *T* and value $e^{-r(T-t)}$ at time *t* as a numeraire.

$$\pi_t^Y = e^{-r(T-t)}E_Q[Y_T - D + \max(0, D - cY_T)]$$

$$\pi_t^Y = e^{-r(T-t)}E_Q[Y_T] - e^{-r(T-t)}D + e^{-r(T-t)}DE_Q[1_A] - e^{-r(T-t)}cE_Q[Y_T 1_A],$$

$$1_A = \begin{cases} 1 \text{ if } D > cY_T \\ 0 \text{ otherwise} \end{cases}$$

The problem in the above equation is to find $E_Q[Y_T 1_A]$. This can be solved by using *Y* itself as a numeraire instead of the risk-free asset. This changes the last part to $cY_0E_Y[1_A]$, where E_Y denotes the expected value in a world that is forward risk neutral with respect to *Y*. The solution is then given by

$$\pi_t^Y = Y_0 - e^{-r(T-t)}D + e^{-r(T-t)}DP_Q(D > cY_T) - cY_0P_Y(D > cY_T)$$

where P_Q and P_Y denotes the probability of an event in a risk-neutral world and when you are forward risk-neutral with respect to *Y* respectively. By using the expression developed for *Y* in chapter 2 these probabilities can be transformed into standard normal probabilities.

$$P_Q\left(D > cY_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\epsilon}\right) = P_Q\left[\frac{\ln\frac{D}{cY_t} - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} > \epsilon\right]$$

and

$$P_Y\left(D > cY_t e^{\left(r+\sigma^2-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\epsilon}\right) = P_Y\left[\frac{\ln\frac{D}{cY_t} - \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} > \epsilon\right]$$

The value at time t is then given by

$$(3.1.2)\ \pi_t^Y = Y_t - e^{-r(T-t)}D + e^{-r(T-t)}DN(-h2) - cY_tN(-h1)$$

where $h1 = \frac{\ln\frac{cY_t}{D} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, $h2 = h1 - \sigma\sqrt{T-t}$ and $N(x)$ gives the cumulative standard normal distribution.

A.2 Equation (3.1.4)

Equation (3.1.3) is given by

$$(3.1.3)\ \pi_T^D = D1_A, \qquad 1_A = \begin{cases} 1 \text{ if } D < cY_T \\ 0 \text{ otherwise} \end{cases}$$

Using the equivalent martingale measure result with a risk-free asset with value 1 at time T and value $e^{-r(T-t)}$ at time t as a numeraire, the value at time t is given as

$$\pi_t^D = e^{-r(T-t)}E_Q[D1_A]$$

$$\pi_t^D = e^{-r(T-t)}DP_Q(D < cY_T)$$

Inserting the expression for Y_T gives

$$\pi_t^D = e^{-r(T-t)}DP_Q\left(\epsilon < \frac{\ln\frac{cY_t}{D} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$\Rightarrow (3.1.4)\ \pi_t^D = e^{-r(T-t)}DN(h2)$$

where $h2$ is given in above.

B Tables

Table 3 Ratio of short term general government debt to GDP Euro 16 area

Ratio of short term general government debt to GDP						
Year	2005	2006	2007	2008	2009	2010
Debt/GDP	0.15	0.14	0.15	0.19	0.21	0.22

Table 4 Year-to-year change in GDP

	Portugal	Ireland	Greece	Spain
2001	5,59 %	11,52 %	7,41 %	8,00 %
2002	4,47 %	11,44 %	6,97 %	7,13 %
2003	2,07 %	7,28 %	10,09 %	7,36 %
2004	4,06 %	6,64 %	7,48 %	7,42 %
2005	3,29 %	8,71 %	5,13 %	8,06 %
2006	4,29 %	9,24 %	8,47 %	8,31 %
2007	5,61 %	6,82 %	7,48 %	7,03 %
2008	1,59 %	-4,96 %	4,32 %	3,28 %
2009	-1,98 %	-11,33 %	-0,80 %	-3,14 %
2010	2,31 %	-3,57 %	-2,04 %	0,83 %

Table 5 CDS prices (higher volatility)

	Portugal		Ireland		Greece		Spain	
Date	**Real**	**Estimates**	**Real**	**Estimates**	**Real**	**Estimates**	**Real**	**Estimates**
Q1 2005	NA	0.00	1.12	0.00	1.60	0.00	NA	0.00
Q2 2005	NA	0.00	4.84	0.00	2.05	0.00	2.50	0.00
Q3 2005	2.79	0.00	8.19	0.00	2.03	0.00	1.28	0.00
Q4 2005	3.10	0.00	3.77	0.00	1.92	0.00	6.05	0.00
Q1 2006	4.16	0.00	0.99	0.00	3.32	0.00	1.17	0.00
Q2 2006	4.50	0.00	1.06	0.00	3.32	0.00	1.01	0.00
Q3 2006	3.85	0.00	1.05	0.00	3.29	0.00	1.10	0.00
Q4 2006	2.80	0.00	2.06	0.00	2.22	0.00	1.33	0.00
Q1 2007	2.74	0.00	4.07	0.00	1.68	0.00	1.18	0.00
Q2 2007	2.37	0.00	2.91	0.00	1.24	0.00	1.15	0.00
Q3 2007	4.70	0.00	3.50	0.00	1.54	0.00	2.06	0.00
Q4 2007	5.59	0.00	4.55	0.00	7.15	0.00	5.88	0.00
Q1 2008	19.16	0.31	17.65	0.30	22.60	2.38	20.55	0.22
Q2 2008	16.62	0.24	16.62	0.24	23.52	1.90	17.43	0.17
Q3 2008	14.89	0.23	20.86	0.23	24.06	1.86	15.51	0.17
Q4 2008	47.96	0.94	81.07	0.93	99.02	6.31	50.07	0.70

Q1 2009	80.91	55.38	210.49	54.95	190.31	199.42	90.29	45.19
Q2 2009	53.02	61.25	163.75	60.77	123.47	216.67	66.97	50.10
Q3 2009	39.67	71.79	122.60	71.24	96.75	246.88	51.10	58.96
Q4 2009	38.86	69.09	105.28	68.56	96.83	239.23	39.03	56.69
Q1 2010	140.27	259.84	109.68	258.24	402.21	688.77	95.95	221.69
Q2 2010	306.85	252.15	201.49	250.58	867.20	672.60	187.26	214.92
Q3 2010	335.95	251.46	345.16	249.90	945.89	671.15	237.38	214.32

C Figures

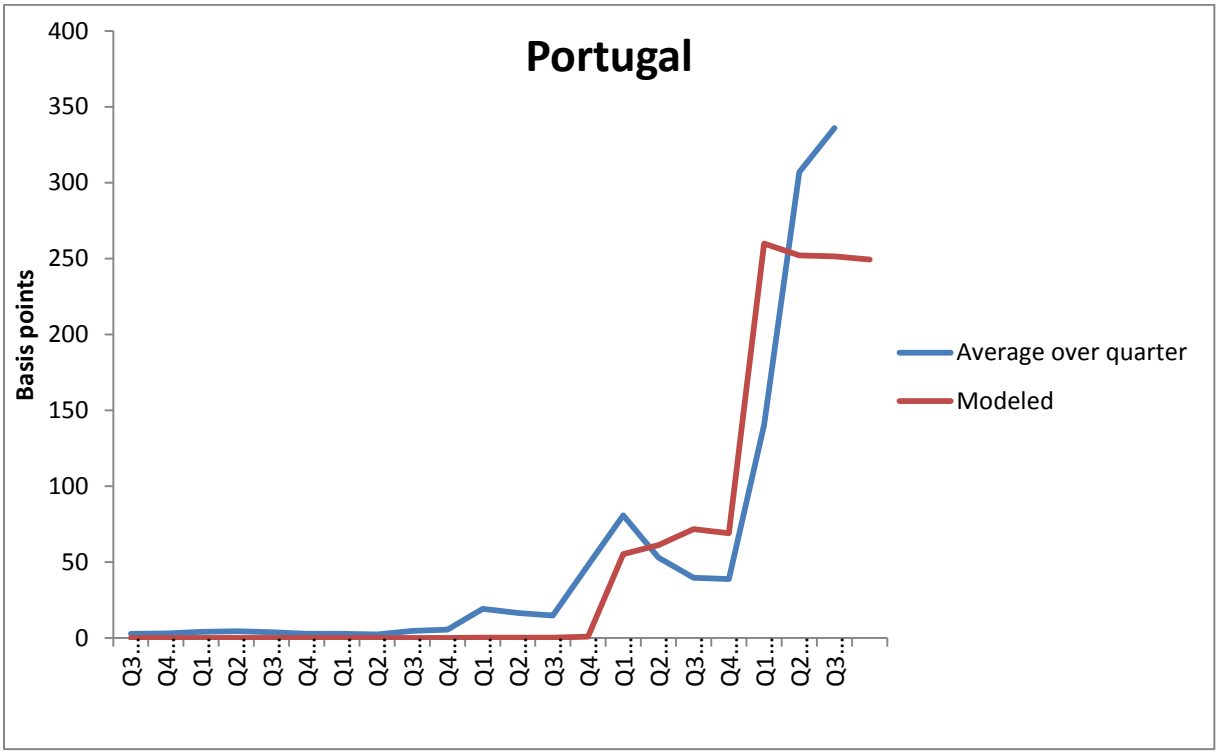

Figure 9 One-year CDS prices Portugal (higher volatility)

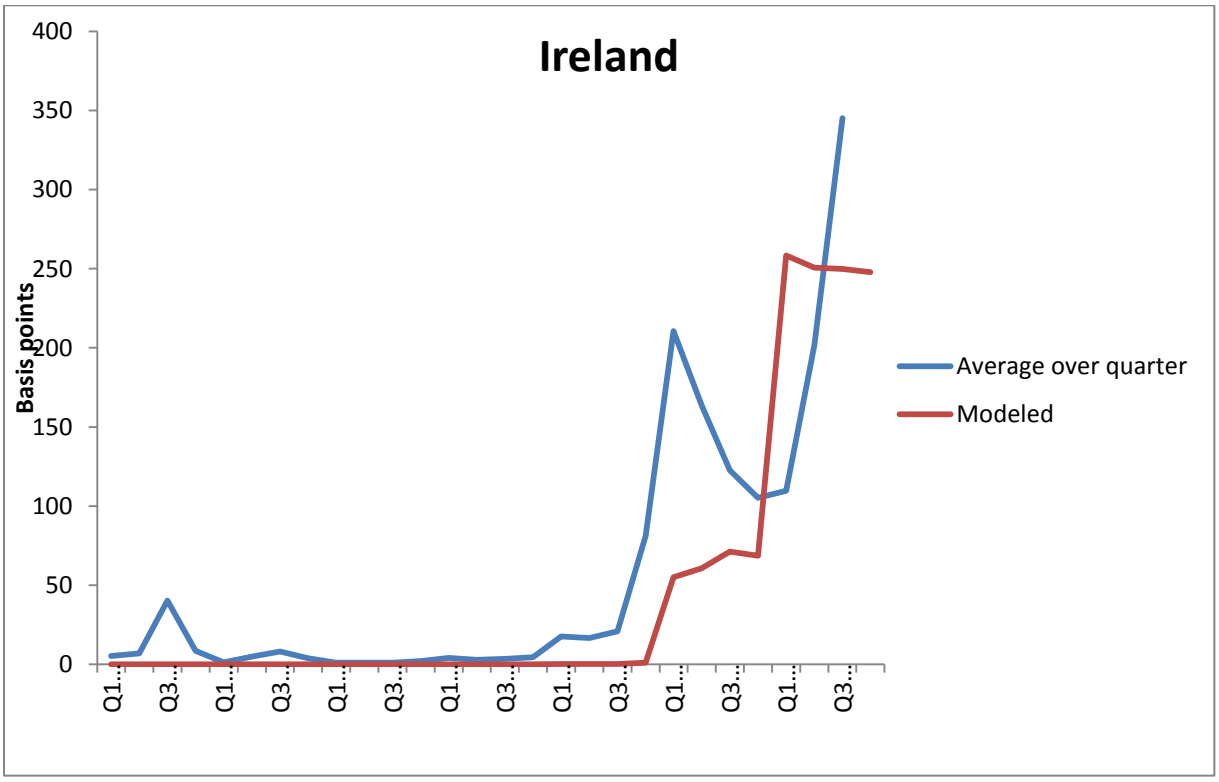

Figure 10 One-year CDS prices Ireland (higher volatility)

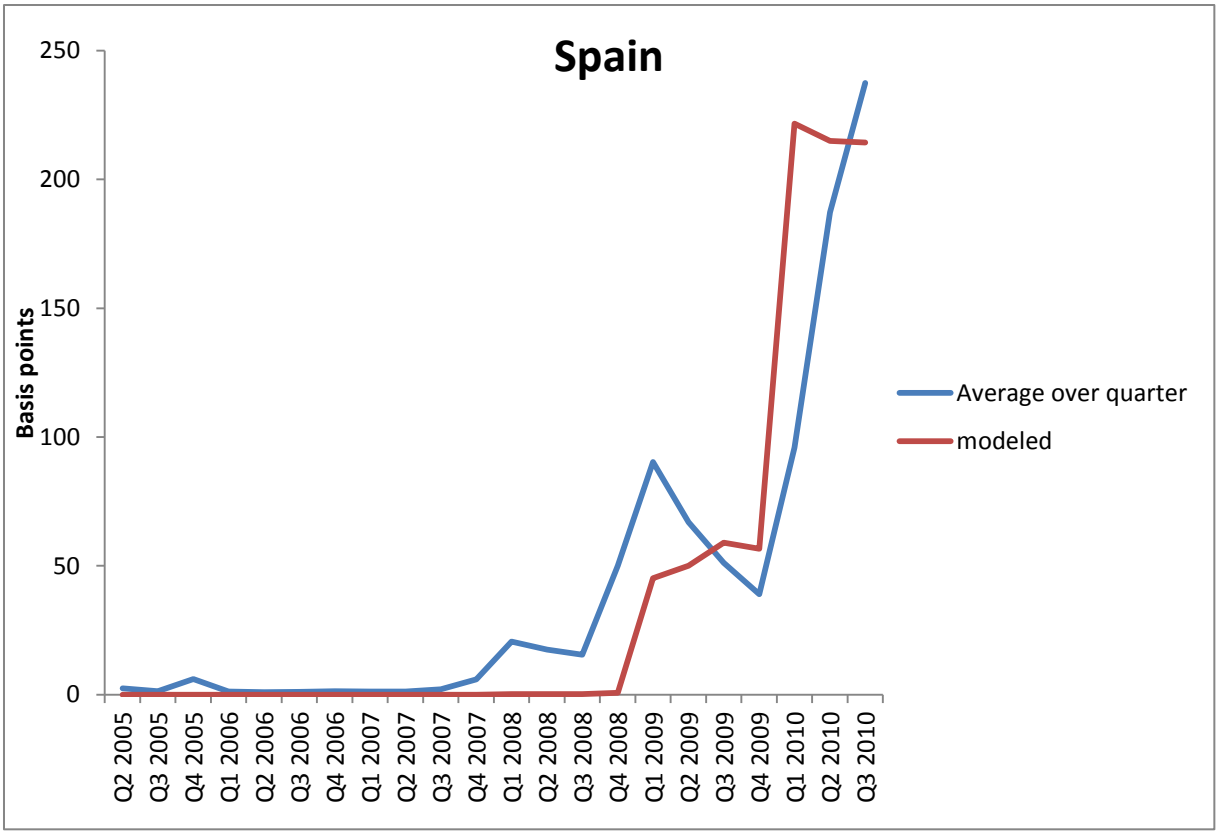

Figure 11 One-year CDS prices Spain (higher volatility)