Hedging Errors for Static Hedging Strategies

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Preface

This thesis completes the two-year Master of Science in Financial Economics program at NTNU. Writing the thesis has not only been a challenging, but also an exciting process. My main inspiration was to understand how theoretical results we learnt in class were applied in practice. In that way, the topic of the thesis perfectly corresponded with my interests. Writing the thesis, I have developed a much deeper understanding of derivatives and the underlying processes, probability theory and statistics. Moreover, I have used the chance to learn programming which has always been mysterious to me.

I would like to thank my supervisor Snorre Lindset for guidance and for always answering my questions. He gave me motivation and led me over a number of stumbling stones.

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1 Introduction

The derivatives market has evolved dramatically over the last decades. Sophisticated derivatives are taking place of plain vanilla options in popularity, becoming a solution that satisfies increasing expectations from investors. They help to achieve a complex payoff profile and address investors with different market expectations and investment strategies. Barrier options are probably the most popular type of these exotic options which were designed to provide the insurance component of an ordinary option without charging a high premium. These options give payoffs identical to those of vanilla options provided that a certain barrier was or was not reached. This feature implies that the option costs less since the payout is narrowed by some restrictions. This characteristic has proved to be attractive for those market participants who have a certain expectation over the asset's future. For example, an investor might believe that the underlying asset will rise, but never above a certain barrier. In that case, it might be reasonable to buy an up-and-out call option which gives a desired payoff (like an ordinary call) until the underlying is under the barrier. As soon as the barrier is breached, no matter what follows, the option becomes worthless. Since the investor believes that the chance of reaching the barrier is low, the option might be of great interest for him because it is less expensive.

Barrier options can have a wide range of underlying assets, from traditional asset classes as currencies, equities or indices to very specific as oil, gold or other commodities. Times to maturity also differ, from a few months, usually for high liquidity assets, to one year or more. Actually, any type of a barrier option with any underlying or time to maturity can be designed if there is a demand for it.

These options are usually traded over-the-counter, few of them are listed on exchanges. This is due to that specifications of the options as underlying, volume, strike price, barrier level and maturity date are very diverse, depending on a specific need of a customer. In addition to being traded separately, barrier options are often embedded in so called structured products, i.e. pre-packaged investment strategies which have become extremely popular in recent time. Financial companies offer a variety of such products where financial engineers work on how to combine different securities into one financial instrument to obtain a specific payoff profile. Many of these products include knock-out components where barrier options become useful. An example of such a product can be, according to Swiss Structured Products

Association (2011), "Capital protection with knock-out". [1] Capital protection is one of the most conservative strategies aiming to preserve the initial investment. However, in exchange for bearing a moderate risk on their initial capital, investors can have an opportunity to receive enhanced returns. Here, an up-and-out call option can be used to create such a payoff that an investor participates in a positive performance of the underlying until knock-out level.

The remarkable growth of structured products sector is probably one of the reasons why barrier options have experienced an increasing demand. Therefore it becomes crucial to know how to price and hedge these options properly. One of the simplest solutions for hedging barrier options was developed by Bowie and Carr (1994). They derive a relationship between scaled calls and puts with different strikes, called *put-call symmetry*, and show how this result can be used for hedging the options using a *static hedge*. Under static hedging one does not need to rebalance the hedging portfolio continuously, as it is for, for example, well-known delta-hedging. The portfolio is bought and simply kept unchanged until the barrier is reached. The benefit is obvious: a hedger can save large transaction costs.

However, one of the assumptions of put-call symmetry result, and, hence, static hedging is that the underlying has zero drift, which is not always the case in real life. The assumption is innocuous for options written on the forward or futures prices of an underlying, but for options written on the *spot* price, it implies zero carrying costs. It means that, for stocks or stock indices, the dividend yield must equal risk-free interest rate. For currencies, the foreign interest rate must equal the domestic rate, and for commodities, the convenience rate must equal the risk-free rate. Since these conditions are rarely fulfilled in practice, it is interesting to investigate what *hedging error* occurs when cost of carry is not zero. This is the aim of our work: we examine the hedging error which results from relaxing the assumption of zero drift of the underlying. As an example, we take a European down-and-out call written on a stock which pays no dividends, i.e. cost of carry is equal to risk-free interest rate. The underlying economic model for the discussion is a standard Black and Scholes model where a stock price is assumed to follow a geometric Brownian motion under a risk-neutral measure. To estimate the error, we use Monte Carlo simulation. In such a way, we model a stock price path and determine the error in case the barrier is reached. The error is stochastic, so it is not only the average value, but also its probability distribution which is a topic of interest.

[1] The strategy is offered by a number of financial firms, but we do not refer to any particular company because of information access legal restrictions.

The rest of the discussion is organized as follows. Chapter 2 specifies the underlying economic model, including application to forward contracts. Chapter 3 introduces barrier options in detail and discusses static hedging of barrier options vs. dynamic hedging. Also, a brief literature review on static hedging strategies is given. In addition, put-call symmetry result and the way it is used for static hedging are presented. In Chapter 4 we focus on relaxing the assumption of zero drift of the underlying, and develop equations for the error. In Chapter 5 we present the way we applied Monte Carlo simulation for the problem, and discuss our numerical results. The results are summarized in Conclusions and Appendix contains a self-written code which was used for the simulation.

2 Economic model and preliminaries

2.1 Geometric Brownian motion

To begin with we give an outline of the economic model we are operating in. In the study of derivatives, a price of a non-dividend-paying stock and other asset prices are commonly assumed to follow a *stochastic process* called *geometric Brownian motion*. To understand an underlying theory and to keep a logical order, we proceed step by step from a definition of a stochastic process to geometric Brownian motion by defining different processes on the way. The following is based on Hull (2009) and McDonald (2006).

A stochastic process is a process where a variable's value changes over time in an uncertain way, and this random process is a function of time. Indeed, it is quite intuitive to assume that stock prices wander through time randomly, but they follows some "pattern" in their randomness.

A *Markov process* is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way how the present has emerged from the past are irrelevant. Predictions for the future are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

A *Wiener process* is a particular type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per unit of time, for instance per year. In that way, a

Wiener process describes the evolution of a normally distributed variable. It means that, if the value of the variable z at time 0 is z_0, then its expected value at any time T is equal to its current value, and z_T is normally distributed with a mean z_0 and a variance 1.0×T = T (and a standard deviation $\sqrt{T}$), since it depends on time. The change Δz during a small period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t}, \tag{1}$$

where ϵ has a standardized normal distribution $\mathcal{N}(0,1)$. A Wiener process is usually assumed to be continuous-time and continuous-variable. When we say continuous-time, we mean that changes in the underlying variable's value can take place at any time. By continuous-variable we mean that the variable can take any values within a certain range. A Wiener process is often referred to as *Brownian motion*.

To take into account a possible non-zero *drift rate* of an underlying and a *volatility rate* different from 1.0, we come to a *generalized Wiener process* that is defined as

$$dx = adt + bdz, \tag{2}$$

where a and b are constants. a is an expected drift rate of x per unit of time and the bdz term can be regarded as adding noise or variability to the path followed by x. The amount of this noise is b times a Wiener process.

Talking about non-dividend-paying stocks, it can be written as

$$dS = \mu dt + \sigma dz, \tag{3}$$

where S is a stock price, μ is a rate of return on the stock and σ is a volatility of the stock (standard deviation). This process is also called *arithmetic Brownian motion*.

It is quite intuitive to suppose that things are a bit more complicated than the previous equation suggests, and that the drift rate and the volatility rate are proportional to the value of the underlying. When we say that a stock's expected rate of return is 0.15, we know that it does not depend on the initial stock price. Either the initial price is \$10 or \$100, we do not get the expected return of \$0.15 on it, but rather 0.15 as 15% on the initial price. So, if we allow parameters μ and σ to depend on the price of the stock S (and, consequently, on time t) and be proportional to it, we come to a process called geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dz \tag{4}$$

or

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz,$$

where, again, S is the stock price, μ is the expected rate of return on the stock, σ is the volatility of the stock return, and dz is the increment of a Wiener process. Equation (4) is the most widely used model of stock price behavior, it underlies the majority of technical option pricing discussions, including the original paper of Black and Scholes (1973).

2.2 Lognormality

An important implication of this equation is lognormality of a distribution of the stock price. To understand why it is so and to see what it gives us, we need a result called *Itô's lemma*. As we know, the price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the underlying and time. If the underlying stock follows a geometric Brownian motion as in (4) and a derivative written on this stock is a function G of stock price S and time t, Itô's lemma shows us that G follows the process

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S dz. \tag{5}$$

We see that G has the same source of uncertainty, dz, as the stock itself.

Itô's lemma should work for all the derivatives, so let us find a process followed by a continuously compounded return on the stock, $\ln S$, as a derivative of S

$$G = \ln S. \tag{6}$$

By plugging

$$\frac{\partial G}{\partial t} = 0, \qquad \frac{\partial G}{\partial S} = \frac{1}{S} \quad \text{and} \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \tag{7}$$

into Itô's lemma, we receive that the process followed by G is

$$dG = \Big(\mu - \frac{1}{2}\sigma^2\Big)dt + \sigma dz. \tag{8}$$

Since μ and σ are constants, the equation indicates that $G = \ln S$ follows a generalized Wiener process. It has a constant drift rate $\mu - \frac{1}{2}\sigma^2$ and a constant variance rate σ^2.

So, as any Wiener process, the change in the stock price has a normal distribution

$$\ln S_T - \ln S_0 \sim \mathcal{N}\Bigg((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T\Bigg)$$

or

$$\ln S_T \sim \mathcal{N}\Bigg(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T\Bigg), \tag{9}$$

from which we conclude that S is lognormally distributed. Lognormality is an important result since it appears to be a convenient model to build any further valuation on. First of all, a lognormally distributed variable can not take negative values, in contrast to a normally distributed variable that can. Second, the distribution is skewed to the right which means that, theoretically, the stock price can take any positive values, while the probability of a moderate return is much higher than the probability of a very high return. Both observations correspond nicely with what is observed in practice.

The change in $\ln S$ between time 0 and T has the following integral representation

$$\ln S_T - \ln S_0 = \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dz. \tag{10}$$

Since μ and σ are constants we have that

$$\ln S_T = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma z, \tag{11}$$

from which we get

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma z}. \tag{12}$$

This is an important result which allows us to simulate a behavior of the stock price. When dealing with barrier options, it is not only the terminal underlying stock price but also its path which matters. So, by dividing T into many small steps and simulating sequentially the price on these steps, we can observe if the price has crossed the barrier or not. There are two parameters which need to be estimated: volatility σ and expected rate of return μ, which might be somewhat difficult. As for volatility, market participants usually use either historical volatility or implied volatility from similar securities, but when it comes to the expected return, there is a way to avoid estimating it at all. This is to assume a *risk-neutral world*, and "substitute" the expected rate of return μ by risk-free rate r (it will be shown how in chapter 2.4).

2.3 Application to Forward contracts

Now let us consider a forward contract as, again, a derivative of the stock. We know (for example, from Hull, 2009) that the forward price of the stock at time t, for delivery at time T, is

$$F_t = S_t e^{r(T-t)}, \tag{13}$$

where F_t is the forward price at time t, S_t is the underlying stock's price at time t (spot price), r is the constant risk-free rate between time t and T, and T is the time to maturity of the forward contract. Since the derivative's price is, as before, a function of the underlying's price and time, we can use Itô's lemma again. The partial derivatives are

$$\frac{\partial F}{\partial t} = -rSe^{r(T-t)}, \qquad \frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0. \tag{14}$$

Plugging them into (5) and then substituting F for $Se^{r(T-t)}$ we get

$$dF = (\mu - r)F\,dt + \sigma F\,dz. \tag{15}$$

We see that the forward contract follows geometric Brownian motion as well as the underlying stock. Reasoning with normal distribution, we can, similarly to (12) for stocks, write the forward price as

$$F_T = F_0 e^{(\mu-r)T+\sigma z}. \tag{16}$$

This is a basis for simulating a forward price. We will use this result in the following discussion.

2.4 Risk-neutral valuation

To move any further we need to outline a risk-neutral world which is the basis for all the Black and Scholes valuation formulas and for our Monte Carlo simulation. As already mentioned, the main advantage of assuming risk-neutrality is that we do not need to estimate the expected rate of return, μ, on an asset anymore. The risk-neutral valuation principle states that a derivative can be valued by a) calculating the expected payoff of the underlying asset based on the assumption that it is equal to the risk-free interest rate and b) discounting the expected payoff at this risk-free interest rate (Hull, 2009). We will first define a parameter known as the *market price of risk* and then show how it contributes into our Monte Carlo simulation.

Suppose that we have an underlying asset, θ, which follows a geometric Brownian motion, and two derivatives on it, with values f_1 and f_2, which also follow a geometric Brownian motion

$$df_1 = \mu_1 f_1 dt + \sigma_1 f_1 dz \tag{17}$$

and

$$df_2 = \mu_2 f_2 dt + \sigma_2 f_2 dz, \tag{18}$$

where μ_1, μ_2, σ_1 and σ_2 are functions of the underlying θ and t (this and the following is taken from Hull, 2009). We construct a portfolio with value Π consisting of $\sigma_2 f_2$ number of the derivative f_1 and $-\sigma_1 f_1$ number of the derivative f_2. The portfolio has the following value

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2, \tag{19}$$

and the process followed by Π is

$$d\Pi = \sigma_2 f_2 df_1 - \sigma_1 f_1 df_2. \tag{20}$$

Plugging (17) and (18) into (20) we get that $d\Pi$ is a process without any uncertainty in it

$$d\Pi = (\mu_1\sigma_2 - \mu_2\sigma_1) f_1 f_2 dt, \tag{21}$$

because the dz term has dissapeared. It means that, with no arbitrage, the portfolio has to earn only risk-free rate on it and can be written as

$$d\Pi = r\Pi dt. \tag{22}$$

Substituting from (19) and (21) into this equation gives us the ratio

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}. \tag{23}$$

For no arbitrage, this ratio has to be fulfilled for *any* derivative f which has θ as an underlying. That is why we can drop the subscripts, and λ is defined as follows

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \frac{\mu - r}{\sigma} \equiv \lambda. \tag{24}$$

The parameter λ is known as the market price of risk of θ. It can be dependent on both θ and t, but it is not dependent on the nature of the derivative f. From that we also have that

$$\mu = \lambda\sigma + r. \tag{25}$$

The process followed by a derivative f is

$$df = \mu f dt + \sigma f dz. \tag{26}$$

The value of μ depends on the risk preferences of investors. In a world where investors are risk-neutral and the market price of risk is zero, λ is equal to zero. In that case, from (25) we get that $\mu = r$ and the process followed by f is then

$$df = r f dt + \sigma f dz. \tag{27}$$

This is referred to as the *traditional risk-neutral world.*

This entitles us to modify in a similar way the equations for the stock return (8) and the forward contract (15) from the previous discussions

$$dG = \Big(r - \frac{1}{2}\sigma^2\Big)dt + \sigma dz \tag{28}$$

and

$$dF = (r - r)F\,dt + \sigma F\,dz = \sigma F\,dz. \tag{29}$$

That is why a forward contract is also called a derivative with zero drift.

Analogously to (12) and (16), the following equations can be derived

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma z} \tag{30}$$

and

$$F_T = F_0 e^{\sigma z}. \tag{31}$$

These are very important results which we will use as a basis for our numerical simulations in Chapter 5 where we will simulate the behavior of the stock and the forward contract, and based on this investigate a static hedging strategy.

2.5 Other assumptions

Having developed the basic and the most important assumption that the stock follows geometric Brownian motion in the risk-neutral world, we present here a set of additional assumptions which led to the derivation of the initial Black and Scholes options formulas, and which underlies other results that we use in our work, as put-call symmetry. As in Dubofsky and Miller (2003), they are:

1. Capital markets are perfect. That is, there are no transaction costs or taxes, and there are no arbitrage opportunities. There are no short selling constraints, and investors get full use of the proceeds from short sales. All assets are infinitely divisible.

2. All investors can borrow and lend at the same risk-free interest rate, which is constant.

3. The stock pays no dividends.

4. Markets are always open, and trading is continuous.

In this chapter we have developed all the fundamental assumptions which are needed for the further discussion. Next we will consider in detail other theory we need: barrier options and static hedging.

3 Other theoretical background

3.1 Barrier options

Barrier options are the most popular form of exotic options or, as they are also called, second generation options. They have been increasingly traded in the last decades in many over-the-counter markets. This popularity is due to the additional flexibility that they give to their holders and due to their lower price compared to plain vanilla options.

A barrier option is an option with a payoff depending upon whether, over the life of the option, the price of the underlying asset reaches a certain barrier, i.e. a certain predetermined value. In that way, it is not only the terminal underlying's price, but also its path which is important for determining the payoff. That is why these options are also called path-dependent. This feature implies certain difficulties in their valuation and hedging.

Barrier options either come into existence (knock-in options) or go out of existence (knock-out options) the first time the asset price reaches the barrier. Down-and-out call (DOC) is one type of knock-out option where the underlying has to fall down in order that the option "knocks out". It means that the barrier lies under the initial asset price (spot price). If the barrier is hit before the time of expiration, the option becomes worthless and pays out nothing. If the barrier is not hit, the option gives exactly the same payoff as an ordinary call, i.e. $\max(S_T - K, 0)$, where T is expiration time, S_T is the underlying's price at time T and K is a strike price.

There are a lot of variations of barrier options: options with rebates (certain amounts which are paid if the options expire worthless), double barrier options when, for example, the knock-out option knocks out the moment the asset reaches either the upper or the lower barrier, and so on. As any other exotics, all these options that can be found, reflect some specific needs of the customer. In our work we will focus on an ordinary European barrier option with no rebates, specifically down-and-out call. A typical example of such an option can be a three-months DOC on the S&P 500 index with the strike at-the-money and the barrier set at 90% of this value.

If no rebates are paid, the following parity applies (see McDonald (2006), for instance)

$$\textit{in-option} + \textit{out-option} = \textit{ordinary option}. \tag{32}$$

Indeed, since barrier options never give a higher payoff then ordinary options, they can not be more expensive. This advantage is used a lot by those market participants who have a strong view on the asset's market perspectives, either it will go up or down.

An interesting thing about barrier options is how they can be hedged using a *static hedge*. This is the main topic of our work, but let us say first a few words about hedging in general.

3.2 Hedging

Hedging is a way of reducing risk. Talking about a derivative as an asset to be hedged, there are primarily financial institutions who are concerned about it. Market-makers sell different derivatives to their customers or buy those from them, and in order not to hold naked positions they usually hedge with other instruments which means they *replicate* the payoff with other assets to offset the risk.

The classical approach to hedge the derivatives involves maintaining an ever-changing position in the underlying asset and investing risk-free the rest. This is called *dynamic hedging* and the hedge position depends on the delta of the option which changes with any movement in the price of the underlying or with a change in other characteristics. It means that the hedge position itself needs to be rebalanced every time the asset price changes, i.e. in most cases continuously. Ideally, one has to maintain not only delta-, but also gamma- and vega-neutrality. There are three main difficulties with this hedging method. First, continuous trading is impossible, and traders adjust their position at discrete points in time. This causes small errors that compound over the life of the option, and the accuracy decreases. Generally, the higher the frequency of hedging, the higher the accuracy. Moreover, when it comes to barrier options, they often have regions with high gamma (when the underlying is close to the barrier) which can be catastrophic for the trader (Derman, Ergener and Kani, 1995). Second, there are transaction costs associated with adjusting the portfolio, which grow with frequency of adjustment and can overwhelm the profit margin of the option. These are not only spreads and commissions, but also the cost of paying individuals to monitor the position continuously. Consequently, traders have to compromise between accuracy and costs. In practice, when managing a large portfolio which depends on a single underlying asset with high liquidity, traders usually make delta zero, or close to zero, at least once a day. Other characteristics as gamma or vega are monitored, but are not usually managed on a daily basis. When it comes to small portfolios, maintaining delta neutrality by trading daily is usually not economically feasible (Hull, 2009). The third difficulty of dynamic hedging is that it requires estimation of dividends and volatility of the underlying asset which is another piece of work. The error arising from using the wrong volatility is directly proportional to the option's vega, which, again, is often high for barrier options

(Carr and Chou, 1997).

For some derivatives, mainly path-dependent, it turns out that it is possible to construct a hedge which does not involve continuous rebalancing. Such static hedges normally involve setting up a portfolio of simple, European options (typically calls and puts) that together guarantee to match the payoff of the hedged instrument (Andersen, Andreasen and Eliezer, 2000). Once the hedge has been initiated, it either does not need to be adjusted at all (so called "hedge-and-forget") or it needs to be adjusted occasionally (when the hedge "expires", i.e the underlying reaches the barrier) which usually occurs over large periods of time. In that way, static hedging saves the hedger both transaction costs and headaches. Answering the third mentioned drawback of dynamic hedging, static replication needs only the implied volatility of the ordinary options at the entry of the static hedge and when the underlying is at the barrier. The volatility realized during the life of the hedge is of no consequence, except to the extent that it affects implied volatility (Carr and Chou, 1997). However, some researches say that the static hedging is less robust to the misspecification of the model. The put-call symmetry, which we will consider next, suggests that among others, out-of-the-money options should be used for the hedging. Nalholm and Poulsen (2006) argue that since Black and Scholes formula's mispricing is large for those options, quoting from the paper, "the static hedges are relatively more model risk sensitive, meaning that performance is strongly affected by the extent to which theoretical model characteristics or market data (skew or smile) are taken into account in their implementation". Nevertheless, they agree that the net effect of possible model misspecification is still lower for static hedges.

Despite all these advantages, there are a few *practical* disadvantages of static hedging, among which we can mention that standard options' market is of less liquidity compared to the market for the underlying asset. In means that when a hedger seeks for plain vanilla options with a certain strike and maturity, he might not find them at competitive prices. Another error, but rather small, might appear when a number of options to be bought (according to the put-call symmetry) is not an integer number.

So, we see that while both dynamic and static hedging strategies work perfectly in theory, a static hedge offers some benefits over dynamic one in practice and *is*, indeed, much more used in practice. Static hedging is likely to be less sensitive to the assumption of zero transaction costs and to the model's specifications. The impact of relaxing these assumptions (imposing discrete trading opportunities, transaction costs and stochastic volatility) was examined in a paper by Tompkins (1997). Using simulation, he compares the Bowie and Carr static hedge of a down-and-out call (exactly that one which we consider in our work) with the standard dynamic hedge. He finds that transaction costs add 8.5% of the theoretical premium to the hedging cost in the dynamic hedge, while they add only 0.06% of this premium to the hedging cost in the static hedge. There are several similar analyses which also come to the conclusion that the static hedging is advantageous over the dynamic one (for the overview of the analyses, see Carr and Picron, 1999).

Two main approaches have been developed within static hedging itself. The one we will be focused on, is by Bowie and Carr (1994) who show how to create static portfolios of plain vanilla options with different strikes whose values match the payoffs of the path-dependent option at expiration and along the barrier. Another, so called calendar spreads static hedging, was developed by Derman, Ergener and Kani (1995) who instead of using ordinary options with different strikes, suggested using options with different maturity dates, but all with the same strikes. Following the first method, put-call symmetry is a result that helps us find the right number of standard options and their strike prices to hedge the barrier option.

3.3 Put-call symmetry

Put-call symmetry (PCS) is a relationship between scaled calls and puts which can have different strike prices. It can be viewed as both an extension and a restriction of the widely known put-call parity. The generalization involves allowing the strikes to differ in a certain manner. The restrictions sufficient to achieve this result are that the underlying price process has both zero risk-neutral drift and a symmetric and deterministic volatility function (Carr, Ellis and Gupta, 1998). The relationship has been derived by Bowie and Carr (1994) based on earlier work by Bates, and even better explained in Carr, Ellis and Gupta (1998).

European put-call symmetry: given frictionless market, no arbitrage, zero drift, and the symmetry volatility conditions, the following relationship between a call option C and a put option P holds

$$\frac{C(K_{call})}{\sqrt{K_{call}}} = \frac{P(K_{put})}{\sqrt{K_{put}}}, \tag{33}$$

where the geometric mean of the call strike K_{call} and the put strike K_{put} is the forward price

F

$$\sqrt{K_{call}K_{put}} = F. \tag{34}$$

For proof and more discussion, see Carr, Ellis and Gupta (1998).

The assumption of zero risk-neutral drift is not crucial for options written on the forward or the futures price of an underlying asset. So we will see first how PCS works for hedging the DOC which is written on a forward. Suppose we have a long position in such a down-and-out call, DOC(K, H). The option is issued at time 0 and expires at time T. Strike price is K, barrier is H and $H < K$. The option knocks out when the forward price reaches the barrier. In order to hedge it we need to match the terminal payoff *and* the payoff along the barrier. The first match is simply done with purchasing a standard call with the same strike price and the same maturity, $C(K)$ or $C(K_{call})$. We will mean K_{call} when we write K, and only where we have both calls and puts, we will distinguish between K_{call} and K_{put}. So, if the barrier is not hit, both calls are alive and give identical payoffs. However, when barrier is hit ($F = H$), the DOC becomes worthless while our current hedge has positive value (Carr, Ellis and Gupta, 1998). Here we might have an idea to have some instrument short that has exactly the same value to offset the long call when $F = H$. That instrument could be an ordinary put option. We need only to find how many puts we need and with which strike. Suppose the barrier is hit and $F = H$, then the equation (34) can be written as

$$\sqrt{K_{call}K_{put}} = H,$$

following that

$$K_{put} = \frac{H^2}{K_{call}}. \tag{35}$$

From (33) we have that

$$C(K_{call}) = \sqrt{\frac{K_{call}}{K_{put}}}P(K_{put}). \tag{36}$$

Setting here the expression for K_{put} (35) we get that

$$C(K_{call}) = \frac{K_{call}}{H}P\left(\frac{H^2}{K_{call}}\right),$$

or, with simpler notations,

$$C(K) = \frac{K}{H}P\left(\frac{H^2}{K}\right). \tag{37}$$

This ratio ensures that the puts have the same value as the vanilla call whenever the forward price is at the barrier. In that way, PCS suggests that we buy one plain vanilla call $C(K)$ and sell K/H number of puts struck at H^2/K in order to replicate the long position in DOC

$$DOC(K,H) = C(K) - \frac{K}{H} \cdot P\left(\frac{H^2}{K}\right). \tag{38}$$

Let us try different scenarios and see how well the hedge works.

- In case the barrier is not hit at any time during the DOC's lifetime $F \neq H \;\; \forall t \in [0,T]$, the DOC has the payoff of $\max(F_T - K, 0)$, the ordinary call has the same payoff $\max(F_T - K, 0)$ and puts expire out-of-the-money since $H^2/K < H < F_T$.

- If the barrier is hit at some time t, the DOC knocks out and gives no payoff. In that case we need to close our hedge position immediately, i.e. to sell vanilla call and buy back puts. According to (37), PCS implies that they have exactly the same values, hence, it does not cost us anything.

As we see, PCS offers us a theoretically perfect hedge where the payoff from the DOC is replicated in either scenario.

4 Relaxing the assumption

PCS works perfectly under the model's assumptions, including the requirement of a zero-drift underlying asset. However, it is interesting to see what happens if we deal with a barrier option on a *stock* which does *not* fulfill this particular requirement. Other assumptions which underlie this hedging strategy are quite crucial, so we will only concentrate on relaxing this particular assumption about zero drift.

In the same paper where PCS and subsequent static hedging were introduced, Bowie and Carr (1994) make an attempt to relax the assumption of zero drift of the underlying. Although it is no longer possible to obtain exact static hedge, they calculate tight bounds on option values. Another approach, already mentioned before, was developed by Derman, Ergener and Kani (1994) who introduce an algorithm for hedging barrier options in a binomial model using calendar spreads where non-zero drift is also possible. The method suggests that, generally, infinite number of vanilla options should be bought to get a perfect hedge, and accuracy decreases as number of options becomes limited (Derman, Ergener and Kani, 1995). Our work, in contrast, aims to determine if an error, associated with hedging a barrier option written on stock, is substantially large. We use as an example a down-and-out call, which was earlier explained in detail.

Suppose we have a long position in a down-and-out call, DOC(K, H), written on a stock with the initial price S_0. The DOC knocks out provided the *stock* price reaches the barrier H. The option is issued at time 0 and expires at time T. Strike price is K, barrier is H and $H < K$. We suppose that we use PCS exactly the same way as for forward contracts: we sell one standard call with the same strike and buy K/H puts with the strike H^2/K, all of them written on the stock and with the same maturity as the DOC. Let us again consider different scenarios and see how well the hedge works:

- In case the barrier is not hit at any time during the DOC's lifetime $S \neq H \;\; \forall t \, \in [0, T]$, the DOC has the payoff of $\max(S_T - K, 0)$, the ordinary call has the same payoff $\max(S_T - K, 0)$ and puts expire out-of-the-money since $H^2/K < H < S_T$.
- If the barrier is hit at some time t, the DOC knocks out and gives no payoff. In that case, as before, we need to close our hedge position immediately, but the question is if we can close it with no out-of-the-pocket expenses. PCS can not guarantee us this time that the call and the puts have the same value. A similar problem might appear when we *enter* into the hedge, i.e. if the sum of the initial costs for the DOC and the hedge portfolio is equal to zero as it should be.

So we see that we have no problems with replicating the payoffs, but costs of buying and selling the hedge portfolio is a topic to be discussed. Let us call them *hedging errors* and try to find them analytically.

4.1 Hedging errors

As we discussed, to *replicate* the long position in DOC we buy the vanilla call and sell the number of puts. However, to *hedge* it meaning insurance as, for example, seen from market-maker's point of view, we need to *sell* the replicating portfolio, i.e. to do it the opposite way: to sell the vanilla call and to buy the number of puts. It is easy to think in terms of money we spend on this:

1. Time 0: we buy the DOC, sell the vanilla call and buy the puts. We define an *initial error* by what we *spend*

$$Initial\,error = -DOC(K,H) + C(K) - \frac{K}{H} \cdot P\left(\frac{H^2}{K}\right). \tag{39}$$

2. Time t when the barrier is hit in case it *is* hit: we sell the puts and buy back the call. We define an *ending error* again by what we spend

$$Ending\,error = \frac{K}{H} \cdot P\left(\frac{H^2}{K}\right) - C(K). \tag{40}$$

   Note that the ending error occurs only if the barrier is reached. In other case it is zero since we do not need to sell the hedge portfolio.

With a perfect hedge both of the errors must be equal to zero, or, as in case with dynamic delta-hedging, the proceeds from investing risk-free the first error must offset the second error. The point is that it is not certain that the second scenario will take place and the ending error will occur. Let us begin with showing that both errors are equal to zero in the original case with the forward price as underlying. We need to have all the formulas and simply plug them into the equations (39) and (40).

When $H < K < F_0$, the value of a down-and-in call, DIC, written on forward at time zero is (Haug, 2007), using our notations

$$DIC(K,H) = e^{-rT}\Big(HN(y) - \frac{F_0 K}{H} N(y - \sigma\sqrt{T})\Big), \tag{41}$$

where

$$y = \frac{\ln \frac{H^2}{F_0 K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

We can use the in-out parity (32), and calculate the price of the down-and-out call, DOC, which we are interested in, as

$$DOC(K,H) = C(K) - DIC(K,H). \tag{42}$$

A Black and Scholes formula of an ordinary call written on forward is (Hull, 2007)

$$C(K_{call}) = e^{-rT}\Big(F_0 N(d_{1call}) - K_{call} N(d_{2call})\Big), \tag{43}$$

where

$$d_{1call} = \frac{\ln \frac{F_0}{K_{call}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_{2call} = d_{1call} - \sigma\sqrt{T}.$$

A put written on forward has the price

$$P(K_{put}) = e^{-rT}\Big(K_{put} N(-d_{2put}) - F_0 N(-d_{1put})\Big), \tag{44}$$

where

$$d_{1put} = \frac{\ln \frac{F_0}{K_{put}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_{2put} = d_{1put} - \sigma\sqrt{T}.$$

Using in-out parity (42) and setting the formulas (41) and (44) into the equation for the initial error (39), we get that the error is equal to zero

$$\begin{aligned}
Initial\,error &= -DOC(K,H) + C(K) - \frac{K}{H}\cdot P\left(\frac{H^2}{K}\right) \\
&= DIC(K,H) - \frac{K}{H}\cdot P\left(\frac{H^2}{K}\right) \\
&= e^{-rT}\left(HN(y) - \frac{F_0 K}{H}N(y-\sigma\sqrt{T})\right) - \frac{K}{H}e^{-rT}\Big(K_{put}N(-d_{2put}) - F_0 N(-d_{1put})\Big) \\
&= e^{-rT}\left(HN(y) - \frac{F_0 K}{H}N(y-\sigma\sqrt{T})\right) - \frac{K}{H}e^{-rT}\left(\frac{H^2}{K}N(-d_{2put}) - F_0 N(-d_{1put})\right) \\
&= e^{-rT}\left(HN(y) - \frac{F_0 K}{H}N(y-\sigma\sqrt{T}) - \left(HN(-d_{2put}) - \frac{F_0 K}{H}N(-d_{1put})\right)\right) \\
&= e^{-rT}\left(H\Big(N(y) - N(-d_{2put})\Big) + \frac{F_0 K}{H}\Big(N(-d_{1put}) - N(y-\sigma\sqrt{T})\Big)\right) \\
&= 0,
\end{aligned} \tag{45}$$

since

$$d_{1put} = \frac{\ln \frac{F_0}{K_{put}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln \frac{F_0 K}{H^2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\left(\frac{\ln \frac{H^2}{F_0 K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) = -(y - \sigma\sqrt{T})$$

and

$$d_{2put} = d_{1put} - \sigma\sqrt{T} = -(y - \sigma\sqrt{T}) - \sigma\sqrt{T} = -y.$$

Analogously, we find an ending error in case it occurs. Suppose that the forward price hits the barrier at time t when the forward price is $F_t = H$. We plug formulas for call (43) and put (44) into the equation for the ending error (40)

$$\begin{aligned}
Ending\,error &= \frac{K}{H} \cdot P\left(\frac{H^2}{K}\right) - C(K) \\
&= \frac{K}{H}\left(e^{-r(T-t)}\left(K_{put}N(-d_{2put}) - F_t N(-d_{1put})\right)\right) \\
&- e^{-r(T-t)}\left(F_t N(d_{1call}) - K_{call}N(d_{2call})\right) \\
&= e^{-r(T-t)}\left(HN(-d_{2put}) - KN(-d_{1put}) - HN(d_{1call}) + KN(d_{2call})\right) \\
&= 0, \qquad (46)
\end{aligned}$$

since, again

$$d_{1put} = \frac{\ln \frac{F_t}{K_{put}} + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} = \frac{\ln \frac{K}{H} + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} = -\left(\frac{\ln \frac{H}{K} - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}\right) = -d_{2call}$$

and

$$d_{2put} = d_{1put} - \sigma\sqrt{T-t} = -d_{2call} - \sigma\sqrt{T-t} = -(d_{1call} - \sigma\sqrt{T-t}) - \sigma\sqrt{T-t} = -d_{1call}.$$

So we see that both hedging errors are equal to zero. However, in real life, when the barrier is being hit, it is not necessarily crossed over at exactly H-level since we do not observe continuous trading. The underlying price moves in small jumps and the first price we observe under the barrier might be slightly or significantly lower than the barrier. In that case we will get some ending error anyway, but it must not be large, unless we have an underlying with low liquidity where the price moves in big jumps.

4.2 Errors in case of a stock as an underlying

In case of a DOC written on a stock, we generally do *not* observe zero errors. Let us first derive equations for the errors. After that we will consider them numerically.

As already said, we suppose that we use PCS exactly the same way as we did for DOC on forward, i.e. we sell one standard call with the same strike and buy K/H puts with the strike H^2/K. We will need the same formulas, but for options written on stock. In Haug (2007):

$$DIC(K,H) = S_0 \left(\frac{H}{S_0}\right)^{2(\nu+1)} N(y) - Ke^{-rT} \left(\frac{H}{S_0}\right)^{2\nu} N(y - \sigma\sqrt{T}), \tag{47}$$

where

$$\nu = \frac{r - \sigma^2/2}{\sigma^2}$$

and

$$y = \frac{\ln \frac{H^2}{S_0 K}}{\sigma\sqrt{T}} + (1+\nu)\sigma\sqrt{T} = \frac{\ln \frac{H^2}{S_0 K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Black and Scholes formula for call is

$$C(K_{call}) = S_0 N(d_{1call}) - K_{call} e^{-rT} N(d_{2call}), \tag{48}$$

where

$$d_{1call} = \frac{\ln \frac{S_0}{K_{call}} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_{2call} = d_{1call} - \sigma\sqrt{T},$$

while for put is

$$P(K_{put}) = K_{put} e^{-rT} N(-d_{2put}) - S_0 N(-d_{1put}), \tag{49}$$

where

$$d_{1put} = \frac{\ln \frac{S_0}{K_{put}} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_{2put} = d_{1put} - \sigma\sqrt{T}.$$

Using (47) and (49) we show now that the initial error is not zero

$$\begin{aligned}
Initial\,error &= -DOC(K,H)+C(K)-\frac{K}{H}\cdot P\left(\frac{H^2}{K}\right)\\
&= DIC(K,H)-\frac{K}{H}\cdot P\left(\frac{H^2}{K}\right)\\
&= S_0\left(\frac{H}{S_0}\right)^{2(\nu+1)}N(y)-Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\nu}N(y-\sigma\sqrt{T})\\
&- \frac{K}{H}\Big(K_{put}e^{-rT}N(-d_{2put})-S_0N(-d_{1put})\Big)\\
&= S_0\left(\frac{H}{S_0}\right)^{2(\nu+1)}N(y)-Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\nu}N(y-\sigma\sqrt{T})\\
&- \frac{K}{H}\Big(\frac{H^2}{K}e^{-rT}N(-d_{2put})-S_0N(-d_{1put})\Big)\\
&= S_0\left(\frac{H}{S_0}\right)^{2(\nu+1)}N(y)-Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\nu}N(y-\sigma\sqrt{T})\\
&- He^{-rT}N(-d_{2put})+\frac{S_0K}{H}N(-d_{1put}), \qquad (50)
\end{aligned}$$

where

$$y=\frac{\ln\frac{H^2}{S_0K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$y-\sigma\sqrt{T}=\frac{\ln\frac{H^2}{S_0K}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_{1put}=\frac{\ln\frac{S_0}{K_{put}}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}=\frac{\ln\frac{S_0K}{H^2}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}=-\Big(\frac{\ln\frac{H^2}{S_0K}-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big),$$

and

$$d_{2put}=d_{1put}-\sigma\sqrt{T}=\frac{\ln\frac{S_0K}{H^2}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}=-\Big(\frac{\ln\frac{H^2}{S_0K}-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big).$$

This expression can not be simplified more since, in general, d_{1put} is neither equal to $-y$ nor to $-(y-\sigma\sqrt{T})$. The same is with d_{2put}. This is because of r in formulas for stock which does not allow parts of the equation to cancel out the others.

Let us see what happens with an ending error. Suppose again that the barrier is being crossed at time t when the stock price is S_t.

$$Ending\,error = \frac{K}{H}\cdot P\left(\frac{H^2}{K}\right)-C(K)$$

$$
\begin{aligned}
&= \frac{K}{H}\Big(K_{put}e^{-r(T-t)}N(-d_{2put}) - S_tN(-d_{1put})\Big) \\
&- \Big(S_tN(d_{1call}) - Ke^{-r(T-t)}N(d_{2call})\Big) \\
&= He^{-r(T-t)}N(-d_{2put}) - \frac{S_tK}{H}N(-d_{1put}) \\
&- S_tN(d_{1call}) + Ke^{-r(T-t)}N(d_{2call}), \qquad (51)
\end{aligned}
$$

where

$$d_{1call} = \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_{2call} = \frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_{1put} = \frac{\ln\frac{S_t}{K_{put}} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\frac{S_tK}{H^2} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -\Big(\frac{\ln\frac{H^2}{S_tK} - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big),$$

and

$$d_{2put} = \frac{\ln\frac{S_tK}{H^2} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -\Big(\frac{\ln\frac{H^2}{S_tK} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big).$$

If, ideally, the stock price hits the barrier at exactly H-level, then using $S_t = H$ we can simplify the previous equation

$$Ending\,error = He^{-r(T-t)}N(-d_{2put}) - KN(-d_{1put}) - HN(d_{1call}) + Ke^{-r(T-t)}N(d_{2call}), \quad (52)$$

where

$$d_{1call} = \frac{\ln\frac{H}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_{2call} = \frac{\ln\frac{H}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_{1put} = \frac{\ln\frac{S_t}{K_{put}} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\frac{K}{H} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -\Big(\frac{\ln\frac{H}{K} - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big),$$

and

$$d_{2put} = \frac{\ln\frac{K}{H} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -\Big(\frac{\ln\frac{H}{K} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big).$$

We see again that r in the expressions for d_1 and d_2 prevents both (51) and (52) from simplifying further: in general d_{1put} is neither equal to $-d_{1call}$ nor to $-d_{2call}$. Neither is d_{2put}.

If the barrier is reached, an investor has a *total error*, discounted to time 0, of

$$Total\,error = initial\,error + e^{-rt} ending\,error. \tag{53}$$

Generally, the expected total error is

$$Expected\ total\,error = initial\,error + E_Q[e^{-rt} ending\,error_t]. \tag{54}$$

The initial error is deterministic while the ending error is stochastic since it depends on time t when the barrier is hit and the stock price at this time which might be slightly lower than the barrier.

5 Numerical investigations

We go now further and try to see numerically how good our hedge works and if the hedging error is large. We set parameters as follows: initial stock price $S_0 = 100$, strike $K = 90$, barrier $H = 80$, risk-free rate $r = 0.06$, volatily $\sigma = 0.3$ and time to expiration $T = 0.25$. So, the strike price is set at 90% of the spot price and the barrier - at 80% of it. The stock price, the strike and the barrier are chosen arbitrarily while the parameters r and σ appear to be market averages. Time to expiration $T = 0.25 = 3$ months is chosen as probably the most popular for this kind of options on a liquid underlying. However, options with longer time to maturity are also traded, so we will consider those as well. We split up the total period of length T into $n = 10,000$ small intervals. One interval, h , has a length of $h = T/n$. By that we approach the assumption of continuous trading quite close, but maybe not closer than the real life. To achieve consistency in the results and to be able to compare those for different values of T we use a constant h rather than a constant n. This means that for $T = 0.5$ we take $n = 20,000$ and for $T = 1$ we take $n = 40,000$.

In Chapter 2.4 we derived the equations (30) and (31) which we will extensively use now. Let us begin with a stock. As in (30), the stock price S can be modeled as

$$S_{h+1} = S_h e^{(r-0.5\sigma^2)h+\sigma z}$$

or

$$S_{h+1} = S_h e^{(r-0.5\sigma^2)h+\sigma\sqrt{h}\epsilon} \tag{55}$$

since z is a Wiener process and, according to (1), is equal to ϵ times the square root of time, where $\epsilon \sim \mathcal{N}(0,1)$. By beginning with S_0 and making n steps, we get a stock path between time 0 and time T, which allows us to see if the barrier is crossed or not.

$$S_1 = S_0 e^{(r-0.5\sigma^2)h+\sigma\sqrt{h}\epsilon},$$

$$S_2 = S_1 e^{(r-0.5\sigma^2)h+\sigma\sqrt{h}\epsilon},$$

and so on until S_n

$$S_n = S_{n-1} e^{(r-0.5\sigma^2)h+\sigma\sqrt{h}\epsilon}. \tag{56}$$

The last simulated price, $S_{10,000}$, will be the stock price at expiration which is used for calculating the payoff. In each simulation we check if the stock price is equal or less than the barrier (for the first time in this particular stock path) and if it is, we use (40), actual time t when the barrier is passed and actual stock price S_t at this time to calculate the ending error. After that we discount it to time 0 and add the initial error which is constant. By that we get the total error. In those stock paths where the barrier was not reached, we leave total error = initial error.

To have a good estimate of the errors we need to have quite many simulations of the stock path. We carry out $m = 1,000,000$ simulations where in each of them we model a new stock path. In this way we are able to

1. estimate the probability that the barrier is reached,
2. estimate the ending error distribution, with its mean, median and variance,
3. estimate the total error distribution, with its mean, median and variance,
4. estimate distribution of the time when the barrier is crossed, and its mean and median,
5. estimate distribution of the stock price when the barrier is crossed,
6. draw histograms for all these frequency distributions, and
7. draw a histogram of a joint frequency distribution of the time when the barrier is crossed and the stock price at that time.

We did it the following way:

1. To estimate the probability of reaching the barrier we simply count how many times, p, out of m simulations the barrier was reached. The probability is then $P = p/m$.

2. - 5. As we carry out all n×m simulations we save the resulting ending errors, discounted ending errors, total errors, time and stock prices in a matrix and then we find maximum, minimum, average values and medians, and the variance.

6. To see how these are distributed we split up the range for each of them (from minimum value to maximum value) into a small intervals and see how many times the values hit those intervals. We use $a = 100$ for t and S_t distributions and $a = 300$ for the errors distributions. We do not bring them to estimated *probability* distributions (although we could since we have quite many observations) since we mostly analyze them graphically, and in that case the *shape* of the frequency distribution curve is similar to that of the probability distribution.

7. The joint frequency distribution is determined by counting how many times time t values hit their intervals *provided* that the stock price at that time is in a particular interval, and so on for all intervals of the stock price (to make it more descriptive, we used fewer intervals here: $a = 40$ for $T = 1$ and $a = 30$ for $T = 0.25$).

In addition, since we have a closed form solution for the initial error, we can examine how it depends on the parameters as time to maturity, risk-free rate and volatility.

The code we used for the stock price simulations, had first been tested on forward contracts, using equation (31) and the formulas for options written on a forward. After we obtained that all the errors are equal to zero, we changed the formulas and applied the code for simulating the stock price.

5.1 Results

First, in Table 1, we present the numerical results for three cases: $T = 0.25, 0.5$ and 1 since we have observed that, among all the parameters, the errors are most sensitive to the change in T. Although many of such barrier options are issued for not more than 3 months, we use larger T to show how dramatically the errors grow as time to maturity grows. Let us analyze the table step by step beginning with the initial error.

5.1.1 Initial error

From the table we see that the initial error grows significantly with time to maturity. To conclude that, we could have taken a derivative of the initial error (50) with respect to T,

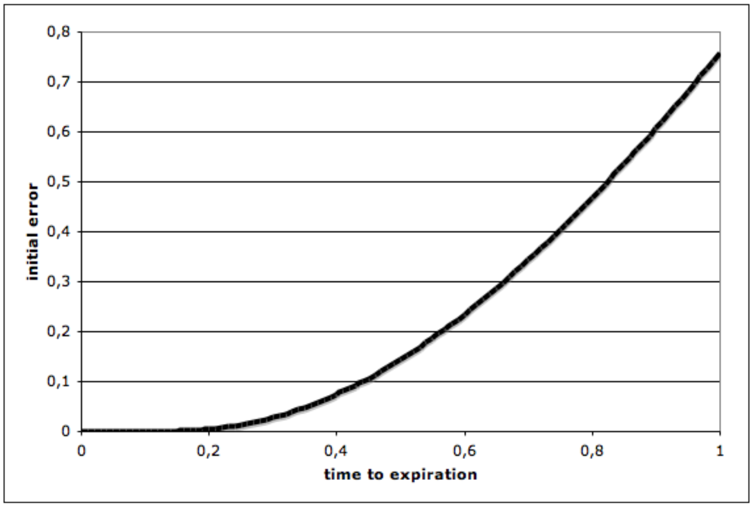

Figure 1: The initial error dependency on the DOC's maturity time T, holding risk-free interest rate $r = 0.06$ and volatility rate $\sigma = 0.3$.

but the result we get is difficult to interpret

$$\begin{aligned}\frac{\partial\, initial\, error}{\partial T} &= S_0 \frac{\sigma}{2\sqrt{T}} \left(\left(\frac{H}{S_0}\right)^{2(\nu+1)} n(y) - \frac{K}{H} n(d_{1put}) \right) \\ &+ rK \left(\frac{H}{S_0}\right)^{2\nu} e^{-rT} N(y - \sigma\sqrt{T}) - rHe^{-rT} N(d_{2put}). \end{aligned} \tag{57}$$

The equation is so complicated that it is not clear if the derivative is positive or negative. However, when considering numerically, we can plot a graph of this dependency holding risk-free rate and volatility rate fixed, see Figure 1. We see that the error grows approximately exponentially with huge growth after $T = 0.2$. Until $T = 0.2$ it is approximately equal to zero (for $T = 0.15$ the error is only 0.001 and for $T = 0.2$ it is 0.005). When T approaches 1 year, the initial error becomes 150 times larger than for $T = 0.2$.

We can also examine how the initial error depends on the risk-free rate. We have plotted this dependency for $T = 0.25$ on Figure 2 and for $T = 1$ on Figure 3 taking the risk-free interest rate range from 0 to 25%. We see that, generally, the higher the interest rate, the higher the error, but the marginal effect decreases with higher interest rates. The speed of growing is approximately the same for $T = 0.25$ and for $T = 1$. A derivative of the error with respect to r is even more complicated than the previous one, so we do not give it here.

It is also interesting to see how the initial error depends on volatility rate, see Figure 4

Table 1: Numerical results for the initial, ending and total errors and other characteristics for $T = 0.25, 0.5$ and 1 holding $r = 0.06$ and $\sigma = 0.3$.

Parameters	for T=0.25	for T=0.5	for T=1
DOC price	12.9725	15.3273	18.3382
Initial error:			
Initial error	0.0136456	0.144044	0.758595
Initial error in % to DOC price	0.11%	0.94%	4.14%
Ending error			
Ending error distribution	[-0.456433, 0.0160046]	[-1.32657, 0.0165925]	[-3.30784, 0.018848]
Average ending error	-0.090961	-0.48675	-1.7076
Ending error median	-0.059620	-0.47174	-1.8533
Average discounted ending error including the cases where the barrier is not reached	-0.01177	-0.13485	-0.73492
Total error:			
Total error distribution	[-0.442074, 0.0294522]	[-1.17992, 0.160192]	[-2.5435, 0.7764]
Max total error in % to DOC price	3.4%	7.7%	13.87 %
Average total error	0.0018756	0.0091921	0.023675
Average total error in % to DOC price	0.01%	0.06 %	0.13%
Variance of total error	0.0020809	0.080217	1.0518
Time t:			
Time t distribution	[0.02465, 0.25]	[0.025, 0.5]	[0.02355, 1]
Time t average	0.16426	0.27163	0.43304
Time t median	0.16698	0.26298	0.38391
Time t "mode", appr.	N/A	0.2	0.2
Others:			
Stock price at time t distribution	[79.5228, 80]	[79.4795, 80]	[79,4751, 80]
Barrier is reached in	13.03% cases	28.02% cases	43.82% cases

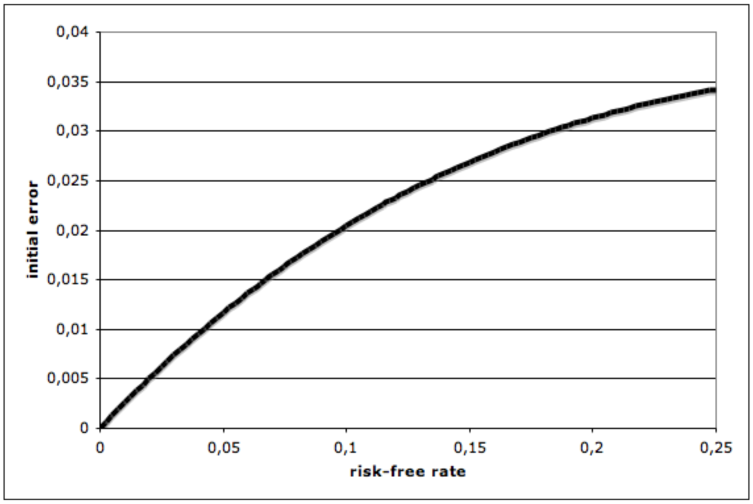

Figure 2: The initial error dependency on the interest rate r, holding time to maturity $T = 0.25$ and volatility rate $\sigma = 0.3$.

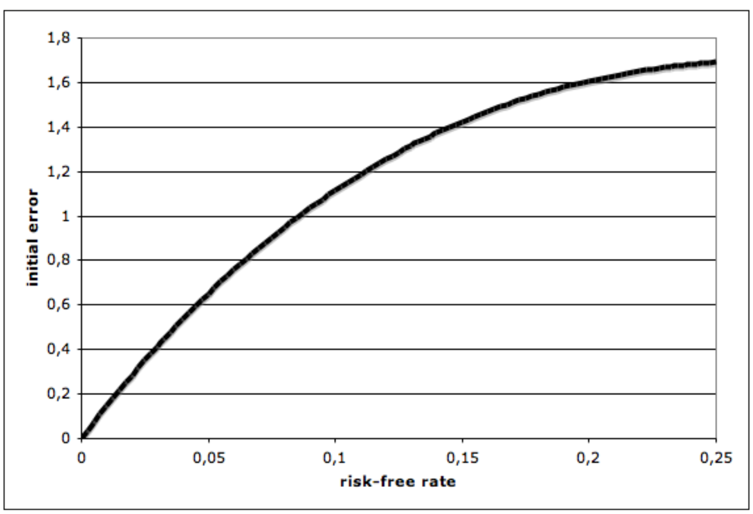

Figure 3: The initial error dependency on the interest rate r, holding time to maturity $T = 1$ and volatility rate $\sigma = 0.3$.

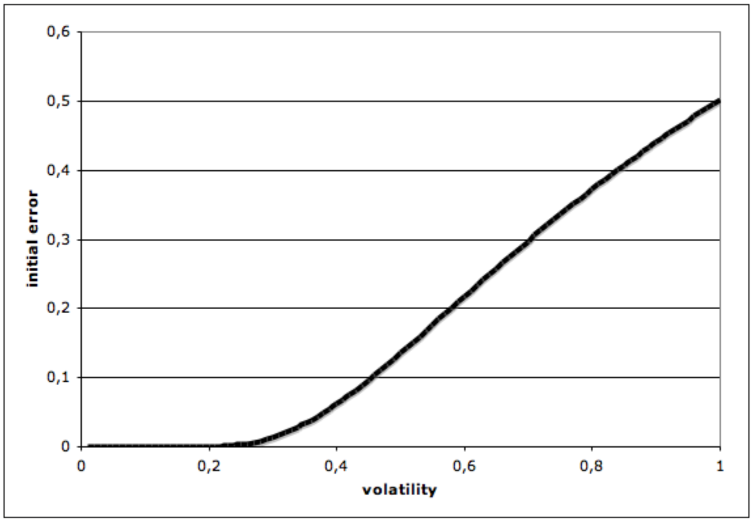

Figure 4: The initial error dependency on the volatility rate σ, holding time to maturity $T = 0.25$ and interest rate $r = 0.06$.

for $T = 0.25$ and Figure 5 for $T = 1$. For the graphs, we took the volatility rate range from 0.01 to 1. This dependency appears to be not that straightforward. Both graphs rise as the volatility rate goes up but the speed of this increase changes from positive to negative, i.e. the second derivative is apparently first positive and then negative after some inflection point. The derivative expression is again too difficult to interpret since, if we look at the equation for the initial error (50), it has not only y, $y - \sigma\sqrt{T}$, d_{1put} and d_{2put}, but also ν which all depend on σ. We also see that the speed of growing is higher for $T = 0.25$ than for $T = 1$. An interesting thing is that up to some level, the volatility almost does not have any effect on the error. We see from Figure 4 that up to about $\sigma = 0.25$ the error is approximately equal to zero (at $\sigma = 0.25$ the error is only 0.003). For $T = 1$ (see Figure 5) this is true for the volatility up to approximately $\sigma = 0.12$.

To conclude, we see that the initial error is growing with respect to all the parameters T, r and σ where T has the largest impact and σ the lowest. For maturities up to 3 months, the error is approximately equal to zero when risk-free rate and volatility have average market values, and its proportion to the hedged option is only 0.1% given $T = 0.25$, $r = 0.06$ and $\sigma = 0.3$. This proportion grows to 4.14% for $T = 1$.

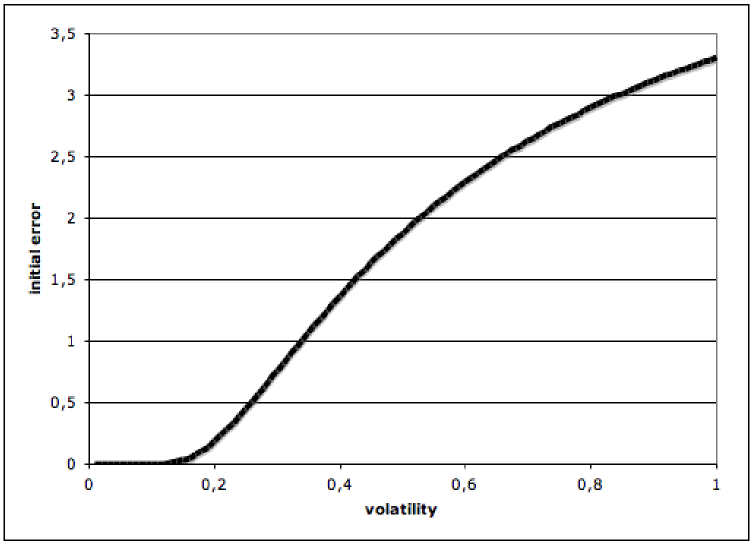

Figure 5: The initial error dependency on the volatility rate σ, holding time to maturity $T = 1$ and interest rate $r = 0.06$.

5.1.2 Time t

It is interesting to find out *when* the barrier is being crossed because the time t when it is crossed influences the ending error and, hence, the total error. From Table 1 we see that $t \in (0.024 - 0.025)$ is a minimum space of time which is needed to come down to the barrier. The fact that this minimum differs slightly among the results for different values of T might be due to inaccuracy of Monte Carlo simulation. However, if volatility is larger than 0.3 assumed here or the barrier is set closer to the spot price, the minimum time required to reach the barrier will be shorter.

Figure 6 displays a frequency distribution of time t for $T = 1$. Time t intervals are represented on x-axis and an absolute number of times when t values hit those intervals (out of p which is a number of times when the barrier was crossed) - on y-axis. We see that the histogram is skewed to the right meaning that more cases occur early rather than late. The frequency (read probability) of reaching the barrier rises sharply from the minimum time $t = 0.024$ up to approximately $t = 0.2$ and then gradually goes down. It is not surprising since Markov property discussed in chapter 2.1 implies that the probability distribution of the next stock's move is the same for any value of the stock, and since the stock has a non-zero drift and, in average, it rises, the probability of falling down to the barrier is lower and lower when time passes. Moreover, the higher the risk-free rate (which is used instead of the drift in our model) the lower the chances to reach the barrier as time goes. So, coming

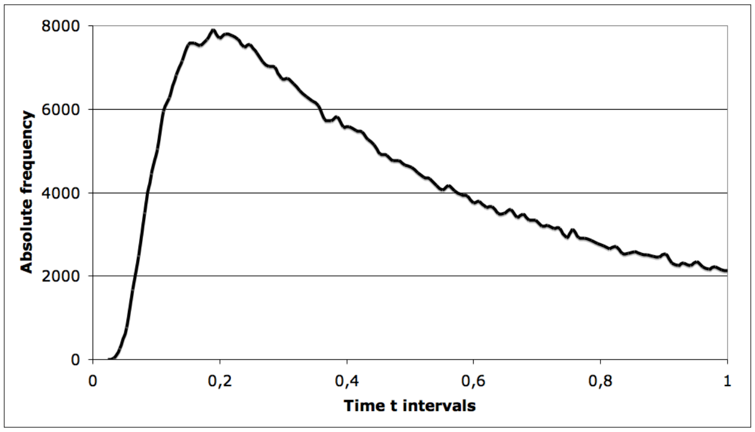

Figure 6: Frequency distribution of the time t when the barrier is reached, for time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

back to our example, if the stock has not reached the barrier until $t = 0.2$, it becomes more and more difficult to reach it after that.

The average $t = 0.433$ for $T = 1$ is probably not so informative, but the median $t = 0.384$ tells us that half the cases of reaching the barrier take place until $t = 0.384$.

On Figures 7 and 8 we plot frequency distributions of t for $T = 0.5$ and 0.25, respectively. Compared to the previous graph for $T = 1$, they seem to repeat the shape of the distribution curve for that part of t from 0 up to $t = 0.5$ and from 0 up to $t = 0.25$ respectively. Indeed, there is no reason for them to change because time to maturity is a parameter which is chosen arbitrarily. The graph for $T = 0.25$ does not have a right tail anymore and the frequency only grows as t grows, only in some last period, from approximately 0.2 to 0.25, it seems to be rather constant. The distributions seem to be quite asymmetric, but we do not consider such a measure as *skewness* here, although it would say more about the symmetry.

Coming back to the distributions for $T = 1$ and $T = 0.5$ (Figures 6 and 7, respectively), we see that the values around $t = 0.2$ are the most frequent ones. This is a *mode* that is one of the measures of central tendency which says that this value occurs most frequently in a data set (we do not present the precise value of the mode since it is of no significance here). Let us just remember that this is the peak of the t frequency distribution and later it will help us understand the frequency distribution of the ending error.

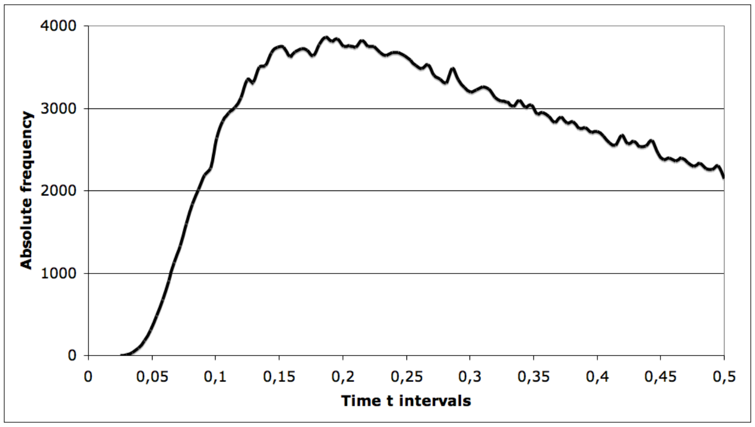

Figure 7: Frequency distribution of the time t when the barrier is reached, for time to maturity $T = 0.5$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

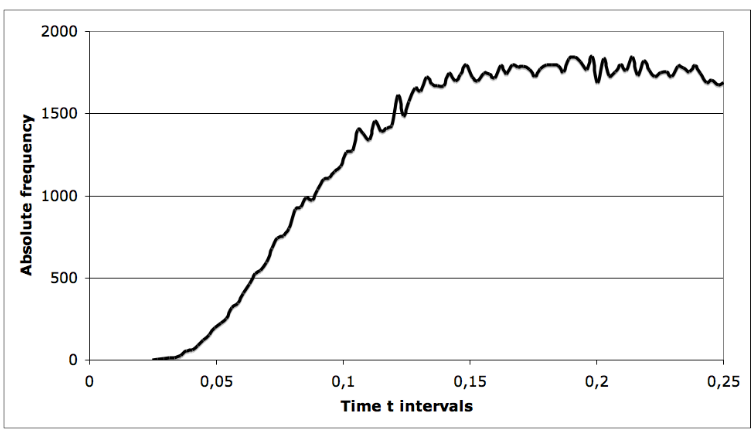

Figure 8: Frequency distribution of the time t when the barrier is reached, for time to maturity $T = 0.25$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

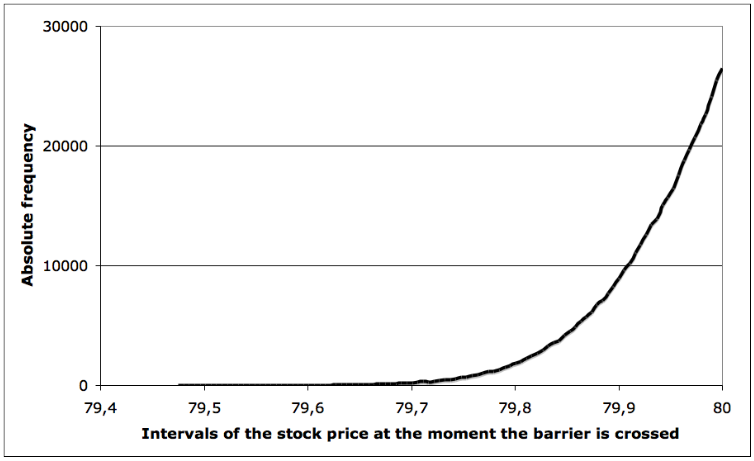

Figure 9: Frequency distribution of the stock price at the moment the barrier is crossed, for time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

5.1.3 Stock price at time t

Another stochastic parameter that influences the ending error is a stock price at the moment when the barrier is crossed, S_t. As already said, it is not always exactly equal to the barrier and might be slightly or significantly lower. In our simulation we observe that the stock price at that time has not varied so much. The lowest price where the barrier has been crossed over is 79,48 and the highest, of course, 80. Actually, the stock price at that moment does not depend on any market parameters. In our Monte Carlo simulation it depends on accuracy, i.e. on step h which is used for simulations, and in practice it mostly depends on a liquidity of a stock and on a minimum step in price which an exchange allows (called a tick size). On Figure 9 we show a frequency distribution of the stock price at the moment the barrier is crossed for $T = 1$ (for other values of T it is the same). It seems that it increases exponentially with the highest chance to cross the barrier at levels very close to 80, and the distribution appears to be reasonable.

The joint frequency distribution of time t and the stock price at that time (for $T = 1$) is shown on Figure 10 which is quite a descriptive histogram. The intervals of t and S_t are represented on horizontal axes while the absolute numbers of *simultaneously* hitting these intervals (again, out of p) are on a vertical axis. It combines the frequency distributions from Figures 6 and 9, discussed above. We see again that, as time goes, the probability of crossing the barrier grows dramatically up to approximately $t = 0.2$ and then decreases smoothly, and, at the same time, the probability of crossing the barrier at $S_t = 80$ is the highest and

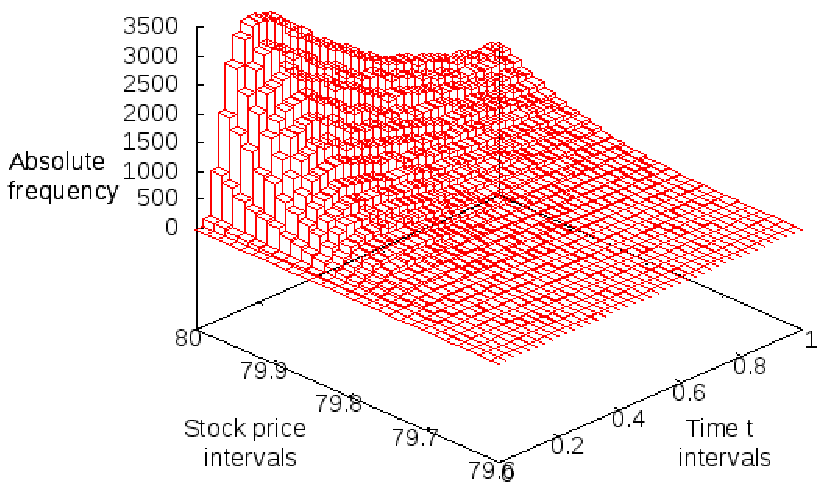

Figure 10: Joint frequency distribution of time t when the barrier is crossed and the stock price at this time, for time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

it goes down exponentially with the stock price going down from 80, for any values of t.

A similar histogram for $T = 0.25$ is shown on Figure 11. As discussed before, the probability of crossing the barrier only grows with time, and the probability of crossing the barrier at the stock price levels lower than 80 decreases exponentially, for any values of t.

5.1.4 Ending error

Thereby, we have two stochastic parameters, time t when the barrier is crossed and stock price S_t at that moment, which influence the ending error. Figure 12 shows the ending error's dependency on these two parameters, for $T = 1$. The ending error values are plotted on a vertical axis against values of time t and stock price on horizontal axes, for all p observations. We see that the dependency is smooth, the ending error increases monotonically when t increases and almost does not depend on S_t . As seen from Table 1, the minimum value of the error is -3.30784 and the maximum is slightly higher than 0. It is important to emphasize that the error *decreases* in absolute value as time t grows. If we had a *short*

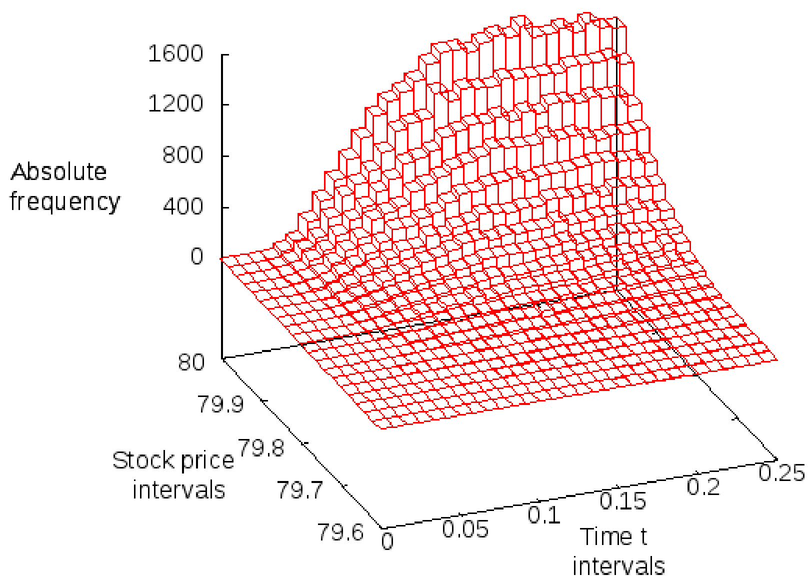

Figure 11: Joint frequency distribution of time t when the barrier is crossed and the stock price at this time, for time to maturity $T = 0.25$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

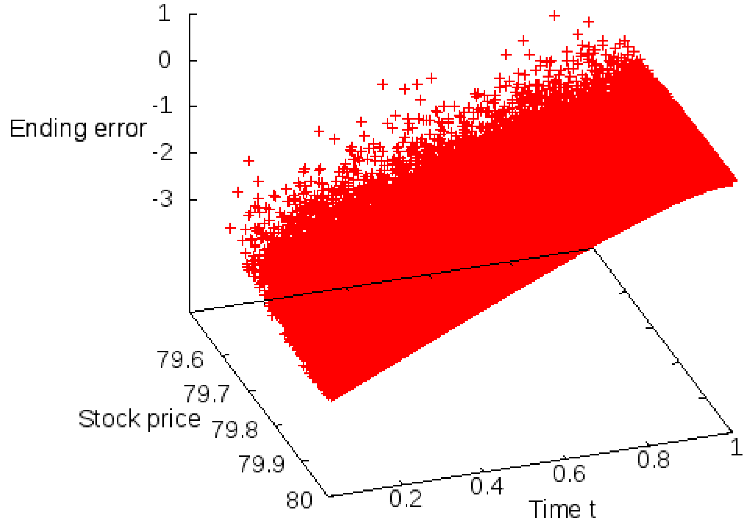

Figure 12: The ending error dependency on time t when the barrier is crossed and the stock price at that time, holding time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility rate $\sigma = 0.3$.

position in a DOC which we needed to hedge, the error would have the opposite sign and would *decrease* as time t goes.

We also see that as t approaches time to maturity, the error change accelerates and goes to zero. This can be explained if we look at the expression for the ending error (51). We see that as t approaches T, the difference $T-t$ goes to 0, making d_{1call} and d_{2call} approach minus infinity ($\ln \frac{S_t}{K} < 0$ since $S_t < K$) and that d_{1put} and d_{2put} approach infinity ($\ln \frac{S_t K}{H^2} \approx \ln \frac{K}{H} > 0$ since $S_t \leq H < K$). So, the cumulative normal distributions $N(d_{1call}), N(d_{2call}), N(-d_{1put})$ and $N(-d_{2put})$ all go to zero, summing up in the ending error which then approaches 0. The fact that the ending error might be slightly above zero (we have observed the maximum value 0.0188) is due to some particular combinations of t and S_t, namely when S_t is on its lower boundary and t is high, for example, $S_t = 79.5$ and $t = 0.9$.

Ending error is an error that occurs when the barrier is reached which may take place at any time during the DOC's lifetime. In spite of the fact that all the ending errors occur

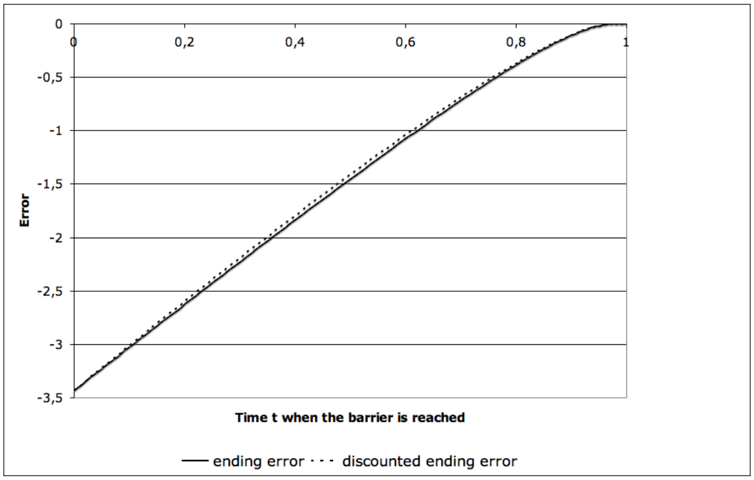

Figure 13: Ending error vs. discounted ending error, for $T = 1$.

at different points in time, we believe that it makes sense to consider its average and distribution since discounting to time 0 does not have a significant effect. Let us first explain why. Consider Figure 13 where we plot the ending error and the discounted ending error depending on time t, for time to maturity $T = 1$. For simplicity we assume that the barrier is hit at exactly H-level, so the stock price becomes of no consequence. We see that the lines are tightly close to each other. When the barrier is reached early, the effect of discounting is small because the time period to be discounted back is small. When the barrier is reached late approaching $t = 1$, the ending error becomes so close to 0 that even multiplied by the discount factor e^{-rt} it is still close to 0. Only for those t in the middle, the difference is visible, but it is still small enough to allow us to neglect it.

Now we come to probably the most interesting part of our analysis, namely the frequency distribution of the ending error. Figure 14 displays this distribution for $T = 1$ where the ending error intervals are on x-axis and the number of times hit in these intervals (again, out of p) are on y-axis. The distribution is not that smooth, and is a *bimodal* one. It means that it has two peaks, one around -2.5 and another around 0. The left part of the distribution until the second mode is skewed to the right and its shape resembles the distribution of time t, see again Figure 6. The mode of the t distribution was approximately 0.2, and plugging $t = 0.2$ and $S_t = 80$ into the equation for the ending error (51) (other parameters are as before $T = 1$, $r = 0.06$ and $\sigma = 0.3$) we obtain that the ending error in this case is equal to -2.63, exactly around the peak we observe. So, we see that the two modes, that for the ending error distribution and that for t distribution, coincide. In that way we can conclude that the shape of the frequency distribution curve of the ending error (until values close to 0) approximately repeats that of the distribution of time t. Indeed, on Figure 13 we saw that the dependency of the ending error on t is approximately linear, not taking into account the interval of t which is close to maturity date, so the shapes of the frequency distributions must approximately coincide. One of the reasons why they coincide *approximately*, but not *precisely*, is that the stock price at time t is not always 80. So, the wider the distribution of the stock price at time t, as for example for low liquidity stocks, the bigger the difference in frequency distributions of the ending error and time t.

When it comes to the second peak around 0, it can be explained by looking again at the expression for the ending error (51). Earlier we showed that when $T - t$ approaches 0, the ending error approaches 0 as well, and the values of the ending error for low $T - t$ become so close to each other, that they all go to those last few intervals in the distribution which are just before or including 0. For example, in our case, we divide the whole range of the ending error [-3.30784, 0.018848] into $a = 300$ intervals. The third and the second intervals from the end are [-0.01442, -0.00333] and [-0.00333, 0.007759] where we cross zero in the second one. The number of times when the value of the error falls into these intervals are 3611 and 7774 respectively (out of $p = 438\,184$), and these numbers are by far higher than for previous intervals. Into the last interval [0.007759, 0.018848] only 9 observations fall, this is because all the values in the interval are above zero and only those peculiar observations discussed above fall into it (lowest possible S_t when t is high). To conclude, we can say that we observe the second peak because the ending error is always approximately equal to zero for very high t.

It is interesting to look at the frequency distributions for $T = 0.5$ and $T = 0.25$, see Figures 15 and 16 respectively. They are both extremely skewed to the left. We see that their shapes up to the right peak only remotely resemble now the shapes of the time t distributions, as if they were flattened. That is because the peak becomes so dramatic that it "grabs the biggest piece of the pie". Such a high peak, higher than before, can be explained by that the proportion of the time interval where $T - t$ approaches 0 (say, this interval is 0.05) to the whole time interval T, is higher for smaller T, so the proportion of zero ending errors is higher.

The average value of the error for $T = 1$ is -1.7076 which is approximately in the middle of the range. As T becomes shorter, the mean and the median go closer to the right peak,

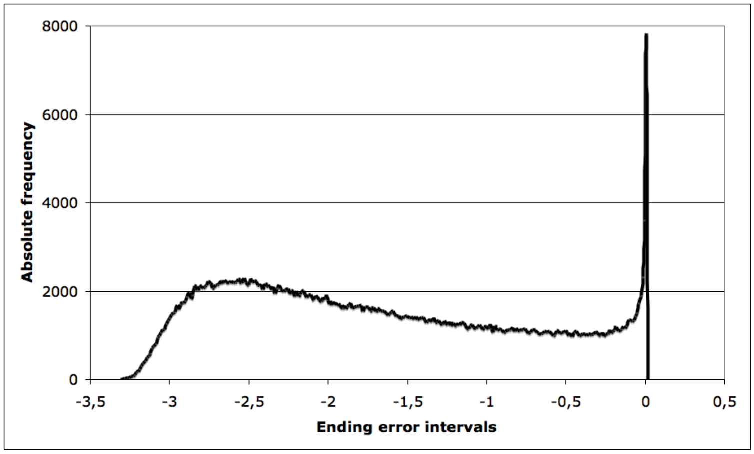

Figure 14: Frequency distribution of the ending error, for time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

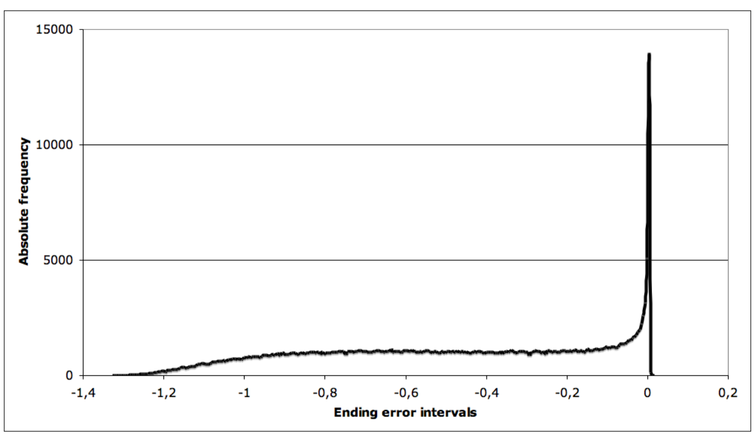

Figure 15: Frequency distribution of the ending error, for time to maturity $T = 0.5$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

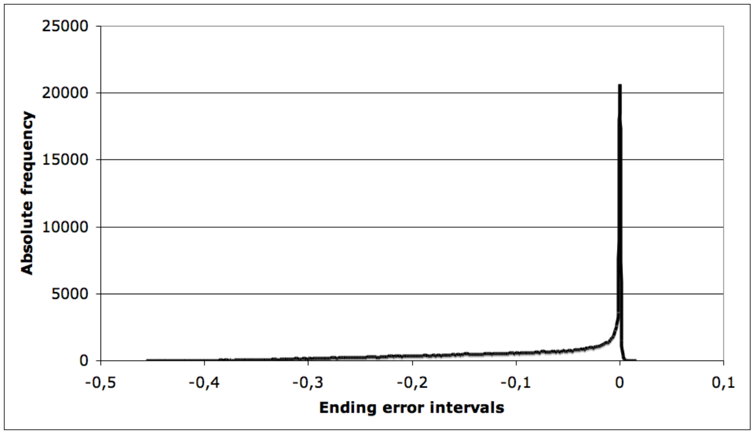

Figure 16: Frequency distribution of the ending error, for time to maturity $T = 0.25$, risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.3$.

approaching it for $T = 0.25$ (see Table 1). It means that the error is approximately equal to zero in up to half the observations. In other words, the probability that the ending error is equal to zero goes to approximately 50% as time to maturity goes down to 0.25 or even lower.

5.1.5 Total error

From Table 1 we see that the average ending error and the initial error have always opposite signs. For $T = 1$, the initial error is equal to 0.758595, that is our expense at time 0. To pay it, we can borrow this amount risk-free and pay it back at time t when the barrier is reached, because the ending error is "advantageous" for us and at time t we receive, in average, 1.7076). However, to calculate the *right* expected ending error, we need to include also those cases when the barrier was not reached and the ending error was zero since we did not need to liquidate our hedge position. Including these, we obtain that the expected ending error, discounted to time 0, is equal to -0.74825, very close to offset the initial error. By that we obtain that our average *total* error is quite small, only 0.023675. We observe the similar for $T = 0.5$ and $T = 0.25$, the initial errors are nearly offset by the discounted ending errors [2], and the total errors are even closer to zero.

Since the total error is equal to the discounted ending error plus a constant initial error, its dependency on stochastic parameters t and St is the same as for the ending error. Consider Figure 17 which shows this dependency for $T = 1$. This is very similar to what

[2]i.e. approximately $Initial\,error = E_Q[e^{-rt} Ending\,error_t]$

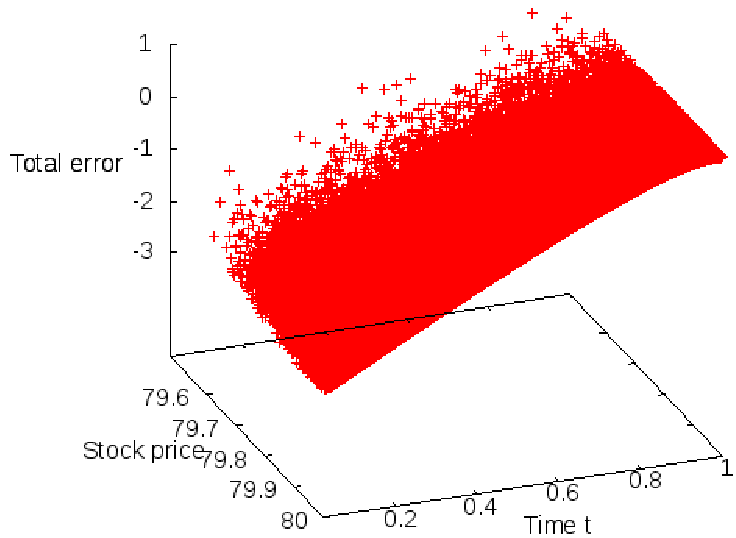

Figure 17: The total error dependency on time t when the barrier is crossed and the stock price at that time, holding time to maturity $T = 1$, risk-free interest rate $r = 0.06$ and volatility rate $\sigma = 0.3$.

we observed from Figure 12: the error decreases in absolute value as time t grows or as the stock price moves away from the barrier, where the impact of the stock price is rather small. The only difference is now that as t approaches time to maturity, the total error approaches 0.758595, the initial error.

The average total error and its distribution are very much affected by the probability that the barrier will be reached. For $T = 1$ this probability is 43.82% (see Table 1) which says that there is a 100 - 43.82 = 56.18% chance that our total error will be 0.758595 (the initial error) and the rest 43.82% are distributed with the ending error distribution, as on Figure 14. The only difference is that the x-axis will be shifted and the right peak will occur around 0.758595. We do not present a histogram of the total error distribution because the peak will overwhelm the rest of the distribution and the graph will not be descriptive at all.

As we see from Table 1, the hitting probability decreases with time to maturity (which is obvious since the stock simply has less time to reach the barrier), and this decreasing probability implies that the peak around the initial error value becomes even higher for short times to maturity.

In the table we also present other measures as a variance of the total error, its average value proportional to the hedged option and its maximum absolute value proportional to the DOC. We see that average total errors in proportion to the DOC are quite small, only 0.01% for a short-term option and 0.13% for a one-year option. The proportion grows as time to maturity grows. The maximum error which a hedger might expect is 3.4% for a three-months option and 13.87% for a one-year option.

To conclude, we see that the errors are significantly small, especially for short-term options. The frequency distributions are not smooth and highly skewed to the left, but, what is advantageous for us, they have a huge peak around the value which is deterministic. This value is the size of the initial error, and it can be calculated before entering into the hedge position.

6 Conclusions

In our work we discussed how put-call symmetry result and the following static hedging strategy can be implemented for hedging a European down-and-out call and investigated a hedging error which occurs in case we relax an assumption of a zero drift of the underlying. We use a standard Black and Scholes economic model where a stock is assumed to follow a geometric Brownian motion under a risk-free measure. To analyze the hedging error we used Monte Carlo simulation to model a stock price behavior and then observe the error. For that a self-written code in OxEdit was used. The error is stochastic, and we obtained an expression for the error as a function of two stochastic parameters. In addition, a closed-form solution for the constant part of the error was obtained. Besides stochastic variables, both the stochastic and the constant part of the error depend on all the usual parameters: risk-free interest rate, stock's volatility, time to maturity of the hedged option, and a proportion among spot price of the stock, strike price and the barrier.

The hedging error consists of two parts: the error which occurs at the initiation of the hedge and the error which might take place at some random moment during the lifetime of the option when the hedger will need to sell the hedging portfolio. There is not any error due to a payoff mismatch, i.e. the DOC and the hedge portfolio have identical payoffs in any scenario. The first, initial, error is deterministic and it generally grows with time to maturity of the option or with a grow in market parameters as risk-free interest rate or volatility. The second, ending error, is stochastic and depends on 1) the probability that the barrier will be hit (it is not the error itself but its expected value which depends), 2) the moment in time when it is hit, and 3) the underlying stock price at this moment. The first dependency is straightforward: the lower the hitting probability the lower the chances to get any ending error at all. This can be advantageous because it reduces the *uncertainty* about the future, but, on the other hand, the ending error (discounted to time 0) is supposed to offset the initial one, since they have opposite signs. The perception of the risk might depend on whether one has a long or a short position in a barrier option. As in our example with a long DOC, the initial error is positive and this is our expense. In that case we are interested in getting an ending error since it almost surely will be negative meaning that this will be our "income". Although the ending error is stochastic, in average, it tends to offset the initial error almost precisely. On the other hand, if we have a *short* position in the DOC, the initial error is our "income", so we might not be interested in getting any ending error since we will need to pay for it.

Talking about stochastic variables, time when the barrier is crossed and stock price at that time, the impact of time is much larger than that of the stock price. Generally, the ending error (and, hence, the total error) decreases in an absolute value when time when the barrier is crossed goes up and as stock price at this time approaches the actual barrier. If time to maturity approaches and the barrier has not yet been hit, even if it will be hit those last moments, the ending error becomes approximately equal to zero *regardless* of the time when the barrier is actually reached. The observed variation of the stock price at the crossing time is so small that we believe, in practice, for high liquidity assets, the price can be assumed constant and equal to the actual barrier, leaving the errors to depend only on one stochastic parameter.

For short-term barrier options both the initial and the ending error become smaller in an absolute value than for long-term options and the average *total* error might be as small as 0.01% of the position to be hedged. This value is obtained for the case where risk-free rate and stock's volatility are market averages, while the barrier is set at 80% of the spot price and the strike price - at 90%. Values of the error for other cases will, of course, differ, but we still believe that total hedging errors can be said to be acceptable. Their averages are close to zero and the probability distributions, although neither symmetric nor normal, seem to be favorable for us. The probability that the error of closing the hedge position is approximately zero is very high, and the lower the time to maturity of the hedged option, the higher this probability. Except for this probability peak of receiving a zero ending error, the rest of the distribution seems to repeat the distribution of the time when the barrier is reached.

A possibility for further investigation would be to consider other types of barrier options and to find out if our conclusions can be generalized.

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8 Appendix

Here we give a self-written code for OxEdit which was used for the simulations.

```
#include<oxstd.h>
#include<oxprob.h>
#include <oxdraw.h>

main()
{
decl S, K, H, r, sigma, T, lambda, y, DOC_price, DIC_price, d_1, d_2,
call_price, number_puts, K_put, put_price, value_in_puts, initial_error;

S = 100;
K = 90;
H = 80;
r = 0.06;
sigma = 0.3;
T = 1;
```

```
//long DOC + hedge

lambda = (r+0.5*sigma^2)/(sigma^2);
y = (log(H^2/S/K))/(sigma*sqrt(T))+lambda*sigma*sqrt(T);

DIC_price = S*(H/S)^(2*lambda)*probn(y)
-K*exp(-r*T)*(H/S)^(2*lambda-2)*probn(y-sigma*sqrt(T));

d_1 = (log(S/K)+(r+0.5*sigma^2)*T)/(sigma*sqrt(T));
d_2 = d_1-sigma*sqrt(T);
call_price = S*probn(d_1)-K*exp(-r*T)*probn(d_2); //hedge element nr.1, short

DOC_price = call_price-DIC_price;

number_puts = K/H;
K_put = H^2/K;
d_1 = (log(S/K_put)+(r+0.5*sigma^2)*T)/(sigma*sqrt(T));
d_2 = d_1-sigma*sqrt(T);
put_price = K_put*exp(-r*T)*probn(-d_2)-S*probn(-d_1);
value_in_puts = number_puts*put_price; //hedge element nr.2, long

initial_error = call_price-DOC_price-value_in_puts;

println("Down-and-out call ", DOC_price);
println("Initial error ",initial_error);
println("Initial error in % to DOC ", initial_error*100/DOC_price, "%");

//simulate price of the stock

decl n, m, h, b = 0, drift, vol, i, j, call_price_new, put_price_new,
value_in_puts_new, error, ending_error, discounted_ending_error,
total_error, sum_stock_price = 0;
decl errorMa, p;
```

```
n = 40000;
m = 1000000;
h = T/n;
errorMa = zeros (m,6);
p = 0;

drift = (r-0.5*sigma^2)*h;
vol = sigma*sqrt(h);

for(j=1;j<=m;j++)
{
// a new stock path simulation
b = 0;
S = 100;
for(i=1;i<=n;i++)
{
S = S*exp(drift+vol*rann(1,1));

if((S<=H)&&(b==0))   //if barrier is hit

{   // sell puts and buy back call

d_1 = (log(S/K_put)+(r+0.5*sigma^2)*(T-h*i))/(sigma*sqrt(T-h*i));
d_2 = d_1-sigma*sqrt(T-h*i);
put_price_new = K_put*exp(-r*(T-h*i))*probn(-d_2)-S*probn(-d_1);
value_in_puts_new = number_puts*put_price_new;
//receive this amount from the puts

d_1 = (log(S/K)+(r+0.5*sigma^2)*(T-h*i))/(sigma*sqrt(T-h*i));
d_2 = d_1-sigma*sqrt(T-h*i);
call_price_new = S*probn(d_1)-K*exp(-r*(T-h*i))*probn(d_2);
//spend this amount for the call

ending_error = value_in_puts_new-call_price_new;
```

```
discounted_ending_error = exp(-r*h*i)*ending_error;
total_error = discounted_ending_error+initial_error;

//save the results in a matrix
errorMa[j-1][0] = h*i;     //t
errorMa[j-1][1] = T-h*i; //T-t
errorMa[j-1][2] = S;
errorMa[j-1][3] = ending_error;
errorMa[j-1][4] = discounted_ending_error;
errorMa[j-1][5] = total_error;

b = 1;
p = p+1;
}
if((i==n)&&(b==0))   //if we reached the expiration and barrier is not hit
{
total_error = initial_error;
errorMa[j-1][5] = total_error;
}
}
sum_stock_price = sum_stock_price + S;

}

decl errorMaWithoutZeros;    //create a matrix without zeros

errorMaWithoutZeros = zeros(p,6);
j=0;
for(i=0;i<=m-1;i++)
{

if(errorMa[i][0] > 0)   //if barrier was hit
{
errorMaWithoutZeros [j][0] = errorMa[i][0];
errorMaWithoutZeros [j][1] = errorMa[i][1];
```

```
errorMaWithoutZeros [j][2] = errorMa[i][2];
errorMaWithoutZeros [j][3] = errorMa[i][3];
errorMaWithoutZeros [j][4] = errorMa[i][4];
errorMaWithoutZeros [j][5] = errorMa[i][5];

j = j+1;
}
}

 //Frequency distributions:

 //t distribution

decl t_step, t_intervals, t_number_intervals, t_distribution;
t_number_intervals = 100;
t_step = (max(errorMaWithoutZeros[:][0])-min(errorMaWithoutZeros[:][0]))
/t_number_intervals;
t_intervals = zeros(t_number_intervals+1, 1);
t_intervals[0][0] = min(errorMaWithoutZeros[:][0]);


for(i=1;i<=t_number_intervals;i++)
//determine all intervals and save in Excel file
{
t_intervals[i][0] = t_intervals[i-1][0]+t_step;
}
savemat("t_intervals.xls", t_intervals);

t_distribution = countc(errorMaWithoutZeros[:][0], t_intervals);
savemat("t_distribution.xls", t_distribution);


//St distribution
decl St_step, St_intervals, St_number_intervals, St_distribution;
St_number_intervals = 100;
```

```
St_step = (max(errorMaWithoutZeros[:][2])-min(errorMaWithoutZeros[:][2]))
/St_number_intervals;
St_intervals = zeros(St_number_intervals+1, 1);
St_intervals[0][0] = min(errorMaWithoutZeros[:][2]);


for(i=1;i<=St_number_intervals;i++)
//determine all intervals and save in Excel file
{
St_intervals[i][0] = St_intervals[i-1][0]+St_step;
}
savemat("St_intervals.xls", St_intervals);

St_distribution = countc(errorMaWithoutZeros[:][2], St_intervals);
savemat("St_distribution.xls", St_distribution);


  //ending error distribution
decl ending_step, ending_intervals, ending_number_intervals, ending_distribution;
ending_number_intervals = 300;
ending_step = (max(errorMaWithoutZeros[:][3])-min(errorMaWithoutZeros[:][3]))
/ending_number_intervals;
ending_intervals = zeros(ending_number_intervals+1, 1);
ending_intervals[0][0] = min(errorMaWithoutZeros[:][3]);


for(i=1;i<=ending_number_intervals;i++)
//determine all intervals and save in Excel file
{
ending_intervals[i][0] = ending_intervals[i-1][0]+ending_step;
}
savemat("ending_intervals.xls", ending_intervals);

ending_distribution = countc(errorMaWithoutZeros[:][3], ending_intervals);
savemat("ending_distribution.xls", ending_distribution);
```

```
  //total error distribution
decl total_step, total_intervals, total_number_intervals, total_distribution;
total_number_intervals = 300;
total_step = (max(errorMa[:][5])-min(errorMa[:][5]))/total_number_intervals;
total_intervals = zeros(total_number_intervals+1, 1);
total_intervals[0][0] = min(errorMa[:][5]);


for(i=1;i<=total_number_intervals;i++)
//determine all intervals and save in Excel file
{
total_intervals[i][0] = total_intervals[i-1][0]+total_step;
}
savemat("total_intervals.xls", total_intervals);

total_distribution = countc(errorMa[:][5], total_intervals);
savemat("total_distribution.xls", total_distribution);



println("The barrier is hit in ", p*100/m, "% cases");

println("TOTAL ERROR:");
println("Total error distribution [", min(errorMa[:][5]),
",", max(errorMa[:][5]), "]");
println("Average total error ", meanc(errorMa[:][5]));
println("Total error variance ", varc(errorMa[:][5]));

println("ENDING ERROR:");
println("Ending error distribution [", min(errorMaWithoutZeros[:][3]),
",", max(errorMa[:][3]), "]");
println("Average ending error ", meanc(errorMaWithoutZeros[:][3]) );
println("Expected ending error including zeros ", meanc(errorMa[:][3]) );
```

```
println("Ending error variance ", varc(errorMaWithoutZeros[:][3]));

println("TIME t:");
println("Time t distribution [", min(errorMaWithoutZeros[:][0]),
 ",", max(errorMaWithoutZeros[:][0]), "]");
println("Average time t ", meanc(errorMaWithoutZeros[:][0]) );
println("Time t variance ", varc(errorMaWithoutZeros[:][0]));

println("STOCK PRICE:");
println("St distribution [", min(errorMaWithoutZeros[:][2]),
",", max(errorMaWithoutZeros[:][2]), "]");
println("Average St ", meanc(errorMaWithoutZeros[:][2]));
println("St variance ", varc(errorMaWithoutZeros[:][2]));

println("Average stock price at expiration ", sum_stock_price/m);

println("Moments for the matrix with zeros ", moments(errorMa));
println("Moments for the matrix without zeros ", moments(errorMaWithoutZeros));
println("Medians for the matrix with zeros ", quantilec(errorMa));
println("Medians for the matrix without zeros ", quantilec(errorMaWithoutZeros));

savemat("ErrorMa.txt", errorMa);
savemat("ErrorMaWithoutZeros.txt", errorMaWithoutZeros);

}
```