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Supply Chain Optimization in the LNG Business

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Oppstartsdato	Innleveringsfrist
16. jan 2012	**11. jun 2012**
Oppgavens (foreløpige) tittel **Supply chain optimization in the LNG business**	
Oppgavetekst/Problembeskrivelse The purpose is to develop optimization models and methods which can support a downstream actor in the LNG business. Problems within a tactical planning horizon will be considered taking ship routing and scheduling, inventory management and contract handling into account. Main contents: 1. Description of the problem 2. Development of mathematical model(s) and method(s) for the problem 3. Implementation of mathematical model(s) and method(s) using commercial software 4. Testing of model(s) and method(s) with relevant data 5. Discussion of the results and the usefulness of the model(s) and method(s)	
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Oppgavens (foreløpige) tittel **Supply chain optimization in the LNG business**

4. Bedømmelse

Kandidatene skal ha *individuell* bedømmelse ☐
Kandidatene skal ha *felles* bedømmelse ☒

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Sammendrag

I distribusjonsnettverket for flytende naturgass (LNG) kan store aktører kontrollere flere ledd. Dette fordrer koordinerte beslutninger og et system for a fasilitere integrerte operasjoner. Denne masteroppgaven løser et kombinert lager- og ruteplanleggingsproblem (IRP) med kontraktshåndtering for en taktisk planleggingshorisont. Formålet med oppgaven er å utvikle optimeringsmodeller- og metoder som kan støtte en nedstrømsaktør i distribusjonsnettverket for LNG.

Skip frakter LNG fra produksjonsanlegg til regasifiseringsterminaler. Henting av gass må oppfylle et sett av kjøpskontrakter, mens lagerstyring må håndteres i regasifiseringsterminalene. Kontraktene definerer øvre og nedre grenser for volumet av LNG som hentes. Slike grenser gjelder både for den totale planleggingsperioden og for kortere tidsintervaller. Videre kan en kontrakt oppgi én eller flere primærhavner hvor en gitt andel av hentevolumet må leveres. For å sørge for en jevn fordeling av henting kan kontraktene kreve en minstetid mellom hvert skipsanløp. Lagerhåndtering i regasifiseringsterminalene handler om å sørge for at lagernivået holdes innenfor definerte grenser. Tidspunkter for leveranse, salg og kjøp er avgjørende for å overholde lagerbetingelser. Problemets planleggingshorisont er mellom seks og ti uker, og målet er å oppnå størst mulig profitt.

Erfaring fra litteraturen indikerer at LNG-IRP er et svært komplekst problem som er beregningsmessig vanskelig å løse. Prosjektoppgaven vår (Emanuelsen and Thorvaldsen, 2011) antyder at det å introdusere kontrakter kompliserer problemet ytterligere. Forbedringer av den matematiske modellen fra prosjektet har blitt gjort og i tillegg blir to nye modeller presenterert. Den første modellen er en *arc flow*-formulering som benytter ulike variable til å representere alle beslutninger eksplisitt. Deretter følger en *path flow*-formulering, som bruker én variabel for å representere en fullstendig path. En path innbefatter en ruteplan med tidsangivelser og kvantumet som lastes og losses i hver havn. Den siste modellen er en *duty flow*-formulering hvor paths er erstattet med duties. En duty kan betegnes som en delvis path, siden ruten i en duty kun omfatter seilas fra en produksjonshavn til en leveringshavn og tilbake til en produksjonshavn.

Modellene er implementert som et blandet heltallsprogram (MIP) ved bruk av kommersiell programvare. Et verktøy har også blitt utviklet for å generere alle kolonner a priori for path flow og duty flow-modellene. For å teste modellene har flere instanser blitt utviklet basert på virkelige data. Målet er å løse problemet med de tre ulike modellene og undersøke deres yteevne relativt til hverandre. Ulike kontraktsparametres innvirkning på lønnsomheten vil også bli analysert.

Resultater fra testingen viser at arc flow-modellen løser de fleste av instansene til

optimalitet, men ikke de aller største. Å bevise optimalitet kan imidlertid være tidkrevende. For path flow-modellen oppleves minnekapasiteten som et stort hinder, siden antallet paths blir svært høyt ved lange planleggingshorisonter. Derfor løses bare små instanser til optimalitet. I motsetning til dette løser duty flow-modellen nesten alle instanser på relativt kort tid. Duties er mindre tallrike enn paths og formuleringen kombinerer duties på en effektiv måte. For å undersøke potensialet for denne formuleringen videre blir modellen også implementert som en dynamisk modell. Videre utvikling av denne er dog nødvendig for å avgjøre hvorvidt den yter bra i løsning av problemet.

Supply Chain Optimization in the LNG Business

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Department of Industrial Economics and Technology Management

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NTNU – Trondheim
Norwegian University of Science and Technology

Preface

This is the final thesis of our Master's degree from the Norwegian University of Science and Technology, Department of Industrial Economics and Technology Management. It has been prepared over a period of 20 weeks during the spring of 2012.

The thesis is within the field of operations research, and presents a planning problem from the LNG supply chain concerning ship routing, inventory management and contract handling. Mathematical models for solving the problem are developed, implemented and compared.

We are very grateful to our two academic supervisors; Associate Professor Henrik Andersson and Professor Marielle Christiansen. During the development of the thesis you have provided guidance that has been invaluable for the final result. Thank you very much for your support, advisory and involvement in the process.

Trondheim, June 2012

Kristian Emanuelsen Sondre Thorvaldsen

Summary

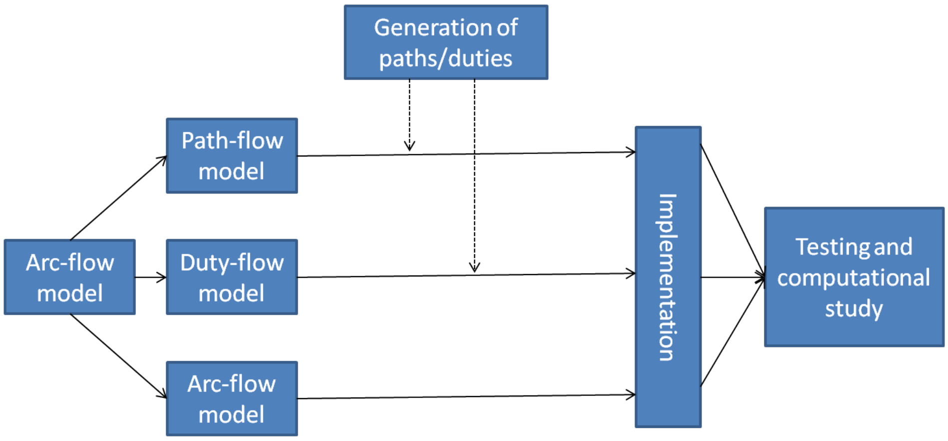
Large players can control several tiers in the liquefied natural gas (LNG) supply chain, which calls for coordinated decisions and a system to facilitate integrated operations. This thesis is concerned with solving a combined inventory and routing problem (IRP) with contract handling within a tactical planning horizon. The purpose is to develop optimization models and methods which can support a company downstream in the LNG supply chain.

Ships transport LNG from liquefaction plants, where the LNG is produced, to regasification terminals. Pick-up of gas must fulfill a set of purchase contracts, whereas inventory management prevails at the regasification terminals. The contracts state upper and lower limits for the volume of LNG that is picked up. Such limits apply for the total planning period and for defined subperiods. Additionally, contracts may define one or more primary ports, to where a given fraction of the LNG must be delivered. To get an even spread on the pick-ups, contracts introduce inter-arrival gaps that state a minimum number of days between pick-ups. Inventory management at the regasification terminals is about ensuring the storage level to be within specified limits. The timing of deliveries, sales and purchases is crucial to comply with inventory requirements. Planning periods of six to ten weeks are considered, and the objective is to obtain maximum profit from operations.

Experience in the literature indicates that the LNG-IRP is very complex and computationally hard to solve. Our project work (Emanuelsen and Thorvaldsen, 2011) suggests that introduction of contracts complicates the problem even more. The mathematical model from the project is improved and we develop two additional models. The first model is an arc flow formulation that use designated variables to represent all decisions explicitly. We then progress with a path flow formulation, which uses one variable to represent a complete path. A path incorporates the schedule for a ship, combined with the quantities loaded and unloaded in each port. The last model is a duty flow formulation, where paths are replaced by duties. Duties are essentially partial paths that cover sailings only from a pick-up port to one or more delivery ports, and back to a pick-up port.

The models are implemented as mixed-integer programs (MIP) by use of commercial optimization software. For the path and duty flow models, a tool is also developed to generate all columns a priori. In order to test the formulations, instances are created based on real-life data. The aim is to approach the problem with these three formulations, and investigate their performance relative to each other. Further, the influence of contractual terms on the profit is analyzed. Figure 1 illustrates the process of developing, implementing and testing the models.

Figure 1: Chart of the work progress



Computational results show that the arc flow model is able to solve most of the instances to optimality. However, proving that the solutions are optimal can be time-consuming. For the path flow model, memory capacity proves to be an obstacle, since the number of paths becomes huge with long planning horizons. Therefore, only small instances are solved to optimality. The duty flow model, in contrast, solves nearly all instances within a relatively short amount of time. Duties are significantly less numerous than paths, and the formulation manage to connect the duties effectively. In order to explore the potential of the duty flow formulation further, a dynamic implementation is also developed. However, further research and development are necessary to conclude on its performance potential.

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1 Introduction

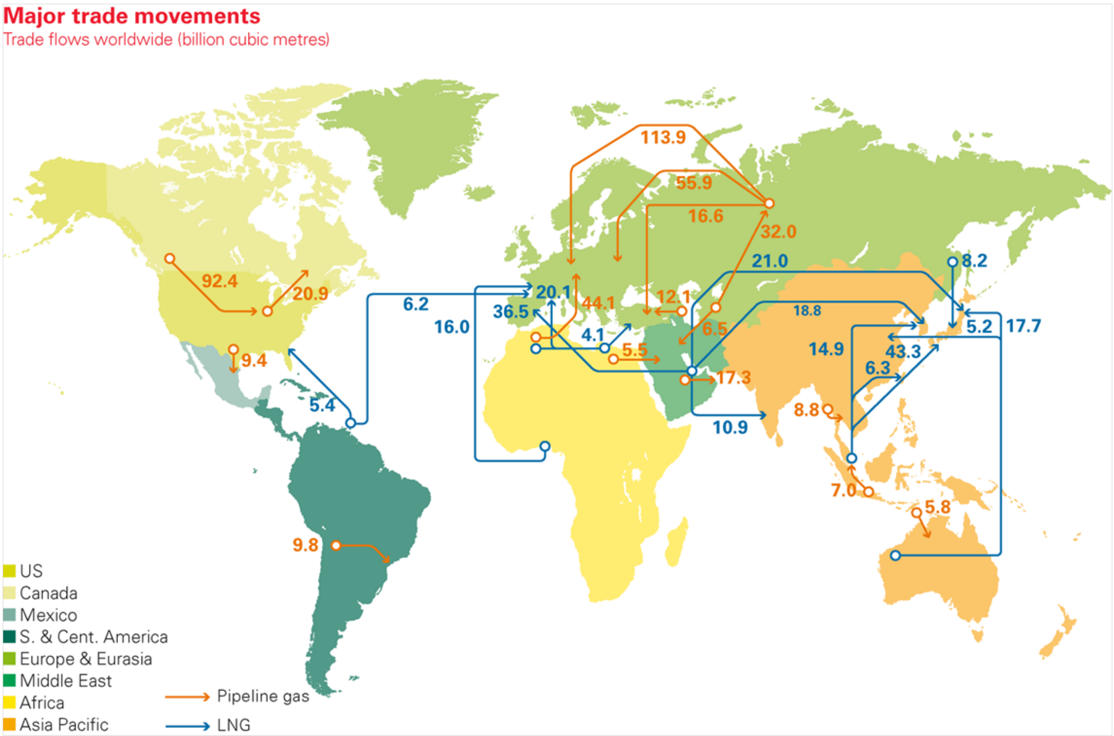
Natural gas is an increasingly important source of energy worldwide. According to the U.S. Energy Information Administration (EIA, 2011), it now accounts for more than one fifth of the global energy consumption. The main areas of application for natural gas are in power generation, in the industrial sector, as well as heating, cooling and cooking in residential and commercial sectors. Following recent advances in drilling techniques, the market has seen a significant increase in natural gas from unconventional gas resources which were previously not economically feasible. Consequently, prices have dropped to levels that are very competitive to other fossil fuels. As natural gas is also favoured among the fossil fuels with regard to environmental concerns, expectations are that growth in demand and supply will continue (EIA, 2011).

Traditionally, natural gas has been transported by pipeline from reservoir to market. As a consequence, major pipeline networks have emerged, especially in Eurasia and North America. However, there exists a geographical mismatch between areas with natural gas resources and areas with high demand. For remote areas far away from the market, transportation by ships is often a more cost-efficient solution than transportation by pipelines (Subero et al., 2004). In order to transport the natural gas by ships, it is cooled down to a liquid state. The product is known as liquefied natural gas (LNG). Figure 2 shows the major trade movements of natural gas, and indicates that ships are the primary mode of transportation for long overseas distances.

After a few years with stagnation and decline in the natural gas market, 2010 again saw an increase of 7.4%. Looking at LNG alone, the trade grew by 25% in 2010. LNG now constitutes 9% of total natural gas demand (IEA, 2011) and accounts for about 30% of global gas trade (BP, 2011). To facilitate this growth, there has been a 40% increase in the world LNG production capacity the last five years (EIA, 2011). The International Energy Agency (IEA, 2011) estimates that the production capacity for LNG will increase by another 21% in the period 2011 - 2016 and nearly double by 2035. Supply of LNG is expected to grow by one third in the same period, and most of the new supply is already contracted. According to the EIA (2011), the largest increase will come from Middle East and Australia, from where the distance to customers is vast. In addition to growth in volumetric terms, the market for LNG is also growing geographically. By the end of 2011, there were 18 countries exporting LNG and 25 countries estimated to import LNG (J.P. Morgan Cazenove, 2012). For reference, before year 2000 the figures were 12 countries both on export and import side (EIA, 2011).

LNG is a cryogenic liquid, with a boiling temperature of ≈ -162°C at atmospheric

Figure 2: Major global trade movements (BP, 2011)



pressure. Compared to natural gas, the volume of LNG is reduced by a factor of more than 600. This enables transportation of LNG by specially designed LNG carriers. Conventional carriers' capacity is in the range 130,000 - 150,000 m^3 LNG, but the last decade has seen tens of carriers with capacity well above 200,000m^3. By the end of 2011 the global LNG fleet was 363 ships, and additionally 33 in the order books (Wang and Notteboom, 2011). In comparison, the fleet of active LNG ships numbered 220 only five years ago (Koren and Richardsen, 2007).

The LNG liquefaction plants are typically located at remote places and transportation time is often in the magnitude of weeks. When the ship reaches its destination, the LNG is either stored in special containers or heated and transformed back to gaseous phase. When needed, the natural gas is distributed to end users through pipelines or by trucks. Since LNG requires thorough processing from reservoir to consumption, the LNG supply chain comprise many tiers. Several of these tiers are often controlled by one large player, which calls for an efficient system to facilitate and manage integrated operations. This has given rise to the LNG-Inventory routing problem (LNG-IRP), which was introduced by Grønhaug and Christiansen (2009).

The LNG-IRP concerns routing and scheduling of a fleet of LNG ships between liquefaction plants and regasification terminals, while simultaneously managing inventory in both types of facilities. This thesis regards a variation of the LNG-IRP where inventory management at liquefaction plants is replaced by purchase contracts that specify upper and lower pick-up limits for LNG. Contracts may also include an *origin-destination clause*, which dictates a minimum share of volume from a contract that must be delivered to a set of designated *primary ports*. Moreover, the contracts state an *inter-arrival gap* to ensure a fairly even spread of pick-ups during the planning period. This indicates a minimum number of days between pick-up from the contract. At regasification terminals, the inventory level must remain within its upper and lower limits throughout the planning period. Sales of gas from these terminals are subject to a daily demand that must be satisfied. From the viewpoint of a company which controls ships and regasification terminals, we aim at maximizing the profit.

Although several papers have been written on the LNG-IRP recent years, the literature concerning this problem is still scarce (Rakke et al., 2012). Furthermore, few have considered contract handling in the LNG-IRP, which is relevant for many downstream actors in the LNG supply chain. This motivates the search for ways to model the LNG-IRP with contracts. The problem is very complex; however, our previous work (Emanuelsen and Thorvaldsen, 2011) suggests that it can be solved by use of exact solution methods.

The purpose of the thesis is to develop optimization models and methods which can support companies downstream in the LNG supply chain in their decision making. Our most important research contributions are; 1) Development, implementation and comparison of three mathematical models for the LNG-IRP with contract handling, and; 2) An innovative and computationally strong duty flow model which appears to have never before been applied to the IRP in a maritime context.

In order to put the research in this thesis in perspective, a brief overview of relevant literature is provided in Section 2. Then we elaborate on the details concerning the problem and state necessary assumptions in Section 3, before proposing three mathematical models based on exact solution methods in Section 4. To provide a better understanding of the models, a detailed description of a solution from all three models are presented in Section 5. Section 6 describes the implementation processes for the mathematical models and the route generation tool. Results from testing and a discussion around these are provided in Section 7. The thesis ends with concluding remarks in Section 8.

2 Literature Overview

The problem described in this thesis shares similarities with some previous work in the area. Reviewing existing literature has therefore provided a good understanding of the LNG business in general and the specific problem at hand. This section provides an overview of relevant literature.

We start with a selection of papers on maritime inventory routing problems (MIRP). Then, the scope is narrowed down to literature concerning MIRP for the LNG industry specifically, called LNG-IRP. Both sections focus on the problems that are addressed in the papers. Papers in the MIRP section are grouped and organized according to key problem characteristics, while the LNG-IRP section presents relevant papers more thoroughly one by one. Finally follows a section that elaborates on solution methods applied to solve the planning problems.

Applicability and comparability for the problem in this thesis is emphasized when selecting literature. Hence, only exact solution methods are discussed. Furthermore, arc flow formulations, path flow formulations and dynamic solution methods are the main focus areas. To the knowledge of the authors, duty flow formulations have not yet been applied in maritime transportation. In order to shed light on this formulation, examples from the airline and railway industries are included. Notice that some papers may be mentioned in different sections; however, with a different focus.

2.1 Maritime Inventory Routing Problems

Increased competition in the global LNG market has called for efficient integration of the different tiers in the supply chain (Andersson et al., 2010b). Moreover, larger profits can be realized through coordinated decisions. This has given rise to the MIRP, which deals with handling distribution of products while simultaneously managing inventory in a number of ports. Although widely found in road-based problems, the inventory routing problem is not as often discussed in a maritime context. We will focus primarily on models with real life application to industrial cases.

Cases with industrial application tend to vary greatly, both in terms of modeling and the underlying problem. Hence, we find a wide range of variants of the inventory routing problem. As noted by Andersson et al. (2010b) a new version is often presented in each paper published. One reason pointed out is that industrial cases often have a uniqueness that calls for changes in already existing models. Among the most important dimensions of the problems are:

Table 1: Problem characteristics in selected papers

Paper	Fleet	Products	Prod/con rate	Pickup ports	Inventory ports	Contracts	Horizon(days)
Grønhaug and Christiansen (2009)	Fixed	Single	Variable	Multiple	All	No	< 60
Fodstad et al. (2011)	Charter	Single	Variable	Multiple	Selected	Yes	< 181
Christiansen (1999)	Fixed	Single	Fixed	Multiple	Internal	No	36
Al-Khayyal and Hwang (2007)	Fixed	Multiple	Fixed	Multiple	All	No	10
Ronen (2002)	Fixed	Multiple	Variable	Multiple	All	No	30
Dauzère-Pérès et al. (2007)	Fixed	Multiple	Variable	Single	All	No	28
Stålhane et al. (2012)	Charter	Multiple	Variable	Single	Pickup	Yes	366

- whether the fleet of ships is fixed or if chartering of additional ships is allowed
- whether a single or multiple products are transported
- whether the production and consumption rate is fixed or variable
- whether it is a single or multiple production ports
- whether inventory management is considered in all or just some ports
- whether contract handling is taken into account
- length of planning horizon

Many papers share common characteristics and only a few papers on different problems are therefore highlighted. Table 1 organizes the selection of papers according to the dimensions. The papers are selected to give relevant examples that describe the typical characteristics of MIRP.

The fleet of ships may be either of fixed size or include possibilities for chartering of additional ships. In both occasions the fleet is usually heterogeneous. This refers to differences in ship characteristics such as size, speed, costs and number of tanks. The counterpart is referred to as homogeneous, which has been more common in road-based IRP problems (Andersson et al., 2010b). Grønhaug and Christiansen (2009) present a problem which involves a fixed heterogeneous fleet. The same ships are thus available throughout the whole planning period. Each of the ships has a given speed, capacity and product flow rate, and the possibility of breakdowns or maintenance during the planning period is not regarded in the problem. In contrast, the problem presented by Fodstad et al. (2011) allows for chartering of ships at a daily rate if that is beneficial or necessary. The starting fleet of ships is also scheduled for maintenance during the planning period. As in all other papers reviewed, the fleet consists of several heterogeneous ships.

The problem might include multiple products, as opposed to a single product. For instance, Christiansen (1999) deals with ammonia as the sole product to be produced and shipped. Al-Khayyal and Hwang (2007), on the contrary, face a problem with multiple products. Their problem is an extension of the one regarded by Christiansen (1999). The main difference is inclusion of multiple products. Like Al-Khayyal and Hwang, Ronen (2002) presents a MIRP with multiple products where each ship has separate compartments so that several products can be shipped simultaneously.

Production and consumption can be of either fixed or variable rate. With a variable rate, some problems include the freedom to decide the optimal rate with a variable. In other problems, the rates are pre-defined as parameters and may vary due to reasons such as seasonal variations or planned maintenance. Among others, Ronen (2002) and Dauzère-Pérès et al. (2007) consider variable production and consumption rates, respectively. The alternative is fixed production and consumption rate, which is the case in the problem from the ammonia business presented by Christiansen (1999) and further developed by Al-Khayyal and Hwang (2007).

A company can deal with a single or multiple production ports from where products are shipped. When there is a single production port, all routes must visit this port and the structure is like a vehicle routing problem with a central depot. For multiple production ports there are potentially many routing combinations between pick-up and delivery ports, which resembles a network structure. Inventory management may prevail in all or only in some of the ports in the problem. The former is the general case and can be found in most of the literature mentioned here. The latter is the case if some ports are owned by the company while others are controlled by external actors. The problem presented by Stålhane et al. (2012) concerns one large pick-up port. Nevertheless, numerous delivery ports and long-term contracts cause a complex problem. Grønhaug and Christiansen (2009) include multiple pick-up and delivery ports, and regard inventory management in all of them. Similarly, Christiansen (1999) presents an inventory routing problem for a large ammonia producer with multiple production and consumption ports. Inventory in ports where the producer itself has a factory must be handled to maintain operations. These ports are labeled internal ports. For ports where the company does not own facilities, the quantity and time of delivery is determined through negotiations, and inventory management is not considered.

Real-life situations most often include some kind of contract arrangements between actors in the supply chain. Therefore, contract handling is included in some problems. Contracts state limits for pick-ups or deliveries of cargo in specific ports. For such problems the contract arrangements may replace the traditional inventory management for the planner. Some contracts also include special conditions, like an origin-destination clause and a price scheme. Fodstad et al. (2011) present a problem with contractual conditions for purchases, sales, prices and destinations.

The planning period, that is the time span for which the MIRP is solved, differs. Long sailing times call for long planning horizons, which can result in very large problems and lead to substantial computation time. Furthermore, uncertain factors will have more influence on the real life problem over a long planning horizon, and results may thus be inaccurate. However, in order to satisfactorily model real scenarios, one cannot approach the problems with too short planning periods, although it might ease the computational complexity. Most of the problems presented concern tactical planning in the range of months. Nevertheless, Stålhane et al. (2012) create an annual delivery program, thus handling up to 366 days. This is regarded as a long planning period within tactical planning.

2.2 LNG - Inventory Routing Problems

The term LNG-IRP, which is a MIRP for the LNG business, was introduced by Grønhaug and Christiansen (2009). In the wake of its introduction, several papers have been written on the LNG-IRP subject. The following regards the most important papers for the development of this thesis.

Grønhaug and Christiansen (2009) and Grønhaug et al. (2010) both consider a tactical LNG-IRP. They are concerned with routing and scheduling of specialized ships for LNG, as well as handling inventory of this product in a number of ports. Both consider routing of ships carrying a single product and a fleet of heterogeneous ships without the possibility for chartering. Boil-off of LNG from the cargo tanks is taken into account, and certain sailing limitations are defined. For instance, a ship is required to load full shiploads and it can only call two delivery ports in succession.

Andersson et al. (2010a) consider two different planning problems; one for a producer of LNG, the other for a vertically integrated company in the LNG business. The former controls one liquefaction plant and serves a number of regasification terminals with a fleet of ships. The latter is in control of both liquefaction plants and regasification terminals, in addition to the transportation, much like the problem presented in Grønhaug and Christiansen (2009) and Grønhaug et al. (2010).

Fodstad et al. (2011) generally covers the LNG-IRP more thoroughly than what has been done before. The problem is expanded from the standard LNG-IRP, and introduces purchase contracts, chartering of ships, maintenance of ships, partial loading and spot markets to the problem. Also, three levels of terminal influence are introduced, in order to make the problem more realistic. One group of terminals are under full control of the planner. For a second group of terminals, the planner only knows the inventory levels and must ensure that cargoes do not violate the bounds. For yet another group, the planner only buys or sells LNG, without having to deal with inventories or other operational issues. The same problem is considered in Uggen et al. (2011), though with a different solution approach.

The master thesis by Horgen and Frich (2004) considers a LNG-producer problem like the one addressed by Andersson et al. (2010a). LNG is to be distributed from a single supplier to several buyers, while maintaining inventory limits and securing cargo delivery. The problem is focused on Statoil's facility at Melkøya in connection to the Snøhvit project.

The master thesis by Moe et al. (2008) presents models to create an annual delivery program (ADP) for an LNG-operator. In the problem only one loading port exists and a ship can only visit one regasification terminal on each voyage. A voyage is the sailing from the liquefaction plant to a regasification terminal, and back to the plant. Contracts state limits for delivery in the regasification terminals, and a hub and spoke network is created. The loading port is the hub, and the combination of long term contracts, types of LNG and regasification terminals make up the demand nodes. The same problem is dealt with by Rakke et al. (2011) and by Stålhane et al. (2012), who both develop further the solution methods from the thesis.

Halvorsen-Weare and Fagerholt (2009) also consider a large scale problem with one single loading port serving several delivery ports. Like Moe et al. (2008) their goal is to create an ADP, which specifies when customers will receive their shipments during the year. Deliveries are decided by contracts, which state limits for the volume of LNG and time windows for arrival. If production is higher than the contracted delivery volume, the excess LNG can be sold in the spot market and transported by external ships without any profit.

2.3 Models and Solution Methods

While the last sections have focused on the problems, this section focuses on solution methods and models utilized in selected papers. Variations of arc flow and path flow formulations will primarily be discussed.

Table 2: Models and solution methods in selected papers

Paper	Arc flow	Path flow: A priori	Path flow: Dynamic	Valid inequalities
Grønhaug and Christiansen (2009)	x	x		
Grønhaug et al. (2010)			x	
Fodstad et al. (2011)	x			
Halvorsen-Weare and Fagerholt (2009)	x			
Christiansen (1999)	x		x	
Andersson (2011)			x	
Bredström and Rönnqvist (2010)		x		
Persson and Göthe-Lundgren (2005)			x	x
Agra et al. (2012)	x			x

Arc flow models use binary sailing variables to represent whether ship v traverses arc (i, j) in a specific time period. That is, if it sails from port i to port j in time period t. Sailing is subject to arc constraints which ensure correct routing. For example, node balance is required in each port and time period, so that a ship must depart the same port in which it arrived. In the arc flow model, feasible routes can be expressed explicitly with the sailing variables. Path flow models do not have this property. Rather, they require defined routes as input to the model. The routes often specify a full schedule for the ship, and define arrival and departure times along with loading and unloading quantities. Then we denote them *paths*. Each path is represented with a single variable that indicates whether a ship make use of the specific path. (Christiansen et al., 2007)

Grønhaug and Christiansen (2009) present an arc flow model for the LNG-IRP. Two complicating factors in the model are separation of the hold of the ships into a number of tanks, and handling boil-off in the tanks. Consequently, a large part of the model handles the ship inventory, where the boil-off is taken into account. Sales and production are decided by variables defined by upper and lower limits, instead of parameters which is the traditional approach in MIRPs. Fodstad et al. (2011) present a richer representation of the LNG-IRP compared to what has been done earlier, and takes a larger part of the supply chain into account. It includes for instance spot trading and contract conditions, and allows for partial loading. This challenges the industry standard of always loading ships to its capacity. The arc flow model by Halvorsen-Weare and Fagerholt (2009) is different in that it has time windows for deliveries and only one production port. The nodes in the model are pre-defined cargoes to be transported. Traditionally nodes represent ports in the model, but since the problem has only one production port, this alteration can be made. Because the model is only able to solve small instances, a decomposition scheme is suggested. It splits the problem into one or more routing subproblems along with a feasibility subproblem, which makes sure solutions are feasible according to berth constraints and inventory levels.

In addition to the arc flow formulation, Grønhaug and Christiansen (2009) also create a path flow formulation. The two models are compared, but neither consistently outperforms the other. To create the paths, a recursive algorithm is developed and implemented. However, due to poor scaling capabilities in the path flow formulation, enumeration of the largest instances proved impossible and the optimizer ran out of memory. Grønhaug et al. (2010) use the same path flow formulation and deploy a branch-and-price method on it. The branch-and-price method consists of a branch-and-bound algorithm where new columns are generated in each node. Initially the column generation starts with a few feasible columns, and iteratively adds new columns to the master problem until no new profitable column can be found. This implies that the optimal linear program (LP) solution is found. Columns are generated by sending dual values to a sub problem, which in turn uses these values to calculate the best feasible route; i.e. the route with the highest reduced cost. That route is returned to the master problem. Testing proves that this method gives a lot better results, on average as much as ten times faster, than with complete enumeration of routes a priori. However, due to the complexity of the LNG-IRP, it is not able to prove optimality for all instances. Andersson et al. (2010a) present a similar path flow model and propose two different methods for solution; a branch-and-price and a rolling horizon heuristic.

Christiansen (1999) solves a MIRP from the ammonia business using a Dantzig-Wolfe decomposition. First, an arc flow formulation is developed. Due to the complexity of the problem, this model is only able to solve small-sized data instances. Therefore, it is reformulated and transformed to a Dantzig-Wolfe decomposition. The problem is split into a master problem and several subproblems. Specifically, a subproblem is created for every ship and for every port where the company owns a facility. They create the ship routes and port visit sequences with the least reduced cost in the master problem. The master problem is a path flow formulation, but different from the ones studied earlier in that it contains two path flow variables, each corresponding to its type of subproblem. Branching is required to obtain the optimal solution with integer requirements. The LP-relaxed solution approach is thus embedded in a branch-and-bound search. Christiansen and Nygreen (1998a,b) are complementary contributions to the paper. Christiansen and Nygreen (1998a) give a detailed description of the solution approach. The master and subproblems are presented, and methods for problem reduction applied. Then the problem is solved by Dantzig-Wolfe column generation along with a branch-and-bound search to obtain integer solutions. The creation and solution of subproblems in the Dantzig-Wolfe decomposition are discussed in Christiansen and Nygreen (1998b). Both the ship routing and the port visiting subproblems are presented in detail.

Andersson (2011) study a maritime pulp distribution problem. A path flow model is presented, and solved by dynamic generation of columns in a branch-and-price tree search. A similar problem is studied by Bredström and Rönnqvist (2010). However, they propose a different solution approach. A path flow model with an a priori route generator is combined with a rolling horizon heuristic. The route generator create partial paths and complete paths are sequentially combined in the master problem. Persson and Göthe-Lundgren (2005) consider a problem concerned with shipping of bitumen products. The solution method is based on a Dantzig-Wolfe decompostion of the problem, and further a branch-and-price algorithm to obtain integer solutions. In addition, the LP relaxation is strengthened by introducing valid inequalities.

Several papers in the literature have demonstrated that introducing branch-and-cut and valid inequalities can affect arc flow formulations positively. Among these are Agra et al. (2012) who present a MIRP from the fuel oil industry. An arc flow model is developed and tested. To improve efficiency, the model is extended by a number of variables and constraints are tightened. Further, cuts are introduced by including valid inequalities. Agra et al. (2011) present a standard arc flow formulation along with a fixed charge network flow problem for a MIRP. Both formulations are strengthened by introducing valid inequalities. Andersson et al. (2012) and Rakke et al. (2012) propose solution methods for the ADP, and present branch-and-cut and branch-and-price-and-cut methods, respectively. The former introduce valid inequalities to a previously developed model, while the latter develop a new decomposition approach for this model and solve it by a branch-price-and-cut method.

Duty flow formulations are not yet seen in maritime transportation literature. However, in the airplane and railway industries it has been used for crew scheduling decisions to minimize labor costs. The book chapter by Barnhart et al. (2003) gives an introduction to the field, and presents standard formulations and solution methods used for air crew scheduling. Desaulniers et al. (1997) were some of the first to implement an exact optimization method in the airplane industry when they developed a crew pairing scheme for Air France and solved it using Dantzig-Wolfe decomposition and branch-and-bound. Ernst et al. (2001) present a crew scheduling problem from the railroad industry. Duties work as the keystone to form full crew schedules and rosters. Due to complexity only a relaxation of the problem is solved.

Back to maritime transportation; Andersson et al. (2010a), Rakke et al. (2011) and Stålhane et al. (2012) present formulations which resemble a duty flow formulation. The papers regard a planning problem for an LNG producer, which controls one production port and serves a fleet of ships to distribute the gas. The formulations use the term voyage for a sailing from pick up port, to delivery port and back to pick up port. A binary variable for selection of voyage is introduced, and the model chooses a combination of voyages that minimizes the company's cost. What separates this from a duty flow formulation is that it does not have a connection between voyages. In the duty flow, duties are connected throughout the period. With the voyages presented in the mentioned papers, only one is allowed at any time, but when one ends it is not explicitly stated that a new one will start.

3 Problem Description

This thesis considers a tactical planning problem within the LNG business. The problem combines ship routing, inventory management and contract handling, and has a planning horizon of six to ten weeks. LNG is purchased based on contract regulations at liquefaction plants and transported by ships to regasification terminals, where the inventory must be maintained within acceptable levels. The problem is approached from the perspective of a company operating in the downstream part of the LNG supply chain. The purpose is to aid the company in making the decisions that maximizes profit. The main decisions are when and where ships should sail, what quantities the company should buy and to which regasification terminal it should deliver. There are, however, restrictions on these decisions, given from contracts and physical conditions in ports and ships.

The problem is thoroughly described in five sections. First an overview of the LNG supply chain is presented to familiarize the reader with the different tiers. Focus is then moved to the specific tiers which this problem primarily concerns. Next, a detailed description of the contract conditions is given. The IRP problem itself is considered complex to solve, and to regard contracts as well introduces even greater complexity. Therefore, some assumptions are made to establish the framework properly. These assumptions are presented along with a summary of the problem characteristics in the last two sections.

3.1 The LNG Supply Chain

Natural gas is formed over millions of years from decomposed organic materials. Today, the gas is trapped in porous rocks underground. Such sources of natural gas are known as reservoirs. Advanced geological studies and geophysical testing are used to determine formations that are possible locations for reservoirs. Exploration wells are then drilled to conduct further studies of the formation. Finally, technical formation characteristics combined with economical analysis determine suitable reservoirs for production.

A reservoir may contain both oil and gas (associated gas) or primarily gas (non-associated gas). Production wells lead the gas from the reservoir to production facilities. The gas is a mix of many components, some of which are undesirable. Therefore, it goes through a process where components like carbon dioxide, sulphur, mercury and water are removed. Eventually, the gas consists predominantly of methane mixed with some heavier components.

To make the transportation of gas by seaborne and landborne vessels a viable option, the gas must be liquefied. This reduces the size of the gas by a factor of around 600, and thus increases the energy density substantially. Liquefaction is carried out at plants that cool the gas to below $\approx -160°$C. During this process, the gas undergoes an even stricter removal of contaminants to avoid freezing and damages on processing equipment.

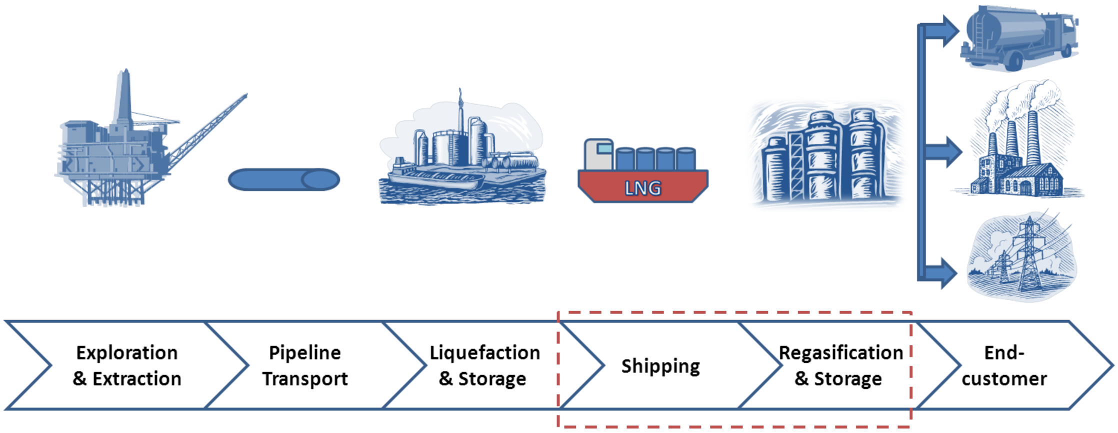
Once liquefied the LNG is loaded directly onto ships or temporarily stored in tanks. In order to ensure predictable supply and demand the ships are often commited to specific projects and serves a set of ports to where the LNG is shipped. They may be owned by the producer or buyer itself, or by designated shipping companies. In addition the in-house fleet, companies may occasionally also employ short-term chartering of ships. Typically, the ships' size are in the range $130,000-150,000\text{m}^3$, but the largest ships today can hold $266,0000\text{m}^3$.

After transportation, the LNG is fed from the ship to a regasification terminal. It can be stored temporarily or used right away. The terminal heats the LNG so that the gaseous phase is again obtained. This process is carefully controlled and monitored. The gas is then compressed and either stored in an underground cavern or sent to a pipeline grid. Storing the gas underground provides flexibility in gas deliveries and works as a buffer for peak demand periods. LNG tanks are expensive to build and to operate, as advanced insulation, cooling and safety measures are required. Caverns can prove more cost-effective, due to lower capital expenses and the possibility to exploit varying electricity rates. LNG can be regasified at low rates and kept in the cavern until further transportation, which also increase available space in the LNG tanks so that ships can unload more often. The pipeline grid takes the gas to its end-user. End-use comprises heating and cooking in the residential and commercial sectors, fuel for gas-turbines in the power generation sector or for a variety of purposes in the industrial sector.

3.2 Liquefaction, Shipping, Storage and Regasification

This section elaborate on the most important supply chain tiers for this thesis and how they relate to the problem at hand. A port where LNG is loaded onto ships is denoted pick-up port. In these ports there is a producer which operates a facility for producing LNG. The produced LNG can be stored in designated tanks and piped to quay, where ships are loaded. For each port there is a defined berth capacity, which determines the maximum number of ships that load or unload simultaneously. A loading operation, including port arrival, docking and departure, can typically take approximately 24 hours. Large storage capacities enable each pick-up port to serve several customers. Inventory is managed by the LNG-producer itself in these ports. Thus, inventory management in pick-up ports is not regarded in

Figure 3: The LNG supply chain



the problem. Rather, pick-up of LNG is regulated through purchase contracts. Contracts are used to ensure predictability in supply, demand and price.

A fleet of heterogeneous ships is available for shipping of the LNG. When the planning period begins, the number of ships is fixed. For this problem, it is therefore not of importance whether the ships are charted or in permanent possession of the company. The ships vary in terms of size, capacity, cost and cruising speed. Moreover, the starting point of each ship is its physical position in the beginning of the planning period. This position may be anywhere on the ocean or in a port. Sailing times between all ports for each ship are based on the distance and the cruising speed of the ship, but not on the load on board.

Physical conditions in ports must comply with certain ship requirements. Due to such ship-port compatibility issues, not all ships can visit all ports. Every ship is assumed to have four tanks that can be stored with LNG. The LNG is stored at atmospheric pressure, but the extremely low temperature calls for tanks that are specially made for the purpose. Since the LNG is at is boiling point during transportation, a small amount is constantly vaporized. This is henceforth referred to as *boil-off*, which is widely utilized as a secondary ship fuel. However, recent technological advances allow ships to be fitted with reliquefaction systems, so that all the cargo remains in a liquid state (Anderson et al., 2009). For this reason, we do not account for boil-off in the problem. Consequently, a ship delivers the exact same volume of LNG as it picked up.

The routing of ships is subject to conditions set in accordance with LNG business practice; 1) A ship will always leave a pick-up port fully loaded and return empty. Successive calls to pick-up ports are thus not permitted. 2) It is possible to unload in succeeding ports, but the number of succeeding delivery ports is limited to two. The reason is that port calls are costly and must therefore be justified with a certain minimum amount of unloaded cargo 3) A ship is permitted to wait maximum two days outside a port before loading or unloading starts. Waiting can be appropriate if a port is already operating at maximum berth capacity, if the current storage level in delivery ports is too high or if interval pick-up limits on contracts in pick-up ports are reached. 4) Since some types of LNG ships experience problems with *sloshing*[1], it is assumed that it is impossible to load and unload partial tanks. Hence, a single tank can be either full or empty on transit, and an integer number of tanks must be loaded and unloaded.

From a pick-up port, the ship carries LNG to regasification terminals, where the LNG is reheated and piped to end-customer. These ports are denoted delivery ports. In delivery ports inventory must be managed to ensure that the storage level is within a given acceptable range. The upper limit represents the maximum storage capacity, whereas the lower limit represents a safety stock. The sales rate is bounded by upper and lower limits given for each day. There should always be sufficient LNG in storage to satisfy daily demand. If the storage is empty, gas must be bought with a price premium directly in the delivery port, in order to comply with demand. Gas prices also vary with time and may differ from port to port. To achieve the best possible profit, gas can therefore be held back and sold at high prices if the capacity allows for it.

3.3 Contracts

Each contract relates to one pick-up port, and several contracts can relate to the same port. Prices and terms in these contracts may vary, but are assumed to be known when the planning period begins. All contracts have a defined start and end time. During this period the contract states upper and lower limits for the volume of LNG that can be purchased and picked up. Occasionally there are also limits within shorter intervals of the total time period. For example, limits can be [300, 400] units for the whole period of 2 months. Yet, volume picked up each month may be allowed within the range of [100, 250] units. The interval pick-up limits are typically less than half the lower total limit and more than half the upper total limit. Such conditions allow for some flexibility in the timing of pick-ups, and thus also flexibility for the routing of ships. If the volume that is picked up does not comply with the defined lower limits, excess gas must be sold at a discounted price directly in the pick-up port.

[1]Sloshing refers to the movement of liquid inside a solid container. Such motions can induce resonance or inertial waves that may inflict destructive forces on the solid.

Contract conditions can define a minimum number of days between pick-ups, to facilitate a fairly even spread on ship arrivals. This is known as the inter-arrival gap. It is particularily important for producers who have limited storage capacity and therefore cannot build up a buffer against abrupt peaks in demand. Another contractual term is the origin-destination clause, which ensures that a given set of ports receive a minimum percentage of the total volume purchased and picked up from the contract. This special set of ports is denoted primary ports. For instance, a company which operatates both liquafaction and regasification terminals, but leaves the shipping to external companies, may want to assure that gas is not sold to a third party through such a clause. This way it can secure supply to its terminals.

The aim for this thesis is to maximize the company's profit, which is a compound calculation. Revenue is generated from sales of gas. Costs are comprised of LNG purchases on contracts, transportation and port fees. Furthermore, the company may be forced to purchase gas in delivery ports to fulfill its sales obligations. Such trade usually requires a price premium compared to gas bought on contract. Transportation costs are assumed to be proportional to the distance covered, whereas the port fee is fixed for each port. The cost of LNG is time and contract dependent. Considering the relatively short planning period of the problem, fixed costs are not taken into account.

3.4 Assumptions

Because of this problem's complexity it has been necessary to make some assumptions. Several of these reflect the LNG business practice, while others are made to establish a fixed framework within which the problem is solved.

- A ship must be loaded to its full capacity in pick-up ports. Thus, it must sail empty from delivery to pick-up port.
- A ship cannot call more than one pick-up port in succession.
- A ship can call up to two delivery ports in succession.
- In delivery ports an integer number of tanks must be unloaded, so that each tank is either full or empty during sailing.
- A ship that calls a delivery port must unload at least one tank.
- A ship that calls a delivery port must unload all the cargo it carries for that port.

- A ship can only wait outside a port before it carries out a port operation (loading or unloading). Thus, after loading or unloading the ship must sail on directly.
- All port operations are assumed to last 24 hours. However, this is included in the sailing time, so port operations take place on the same day as sailing.
- Sailing times, transportation costs and gas prices for purchases and sales are known, but may vary with time.
- There are no costs associated with waiting outside a port. Since the fleet of ships is fixed, the cost of having the ships in disposal is regarded as fixed and independent of whether the ship operates or not. Variable costs associated with daily operations, such as fuel, crew and port fees, are incorporated in the transportation costs.
- The implementation is founded on the following assumptions, but the model is not dependent on these:
  - All ships have four tanks.
  - Routes always have a delivery port as last port call
  - Routes always start in pickup port, and ships are always empty when the planning period starts
  - Storage in delivery ports can not be below a specified lower limit in the end of the planning period, so as to avoid adverse end-of-period effects.

3.5 Summary of Problem

The following list aims to provide a brief overview of the main problem characteristics.

- Maximize profit
  - Purchase, transportation and sales of a single product
  - Planning horizon of six to ten weeks
- Contracts in pick-up ports:
  - Multiple pick-up ports with multiple contracts
  - Pick-up limits for total planning period and shorter intervals
  - Origin-destination clause

  - Inter-arrival gap
  - Purchases at varying, known prices
  - Sales of excess contractual volume below the lower limit at varying, known prices
- Routing of ships:
  - Fixed fleet of heterogeneous ships
  - Known sailing distances and times
  - Sailing and loading rules defined in Section 3.4
- Inventory management in delivery ports:
  - Multiple delivery ports with inventory management in all
  - Upper and lower storage limits
  - Sales of gas at varying, known prices
  - Varying demand for gas

4 Model Description

This section will present three different MIP formulations of the problem; an arc flow, a path flow and a duty flow formulation. First, the arc flow formulation is presented. It describes all aspects of the problem explicitly, because every decision is represented with a designated variable. Therefore, it provides a useful assurance that the problem is modeled in a correct manner, and improvements, extensions or corrections can easily be applied as well. However, experience has shown that it can be time consuming to solve large problem instances with this formulation. Therefore, two alternative formulations are developed, which aim to solve larger problem instances and achieve optimal solutions faster.

The alternative formulations are in principle Dantzig-Wolfe (DW) decompositions where routing and scheduling is handled in a subproblem. Integer and binary requirements are ensured for variables in the subproblem, which is formulated as a MIP. This strengthens the LP relaxation of the models and will possibly enhance their computational efficiency. Instead of using designated variables to represent all the ships' actions, the path flow and duty flow models use one variable to represent a path or a duty. A path contains information about the complete physical sailing route, the times for departure and arrival, the loading and unloading patterns and the costs associated with sailing the route. A duty contains the same information as a complete path, but its sailing route consists only of sailing from one pick-up port via one or two delivery ports, and back to a pick-up port or to the artificial destination node.

One issue regarding path flow formulations is poor scalability. The number of possible paths grows exponentially when the planning period is extended. Hence, it can become demanding to enumerate all the paths. Chances are, however, that the optimization run can compensate for the tedious enumeration process and overall make the path flow formulation favorable. In order to avoid the exponential growth of paths with time, the third formulation replaces paths with duties. This change reduces the number of columns[1] in the problem substantially, and may by that improve the model performance. Although the concept of duties has been applied in other optimization problems, it is not yet applied within the field of the maritime routing as far as the authors know.

The arc flow model was developed first, and has been the basis for both the path and duty flow models. Similarities between the models exist and some constraints and variables are exactly the same. What primarily separates the three models is how the ship routing is modeled. For routing in the arc flow model, only sailing

[1]A column refers to a variable in the problem.

time, sailing cost, port-to-port and port-to-ship feasibility are sent as parameters to the model. The path flow and duty flow models require significantly more input. That is, the master problem is fed with information about each path's or duty's sailing, loading, unloading and costs, which is generated a priori, or dynamically in a subproblem. One decision variable in the path and duty flow models therefore correspond to a set of decision variables in the arc flow model.

The problem to be solved does not have a finite time horizon, whereas the model is proposed as a tactical planning tool for planning periods of six to ten weeks. Therefore, all three models are intended to eventually be applied for planning based on a rolling-horizon principle. Such planning implies to re-run the model with up-to-date parameter values before the initial planning period is over. By doing so, the start of one period will overlap with the end of the previous period, and the solution will then be based on more recent information. This can remove adverse end-effects that often appear in the end of single planning periods. End-effects are decisions that cause distortions to the solution since they do not consider time beyond the defined planning period.

4.1 Arc Flow Formulation

An arc flow formulation must ensure flow balance in every node in all time periods. In this problem, every port represents a node. If a ship calls a port, the same ship will have to leave that port. Time is modeled discretely in time periods and one time period represents one day. We assume that waiting, loading and unloading last one time period. Each time period a ship starts sailing from a port it is also loading or unloading, depending on the type of port it leaves. This way of modeling enables use of time as a common index for several variables and parameters, and makes it easier to keep control of the state of the problem in each time period. On the contrary, it introduces many variables, as several other indices will be combined with the time index.

With regard to notation, decision variables and indices are represented with lower-case letters, whereas parameters and superscripts are characterized with capital letters. Sets are in capital calligraphic letters. Some notational simplifications are made in the model to enhance readability. Variables which are not defined, are regarded as eliminated from the model. For example, $\sum_{i \in N_v | (i,j) \in A_v} x_{ijvt}$ is written $\sum_{i \in N_v} x_{ijvt}$. Furthermore, indices are assumed always to be kept within its defined set. That is for instance, for all values of t in $\sum_{\tau=t}^{t+T^W} w_{iv\tau}$ it is assumed that τ does not run out of range. Some constraints are presented as non-linear to ease readability, but are linearized Section 4.1.4.

The arc flow model was developed in the project thesis written by the same authors; "Optimization of the LNG supply chain with contracts handling" (2011). The mathematical model presented here is based on this previous work.

4.1.1 Sets, Indices, Parameters and Variables

Table 3 provides information about the elements in the arc flow model. The elements are grouped according to type and described briefly. For a more comprehensive explanation of the variables specifically, the reader is referred to Section 4.1.2.

Table 3: Elements in the arc flow model

Sets	
$\mathcal{N}^{\mathcal{D}}$	delivery ports
$\mathcal{N}^{PD}$	pick-up and delivery ports
$\mathcal{V}$	all ships
$\mathcal{N}_v$	all ports ship v can visit
$\mathcal{N}_v^P$	all pick-up ports ship v can visit
$\mathcal{N}_v^D$	all delivery ports ship v can visit
$\mathcal{A}_v$	all arcs ship v can follow
$\mathcal{T}$	time periods
$\mathcal{T}^{SUB}$	a shorter time interval for which contracts may state pick-up limits
$\mathcal{C}$	all contracts
$\mathcal{C}_i$	contracts in port i
$\mathcal{N}_c^A$	primary ports for contract c
Indices	
$o(v)$	origin node for ship v
$d(v)$	destination node for ship v
i, j	port
v	ship
t, τ	time period
Parameters	
T^W	maximum number of successive waiting days outside a port
T_{ijv}	transportation time from port i to port j with ship v
W_v	number of cargo tanks on ship v
B_i^{CAP}	berth capacity in port i
I_{iv}^0	number of full tanks for port i on ship v when the planning period starts
T_c^{INT}	minimum time interval between pick-ups on contract c
Q_v	capacity of ship v

Q_v^W	capacity of one tank on ship v
L_c	minimum fraction of total cargo picked up on contract c that has to be shipped to primary ports
R_{it}^{PORT}	revenue from sales of a unit gas in delivery port i in time period t
R_{ct}^{CON}	revenue from sales of a unit gas from contract c in time period t
C_{ijv}^{SAIL}	cost of sailing from port i to j with ship v, including port fees in port i
C_{ct}^{CON}	cost of purchase of one unit gas from contract c in time period t
C_{it}^{PORT}	cost of purchase of one unit gas in delivery port i in time period t
S_i^0	storage in port i when the planning period starts
$\overline{S}_{it}$	upper limit for storage level in port i in time period t
$\underline{S}_{it}$	lower limit for storage level in port i in time period t
$\overline{Y}_{it}$	upper limit for sales in port i in time period t
$\underline{Y}_{it}$	lower limit for sales in port i in time period t
$\overline{P}_c^T$	upper limit for pick-up from contract c in the planning period
$\underline{P}_c^T$	lower limit for pick-up from contract c in the planning period
$\overline{P}_c^{SUB}$	upper limit for pick-up from contract c in an interval of the planning period
$\underline{P}_c^{SUB}$	lower limit for pick-up from contract c in an interval of the planning period

Variables

x_{ijvt}	$\begin{cases} 1 & \text{if ship } v \text{ starts sailing from port } i \text{ to port } j \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
w_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ waits outside port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
o_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ performs a port operation in port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
p_{cvt}	$\begin{cases} 1 & \text{if cargo is picked up on contract } c \text{ with ship } v \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
z_{ivt}^D	integer number of tanks that are being unloaded in delivery port i from ship v in time period t
z_{ivt}^I	integer number of tanks containing cargo for delivery port i on ship v in time period t
z_{ivt}^P	integer number of tanks that are being loaded for delivery port i on ship v in time period t
f_{civt}	amount of gas purchased from contract c and loaded with destination i on ship v in time period t
y_{ct}^P	amount of gas from contract c sold in pick-up port in time period t
y_{it}^D	amount of gas sold in delivery port i in time period t

e_{it}	amount of gas bought in delivery port i in time period t
s_{it}	amount of gas stored in delivery port i in the end of time period t

4.1.2 Overview of Variables

Simultaneous contract handling, ship routing and inventory management requires many decision variables to model the problem appropriately. To familiarize the reader with main concepts in the model, an overview of the variables and their interconnectivity is presented before moving on to the model formulation. Figure 4 provides a graphical depiction of all the variables.

It is assumed that ships are either sailing towards a port, waiting outside a port or calling a port when the planning period starts. The ship's location in time period 1 is denoted *origin*. When a ship starts sailing, the binary sailing variable x_{ijvt} is assigned value 1. The ship may wait outside a port at arrival. Note that waiting is only possible after a sailing is complete and before a ship calls the port to load or unload. Waiting is represented with the binary variable w_{ivt} and is equal to 1 if the ship waits. However, a ship is only allowed to wait for a maximum number of consecutive days.

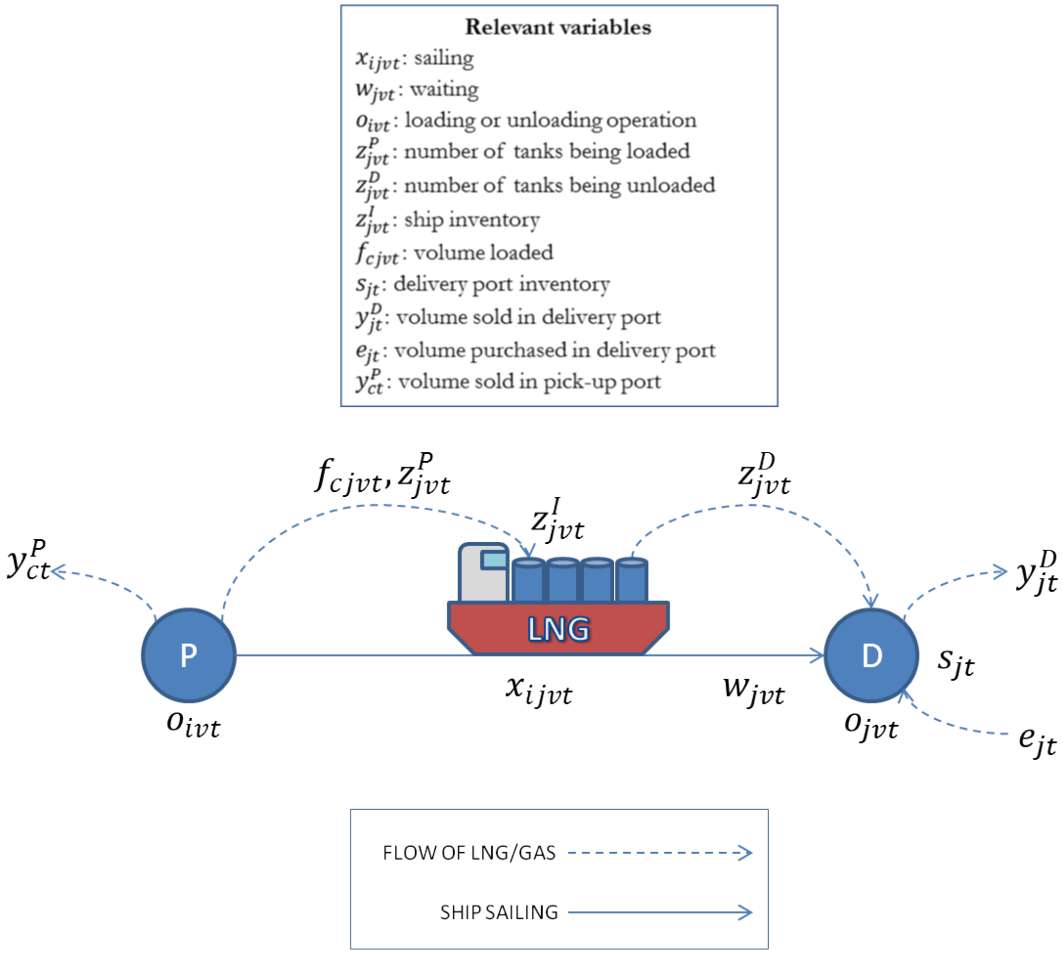
Next, the ship will sail to another port. The same time period that a ship leaves a port, it is required to either load (z^P_{ivt}) or unload (z^D_{ivt}) cargo, depending on which type of port it leaves. A port operation is represented with the binary variable o_{ivt}. In delivery ports ships may unload any integer number of tanks. Each tank is loaded with cargo for a specific port, and the ship can only unload cargo for its intended port. If the ship leaves a pick-up port, it is required to load to its full capacity.

Loading is constrained by requirements stated in purchase contracts. In addition to picking up cargo with the fleet of ships, gas can be sold in the pick-up port. These sales are denoted with the continuous variable y^P_{ct}. This may be profitable or necessary in order to fulfill contract limits. The continuous variable f_{civt} represents the amount of cargo that is being loaded on a ship. It is indexed with the contract for purchase, port of delivery, ship for transportation and time of pick-up. The reason for this is to be able to control several necessities:

1. Ships have to unload an integer number of tanks, and cargo in one tank must therefore be for one destination port only.

2. There are defined limits for the amount of cargo that can be picked up. Such limits can apply both for the total duration of the contract and for shorter time intervals.

Figure 4: Variables illustrated



3. Contracts often state that a certain percentage of the cargo must be delivered to a set of primary ports.

The ship inventory variable z^I_{ivt} represents the number of loaded tanks for a specific port onboard a ship at any given time. If a ship loads in one time period, the variable z^P_{ivt} represents the number of tanks that is being loaded. The variable z^I_{ivt} is then increased accordingly. Equivalently, if a ship unloads, the variable z^D_{ivt} represents the number of tanks that is being unloaded, and inventory variable z^I_{ivt} is decreased accordingly. In a time period where no port operations take place, the variable z^I_{ivt} will stay unchanged and maintain its value from the previous time period. Note that z^P_{ivt}, z^D_{ivt} and z^I_{ivt} are all attached with an address index i for the cargo. This facilitates delivery of cargo to the correct ports.

The inventory level s_{it} is updated every time period. Initial inventory, unloading from ships, sales and purchases of gas in the port determines the inventory level. It increases with $Q_v z^D_{ivt}$ when ships arrive and deliver their cargo. If that is not sufficient to maintain a level above the lower limit, additional gas can be bought directly in the port. The continuous variable e_{it} represents such purchases. Likewise, the inventory level decreases when gas is sold. Sales are represented with the continuous variable y^D_{it}.

4.1.3 Mathematical Model

For enhanced readability and overview, constraints in the mathematical model are grouped together and described in sequence. The full model without any comments or descriptions is provided in Appendix A.

$$\sum_{j \in \mathcal{N}_v} x_{o(v)jv1} = 1 \qquad \forall v \in \mathcal{V} \tag{1}$$

$$\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}_v} x_{id(v)vt} = 1 \qquad \forall v \in \mathcal{V} \tag{2}$$

All ships are initialized by sailing from an artificial origin node in the start of the planning period. This node indicates the ship's initial position, which can be somewhere at sea or in a port. Constraints (1) make sure this sailing is undertaken in the first time period. To indicate that a route ends, the final sailing for every ship is to an artificial destination node. This sailing can be done at any time in the planning period, as given by constraints (2).The sailing to the destination node is not an actual sailing. Instead, it acts as a mechanism for the ships to exit the problem. The last port of call is the ship's physical position when the planning period ends.

$$\sum_{j \in \mathcal{N}_v} x_{ijvt} + w_{ivt} - \sum_{j \in \mathcal{N}_v} x_{jiv(t-T_{jiv})} - w_{iv(t-1)} = 0 \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{3}$$

$$\sum_{\tau=t}^{t+T^W} w_{iv\tau} \leq T^W \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{4}$$

Ships can sail between ports given from each ship's set of possible arcs, A_v. Sailing between ports is regulated by constraints (3). They state that a ship has two options if it arrives a port. The first option is to wait outside the port. Constraints (4) limits the number of successive waiting days. The second option is to call the port and sail further in the same time period.

$$\sum_{j \in \mathcal{N}_v} x_{ijvt} - o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (5)$$

$$\sum_{v \in \mathcal{V}} o_{ivt} \leq B_i^{CAP} \qquad \forall i \in \mathcal{N}^{PD}, t \in \mathcal{T} \qquad (6)$$

$$z_{ivt}^D - W_v o_{ivt} \leq 0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (7)$$

$$z_{ivt}^D - o_{ivt} \geq 0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (8)$$

Loading or unloading is carried out in the same time period as the ship leaves the port. This requirement is controlled by constraints (5). All ports have a given berth capacity, which indicates the maximum number ships performing port operations at the same time. Constraints (6) make sure this capacity is not exceeded. When a ship is unloading, it is ensured that it unloads at least one tank and at most the total number of tanks on the ship. The unloading requirements are complied with through constraints (7) and (8).

$$z_{iv1}^I + z_{iv1}^D - z_{iv1}^P = I_{iv}^0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V} \qquad (9)$$

$$z_{ivt}^I - z_{iv(t-1)}^I + z_{ivt}^D - z_{ivt}^P = 0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \setminus \{1\} \qquad (10)$$

In the start of the planning period a ship may be fully loaded, partly loaded or empty. The number of tanks initially loaded on a ship for a specific port is given by the parameter I_{iv}^0, which initializes the ship inventory in constraints (9). Further, the ship inventory is adjusted throughout the planning period by constraints (10). Ship inventory is updated in the end of every time period, i.e. all loading and unloading that has taken place during that period is included in the ship inventory.

$$o_{ivt} z_{ivt}^I = 0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (11)$$

$$\sum_{j \in \mathcal{N}_v^P} x_{ijvt} \sum_{j \in \mathcal{N}_v^D} z_{jvt}^I = 0 \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (12)$$

$$x_{ijvt} \left(W_v - z_{ivt}^D - \sum_{\tau=t}^{t+T^W} z_{jv(\tau+T_{ijv})}^D \right) = 0 \qquad \forall i, j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (13)$$

All loaded cargo tanks on a ship are fitted with an address label that tells us to what port the cargo is designated. Ships sail to the ports which they are loaded for, and constraints (11) make sure that they unload all tanks addressed to the port of call. The requirement for ships to arrive pick-up ports empty in order to load a full ship is controlled by constraints (12). This implies that a ship must sail directly to a pick-up port or its destination node after unloading its last cargo. According to industry practice, a maximum of two delivery ports can be visited in succession. Constraints (13) make sure of this.

$$\sum_{c \in \mathcal{C}_i} \sum_{j \in \mathcal{N}_v^D} f_{cjvt} - Q_v o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \tag{14}$$

$$\sum_{c \in \mathcal{C}} f_{cjvt} - Q_v^W z_{jvt}^P = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{15}$$

Ships must load all its tanks in pick-up ports, which is controlled by constraints (14). Further, to comply with industry practice and previous requirements ships must unload an integer number of tanks in delivery ports. Consequently, only cargo with the same destination can be loaded into the same tank, as specified in constraints (15).

$$\underline{P}_c^T \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^T \qquad \forall c \in \mathcal{C} \tag{16}$$

$$\underline{P}_c^{SUB} \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}^{SUB}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^{SUB} \qquad \forall c \in \mathcal{C}, \mathcal{T}^{SUB} \subset \mathcal{T} \tag{17}$$

$$\sum_{i \in \mathcal{N}_c^A} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} f_{civt} - L_c \left(\sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{ct}^P) \right) \geq 0 \qquad \forall c \in \mathcal{C} \tag{18}$$

All contracts have given upper and lower limits for how much that can be picked up during the planning period. Occasionally, contracts also state limits for shorter time intervals within the planning period. The total limits for pickups are given by constraints (16), while interval limits are controlled by constraints (17). In addition to pick-up limits, contracts often specify a set of primary delivery ports to where a defined share of volume purchased and picked up must be shipped. This is referred to as an origin-destination clause, and is controlled by constraints (18).

$$\sum_{i \in \mathcal{N}_v^D} f_{civt} - \overline{P}_c^T p_{cvt} \leq 0 \qquad \forall c \in \mathcal{C}, v \in \mathcal{V}, t \in \mathcal{T} \tag{19}$$

$$\sum_{v \in \mathcal{V}} \sum_{\tau = t}^{t + T_c^{INT}} p_{cv\tau} \leq 1 \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \tag{20}$$

Inter-arrival gap is another contract condition to be handled. Here, it is ensured by constraints (19) and (20). The former check if cargo is picked up from a contract in a certain time period, while the latter ensure that the time between two pick-ups on each contract satisfies the inter-arrival gap.

$$s_{i1} - \sum_{v \in \mathcal{V}} Q_v^W z_{iv1}^D + y_{i1}^D - e_{i1} = S_i^0 \qquad \forall i \in \mathcal{N}^D \tag{21}$$

$$s_{it} - s_{i(t-1)} - \sum_{v \in \mathcal{V}} Q_v^W z_{ivt}^D + y_{it}^D - e_{it} = 0 \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \mid t \neq 1 \tag{22}$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \tag{23}$$

$$\underline{Y}_{it} \leq y_{it} \leq \overline{Y}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \tag{24}$$

Inventory management is carried out in all delivery ports. Gas can be sold, bought or delivered in every time period, and the amount in storage always has to satisfy specified limits. Constraints (21) and (22) initialize and update the storage variable throughout the planning period. The inventory has an initial level when planning period starts, and is then updated in the end of every time period. Thus, all flows in or out of storage in a specific time period is reflected in the storage level for the same time period. The storage level and the amount sold must be within upper and lower limits for all time periods. Constraints (23) and (24) specify these limits.

$$x_{ijvt} \in \{0, 1\} \qquad \forall (i, j) \in \mathcal{A}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{25}$$

$$w_{ivt} \in \{0, 1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{26}$$

$$o_{ivt} \in \{0, 1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{27}$$

$$z_{ivt}^D \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{28}$$

$$z_{ivt}^P \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{29}$$

$$z_{ivt}^I \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{30}$$

$$p_{ct} \in \{0, 1\} \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \tag{31}$$

A number of variables in the formulation is subject to binary and integer requirements, given in constraints (25) - (31). Variables not listed above are all defined as continuous and non-negative.

$$
\begin{aligned}
\max \pi = & \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} R^{PORT}_{it} y^D_{it} + \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} R^{CON}_{ct} y^P_{ct} \\
& - \sum_{(i,j) \in \mathcal{A}_v} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} C^{SAIL}_{ijv} x_{ijvt} - \sum_{c \in \mathcal{C}} \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} C^{CON}_{ct} (f_{civt} + y^P_{ct}) \\
& - \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} C^{PORT}_{it} e_{it}
\end{aligned} \tag{32}
$$

The objective function (32) represents the total profit from all operations. It comprises revenues from sales and costs related to purchase and sailing. Revenues from regular sales and sales in pick-up ports are represented in the first and second term. Operational costs of sailing between two ports are cumulated in one parameter C^{SAIL}_{ijv} and accounted for in the third term. The last terms are costs related to purchase in pick-up and delivery ports.

4.1.4 Linearization of Non-Linear Constraints

Constraints (11), (12) and (13) are non-linear. These were listed in a non-linear manner to facilitate the readability. Here, we have formally linearized them in order to ease the computations and to be able use available commercial software for MIP problems.

$$
o_{ivt} z^I_{ivt} = 0 \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{11}
$$

$$
z^I_{ivt} - W_v(1 - o_{ivt}) \leq 0 \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{33}
$$

Constraints (33) are the linearized reformulation of constraints (11).

$$
\sum_{j \in \mathcal{N}^P_v} x_{ijvt} \sum_{j \in \mathcal{N}^D_v} z^I_{jvt} = 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{12}
$$

$$
\sum_{j \in \mathcal{N}^D_v} z^I_{jvt} - W_v(1 - \sum_{j \in \mathcal{N}^P_v} x_{ijvt}) \leq 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{34}
$$

Constraints (34) are the linearized reformulation of constraints (12).

$$
x_{ijvt} \left(W_v - z^D_{ivt} - \sum_{\tau = t}^{t + T^W} z_{jv(\tau + T_{ijv})} \right) = 0 \quad \forall i, j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{13}
$$

$$\left(W_v - z^D_{ivt} - \sum_{\tau=t}^{t+T^W} z^D_{jv(\tau+T_{ijv})}\right) - W_v(1 - x_{ijvt}) \leq 0 \quad \forall i,j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \quad (35)$$

Constraints (35) are the linearized reformulation of constraints (13).

4.2 Path Flow Formulation

The path flow formulation make use of one variable which incorporates all the ship's actions. This variable indicates if a specific ship utilize one specific path. A path includes information about a ship's geographical routing, the times for port operations and sailings, the times and quantities for loading and unloading in pick-up and delivery ports, as well as the costs related to the path. This information is represented with parameters in the model. Many of the variables from the arc flow model are thus recognized as parameters in the path flow model.

We have developed a path generation tool to create all the possible paths. The paths are then fed to the optimization problem, known as the master problem. The path generator ensures that the paths satisfy constraints regarding routing, loading and unloading. These constraints can therefore be removed in the master problem. As a result, there are fewer constraints in the path flow formulation compared to that of the arc flow. However, there will be considerably more variables since each possible path corresponds to a variable.

4.2.1 Sets, Indices, Parameters and Variables

For this model, the path generator returns four matrices. Three of these correspond to variables in the arc flow model; whether port operations are taking place (O_{ivtr}), the number of tanks loaded (Z^P_{jvtr}) and the number of tanks unloaded (Z^D_{jvtr}). The last matrix gives the total cost for a particular path (C^{SAIL}_{vr}). Furthermore, the path variable λ_{vr} is introduced, where the r-index represents a unique path. The model is designed to select a single optimal path or an optimal combination of paths, subject to contractual requirements. The list of sets, indices, parameters and variables from the arc flow model complements Table 4.

Table 4: New elements in the path flow model

Sets	
$\mathcal{R}_v$	paths that ship v can utilize
Indices	
r	path
Parameters	
O_{ivtr}	$\begin{cases} 1 & \text{if port operations take place in port } i \text{ with ship } v \text{ on path } r \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
Z^P_{ivtr}	number of tanks loaded for port i on ship v on path r in time period t
Z^D_{ivtr}	number of tanks unloaded in port i from ship v on path r in time period t
C^{SAIL}_{vr}	cost associated with utilizing path r with ship v
Variables	
λ_{vr}	indicates if ship v utilize path r

4.2.2 Mathematical Model

The model comprises some constraints that are identical to those in the arc flow formulation, some constraints that are slightly adjusted and some that are completely new. Specifically, in the adjusted constraints, the variables o_{ivt}, z^P_{ivt} and z^D_{ivt} in the arc flow model are replaced by the corresponding parameters multiplied by the path variable. For instance, where the arc flow model use variable o_{ivt}, the path flow model will use $O_{ivtr}\lambda_{vr}$. To clarify this link in the text, constraints in the arc flow model will be numbered in brackets after the corresponding constraints in the path flow model. The complete model is listed in Appendix A.

$$\begin{aligned} \text{constraints } (16) - (20) \\ (23) - (24) \end{aligned}$$

These constraints are re-used from the arc flow model and refers to; the constraints (16) and (17) for pick-up limits; constraints (18) for origin-destination clause; constraints (19) and (20) for inter-arrival gap; and constraints (23) and (24) for inventory and sales limits.

$$\sum_{r\in\mathcal{R}_v}\sum_{v\in\mathcal{V}} O_{ivtr}\lambda_{vr} \leq B_i^{CAP} \qquad \forall i \in \mathcal{N}^{PD}, t \in \mathcal{T} \qquad (36)$$

$$\sum_{c\in\mathcal{C}_i}\sum_{j\in\mathcal{N}_v^D} f_{cjvt} - Q_v \sum_{r\in\mathcal{R}_v} O_{ivtr}\lambda_{vr} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \qquad (37)$$

$$\sum_{c\in\mathcal{C}} f_{cjvt} - Q_v^W \sum_{r\in\mathcal{R}_v} Z_{jvtr}^P\lambda_{vr} = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (38)$$

Berth capacity in the ports is satisfied by constraints (36) [(6)]. It is required to load a full ship when loading, which is taken care of by constraints (37)[(14)]. Further, since a single tank can only be either full or empty, constraints (38)[(15)]ensure that an integer number of tanks is loaded to delivery ports.

$$s_{i1} - \sum_{v\in\mathcal{V}}\sum_{r\in\mathcal{R}_v} Q_v^W Z_{iv1r}^D\lambda_{vr} + y_{i1}^D - e_{i1} = S_i^0 \qquad \forall i \in \mathcal{N}^D \quad (39)$$

$$s_{it} - s_{i(t-1)} - \sum_{v\in\mathcal{V}}\sum_{r\in\mathcal{R}_v} Q_v^W Z_{ivtr}^D\lambda_{vr} + y_{it}^D - e_{i1} = 0 \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \mid t \neq 1 \quad (40)$$

Constraints (39)[(21)]initiate the storage variable, and constraints (40)[(22)]ensure storage balance throughout the planning period.

$$\sum_{r\in\mathcal{R}_v} O_{ivtr}\lambda_{vr} \in \{0,1\} \qquad \forall i \in \mathcal{N}^{PD}, v \in \mathcal{V}, t \in \mathcal{T} \qquad (41)$$

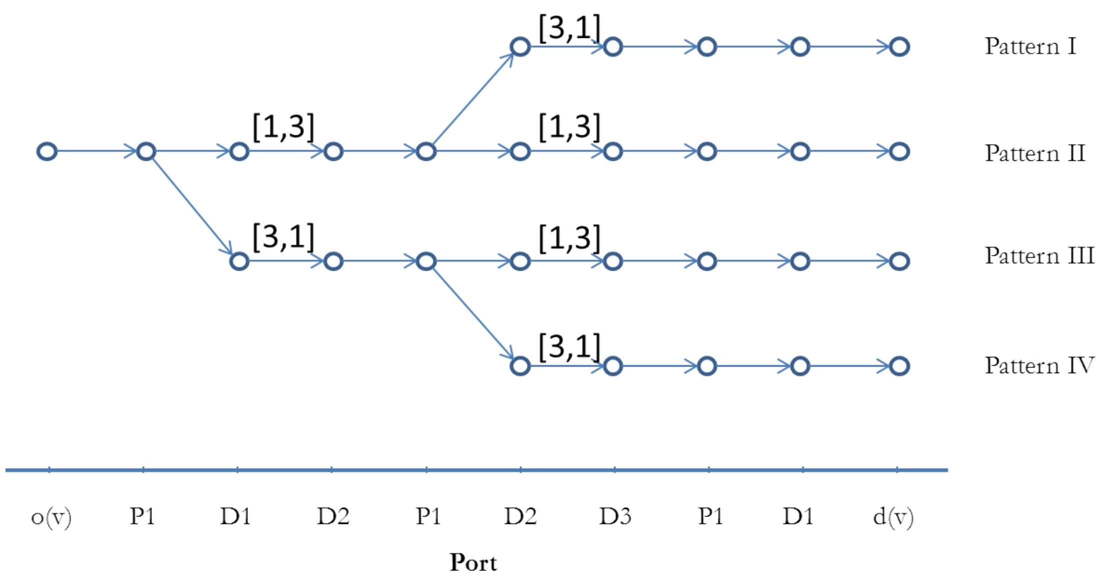
$$\sum_{r\in\mathcal{R}_v} Z_{ivtr}^P\lambda_{vr} \in \{0,...,W_v\} \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (42)$$

$$\sum_{r\in\mathcal{R}_v} \lambda_{vr} = 1 \qquad \forall v \in \mathcal{V} \qquad (43)$$

Constraints (41) ensure that the binary requirement on port operations is satisfied, and constraints (42) make sure that only an integer number of tanks can be loaded to every delivery port. The convexity constraints (43), in combination with (41) and (42), limit the ships to utilize either a single path or two paths combined.

The reason why constraints (41), (42) and (43) are introduced is elaborated in the following. Since an integer number of tanks is loaded and unloaded, a ship's four tanks can be distributed in three different patterns each time a ship calls two subsequent delivery ports; [1,3], [2,2] or [3,1]. This means that the ship can have; 1) One tank for the first port and three tanks for the second port; 2) Two tanks

Figure 5: Increasing number of patterns with calls to delivery ports



for each of the ports, or; 3) Three tanks for the first port and one tank for the second. When generating possible paths, only the extremal loading and unloading pattern [1,3] and [3,1] are considered. From this, a convex combination of paths that have the exact same sailing schedule, i.e. geographical route and timing, can form the pattern [2,2], due to constraints (41), (42) and (43). Resultantly, a path variable's value is in the set $\{0, 0.5, 1\}$.

Figure 5 illustrates the expansion of patterns when two delivery ports are called in succession. The route is $o-P-D-D-P-D-D-P-D-d$, where o is origin, P is pick-up port, D is delivery port and d is destination. Generally, the number of unique patterns are 3^n, where n is the number of times two delivery ports are visited in a row. Thus, there are nine unique loading patterns for this route. However, as the figure illustrates, we only need to generate four of these patterns to obtain all patterns through convex combinations. The number of patterns we generate is thus reduced to 2^n. As an example, if pattern I and II are combined, the first two delivery ports will have one and three tanks, respectively, whereas the two next delivery ports will have two tanks each. Likewise, pattern II and IV combined will result in two tanks unloaded in all delivery ports, except for the last port. The last port will have four tanks delivered in all patterns, since it is the only delivery port visited on that particular duty.

The path variable λ_{vr} could also have been defined as binary. Defining it binary, however, would require generation of all possible loading and unloading patterns. Since the number of unique paths grow exponentially with the number of subsequent calls to delivery ports, this would result in a large number of paths. Generating a large number of paths has shown to be time and memory consuming, and it is also demanding for the optimization software to handle a large number of path variables.

$$\max \pi = \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} R_{it}^{CON} y_{it}^{D} + \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} R_{ct}^{CON} y_{ct}^{P} - \sum_{v \in \mathcal{V}} \sum_{r \in \mathcal{R}_v} C_{vr}^{SAIL} \lambda_{vr} - \sum_{c \in \mathcal{C}} \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} C_{ct}^{CON} (f_{civt} + y_{ct}^{P}) - \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} C_{it}^{PORT} e_{it} \quad (44)$$

Profit is maximized in the objective function (44). The third term differs from the arc flow model. Here, sailing costs are related to the selected path, while in the arc flow model it represents the sum of costs for every arc that is traversed.

4.3 Duty Flow Formulation

A variation of the path flow formulation is to look at duties instead of paths. Grønhaug and Christiansen (2009) define a duty as

> "(...) a journey which starts when a ship either loads all its cargo tanks in a pick-up port or leaves the origin node, and the duty ends immediately before the ship starts loading in the next pick-up port or when the ship reaches the destination node."

In practice, a path is a combination of duties and all paths generated in the path flow formulation are possible to create in the duty formulation by combining the right duties. However, Grønhaug and Christiansen (2009) still generate complete paths and use duties only as a mean to handle ship inventory and boil-off properly. Specifically, when a path is generated, all duties within the path is identified to calculate the correct amount of cargo that is unloaded in each port. In this thesis, on the contrary, we will completely substitute paths with duties and attach information about sailing, loading, unloading and cost to each duty.

4.3.1 Sets, Indices, Parameters and Variables

The alteration from complete paths to duties requires some modifications both in the mathematical formulation and in the route generator. Both formulations

Table 5: New elements in the duty flow model

Sets	
$\mathcal{D}_v$	all possible duties for ship v
$\mathcal{D}_v^S$	possible start duties for ship v
Indices	
d	duty
Parameters	
H_{ivtd}	$\begin{cases} 1 & \text{if duty } d \text{ for ship } v \text{ starts in time period } t \text{ in port } i \\ 0 & \text{otherwise} \end{cases}$
E_{ivtd}	$\begin{cases} 1 & \text{if duty } d \text{ for ship } v \text{ ends in time period } t \text{ in port } i \\ 0 & \text{otherwise} \end{cases}$
Variables	
λ_{vd}	indicates if ship v utilize duty d

parametrize the decision variables from the arc flow model concerning sailing route and loading and unloading patterns, i.e. X_{ijvt} O_{ivt} Z_{ivt}^P Z_{ivt}^D. These parameters' values are determined in the path generation. The difference is that duties can be combined in order to form a complete ship schedule. Combining duties is eased by introducing another two boolean parameters; start-of-duty H_{ivtd} and end-of-duty E_{ivtd}. H_{ivtd} indicates the port and time period in which duty d for ship v starts, whereas E_{ivtd} indicate the port and time period in which duty d for ship v ends.

4.3.2 Mathematical Model

For the sake of brevity, only the most characterizing constraints are described here. Notice that the path variable is substituted with a duty variable λ_{vd}. The duty variable is continuous and indicates whether ship v make use of duty d. The complete mathematical duty flow model is found in Appendix A.

$$\begin{aligned} \text{constraints } (16) - (20) \\ (23) - (24) \end{aligned}$$

The duty flow formulation include some constraints re-used from the arc flow formulation; the constraints (16) and (17) for pick-up limits; constraints (18) for origin destination clause; constraints (19) and (20) for inter-arrival gap; and constraints (23) and (24) for inventory and sales limits.

$$\text{constraints } (36) - (42)$$

Furthermore, a range of constraints are re-used from the path flow formulation, with the set of routes R_v, indexed r, replaced by the set of duties D_v, indexed d. These are constraints (36) for berth capacity, (37)-(38) for loading, (39)-(40) for port inventory, and (41)-(42) for binary and integer requirements on port operations and loading.

$$\sum_{d \in \mathcal{D}_v^S} \lambda_{vd} = 1 \qquad \forall v \in \mathcal{V} \tag{45}$$

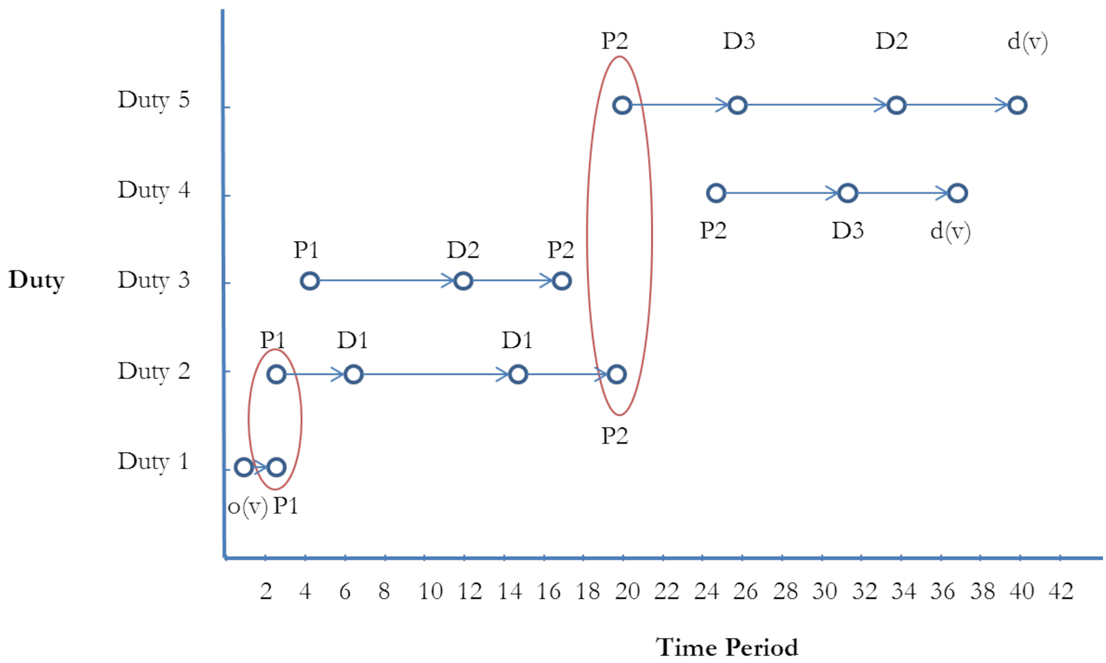
$$\sum_{d \in \mathcal{D}_v} E_{ivtd} \lambda_{vd} - \sum_{d \in \mathcal{D}_v} H_{ivtd} \lambda_{vd} = 0 \qquad \forall i \in \mathcal{N}^P, v \in \mathcal{V}, t \in \mathcal{T} \tag{46}$$

In order to allow for a ship to utilize a sequence of duties rather than one complete path, constraints (43) from the path flow model are replaced with constraints (45) and (46). Constraints (45) ensure that each ship starts with a duty chosen from a set of possible starting duties. That is, duties from the origin node to the first port of call. The linking of duties is handled by constraints (46). For two duties to be linked, the end-of-duty for one duty must be in the same time period and port as the start-of-duty for the next duty. Linking of duties can only occur in a pick-up ports.

A linked sequence of duties is graphically depicted in Figure 6. In addition some duties not included in the selected sequence are illustrated. *Duty 1* is the only duty starting in the origin node. Hence, this has to be chosen as the starting duty. Apart from *duty 1*, all duties start with sailing from a pick-up port. This implies that the ship undertakes a port operation and loads all its tanks in the starting time period of the duty. *Duty 1* has the end-of-duty in $P1$ in the same time period as *duty 2* has the start-of-duty in $P1$. *Duty 3* also has the start-of-duty in $P1$, but in a later time period. *Duty 1* can thus not be linked to *duty 3*. When *duty 2* ends, it is linked with *duty 5* in $P2$. *Duty 5* ends in the destination node. Therefore, it cannot have a subsequent duty. Duties that end in the destination node are the only ones allowed as the last duty of the ship.

Similar to the path flow model, constraints (41) and (42) make sure that only duties with the same schedule can be combined. For duties with subsequent delivery port calls, only patterns which unload one tank to the first port of call and three tanks to the second, or vice verse, are generated. In the generation of duties, care has

Figure 6: Linking of duties



been taken to ensure that the maximum number of waiting days outside a port is not violated. Furthermore, waiting outside a pick-up port is allowed only in the end of a duty. Hence, up to T^W waiting days are followed by the end-of-duty. The next duty then starts immediately with sailing from the port.

As an alternative to a priori generation, columns can be generated dynamically. This approach is based on an iterative process where one optimization problem, known as the subproblem, generates the currently best new column and then feeds it to the master problem. Generation of this new column is based on information sent from the master problem to the subproblem. The iterative cycle ends when no column can be found that has a beneficial reduced cost. This implementation approach is tested for the duty flow formulation and described in Section 6.3.

5 Illustration of the Models

To provide a better understanding of the models, this section presents a solution from each model. We have arbitrarily selected the solutions for problem instance $A1$ (see Section 7.1). That instance contains one ship, denoted $S1$, one pick-up port, denoted $P1$, and three delivery ports, denoted $D1$, $D2$ and $D3$. The length of the planning period is 42 time periods, which is further split in two time intervals of equal length. Each time period represents one day. Three contracts for pick-up of LNG are included; $C1$ in the first time interval; $C2$ in the second; and $C3$ which extends to the whole planning period.

Optimal decisions for ship scheduling are presented separately for each model. First, we present the solution from the arc flow model and elaborate in detail on the variable values. Second, the optimal path in the path flow model is provided, along with an explanation of the pre-generated matrices. Third, optimal duties from the duty flow model are given, and we explain how they are linked to form a complete path. Last, the common denominators in the models are presented; namely, inventory management and contract handling associated with the solutions.

The solutions obtained by each model are identical by objective function value. However, small differences in the schedules may occur due to the timing of waiting days. Since there is no cost associated by waiting, there exist many symmetrical solutions and, as we will see from the different solutions, the models do not always find the exact same one.

5.1 The Arc Flow Model

Routing and scheduling in the arc flow model are governed by multiple different variables. Specifically, there are variables for sailing, port operations, ship inventory, loading, unloading and waiting. The following provides an overview of these variables in the solution of problem instance $A1$.

Figure 7 illustrates the optimal schedule through a time-space network, where sailing (x_{ijvt}) and waiting (w_{ivt}) variables are noted explicitly. The schedule is comprised of two visits to $P1$, and one visit to each of the delivery ports. Three waiting days are scheduled, all in the first time interval, which lasts until day 21. These waiting days are scheduled in order to visit $P1$ only once every time interval, so as to satisfy contract conditions for interval pick-up limits.

To give a closer explanation of the rest of the variables, the first duty of the schedule is represented in detail in Figure 8. In accordance with assumptions previously stated for this problem, a ship can never visit a port without undertaking a port operation. Therefore, a port operation (o_{ivt}) is carried out in each time period that a ship departs a port. Essentially, this means that the ship loads and unloads when leaving a pick-up and delivery port, respectively. Port operations are conducted in; 1) $P1$, when $S1$ sails for $D2$; 2) $D2$, when $S1$ sails for $D1$; and 3) $D1$ when $S1$ sails back to $P1$.

The loading (z^P_{ivt}) and unloading (z^D_{ivt}) variables govern the ship inventory (z^I_{ivt}) variable through constraints (9) and (10)). These variables are indexed with an address i, which indicates where the cargo is to be delivered. When the ship is fully loaded in $P1$ on day 2, the values of the loading variable show that one tank is loaded to $D1$ and the other three tanks to $D2$. Thus, the ship inventory variable is updated accordingly on day 2. The ship inventory remains unchanged until day 10, when three tanks are unloaded in $D2$. Likewise, the ship inventory changes when the last tank is unloaded in $D1$ on day 14. Then, the ship returns empty

Figure 7: Solution of instance $A1$ for the arc flow model

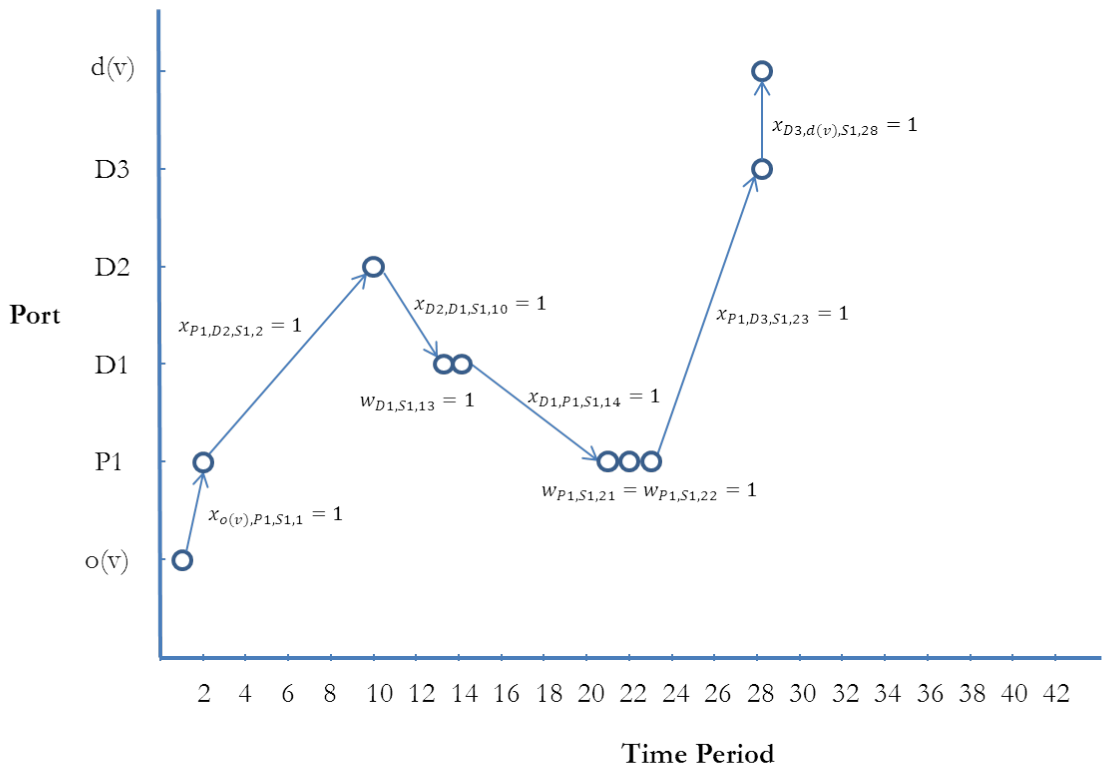
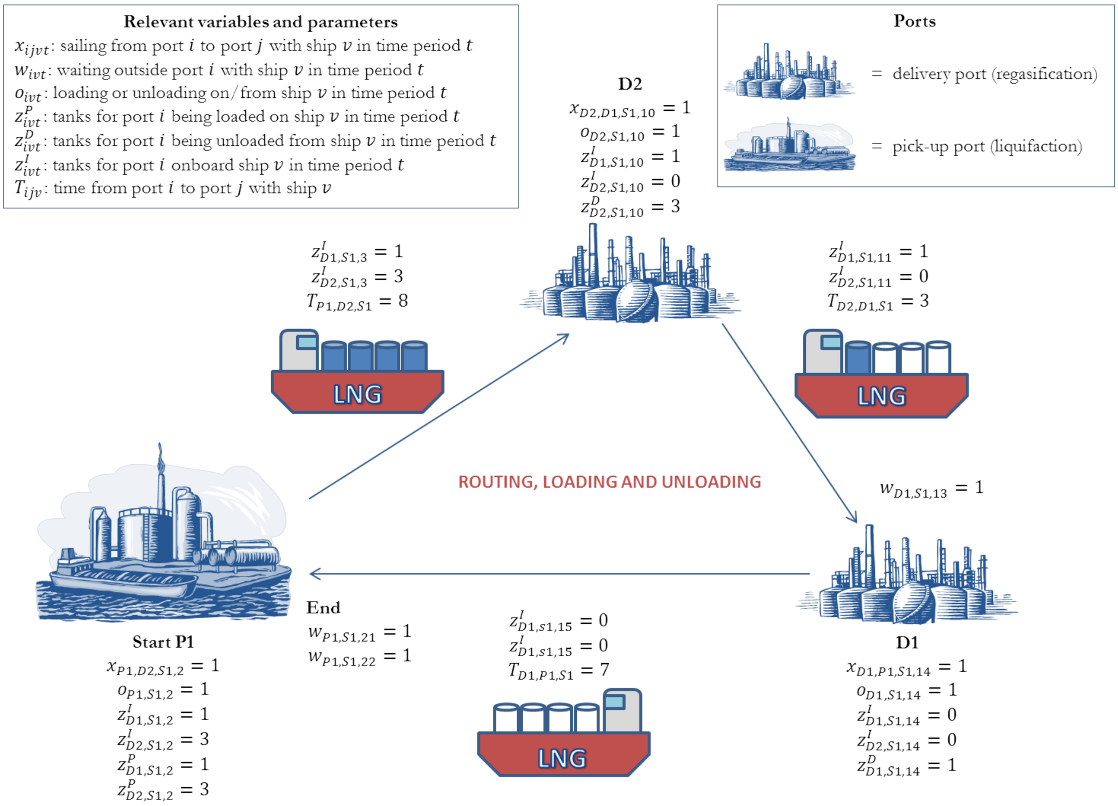


Figure 8: First duty in the solution from the arc flow model

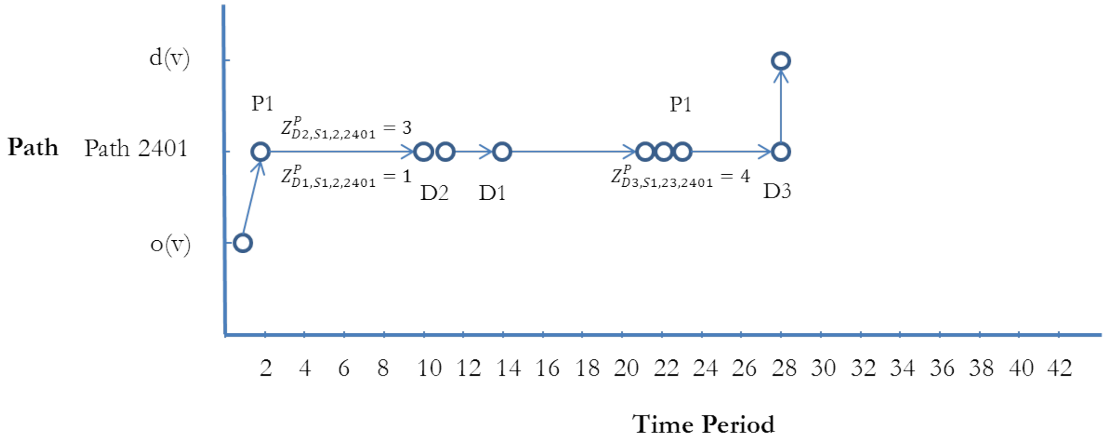


to $P1$. Notice that the ship inventory variable is updated at the end of each time period. This implies that loading or unloading in one time period is reflected in the ship inventory variable for the same time period. If no port operation takes place, the value of the ship inventory variable is carried forward to the next time period. For illustrative purposes, its value is indicated only for the time period of a port call and the subsequent time period in the figure.

5.2 The Path Flow Model

The schedule from the path flow model is almost identical to the one from the arc flow model. The only difference is that one waiting day is moved. Here, a waiting day occurs on day 10 outside $D2$, instead of day 13 outside $D1$. All decisions regarding routing, loading and waiting in the path flow model are governed by the path variable (λ_{vr}). Each unique path is represented with a variable, which for this particular instance results in more than 65,000 path variables.

Figure 9: Solution of instance $A1$ for the path flow model



Information about each path is fed to the model as parameter matrices. The selected, optimal path is number 2401. Thus, variable λ_{vr} equals 1 for $r = 2401$ and 0 for all other values of r. Index $r = 2401$ in the port operations matrix (O_{ivtr}) represents the ship's schedule for this path, whereas index $r = 2401$ in the loading (Z^P_{ivtr}) and unloading (Z^D_{ivtr}) matrix governs to the quantity, time and place of cargo delivery. Since there is only one ship in this problem instance, index v is always $S1$. In Figure 9, the quantities loaded are specified. To illustrate, these are the matrix values for the first and second port calls in $P1$:

First call	**Second call**
$O_{P1,S1,2,2401} = 1$	$O_{P1,S1,23,2401} = 1$
$Z^P_{D1,S1,2,2401} = 1$	$Z^P_{D1,S1,23,2401} = 0$
$Z^P_{D2,S1,2,2401} = 3$	$Z^P_{D2,S1,23,2401} = 0$
$Z^P_{D3,S1,2,2401} = 0$	$Z^P_{D3,S1,23,2401} = 4$

In the first port call, one tank is loaded for $D1$ and three tanks are loaded for $D2$, whereas no tanks are loaded for $D3$. On the contrary, all the tanks are loaded for $D3$ in the second call to $P1$.

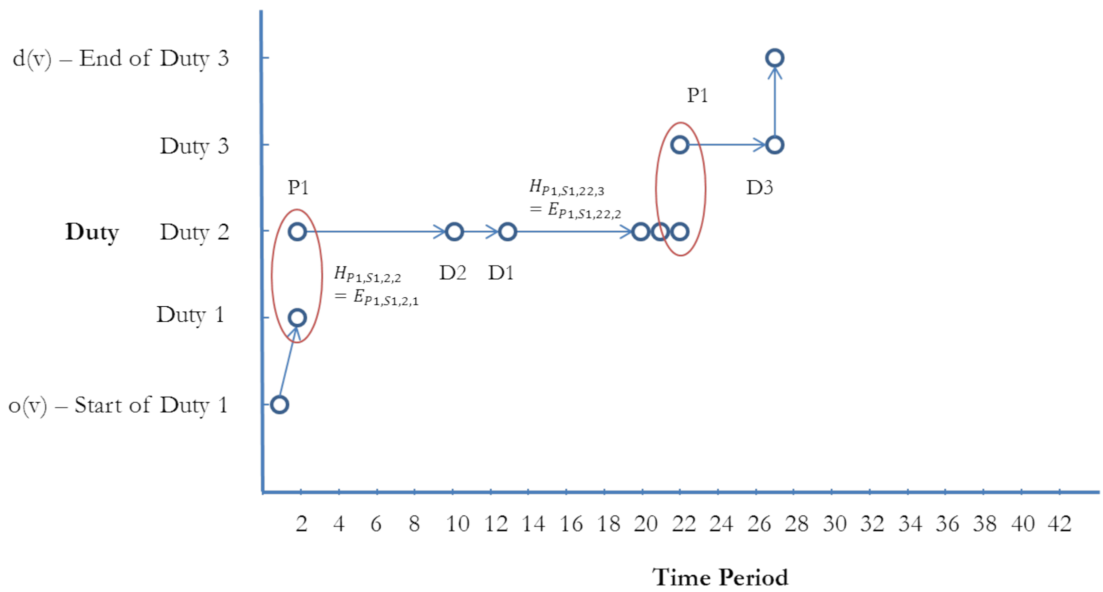
5.3 The Duty Flow Model

Similar to the path flow model, routing and scheduling in the duty flow model are governed by the duty variable λ_{vd}. The difference is that in this model, more than one duty can be selected, and thereafter linked in order to form the complete schedule. Figure 10 illustrates the selected duties in this solution, and shows how they are linked.

The solution contains three unique duties out of 5,500 possible duties. For simplicity, these duties are denoted *duty 1*, *duty 2* and *duty 3*. The first duty is from the origin node $o(v)$ to port $P1$. The next is from $P1$, via $D2$ and $D1$, and back to $P1$. Then, the last duty takes the ship from $P1$ to $D3$, and ends in the destination node $d(v)$. This linking of duties represents the exact same route as for the arc flow and path flow solutions. However, a waiting day outside $D1$ is dropped, and the schedule is therefore brought forward by one day after the visit in $D1$.

Figure 10 illustrates how the linking of two duties is worked out, and we will explain this by looking at the combination of *duty 2* and *duty 3*. The ship is still on *duty 2* while waiting in day 20 and 21 outside $P1$. However, when calling the port on day 22, the end-of-duty parameter indicates that this is the last port of call on the duty ($E_{P1,S1,22,2} = 1$). Constraints (46) then force another duty to start. The start-of-duty in *duty 3* ($H_{P1,S1,22,3} = 1$) matches the end-of-duty in *duty 2*. This duty is thus allowed to start. There is no end-of-duty in *duty 3* ($E_{ivt3} = 0$ $\forall i, v, t$), so it belongs to the group of duties that can end the ship's schedule. After operating in $D3$, the schedule ends in the destination node $d(v)$. The loading and unloading patterns in this duty flow solution corresponds exactly to those of the

Figure 10: Solution of instance $A1$ for the duty flow model



arc flow and path flow solutions.

5.4 Inventory Management and Contract Handling

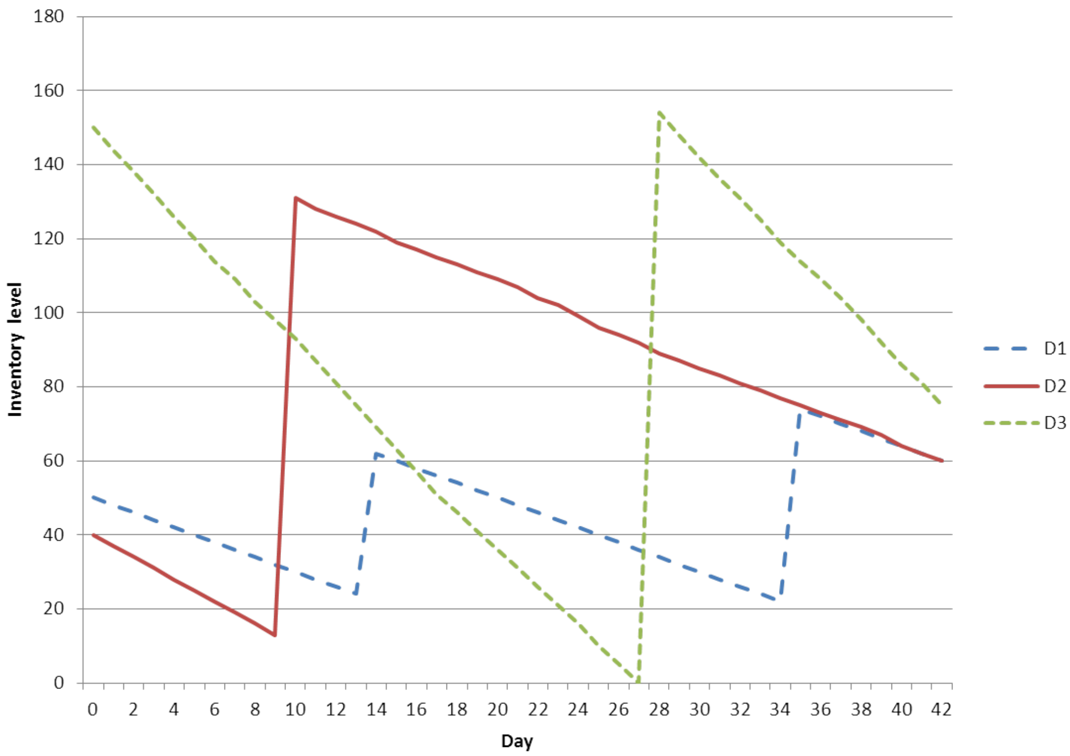
The inventory management and contract handling parts in all three models are governed by the same constraints and variables. Hence, the decision variables have nearly identical values in the three solutions. However, some differences still exist because of symmetrical solutions, but they do not affect the objective value. Decision variables in this part of the solutions are therefore presented independent of the models. Arbitrarily chosen, the values correspond to the arc flow solution.

Figure 11 illustrates the inventory levels in all three delivery ports throughout the planning period. Port $D1$ and $D2$ have a quite similar sales rate (2-3 units a day), while $D3$ has a considerably higher sales rate (5-6 units a day). It is clear that $D1$ and $D2$ start the planning period with significantly less in storage than $D3$. Because of this, it makes sense to deliver a shipload to $D1$ and $D2$ first. $D1$ receives one tank of LNG, whereas $D2$ receives three tanks. Observe that the one, single ship is not able to fully serve the three delivery ports all by itself, and a purchase of gas externally has to be made in $D1$. This explains the increase in inventory on day 35. All the other increases in inventory levels are caused by ship deliveries. During the implementation, a lower limit is introduced for the inventory level in the end of the planning period. The limit is introduced in order to avoid adverse end-of-period effects (see section 6.1), and has been set to 30% of the storage capacity. The end of period inventory levels are 60 units for $D1$ and $D2$, and 75 units for $D3$. These amounts are equal to the lower limit.

In this problem instance, three contracts exist; one in the first time interval, one in the second and one that stretches over the whole planning period. The two former, $C1$ and $C2$, have conditions for upper and lower pick-up limits, in addition to a minimum inter-arrival gap. The latter, $C3$, have limits for pick-up in each of the time intervals and for the total planning period. It also requires that minimum 60% of the volume purchased and picked up to is delivered to port $D3$.

The ship loads in $P1$ two times; on day 2 and day 23. On day 2, 96 units are purchased from $C1$ and 64 units are purchased from $C3$. That satisfies the limits for $C1$, and the interval limits for $C3$. However, none of the purchased gas is shipped to $C3$'s primary port. When loading on day 23, 63 units of gas is purchased from $C2$. This number equals $C2$'s lower limit for pick-up. The remaining 97 units of the ship's capacity is picked up from $C3$. All this cargo is delivered to $D3$, which is $C3$'s primary port. Thus, the origin-destination clause is fulfilled, since $(97/161) = 60\%$. Moreover, the last loading ensures that pick-up limits in $C3$ for

Figure 11: Inventory levels throughout the planning period



the total planning period and the intervals are satisfied.

Finally, it is worth noting the interconnectivity between decisions in the problem. All aspects of the problem are tightly linked and depend on each other. Ship routing is not only concerned with the ship routing constraints; also the inventory levels and contract conditions have to be satisfied. As seen in this example, the inventory levels in the start of the planning period drive the ship to visit $D1$ and $D2$ initially. Afterwards it is also forced to $D3$, in order to maintain the origin destination clause, as well as the decreasing inventory level that cannot fall below the specified limit.

6 Implementation

All the three models have been implemented using commercial off-the-shelf software. First some comments are provided regarding how they are implemented in a mathematical programming language with optimization software. Then, we will elaborate on the process of generating paths and duties for the models a priori, which is performed in a separate program using a general-purpose programming language. The last section presents a Dantzig-Wolfe (DW) decomposition and the implementation of this dynamic approach for the duty flow formulation.

6.1 MIP in Commercial Software

The model implementations have been written in Mosel mathematical programming language, using the Xpress v.7.2 optimizer and interface. All tests have been conducted on a Pentium i7 3.4 gHz processor with 16 GB RAM, running Windows 7 64-bit. We have performed problem reductions and model modifications for enhanced computational efficiency before conducting the tests.

Problem Reduction

Dynamic declarations have been applied on all variables and constraints. There are two reasons for this; first and foremost so that only feasible elements are created in order to obtain correct solutions. For instance, sailing variable x_{ijvt} is not created for arcs (i, j) that is not possible to traverse for ship v in time period t. This allows for a change in the implementation from the mathematical model. In the mathematical model, we sum over all arcs which a ship is allowed to traverse. In the implementation, however, by creating only the sailing variables that are feasible, we can instead sum over *all* arcs. Ultimately, this is equivalent to summing over the arcs that can be traversed. The second reason for dynamic declarations is that it enables further reduction of the problem matrix by removing excessive elements, i.e. elements which removal does not affect the solution. An example of such elements are port operations (o_{ivt}), loading (z^P_{ivt}), unloading (z^D_{ivt}) and waiting (w_{ivt}) variables for port i if ship v cannot sail to that port in time period t. These variables are not created.

Testing has been carried out to identify which variables and constraints are excessive, and how they affect the model. Eliminations like the one previously described make the model more effective. Other eliminations, however, do not consistently improve the computational performance. One such example is constraints that could have been removed for all time periods t when ship v cannot call port i, like constraints (3) and (4). This is experienced because the Xpress solver uses a sophisticated pre-solve function. The pre-solve can utilize constraints and variables otherwise deemed as excessive to create valid inequalities or favorable cuts. To exploit this, selected excessive variables and constraints are still created in the implementations. Selection is based on testing in order to obtain maximum efficiency in the models.

Model modifications

During implementation and testing, a few modifications of the presented models have been made. Constraints (11), which require a ship that calls a port to unload all the tanks it carries for that port, have been identified as redundant. This requirement is satisfied by constraints (8), (12) and (13). In the arc flow model many variables were declared as binary or integer. Not all of those requirements have proved to be necessary. Specifically, the port operations (o_{ivt}) and the ship inventory (z^I_{ivt}) variables can be declared continuous due to constraints (5), (9) and (10). From these constraints it can be observed that o_{ivt} and z^I_{ivt} are essentially governed by other variables, which force binary and integer values. Similar reasoning applies for the schedule and duty variables. In fact, these variables can only take values 0, 0.5 or 1 due to constraints (41) and (42). Furthermore, if it takes value 0.5 the same constraints force the neighbour[1] path or duty variable to take the same value (see Section 4.2.2).

$$\sum_{r \in R_v} O_{ivtr} \lambda_{vr} = bin_{ivt} \qquad \forall i \in N^{PD}, v \in V, t \in T \qquad (47)$$

$$\sum_{r \in R_v} Z^P_{ivtr} \lambda_{vr} = int_{ivt} \qquad \forall i \in N^{D}, v \in V, t \in T \qquad (48)$$

$$bin_{ivt} \in \{0, 1\} \qquad \forall i \in N^{PD}_v, v \in V, t \in T \qquad (49)$$

$$int_{ivt} \in \{0, 1 \ldots, W_v\} \qquad \forall i \in N^{D}_v, v \in V, t \in T \qquad (50)$$

Constraints (41) and (42) in the path and duty formulations are implemented by introducing two new variables for the right hand side, as given in constraints (47) and (48). These variables are binary and integer, respectively. Following the introduction of these variables, the branch-and-bound tree is more balanced

[1]Paths with the exact same schedule are defined as neighbours. Neighbours have different loading patterns, and can form new patterns by convex combinations. Figure 5 illustrates a neighbourhood of paths.

because of branching on the new variables. Prior to the introduction the tree was one sided, and the model as a whole not as effective as after.

Upper and lower limits for storage and sales are not determined from constraints in the implementation. Instead, the valid range of values is specified when creating the variables. Consequently, the number of constraints is reduced. This is possible since the constraints' only purpose is to define an upper and lower limit on the variables. To avoid adverse end-of-period effects, requirements are set on the minimum inventory level in delivery ports at the end of the planning period. Contrary to previously described modifications, this will have an effect on the solution.

For the instances tested, a specified limit force the inventory level to be above or equal to 30% of storage capacity when the planning period ends. Without the requirement it was observed that most inventories were empty at the end of the planning period. The reason is that it will usually be profitable to end the planning period with little inventory, since this implies large revenues from selling, and small transport and purchasing costs. Such planning favours the short-run profit, but may negatively affect profit in the long run. Introducing a limit will force the model to keep the next period in mind when planning, which is helpful when using the model for continuous operations or rolling horizon. However, the strict storage requirement for each port might not be appreciated by all decision makers, as it may quite heavily guide the routing of ships. For example, it may force the model to reject otherwise profitable solutions where ships would plan to arrive shortly after the end of the planning period. An alternative way to handle the end-of-period effects is to assign a value to the inventory in delivery ports. Including this in the objective function will reflect the potential future profit of the gas and make it more attractive to end the period with gas in storage. One could also require a miminum aggregated storage in the ports; $\sum_{i \in N^D} s_{iT^{MX}} \geq Const.$ The disadvantage of this alternative is primarily that the final storage level in delivery ports may be imbalanced. That is, storage may end at very high levels in one port, while completely empty in another.

Test instances

For test purposes 18 instances are created based on representative data from the LNG business. These instances are in some fashions similar to those tested by Grønhaug and Christiansen (2009) and later Fodstad et al. (2011). The physical network of ports, i.e. the position and number of pick-up and delivery ports, is identical. Morover, the number of ships, their size and inventories, as well as sailing distances and costs correspond to their test instances. This thesis, however, apply other lengths of the planning period and replace inventory management with contract handling in pick-up ports. The planning period comprises a given number of weeks. Periods of six, eight and ten weeks are tested. A period is partitioned in two intervals of equal length. Three contracts are attached to each pick-up port. One contract spans the whole planning period, whereas two contracts apply for each of the intervals only.

All contracts have limits for total pick-up volume. These limits are set in accordance with the rate of production in the pick-up port for which it belongs. Moreover, contracts which apply for the whole period include lower and upper pick-up levels in each of the two intervals. In these contracts an origin-destination clause is also defined. The primary port is selected manually. It is preferably a port that can be reached by all ships and has a large storage capacity and high sales rate. Since sailing times can be fairly long compared to the interval length, origin-destination clauses on the short contracts would force many sailings to a predetermined port. To allow for some flexibility in the routing, only the contracts that span the whole planning period have this clause.

It is assumed that every ship has four tanks. Furthermore, in the instances tested all ships carry no cargo when the planning period starts. Therefore a route always starts with sailing from the origin node to a pick-up port. This assumption is made in order to obtain comparable results for the different instances. This first sailing is defined to last zero, one or two days, which means that the ship's initial position is either in the first port, or maximum two days away from that port. Extensive testing has shown that small changes in these initial conditions vastly influence the solutions, both in terms of computational time and objective value.

6.2 Generating Paths and Duties

A path or duty is also denoted by the term column. The path flow model and the duty flow model can use pre-generated columns as input. All possible columns are then generated a priori and information about each column is fed to the master problem. The process of generating all paths or duties is also known as the enumeration process. Enumeration of all columns is handled in a C++ written code and implemented in Visual Studio 2010. The objective when enumerating is to form all feasible and unique combinations of; a) What port to visit in which time period; b) What quantity to load to which port in which time period, and; c) What quantity to unload in which port in which time period. Item a) is known as the ship schedule, whereas item b) and c) forms what we call the loading pattern. Combinations of schedules and loading patterns, which together constitute the path or duty, are thus generated for each ship.

In addition to the assumptions listed in Section 3.4, another assumption is added when generating paths and duties. We assume that all ships have a delivery port as their last port of call. There are several reasons for this assumption to be made. First, the model is proposed as a rolling-horizon planning tool. This implies that the last part of the planning period will be re-run before final desicions are made. Therefore, we do not want a late visit to a pick-up port govern whether the pick-up limits are fulfilled. Second, generation of paths is a complicated procedure requiring a lot of computer memory. Applying this assumption will significantly reduce the number of paths to be created, which facilitates solving of larger data instances. Third, experience from the arc flow model, where it is possible to end the sailing route in a pick-up port, shows that ships tend to always have a delivery port as their last port of call. Note that this assumption is added in the generation tools and that the models are fully capable of handling ship routes that end in a pick-up port.

In the path flow model, each column represents a loading pattern and a complete ship schedule. The ship schedule includes the geographical ship route, waiting days and times of departure. In the duty flow model, each column represents a loading pattern and a partial ship schedule, i.e. the ship's journey from one pick-up port to another pick-up port. On its way the ship have to call one or two delivery ports. Information about waiting days, times of departure and loading patterns is also included.

Algorithm 1 describes the enumeration process. It is not an exact representation of the implemented code. Rather, it is a much simplified representation, for the purpose of illustrating the general idea. The algorithm is separated in three procedures. Calculations from one procedure is used as input in the next. First, all routes are created in Procedure 1. This procedure use a recursive call to have it run through all feasible combinations of port visits and waiting days outside each port. Simultaneously, the cost of each route is calculated. The pseudocode presented applies for generation of paths. When generating duties, one extra condition is applied; the route ends if the ship returns to a pick-up port or sails to the destination node after calling one or two delivery ports. Procedure 2 then use all the routes that are generated to deduce the port operations matrix, which is denoted O_{ivtr} in the mathematical model. Each unique row[1] in the port operations matrix possibly translate to several unique loading patterns. These patterns are denoted Z^P_{ivtr} and Z^D_{ivtr} in the model. Procedure 3 creates the loading patterns based on the port operations matrix. In order to obtain all patterns, a recursive

[1]The term *row* is in this section used to describe the contents in the pre-generated matrices that correspond to each path or duty, not to be mistaken for rows as constraints in the optimization problem

Algorithm 1 Generate all paths

$Route = \emptyset$
$AllRoutes = \emptyset$
for all v in Ships **do**
  createRoutes(v, $o(v)$, $o(v)$, $Route$, $AllRoutes$)
end for

$PortOperations = \emptyset$
for all r in $AllRoutes$ **do**
  createPortOperations(r, $PortOperations$)
end for

$row = 1$ {first row}
$o = 1$ {first operation}
$Pattern = \emptyset$
$AllPatterns = \emptyset$
for all row in $PortOperations$ **do**
  createPatterns(o, row, 0, 0, $Pattern$, $AllPatterns$)
end for

Route, AllRoutes, PortOperations, Pattern and AllPatterns are sets

call is used also in this function.

There is one critical difference between generating paths and duties; the number of paths grows rapidly when extending the planning period. Duties, on the contrary, do not get as numerous. For paths an increase in the length of the planning period can make it possible to add one or multiple port calls to all existing schedules. This, in turn, may give rise to even more possible loading patterns. For instance, if two delivery ports can be added to an existing schedule, several new loading patterns will also be created. Considering that a path is combined of schedules and loading patterns, the increase in number of paths is clear. For duties, an increase in the length of the planning period will rarely make it possible to add new port calls. A duty is normally shorter than the planning period, so all possible routes are already created. The increase in number of duties will mainly come from the possibility for each route to be started in more time periods, and possibly to add waiting days. For instance, we may consider a route $P-D-D-P$ that takes 20 days. For a planning period of 42 days, this means that the route can be started in time period $t \in \{1, 2, \ldots, 22\}$, which make up 22 schedules if we disregard waiting days. Even if we extend the planning period by 14 days, we are not able to extend

Procedure 1 createRoutes(v, i, j, *Route*, *AllRoutes*) $\rightarrow$ Create all routes

```
Route= Route ∪ {i} // add port i to Route
C_vr+ = C_ijv // update cost of sailing Route
T_vr+ = T_ijv // update time to sail Route
for all j in Ports do
  if j ≠ d then // check if port ≠ destination node
    if (i, j) ∈ A_v
        and T_vr + T_ijv ≤ T^MX
        and # waiting days < T^W
        and # successive delivery port calls ≤ 2
        and # successive pick-up port calls ≤ 1 then
          call createRoutes(v, j, i, Route, AllRoutes)
    end if
  else
        Route= Route ∪ {d} // all routes end in destination node
        AllRoutes = AllRoutes ∪ Route
  end if
end for
```

the route by another port call, since a duty can maximum comprise two delivery ports. Neither can we increase the number of loading patterns. However, we can now start the route in time period $t \in \{1, 2, \ldots, 36\}$. The example illustrates the additive nature of duties, which strongly contrast the exponential nature of paths.

Information from the procedures must be fed as parameters to the optimization model. The C++ code therefore uses the generated data to write matrices in an appropriate format to file: O_{ivtr} for port operations, Z^P_{ivtr} for loading, Z^D_{ivtr} for unloading and C^{SAIL}_{vr} for costs. Furthermore, the matrices H_{ivtd} and E_{ivtd} for the duty model are both deduced from O_{ivtd}. H_{ivtd} represents the start-of-duty, which corresponds to the first port operation from O_{ivtd} on every duty. E_{ivtd} is the end-of-duty, and indicates that the duty ends. The last port operation for every duty is moved from the O_{ivtd} matrix to the E_{ivtd} matrix, so that no loading occurs when the duty ends.

The number of unique patterns is kept so that the matrices are of equal size. If, for example, a route goes from one pick-up port, through two delivery ports and back to a pick-up port, two unique loading patterns exist. These two patterns both correspond to the same sequence in the port operations matrix. In order to have each route number r correspond to one row in all the matrices, the sequence is duplicated. Resultantly, the port operations matrix has two identical rows. Each of these rows correspond to a unique loading pattern.

Procedure 2 createPortOperations(r, $PortOperations$) → Create the port operations matrix

create *row* // row in operations matrix corresponding to route r
for all ports $\in r$ **do**
  if current port $\neq$ next port **then** // i.e. the ship is not waiting in port
    assign port to new operation
    assign time to new operation
    assign ship to new operation
    add new operation to *row*
  end if
end for
$PortOperations = PortOperations \cup row$

Procedure 3 createPatterns(o, row, $Load$, $Unload$, $Pattern$, $AllPatterns$) → Create all loading patterns

if $Load \neq 0$ **and** $Unload \neq 0$ **then**
  $Pattern$=$Pattern \cup \{Load\} \cup \{Unload\}$
end if
while $o \leq$ sizeof(row) **do**
  if o takes place in a pick-up port **then**
    if one delivery port in the duty **then**
      $Load \leftarrow$4 tanks to port A
      $Unload \leftarrow$ 4 tanks to port A
      $Pattern$=$Pattern \cup \{Load\} \cup \{Unload\}$
    else // two delivery ports in the duty
      duplicate *row* in *PortOperations*
      $Load \leftarrow$ 3 tanks to port A, 1 tank to port B
      $Unload \leftarrow$ 3 tanks to port A, 1 tank to port B
      createPatterns($o + 1$, row, $Load$, $Unload$, $Pattern$, $AllPatterns$)
      $Load \leftarrow$ 1 tank to port A, 3 tanks to port B
      $Unload \leftarrow$ 1 tank to port A, 3 tanks to port B
      $Pattern$=$Pattern \cup \{Load\} \cup \{Unload\}$
    end if
  end if
  $AllPatterns$=$AllPatterns \cup Pattern$
end while

In the duty model, constraints (45) make sure that a ship sails from the origin node only once during the planning period. The set $\mathcal{D}_v^S$ contains all possible start duties for each ship. All start duties are from the origin node to the first port of call. The first port of call is decided by the instance generator, and is always a pick-up port. Since two waiting days are allowed, three different duties are created from the origin node to the first port. One without waiting, one with one waiting day, and one with two waiting days. Additionally, a duty is included to allow for ships not to be used at all, i.e. to go directly from the origin to destination node. Thus, the set $\mathcal{D}_v^S$ contains four duties for every ship v.

6.3 Dynamic Duty Flow Model

Instead of generating all duties before solving the optimization problem, it is possible to start with only a small set of duties and iteratively add more. The motivation for this dynamic approach is that only a tiny fraction of all possible duties are actually part of the solution. More specifically, the number of possible duties are typically tens or hundreds of thousand, wheras a solution normally comprise of a single-digit number of duties. We have implemented a dynamic approach to explore its potential, but a complete program has not been written. This means that the implementation is partly based on the built-in branch-and-bound algorithm in Xpress, and that optimal solutions are not guaranteed.

The basis of the dynamic duty flow model is a Dantzig-Wolfe decomposition (DW), as presented by Dantzig and Wolfe (1961). Based on the arc flow model, the problem is decomposed into a master problem and several subproblems. Since all routing constraints from the original model apply for each ship, these constraints are moved to the subproblems. Resultantly, there is one subproblem for each ship in the problem instance, and the subproblems' purpose is to generate duties. In order to do this, the subproblem uses dual values from each optimization run of the master problem to generate new columns. The master problem is an LP relaxation of the duty flow model presented in Section 4.3. However, only the set of start duties is fed to the model initially. Dual values that correspond to each constraint in the master problem describe the change in objective function value if the right hand side in the constraint changes by one unit. From these values, one may calculate the reduced cost of any duty. The reduced cost refers to the change in objective function value in the master problem if this duty is taken into the basis. In the case of a maximization problem, we want to include the duty with the largest reduced cost, which is the duty that increases the objective function the most. Each time a new column is added to the master problem, it is re-run and dual values are sent back to the subproblem. The cycle ends when no more

Table 6: New elements in the subproblem

Parameters	
α_{it}	dual value for constraints (59)
β_{ivt}	dual value for constraints (60)
γ_{ivt}	dual value for constraints (61)
σ_{it}	dual value for constraints (62) and (63)
μ_{ivt}	dual value for constraints (46)
Variables	
e_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ have end-of-duty in port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
z^H_{ijvt}	integer number of tanks that are being loaded in pick-up port i for delivery port j on ship v in time period t

columns with positive reduced cost can be produced.

Mathematical Model

The mathematical model for the subproblem is similar to the routing and scheduling part of the arc flow model from Section 4.1. Only new parameters, variables and constraints are presented mathematically in this section, but the complete model is listed in Appendix A.

A subproblem is created for every ship in the planning problem. Each subproblem comprises the following constraints:

$$\begin{aligned} \text{constraints } &(4) - (5) \\ &(7) - (13) \end{aligned}$$

From the arc flow model a number of constraints are unchanged; constraints (4) for the maximum number of consecutive waiting days; constraints (5) connecting sailing to port operations; constraints (7) and (8) forcing ships to unload at least one tank and at most the number of tanks on the ship when visiting a delivery port; constraints (9) and (10) for ship inventory; constraints (11) ensuring all tanks for the port of call are unloaded; constraints (12) forcing the ships to be empty when calling a pick-up port; and constraints (13) limiting the maximum number of delivery ports per duty to two.

$$\sum_{i \in \mathcal{N}_v^P} \sum_{j \in \mathcal{N}_v^D} \sum_{t \in \mathcal{T}} x_{ijvt} = 1 \qquad \forall v \in \mathcal{V} \tag{51}$$

$$\sum_{i \in \mathcal{N}_v^D} \sum_{j \in \mathcal{N}_v^P \cup d(v)} \sum_{t \in \mathcal{T}} x_{ijvt} = 1 \qquad \forall v \in \mathcal{V} \tag{52}$$

All duties that are generated in the subproblem start from a pick-up port, and only one sailing from a pick-up port is allowed within one duty. For a duty to end, a sailing from delivery port to either a pick-up port or the destination node is required. Constraints (51) and (52) ensure the duties' start and end requirements are satisfied.

$$\sum_{j \in \mathcal{N}_v} x_{ijvt} + w_{ivt} + e_{ivt} - \sum_{j \in \mathcal{N}_v} x_{jiv(t-T_{jiv})} - w_{iv(t-1)} = 0 \quad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{53}$$

$$\sum_{i \in \mathcal{N}_v^D} \sum_{t \in \mathcal{T}} e_{ivt} = 0 \qquad \forall v \in \mathcal{V} \tag{54}$$

The new variable e_{ivt} is introduced for the flow balance. The variable indicates whether a duty has reached its end-of-duty. Constraints (53) state that a ship has three options if it arrives a port. The first option is to wait outside the port. The second option is to call the port and sail further in the same time period. The last option is to end the duty in the port. End-of-duty is only allowed in pick-up ports, as indicated by constraints (54).

$$\sum_{j \in \mathcal{N}_v^D} z_{ijvt}^H - Q_v o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \tag{55}$$

$$\sum_{i \in \mathcal{N}_v^P} z_{ijvt}^H - z_{jvt}^P = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{56}$$

Constraints (55) ensure that the ship is fully loaded, whereas constraints (56) ensure that tanks are only loaded with cargo to one delivery port. The variable z_{ijvt}^H is introduced in order to interconnect the constraints.

requirements (25) – (30)

$$z^H_{ijvt} \in \{0, ..., Q_v\} \qquad \forall i \in \mathcal{N}^P_v, j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{57}$$

Integer and binary requirements are maintained for the variables $x_{ijvt}, w_{ivt}, o_{ivt}, z^P_{jvt}, z^D_{jvt}$, z^I_{jvt} and z^H_{ijvt} through constraints (25)–(30) and (57).

$$\begin{aligned}\max \bar{c} = \Bigg\{ &- \sum_{(i,j)\in\mathcal{A}_v} \sum_{t\in\mathcal{T}} C^{SAIL}_{ijv} x_{ijvt} - \sum_{i\in N^{PD}} \sum_{t\in\mathcal{T}} \alpha_{it} o_{ivt} + \sum_{i\in\mathcal{N}^P_v} \sum_{t\in\mathcal{T}} \beta_{ivt} Q_v o_{ivt} \\ &+ \sum_{i\in\mathcal{N}^D_v} \sum_{t\in\mathcal{T}} \gamma_{ivt} Q^W_v z^P_{ivt} + \sum_{i\in\mathcal{N}^D} \sum_{t\in\mathcal{T}} \sigma_{it} Q^W_v z^D_{ivt} - \sum_{i\in\mathcal{N}^P} \sum_{t\in\mathcal{T}} \mu_{ivt}(e_{ivt} - o_{ivt}) \Bigg\}\end{aligned} \tag{58}$$

The objective in the subproblem is to maximize the reduced cost, as given in function (58).

Implementation of the Model

The decomposed model is only able to obtain the optimal solution for the LP relaxation. However, there are integer and binary requirements on several variables in the original problem. Such requirements are usually complied with by built-in branch-and-bound algorithms in the optimizer. It is possible to use this functionality to obtain a MIP solution after the column generation process, but optimality is not guaranteed. The reason is that columns are generated based on reduced costs for continuous variables, and they are therefore not necessarily the most profitable with integer requirements applied.

The dynamic approach has been implemented to study if the duty flow model could achieve even better performance than with pre-generated duties. In the implementation, the subproblem and master problem are part of the same model file. Unfortunately, testing of the implementation has not given as promising results as hoped for. One issue is that it takes a long time to obtain the optimal LP solution. Symmetry may be an important reason for this, since numerous equally good duties are generated. Furthermore, when the LP solution is found, the optimizer has a low success rate of obtaining the actual optimal MIP solution, i.e. the MIP solution is worse than previously found in other implementations. This implies that the column generation process does not create some or all of the duties that are part of the optimal MIP solution. One attempt to handle these issues has been to generate multiple columns in each iteration. In practice, the solver then saves a user-specified number of solutions for each subproblem, to increase the chances of creating the optimal columns. Although substantially more columns are added throughout the process, test results indicate little improvement in the MIP solution.

Further development and testing is required to investigate if the dynamic approach can outperform the other implementations. If so, a natural progress would be to develop a complete branch-and-price algorithm. This requires procedures for creating the node tree, for branching and cutting it, and for simultaneous generation of columns. The reader is referred to Barnhart et al. (1998) for a wider understanding of how this can be accomplished.

7 Computational Results

This section presents results from testing of the three models. The testing has been conducted on computers with specifications as reported in Section 6. The purpose is to evaluate and compare the models. Results from the technical testing are presented first. This involves measurements of computational performance and an analysis of how the performance is affected by the modeling and implementation of waiting days. Then follows a section with economical considerations, where we evaluate selected aspects of the problem and assumptions that are made to solve it. Specifically, contract conditions, the possibility for partial loading and an analysis of solution robustness is included.

7.1 The Framework of Testing

Testing of computational efficiency is carried out on a number of real-life problem instances. The number of ships, ports and contracts for the instances are presented in Table 7. Instance D and E appear identical because they represent the same physical network of ports. However, there are differences in the sailing times and the ship-to-port compatibilities. All instances are tested for three lengths of planning period; six, eight and ten weeks. The instances are later referred to by a letter followed by a number, for example $A1$. The letter represents the physical network of ports in the instance, whereas the number indicates the length of the planning period. Number 1 corresponds to six weeks, 2 to eight weeks, and 3 to ten weeks planning horizon. A more thorough description of the data instances is provided in Section 6.1

Table 7: Test instances

		Ports		
Instance	Ships	Pick-up	Delivery	Contracts
A	1	1	3	3
B	2	1	2	3
C	2	2	2	6
D	2	2	3	6
E	2	2	3	6
F	3	2	2	6

7.2 Technical Testing

The models are first compared with respect to problem size and solution efficiency. Figure 12 depicts the number of rows, continuous variables and global entities for the arc flow and the duty flow model. The duty flow and path flow models are compared with respect to the same details in Figure 13. Due to memory limitations and scaling in the path flow model, the computer is able to generate all paths only for the instances with the shortest planning period. Thus, only the six smallest instances are compared in the figure. The information is presented two separate figures because of different orders of magnitude between the models. Together, continuous variables and global entities constitute all the columns in the problem. Global entities are columns with integer or binary requirements.

In general, the problem size grows rapidly when more time periods are added. This is seen from the increase in number of rows and columns. In the arc flow model, the problem matrix has about the same number of rows and columns. More than half of the columns have integer or binary requirements. The figures are different for the path flow and duty flow model, where the number of columns

Figure 12: Instance dimensions in the arc and duty flow models

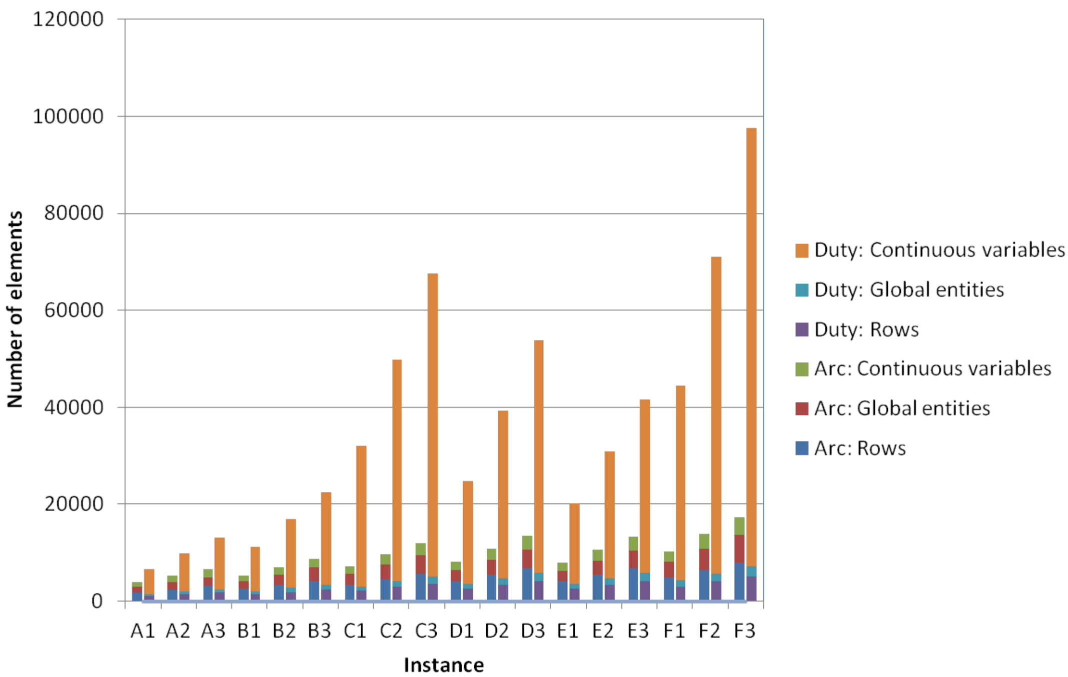
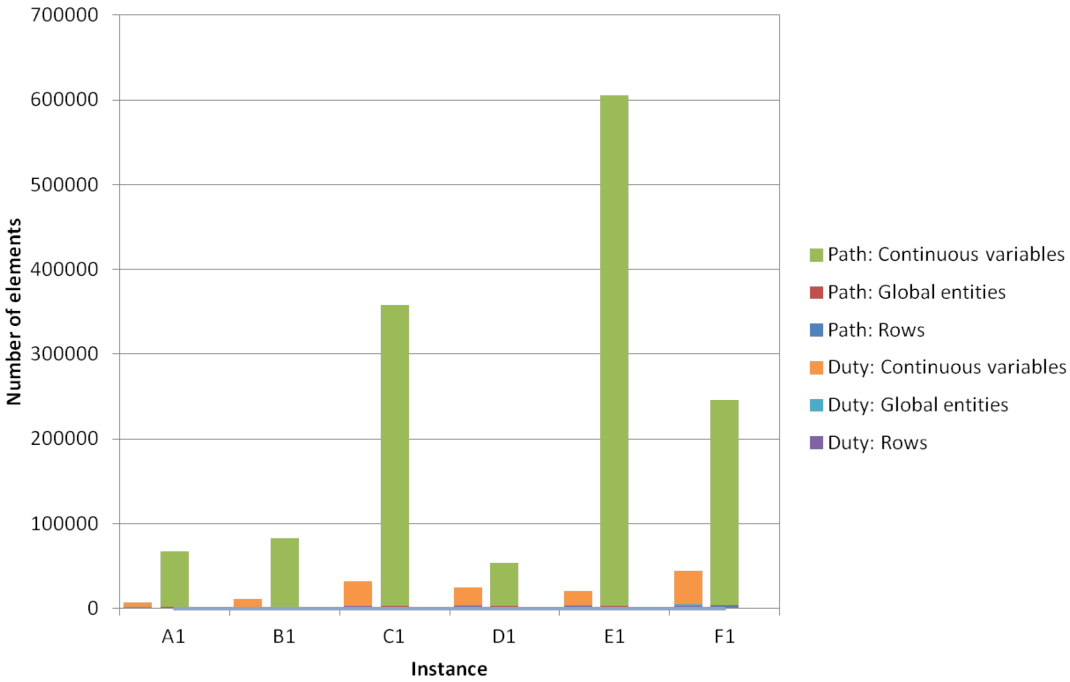


Figure 13: Instance dimensions in the duty and path flow models



is significantly higher than the number of rows. In these models, the number of columns is high because every possible path or duty in the model is represented with a variable. Paths are even more numerous than duties, due to the possibility to combine routing and loading patterns over the entire planning period.

The number of global entities in the path and duty models are equal for all instances. The reason is that all variables with integer or binary requirement are included in constraints that are part of both models. These are the variable for the inter-arrival gap constraints, and the integer and binary variables which are introduced to ensure that an integer number of tanks are loaded and that neighboring paths can be combined (see Section 6.1).

In the arc flow formulation, the routing and scheduling are governed by designated constraints and variables. In contrast, the other two models depend on columns that are fed from the generation tool. This tool makes sure that all the generated columns are in accordance with routing and scheduling constraints from the arc flow model. Since feasibility for all columns is ensured in the generation, these constraints are removed in the path and duty flow models. Therefore, there are fewer rows in these models, compared with the arc flow model.

7.2.1 Model Efficiency

The presented instances are tested with all three models. A maximum runtime of 36,000 seconds (ten hours) is set, but nearly all instances are either solved or out of memory before the limit is reached. If a model is not able to solve an instance to optimality, we obtain a measure of the solution quality in the optimality gap. The gap represents in percentage how much the best feasible solution deviates from the upper bound on the solution at the time when the test stops, i.e. $\frac{UB-MIP}{MIP}$. Parameter values are based on real-life information, but scaled down to get more manageable values throughout the solution process. Hence, the objective value represents a downscaling of the real profit from the planning problem.

Figure 14 depicts the solution times for the three models. Note that the time axis is logarithmic and that only instances solved to optimality are depicted. The arc flow model is able to solve 13 of the 18 test instances within reasonable time. Insufficient memory causes the last five to stop before optimality has been proved. Instances $D3$ and $F1$ are left with optimality gaps around 7%, whereas $B3$ stopped at 14%. The instances with the largest amount of rows and columns, $F2$ and $F3$, have optimality gaps above 30%. None of the models, however, are able to solve these two instances to optimality. For the path flow model, only results from instances with six weeks planning period are obtainable, of which five out of six are solved to optimality. Instance $F1$ terminated with a remaining optimality gap of 7%. The duty flow model is able to solve 15 of the 18 instances, and leaves only small optimality gaps on the remaining three. The largest gap is for instance $F2$ with 14%, while $F3$ and $D3$ have optimality gaps of 5% and 1%, respectively.

A comparison between the three models leaves the duty flow model as the most efficient. Out of the 13 instances that more than one model is able to solve, the duty flow model solves 12 instances fastest. More specifically, the solution time for these 12 instances are on average 83% shorter than that of the arc flow model, which is the second most efficient model. Furthermore, the duty flow model does not experience the same scaling issues as the other models, and runs out of memory only for the largest data instance – $F3$. The path flow model, in addition to requiring the most memory for pre-generation, is consistently the least effective model. Solution times are on average 98% shorter with the duty flow model, and 82% shorter with the arc flow model. In the path flow model, the number of columns is very large compared with the other two. Further, the built in pre-solve feature in the optimizer is not able to reduce the number of columns of the path flow model. The simplex method performed in each node of the branch-and-bound tree then becomes increasingly time consuming. For comparison, in the arc and duty flow models the number of columns is reduced by 33% and 49% on average.

Approximately 40% of the rows are removed in all three models after pre-solve.

When solved to optimality, all models obtain the same MIP solution objective value. This provides a good assurance of consistent modeling in the three formulations. The MIP solution is the solution when one or more variables have integer or binary requirements. What LP solutions concern, the arc flow model yields higher values compared with the path flow and duty flow models. LP solutions are solutions obtained when no variables have binary or integer requirements. The gap between the MIP and LP solutions is depicted in Figure 15. Only results from the arc and duty flow models are presented, as we have limited results from the path flow model. This gap is given as a percentage of the MIP solution, calculated by the formula: $\frac{LP-MIP}{MIP}$. A smaller gap is observed for the duty flow model than for the arc flow model, which indicates that the duty flow model is a tighter formulation. On average, the duty flow model closes 30% of the gap observed in the arc flow model. The explanation for this is that the ship routing and schedul-

Figure 14: Solution times for the three models

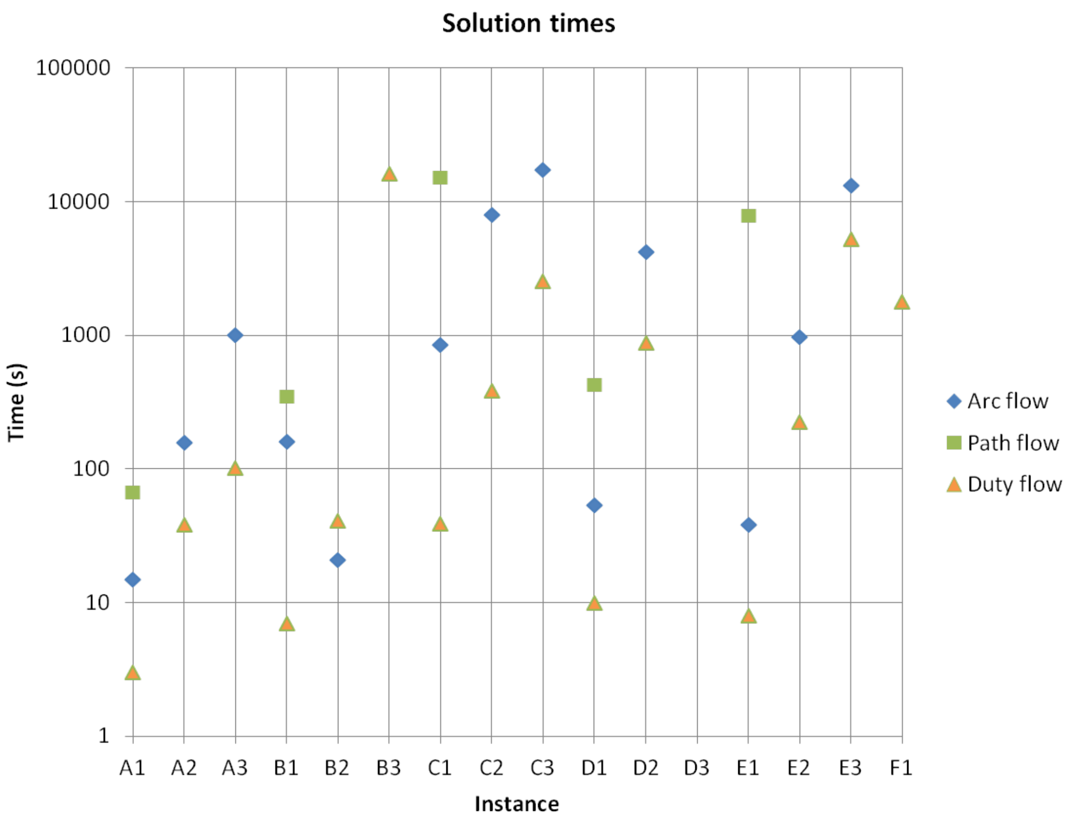
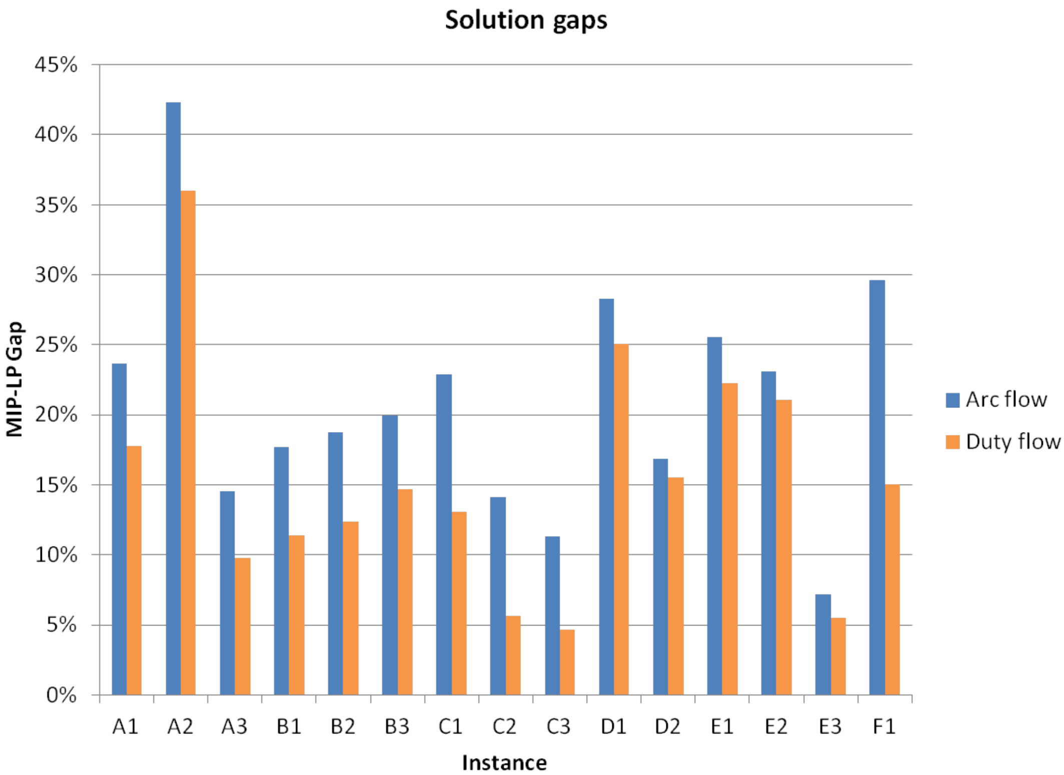


Figure 15: Gap (%) between MIP and LP solution for the arc and duty flow model



ing constraints from the arc flow model are handled differently in the duty flow model. The duty flow model is in principle a DW decomposition where routing and scheduling are handled in subproblems for each ship. Each subproblem, which is either solved in the column generation tool or the optimization subproblem, is a MIP problem that ensures binary and integer requirements for the generated columns. Therefore, the generated columns represent the convex hull of feasible points. Constraints in the arc flow model does not necessarily enclose a space with the same integrality property. Usually they enclose a larger solution space which results in a weaker LP relaxation. The path flow model is centered around the same principle as the duty flow model and these models obtain the same gap.

In addition to solving the optimization problem, the path and duty flow models require some time to pre-generate paths and duties. However, it is necessary to generate the columns only once. After this the models can be re-run even if parameter values change, as long as it does not affect the possible routes. For instance, if pick-up limits or berth capacities were to change, the model can run on the same set of pre-generated columns as before. We therefore report the time and number of columns generated separately in Table 8. For cells where no value is specified, the computer ran out of memory during the generation.

Table 8: Generation of paths and duties

	Path Flow		Duty Flow	
Instance	Time(s)	# Paths	Time(s)	# Duties
A1	9	65,259	0	4,363
A2	390	3,101,008	0	6,883
A3	–	–	0	9,403
B1	13	80,418	0	8,558
B2	–	–	0	13,262
B3	–	–	1	17,966
C1	44	353,810	1	27,896
C2	–	–	2	44,360
C3	–	–	3	60,824
D1	8	48,689	1	19,742
D2	–	–	2	32,510
D3	–	–	2	45,278
E1	45	600,575	1	15,116
E2	–	–	1	24,104
E3	–	–	2	33,092
F1	30	240,541	2	38,694
F2	–	–	3	63,390
F3	–	–	5	88,086

Duties are generated in an insignificant amount of time compared with paths, and the number of duties is considerably lower than the number of paths. With regard to the reported solution times, the duty flow is still superior to the other models when the time to generate columns is included. Notice the large increase in number of paths from $A1$ to $A2$, compared with the increase in duties. While the number of duties grow additively when the planning period increases, paths have an exponential growth (see Section 6.2). The number of paths in $A2$ is too much for the optimizer to handle, and it runs out of memory before the optimization can start.

The reader is referred to Appendix B for further results from the testing of computational performance. This includes complete tabular representations of problem size before and after pre-solve is performed, total solution time, time to first and best solution, MIP and LP objective values, MIP-LP and optimality gaps, as well as the number of nodes in the branch-and-bound tree search.

7.2.2 Removing Symmetry by Introducing Waiting Costs

Permitting ships to wait outside a port provides flexibility in the solutions. However, it also complicates the model considerably. The reason is that many equally good solutions will be produced, since there are no extra costs associated with waiting. For instance, if a solution is for one ship to wait two days outside the second port of call, an equally good solution can be provided by rather waiting one day outside the second and one day outside the third port of call. This phenomenon is likely to occur when sales rates and revenues in the ports vary little throughout all time periods, since the time of waiting then becomes less important for the objective value. Merely securing that ships call delivery ports, so as to never empty the storage, will assure stable revenues. Consequently, exactly when and where the ship waits is not of great importance, as long as it waits the number of days that may be necessary to satisfy all constraints.

Due to short planning periods in this tactical problem, fixed costs are not considered. Further, since the fleet of ships is fixed, the cost of having the ships in disposal is regarded as fixed. This fixed cost incurs whether they are sailing or not. That is the reason why costs associated with waiting outside a port are neglected thus far. When waiting does not imply any extra cost, many symmetrical solutions exist. Thus, the solver may spend a long time proving that a solution is optimal. Further testing has therefore been conducted to investigate if optimal solutions can be found in a shorter time.

In order to differentiate solutions that would otherwise be equally profitable a waiting cost is introduced. Waiting then incurs different costs, depending on which port the ship waits outside. The aim is to remove some of the symmetry in the solution space. It has been implemented as a small imaginary cost, but could just as well have been included as a real cost to cover a daily pay-rate. Regardless of how it is seen, a waiting cost will remove many symmetrical solutions. This is tested for the arc and duty flow models, and results are presented in Table 9. Since we have seen the cost as imaginary, the objective values are the same as presented earlier. Therefore, only solution times are given.

Results indicate that the models' efficiency is positively affected by the introduction of a cost associated with waiting. For the arc flow model, 11 of 13 instances are solved in less time, of which many are solved significantly faster than before. However, the model is not able to prove optimality for additional instances, com-

Table 9: Solution times from the arc and duty flow models with waiting cost

Instance	Time(s)			
	Arc flow		Duty flow	
	Without waiting cost	With waiting cost	Without waiting cost	With waiting cost
A1	15	5	3	2
A2	158	105	38	23
A3	1,010	277	101	84
B1	161	107	7	12
B2	21	37	41	124
B3	–	–	16,271	1,477
C1	843	437	39	37
C2	7,940	806	383	221
C3	17,484	2,322	2,527	1,174
D1	54	27	10	11
D2	4,215	5,083	872	501
D3	–	–	–	5,608
E1	38	26	8	7
E2	975	121	226	141
E3	13,261	9,968	5,270	6,278
F1	–	–	1,794	554
F2	–	–	–	–
F3	–	–	–	–

pared with earlier. The duty flow model solves 11 of 15 instances in less time, and is able to prove optimality for one more instance – $D3$. Furthermore, when the solution time is reduced we often see a significant reduction, while the difference is small when no improvement is made. Resultantly, the net effect of waiting costs is highly favorable for the efficiency of the models. Since the cost is imaginary, the models provide the same solutions as earlier. In total, this suggests that a waiting cost should be implemented for future use of the model.

7.2.3 Maximum One Waiting Day Outside Ports

Another way to handle the computationally difficult waiting days is to simply allow for less waiting. This implies a tightening of the waiting constraints, and could lead to a reduction of the objective value. However, it does not necessarily yield solutions much worse than those previously found, since it appears that two consecutive waiting days are not exploited very often. In Section 7.2.1 we observed that scaling is particularly troublesome with the path flow model, and enumeration of paths was only possible for the instances of six weeks. Therefore, we investigate if this model can produce decent solutions for larger instances with only one waiting day allowed outside each port, as this will reduce the number of paths.

Table 10 presents the solution time, optimality gap and number of columns when only one waiting day outside each port is allowed. With two waiting days allowed outside each port, $A2$ is the largest instance for which we can generate all columns before running out of memory. This results in more than three million columns. The number of columns in the $A2$-instance is reduced to $250,000$, which gives us an idea of how the extra waiting day complicates the problem. Despite the vast reduction of columns, only two of the six instances are solved to optimality. The solution times are still long, compared with both the arc flow and duty flow models. However, they produce the same objective values as with two consecutive waiting days allowed. For instance $E2$ the optimizer runs out of memory when reading the information from the path generator. For instances of ten weeks planning period we were not able to enumerate the paths due to insufficient memory.

Table 10: Path flow model with one waiting day allowed

Instance	Time(s)	Optimality gap (%)	# Columns
A2	7,165	0	246,834
B2	6,652	0	324,740
C2	5,235*	17.6	2,037,782
D2	14,736*	10.7	270,005
E2	1*	-	2,699,381
F2	2,885*	852.9	1,465,068

7.3 Economical Testing

In addition to the technical testing, we have conducted tests to investigate how the objective value is affected by changes in selected parameters, constraints and the number of intervals in the planning period. First, tests are carried out on the contract conditions to give a decision-maker valuable insight in future contract negotiations. The contract parameter values that were used during the technical testing are denoted base case, and objective values are compared to this case when adjusting the parameter values. Second, the industry standard to always fully load a ship is challenged when the model is tested with partial loading allowed. Partial loading means to load an integer number of tanks lower than the ship's capacity. Last, the solution robustness is investigated by including a third time interval in the planning period. The shortest instances are used for this testing. For a solution to be robust, decisions conducted early in the planning period should not be much influenced by adding additional time periods to the planning period.

7.3.1 Contract Conditions

Contracts differentiate the problem in this thesis from the LNG-IRP. Experience has indicated that contracts add complexity to the problem, and that small adjustments in the contract parameters may have significant impact on the solvability of the problem. Testing has therefore been conducted to gain awareness of which parameters drives the profitability. Such knowledge may improve the likelihood of making better decisions.

Tests are carried out on data instances $D1$ and $D2$. The instances have a reasonable size for the testing with two ships, two pick-up ports and three delivery ports. Of the six contracts in each instance, two extend to the whole planning period; one in each pick-up port. The most influential contract parameters will be investigated; the interval pick-up limits, the primary ports and the origin-destination fraction. However, only the longest contracts feature interval pick-up limits and origin-destination clauses. Consequently, the testing only regards these contracts. The parameter values are changed to lower and higher values compared with the base case scenario, first separately and then all parameters together.

Interval pick-up limits

Table 11 shows changes in the objective value when interval pick-up limits are adjusted. The value in columns *Lower* and *Upper* are factors that indicate how tight the limits are, and the volumetric limits the factor translate to. To obtain the volumetric interval pick-up limits, the limits for the total planning period are first divided by the number of intervals and then multiplied by the respective factor. That is, when there are two intervals in the planning period, factors of value 1 indicate that the upper and lower interval pick-up limits are exactly half of the total pick-up limits. Where two values are given in the volume column, the first represents contract 1 and the second contract 2. A large gap between the factors represents loose interval pick-up constraints, and a small gap tightens the constraints.

Table 11: Influence of interval pick-up limits for instance $D1$

	Interval pick-up limits					
	Lower		Upper			
	Factor	Volume	Factor	Volume	Obj. value	Change (%)
Base case	0.5	31/28	1.3	118	1,248	–
I	0.5	31/28	1.1	110	1,152	-7.7
II	0.9	57/50	1.3	118	1,171	-6.2
III	0.9	57/50	1.1	110	727	-41.7
IV	0	0	2	182	1,259	+0.8

A numerical example can describe this further; a contract has total pick-up limits of [100, 200] and multipliers of [0.5, 1.3]. The pick-up limits for each interval are then $[\frac{100}{2}, \frac{200}{2}]*[0.5,\ 1.3]^\top=[25,\ 130]$. In practice, multipliers of [0, 2] remove any requirement for timing of pick-ups and leaves only the total pick-up limits effective.

The objective value of instance $D1$ is most influenced by tightening of the constraints. The influence is particularly strong when tightening both the upper and lower limits, which reduces the objective value by close to 42%. On the contrary, relaxation of the constraints has almost no impact, whereas tightening only one of the limits cause a drop in objective value by 6-7%. This is an indication that the optimal pick-up volumes are close to the lower limit in one interval and close to the upper limit in the other. Only tightening one limit leaves some flexibility in the routing decisions, as indicated by the slight reduction in objective value compared with the base case.

Instance $D1$ comprise of two intervals of three weeks length, while instance $D2$ comprise of two intervals of four weeks length. Testing on instance $D2$ indicates that when the intervals are longer, the interval pick-up limits have less influence on the objective value. Specifically, only changes of 1% in the objective value are observed when applying the parameter changes presented in Table 11 on instance $D2$. This can be explained by a relatively large increase in the total pick-up limits. The total pick-up limits are based on the production rate in the pick-up port. If the production rate is high, the additional contractual volume from extending the planning period can be more than ships have time to pick up during the time added. For problem instance $D2$ this results in more flexibility in the interval pick-up limits. That is, tight interval limit factors in the $D1$ instance, translate to loose volumetric interval limits in $D2$.

Based on the results, negotiations should focus on avoiding tight limits, particularly if both sides are tight and the interval length is short. Sacrificing a lot in order to relax the limits is not worthwhile, since little profit can be gained from it. An alternative is to negotiate price reductions as a compensation for tight limits. This can be illustrated by an example. Assume that the base case profit is \$1,000,000. With the tightest limits the profit would be \$584,000. If one were to blindly follow these figures, it would be reasonable to agree to tight limits only if they were compensated with smaller costs. In particular, if the purchased volume on the contract is 416,000 units of LNG, at least \$1 discount per unit is needed to consider this option. Similar reasoning can be done for all parameters when negotiating contracts. Although the calculations are simple, they can provide useful guidelines.

Origin-Destination Clause and Primary Ports

The origin-destination clause is a mean for the seller to assure stable delivery of LNG to certain ports. This is given in the model as a fraction value, which states the minimum fraction of the total volume from one contract that must be delivered to its primary ports. The total volume includes cargo that is picked up by ships and volume that is sold directly in the pick-up port. That is, a value of 0.6 indicates that 60% has to be delivered to the primary ports. When selecting primary ports in the instances some prerequisites were defined; all ships must be able to call the port and sailing must be allowed from all pick-up ports. In order to emphasize solvability, for most instances the primary port is also a delivery port with a high storage capacity and sales rate.

Table 12 shows the results of testing that concerns the fraction value for instance $D1$. In the base case, the fraction value has been set to 0.6. However, tightening and relaxing of the fraction value have limited effect on objective value. An explanation for this is that the selected primary port has a high sales rate, and large storage capacity. Independent on contractual terms it will thus be profitable to deliver large volumes to this port. However, if the fraction is set to 1, all cargo

Table 12: Influence of origin-destination fraction for instance $D1$

	Fraction	Obj. value	Change (%)
Base case	0.6	1,248	–
I	0	1,298	+4.0
II	0.3	1,257	+0.7
III	1	1,166	-6.6

picked up from the contract must be delivered to the primary port. This force more expensive extra purchases in the other ports, to meet demand and comply with the lower inventory limits. Resultantly, the objective value is reduced.

Selection of primary ports may have a significant influence on the objective value. Test results from changing the primary ports are presented in Table 13. The instance contains three delivery ports. Port $D3$ has the highest upper limits for both storage and sales, and is therefore selected in the base case. The objective value is slightly reduced when port $D1$ is selected, while selection of port $D2$ more than halves the value. Port $D2$ has a small storage capacity compared with $D1$ and $D3$, which may explain the drastic reduction. Since 60% of all gas must be delivered to this port, the total volume that ships can pick up is restricted by the small capacity and little gas can be delivered elsewhere. Including more than one primary port is beneficial, as this relaxes the problem and allows for more flexibility in the routing decisions. Hence, decisions can to a larger extent be driven by profit and less by contractual conditions. When all ports are defined as primary the solution in this case corresponds to a fraction value of 0, since the entire volume purchased from contracts is shipped, and nothing is sold directly in the pick-up port.

Table 13: Influence of primary ports for instance $D1$

D1	Primary ports	Objective value	Change (%)
Base case	D3	1,248	–
I	D1	1,145	-8.2
II	D2	472	-62.2
III	D1 & D2	1,288	+3.2
IV	All	1,298	+4.0

The selection of primary ports is often not a negotiable term due to ownership and co-operation agreements. However, the fraction value can be negotiated. Results from Tables 12 and 13 show that the properties of the primary ports, such as storage capacity, sales rate and price, are important for the profit. A high fraction value is acceptable if the primary ports are ports which are accessible for a wide range of ships, has good storage capacity and high sales prices. It would be profitable to deliver a large volume to this port anyway, and negotiation efforts could in that case be put in elsewhere. On the contrary, if the primary port does not have these properties, it is beneficial to negotiate a low fraction value.

Simultaneous parameter changes

Finally, tests when all parameters are changed simultaneously are presented in Table 14. The first four cases involve very tight contract conditions and little flexibility, whereas the last case in practice removes the interval pick-up limits and the origin-destination clause. Tightening all contract constraints results in a reduction of the objective value by nearly 50%. Tight conditions also give infeasible solutions when changing the primary port. When all parameters are relaxed the objective value improves by 5%.

All results indicate that the contractual conditions in the base case allow for much flexibility in the routing, since only small improvements are made when conditions are set as loose as possible. Conditions have been set loose intentionally, to obtain comparable results during the technical testing. The reason is that it was desirable that the technical testing depended mainly on problem size and structure, not so much on individual contract conditions in each instance.

Table 14: Influence of contracts for instance $D1$

	Interval pick-up limits			Primary	Objective	
	Lower	Upper	Fraction	ports	value	Change (%)
Base case	0.5	1.3	0.6	D3	1,248	
I	0.9	1.1	1	D3	669	-46.4
II	0.9	1.1	1	D1	Infeasible	
III	0.9	1.1	1	D2	Infeasible	
IV	0.9	1.1	1	D1&D2	721	-42.2
V	0	2	0	All	1,307	+4.7

7.3.2 Partial Loading

Due to the costs and time usage associated with loading LNG ships, the industry practice has been to always load ships to capacity. However, this practice was challenged by Fodstad et al. (2011). They allowed partial loading, and experienced that the increase in transportation costs in many cases are outweighed by increased revenue. This is experienced due to; 1) The ability to perform more trips and; 2) The ability to respond to changing gas prices and exploit price spikes. In this thesis, we have conducted tests that validate these findings. The tests are carried out with a relaxation of constraints (14) from $=$–requirement to $\leq$–requirement. This means that a ship must load an integer number of tanks, but not necessarily load all its tanks. Results are presented in Table 15.

Table 15: Objective values when allowing partial loading

Instance	Full loading	Partial loading	Change (%)
A1	851	851	0
A2	926	1,113	20.2
B1	711	728	2.4
B2	823	900	9.4
C1	742	742	0
C2	1,025	1,025	0
D1	1,248	1,356	8.7
D2	1,588	1,709	7.6
E1	1,433	1,462	2.0
E2	1,555	1,678	7.9

When allowing partial loading, it is also possible to allow for subsequent calls to pick-up ports. In such a way, it would be possible for a ship to combine loading in two or several ports to fill its capacity. As with partial loading, this would challenge normal practice in the LNG business. However, in this thesis we do not regard that possibility.

The contracts state an upper limit for purchase quantity. Because ships differ in loading capacity, it is not given that this upper limit corresponds to an integer number of full shiploads. What is often observed when partial loading is allowed, is that the amount picked up is closer to the contracts' upper limit. In that case, an additional duty that applies partial loading is included some time during the planning period. The reason is that a ship may undertake a loading that would exceed the pick-up limit if it was required to load all its tanks. Consequently, a higher supply in the delivery ports is ensured, which in turn implies a higher sales rate and a higher profit. Moreover, by delivering a higher volume to delivery ports, the need to buy extra gas directly in the delivery port is reduced.

7.3.3 Solution Robustness

So far, all instances have been split in two time intervals. To test the robustness of the solutions, a third interval has been added at the end of the planning period. A robust solution will not be much affected when the planning period is extended. That is, decisions early in the planning period should be similar for different lengths of planning period. We use as basis the test instances with intervals of length three weeks. That gives a total length of nine weeks for the new test instances. Limits for pick-up, storage and sales in the intervals are the same as with two intervals. Five instances have been tested and the solutions compared.

The overall tendency is that adding an interval to the test instance will not dramatically change the decisions in the start of the period. What is observed when the planning period consists of three intervals is that decisions in the first and last interval are similar to the respective test instance with two intervals. The first decisions are often focused on deliveries to the ports with low initial inventory, regardless of the number of intervals and length of planning period. Such deliveries are fulfilled in order to avoid purchases of gas directly in the delivery port, since this is expensive. With three intervals, the new routing decisions are most often produced in the middle interval. These seem to be profit-driven decisions and less bound by contractual terms. Decisions in the last interval focus on fulfilling the contract obligations both when the planning period consists of two and three intervals.

An alternative explanation is derived from a study of the solutions in the planning period with two intervals. Ships often sail to the destination node early in the second interval, since contractual terms are fulfilled and there is enough gas in storage to satisfy the demand for the rest of the planning period. However, when adding the third interval the ships have to remain in operation in the second interval in order to fulfill contractual terms in the last interval. It is observed that this can force decisions to be made on the basis of keeping ships in operation. An example of such decisions can be to perform sailings with long transit times, to avoid numerous pick-ups which would violate the upper pick-up limit.

The data instances themselves may have a deciding factor in this. They are initially designed for two intervals, and most of them provide tight pick-up limits. Consequently, little flexibility in the number of visits to a port during an interval exists. Further, they are relatively small, with 1-2 pick-up ports and 2-3 delivery ports. Similarities between the solutions are thus likely to appear, regardless of the number of intervals and length of the planning period.

8 Concluding Remarks

In this thesis we have considered a combined inventory and routing problem for the LNG business (LNG-IRP), and introduced purchase contracts in pick-up ports. Routing of ships must comply with contractual terms and ensure that the inventory in delivery ports is within acceptable levels. Contractual terms include limits for pick-up volume, a minimum time between pick-ups and a clause that dictate where a defined share of the gas must be delivered. In delivery ports, gas is subject to a varying demand that has to be satisfied.

The objective was to develop optimization models and methods which can support a downstream actor in the LNG supply chain. During the research three models have been formulated; an arc flow, a path flow and a duty flow model. Paths are complete ship schedules combined with the patterns for loading and unloading in ports. A duty is similar to a path, but its schedule consists only of sailing from a pick-up port to one or two delivery ports, and back to a pick-up port. As far as we know, a duty flow formulation is never before applied to a maritime transportation problem.

All models have been implemented in Mosel mathematical programming language using the Xpress optimizer. For the path flow and duty flow models, a column generation tool has also been developed and implemented in C++. Technical and economical testing have been conducted with a set of 18 problem instances based on a real-life planning problem.

The arc flow model was able to solve 13 of the problem instances to optimality within the fixed limit of ten hours. Compared with the other models it produces a higher LP-MIP gap and it requires more time than the duty flow model to prove optimality. The advantage of this formulation is that all decisions are represented with designated variables, which facilitates a good overview of details in the solutions. Moreover, since each decision is modeled explicitly, it is easy to understand, modify and extend the model.

For the path flow model, testing was carried out with all columns generated a priori. Due to intense scaling of the problem when increasing the number of time periods, memory capacity was experienced as a large obstacle. Therefore, we were only able to generate columns for the seven smallest problem instances, of which the largest comprised 3,000,000 paths. Five of these were solved to optimality, but in considerably longer time than with the other models. We tried to allow for maximum one waiting day outside each port to facilitate generation of larger instances. Still, the optimizer could solve only two additional instances, and the path generation tool were not able to enumerate columns for instances with the longest planning horizon.

The duty flow model was superior to the other models in terms of computational performance. It solved 15 of the instances to optimality and was the fastest model for 12 of 13 instances which more than one model could solve. For these instances, the solution times were on average 83% and 98% shorter compared with the arc flow and path flow models, respectively. Furthermore, there were no problems concerning generation of duties, neither with respect to memory nor time consumption. In contrast to paths, the number of duties did not become very large when extending the planning horizon. Specifically, the number of duties range from 5,000 to 100,000 in the test instances. Our experience with the duty flow formulation indicates that linking of duties is an effective way to model routing and scheduling.

Although the three models yield the same objective value for our test instances, the optimal decisions differ because of symmetry in the solution space. We introduced imaginary costs associated with waiting outside a port so as to reduce this symmetry. Solution times were then reduced in 11 of the instances, both for the arc flow and duty flow models. The reduction was for most instances significant. Furthermore, one additional instance was solved with the duty flow model. We suggest further reduction of symmetry as a topic for future work in order to enhance the models' efficiency.

We have investigated the profit's sensitivity to changes in contract parameters, and then how it is affected by allowance of partial loading. Two contract parameters have shown to be of great importance; interval pick-up limits and primary ports. Tight limits restrict the routing and can reduce the profit considerably. Our tests also indicate that allowance of partial loading may result in a higher profit, since the ships achieve a larger total pick-up volume. They are thus able to better satisfy demand in delivery ports. These results, however, depend heavily on the problem instances and should not be generalized without testing on different datasets.

None of the models were able to obtain optimal solutions for all instances, but the duty flow model outperformed the more traditional formulations consistently. Therefore, this model is suggested for future work on the problem. We have already experimented with an implementation of the Dantzig-Wolfe (DW) decomposition where columns are generated dynamically in a subproblem. Initial testing indicates that many columns must be generated to obtain good LP solutions, which have been time consuming. However, further development and testing is required to evaluate if this dynamic approach is favourable to the pre-generation of columns. The proposed DW decomposition can be combined with a branch-and-bound algorithm to obtain MIP solutions, which is known as a branch-and-price method.

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A Complete Mathematical Models

In this appendix, all models will be presented in their entirety. First all sets, indices, parameters and variables used in all models are listed. Then follows the arc flow, path flow and duty flow models. Decision variables and indices are represented with lower-case letters, whereas parameters and superscripts are characterized with capital letters. The exceptions are the path and duty variables which are represented with the greeks λ_{vr} and λ_{vd}. Sets are in capital caligraphic letters. For the models, variables not listed with requirements are all defined continuous and non-negative.

A.1 Sets, Indices, Parameters and Variables for All Models

Table 16: Elements in the models

Set	
$\mathcal{N}^{\mathcal{D}}$	delivery ports
$\mathcal{N}^{PD}$	pick-up and delivery ports
$\mathcal{V}$	all ships
$\mathcal{N}_v$	all ports ship v can visit
$\mathcal{N}_v^P$	all pick-up ports ship v can visit
$\mathcal{N}_v^D$	all delivery ports ship v can visit
$\mathcal{A}_v$	all arcs ship v can follow
$\mathcal{T}$	time periods
$\mathcal{T}^{SUB}$	a shorter time interval for which contracts may state pick-up limits
$\mathcal{C}$	all contracts
$\mathcal{C}_i$	contracts in port i
$\mathcal{N}_c^A$	primary ports for contract c
$\mathcal{R}_v$	all possible paths for ship v
$\mathcal{D}_v$	all possible duties for ship v
$\mathcal{D}_v^S$	start duties for ship v
Indices	
$o(v)$	origin node for ship v
$d(v)$	destination node for ship v
i, j	port
v	ship
t, τ	time period
r	path
d	duty

Parameters

T^W	maximum number of successive waiting days outside a port
T_{ijv}	transport time from port i to port j with ship v
W_v	number of cargo tanks on ship v
B_i^{CAP}	berth capacity in port i
I_{iv}^0	number of full tanks for port i on ship v when the planning period starts
T_c^{INT}	minimum time interval between pick up on contract c
Q_v	capacity of ship v
Q_v^W	capacity of one tank on ship v
L_c	minimum fraction of total cargo picked up on contract c that has to be shipped to primary ports
R_{it}^{PORT}	revenue from sales of a unit gas in delivery port i in time period t
R_{ct}^{CON}	revenue from sales of a unit gas from contract c in time period t
C_{ijv}^{SAIL}	cost of sailing from port i to j with ship v, including port fees in port i
C_{ct}^{CON}	cost of purchase of a unit gas from contract c in time period t
C_{it}^{PORT}	cost of purchase of a unit gas in delivery port i in time period t
S_i^0	initial storage in port i
$\overline{S}_{it}$	upper limit for storage level in port i in time period t
$\underline{S}_{it}$	lower limit for storage level in port i in time period t
$\overline{Y}_{it}$	upper limit for sales in port i in time period t
$\underline{Y}_{it}$	lower limit for sales in port i in time period t
$\overline{P}_c^T$	upper limit for pick up from contract c in the planning period
$\underline{P}_c^T$	lower limit for pick up from contract c in the planning period
$\overline{P}_c^{SUB}$	upper limit for pick up from contract c in a part of the planning period
$\underline{P}_c^{SUB}$	lower limit for pick up from contract c in a part of the planning period
O_{ivtr}	$\begin{cases} 1 & \text{if port operations take place in port } i \text{ on ship } v \text{ on path } r \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
Z_{ivtr}^P	number of tanks loaded for port i on ship v on path r in time period t
Z_{ivtr}^D	number of tanks unloaded in port i from ship v on path r in time period t
C_{vr}^{SAIL}	cost associated with sailing path r with ship v
O_{ivtd}	$\begin{cases} 1 & \text{if port operations take place in port } i \text{ on ship } v \text{ on duty } d \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$

Z^P_{ivtd}	number of tanks loaded for port i on ship v on duty d in time period t
Z^D_{ivtd}	number of tanks unloaded in port i from ship v on duty d in time period t
C^{SAIL}_{vd}	cost associated with sailing duty d with ship v
H_{ivtd}	$\begin{cases} 1 & \text{if duty } d \text{ for ship } v \text{ starts in time period } t \text{ in port } i \\ 0 & \text{otherwise} \end{cases}$
E_{ivtd}	$\begin{cases} 1 & \text{if duty } d \text{ for ship } v \text{ ends in time period } t \text{ in port } i \\ 0 & \text{otherwise} \end{cases}$
α_{it}	dual value for constraints (59)
β_{ivt}	dual value for constraints (60)
γ_{ivt}	dual value for constraints (61)
σ_{it}	dual value for constraints (62) and (63)
μ_{ivt}	dual value for constraints (46)

Variables

x_{ijvt}	$\begin{cases} 1 & \text{if ship } v \text{ starts sailing from port } i \text{ to port } j \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
w_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ waits outside port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
o_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ performs a port operation in port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
z^D_{ivt}	integer number of tanks that are being unloaded in delivery port i from ship v in time period t
z^I_{ivt}	integer number of tanks containing cargo for delivery port i on ship v in time period t
z^P_{ivt}	integer number of tanks that are being loaded for delivery port i on ship v in time period t
f_{civt}	amount of gas purchased from contract c and loaded with destination i on ship v in time period t
p_{cvt}	$\begin{cases} 1 & \text{if cargo is picked up on contract } c \text{ with ship } v \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
y^P_{ct}	amount of gas from contract c sold in pick-up port in time period t
y^D_{it}	amount of gas sold in delivery port i in time period t
e_{it}	amount of gas bought in delivery port i in time period t
s_{it}	amount of gas stored in delivery port i in the end of time period t
λ_{vr}	indicate if ship v utilize path r

λ_{vd}	indicate if ship v utilize duty d
e_{ivt}	$\begin{cases} 1 & \text{if ship } v \text{ have end-of-duty in port } i \text{ in time period } t \\ 0 & \text{otherwise} \end{cases}$
z^H_{ijvt}	integer number of tanks that are being loaded in pick-up port i for delivery port j on ship v in time period t

A.2 Arc Flow Model

The following provides the complete arc flow model, with all constraints and requirements. Linearizations are performed for non-linear constraints – (11), (12) and (13).

$$\sum_{j \in \mathcal{N}_v} x_{o(v)jv1} = 1 \qquad \forall v \in \mathcal{V} \qquad (1)$$

$$\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}_v} x_{id(v)vt} = 1 \qquad \forall v \in \mathcal{V} \qquad (2)$$

$$\begin{aligned} &\sum_{j \in \mathcal{N}_v} x_{ijvt} + w_{ivt} \\ &- \sum_{j \in \mathcal{N}_v} x_{jiv(t-T_{jiv})} - w_{iv(t-1)} = 0 \end{aligned} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (3)$$

$$\sum_{\tau=t}^{t+T^W} w_{iv\tau} \le T^W \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (4)$$

$$\sum_{j \in \mathcal{N}_v} x_{ijvt} - o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (5)$$

$$\sum_{v \in \mathcal{V}} o_{ivt} \le B_i^{CAP} \qquad \forall i \in \mathcal{N}^{PD}, t \in \mathcal{T} \qquad (6)$$

$$z^D_{ivt} - W_v o_{ivt} \le 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (7)$$

$$z^D_{ivt} - o_{ivt} \ge 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (8)$$

$$z^I_{iv1} + z^D_{iv1} - z^P_{iv1} = I^0_{iv} \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V} \qquad (9)$$

$$z^I_{ivt} - z^I_{iv(t-1)} + z^D_{ivt} - z^P_{ivt} = 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \setminus \{1\} \qquad (10)$$

$$z^I_{ivt} - W_v(1 - o_{ivt}) \le 0 \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (33)$$

$$\sum_{j \in \mathcal{N}^D_v} z^I_{jvt} - W_v(1 - \sum_{j \in \mathcal{N}^P_v} x_{ijvt}) \le 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (34)$$

$$\begin{aligned} &W_v - z^D_{ivt} - W_v(1 - x_{ijvt}) \\ &- \sum_{\tau=t}^{t+T^W} z^D_{jv(\tau+T_{ijv})} \le 0 \end{aligned} \qquad \forall i, j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (35)$$

$$\sum_{c\in\mathcal{C}_i}\sum_{j\in\mathcal{N}_v^D} f_{cjvt} - Q_v o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \quad (14)$$

$$\sum_{c\in\mathcal{C}} f_{cjvt} - Q_v^W z_{jvt}^P = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \quad (15)$$

$$\underline{P}_c^T \leq \sum_{i\in\mathcal{N}^D}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^T \qquad \forall c \in \mathcal{C} \quad (16)$$

$$\underline{P}_c^{SUB} \leq \sum_{i\in\mathcal{N}^D}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}^{SUB}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^{SUB} \qquad \forall c \in \mathcal{C}, \mathcal{T}^{SUB} \subset \mathcal{T} \quad (17)$$

$$\sum_{i\in\mathcal{N}_c^A}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}} f_{civt} - L_c \left(\sum_{i\in\mathcal{N}^D}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}} (f_{civt} + y_{ct}^P)\right) \geq 0 \qquad \forall c \in \mathcal{C} \quad (18)$$

$$\sum_{i\in\mathcal{N}_v^D} f_{civt} - \overline{P_c^T} p_{cvt} \leq 0 \qquad \forall c \in \mathcal{C}, v \in \mathcal{V}, t \in \mathcal{T} \quad (19)$$

$$\sum_{v\in\mathcal{V}}\sum_{\tau=t}^{t+T_c^{INT}} p_{cv\tau} \leq 1 \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \quad (20)$$

$$s_{i1} - \sum_{v\in\mathcal{V}} Q_v^W z_{iv1}^D + y_{i1}^D - e_{i1} = S_i^0 \qquad \forall i \in \mathcal{N}^D \quad (21)$$

$$s_{it} - s_{i(t-1)} - \sum_{v\in\mathcal{V}} Q_v^W z_{ivt}^D + y_{it}^D - e_{it} = 0 \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \mid t \neq 1 \quad (22)$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \quad (23)$$

$$\underline{Y}_{it} \leq y_{it} \leq \overline{Y}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \quad (24)$$

$$x_{ijvt} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{A}_v, v \in \mathcal{V}, t \in \mathcal{T} \quad (25)$$

$$w_{ivt} \in \{0,1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \quad (26)$$

$$o_{ivt} \in \{0,1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \quad (27)$$

$$z_{ivt}^D \in \{0,1\ldots,W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \quad (28)$$

$$z_{ivt}^P \in \{0,1\ldots,W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \quad (29)$$

$$z_{ivt}^I \in \{0,1\ldots,W_v\} \qquad \forall i \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \quad (30)$$

$$p_{ct} \in \{0,1\} \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \quad (31)$$

Objective function – represents the total profit from all operations

$$\begin{aligned}\max \pi = &\sum_{i\in\mathcal{N}^D}\sum_{t\in\mathcal{T}} R_{it}^{PORT} y_{it}^D + \sum_{c\in\mathcal{C}}\sum_{t\in\mathcal{T}} R_{ct}^{CON} y_{ct}^P \\ &- \sum_{(i,j)\in\mathcal{A}_v}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}} C_{ijv}^{SAIL} x_{ijvt} - \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{N}^D}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}} C_{ct}^{CON}(f_{civt} + y_{ct}^P) \\ &- \sum_{i\in\mathcal{N}^D}\sum_{t\in\mathcal{T}} C_{it}^{PORT} e_{it}\end{aligned} \quad (32)$$

A.3 Path Flow Model

The following provides all constraints and requirements in the path flow model.

$$\underline{P}_c^T \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^T \qquad \forall c \in \mathcal{C} \qquad (16)$$

$$\underline{P}_c^{SUB} \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}^{SUB}} (f_{civt} + y_{vt}^P) \leq \overline{P}_c^{SUB} \qquad \forall c \in \mathcal{C}, \mathcal{T}^{SUB} \subset \mathcal{T} \qquad (17)$$

$$\begin{aligned} &\sum_{i \in \mathcal{N}_c^A} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} f_{civt} \\ &- L_c \left(\sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{ct}^P) \right) \geq 0 \end{aligned} \qquad \forall c \in \mathcal{C} \qquad (18)$$

$$\sum_{i \in \mathcal{N}_v^D} f_{civt} - \overline{P}_c^T p_{cvt} \leq 0 \qquad \forall c \in \mathcal{C}, v \in \mathcal{V}, t \in \mathcal{T} \qquad (19)$$

$$\sum_{v \in \mathcal{V}} \sum_{\tau = t}^{t + T_c^{INT}} p_{cv\tau} \leq 1 \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \qquad (20)$$

$$\sum_{r \in \mathcal{R}_v} \sum_{v \in \mathcal{V}} O_{itr} \lambda_{vr} \leq B_i^{CAP} \qquad \forall i \in \mathcal{N}^{PD}, t \in \mathcal{T} \qquad (36)$$

$$\sum_{c \in \mathcal{C}_i} \sum_{j \in \mathcal{N}_v^D} f_{cjvt} - Q_v \sum_{r \in \mathcal{R}_v} O_{ivtr} \lambda_{vr} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \qquad (37)$$

$$\sum_{c \in \mathcal{C}} f_{cjvt} - Q_v^W \sum_{r \in \mathcal{R}_v} Z_{jvtr}^P \lambda_{vr} = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (38)$$

$$\begin{aligned} &s_{i1} + y_{i1}^D - e_{i1} \\ &- \sum_{v \in \mathcal{V}} \sum_{r \in \mathcal{R}_v} Q_v^W Z_{iv1r}^D \lambda_{vr} = S_i^0 \end{aligned} \qquad \forall i \in \mathcal{N}^D \qquad (39)$$

$$\begin{aligned} &s_{it} - s_{i(t-1)} + y_{it}^D - e_{i1} \\ &- \sum_{v \in \mathcal{V}} \sum_{r \in \mathcal{R}_v} Q_v^W Z_{ivtr}^D \lambda_{vr} = 0 \end{aligned} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \mid t \neq 1 \qquad (40)$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \qquad (23)$$

$$\underline{Y}_{it} \leq y_{it} \leq \overline{Y}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \qquad (24)$$

$$\sum_{r \in \mathcal{R}_v} O_{ivtr} \lambda_{vr} \in \{0, 1\} \qquad \forall i \in \mathcal{N}^{PD}, v \in \mathcal{V}, t \in \mathcal{T} \qquad (41)$$

$$\sum_{r \in \mathcal{R}_v} Z_{ivtr}^P \lambda_{vr} \in \{0, ..., W_v\} \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (42)$$

$$\sum_{r \in \mathcal{R}_v} \lambda_{vr} = 1 \qquad \forall v \in \mathcal{V} \qquad (43)$$

$$p_{ct} \in \{0,1\} \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \qquad (31)$$

Objective function – represents the total profit from all operations

$$\begin{aligned}\max \pi = & \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} R_{it}^{CON} y_{it}^{D} + \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} R_{ct}^{CON} y_{ct}^{P} - \sum_{v \in \mathcal{V}} \sum_{r \in \mathcal{R}_v} C_{vr}^{SAIL} \lambda_{vr} \\ & - \sum_{c \in \mathcal{C}} \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} C_{ct}^{CON} (f_{civt} + y_{ct}^{P}) - \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} C_{it}^{PORT} e_{it} \end{aligned} \qquad (44)$$

A.4 Duty Flow Model

The following provides all constraints and requirements in the duty flow model.

$$\underline{P}_c^T \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{vt}^{P}) \leq \overline{P}_c^T \qquad \forall c \in \mathcal{C} \qquad (16)$$

$$\underline{P}_c^{SUB} \leq \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}^{SUB}} (f_{civt} + y_{vt}^{P}) \leq \overline{P}_c^{SUB} \qquad \forall c \in \mathcal{C}, \mathcal{T}^{SUB} \subset \mathcal{T} \qquad (17)$$

$$\begin{aligned} & \sum_{i \in \mathcal{N}_c^A} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} f_{civt} \\ & -L_c \left(\sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} (f_{civt} + y_{ct}^{P}) \right) \geq 0 \end{aligned} \qquad \forall c \in \mathcal{C} \qquad (18)$$

$$\sum_{i \in \mathcal{N}_v^D} f_{civt} - \overline{P_c^T} p_{cvt} \leq 0 \qquad \forall c \in \mathcal{C}, v \in \mathcal{V}, t \in \mathcal{T} \qquad (19)$$

$$\sum_{v \in \mathcal{V}} \sum_{\tau = t}^{t + T_c^{INT}} p_{cv\tau} \leq 1 \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \qquad (20)$$

$$\sum_{d \in \mathcal{D}_v} \sum_{v \in \mathcal{V}} O_{ivtd} \lambda_{vd} \leq B_i^{CAP} \qquad \forall i \in \mathcal{N}^{PD}, t \in \mathcal{T} \qquad (59)$$

$$\sum_{c \in \mathcal{C}_i} \sum_{j \in \mathcal{N}_v^D} f_{cjvt} - Q_v \sum_{d \in \mathcal{D}_v} O_{ivtd} \lambda_{vd} = 0 \qquad \forall i \in \mathcal{N}_v^P, v \in \mathcal{V}, t \in \mathcal{T} \qquad (60)$$

$$\sum_{c \in \mathcal{C}} f_{cjvt} - Q_v^W \sum_{d \in \mathcal{D}_v} Z_{jvtd}^P \lambda_{vd} = 0 \qquad \forall j \in \mathcal{N}_v^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (61)$$

$$\begin{aligned} & s_{i1} + y_{i1}^D - e_{i1} \\ & - \sum_{v \in \mathcal{V}} \sum_{d \in \mathcal{D}_v} Q_v^W Z_{iv1d}^D \lambda_{vd} = S_i^0 \end{aligned} \qquad \forall i \in \mathcal{N}^D \qquad (62)$$

$$\begin{aligned} & s_{it} - s_{i(t-1)} + y_{it}^D - e_{i1} \\ & - \sum_{v \in \mathcal{V}} \sum_{d \in \mathcal{D}_v} Q_v^W Z_{ivtd}^D \lambda_{vd} = 0 \end{aligned} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \mid t \neq 1 \qquad (63)$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \qquad (23)$$

$$\underline{Y}_{it} \leq y_{it} \leq \overline{Y}_{it} \qquad \forall i \in \mathcal{N}^D, t \in \mathcal{T} \qquad (24)$$

$$\sum_{d \in \mathcal{D}_v} O_{ivtd} \lambda_{vd} \in \{0, 1\} \qquad \forall i \in \mathcal{N}^{PD}, v \in \mathcal{V}, t \in \mathcal{T} \qquad (64)$$

$$\sum_{d \in \mathcal{D}_v} Z^P_{ivtd} \lambda_{vd} \in \{0, ..., W_v\} \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \qquad (65)$$

$$\sum_{d \in \mathcal{D}^S_v} \lambda_{vd} = 1 \qquad \forall v \in \mathcal{V} \qquad (45)$$

$$\sum_{d \in \mathcal{D}_v} E_{ivtd} \lambda_{vd} - \sum_{d \in \mathcal{D}_v} H_{ivtd} \lambda_{vd} = 0 \qquad \forall i \in \mathcal{N}^P, v \in \mathcal{V}, t \in \mathcal{T} \qquad (46)$$

$$p_{ct} \in \{0, 1\} \qquad \forall c \in \mathcal{C}, t \in \mathcal{T} \qquad (31)$$

Objective function – represents the total profit from all operations

$$\begin{aligned} \max \pi = & \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} R^{CON}_{it} y^D_{it} + \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} R^{CON}_{ct} y^P_{ct} - \sum_{v \in \mathcal{V}} \sum_{d \in \mathcal{D}_v} C^{SAIL}_{vd} \lambda_{vd} \\ & - \sum_{c \in \mathcal{C}} \sum_{i \in \mathcal{N}^D} \sum_{v \in \mathcal{V}} \sum_{t \in \mathcal{T}} C^{CON}_{ct} (f_{civt} + y^P_{ct}) - \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} C^{PORT}_{it} e_{it} \end{aligned} \qquad (66)$$

A.5 Subproblems from Dantzig Wolfe Decomposition

For the dynamic generation of duties, the problem is separated in one master problem and several subproblems. The master problem is an LP-relaxation of the duty flow model. Subproblems are created for every ship. Their purpose is to calculate the best available duty and feed it to the master problem. The column generation is based on the reduced cost derived from dual values from the master problem.

$$\sum_{i \in \mathcal{N}^P_v} \sum_{j \in \mathcal{N}^D_v} \sum_{t \in \mathcal{T}} x_{ijvt} = 1 \qquad \forall v \in \mathcal{V} \qquad (51)$$

$$\sum_{i \in \mathcal{N}^D_v} \sum_{j \in \mathcal{N}^P_v \cup d(v)} \sum_{t \in \mathcal{T}} x_{ijvt} = 1 \qquad \forall v \in \mathcal{V} \qquad (52)$$

$$\begin{aligned} & \sum_{j \in \mathcal{N}_v} x_{ijvt} + w_{ivt} + e_{ivt} \\ & - \sum_{j \in \mathcal{N}_v} x_{jiv(t-T_{jiv})} - w_{iv(t-1)} = 0 \end{aligned} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \qquad (53)$$

$$\sum_{i \in \mathcal{N}^D_v} \sum_{t \in \mathcal{T}} e_{ivt} = 0 \qquad \forall v \in \mathcal{V} \qquad (54)$$

$$\sum_{\tau=t}^{t+T^W} w_{iv\tau} \leq T^W \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{4}$$

$$\sum_{j \in \mathcal{N}_v} x_{ijvt} - o_{ivt} = 0 \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{5}$$

$$z^D_{ivt} - W_v o_{ivt} \leq 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{7}$$

$$z^D_{ivt} - o_{ivt} \geq 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{8}$$

$$z^I_{iv1} + z^D_{iv1} - z^P_{iv1} = I^0_{iv} \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V} \tag{9}$$

$$z^I_{ivt} - z^I_{iv(t-1)} + z^D_{ivt} - z^P_{ivt} = 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \setminus \{1\} \tag{10}$$

$$z^I_{ivt} - W_v(1 - o_{ivt}) \leq 0 \qquad \forall i \in \mathcal{N}^D, v \in \mathcal{V}, t \in \mathcal{T} \tag{33}$$

$$\sum_{j \in \mathcal{N}^D_v} z^I_{jvt} - W_v(1 - \sum_{j \in \mathcal{N}^P_v} x_{ijvt}) \leq 0 \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{34}$$

$$\begin{aligned} &W_v - z^D_{ivt} - W_v(1 - x_{ijvt}) \\ &- \sum_{\tau=t}^{t+T^W} z^D_{jv(\tau+T_{ijv})} \leq 0 \end{aligned} \qquad \forall i, j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{35}$$

$$\sum_{j \in \mathcal{N}^D_v} z^H_{ijvt} - Q_v o_{ivt} = 0 \qquad \forall i \in \mathcal{N}^P_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{55}$$

$$\sum_{i \in \mathcal{N}^P_v} z^H_{ijvt} - z^P_{jvt} = 0 \qquad \forall j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{56}$$

$$x_{ivt} \in \{0, 1\} \qquad \forall (i, j) \in \mathcal{A}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{25}$$

$$w_{ivt} \in \{0, 1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{26}$$

$$o_{ivt} \in \{0, 1\} \qquad \forall i \in \mathcal{N}_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{27}$$

$$z^D_{ivt} \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{28}$$

$$z^P_{ivt} \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{29}$$

$$z^I_{ivt} \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{30}$$

$$z^H_{ijvt} \in \{0, 1 \ldots, W_v\} \qquad \forall i \in \mathcal{N}^P_v, j \in \mathcal{N}^D_v, v \in \mathcal{V}, t \in \mathcal{T} \tag{57}$$

Objective function – represents the reduced cost of a duty.

$$\begin{aligned} \max \bar{c} = \Bigg\{ &- \sum_{(i,j) \in \mathcal{A}_v} \sum_{t \in \mathcal{T}} C^{SAIL}_{ijv} x_{ijvt} - \sum_{i \in N^{PD}} \sum_{t \in \mathcal{T}} \alpha_{it} o_{ivt} + Q_v \sum_{i \in \mathcal{N}^P_v} \sum_{t \in \mathcal{T}} \beta_{ivt} o_{ivt} \\ &+ Q^W_v \sum_{i \in \mathcal{N}^D_v} \sum_{t \in \mathcal{T}} \gamma_{ivt} z^P_{ivt} + Q^W_v \sum_{i \in \mathcal{N}^D} \sum_{t \in \mathcal{T}} \sigma_{it} z^D_{ivt} - \sum_{i \in \mathcal{N}^P} \sum_{t \in \mathcal{T}} \mu_{ivt}(e_{ivt} - o_{ivt}) \Bigg\} \end{aligned} \tag{58}$$

B Detailed Test Results

This appendix provides details concerning the problem instances that are tested in this thesis, and a complete overview of computational results from the technical testing. First, the size of the problem instances for all the three models are presented. Then we present solution times, objective values, solution gaps and the size of the branch-and-bound tree. Both sections gives data in a separate table for each model and include figures for all instances tested.

B.1 Problem Dimensions for Each Model

The Tables 17–19 present key figures about the problem instances that have been tested. They provide figures before and after the optimizer has performed the pre-solve methods. The columns *Row*, *Columns* and *Globals* indicate the number of rows, columns and globals in each instance. *Columns* comprise all variables, whereas *globals* explicitly give variables with binary or integer requirement. Each instance is represented with a letter followed by a number. The letter represents the physical network of ports, whereas the number indicates the length of the planning period. Number 1, 2 and 3 corresponds to six, eight and ten weeks planning period, respectively.

For the arc flow model, the number of rows and columns are about the same (Table 17). Through the pre-solve processing of the optimizer, more than one third of all rows and all columns are removed on average. From studying instances with different planning horizons we see that the problem size increase significantly as the number of time periods increases. This model has more globals than the path and duty flow models. The reason is that routing constraints include many binary and integer variables. These constraints are not part of the two latter models, since requirements for the routing is taken care of in the generation of columns.

Only six instances are presented for the path flow model (Table 18), as larger instances could not be generated due to insufficient memory capacity. The number of columns in this model is remarkably higher than the number of rows, since each possible combination of ship schedules and loading patterns is represented with a variable. On average, the pre-solve is able to remove more than 40% of the rows in the problem matrix. However, less than 1% of the columns are removed.

The number of columns in the duty flow model is also high compared to the number of rows (Table 19). However, since the number of duties increase in an additive way when extending the planning period, it is not as numerous as the paths. After pre-solve about 40% of the rows are removed from the problem matrix. Furthermore,

Table 17: Problem dimensions with the arc flow model

	Matrix			Pre-solved		
Instance	Rows	Columns	Globals	Rows	Columns	Globals
A1	1,855	2,053	1,045	1,166	1,443	719
A2	2,468	2,753	1,409	1,694	2,072	1,055
A3	3,078	3,453	1,773	2,220	2,702	1,390
B1	2,529	2,705	1,613	1,653	1,715	1,088
B2	3,366	3,629	2,173	2,392	2,471	1,592
B3	4,200	4,553	2,733	3,129	3,227	2,096
C1	3,370	3,824	2,228	2,222	2,472	1,580
C2	4,484	5,140	3,012	3,196	3,564	2,309
C3	5,592	6,456	3,796	4,164	4,656	3,036
D1	4,127	3,946	2,224	1,890	2,164	1,226
D2	5,493	5,304	3,008	2,893	3,299	1,927
D3	6,853	6,662	3,792	3,888	4,433	2,627
E1	4,043	3,882	2,160	2,014	2,296	1,315
E2	5,381	5,212	2,916	2,978	3,392	1,991
E3	6,713	6,542	3,672	3,939	4,489	2,667
F1	4,801	5,504	3,320	2,846	3,144	2,082
F2	6,391	7,408	4,496	4,328	4,740	3,174
F3	7,975	9,312	5,672	5,625	6,334	4,267

in contrast to the path flow model, about half of the columns can also be removed.

Table 18: Problem dimensions with the path flow model

	Matrix			Pre-solved		
Instance	Rows	Columns	Globals	Rows	Columns	Globals
A1	969	66,350	378	598	50,313	257
B1	1,208	81,676	588	748	81,245	378
C1	1,863	355,740	840	1,123	354,985	537
D1	1,982	51,249	1,008	963	49,816	439
E1	2,012	603,135	1,008	1,044	600,545	475
F1	2,487	243,226	1,260	1,447	241,942	701

Table 19: Problem dimensions with the duty flow model

	Matrix			Pre-solved		
Instance	Rows	Columns	Globals	Rows	Columns	Globals
A1	1,097	5,455	378	558	2,407	228
A2	1,458	8,339	504	852	5,252	356
A3	1,816	11,223	630	1,140	8,110	484
B1	1,433	9,818	588	790	5,334	378
B2	1,906	14,942	784	1,207	9,431	575
B3	2,376	20,066	980	1,621	14,527	771
C1	2,190	29,828	840	1,213	10,731	537
C2	2,912	46,936	1,120	1,853	27,720	822
C3	3,628	64,044	1,400	2,485	44,773	1,103
D1	2,527	22,304	10,08	1,026	4,578	436
D2	3,361	35,926	1,344	1,677	13,767	717
D3	4,189	49,548	1,680	2,329	26,941	1,000
E1	2,527	17,678	1,008	1,110	5,313	465
E2	3,361	27,520	1,344	1,778	13,510	753
E3	4,189	37,362	1,680	2,426	23,147	1,044
F1	3,031	41,382	1,260	1,545	9,375	701
F2	4,033	66,974	1,680	2,518	28,575	1,121
F3	5,029	92,566	2,100	3,486	54,103	1,541

B.2 Computational Performance for Each Model

The Tables 20–22 present key figures from running all instances on the three models. The columns *First*, *Best* and *Total* indicate the time to find the first solution, the best solution and total solution time. A value of 36,000 in column *Total* means that the maximum time of ten hours has been reached, which implies that optimality have not been proven. Another reason for not being able to prove optimality is if the solver runs out of memory. Instances that are not solved to optimality are marked with an asterix(*). The sub-column *Optimality* then indicates the gap that is left when the optimization run stops. This gap is given as a percentage, and calculated from the upper bound (UB) on the optimal solution and the best MIP solution found; $\frac{UB-MIP}{MIP} * 100\%$. The best solution found that satisfies binary and integer requirement is given in the column *MIP*, which is an abbreviation for mixed-integer program. From the column *LP*, short for linear-program, we find the solution obtained in the root node. The difference between the LP and MIP objective value indicates how tight the formulation is. Therefore, we give this gap in the sub-column *LP-MIP*. The gap is calculated from $\frac{LP-MIP}{MIP} * 100\%$. The rightmost column shows the number of nodes in the branch-and-bound tree.

For instances that are solved to optimality with the arc flow model (Table 20), the best solution is on average found after 20% of the total solution time. Although it is time-consuming to prove optimality, these findings indicate that one often has adequately good solutions after a short time. In situations were one has little time to make a decision, for instance in contract negotiations, solutions that are obtained early can be used as a guideline for what makes a profitable decision. For the path flow model (Table 21), the best solution is on average found after about one quarter of the total solution time. The same figure for the duty flow model is 34%.

The arc and duty flow models are able to find a solution quickly. After 5% of the total solution time the arc flow model obtains its first solution on average. The duty and path flow models find the first solution after 15% and 13% of total time. Since the path flow model make use of a vast number of variables, the simplex algorithm in each node of the branch-and-bound tree becomes very time consuming. In absolute numbers, this model use up to ten minutes to find the first solution.

Most of the instances in the tables are solved to optimality. The arc flow model leaves five of the instances with an optimality gap, whereas only three instances are not solved to optimality by the duty flow model. The most significant gaps are for instance $F1$ and $F2$, which none of the models are able to solve. A gap of more than 30% is left for these instances in the arc flow model, compared to 14%

Table 20: Computational results with the arc flow model

	Solution time			Objective value		Gap(%)		
	First	Best	Total	LP	MIP	LP-MIP	Optimality	Nodes
A1	3	7	15	1,052	851	23.6	–	11,931
A2	5	8	158	1,317	926	42.2	–	166,466
A3	7	219	1,010	1,582	1,381	14.6	–	471,156
B1	4	17	161	837	711	17.7	–	139,492
B2	4	16	21	978	823	18.8	–	4,621
B3	28	306	6,301	1,113	928*	–	14.0	–
C1	13	57	843	912	742	22.9	–	271,194
C2	24	147	7,940	1,170	1,025	13.9	–	1,325,043
C3	23	771	17,484	1,427	1,282	11.3	–	2,611,311
D1	5	14	54	1,601	1,248	28.3	–	47,999
D2	14	376	4,215	1,856	1588	16.9	–	1,491,413
D3	11	2,928	6,794	2,122	1,875*	–	7.7	–
E1	5	7	37	1,799	1,433	25.5	–	18,309
E2	11	131	975	1914	1,555	23.1	–	436,634
E3	36	1,695	13,262	2,173	2,027	7.2	–	5,761,759
F1	28	84	23,989	691	533*	–	7.4	–
F2	41	19,376	23,101	872	615*	–	31.7	–
F3	127	1,342	5,848	1,016	715*	–	36.5	–

and 5% in the duty flow model.

The LP-MIP gaps indicate that the duty flow and path flow models are tighter formulations than the arc flow model. On average, 30% of the LP-MIP gap in the arc flow model is closed in the path and duty flow models.

Table 21: Computational results with the path flow model

	Solution time			Objective value		Gap(%)		
	First	Best	Total	LP	MIP	LP-MIP	Optimality	Nodes
A1	20	20	67	1,003	851	17.9	–	597
B1	67	77	346	792	711	11.4	–	4,497
C1	150	4,298	15,160	839	742	13.1	–	13,461
D1	49	164	427	1,561	1,248	25.1	–	10,221
E1	292	293	7,910	1,752	1,433	22.3	–	2,693
F1	586	1,547	12,355	611	533*	–	6.7	–

Table 22: Computational results with the duty flow model

	Solution time			Objective		Gap(%)		
	First	Best	Total	LP	MIP	LP-MIP	Optimality	Nodes
A1	2	3	3	1,002	851	17.7	–	205
A2	2	5	38	1,259	926	36.0	–	33,503
A3	8	9	101	1,516	1,381	9.8	–	38,005
B1	1	3	7	792	711	11.4	–	4,081
B2	1	11	41	925	823	12.4	–	17,963
B3	2	23	1,6271	1,064	928	14.7	–	2,769,623
C1	3	18	39	839	742	13.1	–	9,080
C2	7	103	383	1,083	1,025	5.7	–	15,649
C3	99	307	2,527	1,342	1,282	4.7	–	24,569
D1	4	9	10	1,561	1,248	25.1	–	6,234
D2	10	53	872	1,834	1,588	15.5	–	265,769
D3	40	1,028	36,000	2,095	1893*	–	1.0	–
E1	5	8	8	1,752	1,433	22.3	–	1,703
E2	13	32	226	1,882	1,555	21.0	–	42,195
E3	37	591	5,270	2,139	2,027	5.5	–	332,061
F1	3	114	1,794	613	533	15.0	–	516,890
F2	11	35,398	36,000	785	655*	–	14.4	–
F3	28	17,425	30,883	920	855*	–	4.6	–

C Digital Attachments

Files attached in the .zip file:

- Mosel implementations
  - Arc flow model - ArcFlow.mos
  - Path flow model - PathFlowPreGenerated.mos
  - Duty flow model - DutyFlowPreGenerated.mos
  - Column generation for the duty flow model - DutyFlowDynamic.mos
- Data instances
  - All test instances for the arc flow model
  - All test instances for the duty flow model
  - Test instance $A1$ for the path flow model
  - All test instances of 42 days for column generation in the subproblem
- Column generation tools
  - Code for generation of all possible paths - pathgeneration.cpp
  - Code for generation of all possible duties - dutygeneration.cpp
  - Data files representing the physical network of ports in the test instances
- Instructions for how to use the files - Readme.txt