Discrete Fourier analysis on Lattice Grids

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Abstract. Using group theory we describe the relation between lattice sampling grids and the corresponding non-aliasing Fourier basis sets, valid for all 1-periodic lattice s. This technique enable us to extend the results established in [16]. We also provide explicit formula for the Lagrange functions and show how the FFT algorithm may be used to compute the expansion coefficients.

Keywords: Trigonometric interpolation · Fourier coefficients

1 Introduction

We are interested in interpolating a periodic function f on $[0,1)^s$ by an $s-$dimensional trigonometric polynomial

$$f(\mathbf{x}) \approx \sum_{\mathbf{k}\in S} c_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}},$$

We do so by sampling f in N grid points $\mathbf{x}_j \in [0,1)^s$ such that

$$f(\mathbf{x}_j) = \sum_{\mathbf{k}\in S} c_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}_j} \quad \forall\, \mathbf{x}_j \in \Omega. \tag{1}$$

Ω is called the *sampling grid*, the vector $\mathbf{k} = (k_1, k_2, \cdots, k_s)$ is a multi-index, commonly called wave numbers or Fourier indices, $\mathbf{k}\cdot\mathbf{x} = \mathbf{k}^T\mathbf{x}$ denotes the innerproduct of $\mathbf{k} \in S$ and $\mathbf{x} \in R^s$, and $S \subset \mathbb{Z}^s$ is a finite set with $|S| = N$. S defines the approximation space $\mathscr{H}_S = \{e^{2\pi i \mathbf{k}\cdot\mathbf{x}} \mid \mathbf{k} \in S\}$. We write

$$If = \sum_{\mathbf{k}\in S} c_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}},$$

where I denotes the interpolation operator. For fixed Ω, S and f, (1) defines a linear system of equations for the coefficients $c_{\mathbf{k}}$. If the system is non-singular, the grid is said to be unisolvent with respect to $\mathscr{H}_S$. If sufficient structure is present in the point set, Ω, the FFT-algorithm may be used to solve (1), offering huge savings in computational cost. Unisolvency also ensures that a set of N Lagrange functions satisfying $L_\ell(\mathbf{x}_j) = \delta_{\ell,j}$ exists. If these can be described explicitly, the interpolation may be written: $If = \sum_{\ell=1}^{N} f(\mathbf{x}_\ell) L_\ell(\mathbf{x})$.

The obvious extension to multi-dimensional interpolation is done by taking the tensor product of your favorite one-dimensional interpolation grid. Then the well-known

one-dimensional theory can be straightforwardly extended. However, the exponential increase in cost severely limits this approach and for that reason using non-tensorial sampling grids such as sparse grids [1], [3], [18] and lattice grids [4], [16] have been suggested.

In this paper we will focus on the lattice grid approach. In particular we will establish the relation between a given lattice grid and its corresponding approximation space, and show how to construct the associated Lagrange functions and efficient computation of the expansion coefficients, $c_{\mathbf{k}}$ by the FFT. This was also an issue in [16]. In that paper our proofs were restricted to rank-1 lattices of prime order. In this paper we generalize this result to all 1-periodic lattices. Framing these problems in terms of group theory gives access to the full arsenal of group theoretical tools. This allows us to more precisely describe the decomposition of higher rank lattices into rank-1 lattices, which among other things are the bases for a variable transformation permitting the FFT to be used for fast computation of the interpolation coefficients. Again this has been done before in [14], but there it remains unclear exactly how to relate the computed coefficients with the corresponding basis functions.

For a thorough understanding of the basic properties of 1-periodic integration lattices we recommend [11], [12] and sections 1-4 of [17]. In general, a good understanding of Fourier analysis on lattice grids requires basic knowledge of group theory, especially Abelian and quotient groups. Good references are [2], [13], and [14].

Similar work has been pursued by Li, Sun and Xu, and reported in a series of papers [5], [6], [7], [8] and [9]. Their work is targeting other physical domains in 2 or 3 dimensions such as triangles, hexagons, etc. On the other hand they are not limiting themselves to trigonometric interpolation as they employ variable transformations to obtain Chebyshev's polynomials for algebraic polynomial interpolations. These are then used to develop interpolating quadrature rules.

We first establish the correct correspondence between the interpolating lattice points and the corresponding approximation space. In Section 3 we show how to construct a full set of trigonometric Lagrange functions. In Section 4 we establish the proper variable transformations allowing us to to compute the interpolation coefficients by the FFT.

2 The correspondence between the sampling grid and the index set

An s-dimensional lattice Λ is a finitely generated Abelian group under vector addition. Alternatively it may be viewed as a linear integer combination of s linearly independent basis vectors. When arranging the basis vectors as rows in a matrix, the matrix is said to be an generator matrix for the lattice. In this paper we will consider only $[0,1)^s$-periodic integration lattices, and thus all lattice points may be written $\mathbf{x} = \frac{\mathbf{z}}{N}$, with $\mathbf{z} \in \mathbb{Z}^s$ [10].

Definition 1. *The lattice (sampling) grid T of Λ is*

$$T = \Lambda \cap [0,1)^s.$$

If Λ is periodic on $[0,1)^s$, then T is a group under addition mod 1, and

$$T \simeq \Lambda / \mathbb{Z}^s.$$

See [11] and [17] for details. From here on, any addition of lattice points in T will be tacitly understood to be mod 1. We shall write $N = |T|$ throughout this paper.

Definition 2. *Let* $\mathbf{x} \in T$. *The order* d *of* $\mathbf{x}$ *is the least natural number such that* $d\mathbf{x} = 0$. *The subgroup of* T *generated by* $\mathbf{x}$ *is the set*

$$\{\mathbf{x}\} = \{j\mathbf{x} \mid \quad 0 \le j \le d-1\}.$$

It is clear that $\{\mathbf{x}\}$ is a cyclic group of order d and its periodic extension is a lattice, $\Lambda_{\mathbf{x}}$ and $\Lambda_{\mathbf{x}} \subseteq \Lambda$.

Definition 3. *Let* $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_t \in T$. *We say that* T *has rank* t, *and is generated by* $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_t$ *if*

$$T \simeq \{\mathbf{x}_1\} \oplus \{\mathbf{x}_2\} \oplus \cdots \oplus \{\mathbf{x}_t\}.$$

All finite Abelian groups are direct products of cyclic groups. The orders of the generators of T are called invariants, denoted d_1, d_2,...,d_t. The generators are in general not unique, but the invariants are, under the condition that $d_l | d_{l+1}$ for all l. Elementary group theory implies that $N = \prod_{l=1}^{t} d_l$.

Lemma 1. *Let* $\mathbf{x} = \frac{\mathbf{z}}{N} \in T$, *with* $\mathbf{z} \in \mathbb{Z}^s$, *and* $a = \gcd(\mathbf{z}, N)$, *then the order of* $\mathbf{x}$ *is* N/a.

Proof. Since a divides $\mathbf{z}$, we have

$$\frac{N}{a}\frac{\mathbf{z}}{N} = \mathbf{0}.$$

Let $\mathbf{z}' = \mathbf{z}/a$. If there exist a natural number $b < \frac{N}{a}$ such that b is the order of $\mathbf{x}$, then elementary group theory dictates that we must have $bc = \frac{N}{a}$ for some natural number $c > 1$. Then

$$\frac{\mathbf{z}'}{c} = b\frac{\mathbf{z}}{N} = \mathbf{0},$$

and this implies that c divides all components of $\mathbf{z}'$. But then ac divides $\mathbf{z}$, hence cannot divide N, so $b = \frac{N}{ac}$ is not an integer, and arriving at a contradiction, we conclude that no $b < \frac{N}{a}$ exists. □

It follows that we may always write $\mathbf{x} = \frac{\mathbf{z}}{d}$, with $\mathbf{z} \in \mathbb{Z}^s$, where d is the order of $\mathbf{x}$. If $\mathbf{x} \in T$ has order N, then $\mathbf{x}$ generates T.

We now turn our attention to approximation spaces corresponding to a particular sampling grid T. As stated above this is equivalent to finding index sets $S \in \mathbb{Z}^s$ or $S_{\mathbf{x}}$ for sampling sets T or $\{\mathbf{x}\}$, respectively. Associated with T (or Λ) is the dual lattice.

Definition 4. *The dual lattice of* T *is*

$$\Lambda^{\perp} = \{\mathbf{k} \,:\, \mathbf{k} \cdot \mathbf{x} \in \mathbb{Z} \quad \forall\, \mathbf{x} \in T\}.$$

As $\mathbf{k}, \mathbf{h} \in \Lambda^{\perp} \Rightarrow \mathbf{k} + \mathbf{h} \in \Lambda^{\perp}$, $\Lambda^{\perp}$ is itself a lattice and whenever Λ is 1-periodic, $\Lambda^{\perp}$ is an integer lattice. We may also define

Definition 5. *The dual lattice of* $\{\mathbf{x}\}$ *is*

$$\Lambda_{\mathbf{x}}^{\perp} = \{\mathbf{k} \,:\, \mathbf{k}\cdot\mathbf{x} \in \mathbb{Z} \quad \forall\, \mathbf{x} \in \{\mathbf{x}\}\}.$$

A key observation is that two Fourier modes $e^{2\pi i\mathbf{k}\cdot\mathbf{x}}$ and $e^{2\pi i\mathbf{h}\cdot\mathbf{x}}$ are indistinguishable for $\mathbf{x} \in T$ if $\mathbf{k}-\mathbf{h} \in \Lambda^{\perp}$. $\mathbf{k}$ and $\mathbf{h}$ are said to be aliasing. To be useful for us S must contain only non-aliasing indecies.

Note that $\Lambda_x \subseteq \Lambda$ implies $\Lambda^{\perp} \subseteq \Lambda_{\mathbf{x}}^{\perp}$, and that $\Lambda_{\mathbf{x}}^{\perp} = \Lambda^{\perp}$ if the order of $\mathbf{x} \in T$ is N. The subgroup $\{\mathbf{x}\}$ is treated specifically with respect to aliasing.

Lemma 2. *Let d be the order of* $\{\mathbf{x}\}$*, and let* $\mathbf{z},\mathbf{k},\mathbf{h} \in \mathbb{Z}^s/\{\mathbf{0}\}$*, with* $\mathbf{x} = \frac{\mathbf{z}}{d} \in T$*. Then*

$$\mathbf{k}\cdot\mathbf{z} \equiv \mathbf{h}\cdot\mathbf{z} \pmod{d} \quad \Leftrightarrow \quad \mathbf{k}-\mathbf{h} \in \Lambda_{\mathbf{x}}^{\perp}. \tag{2}$$

Proof. A simple computation yields

$$\begin{gathered} \mathbf{k}\cdot\mathbf{z} \equiv \mathbf{h}\cdot\mathbf{z} \pmod{d} \\ \Updownarrow \\ (\mathbf{k}-\mathbf{h})\cdot\mathbf{z} = m;\ m \in \mathbb{Z}, \end{gathered}$$

and the lemma follows from Definition 5. □

Lemma 3. *Let d be the order of* $\mathbf{x} \in T$*. Then* $|S_{\mathbf{x}}| = d$*.*

Proof. Let $\mathbf{x} = \frac{\mathbf{z}}{d}$ with $\mathbf{z} \in \mathbb{Z}^s$. The equivalence relation

$$\mathbf{k} = \mathbf{h} \quad \text{if} \quad \mathbf{k}\cdot\mathbf{z} \equiv \mathbf{h}\cdot\mathbf{z} \pmod{d},$$

clearly partitions $\mathbb{Z}^s$ into d equivalence classes. From lemma 2 we know that this equivalence relation is equivalent to the equivalence relation

$$\mathbf{k} = \mathbf{h} \quad \text{if} \quad \mathbf{k}-\mathbf{h} \in \Lambda_{\mathbf{x}}^{\perp}.$$

Accordingly, these two equivalence relations partition $\mathbb{Z}^s$ in the same number of equivalence classes, and thus $|S_{\mathbf{x}}| = d$. □

We now turn to the construction of unisolvent approximation spaces on T. Since $\Lambda^{\perp}$ is a normal subgroup of $\mathbb{Z}^s$, we may construct $\mathbb{Z}^s/\Lambda^{\perp}$, which is a group of equivalence classes under the equivalence relation

$$\mathbf{k} = \mathbf{h} \quad \text{if} \quad \mathbf{k}-\mathbf{h} \in \Lambda^{\perp}.$$

Theorem 1. *A non-aliasing Fourier index set for T is*

$$S = \mathbb{Z}^s/\Lambda^{\perp}.$$

Proof. For computations we choose a representative from each equivalence class; it is then evident that no two representatives will alias. Moreover, the set of representatives will be isomorphic to $\mathbb{Z}^s/\Lambda^\perp$ under the addition inherited from $\mathbb{Z}^s/\Lambda^\perp$: if $\mathbf{k},\mathbf{h},\mathbf{l} \in \mathbb{Z}^s$ are representatives for $[\mathbf{k}],[\mathbf{h}],[\mathbf{l}] \in \mathbb{Z}^s/\Lambda^\perp$, then $\mathbf{k}+\mathbf{h}=\mathbf{l}$ if $[\mathbf{k}]+[\mathbf{h}]=[\mathbf{l}]$. □

We write $S_\mathbf{x} = \mathbb{Z}^s/\Lambda_\mathbf{x}^\perp$.

Lemma 4. *The cosets of $S_\mathbf{x}$ partition S.*

Proof. Since $\Lambda^\perp$ is a normal subgroup of $\Lambda_\mathbf{x}^\perp \subset \mathbb{Z}^s$, we know from the fundamental theorem of quotient groups [2] that

$$\frac{\mathbb{Z}^s/\Lambda^\perp}{\Lambda_\mathbf{x}^\perp/\Lambda^\perp} \simeq \mathbb{Z}^s/\Lambda_\mathbf{x}^\perp,$$

which says that the cosets of $\mathbb{Z}^s/\Lambda_\mathbf{x}^\perp$ partition $\mathbb{Z}^s/\Lambda^\perp$. Since $S = \mathbb{Z}^s/\Lambda^\perp$ and $S_\mathbf{x} = \mathbb{Z}^s/\Lambda_\mathbf{x}^\perp$, the result follows. □

3 Trigonometric Lagrange functions for lattice grids

Definition 6. *The Dirichlet kernel of S on $[0,1)^s$ is*

$$D_S(\mathbf{x}) = \sum_{\mathbf{k}\in S} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}. \tag{3}$$

We proceed to prove that the Dirichlet kernel is zero on all $\mathbf{x} \in T$ except at the origin.

Lemma 5. *For $\mathbf{x} \in T$.*

$$D_{S_\mathbf{x}}(\mathbf{x}) = \begin{cases} 0;\, \mathbf{x} \in T\setminus\{\mathbf{0}\} \\ |S_\mathbf{x}|;\, \mathbf{x}=\mathbf{0} \end{cases}.$$

Proof. Let $\mathbf{x} = \frac{\mathbf{z}}{d}$, with $\mathbf{z} \in \mathbb{Z}^s$. We have $\mathbf{k}\cdot\mathbf{x} = \mathbf{k}\cdot\mathbf{z}/d = \frac{m}{d}$ for all $\mathbf{k} \in S_\mathbf{x}$. Now m takes at most d different values, and due to lemma 2, none of them are equal mod d. This implies that $D_{S_\mathbf{x}}(\mathbf{x})$ is a geometric series, and for $\mathbf{x} \neq \mathbf{0}$ we may compute

$$D_{S_\mathbf{x}}(\mathbf{x}) = \sum_{\mathbf{k}\in S_\mathbf{x}} f_\mathbf{k}(\mathbf{x}) = \sum_{\mathbf{k}\in S_\mathbf{x}} e^{2\pi i\mathbf{k}\cdot\mathbf{x}} = \sum_{m=0}^{d-1} e^{\frac{2\pi i m}{d}} = 0.$$

The case $\mathbf{x}=\mathbf{0}$ is trivial. □

Theorem 2. *Let $\mathbf{x} \in T$. Then*

$$D_S(\mathbf{x}) = \begin{cases} 0;\ \mathbf{x} \in T\setminus\{\mathbf{0}\} \\ N;\, \mathbf{x}=\mathbf{0} \end{cases}.$$

Proof. Since $S_\mathbf{x}$ partitions S, we just rearrange the sum in (3) and apply Lemma 5

$$D_S(\mathbf{x}) = \sum_{\mathbf{k}\in S} e^{2\pi i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{h}\in S/S_\mathbf{x}} \sum_{\mathbf{l}\in S_\mathbf{x}} e^{2\pi i(\mathbf{h}+\mathbf{l})\cdot\mathbf{x}} = \sum_{\mathbf{h}\in S/S_\mathbf{x}} e^{2\pi i\mathbf{h}\cdot\mathbf{x}} \sum_{\mathbf{l}\in S_\mathbf{x}} e^{2\pi i\mathbf{l}\cdot\mathbf{x}} = \begin{cases} 0;\ \mathbf{x}\neq\mathbf{0} \\ N;\, \mathbf{x}=\mathbf{0} \end{cases}$$

□

We can now construct a complete set of Lagrange functions.

Corollary 1. *For any $\mathbf{y} \in T$ a trigonometric Lagrange function is given as*

$$L_\mathbf{y}(\mathbf{x}) = \frac{1}{N} D_S(\mathbf{x}-\mathbf{y}).$$

4 Fast Fourier Transform on lattice grids

By utilizing the Smith normal form, the standard FFT-algorithm may be extended in a natural way to lattice grids of any rank. In the following, we shall write $T(A)$ for the sampling grid generated by the generator matrix $A \in \mathbb{R}^{s\times s}$.

Definition 7. *The index set of $T(A)$ is*

$$S(A) = \{k^T = x^T A^{-1} |\, x \in T(A)\} \tag{4}$$

For integration lattices, A^{-1} is an integer matrix, and $N = \det A^{-1}$, see [10]. If A is non-singular, we have the following trivial lemma.

Lemma 6. *$S(A)$ is a group, with $T(A) \simeq S(A)$.*

Proof. Let $\mathbf{x},\mathbf{y},\mathbf{z} \in T(A)$, let $\mathbf{k},\mathbf{h},\mathbf{l} \in S(A)$, with $\mathbf{k} = A^{-1}\mathbf{x}$, $\mathbf{h} = A^{-1}\mathbf{y}$, $\mathbf{l} = A^{-1}\mathbf{z}$ and $\mathbf{x}+\mathbf{y} = \mathbf{z}$. If we agree that $\mathbf{k}+\mathbf{h} = \mathbf{l}$, then under this addition, $S(A)$ is a group, with $T(A) \simeq S(A)$. □

In the preceding section we used the letter S to denote non-aliasing Fourier index sets, and in this subsection we use the same letter to denote lattice index sets. This makes sense because of the following theorem, proved in [16].

Theorem 3. *$S(A^T)$ is a non-aliasing index set for $T(A)$.*

Note that the generator matrix A is not unique. Any matrix UA where U is unimodular (integer matrices with determinant equal ± 1), generates the same lattice and consequently $T(A) = T(UA)$. However, using Definition 7 we see that $S(UA) = \{\mathbf{k}^T U | k \in S(A)\}$ and consequently $S(A) \neq S(UA)$. In [16] we give algorithms for computing good index sets for typical function classes.

To solve (1) efficiently by the multi-dimensional FFT-algorithm the sampling points as well as the index-set must form a hyper-rectangular equidistant grid. This is not the case for $T(A)$ and $S(A^T)$, respectively. However, an appropriate grid/index-set may be obtained by a simple structure preserving variable transformation. The inverse generator matrix A^{-1} may be decomposed by the Smith normal form [15] as

$$\tilde{D} = \tilde{U}A^{-1}\tilde{V}; \qquad \tilde{D} = \mathrm{diag}(d_1,\ldots,d_s); \qquad d_\ell | d_{\ell+1},$$

where $\tilde{U}$ and $\tilde{V}$ are $s\times s$ unimodular matrices, and $\tilde{D} \in \mathbb{Z}^{s\times s}$ is a unique diagonal matrix with $d_\ell | d_{\ell+1}$ for $1 \le \ell < s$, the invariants of $T(A)$. If $t < s$ we may omit the upper $s-t$ rows of U and the leftmost $s-t$ columns of V, writing

$$D = UA^{-1}V, \tag{5}$$

where $U \in \mathbb{Z}^{t\times s}$, $V \in \mathbb{Z}^{s\times t}$, and $D \in \mathbb{Z}^{t\times t}$. Transposition of (5) proves that $T(A)$ and $T(A^T)$ have the same invariants, hence

$$T(A) \simeq T(A^T),$$

and Lemma 6 then implies

$$T(A) \simeq S(A^T).$$

In [12] Lyness and Keast showed that the rows of $D^{-1}U$ generate $T(A)$. Transposition shows that the columns of VD^{-1} similarly generate $T(A^T)$, and hence the columns of $A^{-1}VD^{-1}$ generate $S(A^T)$. The Cartesian grid with d_ℓ points in the ℓ-th coordinate direction may be written $T(D^{-1})$, and its non-aliasing Fourier index set is

$$S(D^{-1}) = \{\mathbf{h}: \quad 0 \le h_\ell < d_\ell \quad 1 \le \ell \le t\}.$$

More formally we may write

$$T(D^{-1}) = \left\{\mathbf{y} : \mathbf{y} = D^{-1}\mathbf{h};\ \mathbf{h} \in S(D^{-1})\right\}.$$

The sampling grid $T(D^{-1})$ and the index space $S(D^{-1})$ are standard regular equidistant t-dimensional grids which allows straightforward use of the FFT for solving equation (1). Now let $\mathbf{x} \in T(A)$, $\mathbf{k} \in S(A^T)$, and $\mathbf{h}, \mathbf{l} \in S(D^{-1})$, with

$$\mathbf{x}^T = \mathbf{h}^T D^{-1} U = \mathbf{y}^T U \tag{6}$$

and

$$\mathbf{k} = A^{-1} V D^{-1} \mathbf{l}. \tag{7}$$

The matrix-vector form for computing all $f(\mathbf{x}_j)$ of (1) becomes

$$\mathbf{f} = F\mathbf{c} \qquad \text{where} \qquad (F)_{j,l} = e^{2\pi i \mathbf{k}_l \mathbf{x}_j};\ \mathbf{k}_l \in S(A^T);\ \mathbf{x}_j \in T(A). \tag{8}$$

If $\mathbf{x}$, $\mathbf{y}$ are related by (6) and $\mathbf{k}$, $\mathbf{l}$ by (7), then by (5) it follows that

$$\exp(2\pi i \mathbf{x} \cdot \mathbf{k}) = \exp(2\pi i \mathbf{y} \cdot \mathbf{l}).$$

Thus the matrix F in (8) is just a permutation of the matrix produced by the $T(D^{-1})$ grid. The matrix obtained by the **y, l** entries correspond to the standard t-dimensional inverse Fourier transform, efficiently carried out by the FFT-algorithm. The permutation is implicitly defined by (6) and (7), and care needs to be taken when matching coefficients with function values in (1).

The computations in (6) and (7) are linear in N. Thus the total complexity is dominated by the FFT which is of order $O(N \log N)$, with a constant factor weekly depending on how N factorize. This complexity stays the same regardless of whether we do a one dimensional FFT of length N (for a rank-1 lattice) or we do a t-dimensional FFT on a $d_1 \times d_2 \times \cdots \times d_t$ array for $N = \prod_{j=1}^{t} d_j$ in the case of a rank-t lattice with N points.

5 Numerical examples

As an example, consider the lattice generated by A when

$$A^{-1} = \begin{pmatrix} 0 & -1 & -4 & -5 \\ 5 & 0 & -1 & -4 \\ 4 & 5 & 0 & -1 \\ 1 & 4 & 5 & 0 \end{pmatrix}$$

Its Smith Normal form, $\tilde{D} = \tilde{U}A^{-1}\tilde{V}$ is:

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&6&0\\ 0&0&0&102 \end{pmatrix} = \begin{pmatrix} 1&-4&4&0\\ 3&4&-5&3\\ -17&3&1&-13\\ -19&-4&9&-16 \end{pmatrix} \begin{pmatrix} 0&-1&-4&-5\\ 5&0&-1&-4\\ 4&5&0&-1\\ 1&4&5&0 \end{pmatrix} \begin{pmatrix} -5&0&-19&263\\ -1&0&-4&55\\ 1&-1&7&-117\\ 0&0&0&1 \end{pmatrix}.$$

This tells us that the lattice has rank 2, with invariants 6 and 102. Utilizing the reduced Smith Normal form, the rows of

$$D^{-1}U = \tilde{D}^{-1}(3:4,3:4)\tilde{U}(3:4,:) = \begin{pmatrix} \frac{1}{6}&0\\ 0&\frac{1}{102} \end{pmatrix} \begin{pmatrix} -17&3&1&-13\\ -19&-4&9&-16 \end{pmatrix},$$

are generators for $T(A)$. The columns of

$$A^{-1}VD^{-1} = \begin{pmatrix} 0&-1&-4&-5\\ 5&0&-1&-4\\ 4&5&0&-1\\ 1&4&5&0 \end{pmatrix} \begin{pmatrix} -19&263\\ -4&55\\ 7&-117\\ 0&1 \end{pmatrix} \begin{pmatrix} \frac{1}{6}&0\\ 0&\frac{1}{102} \end{pmatrix},$$

are generators for $S(A^T)$. To get the lattice points in $[0,1)^s$, needed for practical computation, the points obtained by (6) have to be taken modulo 1. Likewise, members of the non-aliasing set of Fourier coefficients obtained using (7) need to be shifted so that they represent the most significant Fourier modes, typically the lowest frequencies. As an illustration we have computed the interpolation on the above lattice grid for four 1-periodic functions and compared the error to the similar interpolation on a regular Cartesian grid. In all cases the lattice grid produce a more accurate interpolant. More exhaustive experimental results for these testfunctions on lattice grid versus regular grid in 2 and 3 dimension is given in [16], and they show a clear advantage for lattice grid. The advantage seems to increase with the dimension.

function	Lattice grid	regular grid
$f_1(\mathbf{x}) = \prod_{\ell=1}^{s}(x_\ell - 1)^2 x_\ell^2$	0.031	0.038
$f_2(\mathbf{x}) = \prod_{\ell=1}^{s} e^{\sin(2\pi x_\ell)-1}$	0.036	0.128
$f_3(\mathbf{x}) = \prod_{\ell=1}^{s}(2+\mathrm{sign}(x_\ell - \frac{1}{2})\sin(2\pi x_\ell)^p)$	0.250	0.367
$f_4(\mathbf{x}) = e^{-\lambda\prod_{\ell=1}^{s}(x_\ell-1/2)^2}$	0.068	0.084

Table 1: The relative error for the interpolating function when the grid is produced by the lattice in this section or by a regular, equidistant Cartesian grid using $N = 625$ gridpoints. The error is estimated by computing $||f_k(\mathbf{x}) - If_k(\mathbf{x})||_1/||f_k(\mathbf{x})||_1$ on a regular fine grid ($31\times31\times31\times31$ gridpoints). For f_3 we used $p = 3$ and for f_4, $\lambda = 7$.

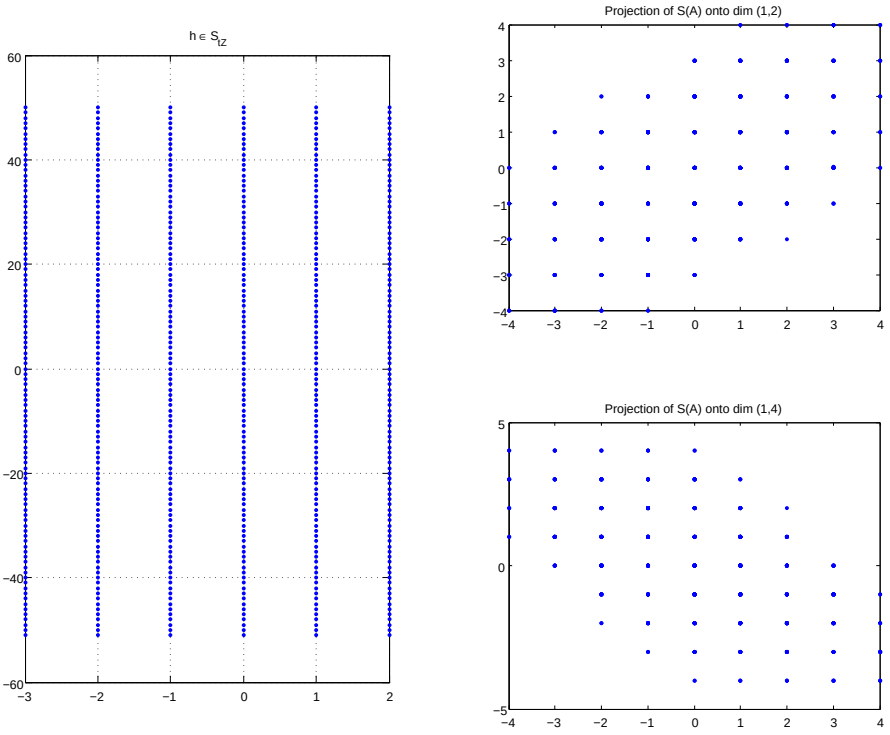

Fig. 1: The left frame displays the set $S(D^{-1})$. In the right frame we display two different 2-dim projections of the 4-dim index array $S(A)$. The 4 remaining 2-dim projections have a similar look.

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