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A New Version of the Hermite–Hadamard Inequality for Riemann–Liouville Fractional Integrals

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Abstract: Integral inequalities play a critical role in both theoretical and applied mathematics fields. It is clear that inequalities aim to develop different mathematical methods. Thus, the present days need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. The concept of convexity plays a strong role in the field of inequalities due to the behavior of its definition. There is a strong relationship between convexity and symmetry. Whichever one we work on, we can apply it to the other one due the strong correlation produced between them, especially in the past few years. In this article, we firstly point out the known Hermite–Hadamard (HH) type inequalities which are related to our main findings. In view of these, we obtain a new inequality of Hermite–Hadamard type for Riemann–Liouville fractional integrals. In addition, we establish a few inequalities of Hermite–Hadamard type for the Riemann integrals and Riemann–Liouville fractional integrals. Finally, three examples are presented to demonstrate the application of our obtained inequalities on modified Bessel functions and q-digamma function.

Keywords: hermite–hadamard inequality; riemann–liouville fractional integrals; special functions

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1. Introduction

Generally, integral inequalities form a strong and thriving field of study within the huge field of mathematical analysis. They have participated in the study of many common fields, e.g., ordinary differential equations, integral equations, and partial differential equations [1,2]. Particularly, they have participated in the field of fractional differential equations, especially fractional integral inequalities, which have been crucial in providing bounds to solve initial and boundary value problems in fractional calculus, and in establishing the existence and uniqueness of solutions for certain fractional differential equations [3–6].

The most efficient branch of mathematical analysis is fractional calculus, which involves integrals and derivatives taken to fractional orders or orders outside of the integer or natural numbers. Here, we present the Riemann–Liouville (RL) definition to facilitate the discussion of the aforementioned operations, which is most commonly used for fractional derivatives and integrals.

Definition 1 ([7,8])**.** *Let $x \in \mathbf{J} := [\alpha_3, \alpha_4]$. Then, for any L^1 function $\bar{G}$ on the interval $\mathbf{J}$, the ηth left-RL and right-RL fractional integrals of $\bar{G}(x)$ are, respectively, defined by:*

$$\mathfrak{I}^{\eta}_{\alpha_3+}\bar{G}(x) = \frac{1}{\Gamma(\eta)}\int_{\alpha_3}^{x}(x-\bar{\chi})^{\eta-1}\bar{G}(\bar{\chi})d\bar{\chi}, \tag{1}$$

and

$$\Im^{\eta}_{\alpha_4-}\bar{G}(x)=\frac{1}{\Gamma(\eta)}\int_x^{\alpha_4}(\bar{\chi}-x)^{\eta-1}\,\bar{G}(\bar{\chi})\mathrm{d}\bar{\chi},$$

where $\mathrm{Re}(\eta)>0$ *and* $\Gamma(\cdot)$ *represents the known gamma function:*

$$\Gamma(\eta)=\int_0^{\infty}\bar{\chi}^{\eta-1}e^{-\bar{\chi}}\mathrm{d}\bar{\chi},\quad \eta>0.$$

The most well-known inequality, which has a strong relationship with the integral mean of a convex function, is the Hermite–Hadamard (HH) inequality [9]:

$$\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\leq\frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4}\bar{G}(\bar{\chi})\mathrm{d}\bar{\chi}\leq\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2},\tag{2}$$

where $\bar{G}:\mathbf{J}\subseteq\mathcal{R}\to\mathcal{R}$ is assumed to be a convex function and $\alpha_3,\alpha_4\in\mathbf{J}$ with $\alpha_3<\alpha_4$.

The HH inequality in Equation (2) gives an approximate from both sides of the mean value of a convex function and ensures the integrability of the convex function as well. In addition, it is a matter of great attention and one has to observe that some of the classical inequalities for means can be obtained from Hadamard's inequality under the usefulness of peculiar convex functions $\bar{G}$. The inequality in Equation (2) plays a crucial role in mathematical analysis and in other areas of pure and applied mathematics as well. Typical applications of the classical inequalities are: probabilistic problems, decision making in structural engineering, and fatigue life.

The right and left part inequality of the inequalities in Equation (2) are called trapezoidal and midpoint inequalities. There are two types of the researchers who have been worked on the inequalities in Equation (2). Many of them have worked only on the trapezoidal type inequality [10–12] or midpoint type inequality [13,14], while others have bee n worked on both of them at the same time [15–17]. Both trapezoidal and midpoint inequalities can be explained in the following definition:

Definition 2 ([15])**.** *Suppose* $\bar{G}:[\alpha_3,\alpha_4]\subseteq\mathcal{R}\to\mathcal{R}$ *is a twice differentiable function on an open interval* (α_3,α_4) *with* $\|\bar{G}''\|_\infty:=\sup_{x\in(\alpha_3,\alpha_4)}\left|\bar{G}''(x)\right|<\infty$*. Then, the trapezoidal and midpoint type inequalities are defined by:*

$$\left|\int_{\alpha_3}^{\alpha_4}\bar{G}(\bar{\chi})\mathrm{d}\bar{\chi}-\frac{\alpha_4-\alpha_3}{2}[\bar{G}(\alpha_3)+\bar{G}(\alpha_4)]\right|\leq\frac{(\alpha_4-\alpha_3)^3}{12}\left\|\bar{G}''\right\|_\infty,\tag{3}$$

and

$$\left|\int_{\alpha_3}^{\alpha_4}\bar{G}(\bar{\chi})\mathrm{d}\bar{\chi}-(\alpha_4-\alpha_3)\,\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right|\leq\frac{(\alpha_4-\alpha_3)^3}{24}\left\|\bar{G}''\right\|_\infty,\tag{4}$$

respectively.

From a complementary viewpoint to Ostrowski type inequalities [18], trapezoidal and midpoint type inequalities provides a prior error bounds in estimating the Riemann integral by a generalized midpoint and trapezoidal formula [15,19]. We know that the development of Ostrowski's inequality has registered an attractive growth in the past decade with more than two thousand papers published on it. Many refinements, generalizations, and extensions in both discrete and integral cases have been discovered (see [16,17]). Generalized versions were discussed, e.g., the corresponding versions on time scales, form-time differentiable functions, for multiple integrals, or vector valued functions (see [15,20]). Many applications in special functions, numerical analysis, probability model, and other fields have been proved (see [16]) as well.

In 2013, Sarikaya et al. [12] generalized the Hermite–Hadamard inequality in Equation (2) to fractional integrals of Riemann–Liouville type. Their result is stated as follows:

$$\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \le \frac{\Gamma(\eta+1)}{2(\alpha_4-\alpha_3)^\eta}\left\{\mathfrak{I}^\eta_{a+}\bar{G}(\alpha_4)+\mathfrak{I}^\eta_{b-}\bar{G}(\alpha_3)\right\} \le \frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}, \tag{5}$$

where $\bar{G}:[\alpha_3,\alpha_4]\to\mathcal{R}$ is assumed to be an L^1 convex function with $\eta>0$. Meanwhile, they obtained some inequalities of trapezoidal type in the same paper.

On the other hand, Sarikaya and Yildirim [13] introduced a new version of the HH-inequality (Equation (2)) for Riemann–Liouville fractional integrals:

$$\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \le \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left\{\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^+}\bar{G}(\alpha_4)+\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^-}\bar{G}(\alpha_3)\right\} \le \frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}. \tag{6}$$

Meanwhile, they obtained some inequalities of midpoint type in the same paper.

There are many papers studying integral inequalities for the Riemann–Liouville fractional integrals and some new relevant generalizations of Hermite–Hadamard type inequalities (see [11,12,19,21–27] for more details).

The purpose of this paper is to introduce a new inequality of Hermite–Hadamard type and to establish a few related inequalities for Riemann–Liouville fractional integrals.

2. The New Hermite–Hadamard Inequality

In view of the HH-inequality in Equation (5) and HH-inequality in Equation (6), we can deduce the following version of the HH-inequality.

Proposition 1. *Let $\bar{G}:[\alpha_3,\alpha_4]\to\mathcal{R}$ be an L^1 convex function on $[\alpha_3,\alpha_4]$ with $\alpha_3<\alpha_4$. Then for $\eta>0$, we have*

$$\frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left[\mathfrak{I}^\eta_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\mathfrak{I}^\eta_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^+}\bar{G}(\alpha_4)+\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^-}\bar{G}(\alpha_3)\right] \le \bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}. \tag{7}$$

Moreover, we have the following new HH-inequality

$$\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \le \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left[\mathfrak{I}^\eta_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\mathfrak{I}^\eta_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right] \le \frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}. \tag{8}$$

Proof. Applying twice the right part of the HH-inequality in Equation (5) with α_3 replaced by $\frac{\alpha_3+\alpha_4}{2}$ and α_4 replaced by $\frac{\alpha_3+\alpha_4}{2}$, obtains

$$\frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left[\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^+}\bar{G}(\alpha_4)+\mathfrak{I}^\eta_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right] \le \frac{1}{2}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\frac{1}{2}\bar{G}(\alpha_4);$$

$$\frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left[\mathfrak{I}^\eta_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\mathfrak{I}^\eta_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^-}\bar{G}(\alpha_3)\right] \le \frac{1}{2}\bar{G}(\alpha_3)+\frac{1}{2}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right).$$

Adding these together, we see that the inequality in Equation (7) is proved directly.

On the other hand, we can deduce from the inequalities in Equations (6) and (7),

$$\frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left[\mathfrak{I}^\eta_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\mathfrak{I}^\eta_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right] \le \frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}, \tag{9}$$

that this gives the right part of the inequality in Equation (8).

By convexity of $\bar{G}$, we can write

$$\bar{G}\left(\frac{\bar{x}+\bar{y}}{2}\right) \leq \frac{\bar{G}\left(\bar{x}\right)+\bar{G}\left(\bar{y}\right)}{2}$$

and for $\bar{x} = \frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4$ and $\bar{y} = \frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4$, write

$$2\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \leq \bar{G}\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right) + \bar{G}\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right), \quad \bar{\chi} \in [0,1]. \tag{10}$$

Multiplying both sides of Equation (10) by $(1-\bar{\chi})^{\eta-1}$, and then integrating both sides with respect to $\bar{\chi}$ over $[0,1]$, we get

$$2\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\int_0^1 (1-\bar{\chi})^{\eta-1}\mathrm{d}\bar{\chi} \leq \int_0^1 (1-\bar{\chi})^{\eta-1}\bar{G}\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} + \int_0^1 (1-\bar{\chi})^{\eta-1}\bar{G}\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi}.$$

By changing the variables in both right side integrals, we obtain

$$\frac{2}{\eta}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \leq \frac{2^{\eta}}{(\alpha_4-\alpha_3)^{\eta}}\int_{\frac{\alpha_3+\alpha_4}{2}}^{\alpha_4}\left(x-\frac{\alpha_3+\alpha_4}{2}\right)^{\eta-1}\bar{G}(x)\mathrm{d}x + \frac{2^{\eta}}{(\alpha_4-\alpha_3)^{\eta}}\int_{\alpha_3}^{\frac{\alpha_3+\alpha_4}{2}}\left(\frac{\alpha_3+\alpha_4}{2}-x\right)^{\eta-1}\bar{G}(x)\mathrm{d}x.$$

This can be expressed as

$$\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \leq \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left[\mathfrak{I}^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \mathfrak{I}^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right], \tag{11}$$

and thus the left part of the inequality in Equation (8) is proved. Therefore, the inequalities in Equations (9) and (11) can be rearranged to the desired inequality in Equation (8). □

Remark 1. *The inequality in Equation (7) with $\eta = 1$ becomes:*

$$\frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4}\bar{G}(x)\mathrm{d}x \leq \frac{1}{2}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \frac{1}{2}\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}.$$

Remark 2. *The inequality in Equation (8) with $\eta = 1$ becomes the inequality in Equation (2).*

Lemma 1. *If $\bar{G} : [\alpha_3, \alpha_4] \to \mathcal{R}$ is an L^1 function, then we have the following trapezoidal formula equality:*

$$\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\mathfrak{I}^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \mathfrak{I}^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}$$
$$= \frac{\alpha_4-\alpha_3}{4}\left\{\int_0^1 (1-\bar{\chi})^{\eta}\bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} - \int_0^1 (1-\bar{\chi})^{\eta}\bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi}\right\}. \tag{12}$$

Proof. By making the use of integrating by parts and then changing the variable, we get

$$\begin{aligned}
\epsilon_1 :&= \int_0^1 (1-\bar{\chi})^\eta \bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} \\
&= -\frac{2(1-\bar{\chi})^\eta}{\alpha_4-\alpha_3}\bar{G}\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\Bigg|_{\bar{\chi}=0}^{1} - \frac{2\eta}{\alpha_4-\alpha_3}\int_0^1 (1-\bar{\chi})^{\eta-1}\bar{G}\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} \\
&= \frac{2}{\alpha_4-\alpha_3}\bar{G}(\alpha_4) - \frac{2^{\eta+1}\eta}{(\alpha_4-\alpha_3)^{\eta+1}}\int_{\frac{\alpha_3+\alpha_4}{2}}^{\alpha_4}\left(u - \frac{\alpha_3+\alpha_4}{2}\right)^{\eta-1}\bar{G}(u)\mathrm{d}u \\
&= \frac{2}{\alpha_4-\alpha_3}\bar{G}(\alpha_4) - \frac{2^{\eta+1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta+1}}\Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\epsilon_2 :&= \int_0^1 (1-\bar{\chi})^\eta \bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} \\
&= \frac{2(1-\bar{\chi})^\eta}{\alpha_4-\alpha_3}\bar{G}\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\Bigg|_{\bar{\chi}=0}^{1} + \frac{2\eta}{\alpha_4-\alpha_3}\int_0^1 (1-\bar{\chi})^{\eta-1}\bar{G}\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} \\
&= \frac{-2}{\alpha_4-\alpha_3}\bar{G}(\alpha_3) + \frac{2^{\eta+1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta+1}}\int_{\alpha_3}^{\frac{\alpha_3+\alpha_4}{2}}\left(\frac{\alpha_3+\alpha_4}{2} - v\right)^{\eta-1}\bar{G}(v)\mathrm{d}v \\
&= \frac{2^{\eta+1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta+1}}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \frac{2^{\eta+1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta+1}}\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right).
\end{aligned}$$

Consequently, we have

$$\frac{\alpha_4-\alpha_3}{4}\Big(\epsilon_1-\epsilon_2\Big) = \frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}.$$

This ends the proof of Lemma 1. □

Remark 3. *Observe that:*

1. *The equality in Equation (12) with $1-\bar{\chi}$ replaced by $\bar{\chi}$ becomes the following midpoint formula equality:*

$$\begin{aligned}
&\frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^\eta}\left\{\Im^{\eta}_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^+}\bar{G}(\alpha_4) + \Im^{\eta}_{\left(\frac{\alpha_3+\alpha_4}{2}\right)^-}\bar{G}(\alpha_3)\right\} - \bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) \\
&= \frac{\alpha_4-\alpha_3}{4}\left\{\int_0^1 \bar{\chi}^\eta \bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} - \int_0^1 \bar{\chi}^\eta \bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi}\right\},
\end{aligned}$$

   which was proved before in [13] by Sarikaya and Yildirim.

2. *The equality in Equation (12) with $\eta = 1$ becomes the following trapezoidal formula equality*

$$\begin{aligned}
&\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4}\bar{G}(x)\mathrm{d}x \\
&= \frac{\alpha_4-\alpha_3}{4}\left\{\int_0^1 (1-\bar{\chi})\bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi} - \int_0^1 (1-\bar{\chi})\bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\mathrm{d}\bar{\chi}\right\}.
\end{aligned}$$

Theorem 1. *If $\bar{G} : [\alpha_3, \alpha_4] \to \mathcal{R}$ is an L^1 function, then, for the convexity of $|\bar{G}'|^{\varrho}, \varrho \geq 1$ on $[\alpha_3, \alpha_4]$ with $\alpha_3 < \alpha_4$, we have the following trapezoidal formula inequality:*

$$\begin{aligned}
&\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}\right| \\
&\quad\leq \frac{\alpha_4-\alpha_3}{4(\eta+1)^{1-\frac{1}{\varrho}}}\left\{\left(\frac{\mathcal{B}(2,\eta+1)}{2}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{2-(\eta+1)\mathcal{B}(2,\eta+1)}{2(\eta+1)}|\bar{G}'(\alpha_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right. \\
&\qquad\left.+\left(\frac{2-(\eta+1)\mathcal{B}(2,\eta+1)}{2(\eta+1)}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{\mathcal{B}(2,\eta+1)}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right\}, \qquad (13)
\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ represents the beta function:

$$\mathcal{B}(\xi_1,\xi_2) = \int_0^1 \chi^{\xi_1-1}(1-\chi)^{\xi_2-1}\mathrm{d}\chi.$$

Proof. At first, we let $\varrho = 1$. Then, by making use of Lemma 1 and the convexity of $|\bar{G}'|$, we have

$$\begin{aligned}
&\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}\right| \\
&\leq \frac{\alpha_4-\alpha_3}{4}\int_0^1(1-\bar{\chi})^{\eta}\left\{\left|\bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3+\frac{2-\bar{\chi}}{2}\alpha_4\right)\right|+\left|\bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3+\frac{\bar{\chi}}{2}\alpha_4\right)\right|\right\}\mathrm{d}\bar{\chi} \\
&\leq \frac{\alpha_4-\alpha_3}{4}\Big(|\bar{G}'(\alpha_3)|+|\bar{G}'(\alpha_4)|\Big)\int_0^1(1-\bar{\chi})^{\eta}\mathrm{d}\bar{\chi} \\
&= \frac{\alpha_4-\alpha_3}{4(\eta+1)}\Big(|\bar{G}'(\alpha_3)|+|\bar{G}'(\alpha_4)|\Big).
\end{aligned}$$

For $\varrho > 1$, we use Lemma 1, power-mean inequality, and the convexity of $|\bar{G}'|^{\varrho}$ to get

$$\begin{aligned}
&\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}\right| \\
&\leq \frac{\alpha_4-\alpha_3}{4}\left(\int_0^1(1-\bar{\chi})^{\eta}\mathrm{d}\bar{\chi}\right)^{1-\frac{1}{\varrho}}\left\{\left(\int_0^1(1-\bar{\chi})^{\eta}\left|\bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3+\frac{2-\bar{\chi}}{2}\alpha_4\right)\right|^{\varrho}\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right. \\
&\left.+\left(\int_0^1(1-\bar{\chi})^{\eta}\left|\bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3+\frac{\bar{\chi}}{2}\alpha_4\right)\right|^{\varrho}\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right\} \\
&\leq \frac{\alpha_4-\alpha_3}{4(\eta+1)^{1-\frac{1}{\varrho}}}\left\{\left(\int_0^1(1-\bar{\chi})^{\eta}\left[\frac{\bar{\chi}}{2}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{2-\bar{\chi}}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right]\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right. \\
&\left.+\left(\int_0^1(1-\bar{\chi})^{\eta}\left[\frac{2-\bar{\chi}}{2}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{\bar{\chi}}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right]\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right\} \\
&= \frac{\alpha_4-\alpha_3}{4(\eta+1)^{1-\frac{1}{\varrho}}}\left\{\left(\frac{\mathcal{B}(2,\eta+1)}{2}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{2-(\eta+1)\mathcal{B}(2,\eta+1)}{2(\eta+1)}|\bar{G}'(\alpha_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right. \\
&\left.+\left(\frac{2-(\eta+1)\mathcal{B}(2,\eta+1)}{2(\eta+1)}|\bar{G}'(\alpha_3)|^{\varrho}+\frac{\mathcal{B}(2,\eta+1)}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right\},
\end{aligned}$$

where the following facts are used:

$$\int_0^1 \frac{\bar{\chi}}{2}(1-\bar{\chi})^{\eta} \mathrm{d}\bar{\chi} = \frac{\mathcal{B}(2,\eta+1)}{2};$$
$$\int_0^1 \frac{\bar{\chi}}{2}(1-\bar{\chi})^{\eta} \mathrm{d}\bar{\chi} = \frac{1}{\eta+1} - \frac{\mathcal{B}(2,\eta+1)}{2}.$$

Thus, we are done. □

Remark 4. *Observe that:*

1. *The inequality in Equation (13) with $\eta = 1$ becomes the following trapezoidal formula inequality:*

$$\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4} \bar{G}(x)\mathrm{d}x\right| \leq \frac{\alpha_4-\alpha_3}{8}\left\{\left(\frac{|\bar{G}'(\alpha_3)|^{\varrho}}{6} + \frac{5|\bar{G}'(\alpha_4)|^{\varrho}}{6}\right)^{\frac{1}{\varrho}} + \left(\frac{5|\bar{G}'(\alpha_3)|^{\varrho}}{6} + \frac{|\bar{G}'(\alpha_4)|^{\varrho}}{6}\right)^{\frac{1}{\varrho}}\right\}. \quad (14)$$

2. *The inequality in Equation (13) with $\eta = 1$ and $\varrho = 1$ becomes the following trapezoidal formula inequality:*

$$\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4} \bar{G}(x)\mathrm{d}x\right| \leq \frac{\alpha_4-\alpha_3}{8}\left(|\bar{G}'(\alpha_3)| + |\bar{G}'(\alpha_4)|\right). \quad (15)$$

Theorem 2. *If $\bar{G} : [\alpha_3, \alpha_4] \to \mathcal{R}$ is an L^1 function, then, for the convexity of $|\bar{G}'|^{\varrho}, \varrho \geq 1$ on $[\alpha_3, \alpha_4]$ with $\alpha_3 < \alpha_4$, we have the following trapezoidal formula inequality:*

$$\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}\right| \leq \frac{\alpha_4-\alpha_3}{4(\eta p+1)^{\frac{1}{\bar{\varrho}}}}\left\{\left(\frac{|\bar{G}'(\alpha_3)|^{\varrho}+3|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}} + \left(\frac{3|\bar{G}'(\alpha_3)|^{\varrho}+|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}}\right\}, \quad (16)$$

where $\frac{1}{\bar{\varrho}} + \frac{1}{\varrho} = 1$.

Proof. Making use Lemma 1, Hölder's inequality, and the convexity of $|\bar{G}'|^{\varrho}$, we get

$$\begin{aligned}
&\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2} - \frac{2^{\eta-1}\Gamma(\eta+1)}{(\alpha_4-\alpha_3)^{\eta}}\left\{\Im^{\eta}_{\alpha_3+}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right) + \Im^{\eta}_{\alpha_4-}\bar{G}\left(\frac{\alpha_3+\alpha_4}{2}\right)\right\}\right| \\
&\leq \frac{\alpha_4-\alpha_3}{4}\left(\int_0^1 (1-\bar{\chi})^{\eta}\mathrm{d}\bar{\chi}\right)^{1-\frac{1}{\varrho}}\left\{\left(\int_0^1 \left|\bar{G}'\left(\frac{\bar{\chi}}{2}\alpha_3 + \frac{2-\bar{\chi}}{2}\alpha_4\right)\right|^{\varrho}\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right. \\
&\left. + \left(\int_0^1 \left|\bar{G}'\left(\frac{2-\bar{\chi}}{2}\alpha_3 + \frac{\bar{\chi}}{2}\alpha_4\right)\right|^{\varrho}\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right\} \\
&\leq \frac{\alpha_4-\alpha_3}{4(\eta p+1)^{1-\frac{1}{\varrho}}}\left\{\left(\int_0^1 \left[\frac{\bar{\chi}}{2}|\bar{G}'(\alpha_3)|^{\varrho} + \frac{2-\bar{\chi}}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right]\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right. \\
&\left. + \left(\int_0^1 \left[\frac{2-\bar{\chi}}{2}|\bar{G}'(\alpha_3)|^{\varrho} + \frac{\bar{\chi}}{2}|\bar{G}'(\alpha_4)|^{\varrho}\right]\mathrm{d}\bar{\chi}\right)^{\frac{1}{\varrho}}\right\} \\
&= \frac{\alpha_4-\alpha_3}{4(\eta p+1)^{\frac{1}{\bar{\varrho}}}}\left\{\left(\frac{|\bar{G}'(\alpha_3)|^{\varrho}+3|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}} + \left(\frac{3|\bar{G}'(\alpha_3)|^{\varrho}+|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}}\right\},
\end{aligned}$$

which rearranges to the proof. □

Remark 5. *The inequality in Equation (16) with $\eta = 1$ becomes the following trapezoidal formula inequality:*

$$\begin{aligned}\left|\frac{\bar{G}(\alpha_3)+\bar{G}(\alpha_4)}{2}-\frac{1}{\alpha_4-\alpha_3}\int_{\alpha_3}^{\alpha_4}\bar{G}(x)\mathrm{d}x\right| &\le \frac{\alpha_4-\alpha_3}{4(p+1)^{\frac{1}{\varrho}}}\left\{\left(\frac{|\bar{G}'(\alpha_3)|^{\varrho}+3|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}}+\left(\frac{3|\bar{G}'(\alpha_3)|^{\varrho}+|\bar{G}'(\alpha_4)|^{\varrho}}{4}\right)^{\frac{1}{\varrho}}\right\}\\ &\le \frac{\alpha_4-\alpha_3}{4}\left(\frac{4}{\bar{\varrho}+1}\right)^{\frac{1}{\bar{\varrho}}}\left(|\bar{G}'(\alpha_3)|+|\bar{G}'(\alpha_4)|\right).\end{aligned}$$

3. Examples

There are many applications to demonstrate the use of integral inequalities, especially applications on special means of the real numbers [10,14,16]. In this section, we present some examples to demonstrate the applications of our obtained results on modified Bessel functions and q-digamma functions. The modified Bessel functions have been shown to play an important role in the Casimir theory of dielectric balls (see, e.g., [28–30]).

Example 1. *Consider the function $\mathcal{I}_{\bar{\varrho}} : \mathcal{R} \to [1, \infty)$, defined by*

$$\mathcal{I}_{\bar{\varrho}}(z) = 2^{\bar{\varrho}}\Gamma(\bar{\varrho}+1)z^{-v}I_{\bar{\varrho}}(z), \quad z \in \mathcal{R}.$$

Here, we consider the modified Bessel function of the first kind $\mathcal{I}_{\bar{\varrho}}$, defined by [31]:

$$\mathcal{I}_{\bar{\varrho}}(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{\bar{\varrho}+2n}}{n!\Gamma(\bar{\varrho}+n+1)}.$$

The first order derivative formula of $\mathcal{I}_{\bar{\varrho}}(z)$ is given by [31]:

$$\mathcal{I}'_{\bar{\varrho}}(z) = \frac{z}{2(\bar{\varrho}+1)}\mathcal{I}_{\bar{\varrho}+1}(z). \tag{17}$$

By making use of Remark 2 and the identity in Equation (17), we can deduce

$$\left|\frac{\mathcal{I}_{\bar{\varrho}}(\alpha_4)-\mathcal{I}_{\bar{\varrho}}(\alpha_3)}{\alpha_4-\alpha_3}\right| \le \frac{\alpha_3\mathcal{I}_{\bar{\varrho}+1}(\alpha_3)+\alpha_4\mathcal{I}_{\bar{\varrho}+1}(\alpha_4)}{4(p+1)}$$

for $\bar{\varrho} > -1, \alpha_3, \alpha_4 \in \mathcal{R}$ with $0 < \alpha_3 < \alpha_4$. Specifically, for $\mathcal{I}_{-\frac{1}{2}}(z) = \cosh(z)$ and $\mathcal{I}_{\frac{1}{2}}(z) = \frac{\sinh(z)}{z}$, we get

$$\left|\frac{\cosh(\alpha_4)-\cosh(\alpha_3)}{\alpha_4-\alpha_3}\right| \le \frac{\sinh(\alpha_3)+\sinh(\alpha_4)}{2}.$$

We can also get the following inequalities for the inequality raised in Remark 1 by the same technique used above:

$$\left|\frac{\mathcal{I}_{\bar{\varrho}}(\alpha_4)-\mathcal{I}_{\bar{\varrho}}(\alpha_3)}{\alpha_4-\alpha_3}\right| \le \frac{\alpha_3+\alpha_4}{8(\bar{\varrho}+1)}\mathcal{I}_{\bar{\varrho}+1}\left(\frac{\alpha_3+\alpha_4}{2}\right)+\frac{\alpha_3\mathcal{I}_{\bar{\varrho}+1}(\alpha_3)+\alpha_4\mathcal{I}_{\bar{\varrho}+1}(\alpha_4)}{8(\bar{\varrho}+1)}.$$

In particular,

$$\left|\frac{\cosh(\alpha_4)-\cosh(\alpha_3)}{\alpha_4-\alpha_3}\right| \le \frac{1}{2}\sinh\left(\frac{\alpha_3+\alpha_4}{2}\right)+\frac{\sinh(\alpha_3)+\sinh(\alpha_4)}{4}.$$

Example 2. *Here, we consider the modified Bessel function of the second kind $\mathcal{K}_{\bar{\varrho}}$, defined by [31]:*

$$\mathcal{K}_{\bar{\varrho}}(z) = \frac{\pi}{2}\frac{\mathcal{I}_{-\bar{\varrho}}(z) + \mathcal{I}_{\bar{\varrho}}(z)}{\sin(\bar{\varrho}\pi)}.$$

Let $\bar{G}_{\bar{\varrho}}(z) := -\left(\frac{\mathcal{K}_{\bar{\varrho}}(z)}{z^{\bar{\varrho}}}\right)'$ with $\bar{\varrho} \in \mathcal{R}$. Consider the integral representation [31]:

$$\mathcal{K}_{\bar{\varrho}}(z) = \int_0^\infty e^{-z\cosh t}\cosh(\bar{\varrho}t)dt, \quad z > 0.$$

It is clear that $z \mapsto \mathcal{K}_{\bar{\varrho}}(z)$ is a completely monotonic function on an interval $(0, \infty)$ for all $\bar{\varrho} \in \mathcal{R}$. Since the product of two completely monotonic functions is also completely monotonic, $z \mapsto \bar{G}_{\bar{\varrho}}(z)$ is a strictly completely monotonic function on the same interval for all $\bar{\varrho} > 1$. Therefore, the function

$$\bar{G}_{\bar{\varrho}}(z) = -\left(\frac{\mathcal{K}_{\bar{\varrho}}(z)}{z^{\bar{\varrho}}}\right)' = \frac{\mathcal{K}_{\bar{\varrho}+1}(z)}{z^{\bar{\varrho}}} \tag{18}$$

is strictly completely monotonic on an interval $(0, \infty)$ for all $\bar{\varrho} > 1$ and thus $\bar{G}_{\bar{\varrho}}$ is a convex function. Then, by making use of Remark 2 and the identity in Equation (2), we can deduce

$$\left|\frac{\alpha_3{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\alpha_4) - \alpha_4{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\alpha_3)}{\alpha_4 - \alpha_3}\right| \leq \frac{\alpha_4{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\alpha_3) + \alpha_3{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\alpha_4)}{2}$$

for each $\bar{\varrho} > 1$ and $\alpha_3, \alpha_4 \in \mathcal{R}$ with $0 < \alpha_3 < \alpha_4$.

We can also get the following inequality for the inequality raised in Remark 1 by the same technique used above:

$$\left|\frac{\alpha_3{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\alpha_4) - \alpha_4{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\alpha_3)}{\alpha_4 - \alpha_3}\right| \leq \frac{1}{2}\left(\frac{2\alpha_3\alpha_4}{\alpha_3 + \alpha_4}\right)^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}\left(\frac{\alpha_3 + \alpha_4}{2}\right) + \frac{\alpha_4{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\alpha_3) + \alpha_3{}^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\alpha_4)}{4}$$

for each $\bar{\varrho} > 1$ and $\alpha_3, \alpha_4 \in \mathcal{R}$ with $0 < \alpha_3 < \alpha_4$.

Example 3. *Consider the q-digamma function Ψ_ϱ, defined by [31]:*

$$\begin{aligned}\Psi_\varrho(z) &= -\ln(1-\varrho) + \ln(\varrho)\sum_{\ell=0}^{\infty}\frac{\varrho^{\ell+z}}{1-\varrho^{\ell+z}} \\ &= -\ln(1-\varrho) + \ln(\varrho)\sum_{\ell=1}^{\infty}\frac{\varrho^{\ell z}}{1-\varrho^{\ell z}}\end{aligned}$$

for $0 < \varrho < 1$, and

$$\begin{aligned}\Psi_\varrho(z) &= -\ln(\varrho-1) + \ln(\varrho)\left(z - \frac{1}{2} - \sum_{\ell=0}^{\infty}\frac{\varrho^{-(\ell+z)}}{1-\varrho^{-(\ell+z)}}\right) \\ &= -\ln(\varrho-1) + \ln(\varrho)\left(z - \frac{1}{2} - \sum_{\ell=1}^{\infty}\frac{\varrho^{-\ell z}}{1-\varrho^{-\ell z}}\right)\end{aligned}$$

for $\varrho > 1$ and $z > 0$.

From those definitions, we see that $z \mapsto \Psi'_\varrho(z)$ is a completely monotonic function on an interval $(0, \infty)$ for all $\varrho > 0$, and consequently $z \mapsto \Psi'_\varrho(z)$ is convex on the same interval.

Let $\bar{G}_\varrho(z) := \Psi'_\varrho(z)$ with $\varrho > 0$ and, therefore, $\bar{G}'_\varrho(z) := \Psi''_\varrho(z)$ is completely monotonic on the interval $(0, \infty)$. Then, from Remark 1, we have

$$\Psi'_\varrho\left(\frac{\alpha_3+\alpha_4}{2}\right) \leq \left|\frac{\Psi_\varrho(\alpha_4)-\Psi_\varrho(\alpha_3)}{\alpha_4-\alpha_3}\right| \leq \frac{\Psi'_\varrho(\alpha_3)+\Psi'_\varrho(\alpha_4)}{2}. \tag{19}$$

Combining the inequalities in Equations (15) and (19), we get

$$\left|\frac{\Psi'_\varrho(\alpha_3)+\Psi'_\varrho(\alpha_4)}{2} - \frac{\Psi_\varrho(\alpha_4)-\Psi_\varrho(\alpha_3)}{\alpha_4-\alpha_3}\right| \leq \frac{\alpha_4-\alpha_3}{8}\left(\left|\Psi''_\varrho(\alpha_3)\right| + \left|\Psi''_\varrho(\alpha_4)\right|\right).$$

4. Conclusions

In this article, we consider the new integral inequality and some related inequalities of the Hermite–Hadamard type for Riemann–Liouville fractional integrals. Integral inequalities form a crucial branch of analysis and have been combined with various type of fractional integrals but never done before in this form. For this reason, we study the inequality of Hermite–Hadamard type and related inequalities via the Riemann–Liouville fractional integrals, generalizing the previous results obtained by Sarikaya et al. [12,13].

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References

1. Lakshmikantham, V.; Leela, S. *Differential and Integral Inequalities: Theory and Applications: Volume I: Ordinary Differential Equations*; Academic Press: New York, NY, USA, 1969.
2. Walter, W. *Differential and Integral Inequalities (Vol. 55)*; Springer: New York, NY, USA, 1964.
3. Agarwal, P.; Tariboon, J.; Ntouyas, S.K. Some generalized Riemann-Liouville k-fractional integral inequalities. *J. Inequal. Appl.* **2016**, *2016*, 122. [CrossRef]
4. Anastassiou, G.A. Opial type inequalities involving Riemann–Liouville fractional derivatives of two functions with applications. *Math. Comput. Model.* **2008**, *48*, 344–374. [CrossRef]
5. Denton, Z.; Vatsala, A.S. Fractional integral inequalities and applications. *Comput. Math. Appl.* **2010**, *59*, 1087–1094. [CrossRef]
6. Zhu, C.; Feckan, M.; Wang, J. Fractional integral inequalities for differential convex mappings and applications to special means and a midpoint formula. *J. Appl. Math. Stat. Inform.* **2012**, *8*, 21–28. [CrossRef]
7. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204, Elsevier Science Inc.: New York, NY, USA, 2006.
8. Miller, S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: New York, NY, USA, 1993.
9. Hadamard, J. Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par, Riemann. *J. Math. Pures. Appl.* **1893**, *58*, 171–215.
10. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
11. Mohammed, P.O. On New Trapezoid Type Inequalities for *h*-convex Functions via Generalized Fractional Integral. *Turkish J. Anal. Number Theory* **2018**, *6*, 125–128. [CrossRef]
12. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]

13. Sarikaya, M.Z.; Yildirim, H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2017**, *17*, 1049–1059. [CrossRef]
14. Kirmaci, U.S. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.* **2004**, *147*, 137–146. [CrossRef]
15. Mohammed, P.O.; Sarikaya, M.Z. On generalized fractional integral inequalities for twice differentiable convex functions. *J. Comput. Appl. Math.* **2020**, *372*, 112740. [CrossRef]
16. Mohammed, P.O.; Abdeljawad, T. Modification of certain fractional integral inequalities for convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 69. [CrossRef]
17. Fernandez, A.; Mohammed, P.O. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Math. Meth. Appl. Sci.* **2020**, 1–18. [CrossRef]
18. Gavrea, B.; Gavrea, I. On some Ostrowski type inequalities. *Gen. Math.* **2010**, *18*, 33–44.
19. Qi, F.; Mohammed, P.O.; Yao, J.C.; Yao, Y.H. Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions. *J. Inequal. Appl.* **2019**, *2019*, 135. [CrossRef]
20. Dinu, C. Hermite-Hadamard Inequality on Time Scales. *J. Inequal. Appl.* **2008**, *2008*, 287947. [CrossRef]
21. Mohammed, P.O. Inequalities of Type Hermite-Hadamard for Fractional Integrals via Differentiable Convex Functions. *Turkish J. Anal. Number Theory* **2016**, *4*, 135–139.
22. Mohammed, P.O. Inequalities of $(k,s),(k,h)$-Type For Riemann-Liouville Fractional Integrals. *Appl. Math. E-Notes* **2017**, *17*, 199–206.
23. Mohammed, P.O. Some new Hermite-Hadamard type inequalities for MT-convex functions on differentiable coordinates. *J. King Saud Univ. Sci.* **2018**, *30*, 258–262. [CrossRef]
24. Mohammed, P.O. Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. *Math. Meth. Appl. Sci.* **2019**, 1–11. [CrossRef]
25. Mohammed, P.O.; Sarikaya, M.Z. Hermite-Hadamard type inequalities for F-convex function involving fractional integrals. *J. Inequal. Appl.* **2018**, *2018*, 359. [CrossRef] [PubMed]
26. Sarikaya, M.Z.; Yaldiz, H. On generalization integral inequalities for fractional integrals. *Nihonkai Math. J.* **2014**, *25*, 93–104.
27. Sarikaya, M.Z.; Yaldiz, H.; Basak, N. New fractional inequalities of Ostrowski-Grüss type. *Le Mat.* **2014**, *LXIX*, 227–235. [CrossRef]
28. Milton, K.A.; Parashar, P.; Brevik, I.; Kennedy, G. Self-stress on a dielectric ball and Casimir-Polder forces. *Ann. Phys. (N. Y.)* **2020**, *412*, 168008. [CrossRef]
29. Parashar, P.; Milton, K.A.; Shajesh, K.V.; Brevik, I. Electromagnetic delta-function sphere. *Phys. Rev. D* **2017**, *96*, 085010. [CrossRef]
30. Brevik, I.; Marachevsky, V.N. Casimir surface force on a dilute dielectric ball. *Phys. Rev. D* **1999**, *60*, 085006. [CrossRef]
31. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1944.