
Bilevel stochastic programming problems:
analysis and application to telecommunications

Adrian S. Werner

Adrian S. Werner
Section of Investment, Finance and Accounting
Department of Industrial Economics and Technology Management
Norwegian University of Science and Technology
N – 7491 Trondheim, Norway

adrian.werner@iot.ntnu.no
<http://websterii.iot.ntnu.no/users/werner/>

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Thesis advisor:
Alexei A. Gaivoronski

Co-advisor:
Jan-Arild Audestad

Evaluation committee:
Abdel Lisser, University of Paris Sud
Sjur D. Flåm, University of Bergen
Mechthild Opperud, Telenor AS
Harald E. Krogstad, Norwegian University of Science and Technology

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Preface

This thesis has been prepared at the Department of Industrial Economy and Technology Management at the Norwegian University of Science and Technology (NTNU) in partial fulfillment of the requirements of the Doktor Ingeniør degree. The work has been carried out in the period from January 2000 to December 2004. The main advisor of the thesis has been Professor Alexei A. Gaivoronski and co-advisor has been Jan A. Audestad. The project has been funded for three years by Telenor R & D whereas the department financed a further half year.

The main subject of the thesis is the analysis of decision problems that are characterised by uncertainty and a hierarchic structure. Both theoretical and practical issues are considered in this thesis. The background of the applications is to be found in telecommunications. One reason for this is the fact that the recent development of the telecommunications sector naturally gives rise to the type of problems considered here. Another reason is that the major part of the project was funded by a telecom company. However, this background is not restrictive and most of the results can easily be generalised. The thesis consists of two parts: the first part is an introduction that discusses the framework of my research and outlines the contents of the papers in part two. The second part consists of the four papers.

The subject of the thesis is located at an intersection of telecommunications, stochastic programming and economic modeling. Thus a broad field is covered providing many impulses for exciting and challenging research. Although there were hard periods filled with frustration and doubts I really enjoyed doing research on this field and overcoming one obstacle after the other. Furthermore, Trondheim is an excellent place to work on a PhD project, especially due to its surroundings. I spent much of my spare time outdoors hiking and staying in small cabins in the mountains or bicycling. Such activities provided both energy and calmness which were important for mastering the everyday challenges.

I am greatly indebted to my supervisor Alexei A. Gaivoronski for pointing out this project for me and for supporting me during the application process. I have to thank him for his patience with me in the beginning when I had to become familiar with doing research in a really scientific manner. He carefully read and commented many drafts of the papers and taught me a lot about stochastic programming. I am grateful for his guidance and good advice; often his questions and suggestions for improvements also inspired further ideas.

I would also like to thank my co-supervisor Jan A. Audestad from Telenor for many interesting discussions. He is a very open-minded and inspiring person and I benefited a lot from his broad knowledge on telecommunications issues. Actually, my research has been significantly motivated by ideas and a problem description emerging from these discussions. I am grateful to Telenor for generously funding this project and for the possibilities this has opened for my work, such as the opportunity to attend several scientific conferences around the world.

Further thanks are due to Stephan Dempe, Freiberg University of Mining and Technology, Germany. I am greatly indebted to him for various interesting discussions and suggestions, especially for sharing his experience with bilevel programming. His support has contributed significantly to the results of this project.

I would like to thank my colleagues at the department for a pleasant and inspiring working atmosphere. Special thanks are due to our departmental secretaries, Guri Andresen and Jorid Øyen, for always being helpful and supportive.

Finally, thanks go to my family and to my friends for their support.

Any errors in this thesis are, of course, entirely my responsibility.

Trondheim, December 2004.

Adrian S. Werner

Abstract

We analyse several facets of bilevel decision problems under uncertainty. These problems can be interpreted as an extension of stochastic programming problems where part of the uncertainty is attributed to the behaviour of another actor.

The field of decision making under uncertainty with bilevel features is quite new and most approaches focus on the interactions and relations between the decision makers. In contrast to these studies, the approach of bilevel stochastic programming pursued here stresses the stochastic programming aspect of the problem formulation. The framework enables a direct application of stochastic programming concepts and solution methods to the bilevel relationship between the actors. Thus more complex problem structures can be studied and the aspect of uncertainty can be treated adequately.

Our analysis covers both theoretical and more practically oriented issues. We study different formulations of one and two stage bilevel stochastic programming problems and state necessary optimality conditions for each of the problem instances. Additionally we present a solution algorithm utilising a stochastic quasi-gradient method. A further study is concerned with the uniqueness of the minima of a convex stochastic programming problem with uncertainty about the decision variables. We state conditions on the distribution of the parameters representing the uncertainty such that the minima of the optimisation problem are unique. We formulate a model of competition and collaboration of two different types of telecom service providers, the owner of a bottleneck facility and a virtual network operator. This represents an application of a bilevel stochastic programming formulation to a liberalised telecommunications environment. Furthermore, the utilisation of the bilevel stochastic programming framework and the developed solution concepts for the analysis of principal agent models is demonstrated. Also here the background of a regulated telecom environment, more specific the relations between a regulator and a regulated telecommunications company, was chosen.

Contents

Introduction	1
1 Motivation	2
2 Hierarchical optimisation under uncertainty	4
2.1 Stochastic programming	4
2.2 Bilevel programming	7
2.3 Hierarchical decision making under uncertainty	9
3 Application background	12
3.1 Virtual Operators	14
3.2 Regulation	15
3.3 Agency theory	17
4 Research contribution and description of the papers	18
5 Conclusions and future research	21
Paper 1: Extending the stochastic programming framework for the modeling of several decision makers: pricing and competition in the telecommunication sector	27
Paper 2: A solution method for bilevel stochastic programming problems	57
Paper 3: Utilisation of stochastic programming methods for the analysis of agency problems	99
Paper 4: Influence of perturbed input data on convexity properties of stochastic programming problems	133

Introduction

The studies in this thesis focus on aspects of decision making under uncertainty when part of the uncertainty derives from actions of another actor. This describes a stochastic programming problem with bilevel structure. The analysis was conducted against the background of a liberalised telecommunications environment since the major part of the work was financed by the Norwegian telecom company Telenor. This background is, however, not restrictive and the results can be applied easily to similar environments such as liberalised electricity markets or general principal agent processes.

The thesis consists of an introductory chapter and four self contained and complementary papers.

In Paper 1, "Extending the stochastic programming framework for the modeling of several decision makers: pricing and competition in the telecommunication sector", a model of competition and collaboration between two different types of telecom service providers in a common market is developed and studied.

This model motivated the analysis in Paper 2, "A solution method for bilevel stochastic programming problems". In this study necessary optimality conditions and a solution algorithm are presented for several variants of bilevel stochastic programming problems.

Considering a principal agent problem of regulation in telecommunications, Paper 3, "Utilisation of stochastic programming methods in the analysis of agency problems", illustrates the application of the framework of bilevel stochastic programming to agency theory.

In Paper 4, "Influence of perturbed input data on convexity properties of stochastic programming problems", we study the effect of uncertainty about the decision variables on the properties of convex optimisation problems, especially on the uniqueness of the optimal solutions.

The remainder of this introductory chapter is organised as follows. First, Section 1 elucidates the motivation of our studies. Section 2 gives an overview over the theoretical context of the work. It comprises the fields of stochastic programming and of bilevel programming, but especially their intersection leading to the framework of bilevel stochastic programming. Section 3 outlines the practical

background of our work motivating the choice of the application examples. We present several aspects of the relationship between independent decision makers which can be found for example in a modern telecom environment. Section 4 explains the research contribution and gives a brief description of each of the papers before in Section 5 main conclusions of the thesis are stated and directions for further research are indicated.

Finally, the four papers constituting the main part of the thesis follow.

1 Motivation

In the recent years industrial sectors such as telecommunications, electricity markets or transportation have been subject to a comprehensive process of reorganisation which is not yet finished. Liberalisation and the rapid pace of technological development fundamentally changed the structures of these sectors. They were transformed from monopolies with a relatively stable and often small range of available products to oligopolies with a broad and constantly changing variety of offered services. At the same time new actors with various different characteristics enter the sector and may, by means of collaboration and competition, form a variety of strategic alliances. Results of this process are for example the absence of perfect markets, complex relationships between the decision makers and a highly dynamic and uncertain environment. This makes the application of traditional microeconomic approaches difficult and alternative concepts are necessary. The rapidly changing environment requires robust strategies and a quick adaptation to new conditions. Furthermore a decision maker must take into account the uncertainty about the environment as well as the influence of his decisions on the behaviour of other actors and vice versa. This suggests the analysis of such models as decision problems under uncertainty taking into account interdependencies of several actors.

Most approaches for strategic decision support under uncertainty focus solely on the uncertainty aspect [BHS92, DST03, RSM98]. They consider the behaviour of other participants in the sector as random events, i.e. as part of the random environment parameters. This simplification ignores the existing interdependencies between the actors and may therefore raise problems as illustrated in

a cautionary note by Haugen and Wallace [HWng]. Another group of approaches puts attention on these interdependencies and utilises concepts of game theory or of bilevel programming. This approach is typically followed in agency theory [GH83, Ros73, Dem95]. However, the development of efficient solution methods is made difficult due to the situation of several interacting decision makers in an uncertain environment and the resulting problem structure. Therefore insights into the mechanisms of such a constellation are often obtained by the study of simplified models with deterministic equivalent formulations of the stochastic components and often only simple or no constraints. This does not regard the effects of the actors' uncertainty about the environment. It is, however, important to treat this uncertainty adequately and to analyse its implications carefully.

The highly dynamic environment makes the utilisation of for example game theoretic concepts inappropriate. Typically, the focus of game theory is on equilibria and a stable environment state. However, stability will never be established in quickly changing environments such as a modern telecommunications sector. Therefore the focus should be rather on the nearest future, on a few subsequent periods where the changes of the environment still can be assessed. This suggests the utilisation of stochastic programming concepts such as recourse problems.

The approach followed in this thesis combines the ideas of stochastic programming and of bilevel programming. We single out the uncertainty about the behaviour of other actors as a separate problem. The interactions between the decision makers can be treated by a suitable methodology such as concepts of bilevel programming. Hence, we enhance the framework of stochastic programming by selected methods from bilevel programming and form a new framework, the methodology of bilevel stochastic programming. Utilising this framework it is possible to take into account the influence of own decisions on the responses of other actors (which in turn affect own decisions) and at the same time to consider the uncertainty about environment parameters which can not be influenced such as certain features of demand behaviour, failure rates, natural phenomena etc. The following section explains basic ideas of this approach.

2 Hierarchical optimisation under uncertainty

This section gives an overview over the theoretical background of my thesis. At first some concepts of stochastic programming are reviewed which will be important for the analysis. Section 2.2 describes relevant notions of bilevel programming. Finally, Section 2.3 is concerned with problems of decision making under uncertainty with a bilevel structure. We survey approaches for the analysis of such problems and present the main principles of the framework of bilevel stochastic programming.

2.1 Stochastic programming

Stochastic programming represents a framework for the analysis of decision problems characterised by uncertainty. Providing techniques for an adequate treatment of this uncertainty, it helps to increase the accuracy and flexibility of solution approaches as well as of found solutions. The methodology has been studied already for some decades but gained increasing popularity in the last decades. A reason for this is the recent state of hardware and software enabling the investigation of more realistic and comprehensive models utilising sophisticated approaches. Introductions can be found in the books by Birge and Louveaux [BL97], by Ermoliev and Wets [EW88] or by Kall and Wallace [KW94]. Previous research shows the capability of the stochastic programming framework for the modelling and analysis of strategic decision problems, for example in telecommunications [ALMP02, BG94a, FGM97, Gai95, Gai04, Rii03, SDC94, TAD⁺98].

Generally a stochastic programming problem can be described as finding a "good" decision without knowing exactly in which state the environment will be when this decision is implemented. This uncertainty is expressed by the help of random variables, say $\omega \in \Omega$, such that a general formulation of a stochastic programming problem is given by

$$\begin{aligned} & \text{"min"} \quad F(x, \omega) & (1) \\ \text{s.t.} \quad & g(x, \omega) \text{"} \leq \text{" } 0 \end{aligned}$$

However, such a problem is not well defined and an evaluation is therefore difficult. Since the decisions x must be found before the actual realisations of the random

parameters ω are known, the meaning of the optimality as well as of the feasibility of a decision x is not clear. For example, the optimality of a problem depends on the context: an objective may be to avoid disastrous decisions but also to do as well as possible under the expected future conditions. In anticipative models decisions must be chosen without taking into account future observations of the random values. Here the feasibility of a decision x can be evaluated by formulations requiring the satisfaction of a constraint in the average

$$\mathbf{E}_\omega\{g(x, \omega)\} \leq 0 \quad (2)$$

or with a certain given level of reliability

$$P\{\omega \in \Omega | g(x, \omega) \leq 0\} \geq \alpha \quad (3)$$

Similar expressions can be utilised for the evaluation of the optimality of a decision. In Paper 3 several such formulations are discussed in the light of agency theory.

Another important formulation is the framework of (multistage) stochastic programming problems with recourse which combines anticipative and adaptive concepts. It reflects a situation where a (long-term) decision is implemented under incomplete knowledge about parameters of the model but there exists a possibility of correcting (short-term) decisions at later stages when information about these parameters reveals. These correcting decisions compensate for example for a violation of constraints involving random parameters. Consequently, the initial or first-stage decision should be determined such that e.g. the costs induced by this decision and the expected (and discounted) costs from the recourse decisions at the later stages are minimised. A model of a two-stage stochastic programming problem with linear constraints can be formulated as follows:

$$\begin{aligned} \min_x \quad & F_1(x) + \mathbf{E}_\omega\{Q(x, \omega)\} \\ & Ax \leq b \\ Q(x, \omega) = \quad & \min_y F_2(x, y, \omega) \\ Wy = \quad & h(\omega) - T(\omega)x \end{aligned} \quad (4)$$

A comprehensive treatment of stochastic programming problems with recourse can be found for example in Ermoliev and Wets [EW88]. Recourse problems

possess a broad field of applications. For example, they enable the analysis of dynamic aspects such as the consideration of an uncertain future. This is especially important for the study of models situated in a highly dynamic environment such as a modern telecom sector. Here, the rapidly changing structures and policies together with a nearly constant emergence of new technologies and the resulting uncertainty require strategies with a high degree of adaptability. In our work the concept was used in order to describe adaptive actions of the considered decision maker.

Solution approaches for stochastic programming problems can be classified into two main types. One class of approaches utilises an approximation by a deterministic nonlinear or linear programming problem whereas the other class employs statistical methods and treats the continuous distributions of the random variables directly.

The original stochastic programming problem can be transformed into a numerically tractable deterministic equivalent problem by expressing the uncertain data through a finite number of scenarios and utilising deterministic equivalent formulations, for example in the shape (2) or (3). Then standard solution techniques for nonlinear or linear programming problems can be applied. This approach is utilised in Paper 1 for the implementation and numerical study of a stochastic programming problem with a specific structure. However, especially for models with dynamic features such as recourse problems or other multistage stochastic programming problems the size of the deterministic equivalent may become quite large and specific structures of the problem should be exploited.

Scenarios describe possible realisations of the random parameters. They occur when the uncertain parameters describe discrete events or phenomena with a countable, finite number of outcomes or when only relatively few events have to be considered. Scenarios can also be generated by a discretisation of a continuous probability distribution of the random variables. In a dynamic setting a scenario describes a set of possible future sequences of outcomes of the random variables. This can be represented by so-called scenario trees, see for example [KW94].

Although the scenarios should be chosen such that they are a good representation of the reality this is not always practicable. The discretisation process of originally continuous variables may therefore be an arbitrary approximation.

Especially for complex problems characterised for example by nonconvexity or nondifferentiability the optimal solutions may be highly sensitive with regard to the problem parameters. This means that small changes of the parameter values on the scenarios can have great effects on the found optima which may undermine the relevance of the found results. It may therefore prove valuable to evaluate the stability of the found solutions, for example through a sensitivity analysis with regard to perturbations of the approximated parameters.

The other class of solution approaches employs statistical techniques such as sampling for a direct consideration of the distribution of the random values. It comprises for example stochastic decomposition [HS91] or stochastic quasi-gradient methods [Erm88, Gai88, Gai04]. The utilisation of statistic estimates of the random data directly in the solution process gives the flexibility to use various representations of the uncertain variables, for example by continuous or discrete but also independent or dependent random variables.

Stochastic quasi-gradient methods represent a generalisation of steepest descent methods. They were developed for the iterative solution of decision problems with complex objective functions and constraints. This makes them applicable to a broad variety of models also beyond the field of stochastic programming problems, including problems with nondifferentiable or nonconvex functions. The main topic of this thesis is the analysis of stochastic programming problems with a bilevel design. Such decision problems exhibit a complicated structure where linearity and convexity properties are typically not present. This motivates the utilisation of a stochastic quasi-gradient method in a solution algorithm which is developed in Paper 2. Paper 3 contains an illustrating example.

2.2 Bilevel programming

Bilevel programming problems represent a system of optimisation problems that consists of two (or more) levels. This structure enables for example the description of decision problems of several actors in a hierarchical relationship from the viewpoint of one of the actors. The upper level decision maker has to find a decision y that optimises some goal under given constraints. However, in order to do so, he must take into account a decision z^* of another decision maker such that

his optimisation problem is

$$\min_{y \in Y} F_U(y, z^*) \quad (5a)$$

$$\text{s.t.} \quad g_U(y, z^*) \leq 0 \quad (5b)$$

The decision z^* represents a response of the lower level decision maker to the upper level decisions y and is thus an optimal solution of the (parametric) decision problem

$$\min_{z \in Z} F_L(y, z) \quad (6a)$$

$$\text{s.t.} \quad g_L(y, z) \leq 0 \quad (6b)$$

This way the upper level decision maker can control the decisions of another actor influencing his decision process. In terms of bilevel programming the upper level decision maker is typically called the leader and the lower level decision maker the follower whereas in agency theory the notions of respectively principal and agent are applied. Often the lower level problem (6) is given in the shape of a parametric nonlinear or linear programming problem having explicit solutions for given upper level decisions, but the optimal response may also be defined implicitly, for example by variational inequalities. This leads to the generalisation of bilevel programming problems as mathematical programs with equilibrium constraints [PW97, PW99].

A prominent example of bilevel relationships are Stackelberg games. Other applications can be found for example in game theory [ER01, FJ03], investigations of oligopolies [FMM02, LS92], network design problems [CP91, Mar86] or traffic management [PR02]. A further large application area is constituted by agency theoretic problems [Dem95, GH83, Mir99]. In this thesis we analyse bilevel relationships with background in telecommunications. Paper 1 describes the bilevel relationship between a Network Owner (representing the leader) and a Virtual Operator (the follower). The examples in Paper 3 represent a principal agent relationship between a telecom regulator and a regulated service provider.

Deterministic bilevel programming problems were intensely studied during the past decades and a variety of solution methods was developed [Dem02, Dem03, VC94]. The problems show inconvenient properties that complicate the development of effective solution algorithms. They are \mathcal{NP} hard even in the linear

case [Bar91]. Taking into account the follower's response, the leader's objective function is generally not convex and neither differentiable. If the leader's constraints depend also on the follower's response (so called connecting upper level constraints) then the region of feasible leader decisions may even be not connected and, consequently, the leader's objective function discontinuous. Therefore it is often assumed that the feasibility of the leader's decisions is not influenced by the follower's behaviour. However, contrary to most studies in our investigations the connecting upper level constraints are explicitly taken into account. This is motivated by a number of applications: Paper 1 describes a case from telecommunications where the customer numbers of one decision maker are influenced also by the decisions of the other actor. Also the participation constraint typically present in a principal relationship represents such a connecting upper level constraint, see Paper 3.

Generally it is conceivable that the follower's decision problem (6) may have nonunique optimal solutions for some decisions of the leader, e.g. due to a lower level objective function which is not strictly convex. In such a case, bilevel theory offers two methods, depending on the degree of control the leader can exert on the follower. The optimistic approach assumes that the leader can direct the follower to the most preferable choice whereas in the pessimistic approach the leader tries to bound the damage from unwelcome responses. Also penalty algorithms can treat nonunique lower level responses [IA92]. In the presence of imperfect information about the follower's decisions the approach studied in Paper 4 is promising. The uncertainty can have an "improving" effect on the convexity of the objective function under some assumptions on the distribution function of the random variables. As a result, the responses of the follower may be uniquely determined taking into account imperfect knowledge even if this was not the case in a deterministic formulation.

2.3 Hierarchical decision making under uncertainty

The problem of decision making under uncertainty with several decision makers creates a new field of research. Depending on the viewpoint on the problem, different approaches evolved combining concepts of decision making under uncertainty

with methods of game theory or of bilevel programming.

Stochastic games [BV00, NS99] can take into account the interplay of the decision makers as well as different types of uncertainty. This framework does not necessarily presuppose a hierarchical relationship between the actors. However, the solution of more complex models involving for example continuous decision variables or nontrivial constraints possibly even depending on the decisions of several actors is very difficult if not even impossible for realistic situations. Furthermore the game theoretic focus on equilibria and related optimality notions can not be applied to problems where dynamic phenomena are important.

The aspect of a decision maker who controls the responses of other actors at least to a certain degree can better be taken into account by employing concepts from bilevel optimisation. Stochastic bilevel programming problems (SBLP) were introduced and studied by Patriksson and Wynter [PW97, Wyn01]. A generalisation is represented by stochastic mathematical programs with equilibrium constraints (SMPEC) [EP04, PW99, Sha04]. They can be interpreted as an extension of the respective deterministic programming problems by allowing for uncertain model parameters such that the focus is on the hierarchical structure of the problem. Suggestions for solution approaches comprise a penalty method [EP04] or the utilisation of a finite number of scenarios and deterministic equivalent formulations [PW99, Sha04]. This results in large deterministic bilevel programming problems which are computationally expensive for problems of a realistic size.

In this thesis a complementary viewpoint is taken. We interpret the decisions of the other actor(s) as a specific type of the uncertainty of the decision maker. This kind of uncertainty has the characteristic that it can be treated by specific methods, namely by the concepts of bilevel programming. Therefore we consider hierarchical decision problems under uncertainty as an extension of stochastic programming problems by a bilevel structure. The resulting bilevel stochastic programming (BLSP) framework underlines the stochastic programming roots of the problem.

The stochastic programming framework is based on techniques of mathematical programming and allows the analysis of problems with complex objective functions and constraints. We consider the bilevel structure as such a complex feature. The BLSP methodology allows then to apply concepts of stochastic programming

directly to complete relationship between the actors and to deal so with the difficulties. The stochastic programming framework seems more flexible and investigations connected with the bilevel structure can be incorporated more efficiently into the stochastic programming context than vice versa. Therefore the BLSP approach can treat the implications of the bilevel features adequately. To our knowledge hierarchical optimisation problems under uncertainty were not studied from that point of view before.

In terms of BLSP a general stochastic programming problem with bilevel structure can be formulated as follows. Expressing the uncertainty about the environment by a random variable $\omega \in \Omega$ the leader wants to find a solution y of his decision problem

$$\text{'' min'' } F_U(y, z^*, \omega) \quad (7a)$$

$$\text{s.t. } g_U(y, z^*, \omega) \text{'' } \leq \text{'' } 0 \quad (7b)$$

where z^* is an optimal solution of the leader's perception of the follower's decision process

$$\text{'' min'' } F_L(y, z, \omega) \quad (8a)$$

$$\text{s.t. } g_L(y, z, \omega) \text{'' } \leq \text{'' } 0 \quad (8b)$$

In order to describe these models more precisely the concepts presented in Section 2.1 can be utilised. Several formulations are presented and studied in the papers. In Paper 1 mainly a scenario formulation was utilised. Paper 2 considers a bilevel one-stage stochastic programming problem and variants of bilevel two-stage stochastic programming problems with a recourse problem in the upper level and one-stage or two-stage problems in the lower level. The utilisation of recourse problems allows to take into account also dynamic features such as an adaptation to a changing environment. Paper 3 gives further formulations of bilevel stochastic programming problems.

In a stochastic programming problem with bilevel structure two main types of uncertainty can be identified. Both decision makers face "natural" uncertainty about the environment. It can be taken into account by concepts from stochastic programming. Furthermore, since the problem is studied from the leader's

viewpoint, he may face "man-made" uncertainty about the follower's response. If he can assume that he is perfectly informed about the follower's decision process he can eliminate this uncertainty completely by solving the lower level decision problem (8). If, however, he assumes or knows that he has only an uncertain perception of the follower's decision process, additional stochastic programming concepts should be utilised for the treatment of this type of uncertainty. An approach is to introduce a random variable η denoting the noise or uncertainty connected with the leader's perception of the follower's response. Then he obtains an estimate z of the follower's response by solving problem (8) but, regarding his uncertainty, he includes an estimation $z + \eta$ of the actually implemented decision in his problem (7). Alternatively, the leader's uncertainty may be incorporated directly in his perception of the follower's decision process. This can be done by solving problem (8) for an uncertain follower decision $z + \omega$ and to utilise the obtained decision z in the leader's decision problem. Such a proceeding is investigated in Paper 4 whereas other ideas of approaching this type of uncertainty are illustrated in Paper 3.

The additional complexity of the bilevel relationship added to the stochastic programming problem makes the development of effective solution methods for BLSPs far more challenging. Difficult properties of bilevel programming problems such as the absence of convexity dominate also the complex structure of stochastic programming problems with bilevel features. However, again stochastic programming approaches are applicable. A direct solution of BLSPs utilises statistical methods for the treatment of the uncertain parameters. The complex nonlinear and nonconvex structure of the problems suggests the application of stochastic quasi-gradient methods. This is demonstrated by the solution approaches presented in Paper 2. The examples stated in Paper 3 illustrate our approach.

3 Application background

This section provides a brief background on the industrial environment that directly motivated the work presented in two of the papers. It highlights only a few topics, for more comprehensive information see Papers 1 and 3 and the literature referred to there. At first the development of the telecom sector is outlined before

the subsections address closer the issues of Virtual Operators, of regulation and of agency theory in telecommunications. They describe different aspects of hierarchical relationships which can be observed in modern telecom environments.

In the recent years the telecommunication sector was subject to fundamental transformations crucially changing its character. Several effects overlap, the main issues being a progressing liberalisation and rapid technology development as well as convergence processes. External influences such as globalisation trends and political restructuring gave additional impulses. As a result, the complexity of the telecommunications industry increased enormously. Before the liberalisation the telecom markets in each country were clearly defined and predictable with a monopolist providing a few quite simple and immutable services. Now there is a multitude of technologies resulting in a great amount of offered services as well as in competition within a country and over its borders. The telecom service providers play various roles and may participate in various strategic alliances.

The problems to be considered in this context can be classified according to three scale levels with increasing degree of aggregation [Gai04]. The technological level focusses on the elements of telecom networks such as switches or routers or on the evaluation of their performance [ACM01, BG94b, Ton04]. Problems located at the network level are concerned with issues of design and planning of the networks [ALMP04, Gai95, LR03, SDC94]. Finally, at the enterprise level strategic decisions such as regulation, pricing policies, the range and amount of the provides services or investments are analysed. The considered enterprise is studied in a larger scale taking into account its placement in the industrial environment, interactions with other actors or heterogeneous customer populations. The topics focussed on in this thesis are located at this enterprise level.

As a result of the transformation processes, the telecom sector shows characteristics that are different from standard economic environments. Most important are a high degree of uncertainty at all areas, a fundamental nonstationarity (implying for example the absence of equilibrium concepts), complex relationships between the actors and the absence of perfect markets. The rapidly changing environment requires robust strategies characterised by a high reaction speed and a nearly continuous adaptation to new conditions. At the same time any given actor must take into account interactions with other decision makers as well as the uncertainty,

for example due to unpredictable market behaviour, emerging new technologies, substitution effects or the nearly constantly changing structures in the sector.

In this focus many interesting problems arise and an important field of research opened that spreads into several directions and involves quite different disciplines. A number of research areas emerges investigating issues of market regulation and licensing, market structures or strategies of cooperation and competition between the operators. The evolution and convergence of technologies is discussed as well as topics of value chain and service convergence. With the start of the liberalisation process the study of interrelations between the different actors became an important topic. The aspects considered there span from problems of regulation and licensing in order to control development and the entry of providers over issues of pricing and cooperation to questions of competition.

3.1 Virtual Operators

Traditionally, a telecom operator possessed his own network. Due to the high sunk costs of essential network facilities which made their duplication unreasonable the telecommunications sector was long considered as a "natural" monopoly. Also long after the initialisation of the liberalisation process these network components are often owned by one or a few licensed providers whereas other service providers buy access to such bottleneck facilities. The latter type of operators is called Virtual Operators. This concept emerged in the early nineties when Virtual Operators often acted as pure resellers with only marginal enhancement of the provided services. In the recent years the issue of Virtual Operators became much more comprehensive. A reason is the emergence of new types of value-added services due to the progress of Internet technologies in combination with Third Generation mobile services. Therefore an exact definition can not be given and the only common denominator is that Virtual Operators act as resellers by buying access to the bottleneck facility. The opinions about their role are very widespread, not at last since they can be designed very differently, ranging from pure resellers to Virtual Operators acting much like licensed Network Operators. Since the issue of Virtual Operators is relatively new the extent of a possible regulatory intervention (comprising for example regulation of prices or conditions for access

to the bottleneck facilities) is still discussed in many countries.

The presence of Virtual Operators may lead to increased competition with regard to prices as well as to service quality, development of new, innovative services or a better utilisation of network capacity with lower average costs of service provision. Since product differentiation is easy, typically the services offered by the single providers are no direct substitutes. The market structure prevailing in such an environment is therefore called an oligopolistic monopoly. Due to their position the providers owning the bottleneck facilities can control the decision process of the Virtual Operators to a certain degree and have often considerable market power. They can discriminate against the Virtual Operators and restrict their access by several methods. Therefore the relationship between the operators is characterised by competition and, at the same time, collaboration. From the point of view of one of the operators (e.g. the Network Operator) this can be interpreted as a problem of decision making under uncertainty. The behaviour of the other decision maker can be singled out as a specific kind of the uncertainty which can be treated by a different methodology. Several aspects of this interpretation are analysed closer in our work. In Paper 1 a modeling framework is developed whereas Paper 2 is concerned with solution approaches for such models.

3.2 Regulation

Regulation describes the interference of an authority with the decisions of actors in a certain (industrial) environment. The activity of a regulator shall create conditions such that competition can take place. Possible scopes of regulation comprise therefore issues of consumer protection, antitrust policy or the encouragement of efficiency and implementation of new technologies. With transformations taking place in the regulated sector also the character of regulation must adapt. During a liberalisation process different stages can be identified, each with according degrees of regulatory activity and challenges for the regulator [HT01]. When the monopoly still exists, regulation must prevent monopolistic behaviour and ensure customer protection. Once the liberalisation process started and competition is introduced, the regulator should encourage and control the entry of new competitors. This comprises control of the relations of incumbents and entrants, of

possible abuse of incumbents' market power or of the access to bottleneck facilities. Finally, competition is established. The regulator can decrease his activity gradually, limiting it to the protection of customers against negative aspects of competition. In the telecom environment these aspects may concern compatibility, privacy and security questions or minimum amount of service provision.

The regulator has to deal with different types of uncertainty. Typically, he is not perfectly informed about the characteristics of the regulated firms. Therefore the effects of an implemented policy, i.e. the response of the regulated firms, can not be evaluated sufficiently. Furthermore, there exists uncertainty about parameters of the environment (e.g. customer demand, technology development, product life cycles). Often the regulated firms have more precise information about these characteristics than the regulator such that an information asymmetry exists. Additionally the interactions of regulatory measures with other fields of public economics such as taxation or licensing must be taken into account. This yields a further source of uncertainty.

Consequently, the determination of a good or even optimal regulatory policy represents a quite complex problem. Since it intervenes with existing or developing mechanisms in an industrial environment, regulation may have promoting but also constraining effects on the economic development. A careful analysis of such effects is therefore indispensable. But typically only simplified models are studied and advice for a general framework is given. This does not take into account the specifics of the considered industry sector. Neither the implications of the uncertainty can be studied sufficiently.

However, taking into account the fundamental uncertainty, the problem can be interpreted as a decision problem under uncertainty. Moreover, it is possible to separate the uncertainty about the response of the regulated firms and about the interactions with other fields of public economy from the uncertainty about the environment. These types of uncertainty can be analysed utilising concepts of game theory or of bilevel programming. Then the regulation problem can be interpreted and analysed along the lines of the framework of bilevel stochastic programming. This is demonstrated in Paper 3. We outline approaches for the determination of optimal regulation policies and study different measures in order to take into account the existing uncertainty.

3.3 Agency theory

The issues discussed in the preceding two subsections, Virtual Operators and regulation, are examples of principal agent problems in a modern telecommunications environment. More generally, agency models describe an asymmetric social or economic interaction of several actors in a common environment. One actor, the principal, delegates the task of decision making to another actor, the agent. In the context of a regulated telecom environment the principal may be represented by a regulator and the agent(s) by the regulated service provider(s). Another example is the relationship between Network Operator (principal) and Virtual Operator (agent). Both actors possess individual utility functions and choose their decisions in order to maximise their expected utility. However, the principal's utility is influenced also by the agent's actions. Therefore he wants to find an incentive schedule inducing the agent to the choice of a decision which is favourable for the principal. This incentive schedule is a function of the agent's decisions and the environment state. Consequently, the agent chooses an action that maximises her utility from this action and the according incentive fee, taking into account the state of the environment. At the same time the principal maximises his utility depending on the agent's action, the according incentive fee and the state of the environment. Often the incentive schedule has a monetary nature but it may also be a success indicator such as reputation or ranking.

Generally it is assumed that the actors have imperfect knowledge about environment parameters and that the principal has limited knowledge about the agent's decision process. Consequently, the principal may initiate a monitoring process in order to decrease his uncertainty about the agent. Often the monitoring turns out to be a difficult and costly task, for example due to a highly complex and dynamic environment, such that an optimal monitoring intensity must be determined.

Although principal agent models are widely studied in economic theory, usually only little effort has been dedicated to the analysis of their mathematical properties, quite simply structured problems are considered and the inherent uncertainty is treated inadequately. However, agency problems can be interpreted as specific problems of hierarchical decision making under uncertainty and are thus amenable to an analysis following the concepts outlined in Section 2.

The model developed in Paper 1 illustrates the agency relationship between a Network Operator and a Virtual Operator. Both decision makers compete in the provision of service to a common customer population. At the same time they must collaborate in order to give the Virtual Operator access to a bottleneck facility owned by the Network Operator. In Paper 3 agency theory is applied to a regulated telecommunications environment and approaches for the determination of optimal incentive schedules utilising stochastic optimisation concepts are outlined. More generally, these studies are applicable also to other oligopolistic environments characterised by mutual dependencies of the actors and a dominating decision maker, combined with possible uncertainty about vital parameters of the analysed model. An example is a liberalised electricity market.

4 Research contribution and description of the papers

This section describes briefly the subject of the single papers contained in this thesis and indicates their research contribution. In all the papers I have done the major part of the research and the writing.

Problems of hierarchical decision making under uncertainty establish a quite new research area which has been studied only recently. The approach applied in this thesis utilising the concept of a bilevel stochastic programming problem was not considered until now.

Paper 1. Extending the stochastic programming framework for the modeling of several decision makers: pricing and competition in the telecommunication sector

This paper was written together with my supervisor Alexei A. Gaivoronski and my co-supervisor Jan A. Audestad.

Most approaches concerned with pricing schemes for access and service in a telecommunication environment consider the market from above, for example by taking on the standpoint of a regulator. Thus the implications on the total or

the social welfare of that environment are studied. The approach pursued in this paper is contrary. The aim is the development of a tool for decision support for one of the actors. Consequently, the viewpoint of that actor is taken on and the implications of his decisions on his welfare are analysed in interaction with the behaviour of competitors. We developed a modeling framework for stochastic programming problems with a bilevel structure. Based on this framework we developed models for network planning and pricing. They describe the competition and cooperation relationship between a Network Operator and a Virtual Operator from the point of view of the Network Operator. Finally, the paper contains an implementation and numerical studies of the properties of such models.

Ideas and results of this work were presented at the 9th International Conference on Stochastic Programming SP01, Berlin, Germany in August 2001 and at the Sixth INFORMS Telecommunications Conference, Boca Raton, Florida, in March 2002. The paper is accepted for publication in the special issue of *Annals of Operations Research* devoted to SP01. An earlier, more popular scientific version of the paper was published in *Telektronikk* 4.2001 (vol. 97) pp. 46–64, a special issue devoted to Mobile Virtual Network Operators. It emphasises modeling issues together with a visualisation and interpretation of model characteristics under different sets of environment parameters.

Paper 2. A solution method for bilevel stochastic programming problems

This paper was written together with my supervisor Alexei A. Gaivoronski. Our approach considers bilevel stochastic programming problems as extension of stochastic programming problems by adding bilevel features. It can thus take into account the stochastic features more adequately than the frameworks of stochastic games (mainly based on game theoretic concepts) or of stochastic bilevel programming (mainly based on bilevel programming concepts) allow. At the same time it enables the consideration of continuous decision variables and of more complex decision problems. In particular it is possible to incorporate so-called connecting upper level constraints depending on follower decisions. We analyse several variants of the bilevel stochastic programming problem and give necessary conditions

for a local optimal solution. Furthermore we propose a solution algorithm utilising a stochastic quasi-gradient method and prove its convergence to a local optimum for different problem formulations.

The paper is submitted for publication in *Mathematical Programming*.

Paper 3. Utilisation of stochastic programming methods in the analysis of agency problems

This paper was written together with my supervisor Alexei A. Gaivoronski. The focus of this study is on another aspect of the interplay of two modern telecommunication actors, the principal-agent relationship of a regulator and a regulated telecom service provider. Agency relationships are widely studied in economic theory, but due to the complex structure of such problems typically only models with quite simple mathematical structures are analysed. Usually these models contain only trivial or no constraints and the inherent uncertainty is not treated adequately. It is often replaced by the expected values of the uncertain parameters which can lead to incorrect results. This paper demonstrates that the application of the bilevel stochastic programming framework helps to consider much more complex models and at the same time to treat the uncertainty adequately.

Paper 4. Influence of perturbed input data on convexity properties of stochastic programming problems

This paper was written together with my supervisor Alexei A. Gaivoronski. The purpose of this paper is to investigate the influence of the stochasticity on properties of the BLSP problem. In particular we study if the properties of the objective function can be improved. In the previous papers we assumed that the follower's response was uniquely determined for all leader decisions. Here we investigate if this assumption can be relaxed when the principal takes into account his uncertainty about the decisions actually implemented by the agent. We found that the uncertainty can improve the quality of the leader's decision process. To our knowledge no similar approach was investigated so far.

5 Conclusions and future research

The work presented in this thesis focuses on a framework for decision making under uncertainty when part of this uncertainty can be attributed to the actions of another decision maker pursuing own goals. This field of research is quite new and some studies have been conducted highlighting either the bilevel or the game theoretic aspects of the relations between the decision makers. However, the studies in this thesis focus on the existent uncertainty and employ stochastic programming techniques.

We developed a framework for modeling stochastic programming problems with a bilevel structure. A solution approach based on stochastic programming methods enhanced by concepts of bilevel programming and game theory is presented and implemented. Additionally, the influence of the uncertainty on properties of the BLSPP problem is investigated. We apply the presented framework to several problems such as the interplay between Network Owner and Virtual Operator, principal agent relationships or regulation issues.

The inclusion of bilevel features into two-stage stochastic programming problems introduced nonlinearity and, especially, nonconvexity of the objective functions as well as of the constraints. The problems studied in this thesis raise a number of questions. In addition to the suggestions made in the single papers, further research may be concerned with a more general analysis of the bilevel stochastic programming framework. The uncertainty has quite dramatic implications on the leader's choice of an optimal solution. Therefore attention should be directed on investigations regarding the effects of the uncertainty. Furthermore alternative solution approaches may be investigated as well as different problem structures, comprising for example recourse problems also in the lower level problem or the case of nonunique follower responses to certain upper level decisions.

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Paper 1

Extending the stochastic programming framework for
the modeling of several decision makers:
pricing and competition in the telecommunication sector

J.-A. Audestad, A.A. Gaivoronski and A.S. Werner

Extending the stochastic programming framework for the modeling of several decision makers: pricing and competition in the telecommunication sector

Jan-Arild Audestad*, Alexei A. Gaivoronski[†] and Adrian S. Werner[‡]

Abstract

We consider the case when part of the uncertainty faced by a decision maker is derived from actions of another independent actor who pursues her own aims. Each party sets its decisions in the next time period in response to the other party's policy. We model this situation by introducing some ideas from game theory, but unlike this theory we do not focus on equilibrium and related optimality notions. Instead, we follow the framework of stochastic programming and take the view of one of the decision makers. Our model is placed in a telecommunication environment with a network owner and operators without their own network facilities. We give an extension to a multiperiod model.

Key words: Stochastic programming, modeling of competition, virtual operators, pricing.

*Telenor AS;

[†]Department of Industrial Economy and Technology Management, Norwegian University of Science and Technology;

[‡]Department of Industrial Economy and Technology Management, Norwegian University of Science and Technology

1 Introduction

We describe a modeling approach to provide decision support for strategy evaluation of an industrial agent in complex relations of competition and collaboration with other agents in the same industrial environment. This is the situation many telecom service providers find now, with a deregulation process and convergence between telecommunications, computer industry and content provision being well under way. The objective of the approach is to provide a set of quantitative decision support tools which would enhance the quality of strategic and tactical decisions.

Microeconomic theory [MW95] provides important theoretical insights in these issues, especially when the studied system is under conditions of equilibrium. However, classical theory often treats uncertainty inadequately. Unfortunately, central features of today's telecommunication environment are the presence of uncertainty and, usually, the absence of equilibria. This makes many established approaches inapplicable. Therefore we employed techniques specially designed to incorporate uncertainty and dynamics in decision models and in particular stochastic programming [EW88, BL97]. On the theoretical level, such techniques have been under development for a few decades, but only relatively recently the state of software and hardware allowed large scale applications. We supplement this by selected ideas from game theory [Bin92] because part of the uncertainty a given decision maker faces results from actions of other decision makers.

Quantitative decision models for a competitive telecommunication environment recently became the subject of intensive research effort. An alternative and complementary approach is constituted by simulation models of systems of interacting agents known as *agent nets* [BEG98, Gai98, Gai99]. Different models which utilise game theoretical concepts were proposed in [LS92, QR01, SL88]. The distinctive feature of the approach presented here is the utilisation of a stochastic programming methodology for the adequate treatment of the uncertainty and the absence of an equilibrium coupled with selected notions from game theory.

Another promising approach for a description of the relations between the providers is the use of stochastic games [BV00], especially in combination with topics of multiagent learning. This framework takes into account both the uncertainty and

the interdependencies between the actors. However, again it is the absence of an equilibrium due to rapid changing technologies (and therefore a rapid changing environment for the agents) that makes such methods of limited applicability to our case.

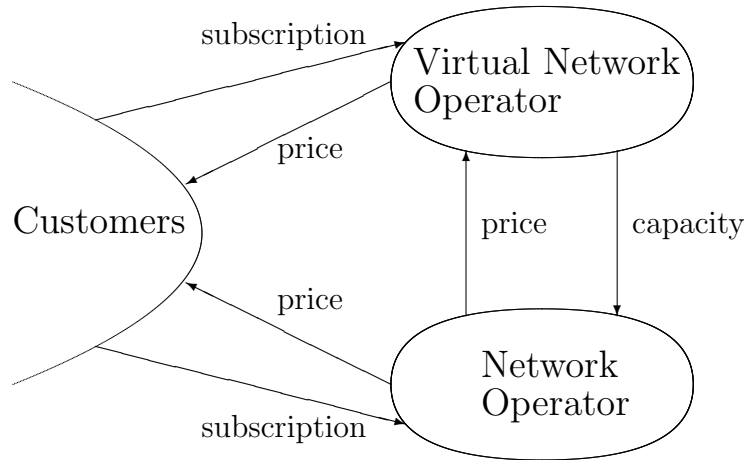


Figure 1: Relations between service providers and customers

We illustrate our approach by a case study which describes relations between service providers and a customer population, see Figure 1. The considered time horizon consists of several time periods. In the simplest case we assume that the operators provide a common market with the same type of service based on a telecommunication network. For delivery of this service they utilise network capacity. Whereas one of the providers owns the network, the other one is a virtual operator without her own network facilities. In order to provide service she must lease capacity from the network owner. We develop a decision model that is decomposed in three submodels: the *customer model*, the *enterprise model* and the *competition model*. These models contain a number of simplifying assumptions, although they are not essential and more specific details can be incorporated easily.

Since the aim is to provide decision support tools for a given actor we do not follow the usual economic view on a market "from above", i.e. the maximisation of a general welfare [MW95]. Instead, our approach adopts the point of view of one of the providers. His main focus lies on maximising his own welfare. We

take the point of view of the network operator, but the virtual operator could be considered similarly. In order to achieve his goal the network provider formulates models describing customer behaviour and predicting his rival's responses to his policy. These models depend on a number of parameters with uncertain values, which makes an adequate treatment of uncertainty particularly important.

Although we placed our approach in a telecommunication setting it is also applicable to other competitive environments where mutual dependencies of actors in an oligopoly may occur, combined with possible uncertainties about vital parameters of the setting.

In the subsequent sections we develop this approach in more detail. In the next section we present more formally the general structure of the model. After that the simplest possible case is considered which deals with one time period and a deterministic setup. Later this model is extended by allowing for uncertainties both about the network owner's policy and about other model parameters. A further extension is made by introduction of a multiperiod model. Theoretical considerations are supplemented by numerical experiments.

2 A general description of the modeling approach

From the point of view of the network owner our modeling approach can be divided into the subproblems *enterprise model*, *competition model* and *customer model* that are connected as illustrated on Figure 2.

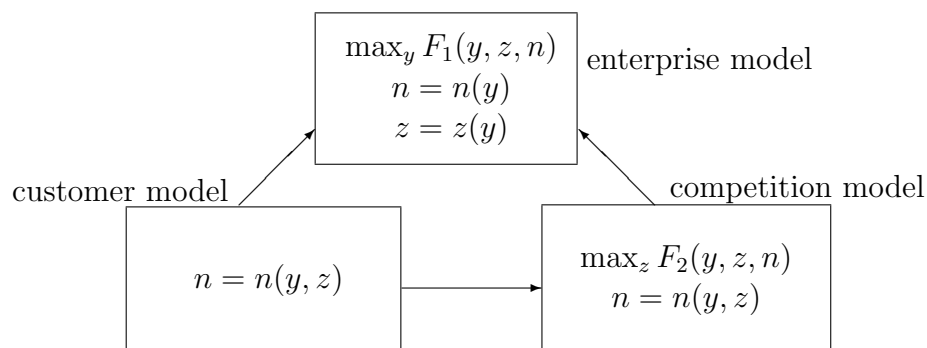


Figure 2: General structure of the approach

At the beginning of each time period the network operator performs the following steps to determine his optimal decision under the current circumstances:

- predict the customer response for a given decision and a given competition response using the *customer model*. This comprises customer numbers for both the network operator and the competition.
- predict the competition response for a given decision using the *competition model*.
- select an optimal policy from the *enterprise model* using as input the predictions of the customer and the competition response obtained in the previous two steps.

The following notations are utilized here:

y – decisions of the network operator (NO): price y_1 for service provision to own customers and price y_2 for capacity leased by his rivals.

z – decisions of the virtual network operator (VNO): price z_1 for service provision and amount z_2 of capacity leased from the NO.

$n = (n_1, n_2)$ – total numbers of customers of the NO and the VNO respectively.

These numbers depend on the respective decisions y and z .

$F_2(y, z, n)$ – objective function of the VNO, depending on both provider's decisions y and z and on the number of her customers $n = n(y, z)$ obtained from the customer model. It comprises the network operator's knowledge about his rival's aims, namely the NO thinks that the VNO chooses his decisions from maximization of this function. More formally, the network operator takes as predictions $z(y)$ for decisions of the virtual operator the solution of the following problem:

$$\max_{z \in Z} F_2(y, z, n(y, z))$$

where Z is the set of admissible decisions of the VNO. Examples of such an objective function could be profit or market share.

$F_1(y, z, n)$ – objective function of the NO, depending on both provider's decisions, y and z , and on the number of his customers $n = n(y, z)$ obtained from the customer model. For a fixed decision y the value of this function is computed using the prediction $z(y)$ of the virtual operator's response and the prediction

$n(y) = n(y, z(y))$ of the network owner's customer number. Consequently, the decision y is found by maximisation of his objective function solving the problem

$$\max_{y \in Y} F_1(y, z(y), n(y))$$

where Y is the set of admissible decisions of the network operator.

In the following this general problem structure is used to develop specific models with special attention to uncertainty and dynamics.

3 Single time period

We start with the simplest possible case which we will use as a benchmark for more complex models, and also because it allows an analytic solution. It employs only one time period and assumes full information of the network owner about the single parameters defining both customer and competition model.

3.1 Deterministic case

The network owner's profit is defined as the difference between revenue earned by service provision and capacity leasing and costs of service provision. Likewise the profit of the virtual operator is the difference between revenue from service provision and costs of capacity leasing and service provision. The decisions about the service and capacity prices and about the leased capacity have to be chosen within some limits. Whereas it is assumed that the network owner always has sufficient capacity to serve all customer demand, there exists a Quality of Service constraint for the virtual operator: the amount of leased capacity should be enough to serve all demand from her customers. However, in this simple model she does not face opportunity costs when she cannot serve all demand.

The decisions of the virtual operator result from her profit maximization model with the mentioned constraints and regarding the network owner's decisions and the customer behaviour. The network owner knows how the virtual operator determines her response to his prices and therefore solves the same model as her. Then he can substitute these decisions into his own problem. Now we describe this model more formally, following the structure given in the previous section.

3.1.1 Customer model

This model represents a module providing detailed input for the complete model. Customers can subscribe to each of the providers for delivery of service. In return, the providers charge a price for this service. We employ a simple and nondifferentiated price structure, although arbitrarily complex schemes can be incorporated similarly considering different customer types. Also other ideas such as an implementation of a customer feedback are not pursued here. At present we assume that the customer behaviour is influenced only by price considerations. Since it also has relevance to the subsequent exposition, we develop the customer model in a general form, which is then somewhat simplified by regarding just one period. A schematic description of the customers' decision process is shown in Figure 3.

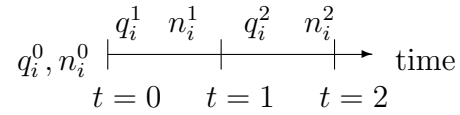


Figure 3: Customer decision process

Here $q_i^t, t = 0, 1, \dots$ denotes the service price charged by provider i at time t and $n_i^t, t = 0, 1, \dots$ the number of customers at the end of period t who utilise the service supplied by provider i . This number n_i^t depends on the providers' prices q_i^t and is structured as follows

$$n_i^{t+1} = n_i^t + m_i^t + m_{ij}^t \quad (1)$$

where m_i^t is the number of first time customers subscribed to provider i at time t and m_{ij}^t is the number of customers who switch from provider j to provider i at time t . Both m_i^t and m_{ij}^t depend on the prices q_i^t . By linearisation of the assumed price/demand relationship in the vicinity of the reference price q we can approximate the relations between customer flow and service price changes:

$$m_i^t = l_i^t + c_i(x_i^{t-1} - x_i^t) \quad (2)$$

$$m_{ij}^t = l_{ij}^t + c_{ij}(x_j^t - x_i^t) \quad (3)$$

with the price structure

$$q_i^t = q + x_i^t$$

Here x_i^t is the price increment of provider i at time t , l_i^t is the amount of new customers who would subscribe to the service of provider i at time t in the absence of price changes, l_{ij}^t is the number of customers who would migrate from provider j to provider i in the absence of price differences between these providers and c_i^t, c_{ij}^t are coefficients to be estimated from market data. The parameters l_i^t and l_{ij}^t model other aspects of customer behaviour besides the response to price changes. In the case of a single period it is convenient to denote $y_1 = x_1^1, z_1 = x_2^1$. Then the number of customers served by the network owner and the virtual operator respectively can be described as follows using relations (1) – (3):

$$n_1(y, z) = k_1 - r_{11}y_1 + r_{12}z_1 \quad (4)$$

$$n_2(y, z) = k_2 + r_{21}y_1 - r_{22}z_1 \quad (5)$$

where the parameters $k_1, k_2, r_{11}, r_{12}, r_{21}, r_{22}$ can be expressed through the parameters found in (1) – (3). Note that the parameters k_1, k_2 also include the initial customer numbers and the initial prices of the respective operators.

3.1.2 Competition model

This model describes the network owner's perception of the virtual operator's profit. Denoting

y_2 – price charged by the NO for a unit of leased capacity;

z_2 – amount of capacity leased by the VNO;

we can express revenue and costs of the VNO as follows:

Revenue. Given that the service price charged by the virtual network operator is $q + z_1$ and her number of customers is given by (5) her revenue is $(q + z_1)n_2(y, z)$.

Costs. They are composed of two components:

– cost for leasing of network capacity y_2z_2 ;

– cost of service provision $g_2 + e_2n_2(y, z)$, where e_2 and g_2 are respectively the VNO's variable service provision cost per customer and fixed service provision cost.

Therefore the virtual operator's profit can be expressed as

$$(q - e_2 + z_1)n_2(y, z) - y_2z_2 - g_2$$

There are two decisions of the VNO that affect her profit in this model:

- the price difference z_1 between the reference service price and the price charged by the VNO;
- the amount of the network capacity z_2 to lease from the NO.

The network operator assumes that the virtual operator maximises her profit under Quality of Service constraints. Then he can predict her decision $z(y)$ as the solution of the following optimisation problem:

Find z_1 and z_2 which maximise

$$(q - e_2 + z_1)n_2(y, z) - y_2z_2 - g_2$$

subject to constraints

$$z_2 \geq dn_2$$

$$n_2(y, z) \geq 0$$

$$\Delta_1 \leq z_1 \leq \Delta$$

$$0 \leq z_2 \leq U_2$$

where d is the average amount of capacity required for service provision of one user with admissible service quality, U_2 the upper limit for the amount of leased capacity and Δ_1, Δ are the respective lower and upper limits for the price change. Note that the solution of this problem depends on the network owner's price decisions y_1 and y_2 . When the unconstrained solution lies within the stated bounds, it can be expressed analytically as follows:

$$z_1(y) = \frac{k_2 - (q - e_2)r_{22} + r_{21}y_1 + dr_{22}y_2}{2r_{22}} \quad (6)$$

$$z_2(y) = \frac{d}{2}(k_2 + r_{21}y_1 + (q - e_2)r_{22} - dr_{22}y_2) \quad (7)$$

which means that the decisions of the VNO depend linearly on the decisions of the NO.

3.1.3 Enterprise model

This model describes the profit of the network operator as dependent on the decisions of the customers and the competitors. It can be determined similarly

to the competition model presented in the previous section. The network owner's revenue and costs are defined as follows:

Revenue. It is composed from two components:

- revenue from service provision to customers. Given that the price charged by the network operator for its service is $q + y_1$ and his number of customers is given by (4) this part of the revenue is

$$(q + y_1)n_1(y, z(y))$$

- revenue from leasing of capacity to the VNO y_2z_2 .

Costs. They are costs of service provision:

$$g_1 + e_1n_1(y, z(y))$$

where g_1 is the network owner's fixed and e_1 the variable cost of service provision per customer. There are two decisions of the NO which affect his profit in this model:

- the price difference y_1 between the reference service price and the price charged by the NO;
- the price y_2 charged to the VNO for a unit of leased capacity.

Furthermore, his profit depends on decisions z_1 and z_2 of the VNO. Now the network operator can substitute the virtual operator's predicted decisions into the expressions for his profit and for his customer number. Note that the network operator's optimisation problem may become infeasible since also his constraints depend on both the virtual operator's and his own decisions. This problem is addressed in Section 6. If the virtual operator's decisions take on the analytical expressions (6) – (7) then, assuming profit maximisation, the network operator's decisions are the solution of the following optimisation problem:

Find y_1 and y_2 which maximise

$$- \left(r_{11} - \frac{r_{12}r_{21}}{2r_{22}} \right) y_1^2 + dr_{12}y_1y_2 - \frac{d^2}{2}r_{22}y_2^2 + a_1y_1 + a_2y_2$$

subject to the constraints

$$n_1(y, z(y)) \geq 0$$

$$\Delta_1 \leq y_1 \leq \Delta$$

$$0 \leq y_2 \leq U_1$$

where a_1 and a_2 are expressed through parameters introduced before, Δ_1, Δ are lower and upper limits for the service price change and U_1 is an upper bound for the price charged for leased capacity fixed by the regulation authorities.

This problem is a simple quadratic programming problem which can be easily solved analytically if it is concave. This, however, is not always the case and concavity conditions should be derived from the parameters which describe customer and competition behaviour.

3.2 Allowing for uncertainties

So far we assumed full knowledge of the providers about all model parameters. However, this is usually not the case. In our further studies we therefore take into account uncertainty of both providers about customer behaviour and incomplete knowledge of the virtual operator about the pricing decisions of the network owner. Other sources of uncertainty such as changing demand or uncertainty about the costs for service provision may be incorporated but are omitted here. Since the virtual operator now cannot predict the exact customer demand, she may face opportunity costs for the unserved demand due to a lack of capacity. We assume that she is also uncertain about the exact level of these opportunity costs. Furthermore now we have to bear in mind that revenue and costs are caused only by the actually served customers whose number is limited by the amount of available capacity.

The network owner's decision process will proceed as presented before. However, it will become more complicated for the network owner to estimate the virtual operator's response. In the competition model he must also find out how the virtual operator perceives his decisions by help of uncertain (random) parameters and his actual prices. Also the customer model includes estimations of the customer numbers. Note that only average or expected profits are maximised due to the described uncertainties.

4 Mathematical description of the modeling approach: general case

For the sake of simplicity, here we consider only two time periods, the studies can be easily extended to the case of more periods. Figure 4 shows the decision process performed for the case of two time periods.

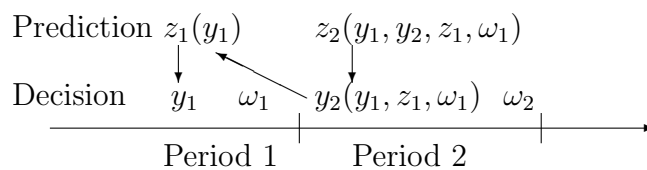


Figure 4: Decision process for two periods in the presence of uncertainty

Here we introduce the following notations:

$\omega_t, t = 1, 2$ – uncertain parameters from the point of view of the NO at time period t . They describe the quantities from the customer and the competition model about which the network operator has uncertain knowledge. The information about these parameters available for the network operator is described by probability distributions. We assume that the values of these parameters become known at the end of period t .

$y_t, t = 1, 2$ – decisions taken by the NO at the beginning of period t before the values ω_t of the uncertain parameters become known. These decisions are taken with the aim to improve some enterprise performance measure averaged with respect to the values of the uncertain parameters.

$z_t, t = 1, 2$ – reaction of the competitors to the decisions of the NO. The network operator forecasts this reaction using the competition model assuming that the competitors take their decisions with the aim to improve some enterprise performance measure averaged with respect to the values of the uncertain parameters. At the beginning of the considered time horizon the network owner implements the decision y_1 . In order to take this decision he must foresee its influence on the decisions taken by the network owner, the competitors and the customers in subsequent periods. This takes place in a decision/prediction process as depicted in Figure 4. Whereas a more specific form is considered in the next section we

will present it here briefly in the most general form which consists of the following steps.

1. Prediction of the competitor's reaction during period 2.

Dependent on y_1, y_2, z_1, ω_1 obtain the prediction $z_2(y_1, y_2, z_1, \omega_1)$ for the decision of the competitors during period 2.

2. Finding the optimal decision for period 2.

Dependent on y_1, z_1, ω_1 and for a given prediction $z_2(y_1, y_2, z_1, \omega_1)$ for the decision of the competitors during period 2 find the optimal decision $y_2 = y_2(y_1, z_1, \omega_1)$ for period 2.

3. Prediction of the competitor's reaction during period 1.

Dependent on y_1 obtain the prediction $z_1(y_1)$ for the decision of the competitors during period 1.

4. Finding the optimal decision for period 1.

Having a prediction $z_1(y_1)$ for the decision of the competitors during period 1, find the optimal decision y_1 of the NO for period 1.

When applied to our model this decision process looks fairly involved. However, given the present state of the art in the optimization methods and the related software it is feasible to build a decision support system based on this approach. In this connection numerical approaches developed in the field of stochastic programming become pivotal. One possible way to proceed consists of the following steps.

- Approximate the probabilistic distributions of the uncertain parameters by a finite number of scenarios which take the form

$$(p_{1i}, \omega_1^i), (p_{2ij}, \omega_2^{ij}), i = 1 : N, j = 1 : M_i \quad (8)$$

where it is assumed that ω_1 takes the value ω_1^i with the probability p_{1i} and ω_2 takes the value ω_2^{ij} with the probability p_{2ij} under the condition that ω_1 takes the value ω_1^i .

- Construct the so-called *deterministic equivalent* of the problem in the last step [EW88, BL97] which makes the problem amenable to solution. A specific form of the deterministic equivalent depends on the structure of the problem and an example can be found in the next section.
- Use commercial software as building blocks for a solution of the deterministic equivalent and for the development of a decision support system.

An alternative stochastic programming approach allows a direct use of continuous distributions by application of sampling techniques and stochastic gradient methods [Gai88].

5 Two stage model

5.1 An intermediary model

A first extension of the one stage model is the following intermediary model. The considered time horizon is extended by a second time period. The providers make all decisions on service and capacity prices and on the amount of leased capacity at the beginning of the first time period on the basis of the expected behaviour of the rival and the customers in both time periods. These decisions are fixed throughout the second stage. However, now we assume that also the network operator has only limited capacity and when the customer demand becomes known at the end of the first time period he has the possibility to extend his network. Furthermore, we refine the model by assuming that the network owner can state an upper limit on the amount of capacity leased by the virtual operator. This gives the contract between the providers a new quality: so far, such a limit was settled by an "outside force" like a regulation authority. Now the network operator can intervene.

Since all decisions depending on the respective rival's policy do not change throughout the considered time horizon both the prediction model and the decision model can be solved by combining the models of the single stages to one model using a discount factor. In the following Section 5.2 we present a general variant of the two stage model with the possibility of a network extension. Therefore we dispense here with an explicit formulation.

5.2 Two period model with uncertainty and investment in infrastructure

Using the general approach described in Section 4 we can now develop a specific decision model for our environment with a network operator and virtual operator(s). It is a further development of the one period deterministic model presented in Section 3.1 including uncertainty, two time periods and investment in the infrastructure as outlined in Sections 3.2 and 5.1, respectively.

Decisions of the network owner:

$y_1 = (y_{11}, y_{12}, y_{13})$ – decisions during period 1;

y_{11} – price to charge for his service to customers during period 1;

y_{12} – price to charge for capacity to the VNO during period 1;

y_{13} – maximal amount of capacity to lease to the VNO during period 1.

$y_2 = (y_{21}, y_{22}, y_{23}, y_{24}, y_{25})$ – decisions during period 2;

y_{21} – price to charge for his service to customers during period 2;

y_{22} – price to charge for capacity to the VNO during period 2;

y_{23} – maximal amount of capacity to lease to the VNO during period 2, we assume that $y_{23} \geq y_{13}$;

y_{24} – amount of capacity to add at the beginning of period 2;

y_{25} – binary variable which equals 1 if the decision to add capacity is taken and 0 otherwise.

Decisions of the virtual operator:

$z_t = (z_{t1}, z_{t2})$ – decisions during period t , $t = 1, 2$;

z_{t1} – price to charge for her service to customers;

z_{t2} – amount of capacity to lease from the NO.

The uncertain parameters ω_t , $t = 1, 2$ determine the customer and the competition model from Sections 3.1.1 and 3.1.2:

$$\omega_t = (r_1^t, r_2^t, \eta^t), r_1^t = (r_{11}^t, \dots, r_{15}^t), r_2^t = (r_{21}^t, \dots, r_{23}^t), t = 1, 2$$

where r_1^t and r_2^t describe the uncertainty related to the enterprise and the competition model respectively. Here $r_{11}^t, r_{12}^t, r_{21}^t, r_{22}^t$ are taken from the relations

$$\begin{aligned} n_1^t &= k_1^t - r_{11}^t y_{t1} + r_{12}^t z_{t1} \\ n_2^t &= k_2^t + r_{21}^t y_{t1} - r_{22}^t z_{t1} \end{aligned}$$

describing the customer model in the case of two periods similar to the model (4) – (5) and $r_{21}^t = r_{12}^t$. Note that the parameters k_1^t, k_2^t include the number of customers and the service price decision of the respective provider in the previous time period. The parameters e_1^t, e_2^t denote the variable costs of service provision per customer taken from the providers' profit expressions

$$\begin{aligned} g_1^t + e_1^t n_1^t(y, z(y)) \\ g_2^t + e_2^t n_2^t(y, z(y)) \end{aligned}$$

similar to Sections 3.1.2 and 3.1.3. The other parameters have the following meaning:

r_{13}^t – opportunity cost of not meeting a unit of demand for the NO during period t ;

r_{14}^t – variable cost of adding a unit of capacity;

r_{15}^t – fixed cost of adding capacity, this parameter together with r_{14}^t is defined only for $t = 2$; r_{23}^t – virtual provider's opportunity cost of not serving one unit of demand during period t ;

The parameter b denotes the current network capacity of the NO and d_1^t, d_2^t are the amounts of capacity required to satisfy demand from one customer of the NO and the VNO respectively;

The variable η^t is used to describe uncertainty in the virtual operator's knowledge about the decisions of the network operator as it is seen by this network operator. More precisely, if the decision of the network owner is y_t then he assumes that the virtual network operator thinks that this decision is

$$\hat{y}_t = y_t + \eta^t$$

and uses this value in his competition model. In order to keep our model simple a relation between the size of the NO's decisions and the VNO's uncertainty is ignored. It is assumed that also \hat{y}_t is in the stated bounds for the decisions of the network owner although it would of course be more correct in the applied context to model this aspect more profoundly.

Generally, the uncertain parameters ω_t are described by continuous probability distributions. Following the scenario approach outlined at the end of Section 4,

we approximate the possible values of the uncertain parameters by a finite number of values with given probabilities as in (8):

$$\omega_1^i = (r_1^{1i}, r_2^{1i}, \eta^{1i}), \omega_2^{ij} = (r_1^{2ij}, r_2^{2ij}, \eta^{2ij}), i = 1 : N, j = 1 : M_i$$

In accordance with the framework described in this section, the decisions of both providers in period 2 depend on the values of the random variables ω_1 . For a finite number of scenarios $i = 1 : N$ this leads to the notions:

y_2^i – decision of the NO in period 2 under scenario i ;

$z_2^i = z_2^i(y_2^i)$ – prediction of the VNO's response to the decision y_2^i of the NO in period 2 under scenario i .

Now we apply the deterministic equivalent of the decision process as outlined in Section 4 to our special environment.

1. **Decision model for the network operator.** This model combines the network operator's decision models for both periods. We assume that he has the predictions $z_1(y_1)$ and $z_2^i(y_2^i)$ of the virtual operator's decisions in period 1 and period 2 under the scenarios $i = 1 : N$ respectively. Using these predictions the network operator tries to find the optimal decisions y_1 for period 1 and y_2^i for period 2 and scenarios $i = 1 : N$ by solving the following problem.

Find y_1 and $y_2 = (y_2^1, \dots, y_2^N)$ which maximize

$$f_1(y_1, z_1(y_1)) + \alpha \sum_{i=1}^N p_{1i} f_2^i(y_2^i, z_2^i(y_2^i)) \quad (9)$$

subject to constraints

$$\Delta_1^1 \leq y_{11} \leq \Delta^1, \quad (10)$$

$$\Delta_1^2 \leq y_{21}^i \leq \Delta^2, i = 1 : N, \quad (11)$$

$$0 \leq y_{12} \leq U_1, \quad (12)$$

$$0 \leq y_{22}^i \leq U_1, i = 1 : N, \quad (13)$$

$$0 \leq y_{13} \leq b, \quad (14)$$

$$y_{13} \leq y_{23}^i \leq b + y_{24}^i, \quad i = 1 : N, \quad (15)$$

$$y_{24}^i \leq Y_{24}^i y_{25}^i, \quad i = 1 : N, \quad (16)$$

$$n_1^{1i}(y_1, z_1(y_1)) \geq 0, \quad i = 1 : N, \quad (17)$$

$$n_1^{2ij}(y_1, y_2^i, z_2^i(y_2^i)) \geq 0, \quad i = 1 : N, \quad j = 1 : M_i \quad (18)$$

The objective function in (9) consists of two terms: the profits of the network owner during period 1 described by function $f_1(\cdot)$ and his average profits during period 2 discounted with the discount coefficient $\alpha \leq 1$ which are described for scenario i by the function $f_2^i(\cdot)$. The function $f_1(\cdot)$ can be expressed as follows.

$$f_1(y_1, z_1(y_1)) = \sum_{i=1}^N p_{1i} \left((n_1^{1i}(y_1, z_1(y_1)) - w_{1i})(q - e_1^1 + y_{11}) - r_{13}^{1i} w_{1i} \right) + y_{12} z_{12}(y_1)$$

where w_{1i} are the potential customers of the network owner which are lost during period 1 under scenario i due to lack of capacity for the service provision:

$$w_{1i} = \max \left(0, n_1^{1i}(y_1, z_1(y_1)) - \frac{1}{d_1^1} (b - z_{12}(y_1)) \right)$$

and \bar{r}_1^1 are the expected values of the uncertain parameters r_1^1 :

$$\bar{r}_1^1 = (\bar{r}_{11}^1, \bar{r}_{12}^1, \bar{r}_{13}^1) = \sum_{i=1}^N p_{1i} r_1^{1i}$$

The first part of the expression for $f_1(\cdot)$ is the profit defined as in the enterprise model of Section 3.1.3. The revenue stems only from the actually served customers. Their number is the number of potential customers minus the number of customers that can not be served. The function $f_2^i(\cdot)$ for the profit during period 2 is very similar to $f_1(\cdot)$ and can be expressed as follows:

$$f_2^i(y_2^i, z_2^i(y_2^i)) = \sum_{j=1}^{M_i} p_{2ij} \left((n_1^{2ij}(y_1, y_2^i, z_2^i(y_2^i)) - w_{2ij})(q - e_1^2 + y_{21}^i) - r_{13}^{2ij} w_{2ij} \right) + y_{22}^i z_{22}^i(y_2^i) - r_{14}^{2ij} y_{24}^i - r_{15}^{2ij} y_{25}^i$$

with w_{2ij} the potential customers lost during period 2 under scenario j due to lack of capacity for service provision provided that the uncertain parameters during period 1 followed scenario i :

$$w_{2ij} = \max \left(0, n_1^{2ij}(y_1, y_2, z_2(y_2)) - \frac{1}{d_1^2}(b + y_{24}^i - z_{22}^i(y_2^i)) \right)$$

and \bar{r}_1^{2i} are the expected values of the uncertain parameters r_1^{2i} conditioned on scenario i of period 1:

$$\bar{r}_1^{2i} = (\bar{r}_{11}^{2i}, \bar{r}_{12}^{2i}, \bar{r}_{13}^{2i}) = \sum_{j=1}^{M_i} p_{2ij} r_1^{2ij}$$

Observe that the costs for the infrastructure upgrade are taken into account in the profit calculation for period 2 through the variables y_{24}^i and y_{25}^i .

The predictions $z_1(y_1)$ and $z_2^i(y_2^i)$ of the virtual operator's decisions which enter this model are obtained through the respective prediction models for periods 1 and 2. However, unlike the network operator's decision model, these prediction models allow the virtual operator only to assess the immediate consequences of her decisions. This gives a further slight bias towards the network owner.

2. **Prediction model for period 1.** For a given y_1 obtain a prediction $z_1(y_1)$ for the decision of the virtual operator during period 1 by solving the following problem.

Find z_1 and $v_1 = (v_{11}, \dots, v_{1N})$ which maximize

$$F_{10}(y_1, z_1, v_1) = \sum_{i=1}^N p_{1i} \left((\tilde{n}_2^{1i}(y_1, z_1(y_1)) - v_{1i})(q - e_2^1 + z_{11}) - r_{13}^{1i} v_{1i} \right) - (y_{12} + \bar{\eta}_2^1) z_{12} \quad (19)$$

subject to constraints

$$v_{1i} \geq \tilde{n}_2^{1i}(y_1, z_1(y_1)) - \frac{1}{d_2^1} z_{12}, \quad i = 1 : N, \quad (20)$$

$$v_{1i} \geq 0, \quad i = 1 : N, \quad (21)$$

$$\Delta_1^1 \leq z_{11} \leq \Delta^1, \quad (22)$$

$$0 \leq z_{12} \leq y_{13}, \quad (23)$$

$$\tilde{n}_2^{1i}(y_1, z_1(y_1)) \geq 0, \quad i = 1 : N, \quad (24)$$

where v_{1i} are the potential customers of the VNO which are lost during period 1 under scenario i due to lack of capacity for service provision. The term \tilde{n}_2^{1i} denotes the customer number of the VNO taken into account her uncertainty about the NO's decisions:

$$\tilde{n}_2^{1i}(y_1, z_1(y_1)) = k_1^{1i} - r_{21}^{1i}(y_{11} + \eta_1^{1i}) - r_{22}^{1i}z_{11}$$

where the parameter k_1^{1i} is also dependent on the initial customer number and the initial service price of the VNO. The structure of the profit function $F_{10}(y_1, z_1, v_1)$ is very similar to the one period competition model from Section 3.1.1. The new elements are the sums containing parameters η_1^{1i} and η_2^{1i} which are used to model imprecise knowledge of the virtual operator about the network owner's decisions. In addition, the last term under the sum represents opportunity costs for not meeting demand.

3. Prediction model for period 2.

For a given scenario i of period 1 and a given decision y_2^i of the network operator during period 2 obtain a prediction $z_2^i(y_2^i)$ for the virtual operator's decision during period 2 by solving the following problem.

Find z_2^i and $v_{2i} = (v_{2i1}, \dots, v_{2iM_i})$ which maximize

$$F_{20}(y_2^i, z_2, v_{2i}) = \sum_{j=1}^{M_i} p_{2ij} \left((\tilde{n}_2^{2ij}(y_2^i, z_1(y_1), z_2^i(y_2^i)) - v_{2ij})(q - e_2^2 + z_{21}^i) - r_{23}^{2ij} v_{2ij} \right) - (y_{22}^i + \bar{\eta}_2^{2i}) z_{22}^i \quad (25)$$

subject to constraints

$$v_{2ij} \geq \tilde{n}_2^{2ij}(y_2^i, z_1(y_1), z_2^i(y_2^i)) - \frac{1}{d_2^2} z_{22}^i, \quad i = 1 : N \quad j = 1 : M_i, \quad (26)$$

$$v_{2ij} \geq 0, \quad i = 1 : N, \quad j = 1 : M_i, \quad (27)$$

$$\Delta_1^2 \leq z_{21}^i \leq \Delta^2, \quad i = 1 : N, \quad (28)$$

$$0 \leq z_{22}^i \leq y_{23}^i, \quad i = 1 : N, \quad (29)$$

$$\tilde{n}_2^{2ij}(y_2^i, z_1(y_1), z_2^i(y_2^i)) \geq 0, \quad i = 1 : N, \quad j = 1 : M_i \quad (30)$$

where v_{2ij} are the potential customers of the VNO lost during period 2 under scenario i of period 1 and scenario j of period 2 due to lack of capacity for

service provision. The term \tilde{n}_2^{2ij} denotes the customer number of the VNO taking into account her uncertainty about the NO's decisions:

$$\tilde{n}_2^{2ij}(y_2^i, z_1(y_1), z_2^i(y_2^i)) = k_1^{2ij} - r_{21}^{2ij}(y_{21}^i + \eta_1^{2ij}) - r_{22}^{2ij} z_{21}^i$$

where the parameter k_1^{2ij} also depends on the customer number and the service price of the VNO in the previous time period. The structure of the profit function and of the constraints (26) – (30) is very similar to the prediction model for period 1.

Observe that both prediction problems are quadratic programming problems which are easily solvable with standard software.

6 Implementation and numerical experiments

The methodology described above underlies a decision support system for the analysis of strategic and tactical decisions in a competitive telecommunication environment which is currently under development. It consists of the following main components.

- *Spread sheet front.* It is used for data entry and the communication with the user.
- *MATLAB engine.* It provides the connection between the blocks, a common implementation platform, a quick prototyping capability for new models and customised top level algorithms for model solution, scenario generation and postprocessing.
- *Model suite.* Contains a set of models based on the approach described in the previous sections.
- *Problem solvers.* This block utilises commercial and custom developed solvers for model solution and analysis, in particular commercial linear and nonlinear programming solvers.

Considering the deterministic one stage case, we conducted studies on concavity and differentiability properties of the function measuring the network owner's performance (here profit). Although this function is quadratic it turns out to have quite a complex shape when the explicit expression for the virtual operator's response to his decisions is substituted. This response is the result of a quadratic programming problem including constraints on her decision variables and a constraint requiring nonnegative customer numbers. Therefore this expression has a rather complex shape. However, we can state exact expressions for the virtual operator's optimal decision depending on the network owner's prices $y = (y_1, y_2)$.

According to which constraints are active at the optimal solution of the virtual operator's problem the domain of the network operator's decisions is divided into regions that are divided further when the constraint on his customer number is taken into account. Not all of these regions exist for the same set of model parameters, exact conditions on the parameter set can be stated for the existence and location of each region in the network operator's domain. Furthermore the network owner's profit function can be specified exactly in each of the regions. This function is concave on most of the regions under quite general assumptions on the parameter set. It suffices to ensure positive r_1, r_2, r_{12} resp. c_1, c_2, c_{12} , i.e. parameters connected with the customer flow dependent on the price changes and the price differences between both providers. More exactly this denotes ordinary behaviour of the customers, where more subscribe to a provider if he charges lower service prices than in the previous period and they move to the provider that charges lower service prices. However, these concavity properties are only valid for the respective region, not over the total domain $[\Delta_1, \Delta] \times [0, U_1]$ of the network owner's profit function. There exist also regions where the objective function is linear, convex or has a saddle point. Furthermore, whereas the network operator's profit function is differentiable on all the single regions it does not show this property on its whole domain.

The two player Stackelberg game considered in our models can be formulated as

a bilevel problem with a general formulation given as follows:

$$\begin{aligned} & \min_y F_1(y, \hat{z}) \\ \text{s.t.} \quad & f_{1i}(y, \hat{z}) \leq 0, \quad i = 1, \dots, m \end{aligned} \quad (31)$$

$$\hat{z} \in \Psi(y) = \arg \min_z \{F_2(y, z) \mid f_{2j}(y, z) \leq 0, j = 1, \dots, n\} \quad (32)$$

In such terms the network operator constitutes the leader whereas the virtual operator represents the follower. The problems in both levels have linear constraints and quadratic goal functions that are concave in the respective decision variable. Note that not only does the feasible area of the follower's problem depend on both providers' decisions but so does the feasible area of the leader's problem, which may have implications on the feasibility of the upper level problem for some decisions of the leader.

We turn our attention again to the deterministic one stage model and try to attack this problem by means of necessary and sufficient optimality conditions as studied for example in [Dem91] or [Out93] and utilizing the special properties of our model. Some of the assumptions necessary for these optimality conditions turn out to be satisfied for each feasible decision pair of the leader and the follower:

Theorem 1. *The virtual operator's problem satisfies the Slater's condition*

$$Z_I = \{z \mid f_{2i}(z, y^0) < 0\} \neq \emptyset \quad (33)$$

for any feasible decision y^0 of the network operator when the model parameters satisfy the conditions

$$\Delta_1 < \frac{k_2}{c_2} \quad (34)$$

$$\Delta > \frac{k_2}{c_2} - \frac{U_2}{dc_2} \quad (35)$$

$$\Delta_1 < \Delta, \quad 0 < U_2. \quad (36)$$

Note that this implies that customers are sensitive to a price change of the operator compared to his initial price ($c_2 \neq 0$).

Proof. When the constraints (36) are satisfied then the domain $[\Delta_1, \Delta] \times [0, U_2]$ of the VNO's problem has interior points. We show now that the complete feasible

area of this problem as formed by all constraints has a nonempty interior for all feasible decisions y^0 of the NO, i.e. that then points $z \in (\Delta_1, \Delta) \times (0, U_2)$ exist which satisfy the constraints which depend on the network operator's decisions

$$0 \leq n_2(y^0, z) = k_2 + r_{21}y_1^0 - r_{22}z_1 \quad (37)$$

$$z_2 \geq dn_2(y^0, z) = dk_2 + dr_{21}y_1^0 - dr_{22}z_1 \quad (38)$$

with strict inequality.

Constraint (37) can be rewritten as

$$z_1 < \frac{k_2}{r_{22}} + \frac{r_{21}}{r_{22}}y_1^0$$

In order to be valid for all feasible NO decisions a feasible z_1 must also exist for the smallest possible right hand side value of this inequality which is taken on for $y_1^0 = \Delta_1$. Then the variable z_1 must satisfy

$$\Delta_1 < z_1 < \frac{k_2}{r_{22}} + \frac{r_{21}}{r_{22}}\Delta_1$$

This holds when the constraint (34) is satisfied. (Note that $r_{22} - r_{21} = c_2$.)

When $n_2(y^0, z) > 0$ then $z_2 \geq dn_2 > 0$ holds automatically, i.e. only $z_2 < U_2$ for any y^0 must still be verified. For $z_2 < U_2$ constraint (38) gives

$$z_1 > \frac{k_2}{r_{22}} + \frac{r_{21}}{r_{22}}y_1^0 - \frac{U_2}{dr_{22}}$$

For any feasible y_1^0 this constraint must be satisfied by a $z_1 \in (\Delta_1, \Delta)$. The right hand side of this inequality has its greatest possible value for $y_1^0 = \Delta$. In order to be in the interior Z_I of the feasible area for this y_1^0 , the variable z_1 must then satisfy

$$\Delta > z_1 > \frac{k_2}{r_{22}} + \frac{r_{21}}{r_{22}}\Delta - \frac{U_2}{dr_{22}}$$

which is true when constraint (35) holds.

Summarizing and taking into account constraint (36) this means that under conditions (34) – (36) for any feasible $y^0 = (y_1^0, y_2^0)$ a point $z = (z_1, z_2)$ exists which satisfies all conditions of the virtual operator's problem with strict inequality, i.e. which is in the interior of the feasible area Z_I of this problem. \square

Theorem 2. *Under a normal customer behaviour with $c_i, c_{ij} > 0$, $i, j = 1, 2$ the virtual operator's problem satisfies the following strong second-order sufficient*

optimality condition (A3) [Dem91] for any feasible decision y^0 of the network operator:

(A3) For each triple (z, d, u) with

$$z \in \Psi(y^0), u \in U(z, y^0)$$

$$\nabla_z f_{2i}(z, y^0)d = 0 \quad \forall i \in J_+(u) := \{j | u_j > 0\}$$

the inequality

$$d^T \nabla_{zz} L(z, y^0, u)d > 0$$

is fulfilled where $U(z, y^0)$ denotes the set of KKT multiplier vectors.

Proof. The Lagrangian of the lower level problem is

$$\begin{aligned} L(z, y^0, u) &= r_{22}z_1^2 - (k_2 - (q - e_2)r_{22} - r_{21}y_1^0)z_1 + y_2^0z_2 \\ &\quad + u_1(r_{22}z_1 - r_{21}y_1^0 - k_2) + u_2(-dr_{22}z_1 - z_2 + d(k_2 + r_{21}y_1^0)) \\ &\quad + u_3(\Delta_1 - z_1) + u_4(z_1 - \Delta) - u_5z_2 + u_6(z_2 - U_2) \end{aligned}$$

Since the Hessian $\nabla_{zz} L(z, y^0, u)$ is positive definite for a normal customer behaviour the condition

$$d^T \nabla_{zz} L(z, y^0, u)d > 0$$

is satisfied for all $d \neq 0$ and thus for all decisions y^0 of the network operator. \square

The conditions (33) and (A3) are fulfilled for all $y \in [\Delta_1, \Delta] \times [0, U_1]$. Therefore $z(\cdot)$ is continuous at each y^0 and directionally differentiable. An alternative consideration of the function $z(\cdot)$ yields that this function is a PC^1 function and has these properties nevertheless. Other necessary assumptions like the Constant Rank Constraint Qualification or the Linear Independence Constraint Qualification raise more problems due to the existence of both nonunique optima and optima not fulfilling a strict complementarity condition in our model. A further complication might also be the dependency of the leader's feasible area on the follower's response. This problem may be overcome by for example adding the constraints of the leader's problem which depend on the follower's decisions also to the follower's problem. By this means all decisions of the follower are excluded

which lead to infeasibility of the leader's problem. Another possibility is to penalize the follower for a choice of such decisions.

However, a deeper study of the theoretic properties of both the one stage and the multistage models and of the effects of the stochasticity as well as suitable solution procedures tailored to the special properties of our models will be the subject of a subsequent paper.

A graphical presentation of the enterprise performance and other model characteristics allows study of the impact on the performance using more information than just the optimal decisions derived from the solution of the model (9) - (18). Furthermore, dependencies on parameters like market behaviour or degrees of uncertainty can be analysed, which is particularly important due to the imprecise nature of the input data. We give here an example of the network operator's enterprise performance in the case of an assumed moderate customer sensitivity to the service prices (Figure 5) and in the case of a very sensitive customer population (Figure 6). Such a presentation helps to give comprehensive interpretations by studying interesting areas in each of the functions. A further analysis of these results in connection with theoretical insights into the problem as sketched above can provide the network owner with advice for profit maximising strategies in specific situations of customer behaviour, information flow or other values of the considered model parameters.

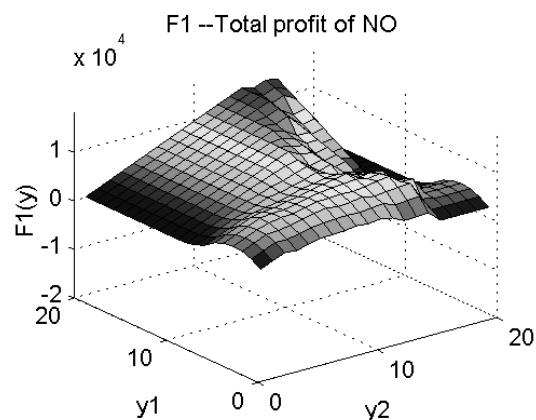


Figure 5: One stage stochastic model: moderate customer sensitivity

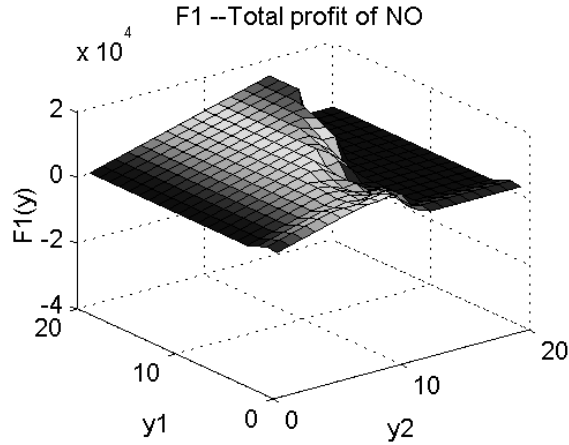


Figure 6: One stage stochastic model: sensitive customer population

7 Maximising market share

In the models presented in the previous sections we used the providers' profit as a measure of the enterprise performance. However, other measures are conceivable, for example market share. Starting from the general definition of market share as the relationship of an enterprise's sales in a defined market to the total sales in that market, we define it in our approach based on the total service sold by the providers, which is the same as the demand served. Assuming that all customers show the same demand we measure then the network owner's enterprise performance using the function

$$F_{M1} = \frac{\text{Number of customers served by the network provider}}{\text{Total number of customers served by both providers}}$$

The market share function of the virtual operator is constructed similarly. Naturally the customer behaviour is uninfluenced by a different choice of the performance measure. However, in the presence of uncertainty the providers must also estimate the rival's customer number in order to calculate their own market share. Therefore the network operator's competition model for the prediction of the virtual operator's response will include both a term for her estimated customer number and a term for the network operator's customer number as perceived by the virtual operator. Depending on the degree of the uncertainty the total amount of served demand may be estimated differently by both providers.

Note that the models show no dependence on the capacity price anymore; their focus lies entirely on the market share and all economic aspects are now ignored. Furthermore the goal functions are no longer quadratic and a solution will become more complicated.

8 Conclusions and future work

We developed a framework for modeling complex competition relationships and for evaluating strategic decisions in the telecommunication environment. The methodological foundation draws upon selected ideas from game theory and stochastic programming. A decision support system based on this framework is under development. In this paper we utilized it for the analysis of interactions between a network operator and a virtual operator for the deterministic problem. Although at present the models have a quite simple structure, they allow us to obtain fairly interesting and nontrivial insights into this situation.

Further work planned is organized along the following interrelated directions.

- A further development of the model suite including the development of more sophisticated customer models with feedback and more differentiated pricing structures.
- The study of mathematical properties of the modeling framework. This will provide insights into the structure of the optimal strategies and facilitate the development of efficient solution techniques.
- The implementation of the architecture of the decision support system. The objective is to create a user friendly and robust tool suite for evaluation of strategic decisions in a competitive telecommunication environment.

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Paper 2

A solution method for
bilevel stochastic programming problems

A.S. Werner and A.A. Gaivoronski

A solution method for bilevel stochastic programming problems

A.S. Werner

A.A. Gaivoronski

Adrian.Werner@iot.ntnu.no

Alexei.Gaivoronski@iot.ntnu.no

Abstract

We analyse stochastic programming problems characterised by a bilevel structure. An additional feature is the presence of connecting constraints in the leader's (upper level) subproblem. Necessary optimality conditions of Fritz John type are given and a solution algorithm utilising a stochastic quasi-gradient method is presented.

Key words: Bilevel stochastic programming, decision making under uncertainty, bilevel programming, principal agent problem.

1 Introduction

In this paper we consider several variants of the bilevel stochastic programming problem (BLSP) with different degrees of uncertainty and complexity. We study necessary optimality conditions and develop an algorithm for the solution of these problems utilising a Lagrange multiplier method.

Bilevel stochastic programming problems can be interpreted as an extension of stochastic programming problems when part of the uncertainty a decision maker faces can be attributed to the decisions of another actor. Traditionally, this type of uncertainty has been interpreted as part of the environment such that the influence of the decision maker on other actors was ignored. However, it can be treated separately by utilising a bilevel problem structure. Interpreting the considered decision maker as leader, this means that he makes his decision taking

into account the response of the other actor, the follower.

Deterministic bilevel programming problems were studied intensely during the past decades and a variety of solution methods was developed [Dem02, VC94]. Such problems show inconvenient properties which complicate the analysis. Taking into account the follower's response, the leader's objective function is generally not convex and neither differentiable. If the leader's constraints include also the follower's response (so called connecting upper level constraints) then the region of feasible leader decisions may even be not connected and, consequently, the leader's objective function discontinuous. Therefore it is often assumed that the feasibility of the leader's decisions is not influenced by the follower's behaviour. In this paper we do not make such an assumption and will take into account the connecting upper level constraints. Our viewpoint is motivated by a number of applications, for example in telecommunications [AGWng], in energy and power management [GR02] or more generally in agency theory [GH83, Mir99, WGng]. We deal with the mentioned implications by first dividing the inducible region into convex segments and then applying a gradient algorithm restricted on such a segment.

Little work is done at the intersection of the fields of decision making under uncertainty and of bilevel programming. Stochastic games [BV00, VM01] analyse the interplay of several actors under uncertainty. Although both the uncertainty and the interdependencies between the decision makers can be regarded, the application of this methodology to more general decision problems under uncertainty with a bilevel structure is complicated. For example, it is difficult to take into account continuous decision variables or nontrivial constraints which is of interest regarding the stochastic programming focus. A viewpoint complementary to our approach are stochastic bilevel programming problems (SBLP) [PW97, Wyn01] and their generalisation, stochastic mathematical programs with equilibrium constraints (SMPEC) [EP04, PW99, Sha04]. They can be interpreted as an extension of the respective deterministic programming problems by allowing for uncertain model parameters. Suggestions for solution approaches comprise a penalty method [EP04] or the utilisation of a finite number of scenarios and deterministic equivalent formulations [PW99, Sha04]. This results in large deterministic bilevel programming problems which are computationally expensive for

problems of a realistic size. In contrast, the BLSP approach considers the problem as an extension of stochastic programming problems [EW88a, RS03] by adding a bilevel structure. This way it is possible to apply the stochastic programming methodology to the complete bilevel relationship. The concept of stochastic programming problems with recourse [EW88b, Wet89] allows to take into account dynamic aspects. In particular, we employ sampling techniques such as stochastic quasi-gradient methods [Erm88, Gai88, Gai04]. This gives the possibility to use various representations of the uncertain variables, for example continuous distributions. Our viewpoint enables therefore a more comprehensive treatment of the uncertainty and more complex problem structures. A related approach is discussed in [BGL04] for the solution of stochastic mathematical programming problems with complementarity constraints. Such problems emerge for example from reformulations of SBLP or SMPEC problems.

The following section defines notations which will be utilised throughout the paper. In Section 3 we analyse the deterministic bilevel programming problem. Necessary optimality conditions are studied and a solution algorithm utilising a gradient method is developed. These considerations are extended in Section 4. Several formulations of bilevel two-stage stochastic programming problems with different complexity are analysed. Also for these problems necessary optimality conditions are given and a solution algorithm is presented utilising a stochastic quasi-gradient method. A numerical illustration of the approach is given in Section 5. Section 6 concludes the paper.

2 Notations and assumptions

Consider the following decision problem

$$\min_{y \in Y} F(y, z) \tag{1a}$$

$$G(y, z) \leq 0 \tag{1b}$$

where the considered decision maker directly controls the variables $y \in Y \subseteq \mathbb{R}^n$. The variables $z \in Z \subseteq \mathbb{R}^m$ denote the response of another decision maker to

these decisions y and are determined by the parametric optimisation problem

$$\min_{z \in Z} f(y, z) \tag{2a}$$

$$g(y, z) \leq 0 \tag{2b}$$

with the parameter y . This represents a bilevel programming problem with the upper level problem (1a) – (1b) and the lower level problem (2a) – (2b). We assume $F, f : R^n \times R^m \rightarrow R^1$, $G : R^n \times R^m \rightarrow R^p$ and $g : R^n \times R^m \rightarrow R^q$. Furthermore we assume that the sets Y and Z are convex and compact.

For a given upper level parameter y^0 denote the Lagrangian function associated with the lower level problem (2a) – (2b) by

$$L(y^0, z^0, \lambda^0, \mu^0) = f(y^0, z^0) + (\lambda^0)^T g(y^0, z^0)$$

and define the following index sets:

$$I(y^0, z^0) = \{i \in \{1, \dots, q\} | g_i(y^0, z^0) = 0\}$$

$$J(\lambda^0) = \{i \in \{1, \dots, q\} | \lambda_i^0 > 0\}$$

Definition 1. *The inducible region denotes the set over which the leader may optimise:*

$$IR = \{y \in Y | \exists z^* \in M(y) : G(y, z^*) \leq 0\} \tag{3}$$

with the lower level solution set $M(y)$ and the feasible lower level area $N(y)$ defined for a given upper level decision $y \in Y$ by

$$M(y) = \arg \min \{f(y, z) | z \in N(y)\}$$

$$N(y) = \{z \in Z | g(y, z) \leq 0\}$$

Definition 2. *The Mangasarian-Fromowitz Constraint Qualification (MFCQ) holds for problem (2a) – (2b) at a point (y^0, z^0) if there exists a direction d with*

$$\nabla_z g_i(y^0, z^0) d < 0, \quad i \in I(y^0, z^0)$$

Definition 3. *The Constant Rank Constraint Qualification (CRCQ) holds for problem (2a) – (2b) at a point z^0 for an upper level parameter y^0 if there exists an open neighbourhood $U_\varepsilon(z^0)$ such that for all subsets $I \subseteq I(y^0, z^0)$ the gradients $\{\nabla_z g_i(y^0, z^0), i \in I\}$ have constant rank for all $z \in U_\varepsilon(z^0)$ and the upper level parameter y^0 .*

Definition 4. *The Linear Independence assumption (LI) holds for problem (2a) – (2b) at a point z^0 with a given upper level parameter y^0 when the gradients of the constraints $\{\nabla_z g_i(y^0, z^0), i \in I(y^0, z^0)\}$ are linearly independent.*

Definition 5. *The Strong Second Order Sufficient Optimality Condition (SSOC) holds for problem (2a) – (2b) at a point z^0 if for all Lagrange multipliers $\lambda^0 \in \mathbf{R}_+^q$ associated with z^0 and for all $d \neq 0$ with*

$$\nabla_z g_i(y^0, z^0)d = 0, \quad i \in J(\lambda^0)$$

the condition

$$d^T \nabla_z^2 L(y^0, z^0, \lambda^0)d > 0$$

holds for a given upper level parameter y^0 .

Assumption (A1). The objective functions $F(y, z)$ and $f(y, z)$ are convex in y and z and at least C^2 (twice continuously differentiable).

The upper level constraints $G_i(y, z), i = 1, \dots, p$, are convex in y and z and at least C^1 . The lower level constraints $g_j(y, z), j = 1, \dots, q$, are linear in y and z .

Assumption (A2). The conditions (MFCQ) and (SSOC) are satisfied at all stationary solutions $z^0 = z(y^0)$ of the lower level problem (2a) – (2b) for an upper level parameter $y^0 \in IR$.

Remark 1. In the following we suppose that Assumption (A2) holds for all $y^0 \in IR$. Then the response $z^0 = z(y^0)$ and the Lagrange multipliers λ^0 are uniquely determined for all upper level decisions.

3 The deterministic problem

We start with an analysis of the deterministic bilevel programming problem. This analysis is especially important due to the connecting upper level constraints which affect the development of a solution algorithm. We consider the following

problem.

$$\min_{y \in Y} F(y, z^*) \quad (4a)$$

$$G(y, z^*) \leq 0 \quad (4b)$$

$$z^* = \arg \min_{z \in Z} f(y, z) \quad (4c)$$

$$g(y, z) \leq 0 \quad (4d)$$

At first necessary optimality conditions for this problem are studied. Then implications of the connecting upper level constraints are discussed in more detail before a solution algorithm is developed.

Theorem 1. *Suppose that*

1. *Assumptions (A1) and (A2) hold,*
2. *for an upper level decision y^0 there exists a solution $z^0 = z(y^0)$ of problem (4c) – (4d) with the Lagrange multipliers $\lambda^0 \in \mathbf{R}_+^p$ such that the point (y^0, z^0, λ^0) satisfies the upper level constraints (4b),*
3. *the sets $I(y^0, z^0)$ of active lower level constraints and $J(\lambda^0)$ of nonzero Lagrange multipliers do not change in the vicinity of the point y^0 .*

Then the following composite problem represents an equivalent formulation of problem (4) which holds in the vicinity of the point (y^0, z^0, λ^0) :

$$\min_{y, z, \lambda} F(y, z) \quad (5a)$$

$$\nabla_z f(y, z) + \lambda^T \nabla_z g(y, z) = 0$$

$$G(y, z) \leq 0$$

$$g_i(y, z) = 0 \quad i \in I(y^0, z^0) \quad (5b)$$

$$\lambda_i = 0 \quad i \notin J(\lambda^0) \quad (5c)$$

$$g_i(y, z) \leq 0 \quad i \notin I(y^0, z^0) \quad (5d)$$

$$\lambda_i \geq 0 \quad i \in J(\lambda^0) \quad (5e)$$

A local optimal solution of problem (4) is also a local optimal solution of (5).

Proof. Due to Assumptions (A1) and (A2) the optimal solution of the lower level problem (4c) – (4d) is unique for given y^0 . Therefore z^0 and λ^0 can be expressed by the Karush Kuhn Tucker optimality conditions of problem (4c) – (4d) and substituted in the upper level problem (4a) – (4b):

$$\min_{y,z,\lambda} F(y^0, z^0) \quad (6a)$$

$$G(y^0, z^0) \leq 0$$

$$\nabla_z f(y^0, z^0) + (\lambda^0)^T \nabla_z g(y^0, z^0) = 0 \quad (6b)$$

$$(\lambda^0)^T g(y^0, z^0) = 0 \quad (6c)$$

$$g(y^0, z^0) \leq 0 \quad (6d)$$

$$\lambda^0 \geq 0 \quad (6e)$$

This system is, however, nonlinear due to the complementarity condition (6c). Therefore this condition is, together with (6d) and (6e), replaced by the constraint

$$\min\{-g(y^0, z^0), \lambda^0\} = 0$$

where the minimum is taken componentwise. Under assumption 3. this constraint is equivalent to constraints (5b) – (5e).

Problem (5) describes a section of the feasible set of problem (4). Each point which is feasible for problem (5) is also feasible for problem (4) and a local optimal solution of problem (5) is locally optimal for (4). \square

The following propositions give necessary optimality conditions of Fritz John type for problem (4). If not stated otherwise, here the gradient is taken with respect to (y, z) .

Proposition 1. *Assume that*

1. *Assumptions (A1) and (A2) are valid,*
2. *(y^0, z^0, λ^0) is a local optimal solution of problem (4).*

Then there exists a nonvanishing vector of multipliers $(\kappa_0, \kappa, \gamma, \zeta, \xi)$ such that the following system is satisfied:

$$\kappa_0 \nabla F(y^0, z^0) + \kappa^T \nabla G(y^0, z^0) + \nabla(\nabla_z f(y^0, z^0) \gamma) + \zeta^T \nabla g(y^0, z^0) = 0 \quad (7a)$$

$$\begin{aligned}
\nabla_z g(y^0, z^0)^T \gamma - \xi^T &= 0 & (7b) \\
g_i(y^0, z^0) \zeta_i &= 0 \quad \forall i \\
\kappa^T G(y^0, z^0) &= 0 \\
\lambda_i^0 \xi_i &= 0 \quad \forall i \\
\zeta_i \xi_i &\geq 0, \quad i \in I(y^0, z^0) \cap J(\lambda^0) \\
\kappa_0, \kappa &\geq 0
\end{aligned}$$

Proof. Dempe [Dem02] considered the bilevel programming problem

$$\min_{y \in Y} F(y, z^*) \quad (8a)$$

$$G(y) \leq 0 \quad (8b)$$

$$z^* = \arg \min_{z \in Z} f(y, z) \quad (8c)$$

$$g(y, z) \leq 0 \quad (8d)$$

$$h(y, z) \leq 0 \quad (8e)$$

without connecting upper level constraints and with equality constraints in the lower level problem. He established the necessary optimality conditions

$$\begin{aligned}
\kappa_0 \nabla F(y^0, z^0) + \kappa^T (\nabla_y G(y^0), 0) + \nabla (\nabla_z L(y^0, z^0, \lambda^0, \mu^0) \gamma) \\
+ \zeta^T \nabla g(y^0, z^0) + \tau^T \nabla h(y^0, z^0) = 0 \quad (9a)
\end{aligned}$$

$$(\nabla_z g(y^0, z^0), \nabla_z h(y^0, z^0))^T \gamma - (\xi, 0)^T = 0 \quad (9b)$$

$$g_i(y^0, z^0) \zeta_i = 0 \quad \forall i$$

$$\kappa^T G(y^0) = 0$$

$$\lambda_i^0 \xi_i = 0 \quad \forall i$$

$$\zeta_i \xi_i \geq 0, \quad i \in I(y^0, z^0) \cap J(\lambda^0)$$

$$\kappa_0, \kappa \geq 0$$

with $L(y^0, z^0, \lambda^0, \mu^0)$ the Lagrangian of the lower level problem (8c) – (8e). For the lower level problem (4c) – (4d) with linear constraints we obtain

$$\nabla (\nabla_z L(y^0, z^0, \lambda^0, \mu^0) \gamma) = \nabla (\nabla_z f(y^0, z^0) \gamma)$$

and constraint (9b) takes on the shape (7b). Furthermore the presence of the connecting upper level constraints $G(y^0, z^0)$ changes the term $\kappa^T (\nabla_y G(y^0), 0)$ in

condition (9a) to

$$\kappa^T(\nabla_y G(y^0, z^0), \nabla_z G(y^0, z^0)) = \kappa^T \nabla G(y^0, z^0)$$

Summarising, system (7) represents the adaptation of the necessary optimality conditions (9) to problem (4). \square

Before a solution algorithm can be presented the structure of problem (4) must be analysed closer. This problem shows two important features prohibiting a direct application of gradient solution methods. The first feature is that the leader's objective function depends on the response of the follower. Therefore the determination of a descent direction at an upper level iterate has to take into account the behaviour of the follower's response. This problem, however, can be overcome by a sensitivity analysis of the lower level problem. Additionally, even if $F(y, z)$ is convex and differentiable with respect to both y and z , the function $F(y, z(y))$ taking into account the optimal lower level response may be nondifferentiable and nonconvex in y . The second important feature is the presence of connecting upper level constraints. Their feasibility can be investigated only after the follower's response was determined. If Assumption (A2) and (CRCQ) hold, then the lower level solution function $z(y)$ is continuous. However, there may exist responses $z(y)$ which do not satisfy the upper level constraints $G(y, z) \leq 0$. A consequence is that the inducible region may not be connected and not convex. Then the upper level objective function $F(y, z(y))$ taking into account the optimal lower level response $z(y)$ may not be continuous in y and the convergence of the solution algorithm can not be guaranteed. However, it is possible to partition the inducible region into a finite number of segments. Such a segment comprises all upper level decisions y with the same characteristic of the response $z(y)$, i.e. with the same indices of active lower level constraints and of nonzero Lagrange multipliers.

Definition 6. *A segment Y^s is defined by*

$$\begin{aligned} Y^s &= \{y \in Y \mid I_L^s(y) = I_1^s, I_C^s(y) = I_2^s\} \\ I_L^s(y) &= \{i \in \{1, \dots, q\} \mid \lambda_i(y) > 0\} \\ I_C^s(y) &= \{i \in \{1, \dots, q\} \mid g_i(y, z(y)) = 0\} \\ I_1^s, I_2^s &\in 2^{\{1, \dots, q\}} \end{aligned} \tag{10}$$

where $2^{\{1, \dots, q\}}$ denotes the family of all subsets of the index set $\{1, \dots, q\}$.

The union of all such segments is the upper level domain Y . However, the inducible region

$$IR = \bigcup_s \{y \in Y^s | G(y, z(y)) \leq 0\} = \{y \in Y | G(y, z(y)) \leq 0\}$$

may be nonconnected. This is demonstrated in the following example.

Example 1. Consider the problem

$$\min_y y + z^* \tag{11a}$$

$$z^* \geq 2 \tag{11b}$$

$$y \geq 0 \tag{11c}$$

$$z^* \in \arg \min_z \{z \in R | y + z \geq 3, y - z \leq 3, z \geq 1\} \tag{11d}$$

The optimal solution of the lower level problem (11d) is

$$z(y) = \begin{cases} y - 3, & 4 \leq y \\ 1, & 2 \leq y \leq 4 \\ -y + 3, & y \leq 2 \end{cases}$$

However, only for $y \in [0, 1] \cup [5, \infty)$ the upper level constraint (11b) is satisfied.

In order to apply a gradient algorithm we need some properties of the segments.

Theorem 2. Assume that the following conditions are satisfied:

1. Assumption (A1) holds.
2. The conditions (LI) and (SSOC) hold for the response $z = z(y)$ to $y \in Y^s$.
3. For $y \in \text{ri } Y^s$ and the response z the Karush Kuhn Tucker conditions

$$\nabla_z f(y, z) + \lambda^T \nabla_z g(y, z) = 0 \tag{12a}$$

$$\lambda^T g(y, z) = 0 \tag{12b}$$

$$g(y, z) \leq 0 \tag{12c}$$

$$\lambda \geq 0 \tag{12d}$$

are satisfied with strict complementarity.

Then the upper level objective function $F(y, z(y))$ is continuously differentiable on the relative interior $\text{ri } Y^s$ of the segment.

Proof. Consider an upper level decision $y^0 \in \text{ri } Y^s$. If the response $z^0 = z(y^0)$ satisfies conditions (LI), (SSOC) and strict complementarity then the Jacobian of the Karush Kuhn Tucker conditions is nonsingular. This means that the function $z(y)$ is uniquely determined and continuously differentiable in the vicinity of the parameter y^0 [Jit84]. Due to Assumption (A1) the upper level objective function $F(y, z)$ is differentiable with respect to z . Therefore, $F(y, z(y))$ is differentiable with respect to y on $\text{ri } Y^s$. \square

Theorem 3. *Assume that*

1. *Assumption (A1) holds and*
2. *the gradient $\nabla_z f(y, z)$ is linear in y and z .*

Then the segment Y^s is convex and compact.

Proof. According to Definition 6 the constraints of the set Y^s are equivalent to the system (12) of Karush Kuhn Tucker conditions characterising z as the optimal lower level response to the upper level parameters $y \in Y^s$. This system describes a convex set if the involved equality constraints are linear and the inequality constraints convex. These conditions are given under Assumption (A1) and condition 2. The compactness of the segment Y^s follows directly from the compactness of Y and from Definition 6. \square

Remark 2. At a point y^b on the boundary of a segment Y^s to an adjacent segment the Karush Kuhn Tucker conditions (12) may not be satisfied with strict complementarity. Therefore the function $F(y, z(y))$ may be nondifferentiable at the boundary between adjacent segments. However, if Assumption (A2) and (CRCQ) hold for all $y \in IR$, the response function $z(y)$ is PC^1 (piecewise continuously differentiable) [Dem02]. This means that $z(y^b)$ and thus also $F(y^b, z(y^b))$ are directionally differentiable into all directions, but the directional derivatives may not coincide.

Utilising the partition into segments, problem (4) can be decomposed into a family of one-level problems:

$$\begin{aligned} \min_{y,z,\lambda} F(y, z) \\ E(y, z, \lambda) &\leq 0 \\ e(y, z, \lambda) &= 0 \end{aligned} \tag{13}$$

with

$$E(y, z, \lambda) = \begin{pmatrix} g_i(y, z), i \notin I_C^s \\ -\lambda_i, i \in I_L^s \\ G(y, z) \end{pmatrix} \tag{14}$$

$$e(y, z, \lambda) = \begin{pmatrix} \nabla_z f(y, z) + \lambda^T \nabla_z g(y, z) \\ g_i(y, z), i \in I_C^s \\ \lambda_i, i \notin I_L^s \end{pmatrix} \tag{15}$$

$$(y, z, \lambda) \in X = Y \times Z \times R_+^q \tag{16}$$

During the solution algorithm the change of the response $z(y)$ under a perturbation of the upper level decision y must be evaluated. Under certain assumptions this can be done by a sensitivity analysis of the optimal lower level response.

Theorem 4. *Assume that the conditions of Theorem 2 hold for the optimal response $z^0 = z(y^0)$ to an upper level decision y^0 . Then*

1. *The lower level response $z^\varepsilon = z(y^0 + \varepsilon)$ to the perturbed upper level decision $y^0 + \varepsilon$ is a locally unique optimal solution of the lower level problem.*
2. *The associated Lagrange multipliers λ^ε are uniquely determined.*
3. *Also z^ε satisfies the assumptions of Theorem 2 and the set of active constraints is not changed for ε near zero.*

An approximation of z^ε and λ^ε is given by the following expression

$$\begin{bmatrix} z^\varepsilon \\ \lambda^\varepsilon \end{bmatrix} = \begin{bmatrix} z^0 \\ \lambda^0 \end{bmatrix} + (M^*)^{-1} N^* \varepsilon + o(\|\varepsilon\|) \tag{17}$$

with

$$\begin{aligned}
 N(\varepsilon) &= \left[-\nabla_{\varepsilon, z}^2 L^T, \lambda_1^0 \nabla_{\varepsilon} g_1, \dots, \lambda_q^0 \nabla_{\varepsilon} g_q \right] \\
 M(\varepsilon) &= \begin{bmatrix} \nabla^2 L & \nabla g_1^T & \dots & \nabla g_q^T \\ -\lambda_1^0 \nabla g_1 & -g_1 & & 0 \\ \vdots & & \ddots & \\ -\lambda_q^0 \nabla g_q & 0 & \dots & -g_q \end{bmatrix} \\
 L &= f(y^0 + \varepsilon, z^0) + (\lambda^0)^T g(y^0 + \varepsilon, z^0) \\
 g_i &= g_i(y^0 + \varepsilon, z^0), \quad i = 1, \dots, q \\
 M^* &= M(0) \quad N^* = N(0)
 \end{aligned}$$

If not stated otherwise, here the gradient ∇ and the Hessian operator ∇^2 are taken with regard to z .

Proof. Assumption (A2) is supposed to hold at the optimal lower level response z^0 . Therefore the associated Lagrange multipliers are unique and the application of the Basic Sensitivity Theorem [Fia76] to the lower level problem (4c) – (4d) yields the theorem. Thereby expression (17) is obtained by applying Corollary 3.2.4 [Fia83] to problem (4c) – (4d). The matrix $M(\varepsilon)$ represents the Jacobian of the Karush Kuhn Tucker conditions of this problem under the perturbed parameter $y + \varepsilon$ with respect to the point $(z^\varepsilon, \lambda^\varepsilon)^T$. The matrix $N(\varepsilon)$ is the negative Jacobian of these conditions with respect to ε . Both are evaluated at $[z^\varepsilon, \lambda^\varepsilon, \varepsilon]$. \square

Now the solution algorithm can be presented:

Algorithm 1: Find local optimum among stationary points on segments.

Step 0. (Initialisation) Find an initial upper level decision y^0 , set $s = 0$.

Step 1. (Determination of segment) Solve the lower level problem (4c) – (4d) with the parameter y^s . This gives the optimal lower level response $z^s = z(y^s)$, the associated Lagrange multipliers λ^s , the index set I_C^s of active lower level constraints and the index set I_L^s of nonzero Lagrange multipliers.

Step 2. (Iteration) Utilising the initial point $x^s = (y^s, z^s, \lambda^s)$, solve problem (13) – (15) with a descent algorithm (Algorithm 2) and obtain a stationary solution $\bar{x}^s = (\bar{y}^s, \bar{z}^s, \bar{\lambda}^s)$.

Step 3. (Optimality test) If the point $\bar{x}^s = (\bar{y}^s, \bar{z}^s, \bar{\lambda}^s)$ satisfies the necessary optimality conditions of Proposition 1 go to Step 5.

Step 4. (Perturbation into feasible descent direction)

If a descent direction d exists which is feasible on the current segment Y^s : perturb \bar{y}^s into that direction

$$y^{s+1} = \bar{y}^s + \beta d$$

with small $\beta > 0$ and estimate the response $(z^{s+1}, \lambda^{s+1}) = (z(y^{s+1}), \lambda(y^{s+1}))$ according to formula (17). Set $I_C^{s+1} = I_C^s, I_L^{s+1} = I_L^s$ and $s = s + 1$.

Go to Step 2.

Otherwise: choose a descent direction d which is feasible on an adjacent segment. Perturb \bar{y}^s into that direction

$$y^{s+1} = \bar{y}^s + \beta d$$

with small $\beta > 0$. Set $s = s + 1$ and go to Step 1.

Step 5. (Termination) The point \bar{y}^s with the optimal lower level response \bar{z}^s is a local optimal solution of problem (4).

Remark 3. 1. Determination of an initial point in Step 0. An initial point y^0 is assumed to be feasible together with the response z^0 , i.e. it may be any $x^0 = (y^0, z^0, \lambda^0) \in Y_1 \times Z \times \mathbf{R}_+^q$ satisfying

$$\nabla_z f(y^0, z^0) + (\lambda^0)^T \nabla_z g(y^0, z^0) = 0 \quad (18a)$$

$$(\lambda^0)^T g(y^0, z^0) = 0 \quad (18b)$$

$$g(y^0, z^0) \leq 0 \quad (18c)$$

$$G(y^0, z^0) \leq 0 \quad (18d)$$

Here, conditions (18a) – (18c) characterise z^0 as optimal lower level response and (18d) denotes the upper level feasibility.

2. In this case the response z^0 and the Lagrange multipliers λ^0 are already determined such that Step 1 in Algorithm 1 is basically completed. It remains only to determine the sets I_C^s and I_L^s .
3. If the set of active lower level constraints changes in the close vicinity of an initial point y^0 this means that this point is located on the boundary of several adjacent segments. In such a case an initial segment may be chosen arbitrarily among these segments.
4. Determination of feasible descent directions in Step 4. The conditions of Theorem 4 are only satisfied if the perturbation $y^{s+1} = \bar{y}^s + \beta d$ is also on the current segment Y^s , i.e. if y^{s+1} and the response $(z^{s+1}, \lambda^{s+1}) = (z(y^{s+1}), \lambda(y^{s+1}))$ satisfy the constraints (14) – (16). In this case the response (z^{s+1}, λ^{s+1}) can be estimated utilising expression (17) and the descent condition on the direction d can be formulated as

$$\begin{aligned}
& \nabla_{(y,z)} F(\bar{y}^s, \bar{z}^s)^T (x^{s+1} - \bar{x}^s) \\
&= \nabla_{(y,z)} F(\bar{y}^s, \bar{z}^s)^T ((y^\varepsilon, z^{s+1}, \lambda^{s+1}) - (\bar{y}^s, \bar{z}^s, \bar{\lambda}^s)) \\
&= \nabla_{(y,z)} F(\bar{y}^s, \bar{z}^s)^T \beta (d, (M^{-1}Nd)^T) < 0
\end{aligned}$$

Due to the linearity of the constraints (14), (15) a feasible descent direction on the current segment Y^s can therefore be determined as solution $d \neq 0$ of the system

$$\nabla_{(y,z)} F(\bar{y}^s, \bar{z}^s)^T \beta (d, (M^{-1}Nd)^T) < 0 \quad (19a)$$

$$E(\bar{y}^s, \bar{x}^s, \bar{\lambda}^s) + \beta E(d, (M^{-1}Nd)^T) \leq 0 \quad (19b)$$

$$e(d, (M^{-1}Nd)^T) = 0 \quad (19c)$$

$$\bar{y}^s + \beta d \in Y_1 \quad (19d)$$

$$(\bar{z}^s, \bar{\lambda}^s)^T + \beta M^{-1}Nd \in Z \times \mathbf{R}_+^q \quad (19e)$$

Constraint (19a) is the descent condition, constraints (19b) and (19c) ensure the satisfaction of the constraints (14), (15) and conditions (19d) and (19e) give the feasibility of the perturbation and the response.

If no feasible descent directions on the same segment exist, the iterate \bar{y}^s

is on the boundary of the segment Y^s and there must be directions of descent into an adjacent segment, say Y^t . However, then the sets of active lower level constraints and of nonzero Lagrange multipliers associated to a response to the perturbed point $y^t = \bar{y}^s + \beta d$ change. Therefore Theorem 4 can not be utilised. The behaviour of $F(y^t, z^t)$ can not be evaluated without solving the lower level problem. However, the perturbed point y^t is in the relative interior of the segment Y^t and it can be assumed that condition (18b) is satisfied with strict complementary slackness. Therefore a possible approach to find a feasible descent direction on another segment is the following.

Determine all possible index sets $I_C^t \neq I_C^s$. Test then for each set I_C^t if the system

$$\nabla_z f(\bar{y}^s + \beta d, z^t) + (\lambda^t)^T \nabla_z g(\bar{y}^s + \beta d, z^t) = 0 \quad (20a)$$

$$g_i(\bar{y}^s + \beta d, z^t) = 0, \quad i \in I_C^t \quad (20b)$$

$$\lambda_i^t > 0, \quad i \in I_C^t \quad (20c)$$

$$g_i(\bar{y}^s + \beta d, z^t) < 0, \quad i \notin I_C^t \quad (20d)$$

$$\lambda_i^t = 0, \quad i \notin I_C^t \quad (20e)$$

$$G(\bar{y}^s + \beta d, z^t) \leq 0 \quad (20f)$$

$$F(\bar{y}^s + \beta d, z^t) - F(\bar{y}^s, \bar{z}^s) < 0 \quad (20g)$$

with small $\beta > 0$ has solutions z^t, λ^t and $d \neq 0$. If this is the case for an index set I_C^t , a feasible descent direction and a new segment $Y^{s+1} = Y^t$ are found. Here, constraints (20b) – (20e) specify the strict complementarity, constraint (20f) ensures the upper level feasibility and constraint (20g) the descent of the direction d .

The stationary point $\bar{x}^s = (\bar{y}^s, \bar{z}^s, \bar{\lambda}^s)$ as solution of problem (13) can be determined in Step 2 of Algorithm 1 by the following gradient projection method:

Algorithm 2: Find stationary point on a segment.

Step 0. (Initialisation) Set $k = 0$, the initial point $\hat{x}^0 = (\hat{y}^0, \hat{z}^0, \hat{\lambda}^0)$ is passed from Algorithm 1.

Step 1. (Objective function) Calculate the values of the objective function $F(\hat{y}^k, \hat{z}^k)$ and of the estimation

$$\bar{F}^k = \frac{1}{k} \sum_{i=1}^k F(\hat{y}^i, \hat{z}^i)$$

Step 2. (Search direction and step size) Determine a search direction

$$d_k = (\nabla_y F(\hat{y}^k, \hat{z}^k), \nabla_z F(\hat{y}^k, \hat{z}^k), 0)^T \quad (21)$$

and a step size α_k according to the Armijo rule.

Step 3. (Update) Determine a new iterate for the upper level decision:

$$\hat{x}^{k+1} = \Pi_{X^s}(\hat{x}^k - \alpha_k d_k)$$

Step 4. (Convergence) If a convergence test is satisfied, for example if

$$|\bar{F}^k - \bar{F}^{k-1}| \leq \varepsilon_c$$

for $k \geq 1$ with a specified precision ε_c , go to Step 5.

Otherwise set $k = k + 1$ and go to Step 1.

Step 5. (Termination) The point $\bar{x}^s = \hat{x}^k$ is a stationary solution of problem (13) – (16), i.e. stationary on the segment Y^s .

Remark 4. Determination of new iterate in Step 3. The operator Π_{X^s} denotes the projection on the feasible region. The obtained iterate $\hat{x}^{k+1} = (\hat{y}^{k+1}, \hat{z}^{k+1}, \hat{\lambda}^{k+1})$ is therefore the optimal solution of the following quadratic programming problem

$$\begin{aligned} \min_x & \|x - (\hat{x}^k - \alpha_k d_k)\|^2 \\ E(x) & \leq 0 \\ e(x) & = 0 \\ x & \in X = Y \times Z \times \mathbf{R}_+^q \end{aligned}$$

where $E(x)$ and $e(x)$ are defined by (14), (15).

Theorem 5. *Assume that*

1. *Assumption (A2) holds,*
2. *the conditions of Theorem 3 are valid,*
3. *the upper level constraints $G_i(y, z)$, $i = 1, \dots, p$, are linear in y and z ,*
4. *the search direction d_k is defined by (21) and the step size α_k is determined by the Armijo rule.*

Then Algorithm 1 utilising Algorithm 2 stops at a local minimum of problem (4).

Proof. The upper level objective function $F(y, z(y))$ as function of y can be discontinuous and nondifferentiable. Therefore the inducible region IR is divided into segments where $F(y, z(y))$ is continuous and differentiable. Step 2 of Algorithm 1 restricts the determination of a local optimum to a search on the segment Y^s . This search is conducted by finding an stationary point solving problem (13) – (16). Under conditions 1. – 3. this problem represents a convex nonlinear optimisation problem subject to linear constraints. The objective function $F(x) = F(y, z)$ is continuously differentiable with respect to y and z . Therefore problem (13) – (16) is amenable to a solution by a standard gradient method such as the Gradient Projection Method implemented in Algorithm 2. With descent directions d_k defined by (21) and step sizes α_k according to condition 4. the direction sequence $\{\bar{d}_k\} = \{\alpha_k d_k\}$ is gradient related [Ber99]. Consequently, every limit point \bar{x}^s of the sequence $\{x^k\}$ generated by Algorithm 2 is stationary on the considered segment Y^s .

If this point \bar{x}^s satisfies the necessary optimality conditions specified in Step 3 of Algorithm 1, a local minimum of problem (4) is found and Algorithm 1 stops. If the optimality conditions are not satisfied then there exist feasible descent directions at the point \bar{x}^s . First, it is then tested if descent directions d exist which are feasible on the current segment Y^s . The index sets I_C^s and I_L^s do not change for the perturbed point $y^{s+1} = \bar{y}^s + \beta d$ and Theorem 4 is applicable for an estimation of the response $(z^{s+1}, \lambda^{s+1}) = (z(y^{s+1}), \lambda(y^{s+1}))$. Therefore the direction d can be found as solution of system (19). If this system has no solution, this means that only directions of descent into adjacent segments exist. These adjacent segments

are characterised by different index sets I_C and I_L . Therefore in this case a direction d can be found as solution of system (20). In both cases the stationary point \bar{y}^s is then perturbed into the found direction and the search is repeated on the according segment Y^{s+1} with the initial point $(y^{s+1}, z^{s+1}, \lambda^{s+1})$. Since the segments describe a finite number of convex optimisation problems, Algorithm 1 stops at a local optimum after a finite number of steps. \square

4 Stochastic two stage problems with bilevel structure

In this section the deterministic problem (4) is extended to a two-stage stochastic programming problem with bilevel structure. Remember that the leader's uncertainty can be divided into two types, his uncertainty about other decision makers and about random model parameters. The first type of uncertainty is treated separately by the bilevel structure and the follower's response can be determined by solving the lower level decision problem. The second type of uncertainty is expressed by a vector $\omega \in \Omega$ of random variables with a given probability distribution and then taken into account by a second stage problem. In a first variant only the leader can take into account a recourse decision. Assuming the case of simple recourse, we state necessary optimality conditions and adapt the solution algorithm developed in the previous section to the new structure. Then the problem formulation is extended such that also the follower's decision problem involves a second stage decision. We analyse show that such a model can be reformulated similarly to the first problem. Thus the presented solution algorithm can be applied also to this more complex problem.

Consider at first the following formulation with a two-stage stochastic programming problem in the upper level and a one-stage stochastic programming problem in the lower level:

$$\min_{y_1 \in Y_1} \{F_1(y_1, z_1^*) + \mathbf{E}_\omega Q(y_1, \omega)\} \quad (22a)$$

$$G(y_1, z_1^*) \leq 0 \quad (22b)$$

$$z_1^* = \arg \min_{z_1 \in Z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) \quad (22c)$$

$$g(y_1, z_1) \leq 0 \quad (22d)$$

$$Q(y_1, \omega) = \min_{y_2 \in Y_2} F_2(y_1, y_2, \omega) \quad (22e)$$

$$W_1(\omega)y_2 = h_1(\omega) - T_1(\omega)y_1 \quad (22f)$$

with $Q : R^n \times \Omega \rightarrow R^1$, $F_2 : R^n \times R^{n_2} \times \Omega \rightarrow R^1$, $W_1 \in R^{n_2} \times R^{p_2}$, $h_1 \in R^{p_2}$, $T_1 \in R^n \times R^{p_2}$ and $Y_2 \subseteq R^{n_2}$. We assume that the second stage objective function $F_2(y_1, y_2, \omega)$ is differentiable in y_1 and y_2 and that the constraints of all subproblems are linear.

Problem (22) represents the most simple formulation of a two-stage stochastic programming problem with bilevel structure. In order to find a first stage decision y_1 , the leader takes into account his recourse decision y_2 and predicts the follower's response z_1^* . We assume that the influence of the follower's decisions is not strong enough to be regarded in the second stage. Therefore this response is not included into the recourse problem here.

In order to describe necessary optimality conditions similar to the conditions stated in Proposition 1, we need the convexity and differentiability of the objective functions $F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega)$ and $\mathbf{E}_\omega f_1(y_1, z_1, \omega)$.

Proposition 2. *If the function $F_1(y_1, z_1)$ is convex in y_1 and $F_2(y_1, y_2, \omega)$ is convex in y_1 and y_2 for all ω then the function $F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega)$ is convex in y_1 . It is differentiable with respect to y_1 almost everywhere. If the random variable ω is absolutely continuously distributed, then the function $F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega)$ is continuously differentiable with respect to y_1 .*

Proof. See for example Birge and Louveaux [BL97]. □

Proposition 3. *Assume that the function $f_1(y_1, z_1, \omega)$ satisfies the following conditions:*

1. *the gradients $\nabla_x f_1(y_1, z_1, \omega)$ and $\nabla_x(\nabla_z f_1(y_1, z_1, \omega))$ are integrable,*

2. at the point $\hat{x} = (\hat{y}_1, \hat{z}_1) \in \mathbf{R}^n \times \mathbf{R}^m$ the residues

$$\begin{aligned}\rho_f(\hat{x}, \omega, t) &= \frac{f_1(\hat{x} + te_i, \omega) - f_1(\hat{x}, \omega)}{t} - \frac{\partial f_1(\hat{x}, \omega)}{\partial x_i} \\ \rho_{gf}(\hat{x}, \omega, t) &= \frac{\nabla_z f_1(\hat{x} + te_i, \omega) - \nabla_z f_1(\hat{x}, \omega)}{t} - \frac{\partial \nabla_z f_1(\hat{x}, \omega)}{\partial x_i}\end{aligned}$$

approach zero for $t \rightarrow 0$ and all $i = 1, \dots, n + m$ where $e_i \in \mathbf{R}^{n+m}$ denotes the i -th unit vector.

Then

1. $\mathbf{E}_\omega f_1(y_1, z_1, \omega)$ is convex and
2. $\mathbf{E}_\omega f_1(y_1, z_1, \omega)$ is at least twice differentiable at (\hat{y}_1, \hat{z}_1) with

$$\begin{aligned}\nabla_z \mathbf{E}_\omega f_1(\hat{y}_1, \hat{z}_1, \omega) &= \mathbf{E}_\omega \nabla_z f_1(\hat{y}_1, \hat{z}_1, \omega) \\ \nabla(\nabla_z \mathbf{E}_\omega f_1(\hat{y}_1, \hat{z}_1, \omega)) &= \mathbf{E}_\omega \nabla(\nabla_z f_1(\hat{y}_1, \hat{z}_1, \omega))\end{aligned}$$

Proof. See for example Kall and Wallace [KW94]. □

The recourse problem affects the first stage problem only through the leader's objective function. Thus problem (22) can be approximated in the vicinity of a feasible point $x^0 = (y_1^0, z_1^0, \lambda^0)$ by a one-level two-stage stochastic programming problem similar to formulation (5):

$$\min_{y_1, z_1, \lambda} F(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega) \tag{23a}$$

$$\nabla_{z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) + \lambda^T \nabla_{z_1} g(y_1, z_1) = 0$$

$$G(y_1, z_1) \leq 0$$

$$g_i(y_1, z_1) = 0, \quad i \in I(y_1^0, z_1^0)$$

$$\lambda_i = 0, \quad i \notin J(\lambda^0)$$

$$g_i(y_1, z_1) \leq 0, \quad i \notin I(y_1^0, z_1^0)$$

$$\lambda_i \geq 0, \quad i \in J(\lambda^0)$$

$$Q(y_1, \omega) = \min_{y_2 \in Y_2} F_2(y_1, y_2, \omega) \tag{23b}$$

$$W_1(\omega) y_2 = h_1(\omega) - T_1(\omega) y_1 \tag{23c}$$

Then the necessary optimality conditions stated in Proposition 1 can be adapted to the problem structure (22). For given first stage decision y_1^0 and observation ω of the random variable the optimal recourse decision is denoted by $y_2^0(y_1^0, \omega)$ and the associated Lagrange multiplier by $v^R(y_1^0, \omega)$.

Proposition 4. *Assume that*

1. *the conditions of Proposition 2 on the leader's objective functions hold*
2. *the point $(y_1^0, z_1^0, \lambda^0)$ is a local minimum of problem (22)*

Then there exists a nonvanishing vector of multipliers $(\kappa_0, \kappa, \gamma, \zeta, \tau, \xi)$ such that the following system is satisfied:

$$\begin{aligned} \kappa_0(\nabla F_1(y_1^0, z_1^0) + \mathbf{E}_\omega\{\nabla F_2(y_1^0, y_2^0(\omega), \omega) - v^R(\omega)T_1(\omega)\}) \\ + \kappa^T \nabla G(y_1^0, z_1^0) + \mathbf{E}_\omega \nabla(\nabla_{z_1} f_1(y_1^0, z_1^0)\gamma) + \zeta^T \nabla g(y_1^0, z_1^0) = 0 \end{aligned}$$

$$\begin{aligned} \nabla_{z_1} g(y_1^0, z_1^0)^T \gamma - \xi^T &= 0 \\ g_i(y_1^0, z_1^0) \zeta_i &= 0 \quad \forall i \\ \kappa^T G(y_1^0, z_1^0) &= 0 \\ \lambda_i^0 \xi_i &= 0 \quad \forall i \\ \zeta_i \xi_i &\geq 0 \quad i \in I(y_1^0, z_1^0) \cap J(\lambda^0) \\ \kappa_0, \kappa &\geq 0 \end{aligned}$$

Proof. Under the assumptions of Proposition 2 the recourse function $E_\omega Q(y_1^0, \omega)$ is differentiable at y_1^0 . For given decision y_1^0 , observation ω and recourse decision $y_2^0(y_1^0, \omega)$ the gradient of the recourse function with respect to (y, z) can be determined using the Lagrangian function of the recourse problem

$$\begin{aligned} \nabla E_\omega Q(y_1^0, \omega) &= \nabla E_\omega L^R(y_1^0, y_2^0(y_1^0, \omega), v^R(y_1^0, \omega)) \\ &= \mathbf{E}_\omega\{\nabla F_2(y_1^0, y_2^0(y_1^0, \omega), \omega) - v^R(y_1^0, \omega)T_1(\omega)\} \end{aligned} \quad (24)$$

Now, keeping in mind that the follower's objective function in problem (23) is $F_1(y_1^0, z_1^0) + E_\omega Q(y_1^0, \omega)$, we apply the necessary optimality conditions stated in Proposition 1 to this problem and obtain the required result. \square

We direct now our attention to a solution method for problem (22). It is possible to proceed as described in the previous section: The original problem is partitioned into a family of stochastic one-level problems described by segments of the upper level domain. Then, a local optimum is found on a segment by a stochastic quasi-gradient method [Erm88, Gai88, Gai04]. However, a direct application of the partitioning strategy yields the following system to be solved by Algorithm 2 on the segment Y^s :

$$\min_{y_1, z_1, \lambda} \{F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega)\} \quad (25a)$$

$$E(y_1, z_1, \lambda) \leq 0 \quad (25b)$$

$$\mathbf{E}_\omega e(y_1, z_1, \lambda, \omega) = 0$$

$$Q(y_1, \omega) = \min_{y_2 \in Y_2} F_2(y_1, y_2, \omega) \quad (25c)$$

$$W_1(\omega)y_2 = h_1(\omega) - T_1(\omega)y_1 \quad (25d)$$

$$(y_1, z_1, \lambda) \in Y_1 \times Z_1 \times \mathbf{R}_+^q$$

with $E(y_1, z_1, \lambda)$ defined by (14) and

$$e(y_1, z_1, \lambda, \omega) = \begin{pmatrix} \nabla_z f_1(y_1, z_1, \omega) + \lambda^T \nabla_z g(y_1, z_1) \\ g_i(y_1, z_1), & i \in I_C^s \\ \lambda_i, & i \notin I_L^s \end{pmatrix} \quad (26)$$

This formulation, however, contains stochastic equality constraints which may complicate a solution by a projection method. Furthermore the second stage problem (25c) – (25d) must be taken into account. Therefore Algorithm 2 is modified utilising a Lagrange multiplier method [NV77] solving the problem

$$\min_x \max_{u \geq 0, v} \mathbf{E}_\omega L(x, u, v, \omega) \quad (27)$$

where

$$L(x, u, v, \omega) = F_1(x) + Q(x, \omega) + uE(x) + ve(x, \omega) \quad (28)$$

is the Lagrangian function of problem (25) with $x = (y_1, z_1, \lambda)$.

Algorithm 3: Find stationary point in a segment utilising Lagrangian; recourse problem.

Step 0. (Initialisation) Set $k = 0$, the initial point $\hat{x}^0 = (\hat{y}_1^0, \hat{z}_1^0, \hat{\lambda}^0)$ is passed from Algorithm 1. The Lagrange multipliers $u^0 \in \mathbf{R}^{m+q}$ and $v^0 \in \mathbf{R}^{p+q}$ are associated to this point \hat{x}^0 .

Step 1. (Recourse decision) Determine a sample $\{\omega^1, \dots, \omega^{N_k}\}$ of observations of the random variable ω .

For each observation ω^ν , $\nu = 1, \dots, N_k$ solve the recourse problem (25c) – (25d) with the first stage iterate $\hat{x}^k = (\hat{y}^k, \hat{z}^k, \hat{\lambda}^k)$ and obtain the recourse decision $y_2^{k,\nu} = y_2(\hat{x}^k, \omega^\nu)$, the Lagrange multipliers $v_R^{k,\nu} = v_R(\hat{x}^k, \omega^\nu)$ and the recourse function $Q(\hat{x}_1^k, \omega^i)$.

Step 2. (Objective function) Calculate an approximation $\tilde{F}(\hat{y}_1^k, \hat{z}_1^k)$ of the objective function and the estimation \bar{F}^k

$$\tilde{F}(\hat{y}_1^k, \hat{z}_1^k) = F_1(\hat{y}_1^k, \hat{z}_1^k) + \sum_{i=1}^{N_k} Q(\hat{x}_1^k, \omega^i)$$

$$\bar{F}^k = \frac{1}{k} \sum_{i=1}^k \tilde{F}(\hat{y}_1^k, \hat{z}_1^k)$$

Step 3. (Search direction and step size) Determine search directions

$$\xi_x^k = \nabla_x F_1(\hat{x}^k) + u^k \nabla_x E(\hat{x}^k)$$

$$+ \frac{1}{N_k} \sum_{\nu=1}^{N_k} \left(v^k \nabla_x e(\hat{x}^k, \omega^\nu) + \nabla_x F_2(\hat{x}^k, y_2^{k,\nu}, \omega^\nu) - v_R^{k,\nu} T_1(\omega^\nu) \right) \quad (29)$$

$$\xi_u^k = E(\hat{x}^k) \quad (30)$$

$$\xi_v^k = \frac{1}{N_k} \sum_{\nu=1}^{N_k} e(\hat{x}^k, \omega^\nu) \quad (31)$$

and step sizes α_x^k, α_u^k and α_v^k satisfying the conditions

$$\begin{aligned}
\alpha_x &\rightarrow 0^+, \quad \sum_{k=1}^{\infty} \alpha_x^k = \infty, \quad \sum_{k=1}^{\infty} (\alpha_x^k)^2 < \infty \\
\alpha_u &\rightarrow 0^+, \quad \sum_{k=1}^{\infty} \alpha_u^k = \infty, \quad \sum_{k=1}^{\infty} (\alpha_u^k)^2 < \infty \\
\alpha_v &\rightarrow 0^+, \quad \sum_{k=1}^{\infty} \alpha_v^k = \infty, \quad \sum_{k=1}^{\infty} (\alpha_v^k)^2 < \infty \\
\frac{\alpha_x^k}{\alpha_u^k} &\rightarrow 0, \quad \frac{\alpha_x^k}{\alpha_v^k} \rightarrow 0
\end{aligned} \tag{32}$$

Step 4. (Update) Determine new iterates for the upper level decision x and the Lagrange multipliers u and v :

$$\begin{aligned}
\hat{x}^{k+1} &= \Pi_{X^s}(\hat{x}^k - \alpha_x^k \xi_x^k) \\
\hat{u}^{k+1} &= \max\{0, \hat{u}^k + \alpha_u^k \xi_u^k\} \\
\hat{v}^{k+1} &= \hat{v}^k + \alpha_v^k \xi_v^k
\end{aligned}$$

The operator Π_{X^s} denotes the projection on the feasible area

$$X^s = \{\hat{x} \in Y_1 \times Z_1 \times R_+^q \mid E(\hat{x}) \leq 0, \frac{1}{N_k} \sum_{i=1}^{N_k} e(\hat{x}, \omega^i) = 0\}$$

with $E(\hat{x})$ defined by (14) and $e(\hat{x}, \omega^i)$ by (26).

Set $k = k + 1$.

Step 5. (Convergence) If a convergence test is satisfied, for example if

$$|\bar{F}^{k-j} - \bar{F}^{k-j-1}| \leq \varepsilon_c, \quad \forall j = 0, \dots, n$$

for $k \geq n + 1$ with given precision ε_c and test horizon $n \geq 0$, go to Step 6.

Otherwise go to Step 1.

Step 6. (Termination) The point $\bar{x}^s = \hat{x}^k$ is a stationary solution of problem (25), i.e. \bar{x}^s is stationary on the segment Y^s .

Remark 5. Convergence test in Step 5. Since the random parameters are approximated by a sample of observations there may occur periods with apparently stationary iterates which are obviously not optimal. Especially if such a period occurs during the first iteration steps the estimation \bar{F}^k seems to converge. In order to avoid the termination of the algorithm in such a case the convergence test evaluates the estimation over a horizon of n iteration steps. (See also the implementation example in Section 5.)

Theorem 6. (*Convergence of Algorithm 3*) Assume that

1. Assumptions (A1) and (A2) hold,
2. the gradient $\nabla_z \mathbf{E}_\omega f_1(\hat{y}_1, \hat{z}_1, \omega)$ is linear in y_1 and z_1 ,
3. the conditions of Proposition 2 are satisfied,
4. the search directions ξ_x^k, ξ_u^k and ξ_v^k are defined by (29) – (31),
5. the step sizes α_x^k, α_u^k and α_v^k satisfy the conditions (32)

Then Algorithm 3 converges with probability 1 to the vicinity of a stationary point \bar{x}^s of problem (25).

Proof. Under Assumption (A1) and Proposition 2 the objective function of problem (25) is convex and continuously differentiable in y_1 and z_1 . Due to Assumption (A2) the response $z_1(y_1)$ is uniquely determined. If condition 2. holds, problem (25) represents then a convex optimisation problem and its optimal solution coincides with the optimal solution of the problem (27). This problem is solved by Algorithm 3 utilising a Lagrange multiplier method.

In order to determine the search directions an estimate of the subgradient of the recourse function is needed. For an iterate \hat{x}^k and an observation ω^ν of the random variable such an estimate is for example the gradient with respect to x of the Lagrangian of the recourse problem:

$$\nabla_x L^R(\hat{x}^k, y_2^{k,\nu}, v_R^{k,\nu}, \omega^\nu) = \nabla_x F_2(\hat{x}^k, y_2^{k,\nu}, \omega^\nu) - v_R^{k,\nu} T_1(\omega^\nu)$$

Taking now into account that the inequality constraints $E(x)$ are deterministic, the search directions (29) – (31) satisfy the stochastic quasi-gradient conditions

$$\begin{aligned}\mathbf{E}_\omega\{\xi_x^k|\hat{x}^0, \dots, \hat{x}^k\} &= \nabla_x \mathbf{E}_\omega L(\hat{x}^k, \hat{u}^k, \hat{v}^k, \omega) \\ \mathbf{E}_\omega\{\xi_v^k|\hat{x}^0, \dots, \hat{x}^k\} &= \nabla_v \mathbf{E}_\omega L(\hat{x}^k, \hat{u}^k, \hat{v}^k, \omega) \\ \xi_u^k &= \nabla_u \mathbf{E}_\omega L(\hat{x}^k, \hat{u}^k, \hat{v}^k, \omega)\end{aligned}$$

Then, with the step size conditions (32), Algorithm 3 converges with probability 1 to the vicinity of a stationary point \bar{x}^s of problem (25) [Erm88]. \square

Theorem 7. *Assume that*

1. *Assumption (A1) holds,*
2. *the gradient $\nabla_z \mathbf{E}_\omega f_1(y_1, z_1, \omega)$ is linear in y_1 and z_1 ,*
3. *the conditions of Propositions 2 and 3 are satisfied,*
4. *the optimality test in Algorithm 1 utilises the necessary optimality conditions stated in Proposition 4.*

Then Algorithm 1 utilising Algorithm 3 stops at a point in the vicinity of a local minimum of problem (22).

Proof. If condition 2. holds, the expectation $\mathbf{E}_\omega \nabla_{(y,z)} f_1(y_1, z_1, \omega)$ is linear in y_1 and z_1 . With assumption 3. the objective functions $F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, \omega)$, $\mathbf{E}_\omega f_1(y_1, z_1, \omega)$ and the gradient vector $\mathbf{E}_\omega \nabla_{(y,z)} f_1(y_1, z_1, \omega)$ are continuously differentiable. Employing then the necessary optimality conditions of Proposition 4 and arguing similarly to the proof of Theorem 5 it is proved that Algorithm 1 stops in the vicinity of a local optimal solution of problem (22). \square

Remark 6. If it is not possible to calculate the expectations, they can be approximated using a sufficiently large sample of observations of the random variable. However, then the satisfaction of the optimality conditions can be verified only with a certain precision such that Algorithm 1 stops at a point in the vicinity of a local optimum of problem (22) only with probability.

Now the two-stage problem (22) is extended by taking into account a reaction of the follower on changed conditions at the second stage. This means that there exists a bilevel relationship between the actors' problems at each stage. However, it is assumed that the follower does not regard the future when she determines her action z_1 in the first stage, i.e. her second stage problem is not interpreted as recourse problem. This way she only adapts her strategy when new information reveals whereas the leader can take into account a later adaptation of his strategy already at the first stage. Furthermore it is assumed that the leader's first stage decision can directly influence the follower's second stage decision. This reflects the case when some of the leader's first stage decisions still are valid for the control of the follower's decision problem, such as certain regulatory obligations on the follower. Such a model can be formulated as follows.

$$\min_{y_1 \in Y_1} \{F_1(y_1, z_1^*) + \mathbf{E}_\omega Q(y_1, z_1^*, \omega)\} \quad (33a)$$

$$G(y_1, z_1^*) \leq 0 \quad (33b)$$

$$Q(y_1, z_1^*, \omega) = \min_{y_2 \in Y_2} F_2(y_1, y_2, z_1^*, \omega) \quad (33c)$$

$$W_1(\omega)y_2 = h_1(\omega) - T_1(\omega)y_1 - U_1(\omega)z_1^* - V_1(\omega)z_2^* \quad (33d)$$

$$z_1^* = \arg \min_{z_1 \in Z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) \quad (33e)$$

$$g(y_1, z_1) \leq 0 \quad (33f)$$

$$z_2^* = \arg \min_{z_2 \in Z_2} f_2(y_1, y_2, z_1^*, z_2, \omega) \quad (33g)$$

$$V_2(\omega)z_2 = h_2(\omega) - T_2(\omega)y_1 - U_2(\omega)z_1^* - W_2(\omega)y_2 \quad (33h)$$

where $Q : \mathbf{R}^n \times \mathbf{R}^{m_2} \times \mathbf{R}^n \rightarrow \mathbf{R}^1$, $f_2 : \mathbf{R}^{2m_2} \times \mathbf{R}^{2n_2} \rightarrow \mathbf{R}^1$, $U_1 \in \mathbf{R}^m \times \mathbf{R}^{p_2}$, $T_2 \in \mathbf{R}^n \times \mathbf{R}^{q_2}$, $U_2 \in \mathbf{R}^m \times \mathbf{R}^{p_2}$, $V_1 \in \mathbf{R}^{m_2} \times \mathbf{R}^{p_2}$, $V_2 \in \mathbf{R}^{m_2} \times \mathbf{R}^{q_2}$, $W_2 \in \mathbf{R}^{n_2} \times \mathbf{R}^{q_2}$, $h_2 \in \mathbf{R}^{q_2}$ and $Z_2 \subseteq \mathbf{R}^{m_2}$.

The leader finds an optimal solution of his first stage problem (33a) – (33b) taking into account the recourse problem (33c) – (33d). For this purpose he predicts the response z_1^* on his first stage decision y_1 and the response z_2^* on his first stage decision y_1 and on his recourse decision y_2 . These responses can be determined by solving the follower's decision problems (33e) – (33f) respective (33g) – (33h). It is assumed that the follower's first stage objective function satisfies the conditions of Proposition 3 such that $\mathbf{E}_\omega f_1(y_1, z_1, \omega)$ is convex and differentiable. Furthermore it is assumed that the follower's second stage response $z_2^* = z_2(y_1, y_2, z_1^*, \omega)$

is uniquely determined for all y_1, y_2, z_1^* and ω and that the matrix $W_2(\omega)$ has full rank for any ω .

Theorem 8. *Assume that*

1. *the second stage objective function $f_2(y_1, y_2, z_1^*, z_2, \omega)$ of the follower is continuously differentiable in z_2*
2. *the gradient $\nabla_{z_2} f_2(y_1, y_2, z_1^*, z_2, \omega)$ is linear in y_2, z_1 and z_2 .*

Then problem (33) can be formulated stochastic programming problem with a structure similar to problem (22).

Proof. The follower's second stage decision is an optimal response to the leader's decisions at both stages. Therefore problem (33) can be reformulated to the following bilevel stochastic programming problem with recourse

$$\min_{y_1 \in Y_1} \{F_1(y_1, z_1^*) + \mathbf{E}_\omega Q(y_1, z_1^*, \omega)\} \quad (34a)$$

$$G(y_1, z_1^*) \leq 0$$

$$z_1^* = \arg \min_{z_1 \in Z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) \quad (34b)$$

$$g(y_1, z_1) \leq 0$$

$$Q(y_1, z_1, \omega) = \min_{y_2, z_2, \mu} F_2(y_1, y_2, z_2, \omega) \quad (34c)$$

$$\nabla_{z_2} f_2(y_1, y_2, z_1, z_2, \omega) + \mu^T V_2(\omega) = 0 \quad (34d)$$

$$W_1(\omega)y_2 + V_1(\omega)z_2 = h_1(\omega) - T_1(\omega)y_1 - U_1(\omega)z_1 \quad (34e)$$

$$W_2(\omega)y_2 + V_2(\omega)z_2 = h_2(\omega) - T_2(\omega)y_1 - U_2(\omega)z_1 \quad (34f)$$

Under condition 2. constraint (34d) can be expressed as

$$A(\omega)y_2 + B(\omega)z_2 + \mu^T V_2(\omega) = c^T(\omega) - D_1(\omega)y_1 - D_2(\omega)z_1 \quad (35)$$

with $A \in R^{n_2} \times R^{m_2}$, $B \in R^{m_2} \times R^{m_2}$, $c \in R^{m_2}$, $D_1 \in R^n \times R^{m_2}$, $D_2 \in R^m \times R^{m_2}$. Resulting, constraints (34d) – (34f) can be collected in one linear constraint

$$\bar{W}(\omega)v_2 = \bar{h}(\omega) - \bar{T}(\omega)v_1$$

with

$$\bar{T}(\omega) = \begin{bmatrix} D_1(\omega) & D_2(\omega) \\ T_1(\omega) & U_1(\omega) \\ T_2(\omega) & U_2(\omega) \end{bmatrix} \quad (36)$$

$$\bar{W}(\omega) = \begin{bmatrix} A(\omega) & B(\omega) & V_2^T(\omega) \\ W_1(\omega) & V_1(\omega) & 0 \\ W_2(\omega) & V_2(\omega) & 0 \end{bmatrix} \quad (37)$$

$$\bar{h}(\omega) = (c(\omega), h_1(\omega), h_2(\omega))^T \quad (38)$$

$$v_1 = (y_1, z_1)^T \quad (39)$$

$$v_2 = (y_2, z_2, \mu)^T \in X_2 = Y_2 \times Z_2 \times \mathbf{R}_+^{q_2} \quad (40)$$

This way all information of the follower's second stage problem is included into the leader's recourse problem. Consequently, problem (33) is similar to problem (22) and necessary optimality conditions for problem (33) are equivalent to the conditions stated in Proposition 4. \square

Reformulating problem (33) in the vicinity of a feasible point $x^0 = (y_1^0, z_1^0, \lambda^0)$ a one-level two-stage stochastic programming problem is obtained which is similar to the formulation (23).

$$\min_{y_1, z_1, \lambda} \{F_1(y_1, z_1) + \mathbf{E}_\omega Q(y_1, z_1, \omega)\} \quad (41a)$$

$$\nabla_{z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) + \lambda^T \nabla_{z_1} g(y_1, z_1) = 0$$

$$g_i(y_1, z_1) = 0 \quad i \in I(y_1^0, z_1^0)$$

$$\lambda_i = 0 \quad i \in J(\lambda^0)$$

$$g_i(y_1, z_1) \leq 0 \quad i \notin I(y_1^0, z_1^0)$$

$$\lambda_i \geq 0 \quad i \notin J(\lambda^0)$$

$$G(y_1, z_1) \leq 0$$

$$Q(y_1, z_1, \omega) = \min_{y_2, z_2, \mu} F_2(y_1, y_2, z_2, \omega) \quad (41b)$$

$$\bar{W}(\omega)(y_2, z_2, \mu) = \bar{h}(\omega) - \hat{T}(\omega)(y_1, z_1) \quad (41c)$$

Then the necessary optimality conditions stated in Proposition 4 can be applied to problem (33) in an equivalent way.

Finally, the following theorem states that Algorithm 1 coupled with Algorithm 3

can be utilised without modification for the solution of the bilevel stochastic two-stage problem (33).

Theorem 9. *Assume that*

1. *Assumption (A1) holds,*
2. *the conditions of Proposition 2 are satisfied,*
3. *the gradient $\nabla_z \mathbf{E}_\omega f_1(y_1, z_1, \omega)$ is linear in y_1 and z_1 ,*
4. *the search directions ξ_x^k , ξ_u^k and ξ_v^k are defined by (29) – (31) and the step sizes α_x^k , α_u^k and α_v^k satisfy the conditions (32)*
5. *the optimality test in Algorithm 1 utilises the necessary optimality conditions stated in Proposition 4.*

Then Algorithm 1 together with Algorithm 3 yields a local optimal solution of problem (33).

Proof. Theorem 8 stated that problem (33) can be reformulated in such a way that it assumes the structure of problem (22):

$$\min_{y_1 \in Y_1} F_1(y_1, z_1^*) + \mathbf{E}_\omega Q(y_1, z_1^*, \omega) \quad (42a)$$

$$G(y_1, z_1^*) \geq 0$$

$$z_1^* = \arg \min_{z_1 \in Z_1} \mathbf{E}_\omega f_1(y_1, z_1, \omega) \quad (42b)$$

$$g(y_1, z_1) \leq 0$$

$$Q(y_1, z_1^*, \omega) = \min_{v_2 \in X_2} F_2(y_1, v_2, \omega) \quad (42c)$$

$$\overline{W}(\omega)v_2 = \overline{h}(\omega) - \overline{T}(\omega)(y_1, z_1) \quad (42d)$$

with $\overline{T}(\omega)$, $\overline{W}(\omega)$, $\overline{h}(\omega)$, v_2 according to (36) – (38) and (40).

The second stage decision variable v_2 of problem (42) comprises the leader's recourse decision y_2 , the follower's second stage decision z_2 as response to y_2 and the Lagrange multipliers μ associated to this response. Since the follower's second stage response is unique for all recourse decisions of the leader the variable v_2 can be determined as optimal solution of the recourse problem (42c) – (42d).

Therefore, following the reasoning of Theorem 6, Algorithm 1 together with Algorithm 3 can be applied to problem (33) without modification. It yields a local optimal solution of the reformulation (42) and thus also a local optimal solution of the original problem (33). \square

5 Numerical studies

This section demonstrates the viability of the presented approach. We applied Algorithm 1 with Algorithm 3 to an example of a principal agent relationship in telecommunications. Such a relationship is described in more detail in [AGWng]. Here, we study a simplified formulation.

We assume that both decision makers maximise their profits from the provision of a similar product (e.g. a telecom service) to a common customer population. The follower is lacking essential infrastructure necessary for the provision of the product and relies on the leader for access to such equipment. The customer demand depends on the decisions of both providers and is subject to several constraints. This means that connecting upper level constraints are present. Both actors make their first stage decisions on the base of estimations of the environment data. When the actually realised values of these data are observable at the end of the first stage, the leader can make a further decision (for example extend the infrastructure).

The following bilevel stochastic programming problem with a recourse problem in the upper level can be formulated. It represents a version of model (33) with simple recourse and no second stage decisions of the follower such that $W_1(\omega) = 1$ and $V_1(\omega) = 0$. The leader's first and second stage decision problems are

$$\max_{y_1} y_1^T C_{11} y_1 + y_1^T C_{12} z + d_{11}^T y_1 + d_{12}^T z + \mathbf{E}_\omega Q(y_1, z, \omega) \quad (43a)$$

$$A_1 y_1 + B_1 z + f_1 \leq 0 \quad (43b)$$

$$y_1 \in Y_1 \subset \mathbf{R}^3$$

$$Q(y_1, z, \omega) = \max_{y_2 \geq 0} q(\omega) y_2 \quad (43c)$$

$$y_2 = h_1(\omega) - T_1(\omega) y_1 - U_1(\omega) z \quad (43d)$$

The follower's response on a first stage decision y_1 is found as an optimal solution of the following problem:

$$\max_z y_1^T C_{21} z + z^T C_{22} z + d_{21}^T y_1 + d_{22}^T z \quad (44a)$$

$$A_2 y_1 + B_2 z + f_2 \leq 0 \quad (44b)$$

$$z \in Z \subset R^3$$

For a given initial point a segment Y^s is described by the sets of nonzero Lagrange multipliers and of active lower level constraints $I_L^s \cup I_C^s$. The second stage problem is not affected by the partitioning since it has no bilevel structure. This results in the following one-level problem.

$$\min_{y_1, z, \lambda} F_1(y_1, z) + E_\omega Q(y_1, z, \omega) \quad (45a)$$

$$E(y_1, z, \lambda) \leq 0 \quad (45b)$$

$$e(y_1, z, \lambda) = 0 \quad (45c)$$

$$Q(y_1, z, \omega) = \min_{y_2 \geq 0} q(\omega) y_2 \quad (45d)$$

$$y_2 = h_1(\omega) - T_1(\omega) y_1 - U_1(\omega) z \quad (45e)$$

with

$$E(y_1, z, \lambda) = \begin{pmatrix} A_1 y + B_1 z + f_1 \\ A_{2i} y + B_{2i} z + f_{2i}, & i \notin I_C^s \\ -\lambda_i & i \in I_L^s \end{pmatrix}$$

$$e(y_1, z, \lambda) = \begin{pmatrix} A_3 y_1 + B_3 z + C_3 \lambda + f_3 \\ A_{2i} y + B_{2i} z + f_{2i}, & i \in I_C^s \\ \lambda_i & i \notin I_L^s \end{pmatrix}$$

The algorithm has been implemented in MATLAB utilising the optimisation toolbox. In order to decrease the computation time we employed two types of iteration steps. In a normal step only one observation of the random data is utilised for the calculations. At regular intervals, a control step is performed utilising a sufficiently large sample of observations. Furthermore at such a step the step sizes are adjusted, either automatically or interactively. In the first case the step sizes are calculated according to a rule satisfying the conditions (32). An interactive step size adjustment allows the user to revise the step size according to his observations of the progress of the iteration. With this strategy the step sizes can

reach arbitrarily small values but do they not approach zero such that only the vicinity of the optimal solution can be reached. Therefore an interactive step size selection can be utilised as an indicator for a good automatic step size strategy. The convergence test performed in the iteration evaluates the behaviour of the estimation \bar{F}^k over the previous three iteration steps.

Generally, the iterates show the following behaviour which is typical for SQG methods: after a period with heavy oscillations the vicinity of the optimal solution is reached quite fast. From that point on the approximation improves only slowly, small oscillations in the vicinity of the optimum persist. More specific, four different sections can be distinguished in our implementation. At first the iterates oscillate heavily around two clusters relatively far away from the optimum, possibly some periods with stable objective function values exist. In the second period, the oscillations shift slowly towards a further cluster in the vicinity of the optimum. A short period of consolidation follows. The variance of the oscillations decreases rapidly and the iterates concentrate more and more in the vicinity of the optimal solution. Finally, the iterates oscillate in the vicinity of the optimum. Especially due to the behaviour of the iterates in the first two periods the estimation \hat{F} of the objective function converges only very slowly. Typically the iteration terminates because a predefined number of iteration steps was reached. The optimality conditions stated in Proposition 4 are often not satisfied and the existence of feasible ascent directions is analysed. Here, a reasonable relaxation of this test is appropriate in order to identify points in a close vicinity of an optimum.

The numerical experiments were conducted with the following specifications. The step sizes were determined according to the rules

$$\alpha_x^k = \frac{C_1}{C_3 + s}, \quad \alpha_u^k = \frac{C_2}{C_4 + s^\gamma} \quad (46)$$

Here k denotes the number of the current iteration step whereas s is the iteration step at which the previous control step was performed. We used a regular review interval of 10 steps such that $s = \lceil k/10 \rceil$, the greatest integer which is smaller than or equal to $k/10$. Furthermore we chose $C_1 = C_2 = 0.1$, $C_3 = C_4 = 1$ and $\gamma = 0.9$. A stationary point \bar{x}^s was identified as optimal when it was within a vicinity of 0.02 % of the actual optimum. We studied uniformly distributed

random data with slight, moderate and high stochasticity. The variances of these data were respectively 0.1 – 1 %, 0.5 – 5 % and 10 – 40 % of the mean values.

Two sets of experiments were performed. At first the segment $I_C^s = \{2, 5, 7\}$ was studied. It is found when for example the initial points $y_1^0 = (0, 3.6, 0)$ or $y_1^0 = (5.2, 6.2, 0)$ are chosen. On this segment the deterministic problem has a local optimum at $y_{1D}^* = (2.533, 4.867, 0)$ with the recourse decision is $y_{2D}^* = 0$, the follower's response $z_D^* = (3.867, 0, 0)$ and the optimal value $F_D^* = 2233.3057$. Here, the step size strategy (46) proved quite efficient. In the case of slightly stochastic data the periods with heavy oscillations were small and a vicinity of 2 % of the optimal solution was reached already after approximately 100 iteration steps and a precision of 0.5 % after further 20 steps. However, even after additional 150 steps the approximation did not increase significantly, the iterates where in a vicinity of about 0.3 % of the optimum. A similar behaviour of the iterates can be observed in the case of more random data. Table 1 compares iterates obtained on this segment using highly stochastic and deterministic data. However, the second set of experiments shows that the good performance of rule (46) can not be generalised. Choosing the initial point $y_1^0 = (0.83, 0, 210)$, the iteration is conducted on the segment $I_C^s = \{2, 7\}$. The local optimum of the deterministic problem is then $y_{1D}^* = (2.533, 0.867, 0)$ with the recourse decision $y_{2D}^* = 0$, the follower's response $z_D^* = (1.867, 1000, 0)$ and the objective value $F_D^* = 1333.297$. Here, the strategy (46) shows a weaker performance. After a few, large initial oscillations a long period of about 300 steps with quite stable iterates follows. During the next 100 steps the vicinity of the optimum is approached with only a few oscillations. Finally, the iterates oscillate in the vicinity of the optimum. For the case of low stochasticity, Figure 1 depicts a typical behaviour of the iterates for this strategy on both segments.

Alternatively, the step size rule

$$\alpha_x^k = \frac{C_1}{C_3^s}, \quad \alpha_u^k = \frac{C_2}{C_4}, \quad k \in [C_3^{s-1}, C_3^s] \quad (47)$$

was tested with the parameters $C_1 = C_2 = 0.25$, $C_3 = 2$, $C_4 = 1.9$. Now, the algorithm performs slightly better for the initial point $y_1^0 = (0.83, 0, 210)$, but with the initial point $y_1^0 = (0, 3.6, 0)$ the performance is worse (see Figure 2).

This observation underlines that a step size strategy which performs equally well

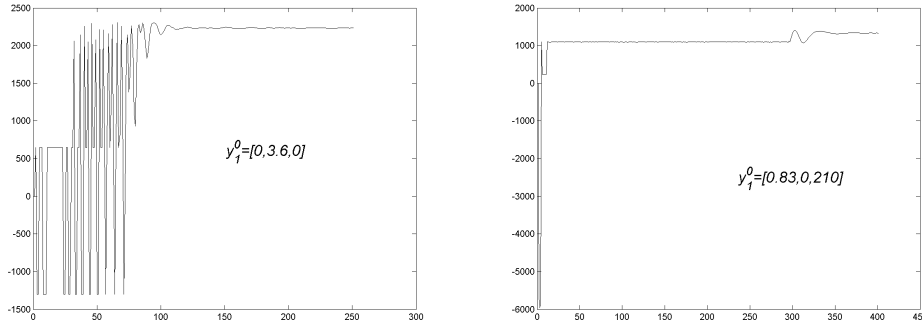


Figure 1: Estimated values of objective function, step size rule (46)

for all problems can hardly be found. Rather, at first the algorithm should be run tentatively in interactive mode in order to obtain a conjecture for a good automatic strategy. Such an automatic strategy can for example be chosen from a toolbox containing several alternatives.

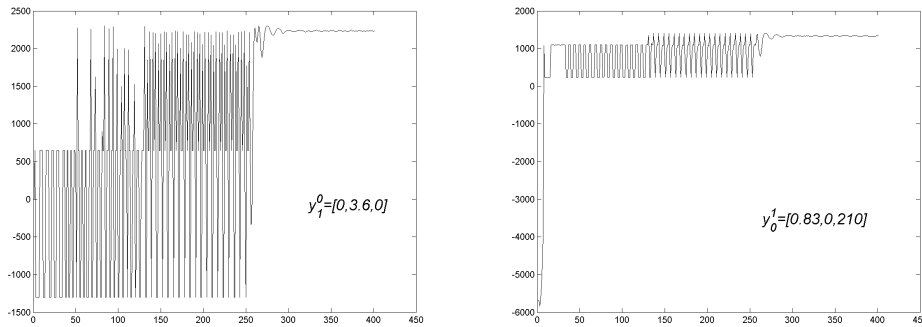


Figure 2: Estimated values of objective function; step size rule (47)

Table 1: Heavily stochastic and deterministic data, $y_1^0 = (0, 3.6, 0)$

k	heavily stochastic data			deterministic data		
	y_1	\hat{F}	F^s	y_1	\hat{F}	F^s
1	(0.00, 3.60, 0)	650.00	650.00	(0.00, 3.60, 0)	650.00	650.00
2	(5.20, 6.20, 0)	-325.00	-1300.00	(5.20, 6.20, 0)	-325.00	-1300.00
3	(5.20, 6.20, 0)	-650.00	-1300.00	(5.20, 6.20, 0)	-650.00	-1300.00
4	(-0.00, 3.60, 0)	-325.00	650.00	(-0.00, 3.60, 0)	-325.00	650.00
5	(-0.00, 3.60, 0)	-260.63	-3.16	(-0.00, 3.60, 0)	-130.00	650.00
6	(-0.00, 3.60, 0)	-108.86	650.00	(-0.00, 3.60, 0)	0.00	650.00
7	(5.20, 6.20, 0)	-279.02	-1300.00	(5.20, 6.20, 0)	-185.71	-1300.00
8	(5.20, 6.20, 0)	-406.65	-1300.00	(5.20, 6.20, 0)	-325.00	-1300.00
9	(5.20, 6.20, 0)	-505.91	-1300.00	(5.20, 6.20, 0)	-433.33	-1300.00
10	(-0.00, 3.60, 0)	-409.86	454.59	(-0.00, 3.60, 0)	-325.00	650.00
100	(3.65, 5.42, 0)	560.52	1408.30	(2.72, 4.96, 0)	898.32	2159.00
150	(3.89, 5.55, 0)	1030.89	1096.62	(2.53, 4.87, 0)	1343.40	2233.79
250	(2.99, 5.09, 0)	1439.17	2008.21	(2.53, 4.87, 0)	1699.36	2233.31
300	(2.78, 4.99, 0)	1561.57	2131.30	(2.53, 4.87, 0)	1788.35	2233.31
350	(2.40, 4.80, 0)	1655.28	2269.37	(2.53, 4.87, 0)	1851.92	2233.31
500	(2.83, 5.01, 0)	1815.00	2104.29	(2.53, 4.87, 0)	1966.33	2233.31
1000	(2.58, 4.89, 0)	2007.75	2216.37	(2.53, 4.87, 0)	2099.82	2233.31
1500	(2.87, 5.03, 0)	2077.38	2081.52	(2.53, 4.87, 0)	2144.32	2233.31
2000	(2.36, 4.78, 0)	2114.53	2278.31	(2.53, 4.87, 0)	2166.56	2233.31

6 Conclusions

We studied a deterministic and several formulations of stochastic programming problems with bilevel structure where connecting upper level constraints are present. Necessary optimality conditions of Fritz John type were stated. Furthermore we developed an algorithm for the solution of the bilevel (stochastic) programming problems. This required a two-stage solution process due to the possible nonconvexity of the inducible region caused by the connecting upper level constraints. We proved that, under certain conditions on the involved functions, the represented solution algorithm yields a local optimal solution of the studied problems. Numerical experiments testify to a reasonable numerical efficiency of the proposed approach.

Future research may include more complex multiperiod problems. For example, the follower's second stage problem may represent a recourse problem instead of the two-stage relationship implemented now. Another conceivable extension takes into account that the leader's perception of the follower's decision process may be imperfect. This means that the leader may obtain certainty about the actually implemented response only at the end of the first stage. Such a consideration of the uncertainty about the lower level decision process is especially important for the analysis of agency problems.

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Paper 3

Utilisation of stochastic programming methods in the
analysis of agency problems

A.S. Werner and A.A. Gaivoronski

Utilisation of stochastic programming methods in the analysis of agency problems

A.S. Werner*

A.A. Gaivoronski

Adrian.Werner@iot.ntnu.no

Alexei.Gaivoronski@iot.ntnu.no

Abstract

We study the application of a stochastic programming framework to the analysis of agency problems. To be more specific, we consider an agency model in a regulated telecommunication environment. This model consists of a regulator or license allocator (the principal) and of a service provider or licensee (the agent) but our results are applicable also to more general agency models. We demonstrate that the utilisation of a stochastic programming framework can help to derive parameters of an incentive schedule inducing the agent to follow regulation or licensing obligations imposed by the principal.

1 Introduction

The analysis of hierarchical relationships between actors in an industrial environment received a great deal of attention in the recent years. Typically, the focus has been on economic aspects and mechanisms and the feature of uncertainty was not regarded adequately. However, the latter is an important feature of agency problems. The objective of this paper is to demonstrate analytical methods of stochastic programming for an adequate treatment of the uncertainty in these problems. Stochastic programming techniques were developed explicitly for decision problems under uncertainty and have found many applications for example

*Norwegian University of Science and Technology, N - 7491 Trondheim, Norway

in finance, telecommunications, production control or transportation. Principal agent relationships are widely studied in economic theory. They describe the social or economic interaction of two (or more) parties in a common environment. One party, called the principal, wants the other one, called the agent, to take certain actions. The actors possess individual utility functions and choose their actions in order to maximise their expected utility. However, the agent's action affects not only her own but also the principal's utility function. Therefore the principal must find an incentive schedule controlling the agent's choice of an action such that it is favourable for the principal. Agency problems occur in a broad field of applications. Insurance theory assumes that the insured's level of caution can not be observed by the insurer; often this issue is studied together with the problem of moral hazard [AS91, RS76, SZ71]. In innovation or employment processes firm owners may not be able to observe the effort researchers or employees exert [AT94, Gua03, HH82, Hol99, Mir76]. Investors are often assumed to have only limited or no investment information. Therefore they may hire an advisor obtaining this information such that the return of a portfolio is optimal. The investor may have imperfect knowledge about the effort the investment advisor applies and the effect of the advisor's effort on the portfolio return must be distinguished from general market effects [BH80, Gol92, Sta87]. An important topic which is considered in this paper is the application of agency theory to regulation issues. Basically, regulation denotes any type of interference of a government with the behaviour of industrial agents and can therefore have positive as well as negative implications. Regulatory methods may constrain the feasibility of the regulated firm's actions. At the same time often a redistribution of wealth is achieved. This can be observed in telecommunications for example in cross subsidisations from long distance to local calls, from urban to rural areas or from business users to domestic calls [HT01]. The field of studies concerning regulation is widespread. However, often only advice or quite general frameworks for regulation policies are given which requires further adjustments to the specifics of the industry sector under consideration. An agency relationship in a regulated telecom environment can be formulated as follows. In order to implement several regulatory goals the regulator develops an incentive schedule that shall induce the regulated firm to appropriate decisions. This schedule may be of a monetary

nature but also a "success indicator" such as reputation or ranking. The topic of licensing is closely connected with the regulation of a liberalised telecom environment. The provision of a right to use telecommunication infrastructure is often tied to technological or economic requirements such as to avoid disturbing interferences of the licensees. Therefore it involves several levels: the process of the license allocation itself, technical specifications, legal enforcement etc. Typically, licenses are allocated to service providers by market based, administrative or hybrid methods. The allocation process follows different guidelines such as stimulation of competition, consumer protection, encouragement of development and implementation of new technologies and services but also the generation of additional wealth for the regulator by cream skimming. Here, we focus on obligations connected with such guidelines.

In the case considered here as well as in many other applications the principal has often limited knowledge about the industry specifics, the agent's decision process and the decisions chosen by the agent. Often a full observation of the actions is either impossible or prohibitively costly. Therefore imperfect estimations must be used when the incentive schedule is designed. This information asymmetry is the source of moral hazard [Hol79, Hol99, Mir99]. The agent may be tempted to provide incorrect or incomplete information. The principal can not rely on the information provided by the agent and needs additional information in order to assess her behaviour. Hence, he starts a monitoring process that helps to reduce the uncertainty. However, this process is expensive and often the additionally observed signals are affected by other random factors. Consequently, compromises have to be found with regard to monitoring and agency costs as well as to the allocation of additional risk and the reduction of uncertainty. It is difficult, if not even impossible, to remove all uncertainty and a certain degree of imperfect information will always persist. This calls for the utilisation of specialised tools for an adequate treatment of the incomplete knowledge.

A framework dealing with imperfect information in decision processes is stochastic programming [BL97, EW88, KW94]. In this paper we apply it to agency models with the aim to increase the accuracy and flexibility of the decisions by exploiting the inherent uncertainty. This is especially important in environments with a high speed of technological and structural changes and a resulting high degree

of uncertainty such as the telecom industry. The invention and implementation of new technologies yield new market structures and providers which again raises new requirements on the regulation policy. The stochastic programming formulation has the capability to deal with an uncertain future, an important issue in the study of agency models placed in a telecommunication environment. Dynamic aspects such as uncertainty in the evaluation of future investments or the future development of customer demand should not be ignored. Another source of uncertainty are inevitably modelling simplifications of the real life relationships between the decision makers. In our work we demonstrate the utilisation of stochastic programming concepts for the derivation of an incentive schedule under different aspects of imperfect information.

Laffont [Laf94] recognised that the problem of regulation is essentially a special case of a decision problem under incomplete information. More exactly, the interrelations between the principal and the agent can be described using concepts of bilevel programming such that agency problems represent bilevel programming problems or Stackelberg games under uncertainty. In this formulation the principal is interpreted as leader and the agent as follower. However, a major difference is that the principal has no or only partial information about the agent's decisions and her decision process. He can only observe the agent's utility as the outcome of this process. Mirrlees [Mir99] underlines the formulation of the agency model as a bilevel programming problem and the implications of this viewpoint. He conducts a fundamental analysis concerning the mathematical structure of agency models in insurance, focussing on the uncertainty of the principal about the agent's decision process. However, although published first recently, the study was completed already in 1975 and well developed bilevel and stochastic programming approaches were not yet available then.

Most of the agency models studied in the literature up to now include no or only a few simple constraints. Then typically first order optimality conditions on the agent's optimal decisions can be employed. This is essentially a reformulation of the bilevel programming problem to a one level nonlinear programming problem which can be analysed easily. However, in order to study more realistic problems it is frequently necessary to control the feasibility of the principal's and especially of the agent's decisions by more complex constraints. Then an analysis of the

one-level programming problem becomes complicated. At this point, the utilisation of bilevel programming concepts can help.

In the analysis of agency models often little effort is dedicated to an adequate treatment of the analytical features of the problems, for example as decision problems under uncertainty. Usually the initial formulations take into account the imperfect information, but subsequent discussions ignore more or less the implications of uncertainty and the uncertain parameters are typically replaced by their expectations. This, however, is not always recommendable. Although this deterministic equivalent formulation reduces the complexity of the model, such a rough treatment may decrease the value of the obtained insights drastically [Wal00, PW97]. With the progressing development of computer technology as well as of powerful algorithms the additional effort tied to an adequate treatment of the uncertainty became rather small. Therefore concepts of stochastic and bilevel programming can be utilised thoroughly.

The central topic of this paper is the treatment of the risk and uncertainty characterising agency models by a stochastic programming framework with bilevel features. This is demonstrated by means of an agency model based on a regularised telecom environment. We restricted ourselves to a model with one agent, considering this as a base for a possible future generalisation to the case of several agents. Section 2 gives some background on agency theory whereas Section 3 describes regulation and licensing with regard to telecom. We turn then over to the application of stochastic programming methods for the analysis of such agency models. First, Section 4 discusses the utilisation of stochastic programming concepts for the treatment of different types of uncertainty. Section 5 outlines then suitable solution approaches. Finally, Section 6 rounds up with conclusions.

2 Agency theory and monitoring

This section provides some background on agency theory focussing on analytical features such as a characterisation of incentive schedules and, especially, issues of uncertainty and monitoring.

Agency theory studies the interdependency of two or more actors with individual utility functions in a common environment. One actor, the principal, delegates

the task of decision making to the other actor, called the agent. Therefore both actors' utility functions depend on the agent's decisions a . But the interests of the principal and of the agent may conflict such that the agent's decision may not be according to the principal's objectives. Consequently, the principal seeks to induce the agent to make a decision which is in his interest. This is often realised by the help of an incentive fee or contract ϕ which is a function of the agent's action. Thus the agent maximises the utility U_A resulting from her own decision and from the according incentive fee whereas the principal maximises his utility U_P depending on the agent's action and the incentive for this choice. Both actors may face restrictions (of technical or other nature) on their decisions typically expressed by constraints. So far, the theoretical analysis was often simplified considerably by ignoring such constraints. Usually only a threshold value for the agent's utility is taken into account reflecting that otherwise the agent would withdraw from a participation in the according environment. In our exposition we will, however, assume explicitly the existence of constraints on both actors' decisions. This opens for the interpretation that the incentive schedule can influence the feasibility (in terms of constraint parameters) as well as the optimality (in terms of penalties or rewards included in the agent's utility function) of the agent's decisions.

The agency relationship can then be formulated in the following model.

$$\begin{aligned} \max_{\phi} U_P(a, \phi) \\ g_P(a, \phi) \geq 0 \end{aligned} \tag{1}$$

where the agent's decision a is an optimal solution of her problem

$$\begin{aligned} \max_a U_A(a, \phi) \\ g_A(a, \phi) \geq 0 \end{aligned} \tag{2}$$

This simple formulation indicates that the principal's incentive schedule may take on a number of different shapes comprising fees or penalties as well as constraints on the agent's decision. In addition to these explicit types also implicit incentives such as reputation (a good performance of the agent improves her future situation) or ratchet effects (discouraging the agent's effort) are conceivable. They become important especially in dynamic formulations of agency models [MV97].

Depending on the nature of the agent's decisions, explicit incentive schedules are often provided as vectors or linear functions affecting the agent's objective function. In Stark [Sta87] two frequent types of schedules and their effects on the agent's behaviour are analysed: bonus performance incentive schedules solely rewarding good performance and symmetric schedules additionally penalising bad performance.

A main feature in the analysis of agency problems is an incorporation of the existing fundamental uncertainty. It is assumed that both actors have imperfect knowledge about the environment. Usually, the parties agree on a contract and the agent decides on an action before the actual state of the environment becomes known. This uncertainty must be therefore taken into account in the process of decision making. Basically both parties may have different perceptions of the uncertain environment parameters. However, for the sake of simplicity it is often assumed that the subjective perceptions coincide although this simplification is again a source of imprecision. Furthermore the agent faces generally a disutility for the provision of information, be it in terms of effort, monetary terms or competitive advantage. Therefore in addition to a greater uncertainty about environment parameters the principal may lack information about the agent's decision process such that only parts of the decision or the outcome of the decision process are observable.

The principal may not be able to evaluate the agent's actions properly due to his imperfect knowledge. He can not distinguish clearly between effects of the agent's actions and effects of random environment events. This problem is referred to as moral hazard. In order to provide a correct incentive schedule the principal is forced to obtain additional information, for example by a monitoring process. Usually such a process causes costs depending on the monitoring intensity. On the other hand may the acquired information help to reduce the costs of the information asymmetry and to induce the agent to better performance. Therefore the principal seeks to determine also an optimal amount of monitoring. Ideally, monitoring does not constrain the agent's decision space but enables the principal to deduce information from already observed characteristics. It may also be designed as part of the incentive schedule, thus inducing the agent to reveal more data about her decision process or about the environment. The monitoring

process can be described as a learning process and becomes therefore especially important in a dynamic setting [Hol99].

Much attention was given to the problem of information asymmetry and monitoring processes in principal agent relationships. Harris and Raviv [HR79] investigated conditions when monitoring yields potential gains, especially to the principal. Guangzhou Hu [Gua03] studied a one-period agency model where the principal finds an optimal amount of monitoring in order to induce the agent to an optimal effort. An important paper concerned with the mathematical treatment of the principal's uncertainty about the agent's decision process is by Mirrlees [Mir99]. Additionally he observed that the study of first-order conditions may not be sufficient for an analysis of more general models. However, he had not yet at hand suitable methods of stochastic and bilevel optimisation in order to extend his considerations.

Mainly focussing on the hierarchical relationship between principal and agent, typically previous studies were not able to take properly into account the prevalent uncertainty. This is even more true when sophisticated agency models are considered. However, this problem can be resolved by concentrating foremost on the feature of uncertainty and first then, in a next step, taking into account the bilevel structure of the problem. The utilisation of the stochastic programming framework supports this proceeding and enables thus the analysis of models with a more complex structure. This approach will be illustrated in Sections 4 and 5.

3 Regulation and licensing in telecommunications

Regulation is becoming a major area of economics because in a world which has given up the debates between socialism and capitalism it is going to be the major battleground of the opposition between more or less governmental interference in economics, ... [Laf94]

This section highlights issues that are important for the study of an agency relationship between a regulator and a regulated firm in a telecommunication environment. Regulation describes any type of interference of a government with

the behaviour of industrial agents. It may therefore possess catalytic but also constraining effects on the economic development. This underlines the necessity of a carefully designed regulation policy and an analysis of all its effects on the considered industry sector. Also interactions with other fields of public economics should be evaluated since regulation is only one dimension of public economics. In telecommunications it is for example closely connected with licensing, taxation and other revenue generating issues.

With the liberalisation of the telecommunication sector in the previous decades the character and the purposes of regulation changed. Before the liberalisation started telecom services were provided in each country by a monopolist. The main purpose of regulation was then customer protection and the observation and control of this monopolist in order to prevent monopolistic behaviour. With progressing liberalisation the former monopoly is softened gradually and more competitors are allowed entry. This calls for an adaptation of the regulatory policy. On the one hand the entry of the new competitors is encouraged but also controlled. Theory states that generally the encouragement of competition in an oligopoly should be one of the main goals of a regulation policy, although physical factors such as high infrastructure costs prohibit an extension of the number of entrants beyond a certain measure. Regulation theory must deal with the behaviour of incumbents and entrants towards each other. On the other hand – since telecommunication services are a public good – customer protection against negative aspects of the competition is important. It includes universal access (compatibility), a minimum amount of service provision (for example in terms of coverage rate) or security and privacy issues. Furthermore, regulatory tools may encourage efficiency and fast implementation of new technology. Finally, when competition is sufficiently established, regulatory activities can be gradually decreased, limited to issues of customer protection. [HT01]

A result of the restructuring process in many countries was an extensive growth of the market for mobile services whereas the capabilities of fixed network services improved only slowly. Future development will be dominated by the Internet and related services. Due to the intrinsic character of the telecom sector the necessity of a national regulation authority and a telecom policy will persist also in the future [HSM03]. Among important issues of this development are questions

of multisector regulation, technology neutral regulation or convergence issues. Multisector regulation denotes the responsibility of a single regulator for diverse industry sectors, for example telecom, energy, water and transportation sectors, mainly due to effects of economies of scope. Another topic is technology neutral regulation. In the past, often different regulation policies were applied to fixed or mobile telecommunication networks or to broadcast networks. Present efforts, however, tend to an equal treatment of these infrastructures. A reason is the progressing convergence such that contents can be provided over different media. Here it is important to distinguish between the notions of technological and regulatory convergence. Technological convergence in telecommunications denotes several issues such as convergence of information technology and telecommunications and the according merging of technologies, fixed-mobile convergence or C4 convergence. This development demands an adaptation of regulation to the new conditions. It raises the widely discussed question of regulatory convergence, taking place in several ways such as mergers of existing regulation authorities or the establishment of completely new regulators. For a more extensive study of multisector regulation, technology neutral regulation or regulatory convergence confer for example Henten, Samarajiva and Melody [HSM03], of technological convergence see Audestad [Aud98].

A licensing process can help to impose regulatory goals such as network roll out, geographic or population coverage and development obligations in order to achieve political, social and economic objectives. Obligations on the license takers comprise efficient use of the spectrum, access to infrastructure and bottleneck facilities, pricing constraints or contributions towards universal service. [Lic01] Usually each license comprises certain frequency capacities and firms can acquire several licenses. Economic considerations together with physical spectrum constraints limit the number of licenses and thus the number of participating firms. Often licenses are allocated for a certain time period ahead, the license fee has to be paid either as a lump sum or with additional periodical fees and resale is prohibited. However, the impact of the fee on the licence takers' policy may be quite serious and restrictive. For example, their decisions about implemented technology and provided services will be motivated rather by short-term considerations in order to recover the paid license fee. Therefore a regulatory intervention

is necessary to direct the policy of the licensees also to long-term goals such as growth of the respective industry sector. [Lic01] However, here we do not want to go into detail about the licensing process and refer rather to a widespread literature such as Bauer [Bau01], Gruber [Gru02], Jehiel and Moldovanu [JM01] or Licensing of Third Generation (3G) Mobile: Briefing Paper [Lic01].

The relationship between a licensing regulator and a license taking service provider in the telecommunication sector constitutes a typical agency problem. The regulator wants the service provider to follow certain policy guidelines tied to the provision of the license. However, the notions of an incentive schedule or fee and of a license fee cannot be equated directly. An *incentive fee* is transferred by the principal to the agent as a reward or a penalty after the agent implemented her decisions and after the principal has evaluated the agent's performance. A *license fee* is an entry fee paid by the agent to the principal usually once and prior to her decisions. Actually, it is necessary in order to implement the agent's decisions, because she is not allowed to provide service otherwise. An inclusion of the licensing process into the agency model is therefore possible by means of obligations the agent has to meet after acquiring a license. The principal may then control the satisfaction of the obligations by an incentive schedule comprising constraints, penalties, rewards or even the withdrawal of the license.

In this agency relationship uncertainty exists at several levels. The telecom sector constitutes a highly dynamic and uncertain environment with a complex and continuously changing structure. Typically, both decision makers may have only imperfect knowledge about vital features such as future technology development, costs or customer behaviour but also the behaviour of other actors. Due to its situation the regulated firm has typically a better knowledge about the mechanisms in the environment than the regulator. Furthermore the regulator lacks exact information about the licensee's characteristics such that an evaluation of the agent's performance or of the satisfaction of license obligations is complicated.

A monitoring process can help to remove part of the information asymmetry. However, it is difficult due to the described features of a modern telecom environment which complicate a thorough analysis of observed data. The provision of a license may imply an obligation to report key data about the agent's or the considered sector's characteristics. But such an obligation raises again a question

of credibility of the obtained information, especially when provided by an actor who may benefit from a provision of biased data. Another measure may be the utilisation of benchmarks by comparing the considered agent's performance or the provided information with similar agents or with insights obtained by general information and theories. However, in a modern telecom environment typically the providers have quite different specifications and suitable benchmarks are hard to obtain. Even the identification of a specific sector may be difficult since often providers may operate across sector bounds and product differentiation is easily possible. Due to this often highly individualistic environment the collection of reliable information for the construction of a benchmark is complicated. Possibly the studied characteristics can be split up into single components with individual benchmarks, for example by evaluating past performance under comparable conditions or providers with similar characteristics.

Again stochastic programming concepts represent important tools for the analysis of the existing uncertainty and of monitoring processes. Dynamic learning and adaptation procedures under uncertainty suggest the application of multistage stochastic programming problems which will be discussed in the subsequent sections. For this purpose we selected typical, yet tractable, examples demonstrating the potential of these concepts.

4 Stochastic programming formulations of agency problems

This section motivates the utilisation of stochastic programming techniques in the analysis of agency problems. We specify model (1) – (2) in the spirit of the preceding exposition and indicate how to treat the inherent uncertainty.

Agency models represent a variant of bilevel programming problems or Stackelberg games. This type of decision problems is quite intricate already in a deterministic version. Taking moreover into account the stochastic nature of several model parameters, especially the partial inobservability of the agent's decisions, the considered problem is highly complex. However, an adequate treatment of the uncertainty is very important. Therefore an interpretation of these models

as decision problems under uncertainty is more suitable. Such problems can be formulated and solved utilising the methodology of stochastic programming. The advances made on the fields of stochastic programming and bilevel programming may contribute to a more serious utilisation of stochastic programming methods in the area of agency theory and to a more sophisticated study of the complex relationships.

Problems containing uncertain parameters are not well defined and their evaluation is therefore difficult. Since the incentive schedule and the agent's response must be found before the actual realisations of the random parameters are known, the meaning of neither the constraints nor the optimality of the objective function(s) is clear at this point of time. It is therefore important to start with a clear characterisation of the random elements. In an agency model two main types can be distinguished, the uncertainty of both actors about environment parameters that realise at the current time period and the uncertainty of the principal about the agent's decision. Especially in the case of very limited information about the agent it is important to distinguish effects which can be attributed to the agent's behaviour from effects caused by the environment state. A "good" quality of service may for example be achieved due to efforts of the service provider such as capacity extension or implementation of more effective transfer technology. It may, however, also be caused by a decreased user demand for services from that provider due to the entrance of other providers or the migration of users to other services.

Sources of uncertainty about the environment may be technological innovation, uncertain demand due to unpredictable user response on new services, quality of service issues (failures etc.). The uncertainty tied to these parameters can be expressed by the help of random variables, say $\omega \in \Omega \subseteq \mathbb{R}^p$, with a known or estimated probability distribution. In the following we demonstrate several formulations regarding an agency relationship between regulator and service provider. We denote the agent's decision by $a \in \mathbb{R}^m$ and assume that the principal's objective is the maximisation of his expected utility composed of the social welfare W_S and the incentive schedule $\phi \in \mathbb{R}^n$. This schedule can be expressed by help of a vector but also, more generally, as a function. The social welfare consists of the customers' welfare W_C and the agent's welfare W_A . The exact values of these

entities depend on uncertain parameters ω such that

$$W_S(a, \phi, \omega) = W_C(a, \omega) + W_A(a, \phi, \omega)$$

or more generally

$$W_S(a, \phi, \lambda, \omega) = \lambda W_C(a, \omega) + (1 - \lambda) W_A(a, \phi, \omega)$$

where the weight $\lambda \in [0, 1]$ represents the importance which the principal attaches to the customers' welfare W_C in comparison to the agent's welfare W_A . In the second variant the principal's task may additionally comprise the determination of an optimal composition of the social welfare. However, this case can be treated similarly to the first version and we do not consider it further here.

The agent maximises the expected utility from her welfare and the according incentive. Her problem is therefore to find an optimal action a taking into account the incentive schedule ϕ whereas the principal wants to determine a schedule ϕ that maximises his expected utility taking into account the agent's reaction. This yields the regulator's decision problem

$$\max_{\phi \in \Phi(a, \omega)} \mathbf{E}_\omega U_P(W_S(a, \phi, \omega), a, \phi, \omega) \quad (3)$$

where the agent's decision a is obtained as optimal solution of the regulator's perception of the agent's problem

$$\max_{a \in A(\phi, \omega)} \mathbf{E}_\omega U_A(W_A(a, \phi, \omega), a, \phi, \omega) \quad (4)$$

The sets $\Phi(a, \omega)$ and $A(\phi, \omega)$ denote the sets of feasible decisions of the principal and the agent, respectively. Typically, they comprise both deterministic and stochastic constraints such that the feasibility of the decisions depends also on the actually realised values of the random parameters. We will describe these sets in more detail below.

Often a regulation policy can be expressed in the form of rules. Some of these rules may have a concrete character such as requirements of a minimum amount of service provision or maximal market share. They can be formulated in the shape of constraints on the provider's decision problem thus influencing the feasibility of her decisions. Other rules may be less concrete, for example inducing a behaviour that is as good as possible. In this case the decisions of the agent may

be controlled by way of penalties or rewards including tax breaks or subsidies, possibly with selective properties such as to encourage investments into unpopular areas. These penalties or rewards affect the agent's utility function and thus the optimality of her decisions.

Consequently, the agent's problem may contain constraints which are determined by the regulator as part of the incentive schedule and which depend on both the agent's decision a and the incentive ϕ . Furthermore the problem comprises typically constraints that are independent of the incentive. Such constraints may concern for example the agent's technology or environment conditions. Also the regulator faces restrictions (of technical or other nature) on components of the incentive schedule. They include the participation constraint $\mathbf{E}_\omega W_A(a, \phi, \omega) \geq W_0$ expressing a withdrawal of the agent if the expected welfare is too low.

Generally, the set of feasible decisions of the regulator can be described by

$$\Phi(a, \omega) = \{\phi | g_P(a, \phi, \omega) \geq 0\}$$

and likewise the set of feasible decisions of the agent by

$$A(\phi, \omega) = \{a | g_A(a, \phi, \omega) \geq 0\}$$

These sets may comprise deterministic and stochastic constraints. The satisfaction of the stochastic constraints involving random parameters ω is contingent on the exact realisation of these parameters. Dependent on the meaning of the constraints several deterministic equivalent formulations are conceivable. The concerning constraint may be satisfied on average, such as coverage rates or certain quality of service requirements. This results in the deterministic equivalent formulation

$$\mathbf{E}_\omega g(a, \phi, \omega) \geq 0 \tag{5}$$

Reliability requirements or coverage issues demand a satisfaction with a given minimal probability α . An according deterministic equivalent formulation is

$$\mathbf{P}\{\omega | g(a, \phi, \omega) \geq 0\} \geq \alpha \tag{6}$$

The satisfaction of other constraints may be required for any realisation of the random variables. This comprises modelling requirements necessary to establish

a concise model of the reality such as nonnegative customer numbers or capacity constraints. Possibly the realisations of the random variables $\omega \in \Omega$ can be expressed by a finite set of scenarios $(\omega_1, \dots, \omega_N)$, obtained for example by observations or by a discretisation of the random variables. Then the following deterministic equivalent formulation can be utilised

$$\begin{aligned} g(a, \phi, \omega_1) &\geq 0 \\ &\vdots \\ g(a, \phi, \omega_N) &\geq 0 \end{aligned} \tag{7}$$

For continuously distributed random variables this type can be interpreted as a special case of formulation (6) with $\alpha = 1$.

An alternative approach is to consider the loss arising from a violation of the random constraints. This loss is determined at a second stage after the values of the random variables realised and is then included in the actor's objective function as a recourse function. Thus the optimality of a solution is balanced against its feasibility. In the agent's problem the loss may be considered as a part of the incentive schedule and includes therefore rewards for a satisfaction of certain constraints. This reflects for example tax breaks. Such a formulation may have the following shape:

$$\max_{\hat{a}} \mathbf{E}_{\omega} \{U_P(W_S(\hat{a}, \phi, \omega) - Q_P(\hat{a}, \phi, \omega), \hat{a}, \phi, \omega)\} \tag{8}$$

$$\hat{a} \in \arg \max_a \mathbf{E}_{\omega} \{U_A(W_A(a, \phi, \omega) - Q_A(a, \phi, \omega), a, \phi, \omega)\} \tag{9}$$

with the recourse functions

$$Q_P(a, \phi, \omega) = h_P(g_i^{P+}(a, \phi, \omega), i = 1, \dots, n^P) \tag{10}$$

$$Q_A(a, \phi, \omega) = h_A(g_j^{A+}(a, \phi, \omega), j = 1, \dots, n^A) \tag{11}$$

where

$$g_{P,i}^+(a, \phi, \omega) = \max\{0, g_{P,i}(a, \phi, \omega)\}, \quad i = 1, \dots, n^P \tag{12a}$$

$$g_{A,j}^+(a, \phi, \omega) = \max\{0, g_{A,j}(a, \phi, \omega)\}, \quad j = 1, \dots, n^A \tag{12b}$$

By spirit this problem is an extension of the stochastic programming problem with simple recourse to a bilevel formulation. However, in the model (8) – (12)

the recourse functions Q_P and Q_A represent only penalties (or rewards) for a violation of the respective constraints. We refrained in this formulation from explicit recourse decisions which, after observing the realised values of the random variables ω , may compensate for a violation of the stochastic constraints and correct the initial decisions ϕ and a , respectively. In such a case the functions Q_P and Q_A would include the costs of these actions.

Imperfect knowledge of the principal about the agent's decision can basically be treated similarly to the uncertainty about environment parameters but the analysis is more elaborate. Generally, the regulator has no insight into the agent's decision process. Often he has also imperfect knowledge about her actually implemented decision. However, it can be assumed that the principal has a certain conjecture about the agent's decision process, obtained by theoretic analysis or by observations. Additionally the agent may have committed to report key data. The formulation of the agent's problem (4) reflects thus the principal's perception of the agent's decision process. Recognising that this formulation is imprecise he utilises an estimation, say $b(\eta) = a + \eta$, of the decision actually implemented by the agent instead of the value a obtained as solution of his formulation of the agent's decision problem. The random parameter η reflects the principal's uncertainty about the quality of his information and its distribution function is assumed to be known. Then the problem to be solved by the principal is

$$\max_{\phi \in \Phi(b(\eta), \omega)} \mathbf{E}_\omega \mathbf{E}_\eta U_P(W_S(b(\eta), \phi, \omega), b(\eta), \phi, \omega) \quad (13)$$

whereas his perception of the agent's problem is represented by problem (4).

An incentive schedule may include the evaluation of the agent's performance by the help of benchmarks. As benchmarks serve for example the performance of comparable providers or characteristics deduced from observations or theoretical considerations. An example is a licensing process where the licence taker's actual performance is evaluated after some time. Assume that the regulator has the possibility to evaluate the agent's behaviour at a second stage after the random state of the environment became known. He compares then the characteristics $X(a, \omega)$ resulting from the agent's decision a against benchmarks $X_P(\omega)$. In this process he takes into account the realised state ω of the environment and imposes a penalty for deviations from the benchmark which is included into the principal's

incentive schedule. The resulting model exhibits again the shape of a stochastic programming problem with recourse, but the bilevel structure is different than in problem (8) – (12).

The regulator maximises his expected utility from the social welfare, taking into account the incentive schedules ϕ_1 and ϕ_2 implemented in the first and second stages (i.e. before and after the random state of the environment realises).

$$\begin{aligned} \max_{\phi_1, \phi_2} \mathbf{E}_\omega U_P(W_S(a, \phi_1, \omega), a, \phi_1, \phi_2, \omega) \\ \phi_1 \in \Phi_1(a, \omega) \\ \phi_2 \in \Phi_2(a, \omega) \end{aligned} \quad (14)$$

The agent finds a decision a such that her expected utility from her own welfare is maximised taking into account the penalty for deviations from the benchmark,

$$\begin{aligned} \max_a \mathbf{E}_\omega U_A(W_A(a, \phi_1, \omega), Q(a, \phi_2, \omega), a, \phi_1, \omega) \\ a \in A(\phi_1, \omega) \end{aligned} \quad (15)$$

where the penalty $Q(a, \phi_2, \omega)$ is determined when the random variable ω becomes known.

$$\begin{aligned} Q(a, \phi_2, \omega) = \min_y \phi_2 y \\ T(\omega)y = X_P(\omega) - X(a, \omega) \end{aligned} \quad (16)$$

In this formulation the penalty function was assumed linear in the deviation, but also nonlinear penalty functions are conceivable. Furthermore, problem (16) can be designed as penalty for bad behaviour (i.e. a deviation $X(a, \omega) < X_P(\omega)$) or as reward for good behaviour.

Monitoring provides additional information. It helps to increase the precision of the regulator's information and so to improve his incentive schedule. However, generally the costs connected with a monitoring process are increasing with the dedicated monitoring effort and the additionally obtained information. Therefore the principal must determine an optimal amount of monitoring in addition to the optimal incentive schedule. A model including a monitoring process can be formulated as follows. Assume that the regulator has a perception of the agent's

decision problem

$$\begin{aligned} & \max_b \mathbf{E}_\omega U_A(W_A, b, \phi, \omega) \\ & b \in B(\phi, \omega) \end{aligned} \quad (17)$$

However, he knows that the obtained solution b is an imperfect estimation of the actually implemented decision a :

$$b = a + \eta \quad \text{with} \quad \eta \sim N(0, \delta^2)$$

The estimation b represents the basis for the incentive fee $\phi(b) = \phi(a + \eta)$ transferred from the regulator to the agent. The purpose of this fee, however, is to control the actually implemented decision a such that the social welfare $W_S(a, \phi, \omega)$ is maximised. The principal starts therefore a monitoring process that, dependent on the monitoring intensity θ , reduces the variance of the noise or estimation error η :

$$\delta^2 = \theta \bar{\delta}^2$$

where $\bar{\delta}^2$ denotes the inherent or original variance of η . The intensity $\theta \in (0, 1]$ is defined such that $\theta = 1$ when no monitoring takes place and higher intensity corresponds to lower values of θ . The monitoring causes costs $c(\theta)$ that are increasing with the intensity and diminish the principal's utility. Concluding, the principal's decision problem can be formulated as

$$\begin{aligned} & \max_{\phi, \theta} \mathbf{E}_\omega \mathbf{E}_\eta U_P(W_S(a, \phi, \omega) - c(\theta), a, \phi, \eta, \omega) \\ & \phi \in \Phi(a, \eta, \omega) \\ & \theta \in (0, 1] \end{aligned} \quad (18)$$

A further aspect of imperfect knowledge of the regulator about the agent's decision process is the possible existence of several optimal responses of the agent to a given incentive schedule. In this case the regulator can not properly evaluate the schedule which induced these nonunique responses. He may then increase the monitoring intensity in order to obtain more precise indications about the agent's decisions. In the case of discrete responses they can be interpreted as possible scenarios of the agent's behaviour and analysed separately. Possibly the incentive

schedule can be refined in order to exclude nonunique responses. Alternatively, the regulator may assume that the agent will choose that decision which is least preferable for him and evaluate his problem for this choice. This method is an adaptation of the pessimistic concept of bilevel programming. An alternative approach is analysed in Werner and Gaivoronski [WGnga]. It takes into account the principal's uncertainty about the actually implemented decisions.

5 Solution approaches

This section discusses approaches for the solution of the described agency problems with focus on the uncertainty. Such problems represent stochastic programming problems with a bilevel feature. Due to the complex structure a nontrivial adaptation of known stochastic programming methods and the development of new methods tailored to this problem type is necessary. For this task the utilisation of concepts of stochastic programming as well as of bilevel programming is of interest. Different solution concepts are applicable depending on the actual formulation of the agency relationship. They can be classified into two main types. One class of approaches utilises a finite number of scenarios and an according construction of deterministic equivalents. The other class employs statistical methods that are capable of treating continuous distributions of the random variables directly.

Utilising a finite number of scenarios, a deterministic equivalent of the agency problem can be formulated, for example as indicated by formulations (3) – (7). Thus the stochastic agency problem is reduced to a (possibly large scale) deterministic bilevel programming problem. Such problems have been studied intensely in the past decades. Therefore we outline here only a few issues. Bilevel programming problems are not easily solvable since they often exhibit some unpleasant properties. Taking into account the agent's response, the principal's objective function is generally not convex and neither differentiable. The participation constraint represents a so-called connecting upper level constraint. It is located in the principal's subproblem but its feasibility depends also on the agent's response. As a consequence the set of feasible principal decisions may not be connected and not convex. Also a possible existence of nonunique responses

of the agent to some principal decisions complicates the evaluation of the principal's problem. In order to deal with these difficulties several concepts can be employed. Among these concepts are penalty methods in the case of connecting upper level constraints or the optimistic and pessimistic approaches for nonunique agent responses. A variety of solution methods for bilevel programming problems has been developed comprising among others descent algorithms [VSJ94], penalty approaches [IA92] or reformulations to one-level nonlinear programming problems [BM90, JF94]. For a broader overview see for example Dempe [Dem02] or the bibliographies [Dem03] and [VC94].

The identification of a finite number of scenarios or the discretisation of the random variables is often a rather arbitrary process. However, due to the bilevel structure small changes of model parameters may have great effects on an optimal response to a given regulatory decision and consequently on the optimality of an incentive schedule. Therefore the stability of a found optimal decision of the principal should be investigated. Sensitivity analysis helps to study the behaviour of the agent's response in dependence on small changes of the random model parameters [PW97]. Further considerations of bilevel programming problems with regard to uncertain model parameters can be found in Wynter [Wyn01].

The second class of solution approaches treats continuous distributions of the random variables directly and allows so for the development of flexible solution methods. This is achieved by the utilisation of statistical methods such as stochastic decomposition [HS96] or stochastic quasi-gradient (SQG) methods [Erm88, Gai88, Gai04]. Stochastic quasi-gradient methods are suitable for the solution of optimisation problems with complex objective functions and constraints. This makes them especially applicable to stochastic programming problems with a bilevel structure as represented by agency relationships. The following example based on the general formulations (3), (4) shall illustrate the main ideas and the potential of this methodology.

Example 5.1. Consider the principal agency relationship between a telecom service provider and a regulator. In this relationship the regulator takes on the role of a principal who wants the service provider (the agent) to choose decisions which comply with a certain regulation policy. We assume that both decision

makers are uncertain about parameters of the environment. This uncertainty is expressed by making these parameters dependent on random variables ω . We consider a one-stage relationship, i.e. the actors can not take into account possible corrective actions after observing the state of the environment.

The agent's decision $a = (a_1, a_2)$ consists of a price a_1 for service provided to customers and of an "effort" a_2 (for example Quality of Service). We assume that the agent's welfare depends only on the revenue from service provision and on the costs due to the chosen level of effort. Simplifying we assume that the agent's revenue from service provision depends linearly on the demand $d(\omega)$ for the service. This demand is not perfectly known.

$$R(a_1, \omega) = d(\omega)a_1$$

The costs due to the chosen effort depend quadratically on the effort.

$$E(a_2, \omega) = c(\omega)a_2^2$$

Hence the agent's welfare is

$$\begin{aligned} W_A(a, \omega) &= R(a_1, \omega) - E(a_2, \omega) \\ &= d(\omega)a_1 - c(\omega)a_2^2 \end{aligned}$$

The customers assess the provided service by means of the price a_1 and the quality a_2 . For the sake of simplicity also this dependence is assumed linear such that

$$S(\hat{a}, \omega) = s_1(\omega)\hat{a}_1 + s_2(\omega)\hat{a}_2$$

where \hat{a}_1, \hat{a}_2 denote the agent's decisions as perceived by the principal. Typically the coefficients are such that $s_1(\omega) \leq 0$ and $s_2(\omega) \geq 0$ for any ω . The customers' welfare is then

$$\begin{aligned} W_C(\hat{a}, \omega) &= S(\hat{a}, \omega) - R(\hat{a}, \omega) \\ &= (s_1(\omega) - d(\omega))\hat{a}_1 + s_2(\omega)\hat{a}_2 \end{aligned}$$

Hence the principal assumes that the social welfare $W_S(\hat{a}, \omega)$ generated by the agent's decisions $\hat{a} = (\hat{a}_1, \hat{a}_2)$ is

$$\begin{aligned} W_S(\hat{a}, \omega) &= W_C(\hat{a}, \omega) + W_A(\hat{a}, \omega) \\ &= s_1(\omega)\hat{a}_1 + s_2(\omega)\hat{a}_2 - c(\omega)\hat{a}_2^2 \end{aligned}$$

The principal's incentive schedule $\phi = (\phi_1, \phi_2)$ consists of both obligations and rewards. The obligations ϕ_1 control the agent's pricing decisions by defining an upper bound on the service price and are thus part of the agent's constraints. Also the principal's choice of the incentive schedule is subject to constraints. Thus the agent's set $A(\phi, \omega)$ of feasible decisions is

$$A(\phi, \omega) = \{a_1, a_2 \in \mathbb{R} : a_1 \geq \mathbf{E}_\omega c(\omega), a_1 \in [a_{1,L}, \phi_1], a_2 \in [a_{2,L}, a_{2,U}]\}$$

and the principal's set $\Phi(\hat{a}, \omega)$ of feasible decisions is

$$\begin{aligned} \Phi(\hat{a}, \omega) = \{ & \phi_1, \phi_2 \in \mathbb{R} : \phi_1 \in [\phi_{1,L}, \phi_{1,U}], \phi_2 \in [\phi_{2,L}, \phi_{2,U}], \\ & h_P(\omega) - n_1(\omega)\hat{a}_1 - n_2(\omega)\hat{a}_2 - m_1(\omega)\phi_1 - m_2(\omega)\phi_2 \geq 0\} \end{aligned}$$

The reward ϕ_2 encourages the agent's effort by affecting the agent's utility. Furthermore it influences also the principal's utility. Consequently, the agent's and the principal's utility functions are

$$\begin{aligned} U_A(a, \phi, \omega) &= W_A(a, \omega) + \phi_2 a_2 \\ &= d(\omega)a_1 - c(\omega)a_2^2 + \phi_2 a_2 \\ U_P(\hat{a}, \phi, \omega) &= W_S(\hat{a}, \omega) - \phi_2 \hat{a}_2 \\ &= s_1(\omega)\hat{a}_1 + (s_2(\omega) - \phi)\hat{a}_2 - c(\omega)\hat{a}_2^2 \end{aligned}$$

For the sake of simplicity we assume that the regulator's and the agent's decision problems contain only one linear constraint each. These constraints are required to be satisfied on average according to formulation (5). Moreover, we assume that the principal is perfectly informed about the agent's decision process such that $\hat{a} = a$. Assuming that both decision makers maximise their utility the following bilevel stochastic programming problem can be formulated.

$$\max_{\phi_1, \phi_2} \mathbf{E}_\omega \{s_1(\omega)a_1 + (s_2(\omega) - \phi_2)a_2 - c(\omega)a_2^2\} \quad (19a)$$

$$\mathbf{E}_\omega \{h_P(\omega) - n_1(\omega)a_1 - n_2(\omega)a_2 - m_1(\omega)\phi_1 - m_2(\omega)\phi_2\} \geq 0 \quad (19b)$$

$$\phi_1 \in [\phi_{1,L}, \phi_{1,U}]$$

$$\phi_2 \in [\phi_{2,L}, \phi_{2,U}]$$

where the agent's response $a = a(\phi, \omega)$ is obtained as the optimal solution of her decision problem

$$\begin{aligned} \max_{a_1, a_2} & \mathbf{E}_\omega \{d(\omega)a_1 + \phi_2 a_2 - c(\omega)a_2^2\} \\ & a_1 \geq \mathbf{E}_\omega c(\omega) \\ & a_1 \geq a_{1,L} \\ & a_1 \leq \phi_1 \\ & a_2 \in [a_{2,L}, a_{2,U}] \end{aligned} \quad (20)$$

This problem can now be solved by a stochastic quasi-gradient (SQG) method (see Appendix A.1). In Werner and Gaivoronski [WGngb] it was proved that this method converges with probability 1 to the vicinity of a local optimal solution. This means that an optimal incentive schedule $\phi^* = (\phi_1^*, \phi_2^*)$ of the regulator and at the same time an optimal response $a^* = a(\phi^*)$ of the agent can be obtained. Consider for example the following case. Assume that the agent has the prospect to receive financial allowances (from another authority than from the principal) subsidising his effort. Furthermore the value of the coefficient n_2 in constraint (19b) is not perfectly known. Therefore the principal distinguishes two scenarios and assumes that scenario 1

$$(s_1, s_2, c(\omega^1), d, h_P, m_1, m_2, n_1, n_2(\omega^1)) = (-5, 25, 7, 50, 150, 2, 3, 4, -100)$$

will realise with probability $p_1 = 2/3$ and scenario 2

$$(s_1, s_2, c(\omega^2), d, h_P, m_1, m_2, n_1, n_2(\omega^2)) = (-5, 25, -7, 50, 150, 2, 3, 4, 0)$$

with probability $p_2 = 1/3$. The bounds on the actors' decisions are

$$\begin{aligned} (\phi_1, \phi_2) & \in [0, 50] \times [0, 50] \\ (a_1, a_2) & \in [.5, \infty] \times [-3, 10] \end{aligned}$$

The negative lower limit on the regulator's reward ϕ_2 symbolises the possibility of a penalty for too low effort of the agent. The traditional approach utilising the expectation of the uncertain parameters

$$\begin{aligned} (s_1, s_2, \mathbf{E}_\omega c(\omega), d, h_P, m_1, m_2, n_1, \mathbf{E}_\omega n_2(\omega)) \\ = (-5, 25, 2.33, 50, 150, 2, 3, 4, -66.67) \end{aligned}$$

yields an optimal schedule $\phi^E = (\phi_1^E, \phi_2^E) = (2.33, 46.67)$ with the agent's response $a = (a_1^E, a_2^E) = (2.33, 10)$. If scenario 1 realises, this schedule will yield a utility

$$U_P(\omega^1, \phi^E) = -5a_1^E + (25 - \phi_2^E)a_2^E - 7(a_2^E)^2 = -928.33$$

whereas in the case of scenario 2 the principal's utility will be

$$U_P(\omega^2, \phi^E) = -5a_1^E + (25 - \phi_2^E)a_2^E + 7(a_2^E)^2 = 471.67$$

An application of the proposed SQG approach taking into account the scenarios ω_1 and ω_2 does not yield a unique schedule but a range of advices. This is due to the utilisation of a finite number of samples reflecting realisations of both scenarios. At each iteration step a new set of such samples is determined. Therefore only a vicinity of the optimal schedule is reached. Utilising a sample of 100 realisations the proposed algorithm determines utility maximising schedules in the range $\phi^S = (\phi_1^S, \phi_2^S) \in [1.8, 3.6] \times [30, 50]$ and responses $a^S = (a_1^S, a_2^S)$ with $a_1^S = \phi_1^S$ and $a_2^S \in [6, 10]$. An example is $\phi^S = (3.03, 34.13)$ with the response $a^S = (3.03, 9.38)$. With this schedule the principal obtains the following utilities under the single scenarios

$$U_P(\omega^1, \phi^S) = -5a_1^S + (25 - \phi_2^S)a_2^S - 7(a_2^S)^2 = -716.31$$

$$U_P(\omega^2, \phi^S) = -5a_1^S + (25 - \phi_2^S)a_2^S + 7(a_2^S)^2 = 514.73$$

For both scenarios these utilities are higher than the utilities obtained by applying the schedule ϕ^E .

This simple example illustrated some principles of the application of concepts of stochastic programming to agency problems. Here, the methodology of bilevel stochastic programming problems was utilised for the determination of an optimal incentive schedule in an agency relationship between a telecom regulator and a service provider. However, an extensive analysis of these concepts, for example with regard to an application to the more complex models of Section 4, is beyond the scope of this exposition. Partly it was considered in [WGngb], partly it will be the subject of further research.

6 Conclusions

In this exposition we studied the utilisation of stochastic programming concepts for the treatment of the uncertainty present in agency models. For this purpose a model of a principal agency relationship between a regulator and license provider and a regulated service provider and licensee in a liberalised telecom environment was considered. We restricted the studies to the case of one agent. However, the models were held quite general, providing the base for an extension to a multi-lateral formulation with several regulated agents.

After we provided some background on agency theory as well as on regulation and licensing in telecommunications we presented a framework for solving agency problems under uncertainty. It is based on concepts of stochastic and bilevel programming. This makes it possible to enhance the flexibility of agency theoretic studies since changes in the environment and other uncertain factors can be taken better into account. However, the purpose of this exposition was not to consider these analytical techniques as a substitution of qualitative economic considerations. Both branches should be conducted on an equal footing and stochastic optimisation approaches should rather be seen as an indispensable supplement. They allow for the study of more intricate models involving a considerable extent of uncertainty at several levels, complex types of objective functions and constraints on the choices of both actors' or model formulations with decisions spanning over several stages such as stochastic programming problems with recourse. Together with the economical analysis powerful interpretations of the obtained results and concepts can be achieved.

Further research may be organised along these lines, combining economic studies with the analytical framework in terms of bilevel stochastic programming concepts. It is for example more appropriate to represent effects caused by the presence of further agents by a multilateral formulation than by uncertain environment parameters. However, at the same time the interactions of the agents with the regulator and the customer population as well as possibly among each other complicate the model further. Therefore, as indicated in Section 4, an extension of the considerations to more complex models including multistage or multilateral formulations with dependent or independent agents is also of interest.

A Appendix

A.1 Algorithm for solution of Example 5.1

In the following we describe main details of an application of the proposed solution algorithm to Example 5.1. A thorough analysis of the method is given in Werner and Gaivoronski [WGngb].

An optimal response $a(\phi, \omega)$ of the agent on a schedule ϕ can be characterised by the Karush Kuhn Tucker optimality conditions of the agent's problem. Then the bilevel stochastic programming problem (19), (20) can be formulated as a nonlinear one-level stochastic programming problem. The decision variables of this problem consist of the actors' decision variables a and ϕ and of the Lagrange multipliers $\lambda = (\lambda_1, \dots, \lambda_5)$ associated to the response a .

$$\max_{a, \phi, \lambda} \mathbf{E}_\omega \{s_1(\omega)a_1 + (s_2(\omega) - \phi_2)a_2 - c(\omega)a_2^2\} \quad (21a)$$

$$\mathbf{E}_\omega \{d(\omega) + \lambda_1 - \lambda_2 + \lambda_5\} = 0 \quad (21b)$$

$$\mathbf{E}_\omega \{-2c(\omega)a_2 + \phi_2 + \lambda_3 - \lambda_4\} = 0 \quad (21c)$$

$$\mathbf{E}_\omega \{\lambda_i g_{A,i}\} = 0 \quad i = 1, \dots, 5 \quad (21d)$$

$$\mathbf{E}_\omega \{h_P(\omega) - n_1(\omega)a_1 - n_2(\omega)a_2 - m_1(\omega)\phi_1 - m_2(\omega)\phi_2\} \geq 0 \quad (21e)$$

$$g_A(a, \phi) \geq 0 \quad (21f)$$

$$\lambda_1, \dots, \lambda_5 \geq 0 \quad (21g)$$

$$\phi_1 \in [\phi_{1,L}, \phi_{1,U}]$$

$$\phi_2 \in [\phi_{2,L}, \phi_{2,U}]$$

where

$$g_A(a, \phi) = (a_1 - a_{1,L}, \phi_1 - a_1, a_2 - a_{2,L}, a_{2,U} - a_2, a_1 - \mathbf{E}_\omega c(\omega))^T$$

This problem (21) comprises the nonlinear equality constraint (21d). In order to deal with this difficulty the algorithm employs a decomposition of the problem into a family of subproblems with linear constraints. With a given initial point $x^0 = (a^0, \phi^0, \lambda^0)$ where a^0 is the optimal response to the decision ϕ^0 and $\lambda^0 = (\lambda_1^0, \dots, \lambda_5^0)$ the associated Lagrange multipliers, such a subproblem can be

formulated as follows.

$$\max_{a, \phi, \lambda} \mathbf{E}_\omega \{s_1(\omega)a_1 + (s_2(\omega) - \phi_2)a_2 - c(\omega)a_2^2\} \quad (22a)$$

$$\mathbf{E}_\omega f_1(a, \phi, \lambda, \omega) \geq 0 \quad (22b)$$

$$\mathbf{E}_\omega f_2(a, \phi, \lambda, \omega) = 0 \quad (22c)$$

where

$$f_1(a, \phi, \lambda, \omega) = \begin{pmatrix} g_{A,i}(a, \phi), & i \in \{i = 1, \dots, 5 : g_{A,i}(a^0, \phi^0) > 0\} \\ \lambda_i, & i \in \{i = 1, \dots, 5 : \lambda_i^0 > 0\} \\ g_P(a, \phi, \omega) \end{pmatrix}$$

$$f_2(a, \phi, \lambda, \omega) = \begin{pmatrix} g_{A,i}(a, \phi), & i \in \{i = 1, \dots, 5 : g_{A,i}(a^0, \phi^0) = 0\} \\ \lambda_i, & i \in \{i = 1, \dots, 5 : \lambda_i^0 = 0\} \\ d(\omega) + \lambda_1 - \lambda_2 + \lambda_5 \\ -2c(\omega)a_2 + \phi_2 + \lambda_3 - \lambda_4 \end{pmatrix}$$

$$g_P = \begin{pmatrix} h_P(\omega) - n_1(\omega)a_1 - n_2(\omega)a_2 - m_1(\omega)\phi_1 - m_2(\omega)\phi_2 \\ \phi_1 - \phi_{1,L} \\ \phi_{1,U} - \phi_1 \\ \phi_2 - \phi_{2,L} \\ \phi_{2,U} - \phi_2 \end{pmatrix}$$

This subproblem (22) is now solved iteratively by a SQG method. At each step k of this method new values for the iterates $x^k = (\phi_1^k, \phi_2^k, a_1^k, a_2^k, \lambda_1^k, \dots, \lambda_4^k)$, u^k and v^k are determined where x^k denotes the current iterates of the decision variables, u^k the Lagrange multipliers of the inequality constraints (22b) and v^k the Lagrange multipliers of the equality constraints (22c). These variables are updated according to the rules

$$x^{k+1} = \Pi_X(x^k - \alpha_x^k \zeta_x^k) \quad (23a)$$

$$u^{k+1} = \max\{0, u^k + \alpha_u^k \zeta_u^k\} \quad (23b)$$

$$v^{k+1} = v^k + \alpha_v^k \zeta_v^k \quad (23c)$$

The operator Π_X denotes the projection on the domain

$$X = \{x = (a, \phi, \lambda) \in \mathbb{R}^9 : \frac{1}{N_k} \sum_{i=1}^{N_k} f_1(x, \omega^i) \geq 0, \frac{1}{N_k} \sum_{i=1}^{N_k} f_2(x, \omega^i) = 0\}$$

where $(\omega^1, \dots, \omega^{N_k})$, $N_k > 0$, is a collection of independent samples according to the distribution of the random variable $\omega \in \Omega$.

The step sizes α_x^k , α_u^k and α_v^k can be determined for example according to

$$\alpha_x^k = \frac{C_1}{C_2 + k}, \quad \alpha_u^k = \frac{C_3}{C_4 + k^{\gamma_1}}, \quad \alpha_v^k = \frac{C_5}{C_6 + k^{\gamma_2}}$$

with nonnegative constants C_1, \dots, C_6 and $\gamma_1, \gamma_2 \in (0, 1)$.

For the determination of the current search directions ξ_x^k , ξ_u^k and ξ_v^k we utilise the Lagrangian function of problem (22) for the iterates x^k , u^k , v^k and given observation ω^i of the random parameters

$$\begin{aligned} L(x^k, u^k, v^k, \omega^i) = & -s_1(\omega^i)a_1^k - (s_2(\omega^i) - \phi_2^k)a_2^k + c(\omega^i)(a_2^k)^2 \\ & - (u^k)^T f_1(a^k, \phi^k, \lambda^k, \omega^i) + (v^k)^T f_2(a^k, \phi^k, \lambda^k, \omega^i) \end{aligned}$$

Then the search directions can be determined by means of statistical estimates of the gradients of this Lagrangian, for example by

$$\begin{aligned} \xi_x^k &= \frac{1}{N_k} \sum_{i=1}^{N_k} \nabla_x L(x^k, u^k, v^k, \omega^i) \\ \xi_u^k &= -\frac{1}{N_k} \sum_{i=1}^{N_k} f_1(a^k, \phi^k, \lambda^k, \omega^i) \\ \xi_v^k &= \frac{1}{N_k} \sum_{i=1}^{N_k} f_2(a^k, \phi^k, \lambda^k, \omega^i) \end{aligned}$$

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Paper 4

Influence of perturbed input data on convexity
properties of stochastic programming problems

A.S. Werner and A.A. Gaivoronski

Influence of perturbed input data on convexity properties of stochastic programming problems

A.S. Werner*

A.A. Gaivoronski

Adrian.Werner@iot.ntnu.no

Alexei.Gaivoronski@iot.ntnu.no

Abstract

We consider convex optimisation problems with nonunique minima and study the effect of uncertainty about the decision variables. Such a type of problems may occur for example in agency theory due to the information asymmetry between the principal and the agent. In the unconstrained case we state conditions on the uncertainty such that the perturbed function has a unique minimum. Under stricter assumptions we prove that this minimum converges to one of the original minima. In the presence of nontrivial constraints the uncertainty affects also the feasibility. We state conditions such that the perturbed problem has a unique minimum.

Key words: Strict convexity, bilevel stochastic programming, nonunique minima.

1 Introduction

In this paper we study how uncertainty about decision variables may affect convexity properties of optimisation problems. This problem evolved from the analysis of a specific class of stochastic programming problems where part of the uncertainty of the considered decision maker can be attributed to the response of another actor. In this sense the decisions of the former actor represent parameters which influence the other actor's decision process and vice versa. This situation

*Norwegian University of Science and Technology, N - 7491 Trondheim, Norway

can be modelled by a stochastic programming problem with bilevel structure. The considered decision maker finds a decision y solving the problem

$$\begin{aligned} & \min_y \mathbf{E}_\xi g(y, z^*, \xi) \\ \text{s.t.} \quad & \mathbf{E}_\xi H(y, z^*, \xi) \leq 0 \end{aligned} \tag{1}$$

taking into account the response z^* of the other actor. This response is determined as optimal solution of the problem

$$\begin{aligned} & \min_z \mathbf{E}_\xi f(y, z, \xi) \\ \text{s.t.} \quad & \mathbf{E}_\xi h(y, z, \xi) \leq 0 \end{aligned} \tag{2}$$

where the uncertain parameters are expressed by the help of a random variable $\xi \in \Xi$. Interpreting the former decision maker as principal or leader and the latter one as agent or follower, problem (1) – (2) can be analysed utilising concepts of bilevel programming in addition to stochastic programming methods.

Decision problems under uncertainty with a bilevel structure can be found in a variety of applications, for example in telecommunications [AGWng], energy and power management [GR02] and especially in agency theory [GH83, Mir99, WGng]. Typical for such models is the uncertainty of the principal about the agent. He has a perception of the agent's decision process but often he is uncertain about which decisions the agent actually implements. A further source of imperfect knowledge is the existence of nonunique responses. In this case the principal does not know exactly which decision the agent will choose even if her decision process may be perfectly known. This complicates the principal's decision process since he can not exactly evaluate the optimality and the feasibility of his decisions. Then in deterministic bilevel programming typically the so-called optimistic or pessimistic concepts are utilised, depending on the degree of control the principal can exert on the agent's choice.

In this exposition we follow a different approach. We investigate if the principal can improve his decision process (1) if he is aware of his imperfect knowledge about the agent and takes it explicitly into account. Much work has been directed to the analysis of stochastic programming problems when the distribution function of the random or uncertain parameters is not completely known

[AW96, DR98, RR02, Sch00, Sha93, Sha94]. These studies focus mainly on stability properties of the optimal solutions and of the solution function in a general setting. However, the effects of perturbations on the convexity properties of the problems and thus on quantitative properties of the solution set have not yet been analysed. Here, we study the effect of noisy decision variables of the agent on the convexity properties of her decision problem (2). Especially we focus on the uniqueness of the optimal solutions. To our knowledge no similar problem was studied so far.

We investigate the properties of the agent's decision problem (2) for a given principal or upper level decision. In the following we ignore therefore the dependency of this problem on the upper level parameter y . For the sake of transparency we assume furthermore that the model parameters are perfectly known. Consequently, our starting point is a convex deterministic function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with a set of nonunique minima

$$A = \arg \min_z f(z) \quad (3)$$

We assume then that the decision variables $z \in \mathbb{R}^n$ are affected with uncertainty expressed by a random variable $\omega \in B \subseteq \mathbb{R}^n$. This means that, instead of the original function $f(z)$, a perturbed function $F(z) = \mathbf{E}_\omega f(z + \omega)$ is minimised

$$\min_z F(z) = \min_z \mathbf{E}_\omega f(z + \omega) \quad (4)$$

Our focus is on conditions on the perturbation ω such that the minimum found in problem (4) is unique. However, the optimum z_p^* of the perturbed problem may be different from the original optima, $z_p^* \notin A$. In a further step we want to know therefore under which conditions the unique minimum of $F(z)$ converges to one of the nonunique minima of the original function. Finally, we consider problems (3) and (4) for the case when the decision variable z is subject to nontrivial constraints $g(z) \leq 0$. In this case the uncertainty about the decision variables affects also the feasible area of the considered problem.

The paper is structured as follows. The following section gives some notations and conventions needed for the further exposition. In Section 3 the unconstrained case is analysed and the main results of the work are presented. Section 4 considers the constrained case. Finally, Section 5 rounds up the paper with conclusions and indicates further research directions.

2 Preparatory material

This section provides definitions and conventions that are utilised throughout the subsequent analysis.

Considering a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote the i -th component of this vector by x_i . Furthermore we denote by e_i the i -th basis vector of \mathbb{R}^n .

We assume that the original function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ considered in our analysis is continuous and convex (see for example [Ber99]). To be more specific, we assume that the set A of the minima of f is bounded and has a nonempty interior. This means that f has nonunique minima and is therefore not strictly convex.

A convex function is differentiable almost everywhere. At the points of nondifferentiability directional derivatives can be utilised.

Definition 1. *The (one-sided) directional derivative of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ at a point $z \in \mathbb{R}^n$ in direction $r \in \mathbb{R}^n$ is defined by*

$$f'(z; r) = \lim_{t \rightarrow 0^+} \frac{f(z + tr) - f(z)}{t}$$

The directional derivatives in directions e_i and $-e_i$ for $i = 1, \dots, n$ are then given by

$$\begin{aligned} D_+^i f(z) &= f'(z; e_i) \\ D_-^i f(z) &= f'(z; -e_i) \end{aligned}$$

Due to the convexity of the considered function f we have then

$$D_-^i f(z) \leq D_+^i f(z)$$

for all $z \in \mathbb{R}^n$. If f is differentiable at the point z then the directional derivatives $D_+^i f(z)$ and $D_-^i f(z)$ in directions e_i and $-e_i$ coincide and form the gradient

$$\nabla_z f(z) = (D^1 f(z), \dots, D^n f(z))$$

$$\text{with } D^i f(z) = D_+^i f(z) = D_-^i f(z), \quad i = 1, \dots, n$$

Definition 2. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is called separable if it is the sum of a finite set of univariate functions*

$$f(z) = \sum_{i=1}^n f_i(z_i) \tag{5}$$

for $z = (z_1, \dots, z_n) \in \mathbb{R}^n$.

The perturbation of the input data $z \in \mathbb{R}^n$ is expressed by the random variable ω which is defined on a probability space (B, \mathcal{B}, P) . Here the set $B \subseteq \mathbb{R}^n$ is a Borel set with nonempty interior, \mathcal{B} is the Borel field of subsets of B and P a probability measure on \mathcal{B} . Furthermore we assume that the density function $h : B \rightarrow \mathbb{R}^1$ of ω exists. Then the perturbed function $F(z)$ is

$$F(z) = \mathbf{E}_\omega f(z + \omega) = \int_B f(z + \omega) P(d\omega) = \int_B f(z + \omega) h(\omega) d\omega$$

Since the original function f is convex also the expectation F is convex. If h is absolutely continuous then F is continuously differentiable [BL97].

Throughout the analysis we assume that the sets A and B have a quite general shape. However, sometimes a more precise description of the sets and especially of their boundary is needed. Traditionally, the boundary of a set is defined as follows.

Definition 3. *Assume that the set $A \subset \mathbb{R}^n$ is convex and compact. The interior of A is the open set*

$$\text{int } A = \{x \in A : \exists \varepsilon > 0 : U_\varepsilon(x) \subset A\} \quad (6)$$

with $U_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$ a neighbourhood of the point x .

The boundary of the set A is then defined by

$$\text{bd } A = A \setminus \text{int } A \quad (7)$$

However, in our analysis we will utilise a slightly different notion which is illustrated by the Examples 1. The equivalence of both definitions is proved in Proposition 1.

Definition 4. *The boundary of a convex and compact set $A \subset \mathbb{R}^n$ is given by*

$$\text{bd } A = \bigcup_{\substack{z \in A \\ i=1, \dots, n}} \{\underline{\alpha}^i(z), \bar{\alpha}^i(z)\} \quad (8)$$

with $\underline{\alpha}^i(z) = (\underline{\alpha}_1^i(z), \dots, \underline{\alpha}_n^i(z))$, $\bar{\alpha}^i(z) = (\bar{\alpha}_1^i(z), \dots, \bar{\alpha}_n^i(z))$, $z = (z_1, \dots, z_n)$ and

$$\underline{\alpha}^i(z) = z + e_i \inf\{y \in \mathbb{R}^1 : z + ye_i \in A\} \quad (9a)$$

$$\bar{\alpha}^i(z) = z + e_i \sup\{y \in \mathbb{R}^1 : z + ye_i \in A\} \quad (9b)$$

for $i = 1, \dots, n$.

This definition can be represented by cuts through the set A at the point $z \in A$ parallel to each of the dimensions $i = 1, \dots, n$ (Figure 1).

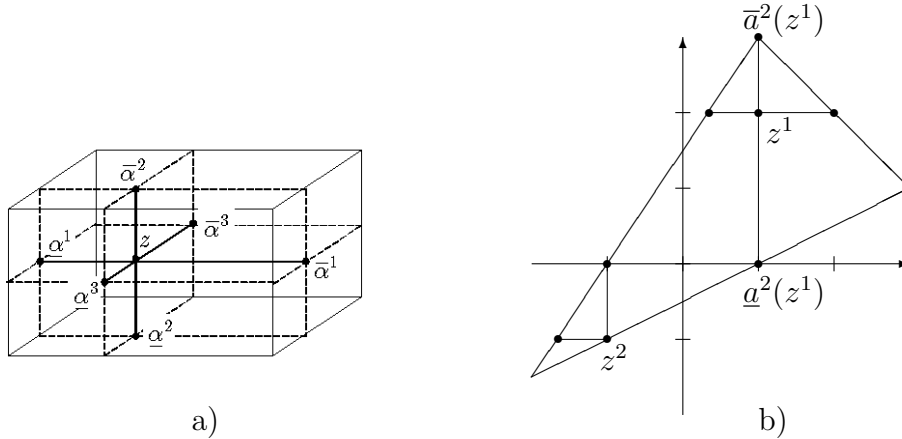


Figure 1: Description of the boundary of a set at a given point

Proposition 1. *Assume that the set $A \subset \mathbb{R}^n$ is convex and compact. Then the definitions (7) and (8) of the boundary of A are equivalent.*

Proof. We prove that a point $x \in \mathbb{R}^n$ is on the boundary of the set A according to definition (7) if and only if x satisfies expression (8). For this purpose we define

$$\underline{y}^i(z) = \inf\{y \in \mathbb{R}^1 : z + ye_i \in A\} \quad (10)$$

$$\bar{y}^i(z) = \sup\{y \in \mathbb{R}^1 : z + ye_i \in A\} \quad (11)$$

for $z \in A$ and a dimension $i \in \{1, \dots, n\}$.

Consider the point $x \in A$ on the boundary of A satisfying expression (8). Then there exist $z \in A$ and $i \in \{1, \dots, n\}$ with

$$x \in \{z + e_i \underline{y}^i(z), z + e_i \bar{y}^i(z)\}$$

We assume at first that $x = z + e_i \underline{y}^i(z)$. Due to the compactness of A there exists no $\varepsilon > 0$ such that $z + e_i(\underline{y}^i(z) - \varepsilon) \in A$. Hence, since

$$z + e_i(\underline{y}^i(z) - \varepsilon) = z + e_i \underline{y}^i(z) - e_i \varepsilon = x - e_i \varepsilon$$

there exists no $\varepsilon > 0$ such that $x - e_i \varepsilon \in A$. Therefore, there is no neighbourhood $U_\varepsilon(x)$ of x with $U_\varepsilon(x) \subset A$. This is equivalent to definition (7).

The case $x = z + e_i \bar{y}^i(z)$ is proved similarly. Since A is assumed compact there

exists no $\varepsilon > 0$ with $z + e_i(\bar{y}^i(z) + \varepsilon) \in A$. This means that there is no $\varepsilon > 0$ such that

$$z + e_i(\bar{y}^i(z) + \varepsilon) = z + e_i\bar{y}^i(z) + e_i\varepsilon = x + e_i\varepsilon \in A$$

Therefore there exists no neighbourhood $U_\varepsilon(x)$ of x with $U_\varepsilon(x) \subset A$ and the point x is on the boundary of A as defined by expression (7).

Finally, assume that the point $x \in A$ does not satisfy expression (8), i.e. that

$$x \notin \{z + e_i\underline{y}^i(z), z + e_i\bar{y}^i(z)\}$$

for all dimensions $i = 1, \dots, n$. This means that there exists some $\varepsilon > 0$ such that $U_\varepsilon(x) \subset A$. Hence $x \in \text{int } A$, i.e. according to definition (7) the point x is no boundary point.

Concluding, Definitions 3 and 4 of the boundary of a convex and compact set A are equivalent. \square

The following examples illustrate the idea of Definition 4.

Example 1. a) Assume that the set A_1 is an n -dimensional rectangle (cf. Figure 1 a)). Then for all $z \in A$ and $i = 1, \dots, n$ we have

$$\underline{\alpha}_i^i(z) = \underline{c}^i \quad \text{and} \quad \bar{\alpha}_i^i(z) = \bar{c}^i$$

with constants $\underline{c}^i, \bar{c}^i \in \mathbb{R}^1$. Consequently,

$$A_1 = [\underline{c}^1, \bar{c}^1] \times \dots \times [\underline{c}^n, \bar{c}^n].$$

b) If A_2 describes a sphere around the origin with radius a

$$A_2 = \{z \in \mathbb{R}^n : \sum_{i=1}^n z_i^2 \leq a^2\}$$

then the points $\underline{\alpha}^i(z)$ and $\bar{\alpha}^i(z)$ are determined for $z \in A_2$ by

$$\begin{aligned}\underline{\alpha}_i^i(z) &= -(a^2 - \sum_{\substack{j=1 \\ j \neq i}}^n z_j^2)^{1/2} \\ \bar{\alpha}_i^i(z) &= (a^2 - \sum_{\substack{j=1 \\ j \neq i}}^n z_j^2)^{1/2} \\ \underline{\alpha}_j^i(z) &= \bar{\alpha}_j^i(z) = z_j, \quad j \neq i \\ i, j &= 1, \dots, n\end{aligned}$$

c) Consider the set A_3 depicted in Figure 1 b). At the point $z^1 = (1, 2)$ the boundary of this set is

$$\begin{aligned}\underline{\alpha}^1(z^1) &= (0.33, 2) & \bar{\alpha}^1(z^1) &= (2, 2) \\ \underline{\alpha}^2(z^1) &= (1, 0) & \bar{\alpha}^2(z^1) &= (1, 3)\end{aligned}$$

whereas at the point $z^2 = (-1, -1)$ we have

$$\begin{aligned}\underline{\alpha}^1(z^2) &= (-1.67, -1) & \bar{\alpha}^1(z^2) &= (-1, -1) \\ \underline{\alpha}^2(z^2) &= (-1, -1) & \bar{\alpha}^2(z^2) &= (-1, 0)\end{aligned}$$

Remark 1. Observe that generally

$$A \neq [\underline{\alpha}_1^1(z), \bar{\alpha}_1^1(z)] \times \dots \times [\underline{\alpha}_n^n(z), \bar{\alpha}_n^n(z)]$$

at any point $z \in A$. However, if the convex function f is separable and has nonunique minima, then the set A of nonunique minima is a rectangle

$$A = [\underline{\alpha}_1^1, \bar{\alpha}_1^1] \times \dots \times [\underline{\alpha}_n^n, \bar{\alpha}_n^n]$$

with $\underline{\alpha}_i^i \leq \bar{\alpha}_i^i$ for all $i = 1, \dots, n$ and $\underline{\alpha}_i^i < \bar{\alpha}_i^i$ for at least one i .

Remark 2. If the support B of the perturbation ω is convex and compact the boundary of B is given in a similar way by

$$\text{bd } B = \bigcup_{\substack{\omega \in B \\ i=1, \dots, n}} \{\underline{\beta}^i(\omega), \bar{\beta}^i(\omega)\} \quad (12)$$

with $\underline{\beta}^i(\omega) = (\underline{\beta}_1^i(\omega), \dots, \underline{\beta}_n^i(\omega))$, $\overline{\beta}^i(\omega) = (\overline{\beta}_1^i(\omega), \dots, \overline{\beta}_n^i(\omega))$, $\omega = (\omega_1, \dots, \omega_n)$ and

$$\underline{\beta}^i(\omega) = \omega + e_i \inf\{y \in \mathbb{R}^1 : \omega + ye_i \in B\} \quad (13a)$$

$$\overline{\beta}^i(\omega) = \omega + e_i \sup\{y \in \mathbb{R}^1 : \omega + ye_i \in B\} \quad (13b)$$

for $i = 1, \dots, n$.

Definition 5. The set $A - B$ denotes the Minkowski sum of the sets A and $(-1) \cdot B$:

$$A - B = \{x \in \mathbb{R}^n : x = z - \omega, z \in A, \omega \in B\} \quad (14)$$

The set $I(z)$ defines a subset of the support B for given $z \in \mathbb{R}^n$:

$$I(z) = \{\omega \in B : z + \omega \in A\} \quad (15)$$

Proposition 2. Assume that the sets A and B are compact and convex and have a nonempty interior. Then there exist $\underline{\alpha}^i(z), \overline{\alpha}^i(z) \in \text{bd } A$, $z \in A$ and $\underline{\beta}^i(\omega), \overline{\beta}^i(\omega) \in \text{bd } B$, $\omega \in B$ defined by expressions (9) and (13), respectively, such that the boundary of the set $A - B$ is given by

$$\text{bd } (A - B) = \bigcup_{\substack{x \in A - B \\ i=1, \dots, n}} \{\underline{\varphi}^i(x), \overline{\varphi}^i(x)\}$$

with

$$\underline{\varphi}^i(x) = \underline{\alpha}^i(z) - \overline{\beta}^i(\omega) \quad (16a)$$

$$\overline{\varphi}^i(x) = \overline{\alpha}^i(z) - \underline{\beta}^i(\omega) \quad (16b)$$

and $x = z - \omega$.

Proof. We study a point $\tilde{\varphi} \in \text{bd } (A - B)$ and show the existence of boundary points $\underline{\alpha}^i(z), \overline{\alpha}^i(z) \in \text{bd } A$ and $\underline{\beta}^i(\omega), \overline{\beta}^i(\omega) \in \text{bd } B$ with $z \in A, \omega \in B$ such that $x = z - \omega$ and relations (16a) and (16b) hold.

Consider the set $C = A - B$. Then, according to definition (8), the boundary of this set is given by

$$\text{bd } C = \bigcup_{\substack{x \in C \\ i=1, \dots, n}} \{\underline{\varphi}^i(x), \overline{\varphi}^i(x)\}$$

with

$$\begin{aligned}\underline{\varphi}^i(x) &= x + e_i \inf\{y \in \mathbb{R}^1 : z + ye_i \in C\} \\ \overline{\varphi}^i(x) &= x + e_i \sup\{y \in \mathbb{R}^1 : z + ye_i \in C\}\end{aligned}$$

This means that a point $\tilde{\varphi} = \varphi(x)$, $x \in C$, is on the boundary of C if and only if there exists a dimension $i \in \{1, \dots, n\}$ such that either $\tilde{\varphi} = \underline{\varphi}^i(x)$ or $\tilde{\varphi} = \overline{\varphi}^i(x)$.

Assume at first

$$\tilde{\varphi} = \underline{\varphi}^i(x) = x + e_i \inf\{y \in \mathbb{R}^1 : z + ye_i \in C\}$$

Since $x \in C = A - B$ there exist $z \in A$ and $\omega \in B$ such that $x = z - \omega$. Then

$$\begin{aligned}\underline{\varphi}^i(x) &= z - \omega + e_i \inf\{y \in \mathbb{R}^1 : z - \omega + ye_i \in A - B\} \\ &= z - \omega + e_i(\inf\{y \in \mathbb{R}^1 : ye_i \in A - \{z\}\} \\ &\quad + \inf\{y \in \mathbb{R}^1 : ye_i \in (-1) \cdot B + \{\omega\}\}) \\ &= z - \omega + e_i(\inf\{y \in \mathbb{R}^1 : ye_i \in A - \{z\}\} - \sup\{y \in \mathbb{R}^1 : ye_i \in B - \{\omega\}\}) \\ &= z + e_i \inf\{y \in \mathbb{R}^1 : z + ye_i \in A\} - (\omega + e_i \sup\{y \in \mathbb{R}^1 : \omega + ye_i \in B\}) \\ &= \underline{\alpha}^i(z) - \underline{\beta}^i(\omega)\end{aligned}$$

A similar analysis can be performed for $\tilde{\varphi} = \overline{\varphi}^i(x)$ such that

$$\begin{aligned}\overline{\varphi}^i(x) &= x + e_i \sup\{y \in \mathbb{R}^1 : z + ye_i \in C\} \\ &= z - \omega + e_i(\sup\{y \in \mathbb{R}^1 : ye_i \in A - \{z\}\} \\ &\quad + \sup\{y \in \mathbb{R}^1 : ye_i \in (-1) \cdot B + \{\omega\}\}) \\ &= z - \omega + e_i(\sup\{y \in \mathbb{R}^1 : ye_i \in A - \{z\}\} - \inf\{y \in \mathbb{R}^1 : ye_i \in B - \{\omega\}\}) \\ &= \overline{\alpha}^i(z) - \underline{\beta}^i(\omega)\end{aligned}$$

Consequently, the assertion of the proposition holds. \square

The following example shows that not for all $z \in A$, $\omega \in B$ with $x = z - \omega$ also relations (16a) and (16b) are satisfied.

Example 2. Assume that the sets A and B are spheres around the origin with the radii $a = 1$ and $b = 2$, respectively (cf. Example 1 b)). The set $C = A - B$ is

a sphere around the origin with the radius $c = 3$.

Consider the point $x = (-(3 + \sqrt{3})/4, 1.5) = (-1.183, 1.5) \in C$. Then

$$\begin{aligned}\underline{\varphi}^1(x) &= (-2.598, 1.5) & \overline{\varphi}^1(x) &= (2.598, 1.5) \\ \underline{\varphi}^2(x) &= (-1.183, -2.757) & \overline{\varphi}^2(x) &= (-1.183, 2.757)\end{aligned}$$

For $z_1 = (-0.25, 0.5) \in A$ and $\omega_1 = ((2 + \sqrt{3})/4, -1) = (0.933, -1) \in B$ we have

$$\begin{aligned}\underline{\alpha}^1(z_1) &= (-0.866, 0.5) & \overline{\alpha}^1(z_1) &= (0.866, 0.5) \\ \underline{\alpha}^2(z_1) &= (-0.25, -0.968) & \overline{\alpha}^2(z_1) &= (-0.25, 0.968) \\ \underline{\beta}^1(\omega_1) &= (-1.732, -1) & \overline{\beta}^1(\omega_1) &= (1.732, -1) \\ \underline{\beta}^2(\omega_1) &= (0.933, -1.769) & \overline{\beta}^2(\omega_1) &= (0.933, 1.769)\end{aligned}$$

Then $x = z_1 - \omega_1$ and relations (16a) and (16b) hold for $i = 1, 2$.

However, consider now the points $z_2 = (0, 1) \in A$ and $\omega_2 = (1.183, -0.5)$ with

$$\begin{aligned}\underline{\alpha}^1(z_2) &= (0, 1) & \overline{\alpha}^1(z_2) &= (0, 1) \\ \underline{\alpha}^2(z_2) &= (0, -1) & \overline{\alpha}^2(z_2) &= (0, 1) \\ \underline{\beta}^1(\omega_2) &= (-1.936, -0.5) & \overline{\beta}^1(\omega_2) &= (1.936, -0.5) \\ \underline{\beta}^2(\omega_2) &= (1.183, -1.613) & \overline{\beta}^2(\omega_2) &= (1.183, 1.613)\end{aligned}$$

Also here $x = z_2 - \omega_2$, but relations (16a) and (16b) are not satisfied.

Definition 6. Consider a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$. The level set of f at the level $c \in \mathbb{R}^1$ is the set

$$L(c) = \{z \in \mathbb{R}^n : f(z) \leq c\} \quad (17)$$

The contour of the function f at the level c is the boundary of the level set $L(c)$

$$\tilde{L}(c) = \{z \in \mathbb{R}^n : f(z) = c\} \quad (18)$$

In this work we consider the class of functions f which are representable by an auxiliary function $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^1$ and a constant $c = \min_z f(z)$ as follows:

$$f(z) = \begin{cases} \hat{f}(z), & z \notin A \\ c, & z \in A \end{cases} \quad (19)$$

We assume that the function \hat{f} is convex and continuously differentiable at least on $A - B$. Furthermore we assume $\hat{f}(z) < c$ for all $z \in \text{int } A$. Then description (19) implies that the minimum of the function \hat{f} is in A . It illustrates furthermore that the set A represents a level set of the function \hat{f} and is therefore convex and compact. Moreover, the expectation function F can be expressed by

$$\begin{aligned} F(z) &= \int_B f(z + \omega)h(\omega)d\omega \\ &= \int_B \hat{f}(z + \omega)h(\omega)d\omega + \int_{I(z)} (c - \hat{f}(z + \omega))h(\omega)d\omega \end{aligned} \quad (20)$$

where the set $I(z)$ is defined according to (15).

The following propositions state properties of the set $A - B$.

Proposition 3. *Assume that*

1. *the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is convex with a set A of nonunique minima*

$$A = \arg \min_z f(z)$$

2. *the support B is convex and compact.*

Then the set $A - B$ is compact and convex.

Proof. The set A represents a level set of the convex function f and is therefore compact and convex. Due to assumption 2. then also $A - B$ is compact.

In order to prove the convexity of $A - B$ we consider points

$$\begin{aligned} x^1 &= z^1 - \omega^1 \\ x^2 &= z^2 - \omega^2 \end{aligned}$$

with $z^1, z^2 \in A$ and $\omega^1, \omega^2 \in B$. Then $x^1, x^2 \in A - B$. Due to the convexity of A and B also

$$\begin{aligned} z &= \lambda z^1 + (1 - \lambda)z^2 \in A \\ \omega &= \lambda \omega^1 + (1 - \lambda)\omega^2 \in B \end{aligned}$$

with $\lambda \in [0, 1]$. Therefore

$$\begin{aligned}\lambda x^1 + (1 - \lambda)x^2 &= \lambda z^1 - \lambda\omega^1 + (1 - \lambda)z^2 - (1 - \lambda)\omega^2 \\ &= z - \omega \in A - B\end{aligned}$$

i.e. the set $A - B$ is convex. \square

Proposition 4. *Assume that $0 \in B$. Then $A \subseteq A - B$.*

Proof. The set $A - B$ is defined by

$$A - B = \{x \in \mathbb{R}^n : x = z - \omega, z \in A, \omega \in B\}$$

Under the assumption $0 \in B$ we have

$$\begin{aligned}A &= \{x \in \mathbb{R}^n : x = z - 0, z \in A, \} \\ &\subseteq \{x \in \mathbb{R}^n : x = z - \omega, z \in A, \omega \in B\} = A - B\end{aligned}$$

such that the assertion of the proposition holds. \square

3 Unconstrained problem

In this section we begin our analysis with the unconstrained case represented by problems (3) and (4). Considering the case of nonunique minima, we state conditions on the random variable $\omega \in \mathbb{R}^n$ such that strict convexity of the perturbed function F is established on the set $A - B$. As Proposition 4 proves, this set is a superset of the set A of the minima of the original function $f(z)$. Furthermore we give conditions on ω ensuring that the unique minimum of $F(z)$ is on $A - B$ or even on A . Our focus is on convexity properties of the considered functions in connection with their minima. Therefore we are less interested in the behaviour of the functions on regions where extremal values can not be expected, as long as the functions are convex and thus the obtained minima are global.

Theorem 1. *Assume that the following conditions are satisfied.*

1. *the set A of nonunique minima of the convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ has a nonempty interior,*
2. *the support B of the random variable is convex and compact,*

3. the function \hat{f} satisfying relation (19) is convex and continuously differentiable on $A - B$,
4. the density function $h : B \rightarrow \mathbb{R}^1$ of the random variable ω is absolutely continuous and $h(\omega) > 0$ for $\omega \in B$,
5. $0 \in B$ and $A \subseteq B$.

Then the perturbed function $F(z) = \mathbb{E}_\omega f(z + \omega)$ is strictly convex on $A - B$.

In order to prove this theorem we need the following relation.

Proposition 5. *Assume that*

1. the conditions of Theorem 1 are satisfied,
2. for $z, y \in A - B$, $z \neq y$ at least one of the sets $I(z)$ and $I(y)$ has a nonempty interior.

Then the relation

$$\int_{I(z)} (c - \hat{f}(y + \omega))h(\omega)d\omega < \int_{I(y)} (c - \hat{f}(y + \omega))h(\omega)d\omega \quad (21)$$

holds where $c = \min_x f(x)$.

Proof. For $z, y \in A - B$ the sets $I(y)$ and $I(z)$ exist and can be split up into the pairwise disjoint subsets

$$S_1 = \{\omega \in B : z + \omega \in A, y + \omega \notin A\}$$

$$S_2 = \{\omega \in B : z + \omega \in A, y + \omega \in A\}$$

$$S_3 = \{\omega \in B : z + \omega \notin A, y + \omega \in A\}$$

such that $I(z) = S_1 \cup S_2$ and $I(y) = S_2 \cup S_3$.

At first we prove that at least one of the sets S_1 and S_3 has a nonempty interior. Assume that S_1 is a null set. If $I(z)$ has an empty interior, then also the set $S_2 = I(z) \setminus S_1$. Then, however, $I(y)$ has a positive measure due to assumption 2. and therefore also the set $S_3 = I(y) \setminus S_2$, i.e. S_3 has a nonempty interior.

If $I(z)$ has a nonempty interior then also S_2 . Assume now at first that $S_3 \neq \emptyset$. Then, since S_2 and $I(y)$ are closed, the set $S_3 = I(y) \setminus S_2$ is not closed and has

therefore a nonempty interior. The case $S_3 = \emptyset$ means that $I(y) = S_2$ and thus $I(y) \subseteq I(z)$. However, for $y \neq z$ and compact set A we have

$$A - \{y\} \neq A - \{z\}$$

and therefore

$$I(y) = B \cap (A - \{y\}) \neq B \cap (A - \{z\}) = I(z)$$

Hence $I(y) \subset I(z)$ if $S_3 = \emptyset$ and therefore $S_1 \neq \emptyset$. However, since $I(z)$ and S_2 are closed, the set $S_1 = I(z) \setminus S_2$ is not closed and, consequently, not a null set which contradicts the above assumption.

Resulting, at least one of the sets S_1 and S_3 has a nonempty interior.

If S_1 has a nonempty interior then $y + \omega \notin A$ and therefore $c - \hat{f}(y + \omega) < 0$ for $\omega \in S_1$. Hence, due to assumption 4. of Theorem 1,

$$\int_{S_1} (c - \hat{f}(y + \omega))h(\omega)d\omega < 0 \quad (22)$$

If S_3 has nonempty interior then

$$\int_{S_3} (c - \hat{f}(y + \omega))h(\omega)d\omega > 0 \quad (23)$$

since $y + \omega \in \text{int } A$ and hence $c - \hat{f}(y + \omega) > 0$ for any interior point $\omega \in \text{int } S_3$. Consequently, we obtain

$$\begin{aligned} & \int_{I(z)} (c - \hat{f}(y + \omega))h(\omega)d\omega \\ &= \int_{S_1} (c - \hat{f}(y + \omega))h(\omega)d\omega + \int_{S_2} (c - \hat{f}(y + \omega))h(\omega)d\omega \\ &< \int_{S_2} (c - \hat{f}(y + \omega))h(\omega)d\omega + \int_{S_3} (c - \hat{f}(y + \omega))h(\omega)d\omega \\ &= \int_{I(y)} (c - \hat{f}(y + \omega))h(\omega)d\omega \end{aligned}$$

and relation (21) holds. □

Now Theorem 1 can be proved.

Proof. The function $F(z)$ is convex and differentiable and therefore

$$F(y) - F(z) \geq \langle \nabla_z F(z), y - z \rangle \quad (24)$$

In the following we verify that this expression is satisfied with strict inequality for $y, z \in A - B$.

The function $\hat{f}(z)$ was assumed convex and differentiable such that

$$\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - \hat{f}(y + \omega) + \hat{f}(z + \omega) \leq 0$$

Since furthermore $I(z) \neq \emptyset$ for all $z \in A - B$ and by definition $I(z) \subseteq B$, we have

$$\begin{aligned} & \int_B (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - \hat{f}(y + \omega) + \hat{f}(z + \omega)) h(\omega) d\omega \\ & \leq \int_{I(z)} (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - \hat{f}(y + \omega) + \hat{f}(z + \omega)) h(\omega) d\omega \end{aligned}$$

Assume now that at least one of the sets $I(y)$ and $I(z)$ has a nonempty interior such that relation (21) holds. Then the following inequality chain can be established:

$$\begin{aligned} & \int_B (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - \hat{f}(y + \omega) + \hat{f}(z + \omega)) h(\omega) d\omega \\ & \leq \int_{I(z)} (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - \hat{f}(y + \omega) + \hat{f}(z + \omega)) h(\omega) d\omega \\ & = \int_{I(z)} (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - c + \hat{f}(z + \omega)) h(\omega) d\omega + \int_{I(z)} (c - \hat{f}(y + \omega)) h(\omega) d\omega \\ & < \int_{I(z)} (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle - c + \hat{f}(z + \omega)) h(\omega) d\omega + \int_{I(y)} (c - \hat{f}(y + \omega)) h(\omega) d\omega \end{aligned}$$

This is the same as

$$\begin{aligned} & \int_B \langle \nabla_z \hat{f}(z + \omega), y - z \rangle h(\omega) d\omega - \int_B \hat{f}(y + \omega) h(\omega) d\omega + \int_B \hat{f}(z + \omega) h(\omega) d\omega \\ & < \int_{I(z)} \langle \nabla_z \hat{f}(z + \omega), y - z \rangle h(\omega) d\omega - \int_{I(z)} (c - \hat{f}(z + \omega)) h(\omega) d\omega + \int_{I(y)} (c - \hat{f}(y + \omega)) h(\omega) d\omega \end{aligned}$$

Recalling the expression (20) of the expectation F , this inequality can be rearranged to

$$\begin{aligned}
F(y) - F(z) &= \int_B \hat{f}(y + \omega)h(\omega)d\omega + \int_{I(y)} (c - \hat{f}(y + \omega))h(\omega)d\omega \\
&\quad - \int_B \hat{f}(z + \omega)h(\omega)d\omega - \int_{I(z)} (c - \hat{f}(z + \omega))h(\omega)d\omega \\
&> \int_B (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle)h(\omega)d\omega - \int_{I(z)} (\langle \nabla_z \hat{f}(z + \omega), y - z \rangle)h(\omega)d\omega \\
&= \langle \nabla_z F(z), y - z \rangle
\end{aligned}$$

This means that for $y, z \in A - B$, $y \neq z$ such that $I(y)$ or $I(z)$ have a nonempty interior expression (24) is satisfied with strict inequality.

The sets $I(y)$ and $I(z)$ have an empty interior for $y, z \in A - B$ only if

$$y, z \in \{x \in A - B : x = \tilde{x} - \tilde{\omega}, \tilde{x} \in \text{bd } A, \tilde{\omega} \in \text{bd } B\}$$

This set is a null set. Since the function F is convex everywhere and strictly convex on $A - B$ except possibly on a null set, it is strictly convex on $A - B$. \square

Remark 3. If the density function $h(\omega)$ of the perturbation is positive on the whole space, i.e. if $B = \mathbb{R}^n$, then also $A - B = \mathbb{R}^n$. Then $F(z)$ is strictly convex on \mathbb{R}^n if all other assumptions of Theorem 1 are satisfied.

If the original function f has a unique minimum, i.e. if the set A is a singleton, then strict convexity of the perturbed function F can be established in a surrounding of this point by considering a perturbation ω with a support $B \supset A$.

The following theorems study the location of the unique minimum z_p^* of the perturbed function $F(z) = \mathbf{E}_\omega f(z + \omega)$. Theorem 2 states conditions such that $z_p^* \in A - B$. Under the conditions of Theorems 3 or 4 we have finally $z_p^* \in A$.

Theorem 2. *Assume that*

1. *the conditions of Theorem 1 hold,*
2. *the function f is separable,*

3. the support B of the perturbation ω is rectangular

$$B = [\underline{\beta}_1^1, \overline{\beta}_1^1] \times \dots \times [\underline{\beta}_n^n, \overline{\beta}_n^n]$$

Then the function $F(z)$ has a unique minimum $z^* \in A - B$.

Proof. At first we prove that for all $i = 1, \dots, n$ points $z = (z_1, \dots, z_n) \in A - B$ exist with a unique component z_i such that $\frac{\partial F(z)}{\partial z_i} = 0$. Then we show that this implies the existence of a unique point $z_P^* \in A - B$ such that $\frac{\partial F(z_P^*)}{\partial z_i} = 0$ for all $i = 1, \dots, n$, i.e. the existence of a unique minimum of F .

Due to the separability of f the set A is rectangular (see Remark 1). Taking additionally into account that also the support B is rectangular we have

$$\begin{array}{lll} \underline{\alpha}_i^i(z) = \underline{\alpha}^i & \overline{\alpha}_i^i(z) = \overline{\alpha}^i & \underline{\alpha}_j^i(z) = \overline{\alpha}_j^i(z) = z_j \\ \underline{\beta}_i^i(\omega) = \underline{\beta}^i & \overline{\beta}_i^i(\omega) = \overline{\beta}^i & \underline{\beta}_j^i(\omega) = \overline{\beta}_j^i(\omega) = \omega_j \end{array}$$

for all $z \in A$ and all $\omega \in B$ and $i, j = 1, \dots, n$ with $j \neq i$. With the expressions (16) for the boundary of $A - B$ we obtain therefore

$$\begin{array}{l} \underline{\varphi}_i^i(x) = \underline{\varphi}_i^i = \underline{\alpha}_i^i - \overline{\beta}_i^i \\ \overline{\varphi}_i^i(x) = \overline{\varphi}_i^i = \overline{\alpha}_i^i - \underline{\beta}_i^i \\ \underline{\varphi}_j^i(x) = \overline{\varphi}_j^i(x) = z_j - \omega_j \end{array}$$

for all $x \in A - B$.

We consider a dimension $i \in \{1, \dots, n\}$ and study at first the point $\underline{\varphi}^i = \underline{\alpha}^i - \overline{\beta}^i$.

Then

$$\underline{\varphi}_j^i + \omega_j \leq \underline{\alpha}_j^i$$

for $j = 1, \dots, n$ and all $\omega \in B$. Hence, due to the convexity of f ,

$$\frac{\partial f(\underline{\varphi}^i + \omega)}{\partial z_i} \leq D_-^i(\underline{\alpha}^i(z)) \leq 0 \quad \forall \omega \in B$$

The first relation is satisfied strictly for all $\omega = (\omega_1, \dots, \omega_n) \in B$ with $\omega_i < \overline{\beta}_i^i(\omega)$.

Consequently, since we assumed that B has a nonempty interior and that the distribution function $h(\omega)$ is strictly positive on B ,

$$\frac{\partial F(\underline{\varphi}^i)}{\partial z_i} = \int_B \frac{\partial f(\underline{\varphi}^i + \omega)}{\partial z_i} h(\omega) d\omega < 0 \quad (25)$$

Now, consider the point $\bar{\varphi}^i = \bar{\alpha}^i - \underline{\beta}^i$. Here

$$\bar{\varphi}_j^i + \omega_j \geq \bar{\alpha}_j^i$$

for $j = 1, \dots, n$ and all $\omega = (\omega_1, \dots, \omega_n) \in B$ and therefore

$$\frac{\partial f(\bar{\varphi}^i + \omega)}{\partial z_i} \geq D_+^i(\bar{\alpha}^i(z)) \geq 0$$

where the first relation is satisfied strictly for all $\omega \in B$ with $\omega_i > \underline{\beta}_i^i(\omega)$. Since B has a nonempty interior and $h(\omega) > 0$ for $\omega \in B$,

$$\frac{\partial F(\bar{\varphi}^i)}{\partial z_i} = \int_B \frac{\partial f(\bar{\varphi}^i + \omega)}{\partial z_i} h(\omega) d\omega > 0 \quad (26)$$

Under the conditions of Theorem 1 the function F is strictly convex on $A - B$. Therefore the inequalities (25) and (26) mean that there exist points $\tilde{z} \in A - B$ with a unique component $\tilde{z}_i = z_i \in [\underline{\varphi}_i^i, \bar{\varphi}_i^i]$ such that $\frac{\partial F(\tilde{z})}{\partial z_i} = 0$.

Since f is assumed separable and the ω_i are not correlated, the above analysis can be conducted for any dimension $i = 1, \dots, n$. This implies that the point $z^* = (z_1^*, \dots, z_n^*) \in A - B$ with $z_i^* = z_i$, $i = 1, \dots, n$ is the unique minimum of the function F . \square

Theorem 3. *Assume that*

1. *the conditions of Theorem 1 are satisfied,*
2. *the function f is separable,*
3. *the support B of the perturbation ω satisfies*

$$A \subseteq B \subseteq A - A \quad (27)$$

Then the function $F(z) = \mathbb{E}_\omega f(z + \omega)$ has a unique minimum $z^ \in A$.*

Proof. We proceed similarly to the proof of Theorem 2. First we prove that for all $i = 1, \dots, n$ there exist points $z = (z_1, \dots, z_n) \in A$ with a unique component z_i such that $\frac{\partial F(z)}{\partial z_i} = 0$. We show then that this implies the existence of a unique minimum $z^* = \arg \min_z F(z) \in A$.

Consider a dimension $i \in \{1, \dots, n\}$. Due to assumption 2. and since the ω_i are

uncorrelated we can express the partial derivative of the function F at a point $z \in \mathbb{R}^n$ with respect to the component z_i by

$$\begin{aligned} \frac{\partial F(z)}{\partial z_i} &= \int_B \frac{\partial f(z + \omega)}{\partial z_i} h(\omega) d\omega \\ &= \int_B \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega \end{aligned} \quad (28)$$

Due to the separability of f we have

$$A = [\underline{\alpha}_1^1, \bar{\alpha}_1^1] \times \dots \times [\underline{\alpha}_n^n, \bar{\alpha}_n^n]$$

such that the boundary of the set A is independent of the choice of a point $x \in A$.

Then condition (27) means

$$\underline{\beta}_j^i(\omega) \geq \underline{\alpha}_j^i - \bar{\alpha}_j^i \quad (29)$$

$$\bar{\beta}_j^i(\omega) \leq \bar{\alpha}_j^i - \underline{\alpha}_j^i \quad (30)$$

for all $\omega \in B$ and $j = 1, \dots, n$. Utilising the pairwise disjoint sets

$$B_1(z) = \{\omega \in B : z_i + \omega_i < \underline{\alpha}_i^i\}$$

$$B_2(z) = \{\omega \in B : z_i + \omega_i \in [\underline{\alpha}_i^i, \bar{\alpha}_i^i]\}$$

$$B_3(z) = \{\omega \in B : z_i + \omega_i > \bar{\alpha}_i^i\}$$

such that $B_1(z) \cup B_2(z) \cup B_3(z) = B$ we find that expression (28) is equivalent to

$$\begin{aligned} \frac{\partial F(z)}{\partial z_i} &= \int_{B_1(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega \\ &\quad + \int_{B_2(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega + \int_{B_3(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega \end{aligned} \quad (31)$$

If the sets $B_1(z)$ and $B_3(z)$ exist they have a nonempty interior since it was assumed that the interior of B is not empty.

For $\omega \in B_2(z)$ we have $\frac{\partial f_i(z_i + \omega_i)}{\partial z_i} = 0$. Therefore the middle term in expression (31) vanishes:

$$\int_{B_2(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega = 0$$

For $\omega \in B_1(z)$ we have $\frac{\partial f_i(z_i + \omega_i)}{\partial z_i} < 0$. Due to the positivity of the density function $h(\omega)$ on $B_1(z) \subseteq B$ and since $\text{int } B_1(z) \neq \emptyset$, it holds therefore

$$\int_{B_1(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega \leq 0$$

Likewise we have $\frac{\partial f_i(z_i + \omega_i)}{\partial z_i} > 0$ for $\omega \in B_3(z)$ and hence, since $h(\omega) > 0$ for $\omega \in B_3(z) \subseteq B$ and $\text{int } B_3(z) \neq \emptyset$,

$$\int_{B_3(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega \geq 0$$

Consider now the point $z^1 = \underline{\alpha}^i$. Then the sets $B_1(\underline{\alpha}^i)$ and $B_3(\underline{\alpha}^i)$ are

$$\begin{aligned} B_1(\underline{\alpha}^i) &= \{\omega \in B : \omega_i < 0\} \\ B_3(\underline{\alpha}^i) &= \{\omega \in B : \omega_i > \bar{\alpha}^i - \underline{\alpha}^i\} \end{aligned}$$

Under condition (30) the set $B_3(\underline{\alpha}^i)$ is empty. Furthermore the set $B_1(\underline{\alpha}^i)$ has a positive measure such that

$$\frac{\partial F(\underline{\alpha}^i)}{\partial z_i} = \int_{B_1(\underline{\alpha}^i)} \frac{\partial f_i(\underline{\alpha}_i^i + \omega_i)}{\partial z_i} h(\omega) d\omega < 0 \quad (32)$$

Similarly we have for $z^2 = \bar{\alpha}^i$

$$\begin{aligned} B_1(\bar{\alpha}^i) &= \{\omega \in B : \omega_i < \underline{\alpha}^i - \bar{\alpha}^i\} \\ B_3(\bar{\alpha}^i) &= \{\omega \in B : \omega_i > 0\} \end{aligned}$$

Therefore $B_1(\bar{\alpha}^i) = \emptyset$ under condition (29) and $B_3(\bar{\alpha}^i)$ has a positive measure.

Hence

$$\frac{\partial F(\bar{\alpha}^i)}{\partial z_i} = \int_{B_3(\bar{\alpha}^i)} \frac{\partial f_i(\bar{\alpha}_i^i + \omega_i)}{\partial z_i} h(\omega) d\omega > 0 \quad (33)$$

Due to the strict convexity of $F(z)$ on $A \subseteq A - B$ there exist thus points $\tilde{z} \in A$ with a unique component $\tilde{z}_i = z_i$ such that $\frac{\partial F(\tilde{z})}{\partial z_i} = 0$. Taking into account the separability of f and that the ω_i are not correlated this implies that the point $z^* \in A$ with $z_i^* = z_i$, $i = 1, \dots, n$ is the unique minimum of the function F . \square

If the condition (27) on the support B of the perturbation ω is not satisfied, it is more difficult to state conditions such that the unique minimum of the expectation F is in A . In this case more information about the original function f and

about the perturbation is required. The following theorem gives such conditions for the case of a piecewise linear function f .

For the characterisation of the perturbation we utilise here additionally the marginal distribution function at a point $x = (x_1, \dots, x_i, \dots, x_n)$

$$H_i(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{x_i} h(\omega) d\omega_i d\omega_j$$

with $j = 1, \dots, n$ and $j \neq i$. Recall that the support B is determined at the random variable $x \in B$ in dimension i by the interval $[\underline{\beta}_i^i(x), \overline{\beta}_i^i(x)]$. Therefore we have

$$H_i(x) = \int_{-\infty}^{\infty} \cdots \int_{\underline{\beta}_i^i(x)}^{x_i} h(\omega) d\omega_i d\omega_j$$

$$H_i(\underline{\beta}_i^i(x)) = 0, \quad H_i(\overline{\beta}_i^i(x)) = 1$$

Theorem 4. *Assume that*

1. *the conditions of Theorem 1 are satisfied,*
2. *the function $f(z)$ is separable and piecewise linear*

$$\frac{\partial f_i(z_i)}{\partial z_i} = \begin{cases} \underline{c}^i < 0, & z_i < \underline{\alpha}_i^i \\ 0, & z_i \in [\underline{\alpha}_i^i, \overline{\alpha}_i^i] \\ \overline{c}^i > 0, & z_i > \overline{\alpha}_i^i \end{cases}$$

for all $i = 1, \dots, n$,

3. *for each dimension $i = 1, \dots, n$ either*

$$\underline{\beta}^i(\omega) \geq \underline{\alpha}^i - \overline{\alpha}^i \tag{34}$$

$$\text{or } \overline{c}^i(H_i(0) - 1) \leq \underline{c}^i H_i(\underline{\alpha}^i - \overline{\alpha}^i) \tag{35}$$

4. *for each dimension $i = 1, \dots, n$ either*

$$\overline{\beta}^i(\omega) \leq \overline{\alpha}^i - \underline{\alpha}^i \tag{36}$$

$$\text{or } \underline{c}^i H_i(0) \leq \overline{c}^i (H_i(\overline{\alpha}^i - \underline{\alpha}^i) - 1) \tag{37}$$

Then the function $F(z) = \mathbb{E}_\omega f(z + \omega)$ has a unique minimum $z_P^* \in A$.

Proof. The function f was assumed separable and the ω_i are uncorrelated. Therefore we can again conduct the analysis for an arbitrary dimension $i \in \{1, \dots, n\}$. Utilising expression (31) for the partial derivative of the expectation function F at the point z we have

$$\frac{\partial F(z)}{\partial z_i} = \int_{B_1(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega + \int_{B_3(z)} \frac{\partial f_i(z_i + \omega_i)}{\partial z_i} h(\omega) d\omega$$

This expression is now studied at $z_1 = \underline{\alpha}^i$ such that

$$\begin{aligned} B_1(\underline{\alpha}^i) &= \{\omega \in B : \omega_i < 0\} \\ B_3(\underline{\alpha}^i) &= \{\omega \in B : \omega_i > \bar{\alpha}_i^i - \underline{\alpha}_i^i\} \end{aligned}$$

If condition (34) is satisfied we can follow the proof of Theorem 3 and obtain $\frac{\partial F(\underline{\alpha}^i)}{\partial z_i} < 0$ with relation (32). If, however, condition (34) is violated we have

$$\begin{aligned} B_1(\underline{\alpha}^i) &= \{\omega \in B : \omega_i < 0\} \\ B_3(\underline{\alpha}^i) &= \{\omega \in B : \omega_i \in [\bar{\alpha}_i^i - \underline{\alpha}_i^i, \bar{\beta}_i^i]\} \end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\partial f_i(\underline{\alpha}_i^i + \omega_i)}{\partial z_i} h(\omega) d\omega_i = \int_{\underline{\beta}_i^i}^0 \frac{\partial f_i(\underline{\alpha}_i^i + \omega_i)}{\partial z_i} h(\omega) d\omega_i + \int_{\bar{\alpha}_i^i - \underline{\alpha}_i^i}^{\bar{\beta}_i^i} \frac{\partial f_i(\underline{\alpha}_i^i + \omega_i)}{\partial z_i} h(\omega) d\omega_i$$

Due to the piecewise linearity of f this gives

$$\begin{aligned} \frac{\partial F(\underline{\alpha}^i)}{\partial z_i} &= \underline{c}^i \int_{-\infty}^{\infty} \dots \int_{\underline{\beta}_i^i}^0 h(\omega) d\omega + \bar{c}^i \int_{-\infty}^{\infty} \dots \int_{\bar{\alpha}_i^i - \underline{\alpha}_i^i}^{\bar{\beta}_i^i} h(\omega) d\omega \\ &= \underline{c}^i (H_i(0) - H_i(\underline{\beta}_i^i)) + \bar{c}^i (H_i(\bar{\beta}_i^i) - H_i(\bar{\alpha}_i^i - \underline{\alpha}_i^i)) \\ &= \underline{c}^i H_i(0) + \bar{c}^i (1 - H_i(\bar{\alpha}_i^i - \underline{\alpha}_i^i)) \end{aligned}$$

Therefore $\frac{\partial F(\underline{\alpha}^i)}{\partial z_i} \leq 0$ under condition (35).

A similar analysis can be performed for $z_2 = \underline{\alpha}^i$. If condition (36) holds, relation (33) in the proof of Theorem 3 yields $\frac{\partial F(\bar{\alpha}^i)}{\partial z_i}$. If condition (36) is violated, we have

$$\begin{aligned} B_1(\bar{\alpha}^i) &= \{\omega \in B : \omega_i \in [\underline{\beta}_i^i, \underline{\alpha}_i^i - \bar{\alpha}_i^i]\} \\ B_3(\bar{\alpha}^i) &= \{\omega \in B : \omega_i > 0\} \end{aligned}$$

such that

$$\int_{-\infty}^{\infty} \frac{\partial f_i(\bar{\alpha}_i + \omega_i)}{\partial z_i} h(\omega) d\omega_i = \int_{\underline{\beta}_i}^{\bar{\alpha}_i - \underline{\alpha}_i} \frac{\partial f_i(\bar{\alpha}_i + \omega_i)}{\partial z_i} h(\omega) d\omega_i + \int_0^{\bar{\beta}_i} \frac{\partial f_i(\bar{\alpha}_i + \omega_i)}{\partial z_i} h(\omega) d\omega_i$$

and therefore due to the piecewise linearity of f

$$\begin{aligned} \frac{\partial F(\bar{\alpha})}{\partial z_i} &= \underline{c}^i \int_{-\infty}^{\infty} \cdots \int_{\underline{\beta}_i}^{\bar{\alpha}_i - \underline{\alpha}_i} h(\omega) d\omega + \bar{c}^i \int_{-\infty}^{\infty} \cdots \int_0^{\bar{\beta}_i} h(\omega) d\omega \\ &= \underline{c}^i (H_i(\underline{\alpha}^i - \bar{\alpha}^i) - H_i(\underline{\beta}^i)) + \bar{c}^i (H_i(\bar{\beta}^i) - H_i(0)) \\ &= \underline{c}^i H_i(\underline{\alpha}^i - \bar{\alpha}^i) + \bar{c}^i (1 - H_i(0)) \end{aligned}$$

Hence $\frac{\partial F(\bar{\alpha}^i)}{\partial z_i} \geq 0$ if condition (37) holds.

Due to the strict convexity of $F(z)$ on $A \subseteq A - B$ there exists then a unique value for the component z_i such that $\frac{\partial F(\tilde{z})}{\partial z_i} = 0$ for all $\tilde{z} \in A$ with $\tilde{z}_i = z_i$. Taking into account the separability of f and that the ω_i are not correlated this implies that the point $z^* \in A$ with $z_i^* = z_i$, $i = 1, \dots, n$ is the unique minimum of $F(z)$. \square

The following examples underline the statements of Theorems 3 and 4.

Example 3. Consider the original function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ with

$$f(z) = \begin{cases} -z - 3, & z < -3 \\ 0, & z \in A = [-3, 3] \\ z - 3 & z > 3 \end{cases}$$

This means that $\underline{\alpha} = -\alpha = -3$ and $\bar{\alpha} = \alpha = 3$. Assume that the decision variable $z \in \mathbb{R}^1$ is perturbed by a random variable $\omega \in \mathbb{R}^1$ with a density function

$$h(\omega) = \begin{cases} \frac{1}{2\beta}(-\frac{\omega}{\beta} + 1), & \omega \in B = [-\beta, \beta] \\ 0, & \omega \notin B \end{cases}$$

such that $\underline{\beta} = -\beta$ and $\bar{\beta} = \beta$.

a) The parameter $\beta = 5 \in [\alpha, 2\alpha]$ satisfies the assumptions of Theorem 3.

Then

$$F(z) = \begin{cases} -\frac{4}{3} - z, & z < -8 \\ \frac{1}{10} \left(\frac{1}{30} z^3 + \frac{4}{5} z^2 - \frac{18}{5} z + \frac{56}{15} \right), & z \in [-8, -2) \\ \frac{1}{10} \left(\frac{1}{15} z^3 + z^2 - \frac{16}{5} z + 4 \right), & z \in [-2, 2) \\ \frac{1}{10} \left(\frac{1}{30} z^3 + \frac{1}{5} z^2 + \frac{2}{5} z + \frac{4}{15} \right), & z \in [2, 8) \\ z - \frac{14}{3}, & z \geq 8 \end{cases}$$

and the unique minimum z_P^* of this function is $z_P^* = -5 + \sqrt{41} = 1.40$ with $F(z_P^*) = 0.17$. In this case $z_P^* \in A$. (See Figure 2 a).)

b) The parameter $\beta = 8$ violates condition (27) of Theorem 3. However, the conditions (35), (37) of Theorem 4 are satisfied.

The unique minimum of the perturbed function $F(y) = \mathbf{E}_\omega f(y + \omega)$ can then be determined as $y_P^* = -8 + \sqrt{119} = 2.908$ with $F(y_P^*) = 0.905$. Also here $y_P^* \in A$.

c) Finally, assume that $\beta = 9$. For this parameter both condition (27) of Theorem 3 and the conditions (35), (37) of Theorem 4 are violated. Here the function $F(z) = \mathbf{E}_\omega f(z + \omega)$ has the unique minimum $z_P^* = -9 + 3\sqrt{17} = 3.37$ with $F(z_P^*) = 1.21$. Although $z_P^* \in A - B = [-12, 12]$ we find that $z_P^* \notin A = [-3, 3]$. (See Figure 2 b).)

4 Constrained problem

Now we extend the analysis to the constrained optimisation problem

$$\begin{aligned} \min f(z) & \\ z \in G = \{z \in \mathbb{R}^n : g_j(z) \leq 0, j = 1, \dots, m\} & \end{aligned} \quad (38)$$

If the decision variable $z \in \mathbb{R}^n$ is afflicted with uncertainty then this uncertainty affects also the feasible area of the considered problem. Therefore the following

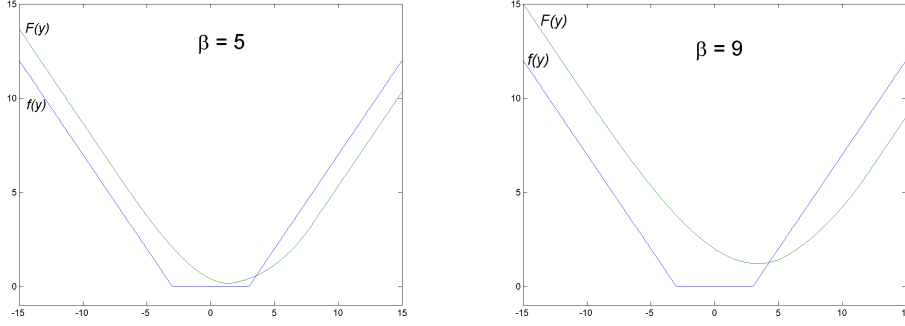


Figure 2: Original and perturbed functions $f(z)$ and $F(z)$ in Example 3 a) with $\beta = 5$ and c) with $\beta = 9$

perturbed optimisation problem is analysed

$$\begin{aligned} \min F(z) &= \min \mathbf{E}_\omega f(z + \omega) \\ z \in H &= \{z \in \mathbb{R}^n : h_j(z) = \mathbf{E}_\omega g_j(z + \omega) \leq 0, j = 1, \dots, m\} \end{aligned} \quad (39)$$

We assume that the uncertainty is expressed by a random variable $\omega \in B \subseteq \mathbb{R}^n$ with an absolutely continuous density function $h : B \rightarrow \mathbb{R}^1$ with convex and compact support B .

Under the conditions stated in Theorem 2 the unconstrained function $F(z)$ takes on a unique minimum on $A - B$. In order to analyse effects of the perturbation on the feasibility of this minimum we distinguish the minima of the unconstrained and of the constrained problems.

Definition 7. We denote by z_D^* and z_P^* the minima of the unconstrained deterministic and perturbed objective functions $f(z)$ and $F(z)$, respectively,

$$\begin{aligned} z_D^* &\in A = \arg \min_z f(z) \\ z_P^* &= \arg \min_z F(z) \end{aligned}$$

and by \tilde{z}_D^* and \tilde{z}_P^* the minima of the constrained deterministic and perturbed problems (38) and (39),

$$\begin{aligned} \tilde{z}_D^* &\in \tilde{A} = \arg \min_z \{f(z), z \in G\} \\ \tilde{z}_P^* &= \arg \min_z \{F(z), z \in H\} \end{aligned}$$

With these notations the feasibility of the unconstrained minima can be described by their location with regard to the feasible sets of the deterministic and the perturbed problems, respectively. The following cases can be distinguished:

(D1): $A \subseteq G$. All nonunique minima of the unconstrained function $f(z)$ satisfy the constraints G of the deterministic problem (38) such that $\tilde{A} = A$.

(D2): $A \not\subseteq G$ and $A \cap G \neq \emptyset$. Some of the minima of $f(z)$ are feasible, $\tilde{A} = A \cap G$.

(D3): $A \cap G = \emptyset$. None of the minima of $f(z)$ satisfy the constraints G , $\tilde{A} \cap A = \emptyset$.

(P1): $z_P^* \in H$. The unique minimum of $F(z)$ satisfies the constraints H of the perturbed problem (39), $\tilde{z}_P^* = z_P^*$.

(P2): $z_P^* \notin H$. The minimum of $F(z)$ does not satisfy the constraints H , $\tilde{z}_P^* \neq z_P^*$.

In the following we study relationships between these cases. Utilising information about the deterministic problem, we give statements about the feasibility of the unique minimum z_P^* of the perturbed function $F(z) = \mathbf{E}_\omega f(z + \omega)$. This analysis is conducted under the assumption that the unique minimum z_P^* of the unconstrained function $f(z)$ is element of A . Therefore we study the location of A with respect to the feasible area H of the perturbed problem. Furthermore we assume that $\mathbf{E}\omega = 0$.

If the constraints g_j are linear it can be proved that case (D1) implies (P1).

Theorem 5. *Assume that*

1. *the constraints g_j , $j = 1, \dots, m$, of the deterministic problem (38) are linear,*
2. *$A \subseteq G$, i.e. all nonunique minima of the unconstrained function $f(z)$ are feasible for the deterministic problem (38) (case (D1)).*

Then the minimum \tilde{z}_P^ of the perturbed problem (39) is unique.*

Proof. The linear constraints of the deterministic problem can be expressed by

$$G = \{z \in \mathbb{R}^n : Cz + B \leq 0\}$$

Consequently, the feasible set of the perturbed problem is

$$\begin{aligned} H &= \{z \in \mathbb{R}^n : \mathbf{E}_\omega\{C(z + \omega) + B\} \leq 0\} \\ &= \{z \in \mathbb{R}^n : Cz + B + C\mathbf{E}\omega \leq 0\} = G \end{aligned} \quad (40)$$

Since $z_P^* \in A$ and $A \subseteq G$, the minimum z_P^* is also feasible for the perturbed problem (39). This characterises case (P1) and the perturbed problem has a unique minimum $\bar{z}_P^* = z_P^*$. \square

If the first assumption of the above theorem is relaxed then it can only be proved that case (P1) follows from (D1) and not from (D2) or (D3). It may, however, be possible that (D1) implies also (P2). This case is demonstrated in Example 4.

Theorem 6. *Assume that*

1. *the constraints g_j , $j = 1, \dots, m$, of the deterministic problem (38) are convex,*
2. *$A \subseteq H$, i.e. all nonunique minima of the unconstrained unperturbed function $f(z)$ satisfy the constraints of the perturbed problem (39),*
3. *in the perturbed problem (39) case (P1) is present.*

Then the constraints g_j were satisfied by all nonunique minima of the unconstrained function f , i.e. case (D1) was present in the original problem.

Proof. Due to the convexity of the constraints $g_j(z)$ Jensen's inequality can be applied. Then, since $\mathbf{E}\omega = 0$,

$$g_j(z) \leq \mathbf{E}_\omega g_j(z + \omega) = h_j(z)$$

for all $j = 1, \dots, m$. This means that

$$\begin{aligned} H &= \{z : h_j(z) \leq 0, j = 1, \dots, m\} \\ &\subseteq \{z : g_j(z) \leq 0, j = 1, \dots, m\} = G \end{aligned} \quad (41)$$

Case (P1) is characterised by $z_P^* \in A \subseteq H$. Due to relation (41) $A \subseteq H$ implies $A \subseteq G$. Concluding, if in the perturbed problem case (P1) occurs then in the deterministic formulation case (D1) was present. \square

This means that cases (D2) and (D3) yield the constellation (P2), i.e. the unique minimum z_P^* of the unconstrained perturbed function $F(z)$ does not satisfy the perturbed constraints. The following example demonstrates that from case (D1) also case (P2) may follow, for example when the deterministic problem (38) has strictly convex constraints.

Example 4. Consider the deterministic optimisation problem

$$\begin{aligned} \min_z f(z) \\ g(z) = z^2 - 13 \leq 0 \\ z \in \mathbb{R}^1 \end{aligned}$$

where

$$f(z) = \begin{cases} 25(z+3)^2, & z < -3 \\ 0, & z \in A = [-3, 3] \\ (z-3)^2, & z > 3 \end{cases}$$

This means that $A \subset G = \{z \in \mathbb{R}^1 : g(z) \leq 0\}$, i.e. case (D1) is present.

Assume that the perturbation ω is uniformly distributed on $B = [-6, 6]$ such that the perturbed optimisation problem is

$$\begin{aligned} \min_z F(z) = \mathbf{E}_\omega f(z + \omega) \\ h(z) = \mathbf{E}_\omega g(z + \omega) = z^2 - 1 \leq 0 \end{aligned} \quad (42)$$

where

$$F(z) = \begin{cases} 25(z+3)^2 + 300, & z < -9 \\ \frac{25}{36}(3-z)^3, & z \in [-9, -3] \\ \frac{1}{36}(25(3-z)^3 + (3+z)^3), & z \in [-3, 3] \\ \frac{1}{36}(3+z)^3, & z \in [3, 9] \\ (z-3)^2 + 12, & z \geq 9 \end{cases}$$

The unique minimum of the function $F(z)$ is $z_P^* = 2$ with $F(z_P^*) = 25/6 = 4.167$. However, this point does not satisfy the constraint h . This represents case (P2). Taking into account the constraint h , problem (42) has the solution $\bar{z}_P^* = 1$ with $F(\bar{z}_P^*) = 22/3 = 7.33$.

Only in the cases (D1) and (D2) the existence of nonunique minima in the deterministic problem (38) can be attributed directly to the convexity properties of the objective function $f(z)$. In case (D3) the nonunique minima of the unconstrained function $f(z)$ are not feasible. Consequently, the minima of the constrained problem (38) are situated on the boundary of the feasible area G . This is illustrated in Figure 3. Here the objective function $f(z)$ is represented by its contours $\tilde{L}(e) = \{z \in \mathbb{R}^n : f(z) = e\}$. Then the solution of problem (38) is

$$e^* = \min\{f(z), z \in G\} = \inf\{q \in \mathbb{R}^1 : f(z) \leq q, z \in G\} \quad (43)$$

If the active constraints of Problem (38) and the contour $\tilde{L}(e^*)$ of the objective function are linear they may coincide on a set \tilde{A} . Then obviously $f(\tilde{z}_D^*) = e^*$ for all $\tilde{z}_D^* \in \tilde{A}$ which means that problem (38) has nonunique minima represented by the set \tilde{A} . The conditions of Theorem 1 ensure strict convexity of the function $F(z)$ on the set $A-B$. However, outside of $A-B$ this function may be linear. If it is additionally subject to linear constraints, the perturbed problem (39) may have nonunique minima in the case (P2). This illustrates that the consideration of a perturbation of the decision variable z may not always eliminate the nonunique minima of the original deterministic problem. A further analysis of this situation, however, requires more information about the exact location of the minimum z_P^* and in particular of the feasible set H in dependence on the perturbation and the original functions. This necessitates a more detailed knowledge of the perturbation as well as of the involved original functions. Such studies are only indirectly connected with the convexity properties of the functions and therefore beyond the scope of the present paper. Also a relaxation of the assumptions of Theorems 5 or 6 makes it more difficult to state conditions on the perturbation such that the perturbed problem (39) has a unique minimum. Again a closer knowledge of the deterministic functions and of the characteristics of the perturbation is necessary.

5 Conclusions

We considered convex optimisation problems with nonunique minima and uncertainty about the decision variables. Such a problem type can be found in

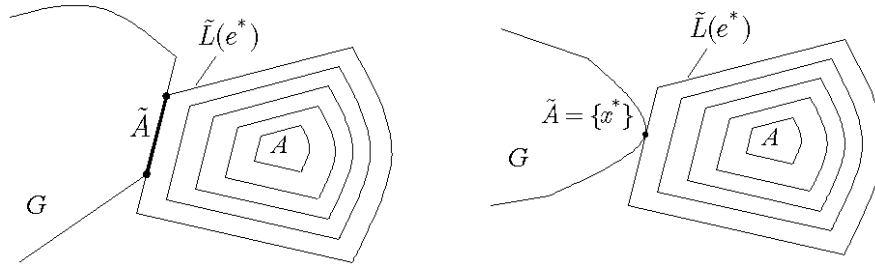


Figure 3: Nonunique and unique minima in case (D3)

stochastic programming problems with a bilevel structure, for example agency models. Our approach is aimed at establishing a third concept for the analysis of bilevel programming problems with nonunique lower level optimal solutions, in addition to the well-known concepts of optimistic and pessimistic formulations of deterministic bilevel programming.

To begin with we analysed an unconstrained convex function $f(z)$ with a set A of nonunique minima. We stated conditions on the uncertainty ω such that the perturbed function $F(z) = \mathbf{E}_\omega f(z + \omega)$ is strictly convex. Additionally we investigated conditions such that the function $F(z)$ will have a unique minimum and when this minimum is in the set A . In a next step we extended the analysis to constrained optimisation problems. In this case the perturbation affects also the feasible area of the considered problems. We specified several cases describing the location of the minima of the unconstrained functions $f(z)$ and $F(z)$ in comparison to the feasible sets. Relations between these cases in the deterministic and the perturbed problems can be established under certain assumptions. A relaxation of the assumptions requires a better knowledge of the considered functions in order to state conditions for unique minima. Further research should focus on this problematic as well as on the case of nonunique minima on the boundary of the feasible area. Also in this case conditions for a unique minimum of the perturbed problem can be given only if the impact of the perturbation on the feasibility of the minima can be evaluated more precisely.

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