SOLITARY WAVE SOLUTIONS TO A CLASS OF WHITHAM–BOUSSINESQ SYSTEMS

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Abstract. In this note we study solitary wave solutions of a class of Whitham–Boussinesq systems which includes the bi-directional Whitham system as a special example. The travelling wave version of the evolution system can be reduced to a single evolution equation, similar to a class of equations studied by Ehrnström, Groves and Wahlén [10]. In that paper the authors prove the existence of solitary wave solutions using a constrained minimization argument adapted to noncoercive functionals, developed by Buffoni [3], Groves and Wahlén [15], together with the concentration-compactness principle.

1. Introduction

This work is devoted to the study of solitary wave solutions of the Whitham–Boussinesq system

$$\begin{aligned} \partial_t \eta &= -K\partial_x u - \partial_x(\eta u) \\ \partial_t u &= -\partial_x \eta - u\partial_x u. \end{aligned} \tag{1.1}$$

A solitary wave is a solution of the form

$$\eta(x,t) = \eta(x-ct),\ u(x,t) = u(x-ct), \tag{1.2}$$

such that $\eta(x-ct), u(x-ct) \longrightarrow 0$ as $|x-ct| \longrightarrow \infty$. Here, η denotes the surface elevation, u is the rightward velocity at the surface, and K is a Fourier multiplier operator defined by

$$\mathcal{F}(Kf)(k) = m(k)\hat{f}(k),$$

for all f in the Schwartz space $\mathcal{S}(\mathbb{R})$. More specifically, we require that

(A1) The symbol $m \in S_\infty^{m_0}(\mathbb{R})$ for some $m_0 < 0$, that is

$$|m^{(\alpha)}(k)| \le C_\alpha(1+|k|)^{m_0-\alpha},\ \alpha \in \mathbb{N}_0.$$

(A2) The symbol $m : \mathbb{R} \to \mathbb{R}$ is even and satisfies $m(0) > 0$, $m(k) < m(0)$, for $k \neq 0$ and

$$m(k) = m(0) + \frac{m^{(2j_*)}(0)}{(2j_*)!}k^{2j_*} + r(k),$$

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for some $j_* \in \mathbb{N}_+$, where $m^{(2j_*)}(0) < 0$ and $r(k) = \mathcal{O}(k^{2j_*+2})$ as $k \to 0$.

As an example we have $m(k) = \tanh(k)k^{-1}$, which yields the bi-directional Whitham (BDW) system, and this choice of symbol is the main motivation for studying (1.1). The BDW system was formally derived in [1,21] from the incompressible Euler equations to model fully dispersive shallow water waves whose propagation is allowed to be both left- and rightward, and appeared in [19, 22] as a full dispersion system in the Boussinesq regime with the dispersion of the water waves system. There have been several investigations on the BDW system: local well-posedness [13, 18] (in homogeneous Sobolev spaces at a positive background), a logarithmically cusped wave of greatest height [11]. There are also numerical results, investigating the validity of the BDW system as a model of waves on shallow water [4], numerical bifurcation and spectral stability [5] and the observation of dispersive shock waves [24]. However there are no results on the existence of solitary wave solutions.

We also mention that one can include the effects of surface tension in the BDW system by choosing $m(k) = \tanh(k)k^{-1}(1 + \beta k^2), \beta > 0$. It was recently shown in [17] that (1.1) is locally well-posed for this choice of symbol. However, the above symbol with $\beta > 0$ is not included in the class of symbols considered in the present work. Moreover, in [6,7,16], other types of fully dispersive Whitham-Boussinesq systems are considered. We also mention the generalized class of Green–Nagdhi equations introduced in [8], which was shown to posses solitary wave solutions in [9].

2. Solitary wave solutions to the Whitham equation

In order to prove existence of solitary wave solutions of (1.1) our strategy will be to reduce this to a problem that is similar to one studied in [10]. For this reason we first discuss the results and methods of that paper. In [10] the authors prove the existence of solitary wave solutions of the pseudodifferential equation

$$u_t + \big(Ku + \tilde{n}(u)\big)_x = 0, \tag{2.1}$$

where K have properties $(A1)$, $(A2)$ and the nonlinearity $\tilde{n}$ satisfies

(A3) The nonlinearity $\tilde{n}$ is a twice continuously differentiable function $\mathbb{R} \to \mathbb{R}$ with

$$\tilde{n}(x) = \tilde{n}_p(x) + \tilde{n}_r(x),$$

in which the leading order part of the nonlinearity takes the form $\tilde{n}_p(x) = c_p|x|^p$ for some $c_p \neq 0$ and $p \in [2, 4j_* + 1)$ or $\tilde{n}_p(x) = c_p x^p$ for some $c_p > 0$ and odd integer p in the range $p \in [2, 4j_* + 1)$, while

$$\tilde{n}_r(x) = \mathcal{O}(|x|^{p+\delta}), \quad \tilde{n}_r'(x) = \mathcal{O}(|x|^{p+\delta-1})$$

for some $\delta > 0$ as $x \to 0$.

In particular, the uni-directional Whitham equation, introduced in [25], belongs to this class of equations (2.1), with $m(k) = \sqrt{\tanh(k)k^{-1}}$. The Whitham equation possesses periodic travelling waves [12] and solitary waves [10], moreover the solitary waves decay exponentially [2]. It was recently confirmed that the Whitham equation possesses a highest cusped wave [14], as conjectured by Whitham.

Under the travelling wave ansatz: $u(t, x) = u(x - ct)$, the equation (2.1) becomes

$$Ku - cu + \tilde{n}(u) = 0. \tag{2.2}$$

The existence of solutions of (2.2) is established via a related minimization problem. Let

$$\tilde{\mathcal{E}}(u) = -\frac{1}{2}\int_{\mathbb{R}} uKu \, \mathrm{d}x - \int_{\mathbb{R}} \tilde{N}(u) \, \mathrm{d}x, \quad \mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 \, \mathrm{d}x$$

with

$$\begin{aligned} \tilde{N}(x) &= \tilde{N}_{p+1}(x) + \tilde{N}_r(x), \\ \tilde{N}_{p+1}(x) &= \int_0^x \tilde{n}_p(s) \, \mathrm{d}s = \frac{c_p x^{p+1}}{p+1}, \text{ or } \frac{c_p x|x|^p}{p+1}, \\ \tilde{N}_r(x) &= \int_0^x \tilde{n}_r(s) \, \mathrm{d}s = \mathcal{O}(|x|^{p+1+\delta}). \end{aligned}$$

Let $q, R > 0$ and

$$V_{q,R} := \{u \in H^1(\mathbb{R}) : \mathcal{I}(u) = q, \ \|u\|_{H^1} < R\}.$$

Minimizers of $\tilde{\mathcal{E}}$ over $V_{q,R}$ (that are not on the boundary) satisfy the Euler-Lagrange equation

$$\mathrm{d}\tilde{\mathcal{E}}(u) + \nu \mathrm{d}\mathcal{I}(u) = 0, \tag{2.3}$$

for a Lagrange multiplier ν, and (2.3) is precisely (2.2), with $c = \nu$. In [10] the authors show that there exist solutions of the minimization problem

$$\arg \inf_{V_{q,R}} \tilde{\mathcal{E}}(u),$$

which by the above argument yields travelling wave solutions of (2.1). The existence of minimizers is established using methods developed in [3,15] and we give here a brief outline of the proof. The functional $\tilde{\mathcal{E}}$ is not coercive and since the domain is unbounded one cannot use the Rellich–Kondrachov theorem. In particular, direct methods cannot be used to obtain a minimizer. Because of this one needs to study a related penalized functional acting on periodic functions. Let $P > 0$ and L^2_P be the space of P- periodic, locally square-integrable functions with Fourier-series representation

$$w(x) = \frac{1}{\sqrt{P}} \sum_{k \in \mathbb{Z}} \widehat{w}(k) \exp(2\pi i k x / P),$$

with

$$\widehat{w}(k) := \frac{1}{\sqrt{P}} \int_{-\frac{P}{2}}^{\frac{P}{2}} w(x) \exp(-2\pi i k x/P)\, \mathrm{d}x.$$

For $s \geq 0$, we define

$$H_P^s := \{w \in L_P^2 : \ \|w\|_{H_P^s} < \infty\},$$

where the norm is given by

$$\|w\|_{H_P^s} := \left(\sum_{k\in\mathbb{Z}} \left(1 + \frac{4\pi^2 k^2}{P^2}\right)^s |\widehat{w}(k)|^2 \right)^{\frac{1}{2}}.$$

The authors [10] studied the following penalized functional

$$\tilde{\mathcal{E}}_{P,\varrho}(u) := \varrho(\|u\|_{H_P^1}^2) + \tilde{\mathcal{E}}_p(u),$$

over the set

$$V_{P,q,R} := \{u \in H_P^1 : \ \mathcal{I}_P(u) = q, \ \|u\|_{H_P^1} < 2R\},$$

where $\tilde{\mathcal{E}}_P$, $\tilde{\mathcal{I}}_P$ are the same functionals as $\tilde{\mathcal{E}}$, $\tilde{\mathcal{I}}$ but where the integration is over $[-P/2, P/2]$, and $\varrho : [0,(2R)^2] \mapsto [0,\infty)$ is a penalization function such that $\varrho(t) = 0$ whenever $t \in [0, R^2]$ and $\varrho(t) \to \infty$ as $t \to (2R)^2$. The penalization function makes $\tilde{\mathcal{E}}_{P,\varrho}$ coercive, and the fact that we are now working in H_P^1 allows the use of the Rellich-Kondrachov theorem. It is then an easy task to show that there exists a minimizer $u_P \in V_{P,q,2R}$, of $\tilde{\mathcal{E}}_{P,\varrho}$. The next step is to show that u_P in fact minimizes $\tilde{\mathcal{E}}_P$ over $V_{q,R}$. This is immediate after showing that

$$\|u_P\|_{H_P^1}^2 \leq Cq,$$

and choosing q sufficiently small. The other key ingredient of the proof is the concentration compactness theorem [20]. In the application of this theorem, the main task is to show that 'dichotomy' does not occur. This is done using proof by contradiction, where the contradiction is arrived at using the strict subadditivity of

$$I_q := \arg \inf_{V_{q,R}} \tilde{\mathcal{E}}(u),$$

as a function of q. The strict subadditivity of I_q is established by using a special minimizing sequence for $\tilde{\mathcal{E}}$, constructed from the minimizers u_P. In addition it is necessary to decompose u into high and low frequencies in order to get satisfactory estimates on $\|u\|_{L^\infty}$, see [10, Corollary 4.5]. It is an easy task to show that 'vanishing' cannot occur either. Therefore, from the concentration compactness theorem, 'concentration' is the only possibility and the existence of minimizers then follows from a standard argument.

Under the additional assumption that

(A4) $\tilde{n} \in C^{2j_*}(\mathbb{R})$ with

$$\tilde{n}_r^{(j)}(x) = \mathcal{O}(|x|^{p+\delta-j}), \ j = 0, \ldots, , 2j_*,$$

it is possible to relate the minimizers of $\tilde{\mathcal{E}}$ to those of $\tilde{\mathcal{E}}_{lw}$, where

$$\tilde{\mathcal{E}}_{lw}(u) = -\int_{\mathbb{R}} \left(\frac{m^{(2j_*)}(0)}{2(2j_*)!}(u^{(j_*)})^2 + \tilde{N}_{p+1}(u) \right) \mathrm{d}x.$$

More specifically,

$$\sup_{u \in \tilde{D}_q} \mathrm{dist}_{H^{j_*}(\mathbb{R})}(S_{lw}^{-1}u, \tilde{D}_{lw}) \to 0, \quad \text{as } q \to 0,$$

where $\tilde{D}_{lw}$ is the set of minimizers of $\tilde{\mathcal{E}}_{lw}$ over the set

$$\{u \in H^{j_*}(\mathbb{R}) : \mathcal{I}(u) = 1\},$$

and $\tilde{D}_q$ is the set of minimizers of $\tilde{\mathcal{E}}$ over $V_{q,R}$ and

$$(S_{lw}u)(x) := q^{\alpha}u(q^{\beta}x)$$

is the 'long-wave test function' with

$$\alpha = \frac{2j_*}{4j_* + 1 - p}, \quad \beta = \frac{p-1}{4j_* + 1 - p}. \tag{2.4}$$

The numbers α and β are chosen in such a way that

$$2\alpha - \beta = 1, \quad (p-1)\alpha = 2j_*\beta.$$

This choice of α, β appear naturally when deriving the long-wave approximation of (2.2). The functional $\tilde{\mathcal{E}}_{lw}$ is related to $\tilde{\mathcal{E}}$ via (see [10, Lemma 3.2])

$$\tilde{\mathcal{E}}(S_{lw}u) = -qm(0) + q^{1+(p-1)\alpha}\tilde{\mathcal{E}}_{lw}(u) + o(q^{1+(p-1)\alpha}),$$

for any $u \in W := \{u \in H^{2j_*}(\mathbb{R}) : \|u\|_{H^{2j_*}} < S\}$ with S being a positive constant.

We mention here a recent work [23] where they use an entirely different approach to prove the existence of small amplitude solitary wave solutions of the Whitham equation.

3. Solitary wave solutions to the Whitham–Boussinesq system

3.1. **Formulation as a constrained minimization problem.** In the present work we seek solitary wave solutions of (1.1), and the idea is to reformulate (1.1) in such a way that the method of [10] can be applied. Under the travelling wave ansatz (1.2), the system (1.1) then becomes

$$c\eta = Ku + \eta u, \tag{3.1}$$

$$cu = \eta + \frac{u^2}{2}. \tag{3.2}$$

It follows from (3.2) that $\eta = u(c - \frac{u}{2})$, and if we insert this into (3.1) then we find that

$$Ku - u(u-c)(\frac{u}{2} - c) = 0. \tag{3.3}$$

We first formally assume that $\|u\|_{L^\infty} \ll c$ to formulate (3.3) into a variational problem. This is no restriction since the constructed solutions will automatically satisfy this smallness condition (see Theorem (3.1)). Let $w = \frac{u}{c}(\frac{u}{c} - 2)$, so that $u = c - c\sqrt{1+w}$. The map $w \mapsto u$ is well-defined, since

$$\|w\|_{L^\infty} \leq \left\|\frac{u}{c}\right\|_{L^\infty} \left\|\frac{u}{c} - 2\right\|_{L^\infty} \lesssim \left\|\frac{u}{c}\right\|_{L^\infty} \ll 1,$$

We then may rewrite the equation (3.3) using the new unknown w as

$$\frac{2}{\sqrt{1+w}} K(\sqrt{1+w} - 1) - \lambda w = 0, \tag{3.4}$$

with $\lambda = c^2$. We now define

$$\mathcal{E}(w) = \underbrace{-\frac{1}{2}\int_{\mathbb{R}} wKw \; \mathrm{d}x}_{:=\mathcal{K}(w)} \underbrace{- \int_{\mathbb{R}} N(w) \; \mathrm{d}x}_{:=\mathcal{N}(w)},$$

where

$$N(w) = 2\Psi(w)Kw + 2\Psi(w)K(\Psi(w)),$$

$$\Psi(w) = \sqrt{1+w} - 1 - \frac{w}{2} = -\frac{w^2}{8} + \Psi_r(w),$$

$$\Psi_r(x) = \mathcal{O}(x^3).$$

To extract the lower-order parts we also write

$$N(w) = N_h(w) + N_l(w),$$

with

$$N_h(w) = -\frac{w^2}{4}Kw, \quad N_l(w) = 2\Psi(w)Kw + 2\Psi(w)K(\Psi(w)).$$

We then note that

$$\mathrm{d}\mathcal{E}(w) + \lambda \mathrm{d}\mathcal{I}(w) = 0$$

is precisely (3.4). Hence, w is a critical point of $\mathcal{E}$ under the constraint $\mathcal{I}(w) = q$, if and only if $u = c - c\sqrt{1+w}$ is a solution of (3.3), with $\lambda = c^2$. We will find critical points of $\mathcal{E}(w) + \lambda\mathcal{I}(w)$ by considering the minimization problem

$$\arg \inf_{V_{q,R}} \mathcal{E}(w).$$

Here we are minimizing a functional $\mathcal{E}$ of almost the same type as in [10], with $p = 2$, but with a slightly different nonlinearity. In our case, the nonlocal operator K appears in the nonlinear term N. However, since K is a bounded smoothing operator, it is not hard to show that the methods used in [10] can be applied to the functional $\mathcal{E}$. However, the results [10, Lemma 2.3, Lemma 3.2, Lemma 3.3] require a bit more care, in particular it is important to know how $\mathcal{N}$ acts under the long-wave scaling, and we therefore include the proofs of these results in the next subsection. We finally have the following existence result:

Theorem 3.1. *There exists $q_* > 0$ such that the following statements hold for each $q \in (0, q_*)$.*

(i) The set D_q of minimizers of $\mathcal{E}$ over the set $V_{q,R}$ is nonempty and the estimate $\|w\|^2_{H^1(\mathbb{R})} = \mathcal{O}(q)$ holds uniformly over $w \in D_q$. Each element of D_q is a solution of the travelling wave equation (3.4)*; the squared wave speed c^2 is the Lagrange multiplier in this constrained variational principle.*

(ii) Let $s < 1$ and suppose that $\{w_n\}_{n\in\mathbb{N}_0}$ is a minimizing sequence for $\mathcal{E}$ over $V_{q,R}$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ of real numbers such that a subsequence of $\{w_n(\cdot + x_n)\}_{n\in\mathbb{N}_0}$ converges in $H^s(\mathbb{R})$ to a function in D_q.

3.2. **Technical results.** In our case the long-wave functional $\mathcal{E}_{lw}$ is given by

$$\mathcal{E}_{lw}(w) := -\int_{\mathbb{R}} \left(\frac{m^{2j_*}(0)}{2(2j_*)!}(w^{(j_*)})^2 - \frac{m(0)}{4} w^3 \right) \mathrm{d}x,$$

and we also recall the long-wave scaling:

$$S_{lw}w(x) = \mu^\alpha w(\mu^\beta x),$$

with

$$\alpha = \frac{2j_*}{4j_* - 1} \quad \text{and} \quad \beta = \frac{1}{4j_* - 1}. \tag{3.5}$$

Note that (3.5) is a special case of (2.4), with $p = 2$.

We first present a result corresponding to [10, Lemma 3.2], which relates $\mathcal{E}$ with $\mathcal{E}_{lw}$.

Lemma 3.2. *Let $w \in W$ with $\|w\|_{L^\infty} \ll 1$ and $\mathcal{I}(w) = 1$. Then*

$$\mathcal{E}(S_{lw}w) = -qm(0) + q^{1+\alpha}\mathcal{E}_{lw}(w) + o(q^{1+\alpha}). \tag{3.6}$$

Proof. Recall the definition

$$\mathcal{E}(S_{lw}w) = \mathcal{K}(S_{lw}w) + \mathcal{N}(S_{lw}w).$$

We first calculate that

$$\begin{aligned}
&\mathcal{K}(S_{lw}w)\\
&= -\frac{1}{2}\int_{\mathbb{R}} q^{2\alpha} w(q^\beta x) K w(q^\beta \cdot)(x)\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{\mathbb{R}} q^{2\alpha} m(k) |\mathcal{F}(w(q^\beta \cdot))(k)|^2\,\mathrm{d}k\\
&= -\frac{1}{2}\int_{\mathbb{R}} q^{2\alpha-\beta} \left(m(0) + q^{2j_*\beta}\frac{m^{(2j_*)}(0)}{(2j_*)!} k^{2j_*} + r(q^\beta k) \right) |\hat{w}(k)|^2\ \mathrm{d}k\\
&= -qm(0) - q^{2\alpha+(2j_*-1)\beta}\int_{\mathbb{R}} \frac{m^{(2j_*)}(0)}{2(2j_*)!}(w^{j_*})^2\ \mathrm{d}x - \frac{q^{2\alpha-\beta}}{2}\int_{\mathbb{R}} r(q^\beta k)|\hat{w}(k)|^2\,\mathrm{d}k,
\end{aligned}$$

and one may continuously estimate the last term as

$$\left|\frac{q^{2\alpha-\beta}}{2}\int_{\mathbb{R}} r(q^\beta k)|\hat{w}(k)|^2\,\mathrm{d}k\right| \lesssim q^{2\alpha+(2j_*+1)\beta}\int_{\mathbb{R}} k^{2j_*+2}|\hat{w}(k)|^2\,\mathrm{d}k,$$

and $\int_{\mathbb{R}} k^{2j_*+2}|\hat{w}(k)|^2 \, \mathrm{d}k$ is uniformly bounded, since $w \in W$. We next consider

$$\mathcal{N}(S_{lw}w) = -\int_{\mathbb{R}} N_h(S_{lw}w) + N_l(S_{lw}w) \, \mathrm{d}x.$$

A direct calculation shows that

$$\begin{aligned}
&-\int_{\mathbb{R}} N_h(S_{lw}w) \, \mathrm{d}x = \int_{\mathbb{R}} \frac{q^{3\alpha}}{4} w^2(q^\beta x) K w(q^\beta \cdot)(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{q^{3\alpha-\beta}}{4} \overline{\mathcal{F}(w^2)(k)} \hat{w}(k) \left(m(0) + q^{2j_*\beta} \frac{m^{(2j_*)}(0)}{(2j_*)!} k^{2j_*} + r(q^\beta k) \right) \mathrm{d}k \\
&= q^{3\alpha-\beta} \int_{\mathbb{R}} \frac{m(0)}{4} w^3 \, \mathrm{d}x + o(q^{3\alpha-\beta}),
\end{aligned}$$

where we again used that $w \in W$ in order to estimate the remaining terms. The term $\int_{\mathbb{R}} N_l(S_{lw}w) \, \mathrm{d}x$ is of lower order and can be estimated in the same way.

Combining all the above estimates yields the identity (3.6).

□

We next move to the corresponding result of [10, Lemma 3.2].

Lemma 3.3. *Let*

$$\mathcal{K}_P(w) = -\frac{1}{2} \int_{-\frac{P}{2}}^{\frac{P}{2}} wKw \, \mathrm{d}x, \quad \mathcal{N}_P(w) = -\int_{-\frac{P}{2}}^{\frac{P}{2}} N(w) \, \mathrm{d}x,$$

$$\mathcal{E}_P(w) = \mathcal{K}_P(w) + \mathcal{N}_P(w),$$

and let $\{\tilde{w}_P\}$ *be a bounded family of functions in* $H^1(\mathbb{R})$ *with* $\|\tilde{w}_P\|_{L^\infty(\mathbb{R})} \ll 1$ *such that*

$$\operatorname{supp}(\tilde{w}_P) \subset (-\frac{P}{2}, \frac{P}{2}) \quad \text{and} \quad \operatorname{dist}\big(\pm\frac{P}{2}, \operatorname{supp}(\tilde{w}_P)\big) \geq \frac{1}{2} P^{\frac{1}{4}},$$

and define $w_P \in H^1_P$ *by the formula*

$$w_P = \sum_{j \in \mathbb{Z}} \tilde{w}_P(\cdot + jP).$$

(i) *The function* w_P *satisfies*

$$\lim_{P\to\infty} \|K\tilde{w}_P - Kw_P\|_{H^1(-\frac{P}{2},\frac{P}{2})} = 0, \quad \lim_{P\to\infty} \|K\tilde{w}_P\|_{H^1(|x|>\frac{P}{2})} = 0.$$

(ii) *The functionals* $\mathcal{E}$, $\mathcal{I}$ *and* $\mathcal{E}_P$, $\mathcal{I}_P$ *have the properties that*

$$\lim_{P\to\infty} \big(\mathcal{E}(\tilde{w}_P) - \mathcal{E}_P(w_P)\big) = 0, \quad \mathcal{I}(\tilde{w}_P) = \mathcal{I}_P(w_P),$$

and

$$\begin{aligned}
&\lim_{P\to\infty} \big\|\mathcal{E}'(\tilde{w}_P) - \mathcal{E}_P'(w_P)\big\|_{H^1(-\frac{P}{2},\frac{P}{2})} = 0, \quad && \lim_{P\to\infty} \big\|\mathcal{E}'(\tilde{w}_P)\big\|_{H^1(-\frac{P}{2},\frac{P}{2})} = 0 \\
&\big\|\mathcal{I}'(\tilde{w}_P) - \mathcal{I}_P'(w_P)\big\|_{H^1(-\frac{P}{2},\frac{P}{2})} = 0, && \big\|\mathcal{I}'(\tilde{w}_P)\big\|_{H^1(|x|>\frac{P}{2})} = 0.
\end{aligned}$$

To prove Lemma 3.3, we need the following technical result of [10, Proposition 2.1].

Proposition 3.4. *The linear operator K satisfies*
(a) K belongs to $C^\infty(H^s(\mathbb{R}), H^{s+|m_0|}(\mathbb{R})) \cap C^\infty(\mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ for each $s \geq 0$.
(b) For each $j \in \mathbb{N}$ there exists a constant $C_l = C(\|m^{(l)}\|_{L^2(\mathbb{R})}) > 0$ such that

$$|Kf(x)| \leq \frac{C_l \|f\|_{L^2}}{\operatorname{dist}\big(x, \operatorname{supp}(f)\big)^l}, \quad x \in \mathbb{R} \setminus \operatorname{supp}(f),$$

for all $f \in L^2_c(\mathbb{R})$.

Proof of Lemma 3.3. The limits in (i) are proved in [10, Proposition 2.1], so we turn to (ii). Using (i) we get that $\mathcal{K}(\tilde{w}_P) - \mathcal{K}(w_P) \to 0$, as $P \to \infty$. Note that

$$\begin{aligned}
\mathcal{N}(\tilde{w}_P) &= -2\int_{\mathbb{R}} \Psi(\tilde{w}_P) K\tilde{w}_P + \Psi(\tilde{w}_P) K(p(\tilde{w}_P))\,\mathrm{d}x \\
&= -2\int_{-\frac{P}{2}}^{\frac{P}{2}} \Psi(w_P) K\tilde{w}_P + \Psi(w_P) K(\Psi(\tilde{w}_P))\,\mathrm{d}x \\
&= -2\int_{-\frac{P}{2}}^{\frac{P}{2}} \Psi(w_P) K(\tilde{w}_P - w_P) + \Psi(w_P) K\big(\Psi(\tilde{w}_P) - \Psi(w_P)\big)\,\mathrm{d}x \\
&\quad + \mathcal{N}_P(w_P).
\end{aligned} \tag{3.7}$$

In light of (i) we have

$$\begin{aligned}
&\left| \int_{-\frac{P}{2}}^{\frac{P}{2}} \Psi(w_P) K(\tilde{w}_P - w_P)\,\mathrm{d}x \right| \\
&\leq \|\Psi(w_P)\|_{L^2(-\frac{P}{2},\frac{P}{2})} \|K(\tilde{w}_P - w_P)\|_{L^2(-\frac{P}{2},\frac{P}{2})} \to 0, \quad \text{as } P \longrightarrow \infty.
\end{aligned} \tag{3.8}$$

Since $\|\tilde{w}_P\|_{L^\infty} \ll 1$, we have $\|w_P\|_{L^\infty} \ll 1$. To estimate the second term on the right hand side of (3.7), one first calculates

$$\begin{aligned}
\Psi(\tilde{w}_P) - \Psi(w_P) &= \sqrt{1+\tilde{w}_P} - \sqrt{1 + \sum_{j\in\mathbb{Z}} \tilde{w}_P(\cdot + jP)} + \frac{1}{2}\sum_{|j|\geq 1} \tilde{w}_P(\cdot jP) \\
&= -\frac{\sum_{|j|\geq 1} \tilde{w}_P(\cdot + jP)}{\sqrt{1+\tilde{w}_P} + \sqrt{1+w_P}} + \frac{1}{2}\sum_{|j|\geq 1} \tilde{w}_P(\cdot + jP) \\
&= \left(\frac{1}{2} - \frac{1}{\sqrt{1+\tilde{w}_P} + \sqrt{1+w_P}}\right) \sum_{|j|\geq 1} \tilde{w}_P(\cdot + jP),
\end{aligned}$$

and then applies Proposition 3.4 to get

$$
\begin{aligned}
&\int_{-\frac{P}{2}}^{\frac{P}{2}} \left|K\big(\Psi(\tilde{w}_P)-\Psi(w_P)\big)\right|^2 \mathrm{d}x \\
&\leq \int_{-\frac{P}{2}}^{\frac{P}{2}} \Big|\sum_{|j|\geq 1} K\left[\tilde{w}_P(\cdot+jP)\big(\frac{1}{2}-\frac{1}{\sqrt{1+\tilde{w}_P}+\sqrt{1+w_P}}\big)\right]\Big|^2 \mathrm{d}x \\
&\lesssim \int_{-\frac{P}{2}}^{\frac{P}{2}} \left(\sum_{|j|\geq 1} \frac{\left\|\tilde{w}_P(\cdot+jP)\left(\frac{1}{2}-\frac{1}{\sqrt{1+\tilde{w}_P}+\sqrt{1+w_P}}\right)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})}}{\operatorname{dist}\big(x+jP,\operatorname{supp}(\tilde{w}_P)\big)^3}\right)^2 \mathrm{d}x \qquad (3.9)\\
&\lesssim \|\tilde{w}_P\|_{L^2} \int_{-\frac{P}{2}}^{\frac{P}{2}} \big(\sum_{|j|\geq 1}\frac{1}{(jP+\frac{1}{2}P^{\frac{1}{4}})^3}\big)^2 \mathrm{d}x \\
&\to 0, \quad \text{as } P\longrightarrow\infty.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&\left|\int_{-\frac{P}{2}}^{\frac{P}{2}} \Psi(w_P)K\big(\Psi(\tilde{w}_P)-\Psi(w_P)\big)\ \mathrm{d}x\right| \\
&\leq \|\Psi(w_P)\|_{L^2(-\frac{P}{2},\frac{P}{2})} \left\|K\big(\Psi(\tilde{w}_P)-\Psi(w_P)\big)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})} \to 0, \quad \text{as } P\longrightarrow\infty.
\end{aligned} \tag{3.10}
$$

From (3.7), (3.8) and (3.10), it follows that $\mathcal{N}(\tilde{w}_P)-\mathcal{N}_P(w_P)\to 0$, which in turn implies that

$$
\mathcal{E}(\tilde{w}_P)-\mathcal{E}_P(w_P)\to 0, \quad \text{as } P\longrightarrow\infty.
$$

The equality $\mathcal{I}(\tilde{w}_P)=\mathcal{I}_P(w_P)$ is immediate.

A direct calculation yields

$$
\mathcal{N}'(w) = -\left(\frac{1}{\sqrt{1+w}}-1\right)Kw - \frac{2}{\sqrt{1+w}}K(\Psi(w)),
$$

so we may estimate

$$
\begin{aligned}
&\left\|\mathcal{N}'(\tilde{w}_P)-\mathcal{N}_P'(w_P)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})} \\
&\leq \left\|\left(\frac{1}{\sqrt{1+w_P}}-1\right)(K\tilde{w}_P-Kw_P)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})} \\
&\quad + \left\|\frac{2}{\sqrt{1+w_P}}K\big(\Psi(\tilde{w}_P)-\Psi(w_P)\big)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})} \\
&\to 0, \quad \text{as } P\longrightarrow\infty,
\end{aligned}
$$

where we have used (i) and (3.9). One can similarly show that

$$
\left\|\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{N}'(\tilde{w}_P)-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{N}_P'(w_P)\right\|_{L^2(-\frac{P}{2},\frac{P}{2})} \to 0, \quad \text{as } P\to\infty.
$$

Hence

$$\left\|\mathcal{E}'(\tilde{w}_P)-\mathcal{E}_P'(w_P)\right\|_{H^1(-\frac{P}{2},\frac{P}{2})}\to 0,\quad \text{as } P\to\infty.$$

Note that $\frac{1}{\sqrt{1+\tilde{w}_P}}-1=0$ for $|x|>\frac{P}{2}$, we calculate

$$\begin{aligned}&\left\|\mathcal{N}'(\tilde{w}_P)\right\|_{L^2(|x|>\frac{P}{2})}\\&=\left\|\left(\frac{1}{\sqrt{1+\tilde{w}_P}}-1\right)K\tilde{w}_P+\frac{2}{\sqrt{1+\tilde{w}_P}}K(\Psi(\tilde{w}_P))\right\|_{L^2(|x|>\frac{P}{2})}\\&=\left\|\frac{2}{\sqrt{1+\tilde{w}_P}}K(\Psi(\tilde{w}_P))\right\|_{L^2(|x|>\frac{P}{2})}.\end{aligned}$$

Since $\operatorname{supp}(\Psi(\tilde{w}_P))=\operatorname{supp}(\tilde{w}_P)$, we have $\|K(\Psi(\tilde{w}_P))\|_{L^2(|x|>\frac{P}{2})}\to 0$. It follows that

$$\left\|\mathcal{N}'(\tilde{w}_P)\right\|_{L^2(|x|>\frac{P}{2})}\to 0,\quad \text{as } P\to\infty.$$

A similar calculation shows that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{N}'(\tilde{w}_P)\right\|_{L^2(|x|>\frac{P}{2})}\to 0.$$

Consequently, we have

$$\left\|\mathcal{N}'(\tilde{w}_P)\right\|_{H^1(|x|>\frac{P}{2})}\to 0,\quad \text{as } P\to\infty.$$

□

Just as in [10, Theorem 6.3] we can relate the minimizers of $\mathcal{E}$ with those of $\mathcal{E}_{lw}$:

$$\sup_{w\in D_q}\operatorname{dist}_{H^{j_*}(\mathbb{R})}(S_{lw}^{-1}w,D_{lw})\to 0,\qquad \text{as } q\to 0,$$

where D_{lw} is the set of minimizers of $\mathcal{E}_{lw}$ over the set

$$\{w\in H^{j_*}(\mathbb{R}):\ \mathcal{I}(w)=1\},$$

and D_q is the set of minimizers of $\mathcal{E}$ over $V_{q,R}$.

We finally include a regularity result for the travelling wave solutions of (3.4) which corresponds to [10, Lemma 2.3].

Lemma 3.5. *Let w be a solution of* (3.4) *in with $\|w\|_{L^\infty}\ll 1$. Then for any $k\in\mathbb{N}_+$, $w\in H^k$ and satisfies*

$$\|w\|_{H^k}\le C(k,\|w\|_{H^1}).$$

Proof. Let

$$f=\sqrt{1+w}-1,$$

then one has $\|f\|_{L^\infty}\ll 1$ due to $\|w\|_{L^\infty}\ll 1$. In view of (3.4), f solves

$$f=\frac{2}{\lambda(1+f)(2+f)}Kf. \tag{3.11}$$

Differentiating in (3.11) yields

$$\partial_x f = \frac{2}{\lambda[(1+f)(2+f)+f(2+f)+f(1+f)]} K\partial_x f. \tag{3.12}$$

The denominator is positive due to $\|f\|_{L^\infty} \ll 1$.

Let $l \in \{1, 2, \cdots, k\}$. For each fixed $f \in H^l$ we define a formula ϕ_f by

$$\phi_f(g) = \frac{2}{\lambda[(1+f)(2+f)+f(2+f)+f(1+f)]} g.$$

Then one now may follow the argument in [EGW, Lemma 2.3] by using the properties of ϕ_f and K to show

$$\|\partial_x f\|_{H^l} \le C(\|f\|_{H^1})\|\partial_x f\|_{L^2}.$$

For completeness, we give its proof here. For any $s \in [0, l]$, it is easy to see that ϕ_f and K define an operator in $B(H^s, H^s)$ and $B(H^s, H^{s+|m_0|})$, respectively. Thus the composition

$$\psi_f = \phi_f \circ K \in B(H^s, H^{s_*}), \quad s_* = \min\{l, s + |m_0|\},$$

and the norm of ψ_f depends upon $\|f\|_{H^l}$. Consequently, any solution g of $g = \psi_f(g)$ belongs to H^{s_*} and satisfies

$$\|g\|_{H^{s_*}} \le C_{l, \|f\|_{H^l}} \|g\|_{H^s}.$$

Applying this argument recursively, one finds that any solution $g \in L^2$ belongs to H^l and satisfies

$$\|g\|_{H^l} \le C(l, \|f\|_{H^l})\|g\|_{L^2}.$$

Since (3.12) is equivalent to $\partial_x f = \psi_f(\partial_x f)$, a bootstrap argument shows that $f' \in H^l$ with

$$\|\partial_x f\|_{H^l} \le C(l, \|f\|_{H^1})\|\partial_x f\|_{L^2}, \ l = 1, 2, \cdots, k.$$

So far we have shown that

$$\|f\|_{H^k} \le C(k, \|f\|_{H^1}).$$

Finally, recalling that $w = f^2 + 2f$ and H^l is an algebra, we therefore obtain

$$\|w\|_{H^k} \le C(k, \|f\|_{H^1}) \le C(k, \|w\|_{H^1}),$$

where we have used $\|w\|_{L^\infty} \ll 1$ in the last inequality.

□

Remark 3.6. *The results of the present work may be extended to a more general class of nonlinearities N. On the one hand, we have that the leading order part of N is cubic, but this could be extended to higher power nonlinearities. On the other hand, the multiplier operator K appearing in N can be replaced by an operator K' belonging to a wider class of Fourier multipliers. For instance, it is not necessary for the symbol of this K' to be of negative order. An example is $K' = Id$, which yields the nonlinearities studied in [10].*

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