SHUNNING ALGEBRAIC FORMALISM: STUDENT TEACHERS AND THE INTRICACY OF PERCENTS

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SANS FORMALISME ALGÉBRIQUE AUCUN : DES ÉLÈVES PROFESSEURS FACE AUX PIÈGES DES POURCENTAGES

Résumé – Cette étude porte sur l'emploi et la maîtrise, par de futurs professeurs du primaire et du début du collège, du formalisme algébrique dans la détermination du taux de croissance de l'aire d'un carré dont le côté croît de *p* %. Les données empiriques consistent en l'enregistrement vidéo d'une séance de travail d'une triade d'élèves professeurs s'efforçant de résoudre ce problème (en interaction avec leur professeur). Les outils d'analyse utilisés relèvent de la théorie des situations didactiques en mathématiques et de la théorie sémiotique. L'analyse montre comment la notation en termes de pourcentages complique la tâche visée et comment se trouve évité tout calcul algébrique. On montre en outre comment une évolution du milieu engendre la création d'un matériel fournissant aux élèves professeurs une représentation du problème relevant d'un autre registre sémiotique, celui des figures géométriques, dont la traduction en pourcentages permet d'arriver à une solution au problème étudié. La discussion des implications didactiques de cette étude porte sur la conception de la tâche considérée, d'une part, et sur l'apport possible en termes de formation des enseignants, d'autre part.

Mots clés: algèbre, pourcentages, milieu matériel, systèmes de représentation, formation des enseignants.

REHUYENDO EL FORMALISMO ALGEBRAICO: LOS PROFESORES EN FORMACIÓN Y EL LABERINTO DE LOS PORCENTAJES

Resumen – Este artículo investiga cómo los estudiantes en formación inicial para profesores de educación primaria y secundaria obligatoria utilizan y dominan el formalismo algebraico al resolver la tarea de determinar la tasa de crecimiento del área de un cuadrado cuando el lado de éste se incrementa un *p*%. El material empírico consiste en una grabación en vídeo de una sesión de trabajo en pequeño grupo en la que un grupo de tres estudiantes resuelve la tarea (interactuando con la formadora). Se utilizan como herramientas de análisis nociones de la teoría de las situaciones didácticas en matemáticas y de la teoría

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semiótica. El análisis muestra cómo la notación de los porcentajes complica la tarea y cómo se evitan los cálculos algebraicos. Se muestra también cómo una evolución del medio da lugar a la creación de manipulativos que proporcionan a los estudiantes representaciones en distintos registros semióticos, el de las figuras geométricas planas, que se transforman en notación porcentual con el fin de resolver la tarea. Se discuten finalmente las implicaciones didácticas tanto en el diseño de tareas de este tipo como en la formación del profesorado.

Palabras-claves: algebra, porcentajes, medio material, sistemas de representación, formación del profesorado.

ABSTRACT

This paper investigates primary and lower secondary school student teachers' use and mastery of algebraic formalism in solving the task of determining the rate of increase of a square's area when the square's side increases by *p*%. The empirical material consists of a video-recorded small-group session where a triad of student teachers was solving this task (with teacher interaction). Notions from the theory of didactical situations in mathematics and semiotic theory are used as analytic tools. The analysis shows how percentage notation complicated the task, and how algebraic calculations were avoided. It is further shown how an evolution of the milieu gave rise to creation of manipulatives that provided the student teachers with representations from a different semiotic register, that of plane geometric figures, which were transformed into percentage notation that enabled them to produce a solution to the task. Didactical implications are discussed regarding design of such a task on the one hand, and teacher education on the other hand.

Key words: algebra, material milieu, percentage, representation systems, teacher education.

INTRODUCTION

Algebra is by many considered to be the language through which generalization of quantities and relationships between quantities can be expressed and manipulated (e.g., Whitehead, 1947; Kieran, 2004). A more recent view is that elementary algebra is a modelling tool for arithmetic phenomena which goes well beyond mere generalization (Bosch, 2015; Ruiz-Munzón, Bosch & Gascón, 2013). Algebraic knowledge is relevant in everyday life and working life, either directly, or as a tool (Katz, 2007; Kendal & Stacey, 2004). However, students find algebra difficult to learn, a situation that is well documented in the research literature (e.g., Herscovics & Linchevski, 1994; Kieran, 1992, 2007; Küchemann, 1981; MacGregor & Stacey, 1997; Strømskag, 2015). Studies also show that algebra is difficult to teach (e.g., Stacey, Chick & Kendal, 2004; Strømskag Måsøval, 2011; Watson, 2009).

In the prevailing mathematics subject curriculum in Norway (Directorate for Education and Training, 2013), *Numbers and algebra* is one of four main subject areas in Grades 5–7 (age 10–13), and one of five main subject areas in Grades 8–10 (age 13–16).[1] Results from international comparative studies show that Norwegian pupils struggle with algebra: In *Trends in International Mathematics and Science Study* (TIMSS 1995, 2003, 2007, 2011), which measures pupils' competency in mathematics and science in Grades 4/5 and 8/9 in about 60 countries, Norwegian pupils have scored considerably lower than average in algebra (Grønmo et al., 2012). TIMSS 2015 shows that Norwegian pupils score *very low* in algebra compared to how they score in numbers, geometry and statistics, subjects in which there has been a positive development from TIMSS 2011 (Bergem, 2016).

Moreover, *The Programme for International Student Assessment* (PISA) measures to what extent pupils of age 15 have achieved competencies in mathematics, reading, and science. In PISA 2015, which focused on science, 72 countries participated, including 35 OECD countries (OECD, 2016). The Norwegian achievements in mathematics have been relatively stable in the five PISA-tests from 2003 to 2015, even if there was an improvement from 2012 to 2015, and for the first time, Norway was (in 2015) above the OECD average in mathematics (Nortvedt & Pettersen, 2016). However, PISA 2012, which focused on mathematics, showed that Norwegian pupils are particularly weak in solving tasks related to the use of mathematical formal competency (Kjærnsli & Olsen, 2013), an area where algebra is an important domain.

In light of the situation described above, it is relevant to study how algebra is taught to student teachers—that is, how prospective teachers are provided with material and non-material tools that enable their facilitation of students' problem solving. This paper reports from a study where I observed and analyzed what was going on in a teacher education program for primary and lower secondary education in Norway on the question of use and mastery of algebraic formalism in solving generalization problems. Furthermore, I was interested in the particular case of student teachers' determining the rate of increase of the square of a quantity, q^2, when the increase of q is known, and how they were handling percents in this case.

Percent is a treasured concept, due to its essential place in the secondary school curriculum, and its frequent use in the popular press and in news broadcasts. However, it is difficult to learn. In a review article on percent, Parker and Leinhardt (1995) provide rich evidence from nearly seven decades of research efforts, which shows that percent is one of the most difficult topics of elementary mathematics. Percent is hard to teach and to learn because it is ambiguous and subtle. Also for Norwegian students, percent is highly problematic. This is substantiated by a prior-knowledge test of beginner university students' level of mathematical knowledge from compulsory school. The test is administered by the Norwegian Mathematical Council every second year since 1984 to students enrolled on programs containing at least 60 ECTS credit points in mathematics. A percent task on the test asks the students to calculate (without using a calculator) the percent of girls at a school, given the number of girls and the number of boys at this school. In 2017, less than half of the students solved this task correctly (Nortvedt & Bulien, 2018). For example, (in 2017) master students in teacher education for Grades 1–7 had the lowest score, 24.5%; master students in teacher education for Grades 5–10 had score 38.8%; and master students in technology (engineering) had the highest score, 68.3%.

In a review of CERME papers on algebraic thinking from 1998 to 2017, Hodgen, Oldenburg and Strømskag (2018) point at the need for a more systematic analysis of mathematical tasks used as instruments in classroom research on algebra. I aim to address this in this paper, where an important part of the study is the *a priori* analysis of the teaching situation, in which the mathematical task plays a crucial role. The research question I have sought to answer is: "What conditions enable or hinder the students to solve the mathematical problem they are confronted with?" In particular, *what material and non-material tools prove relevant or missing in the milieu that they resort to?"*

Following this introduction, I present the theoretical framework for the study, followed by an outline of the methodical approach. Next, I turn to *a priori* and *a posteriori* analyses of the teaching situation, followed by a discussion and conclusion.

A THEORETICAL ORIENTATION TOWARDS ALGEBRA

Algebra and algebraic thinking

The American heritage dictionary of the English language defines *algebra* as: "A branch of mathematics in which symbols, usually letters of the alphabet, represent numbers or members of a specified set and are used to represent quantities and to express general relationships that hold for all members of the set" (Algebra, 2019). This, I believe, is a recognizable definition of algebra as a body of knowledge characterized by the *use of symbols*. In this section, I develop a conceptual framework for algebra and algebraic thinking—essential for the study reported here—where I address the issue of what the use of symbols involve.

The framework for algebra and algebraic thinking I have adopted for this study is a combination of three theoretical elements: 1) a description of the *mental activity* associated with algebra (based on Gattegno, 1988/2010), 2) a characterization of algebraic thinking in terms of two *core activities* (based on Kaput, 2008), and 3) a characterization of the *objects* involved in algebraic thinking (based on Radford, 2018). These elements are what I explain next.

The first element of the framework is Gattegno's (1988/2010) description of algebra as being what we do when we *operate on operations* on objects such as for instance equations, polynomials, and mappings. By operating on operations he means combination, or merging, of two operations by replacing the two by one. For example, when evaluating the expression $4 \times (1 + \frac{p}{100})$, multiplication and addition are combined to coalesce into the operation of addition, $4 + \frac{4 \times p}{100}$ (or, $4 + 0.25 \times p$). Here, the combination of the two operations illustrates the distributive property of multiplication over addition, a quality stated by one of the field axioms (applied here to the field of rational numbers). Following Gattegno (1988/2010), "algebra is another way of speaking of the *mental dynamics* necessary to transform a mental given into another mental form, which is kept related to the first" (p. 78, emphasis added). The above example illustrates the mental dynamics necessary to transform the given polynomial into an equivalent one. Although Gattegno (1988/2010) does not refer to "algebraic thinking", his account of algebra is all about reasoning and can thus be seen to equate algebra and algebraic thinking. I find his explication useful because it distinguishes algebraic thinking from arithmetic thinking.

The second element of the framework is Kaput's (2008) characterization of algebraic thinking in terms of two core activities: a) generalization and expression of generalities in increasingly formal and conventional symbol systems; and b) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. These activities are understood as being embodied in three strands of school algebra: algebra as the study of structures and systems abstracted from computations and relations; algebra as the study of functions, relations, and joint variation; and algebra as the application of a cluster of modelling languages to reason about and express properties of situations being modelled (Kaput, 2008).

Two remarks are in order here. The first remark is that I view the logical bases underlying the two types of algebraic thinking (generalization, and reasoning with symbolic forms) as being different. The logical base underlying generalization is that of justification of the conclusion—it is a proof process, which moves from empirical knowledge to abstract knowledge that is beyond the empirical scope. The logical base of reasoning with symbolic forms, on the other hand, is found in its *analytic* nature-for example, when solving an equation with one unknown, it is assumed that the number sought is known, so it can be manipulated and its identity can be revealed in the end.[2] The second remark is that I conceive of algebraic thinking in terms of generalization as consisting of two phases: the first phase involves generalization and symbolization of the produced generality; the second phase involves justification of that generality.

The third element of the framework is Radford's (2018) characterization of the objects involved in algebraic thinking: 1) the nature of the objects: *indeterminacy*-i.e., the objects are not-known quantities such as unknowns, variables, parameters, generalized numbers; 2) notations to signify the objects: *denotation* of indeterminate quantities-i.e., using alphanumeric symbols or other semiotic systems like natural language, gestures, rhythms, or a mixture of these; and 3) the way to treat the objects: dealing with indeterminate quantities in an *analytic* manner-i.e., although these quantities are not known, they are treated arithmetically as if they were known numbers.

When dealing with symbolic forms, the attention is on the symbols and syntactical rules for changing their form. However, it is possible to act on symbolic forms semantically, where one's action is guided by what one believes the symbols *stand for*. For example, the expression $x^2 + x$ can semantically be interpreted as the sum of a number and its square (arithmetic interpretation), or it can be interpreted as the sum of two areas being a square of side x and a rectangle of sides x and 1 (geometric interpretation). Furthermore, $x^2 + x$ can syntactically be converted into $x(x + 1)$ that semantically can be interpreted as the product of two numbers-one of which exceeds the other by 1-, or $x(x + 1)$ can be interpreted as the area of a rectangle with side lengths x and $x + 1$.

The role of alphanumeric symbolism in algebraic thinking

When the term algebra is used it involves algebraic thinking and algebraic symbolism. There are however different conceptions among scholars about the relationship between the two concepts. In Gattegno's conceptualization of algebra, alphanumeric symbolism is indirectly required through the objects dealt with (polynomials, equations, mappings). Also Kaput, Blanton and Moreno (2008) require alphanumeric symbolism for a symbolic activity to be considered algebraic thinking: "a symbolization activity [is] algebraic if it involves symbolization in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbol systems" (p. 49). In a recent publication, Blanton and her colleagues claim that "algebraic reasoning ultimately involves reasoning with perhaps the most ubiquitous cultural artifact of algebra—the conventional symbol system based on variable notation [letters used to represent variable quantities]" (Blanton, Brizuela, Gardiner, Sawrey & Newman-Owens, 2017, p. 182).

Radford (2018), on the other hand, does not require alphanumeric symbolism for a symbolic activity to be considered algebraic thinking, a stance evident in his characterization of the objects involved presented above. A background for his view is given in (Radford, 2014), where he argues that notations are neither a necessary nor a sufficient condition for algebraic thinking. In the same vein, Squalli (2015) does not require alphanumeric symbolism for a symbolic activity to be considered algebraic thinking. He characterizes a generalization activity as algebraic when the produced generality can be represented in the *algebraic register*, which means that it involves a finite number of binary or *n*-ary operations (determined by internal or external composition laws), numbers, letters, words, symbols—in which the presence of letters is not essential.

A THEORETICAL ZOOM ON PERCENTS

The mathematical task solved by the observed students is a generalization task in which the variable quantity is given in percent notation. In order to analyze and understand the observed classroom situation, it is necessary to establish some background on the concept of percent. In this section, I therefore explain briefly what percent *is* and give some reasons why it is so hard to learn. First, a clarification on terminology: I use the words *base*, *percent* (or *rate*), and *percentage* to refer to the three elements of a percent equation: the base is the reference quantity, the percent is the rate, and the percentage is the proportional quantity determined by the rate.[3] In the expression $p\%$, p is the *percent numeral*. When I use the noun percent in indefinite singular form, I refer to the *concept* of percent.

What is percent?

Percent connects real-world situations and mathematical concepts. On the one hand, percent is a universal, practical topic that has deep roots in the marketplace—it is present in the media, in everyday trade, and in secondary school textbooks; on the other hand, it is part of the mathematical ideas involved in multiplicative structures that go back to Greek proportional geometry. Parker and Leinhardt (1995) have described the evolution of percent from its early commercial roots to its present role as an expression of comparison:

> Percent has changed from a simple monetary amount of tax or interest per hundred to a function used in conjunction with the Rule of Three, to a non-monetary use as a fraction comparing parts to wholes, to a ratio comparison between different objects and sets, and finally, to a number used for comparison of data expressed in relative form. (p. 434)

According to Usiskin and Bell (1983), there are six numeral types that have status as number: counts, measures, locations, ratio comparisons, codes, and derived formula constants. They classify percent as a *ratio comparison number*, of which they give a concise and productive description: "A ratio comparison is a number which can be thought of as a result of dividing two measures or counts with the same unit (but may not have actually been calculated this way)" (Usiskin & Bell, 1983, p. 25). So, a percent is a ratio, a proportional relationship which relates two quantities of the same kind (e.g., Euros per Euros; students per students)—it has no label due to cancelling in the division process (Parker & Leinhardt, 1995). There are different types of comparative situations in which percent is used. In the following I give a short description of three types—it is based on the explication given by Parker and Leinhardt (1995).

Percent as fraction: subsets of sets

This type of situation is where percent is used to compare the size of a subset to the size of the set of which it is a part. The part-whole model of percent belongs here, and percent numerals greater than 100 do not occur within this context. An example is a situation where 9% of the class received an A on the test; here 9% is a relational quantity which relates the number of students who received an A to the total number of students in the class (students per students).

Percent as ratio: separate sets

This type of situation is where we have separate sets, and the percent is used to describe a comparison involving either different sets, different attributes of the same set, or the change in a set from one time to another. Examples of the three situations are: comparison of the number of citizens in one country to the number of citizens in another country, comparison of the number of females to the number of males at a university department, and comparison of the new price to the former price. It is within this context that the nature of percent changes from a part of a whole to a descriptor of the relationship of one set to another, or of the relative amount by which one set differs from another. Percent numerals greater than 100 do occur in these contexts whenever a set of greater size is compared to one of lesser size. The relationship of the sets to one another determines whether the situation is one of *change* or *comparison.* Change is where the size of a set changes over time; comparison is where we have a relationship between two different sets at one point in time.

Percent as a statistic or function

Data reported in media often involve comparisons such as: a single percent statistic to describe a particular proportion (e.g., global vaccination coverage remains at 85%, with no significant changes during the past few years); or, the comparison of two percent statistics (e.g., the unemployment rate in January 2019 was 3.9% for Norway compared to 5.1% for Denmark). In both cases the original data are reduced and made easier to interpret due to the statistical use of percent. Besides this statistical usage, there exists a functional usage: Percent can be used to establish a uniform rate to determine for example final tax amounts, discounts, budget cuts, etc. Percent is here used to quantify the magnitude of a functional operator. Functional use is also what describes operations such as: find 16% of 60.

Some difficulties with percent

The part-whole model of percent is the first students are taught and they must later extend their procedures to situations that are not part-whole in structure. A strong part-whole notion of percent can lead to a serious misconception that makes percent numerals greater than 100 counterintuitive, since a part cannot be greater than the whole. Another problem is transformations between fractions, decimal and percents that share a common numerical value (Parker & Leinhardt, 1995).

Addition and subtraction are opposite and symmetric operations when quantities such as counts and measures are operands. If a price is increased by 4 € and then decreased by 4 €, it will again become the original price. However, when a price is increased by 4% and then decreased by 4%, the symmetry is lost. The amounts of increase and decrease are not the same, because the latter statement is about a multiplicative relationship. An increase by 4% means that the original price is multiplied by the factor (1 + 4%). One way of finding the percent decrease that will make the price get back to the original amount, is to find the multiplying factor (1 – *p*%) that satisfies the equation $(1 + 4\%) \cdot (1 - p\%) = 1$. Expanding the parentheses results in the equation $1 + 4\% - p\% - 4\%p\% = 1$, on which we do some algebraic manipulations and arrive at $p\% + 4\%p\% = 4\%$, which leads to $(1 + 4\%)p = 4$ and therefore, the *percent numeral* of the decrease is given by: $p = 4/(1 + 4\%) \approx 3.85$. This means that if a price is increased by 4% and then decreased by 3.85%, it will again become the original price. What I have demonstrated here provides a link between algebra and percents: percents are treated in calculations as *true numbers*, and further, the indeterminate quantity *p*% is treated in an *analytic* manner, as explained in the previous section.

The fact that increase and decrease in percent are non-symmetric operations makes percent counterintuitive. Even if the wording in the above statements about increase and decrease is analogous, the meaning in the additive and multiplicative situations is not the same, a fact that is quite surprising to most people.[4] Another example is evaluation of the better offer when there is a discount of 20% or two successive discounts of 10% each. The latter offer is likely to be misinterpreted as an additive situation, which makes one believe that the offers are the same.

Percent has undergone a shift from a part-whole meaning, to a proportional meaning, to a concise notational system that has different meanings in different situations (Parker & Leinhardt, 1995). These different meanings are far from easy to express in the natural language system. The mathematical language is so concise that it usually does not include the referent quantities, a convention that makes percent ambiguous. Moreover, the preposition *of* does not always have the same meanings in percent situations as it has in situations with fractions and multiplication. Last, the additive language of *more than, less than, increased by* and *decreased by* hides the multiplicative meanings of percent and hints to a symmetry that does not exist for percent.

SOME MORE THEORETICAL TOOLS

In order to explain the impact of the wide variety of representations used in the analyzed session, and the complexity of going from one representation to another, I have used elements from Duval's theory of semiotics. Further, in order to conceptualize the knowledge progress and explain the role of the system of resources used by the students, I have used elements from Brousseau's theory of didactical situations in mathematics. Here, the knowledge progress refers to the change in status of the solution to the task during the observed session, from an informal, implicit model to a formal, mathematical model.

Semiotic representations

Duval's theory of semiotics is applicable as a framework to make sense of problems of representation in mathematics. Duval (2002) classifies four registers (systems) of semiotic representations that are mobilized in mathematical activity: *natural language* (verbal associations, reasoning), *notation systems* (numeric, algebraic, symbolic), *plane or perspective geometric figures* (configurations of 0-, 1-, 2- and 3-dimensional forms; ruler and compass construction), and *Cartesian graphs* (changes of coordinate systems, interpolation, extrapolation). Duval (2006) distinguishes between two types of transformation of semiotic representations: *treatments* and *conversions*. Treatments are transformations that happen within the same register (e.g., carrying out a calculation while remaining in the same notation system for representing the numbers). Conversions are transformations that consist of changing a register without changing the objects being denoted (e.g., passing from the natural language statement of a relationship to its algebraic representation).

According to Duval (2006), mathematical activity involved in problem solving situations requires the ability to convert representations from one register into another register, either because it is necessary to change to a more suitable presentation of the data, or because two registers must be brought together into play. He argues (Duval, 2006, p. 128): "Changing representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum."

The theory of didactical situations in mathematics (TDS)

Elements from the TDS are used in this paper to analyze a classroom session from the perspective of the environment that is meant to provoke and justify the learning of a particular piece of mathematical knowledge. The TDS' methodology is concerned with two issues: the necessary *conditions* for a situation to implement the knowledge aimed at; and how a situation can be *designed* and its evolution managed, in a given educational institution (Bosch, Chevallard & Gascón, 2005). The following presentation of the TDS is based on (Brousseau, 1997, 2006).

An important element in the TDS is the concept of *milieu* that models the system with which the students interact when solving a problem. The milieu for a situation aiming at a piece of mathematical knowledge may comprise: material or symbolic tools (artefacts, informative texts, data, etc.), students' prior knowledge (relevant for the knowledge at stake), other students, and an arrangement of the classroom and rules for operating in the situation (determining who is supposed to interact with whom, and further, what shall happen if the expected result is not achieved).

An *adidactical situation* is a situation in which the students as a group are capable of reaching an optimal solution to a problem through interaction with the milieu, without significant help from the teacher. An appropriate milieu for a problem has an adidactical potential, which means that the milieu gives feedback to the students whether their responses are adequate or not with respect to the target knowledge (i.e., the solution to the problem). In the TDS, the teacher has two dual roles, in addition to having responsibility for the evolution of adidactical situations, referred to as *regulation* (Mangiante-Orsola, Perrin-Glorian & Strømskag, 2018). The dual roles have to do with the relationships between the adidactical and didactical dimensions of situations: One is *devolution*, where the teacher transfers to the students the responsibility for solving the problem in an adidactical situation. The other role is *institutionalization*, where the teacher attempts to transform the contextualized knowledge (in French, *connaissances*) that the students have developed as a response to the problem given to them, into cultural knowledge (in French, *savoir*) that can be used outside the context in which it is developed.

In the TDS, after devolution, three (intentionally) adidactical situations follow, where the teacher's role and the status of knowledge change: The situation of *action* is where the students construct an implicit solution on the basis of experimentations of successes and failures on a material milieu. In action, the teacher intervenes only if necessary to guide the students so they fully understand the task; the teacher does not give them mathematical tools for the solution, except if students encounter too many difficulties (in which case the adidactical situation collapses). The situation of *formulation* is where the students operate indirectly on the milieu by formulating an explicit solution enabling somebody else to operate on the milieu. In formulation, the teacher's role is to make different formulations "visible" in the classroom. The situation of *validation* is where the students try to make their solution valid or try to verify a conjecture. In validation, the teacher's role is to act as a chair of a scientific debate, and (ideally) intervenes only to structure the debate and make the students express themselves more mathematically precise. The adidactical situations are succeeded by the situation of *institutionalization*, where the teacher (as described above) transforms students' solutions into scholarly and decontextualised forms of knowledge. The four situations described involve phases in the evolution of mathematical knowledge in a classroom setting, from informal knowledge to formal, mathematical knowledge. The progress of knowledge in adidactical situations is governed by the adidactical potential of the milieu.

METHODICAL APPROACH

The aim of the research reported here was to study the phenomenon of student teachers' use and mastery of algebraic formalism, using percent notation. I studied this phenomenon in its naturalistic setting—that is, I had little or no control over the studied events. Further, the study was conducted within a localized boundary of space and time. These features are consistent with a case study design, as described by Yin (2009) and Bassey (1999). The didactical situation has been instrumental to my understanding of the phenomenon studied. Hence, it is an *instrumental* case study (Stake, 2005).

The research participants were three female student teachers (henceforth, 'students'), and two male teacher educators of mathematics who taught them mathematics. The triad of students—Alice, Ida and Sophie (names are pseudonyms)—were in the first academic year of a four-year teacher education program for primary and lower secondary education. They were members of a class with 66 students. Based on the mathematics courses they had taken in upper secondary school, and their achieved marks, Alice, Ida and Sophie can be considered middle-average strong in mathematics.

One of the teacher educators was a professor of mathematics with more than ten years of practice as a mathematics teacher educator, the other was a senior lecturer with more than thirty years of the same practice. They played slightly different roles: the professor was responsible for the lessons on algebra, including the design of tasks—he will be referred to as 'designer'; the senior lecturer had the role of a "teacher assistant" in the orchestration of the students' work and shared with the designer the task of helping students—he will be referred to as 'teacher'.

It is relevant to inform about the structure of the teacher education program on which the observed students were enrolled: Mathematics and didactics of mathematics were taught as integrated elements in this program (where also school-based practice is included). That is, students do not finish with mathematics before they start with didactics of mathematics (this applies for teacher education programs for primary and secondary education throughout Norway). The mathematics course (Mathematics 1), from which the data were collected, was a compulsory course within the teacher education program. I taught a minor part of the course, which did not involve the observed session or any other class dealing with algebra. In order for the observed session to be as naturalistic as possible, I had the role of non-participant observer while video-recording the session. An *a priori* analysis of the teaching situation and its organization is presented in the next section.

As presented in the introduction, the particular research question I set out to answer was: *What material and non-material tools prove relevant or missing in the milieu that the students resort to?* The data consist of the transcript of a video-recorded observation of a session (lasting 90 minutes) in which the triad of students work collaboratively to solve the task presented in Figure 1. The transcript is partitioned into numbered turns, where a turn is defined as a set of utterances made by a person until another person takes a turn. Excerpts from the transcript used in the paper have kept the original numbering of turns. The transcript has been analyzed by a thematic coding process (Robson, 2011). This means that the transcript has been divided into segments, where each segment constitutes a unit of meaning with respect to conditions that enabled or hindered the students to solve the mathematical problem they were confronted with. The analysis of the transcript has been substantiated by the *a priori* analysis of the teaching situation, which is presented next.

The analyzed session shows 1) how percent notation complicated the task and how algebraic calculations were avoided, and 2) how an evolution of the milieu (in terms of geometric figures) enabled the students to solve the task.

A PRIORI ANALYSIS OF THE TEACHING SITUATION

The observed session and its organization

The situation analyzed in this paper is part of an *ordinary* teaching situation in the sense that TDS principles and concepts have neither influenced the design of the task nor the classroom activity. It is a small-group session in mathematics at the university, where students collaborated in groups of three or four to solve mathematical tasks, and two teacher educators were (intermittently) present to observe and interact with them. The session was part of an 'Algebra module' that spanned 24 classes (each of 45-minute duration) over a two-month period. In addition to small-group sessions, the algebra module consisted of lectures, short introductions, and whole-class discussions including reviews of previous tasks solved by the students.

There were several topics included in the algebra module: figurate numbers and shape pattern sequences; different categories of algebraic expressions (formulae—functions—identities); area; polyhedra; investigations, conjectures and proofs; the concept of variable and the role of the equal sign; systems of linear equations; word problems and equations; and mathematical models. The designer of the tasks claimed that one of the aims with the algebra module was to let the students engage with generalization in different contexts, where the achieved generalities should be represented in algebraic notation. An important part of this involved transformation of representations in different semiotic systems (natural language, algebra, geometric figures).

The mathematical task solved by the observed students was part of a collection of six worksheets (each with two tasks) under the headline 'Algebra'. Groups were connected in pairs, where each group chose a worksheet (different from the one chosen by its peer group) for solution followed by presentation to its peer group. The observed students worked only on the second task on the chosen worksheet (possibly for lack of time), the solution to which they presented to the peer group afterwards. To reduce the background noise on the video-recordings, the observed students were placed in a small room adjacent to the classroom where the rest of the students were working. They were informed that, at any time, they could go and ask for help from the teachers or discuss with peer students. Occasionally, the teacher educators, on their own initiative, went to the room in which the observed students worked.

The material milieu available to the students from the start of the session consisted of some writing material, sheets of paper, scissors, calculators, and a mathematics textbook for teacher education, on algebra and functions (Selvik, Rinvold & Høines, 2007). This textbook has nothing on percentages. However, it has two sections entitled, respectively, "Symbols denoting numbers and magnitudes" and "Symbols with which we can calculate and think": they both (unintentionally) point to an essential aspect of the initial deficiencies of the milieu which the students observed had to rely upon. As we will see in the *a posteriori* analysis, the milieu did evolve during the observed session.

The task and its anticipated involved actions

The task with which the observed students engaged is presented in Figure 1.[5]

Imagine that you have a square. Make a new square where the side length has increased by 50%. How many percent has then the area increased? Imagine now that the side length increases by *p*%. How many percent will the area consequently increase?

Figure 1. - The task chosen by the students

The mathematical (adidactical) objective of the task is twofold: First, it is to solve the problem of finding the rate of increase of a square's area when the square's side increases by a given percent. Second, it is to generalize this relationship, using algebraic notation, *p*%, where the percent numeral is a parameter. Further, the (designer's) didactical objective of the task is the students' ability to transform representations into different semiotic registers—here, natural language, numeric notation systems (with particular focus on percents), the algebraic notation system, and plane geometric figures.

An underlying concept in the task is similarity of figures. Two figures are said to be *similar* when all corresponding angles are equal and all distances are increased (or decreased) in the same ratio, called the ratio of magnification or scale factor. A transformation that takes figures to similar figures is called a *similarity*. Essential relationships (scaling laws) between length, area and volume of similar figures can be expressed in terms of these equalities:

$$\text{(ratio of areas)} = \text{(ratio of lengths)}^2$$
$$\text{(ratio of volumes)} = \text{(ratio of lengths)}^3.$$

Instead of presenting the problem as a similarity problem, the task proceeds straight away to percent, which perhaps is a bit unconventional. However, the use of percent is comprehensible with regard to the designer's aim that the students would have to make transformations of different semiotic representations. The mathematical object aimed at in the task is the *percent* (or *rate*) *of increase* of a square's area when the square's side increases by an arbitrary percent, *p*%.

One obstacle to be considered consists in the fact that the nonmathematical, ordinary linguistic handling of percent, which leads to speak of, say, "the current selling price minus 20 percent", is very likely to be resorted to by the students (due to lack of mathematically correct formulation), which may induce them to write expressions like 100 € – 20% (instead of 100 € – 20% · 100 €, etc.). To overcome this cultural obstacle, didactical means might be provided in order to arrive at the conclusion that, if a quantity q increases by $p\%$, then the new value q' is expressed by:[6]

$$q' = q + \frac{p}{100} \cdot q.$$

In the teaching situation, it can be supposed that the teacher considers the basic equality $q' = q + \frac{p}{100} \cdot q$ prior knowledge on the part of the students. More realistically, one can think that 1) the students have encountered this equality in their secondary education, and 2) they have forgotten about it (or even "suppressed" it), as most nonmath people seem to do, provided they did not have any need to use it since they left upper secondary school. If this is so, it could be expected that the teacher would provide a gentle reminder about the basic equality. Such a reminder would be part of the *institutionalization* of the knowledge used or produced during the classroom session.

A crucial aspect of the task assigned to the students is the *semiotic registers* that the students can afford to use. One can suppose that the students involved have studied, not that long ago, the representation system of elementary algebra, which allows one to write, in full generality:

$q' = q + p\% \cdot q = q + \frac{p}{100} \cdot q = q(1 + \frac{p}{100})$, which leads to $q'^2 = q^2(1 + \frac{p}{100})^2$ and therefore, the *rate of increase* is given by:

$\frac{q'^2 - q^2}{q^2} = \frac{q'^2}{q^2} - 1 = (1 + \frac{p}{100})^2 - 1 = \frac{p}{100} \cdot (2 + \frac{p}{100}) = \frac{2p}{100} + \frac{p^2}{100^2}$
$= (2p + \frac{p^2}{100})\%$. If $p = 50$, then $\frac{q'^2 - q^2}{q^2} = (100 + 25)\% = 125\%$.

It can be noted that, if the general problem of the rate of increase of the square q^2 of a quantity q was considered, this problem could be modelled geometrically, using a model as shown in Figure 2. The problem raised in the classroom session is therefore that of the geometric solution to the general problem, which to some extent allows the students to shun algebraic calculations. Figure 2 illustrates that the increase of the area is equal to

$(p\% \cdot q) \cdot q + (p\% \cdot q) \cdot q + (p\% \cdot q)^2 = p\%(2 + p\%)q^2$. Accordingly, the *rate* of increase is $p\%(2 + p\%) = (2p + \frac{p^2}{100})\%$.

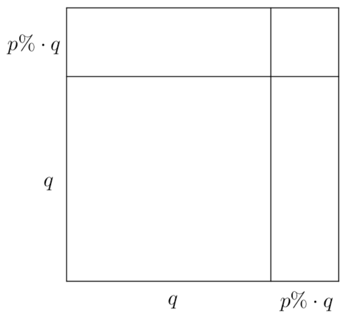


Figure 2. - A geometric model of the solution to the general problem

The task can be interpreted to involve a comparison between the enlargement of a square's side and the enlargement of its area, where the percent has a *functional usage*. It is used to quantify the magnitude of the functional operator denoted by for example the expression $f(p\%) = (2p + \frac{p^2}{100})\%$, which gives the rate of increase of a square's area as a function of the rate of increase of its side.

Two more remarks are in order now. The first remark is that the study of the rate of increase of a square's area can entirely dispense with the use of percents. Given a percent $p\%$, let $r = 1 + p\%$. A square of side q is enlarged into a square of side $q' = rq$. The area of the enlarged square is $q'^2 = (rq)^2 = r^2q^2$ and therefore one gets: $\frac{q'^2 - q^2}{q^2} = \frac{r^2q^2 - q^2}{q^2} = r^2 - 1$. If we take r to be $1 + p\%$, we arrive at $\frac{q'^2 - q^2}{q^2} = (1 + p\%)^2 - 1 = p\%(2 + p\%) = (2p + \frac{p^2}{100})\%$. Contrary to other authors, who completely avoid using percents in this case (see e.g. Lang & Murrow, 1983, chap. 8), the designer's scenario expresses the "multiplying factor" r as a percent, or rather ignores the idea of a multiplying factor completely, which generates specific calculation difficulties that can hardly be overcome during the classroom session.

The second remark is that the use of the multiplying factor conceals the difficulties appended to the use of percents in calculations. For example, in calculating the expression $p\%(2 + p\%)$, few persons would write $p\%(2 + p\%) = 2p\% + (p\%)^2$ and fewer still would dare to write $(p\%)^2 = p^2\%^2$, so that we would arrive at: $p\%(2 + p\%) = 2p\% + (p\%)^2 = 2p\% + p^2\%^2 = p(2 + p\%)\%$. The crux of the problem here is that the "entity" $p\%$ is not definitely regarded *as a number*, to be treated accordingly, so that we have $p\% = \frac{p}{100}$ and (even) 1% = % = $\frac{1}{100}$ = 0.01. This means that, at least at the lowest levels of mathematical practice, at school and outside of school, percents remain ontologically ambiguous entities, which generates difficulties in using them in calculations.[7]

A POSTERIORI ANALYSIS OF THE TEACHING SITUATION

The session was introduced for the whole class by the designer of the tasks, where he informed about the six worksheets planned for work in groups followed by presentation in pairs of groups. There was no introduction to, or institutionalization of, mathematical knowledge (intended for use or developed during the classroom session) before the group-work started. Expected (by the teachers) prior knowledge on the part of the students was that an increase of *p* percent of a quantity corresponds to a multiplying factor of $1+ \frac{p}{100}$.

Conversion made unattainable due to an additive model

Alice, Ida and Sophie had drawn a 2 × 2 cm^2 square, increased its sides by 50%, drawn the resulting 3 × 3 cm^2 square, and then found that the square's area had increased by 9 cm^2 – 4 cm^2 = 5 cm^2. Alice then led them to (incorrectly) take as the rate of increase the ratio 5 / 9, which made Ida wonder:[8]

> 33 Ida: Why do you divide by the largest one then, nine?
>
> 34 Alice: Part divided by whole. It's just a rule I have learnt.

The misconception caused by Alice's part-whole notion of percent—which makes percents greater than 100 counterintuitive—was not recognized by Ida and Sophie. They accepted Alice's explanation, and the triad concluded that the percent sought was equal to 5 / 9 = 0.555... ≈ 56%.

Next, they tried to find a general formula for the rate of increase of the square's area by taking the parameter *p* into account.

> 68 Sophie: Imagine now that the side length increases by *p* percent. ⟨Reads the task⟩.
>
> 69 Alice: Well, but what will happen when we increase this [the side length] by fifty percent? Then we'll get fifty six percent increase of the area, thus, six percent more than what is here [the rate of increase of the side]. So, then it will be *p* percent plus six. But I don't know whether it will be like this, always.

The object Alice dealt with in turn 69 is an indeterminate quantity: a parameter denoted by the letter *p*. But because she did not treat the parameter in an analytic manner, I interpret the generalization process not to be genuinely algebraic thinking. The students had concluded that when the square's side increased by 50%, the square's area would increase by 56%. For Alice, this seemed to evoke an *additive* mental model, where she focused on the fact that 56% is 6% "more than" 50%. The additive model hindered the generalization process that relied on a multiplicative relationship. It resulted in a hypothetical formula for the rate of increase of the square's area: *p*% + 6. This expression was apparently based on the case already calculated (the inaccuracy of which the students were not aware of): 50% increase of the square's side implied 56% increase of the square's area, so that Alice replaced 50% by *p*% and added 6.

Syntactically, the expression *p*% + 6 does not correspond to the observation made by Alice ("6 percent more than *p*%" would correspond to (*p* + 6)%), which is an indication of the complexity of converting statements of percents in the natural language register into the algebraic register of concise mathematical expressions.

Replacing a "part-whole" notion of percent

The students continued on the path opened by Alice and explored a new example, a square of side 5 cm: the new side was therefore 7.5 cm and the new area was 56.25 cm^2. At this point the designer entered the room, while they were arriving once again at (56.25 – 25) / 56.25 = 0.555... ≈ 56%. Through a rather long dialogue (spanning 61 turns), the designer corrected the trajectory of the students. During this exchange he drew the students' attention to the distinction between increasing a quantity from 4 to 9 and reducing a quantity from 9 to 4:

127 Alice: But do I have to, when we do like this, divide it by four then, because that was our starting point?

128 Designer: That is the question… because what you're saying…

129 Ida: Yes, it will be hundred and twenty five [percent]. ⟨Uses the calculator⟩.

[⁝]

134 Designer: And then it, in a way, goes the other way (Alice: yes). Such that… going from… this will be an expression of… well what will this be an expression of? The fifty six percent is pertinent in this respect.

135 Sofie: The increase…

136 Designer: What might a problem look like… so that this number [56%] were pertinent… to consider?

137 Sophie: How much it had been reduced?

138 Designer: Yes, for instance ⟨Ida nods⟩. We go from nine down to four, then we have had a *reduction* of more than the half, indeed a fifty six percent reduction (Alice and Sophie: yes). But now we go upwards [from 4 to 9].

Their exchange confirmed that the students had first and foremost to accept that a percent can be greater than 100%. This was used by Ida (turn 129) to establish that the rate of increase of the square's area is 125% when its side increases by 50%.

A formula derived by didactical reasoning

After the designer left the room, the triad of students went on to find a general formula for the rate of increase of the square's side by taking the parameter *p* into account. Ida was hinting at the concept of multiplying factor when she said "I was thinking that it was one plus or minus *p* hundredths". This was, however, not taken further by any of the others. Alice struggled to represent in alphanumeric notation the increased side length, where she tried to mimic the only calculation they knew to be correct, i.e., (9 – 4) / 4 = 1,25 = 125%.

174 Alice: Two multiplied by *p*… because what we did now, fifty percent of this was one centimetre (Sophie: mm), then we added one centimetre (Ida: yes). So this will be two plus one (Ida: yes)… in order to get that [the square of side 3 cm] (Ida: yes). So therefore you must have two plus *p*… percent… ⟨Alice writes 2 + *p* % in her notepad⟩.

Alice's representation of the increased side length by the expression 2 + *p* % is a manifestation of the cultural obstacle constituted by the nonmathematical, ordinary linguistic handling of percents described in the *a priori* analysis of the teaching situation. Turn 174 is yet another manifestation of the complexity of converting a natural language statement of percents into the algebraic register.

After this, Alice proposed 2.5 · *p*% as a formula for the rate of increase of a square's area when its side increases by *p* percent. The factor 2.5 is the ratio 125 / 50, i.e., the rate of increase of the area to the rate of increase of the side (for the case they had calculated). Ida was doubtful whether this was sensible:

204 Ida: I was just wondering why you took hundred and twenty five over fifty?

205 Alice: Because it was these [numbers] that were present in the example.

206 Ida: But are you sure we should continue with these [numbers] in the rest of the task?

Alice's model, which involved a kind of (invalid) proportional reasoning, can be seen as a result of didactical reasoning, demonstrated in turn 205: the ratio 125 / 50 was just a result of using the numbers from the first part of the task together with p% (without any apparent mathematical rationale). Sophie and Ida wanted to check Alice's formula for $p = 25$, by first, calculating manually what a 25% increase involved; next evaluating the formula for $p = 25$, and then comparing the values. So, Sophie drew a 2×2 cm^2 square, increased its sides by 25%, drew the resulting 2.5×2.5 cm^2 square, and found that the square's area had increased by 6.25 cm^2 – 4 cm^2 = 2.25 cm^2. Ida found that the increase was 56.25% (in the manual case), whereas Alice found that the increase was 62.5% (in the case of the formula). Because the values were different, the triad rejected Alice's model and decided on getting help from one of the teachers.

It was expected by the teachers that the students had in their repertoire a technique to represent the new value obtained when a quantity q increases by p%. However, the students lacked such a technique. The case of the square where the side was increased by 25% became part of the milieu, and enabled the students to conclude that the proposed model was wrong. But the milieu they resorted to did not give further feedback that could help them to solve the task.

In the following, I present how the milieu was changed and how this affected the students' solution to the task.

Evolution of the milieu

Seeing structure leads to invention of manipulatives

The teacher changed the milieu by encouraging the students to draw several figures and then increase their side lengths by 50%. They drew 4×4-, 6×6- and 8×8-squares with corresponding enlargements and discussed how the enlarged squares appeared. From the discussion of the common structure of the three examples, they realized that the square's size was in fact irrelevant and saw the utility of a 1×1 square (independent of denomination). The insight that side lengths of 1 unit represented no loss of generality was made possible by the teacher's change of the milieu. This, in turn, was instrumental for a further evolution of the milieu: the students made paper manipulatives to illustrate what an enlarged square looked like for enlargements by different percents. From sheets of A3 paper they made congruent squares, with which they did the following for a 50% increase of the side: one square was used as the original square, a second square was folded once and torn along the fold (to give two segments of size $0.5 \times$

1), and a third square was folded twice and torn along the folds (to give four segments of size 0.5 × 0.5, one of which was used as illustration). The same principle was used in cases of 25% and 10% increase of the side (though the foldings were different). Note that, in so doing, the triad of students considered implicitly that the unit of length they use is the length of the square's side, whatever the size of the square they consider.

Iconic illustrations of the three cases are presented in Figure 3.

(S1) Result of 50% increase (S2) Result of 25% increase (S3) Result of 10% increase

Figure 3. - Iconic illustrations of the paper manipulatives

In Figure 3, the black segments represent the original squares, and the grey segments represent the enlargements in each case. The students calculated the enlargement of the area of a unit square in the three cases and wrote the results in their notepads, similar to what is shown in Table 2.

Enlargement of side	Enlargement of area
50%	125%
25%	56.25%
10%	21%

Table 2. - Values calculated by the students

Operating within different semiotic registers: geometric figures and three notation systems

The task for the students was to find (in percent notation) the total area of the grey segments as a function of the increase of the square's side. When considering a 50% increase of the side, they observed that each of the rectangular segments covered half the area of the original square. Hence, the issue for them was the area of the remaining segment that was needed to cover the enlarged square. The teacher's focus in the transcript below is the relationship between the enlargement of the original square's side and the area of the quadratic segment up in the right corner of the illustrations (S1, S2, and S3) in Figure 3. These upper corner segments will henceforth be referred to as "small squares". The transcript below shows how operating within different semiotic registers complicated the communication.

> 783 Teacher: When the side got one half longer, it consequently increased by fifty percent. How large did the segment up there get then? ⟨Refers to the small square of S1 in Figure 3⟩.
>
> 784 Ida: One fourth.
>
> 785 Alice: One fourth.
>
> 786 Teacher: When you increased by one fourth, how large did the segment up there get then?
>
> 787 Sophie: Wait a little. ⟨Starts to write in her notepad⟩.
>
> 788 Teacher: []
>
> 789 Alice: One sixteenth.
>
> 790 Teacher: And when you have now increased by one tenth?
>
> 791 Alice: One hundredth.
>
> 792 Ida: yes
>
> 793 Alice: Do we have some…? ⟨Alice looks at Sophie's notepad; Ida uses her calculator⟩.
>
> 794 Sophie: Five times five is twenty-five. Sixteen… how much is one sixteenth?
>
> 795 Ida: One sixteenth, that is zero point zero six two five.
>
> 796 Sophie: And if you multiply that by…?
>
> 797 Ida: You can write [] zero point zero six two five. ⟨Points at the fraction $\frac{1}{16}$ in Sophie's notepad⟩. And that one is zero point twenty-five and that one is zero point zero one. ⟨Points at the fractions $\frac{1}{4}$ and $\frac{1}{10}$ in Sophie's notepad⟩.
>
> 798 Sophie: Zero point zero one?
>
> 799 Ida: Connection?
>
> 800 Teacher: What if you were just thinking in terms of fractions here now? (Ida: yes) One half, what was the segment up there then? ⟨Points at the small square of S1 in Figure 3⟩.

Ida and Alice (turns 784-792) gave correct answers when asked for the area of the small squares when the sides of the original squares were enlarged by fractions (one half, one fourth and one tenth). The teacher's intention (turn 815) was possibly to make the students observe concrete cases that would enable them to generalize the relationship between the enlargement of the side and the area of the small square. He had changed the notation system from percents to fractional notation, without any explanation to the students. This was a treatment, and his reason for doing so may have been that it is easier (by mental calculation) to square the fractions involved, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{10}\}$, than it is to square the corresponding decimal numbers, {0.5, 0.25, 0.1}. However, Sophie and Ida were concerned with decimal notation. They transformed the involved fractions into decimal numbers: $\frac{1}{4} = 0.25$, $\frac{1}{16} = 0.0625$, and $\frac{1}{100} = 0.01$ (turns 794–798). The decimals, 0.25 and 0.0625, are "closer" to the numbers represented by percents, which the students had already calculated (see Table 2). After this, Ida (turn 799) posed an important question about *connection*. But the teacher (turn 800) kept to fractional notation, and apparently missed an opportunity to initiate a treatment of decimals into percents, which was the semiotic register used (and asked for) in the task.

The next excerpt from the transcript shows that Alice and Ida were able to use (on examples) the quadratic relationship aimed at by the teacher and, further, that this relationship eluded Sophie, who followed a false lead.

812 Teacher: When the increase was one half, what was the segment itself? ⟨Refers to the small square of S1 in Figure 3⟩.

813 Ida: One fourth.

814 Alice: One fourth.

815 Teacher: What connection is there between one half and one fourth then?

816 Alice: It is half as large.

817 Teacher: What if the increase was one fourth? Then it was?

818 Alice: One sixteenth.

819 Sophie: It should be one eighth here.

820 Alice: No, because it is twenty-five percent, that is what one fourth is.

821 Ida: Yes.

822 Sophie: Yes, but not… We have to take one eighth there, that makes one half.

828 Teacher: When it was one tenth increase, how large was the small segment then?

829 Alice: One hundredth.

830 Ida: One hundredth.

831 Teacher: What if I increased the side length by one fifth, what would the segment up there look like then? ⟨Pause 4 seconds⟩.

832 Sophie: With this as a starting point? ⟨Points at something in Alice's notepad⟩.

833 Teacher: If I were to take one fifth of the side and increase further here? ⟨Demonstrates on an original square by pointing⟩. How large would the one up here be then?

834 Alice: One twenty-f…One twenty-fifth.

835 Teacher: Yes… that is actually what it is []

836 Sophie: So, the square, the side of it, it is… Is it half of… *p*? ⟨Looks at the teacher⟩.

837 Teacher: [] when you increased by one half?

838 Alice: What did I say? One twenty-fifth? When you multiply by, or when you had five percent increase, then it was, it was… No, what did I say?

839 Sophie: No, I don't remember. ⟨Laughs⟩. But it will be one fourth… of *p*. No of…

When the teacher (turn 815) asked for a *connection* between one half and one fourth, he aimed at formalising the relationship between the enlargement of the side and the area of the small square of S1 in Figure 3—that it was a *quadratic* relationship. Alice (turn 816) responded by saying that it was half as large, which in isolation was appropriate, but the teacher was after the model of squaring. So, didactically, one half was an ambiguous example. In the following, the relationship between enlargement of a side (a) and the area of the resulting small square (b) will be denoted ($a \rightarrow b$). The teacher (turn 817) then provided an unambiguous example ($\frac{1}{4} \rightarrow ?$), to which Alice (turn 818) suggested the relationship ($\frac{1}{4} \rightarrow \frac{1}{16}$). This indicated that she had discovered the quadratic relationship aimed at by the teacher, which (in turns 829 and 834) was further supported by appropriate answers for two more cases, ($\frac{1}{10} \rightarrow \frac{1}{100}$) and ($\frac{1}{5} \rightarrow \frac{1}{25}$). The initial, incorrect model of halving had been replaced by Alice by a correct model of squaring. The change of model was however not explained in plain words, neither was the property 'quadratic'. This presumably made it more difficult for Sophie to replace the model of halving (turns 819 and 822).

In turn 838, Alice confused an increase by one fifth (mentioned by the teacher in turn 833) with an increase by five percent. This was probably due to the variety of notation systems employed and the "appearance" of the numbers (five is involved in both $\frac{1}{5}$ and 5%). In turns 836 and 839, Sophie tried to figure out the area of the small square in the general case, using the letter *p*. But both models (of halving and of quartering) were wrong, and she operated on the wrong value, *p* instead of *p*%.

The students were supposed to present their work to another group. For this reason, they began to worry whether they had anything of value to share. During a break, it was decided that there should be a phase of gathering the threads and trying to find out how the geometric figures could be used to find an expression for the general case with *p*%. This is presented next.

Transformation of geometric figures into percent notation making possible algebraic notation

The students collaborated on the three exemplars of the manipulatives (S1, S2 and S3, shown in Figure 3), and decided that it was necessary to write the area of the grey segments in percent notation (*p*% written as $\frac{p}{100}$). With some input from the teacher on writing it systematically, they expressed the area of each grey segment as a product of side lengths in percent notation. This resulted in the following arithmetic expressions for the enlargements of the original square:

$$2 \cdot (\frac{50}{100} \cdot \frac{100}{100}) + \frac{50}{100} \cdot \frac{50}{100} \text{ (for a 50\% increase);}$$

$$2 \cdot (\frac{25}{100} \cdot \frac{100}{100}) + \frac{25}{100} \cdot \frac{25}{100} \text{ (for a 25\% increase);}$$

$$2 \cdot (\frac{10}{100} \cdot \frac{100}{100}) + \frac{10}{100} \cdot \frac{10}{100} \text{ (for a 10\% increase).}$$

Through inductive reasoning, relying on the form of an expression, the students arrived at the hypothetical general form: $2 \cdot (\frac{p}{100} \cdot \frac{100}{100}) + \frac{p}{100} \cdot \frac{p}{100}$. This was simplified to $\frac{2p}{100} + \frac{p^2}{100^2}$.

When the students presented their work to the other group, they used the geometric figures (S1 and S2) to illustrate the cases when the side was increased by 50% and 25%. Then they presented the expression $\frac{2p}{100} + \frac{p^2}{100^2}$ as the solution to the general problem when the side was increased by *p*%, and drew a model with no measures as shown in

Figure 4. They commented that $\frac{2p}{100} + \frac{p^2}{100^2}$ was derived from the expression $2 \cdot (\frac{p}{100} \cdot \frac{100}{100}) + \frac{p}{100} \cdot \frac{p}{100}$ that had the same structure as the concrete cases presented. Further, the students explained that $2 \cdot (\frac{p}{100} \cdot \frac{100}{100})$ represented the area of the two rectangles and $\frac{p}{100} \cdot \frac{p}{100}$ represented the area of the small square in Figure 4.

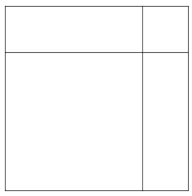


Figure 4. - Model used by the students to explain the general case

It must be emphasized that the route taken by the students—which led them to the expression $\frac{2p}{100} + \frac{p^2}{100^2}$—*avoided* the basic equality ($q' = q + \frac{p}{100} \cdot q$, where q' is the new value obtained when a quantity q increases by p%), since the expression arrived at was not obtained by a calculation using the letter p, but by an inductive reasoning process (for which the supporting evidence consisted of three cases: $p = 10$, $p = 25$, and $p = 50$). To arrive at this conclusion, one must, however, know that the rate of increase of the square's area is given by the formula $\frac{q'^2 - q^2}{q^2}$ (where q is the length of the square's side), maybe under a different guise (e.g., "the increase of the area over the initial value of the area"). The new value of the square's side, q', was *implicitly* taken by the students to be equal to $q + p\% \cdot q$, since they lacked a technique to represent the new value obtained when a quantity q increases by p%.

The inductive process undertaken by the triad of students can be seen as a first phase of algebraic thinking, where they identified the common structure of the three arithmetic expressions. The letter p in the proposed algebraic expression can be understood as a *placeholder* (Ely & Adams, 2012) that stands for an indeterminate number that is to be provided whenever the expression is used to solve a particular problem. This interpretation is consistent with the way the algebraic expression was established in the observed session—it emerged just from the concrete cases, where p was "exchanged with", or "held the place of", the particular numbers 10, 25, and 50.

However, a necessary second phase of algebraic thinking would be to carry out a *mathematical proof* of the hypothesized generality, as explained in the theoretical section on algebra. Although there were fragments of Figure 4 being used by the students as a *generic example* (Balacheff, 1988), in the form of explaining which segment on the model corresponded to which term of the proposed algebraic expression (as explained above), their reasoning lacked rigor in that they did not explain *why* the terms of the algebraic expression did have the form they had. This concerns the same phenomenon as described above—avoiding the basic equality and implicitly taking the new value of the square's side to be equal to $q + p\% \cdot q$.

DISCUSSION

Algebra and percents

In the classroom session I observed, the students were confronted with a clear-cut problem, that of enlarging a square. In the formulation adopted, percents were used, which complicated much the problem. The students were involved in generalization and expression of generalities in symbolic form, but their reasoning with and manipulation of these symbolic forms were inadequate—in fact, algebraic *calculation* remained almost absent from the activity of the students. According to the framework adopted for algebra (drawing on Gattegno, 1988/2010; Kaput, 2008; and Radford, 2018), their activity is therefore not considered full-fledged algebraic thinking. However, algebraic thinking was not completely absent, since the triad of students resorted to some sort of inductive reasoning.

The fact that they did not have a technique to represent the new value q' obtained when a quantity q increases by p% was a condition that hindered their use of algebraic calculations. Further, I want to highlight that what I have labelled "expected prior knowledge"—the fact that an increase of p percent of a quantity corresponds to a multiplying factor of $1+\frac{p}{100}$—was not activated during the session, although it was mentioned by one of the students (before the teacher arrived). It seems that the *multiplicative* expression of the increase of a quantity ($q' = q(1 + p\%)$), which is the key to most percent problems, was ignored or even thoroughly sidelined (by the teacher), to the advantage of its more common *additive* expression ($q' = q + p\% \cdot q$), which is the ordinary but little effective way of handling percents. From the point of view of mathematics education, this is a serious problem raised by the scenario I have tried to account for in this study. It should be noted though that the preference for an additive expression of percents in the analyzed session was likely influenced by the material milieu (the paper cut-outs) used to solve the task.

When the students couldn't solve the task devolved to them, the adidactical situation was turned into a didactical one, where the teacher employed a kind of Socratic method where he elicited the solution from the students, based on a series of questions and answers. An important moment was when the students realized the utility of a unit square, because this enabled them to develop manipulatives in the form of paper cut-outs. This material milieu made it possible for the students—in the situation of *action*—to construct a representation of the situation that served as an implicit "model" that guided them in their decisions. Furthermore, the new milieu provided the students with representations from a different semiotic register (geometric figures), and—in the situation of *formulation* (where the implicit model was made explicit)—they succeeded in converting the geometric figures in each case ($p = 10$, $p = 25$, and $p = 50$) into percent notation.

Through a process of inductive reasoning—based on the structure of the arithmetic expressions written down—they arrived at a hypothetical general form for the rate of increase of a square's area, where the parameter p (the percent numeral of the rate of increase of the square's side) was used as a placeholder. Because the hypothesis was not mathematically justified, the situation of *validation* was incomplete. The session did not contain a situation of *institutionalization*. There was no time left after the presentation to institutionalize the knowledge reached concerning the square's increase, and there was no whole-class session scheduled to institutionalize the solutions to the six worksheets.[9] This means that the status of the knowledge reached remained unsettled—that is, its validity, importance and future were not discussed. From the point of view of mathematics education, this is a clear limitation of the observed session.

Transformations of semiotic representations

The mathematical task dealt with by the observed students is a generalization task in which the variable quantity is given in percent notation. Two aspects complicated the task: the use of percents in the context of change, and the use of an arbitrary percent numeral (a parameter) in generalization of that change. The situation in the task involves a comparison between the enlargement of a square's side and the enlargement of its area, where the percent has a *functional usage*. It is used to quantify the magnitude of the functional operator denoted by e.g. the expression $f(p\%) = (2p + \frac{p^2}{100})\%$, which gives the rate of increase of a square's area as a function of the rate of increase of its side. This functional operator is an algebraic model; it is a compressed expression which conceals the relational features of the quantities at stake.

Percent is a complex concept because it is ambiguous and often appears to have several meanings at once. Percent uses an extremely concise linguistic form and compact notations in which proportional comparisons are hidden. The language used in comparative statements involving percents is additive in form, requires attention to unstated relationships and is often in direct conflict with common everyday language use (Parker & Leinhardt, 1995). These features of percents complicate transformations of representations involving percents, both in terms of treatments and conversions: this is what my analysis bears witness to.

There exists a network of transformation rules for changing between these notation systems: percents, fractions, and decimals. These transformations are referred to as treatments in Duval's (2006) theory, since they happen within the same register. In the analyzed session, these registers were at play: notation systems (percents, fractions, decimals, algebraic symbols), natural language, and geometric figures. The wide variety of notation systems used complicated the students' work. This complication was sharpened by the fact that the communication of the mathematics was, until the very end, mostly done verbally, in natural language. Except for the values calculated for the particular cases, little was written down that could enable the students to confront the handling of numbers and expressions. There are three reasons why natural language does not support percent (Parker & Leinhardt, 1995). First, the referents are often hidden due to the conciseness of the percent language. Second, the preposition *of* has different meanings: in natural language it means *is a part of*; in arithmetic, it indicates a call for *multiplication*; and with percent, it is an indicator of the *direction* of a multiplicative comparison. Third, percent uses additive language in a multiplicative world. These language problems tend to suppress an intuitive understanding of percent, and help explain the problems encountered by the students and teacher in the session analyzed in this paper.

Crucial for the students' progress towards percent notation was the development of manipulatives, where geometric figures (paper cut-outs) were used to represent the enlarged squares in the cases of 10%, 25% and 50% increase of the squares' side. Conversions took place from the register of notation systems (numeric notation with decimals, where the students had calculated the particular cases), to the register of geometric figures, where the quantities involved were conceptualized as areas. Next, the geometric figures were transformed into arithmetic expressions in percent notation. In each of the three cases, the arithmetic expression was a sum, written systematically so as to represent the area of each of the geometric figures that represented the enlarged square. This was a conversion of representations in the register of geometric figures into the register of notation systems (numeric notation with percents). For the sake of efficiency in calculations, as commented on above, a further transformation—a treatment—would have been necessary in order to change the additive expression into a multiplicative one. A transformation like that would belong to a situation of institutionalization, which however was not part of the observed session.

The arithmetic expressions in percent notation were then transformed in terms of a treatment into algebraic notation by an inductive process, as explained above. The algebraic expression in percent arrived at was the solution to the task. There were yet two additional transformations (in terms of conversions) during the students' presentation of their solution to a peer group: the generalization achieved (an algebraic expression using percents and letters) was during the explanation transformed into natural language which in turn was transformed into geometric figures. This last phase is important from the perspective of teacher education—the impact of explaining to others: this requires the activation of several semiotic registers and transformations of representations in and between them. According to Duval (2006), conversions are of particular importance for the learning of mathematics.

Summary of results

I wanted to find out what conditions that enabled or hindered the students to solve the mathematical problem they were confronted with. The particular research question was: *What material and non-material tools prove relevant or missing in the milieu that they resort to?*

Tools that proved missing (or not used) in the milieu were: text(s) that could be studied; the basic equality $q' = q + \frac{p}{100} \cdot q$, where q' is the new value obtained when a quantity q increases by p% (or a technique to arrive at this equality); a technique for transforming representations from one notation system into another (fractions, decimals, percents); and elementary algebraic manipulations.

Tools that proved relevant in the milieu were: calculating and writing down several examples (enabling specializing, conjecturing, generalizing); realization of the utility of a 1 × 1 square (detecting invariance); use of manipulatives (enabling conversions); looking at arithmetic expressions as pre-algebraic forms; looking at the percent numeral as a placeholder; calculators.

Didactical implications of the study

The observation and analysis reported here show that, as seems to be often the case, the activity devolved to the students appears to be too rich, too complex by combining aspects that would aptly be separated, essentially the enlargement problem, which is a geometric model of the rate of increase of a quantity, and percents, which seems to be a topic that could have been avoided, or at least cautiously handled.

One particular aspect of percents is that they still seem to be ontologically ambiguous, so that they are too often not handled in calculations as true mathematical entities, similar to any number or letter that can enter into an expression to be calculated. It thus becomes slippery to develop algebraic calculations in such a context. In conclusion, as far as the learning of algebraic calculations is concerned, the scenario implemented with the students I observed seems to be far from perfect, and in this respect, significant changes (for teacher education) can be advocated.

As a prelude, the basic equality $q' = q + \frac{p}{100} \cdot q$ should be institutionalized and its relationship with the multiplying factor $1 + \frac{p}{100}$ emphasized. Then, in a first stage, the enlargement problem could be stated as a similarity problem, with focus on studying relationships between length, area and volume of similar figures. In this stage, the problem should be stated without reference to percents. In a second stage, percents could be introduced and their meaning examined in order to genuinely integrate them in calculations. Here, students should become familiar with percent notation and be able to handle percents in calculations as true mathematical entities, similar to what I have demonstrated in this paper. It should be emphasized that percent is a *ratio comparison number*, and students should (in each situation with percents) identify which two quantities are compared. Furthermore, this second stage should include tasks on the counterintuitive situations described above: 1) calculating with percent numerals greater than 100;

2) considering an increase by *n*% followed by a decrease by *n*%—*do we get back to the original quantity?*; and 3) comparing a discount of 2*n*% to two successive discounts of *n*% each—*do they give the same price?*

Another point is about justification of conjectures in the classroom: it is critical to provide student teachers with tools that enable them to lead "scientific debates" in the (school) classroom, where different kinds of justification should be discussed and institutionalized. Such tools may include experiences from mathematics sessions involving argumentation and proof at the university, accompanied by a framework for proving that could be used in school, such as the ones proposed by Stylianides (2007) and Balacheff (1988).

Concluding remarks

The observed session highlights the fact that elementary algebra is a modelling tool which historically replaces geometric models which have less flexibility and which appear in the observed session as a surrogate modelling tool. The conclusion we may draw from this analysis is that the students involved have gone almost all the way from the simple use of words and figures to algebraic means, but in fact failed to reach this goal, clinging at the old geometric models which in some way hinders the straight forward use of algebra.

My conclusive statements should certainly be seen in a broader context, namely that of the continuing uncertain relationship between most present-day societies and elementary algebra. Although almost universally taught from an early level, algebra remains a territory unfamiliar to many nonmath people, who sometimes seem to shun it as best they can: this is what my observations bear witness to. One can doubt whether such a long-established constraint is likely to be lifted thanks to classroom activities *only*.[10]

NOTES

[1] In Grades 5–7 in Norway, algebra involves: finding symbolic formulae for structure and change in geometric and numeric patterns; and writing and solving simple equations. In Grades 8–10, algebra involves: manipulating algebraic expressions and using identities of quadratic algebraic expressions; solving linear equations and inequalities, and systems of equations in two unknowns; and using variables in investigations related to theoretical and practical problem solving. In comparison, in France, algebra or, more precisely, algebraic equations (in their simplest form) are traditionally introduced in "cinquième" (i.e., Grade 7, age 12–13), at the *college*.

[2] I am using the adjective *analytic* here in the context of equation solving. It is derived from the noun *analysis* which is explained by the ancient mathematician Pappus of Alexandria this way: "in analysis we assume what is sought as if it has been achieved […]" (Pappus, Collection 7.1:13–18, translated by Jones, 1986, as cited in Rideout, 2008, p. 63).

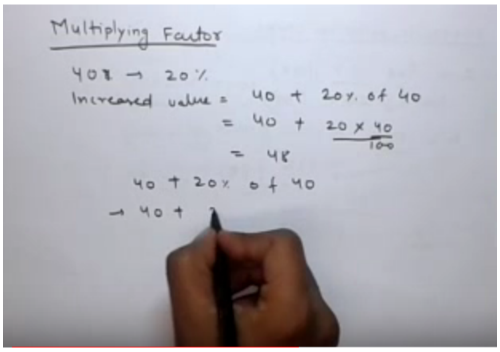
[3] This is inspired by Parker and Leinhardt (1995), who use the mathematical definitions of the terms base, percent, and percentage. In actual use, the words *percent* and *percentage* are frequently interchanged.

[4] An episode described by Heather Krause illustrates this phenomenon: "Last month, I bought a pair of socks for $5.00. Last week, the store had a sale and dropped the price of the socks by 25%. This week, they raised the price by 25%, so now the same socks I bought last month are selling for $4.69. *Wait, what?!*" Retrieved from https://idatassist.com/why-percent-change-is-actually-misleading-most-of-the-time/

[5] The task contained an additional question: "What is the effect on the volume of a cube when its side length increases by *p* percent?" This question was neglected by the observed student, possibly due to lack of time.

[6] For an example, see http://mathforum.org/library/drmath/view/67764.html.

[7] A video at https://www.youtube.com/watch?v=Vk8u1Gk28G8 gives an excellent illustration of this sociomathematical phenomenon. The instructor explains the notion of "multiplying factor". If the number 40 increases by 20%, what will be its new value?



As can be seen on the screenshot above, the instructor first expresses the "increased value" as "40 + 20% *of* 40": he dares not write "40 + 20% × 40" (read “40 plus 20% *times* 40”). On the second line, instead of writing “40 + $\frac{20}{100}$ × 40”, he first writes “40 + 20 × $\frac{40}{100}$”, probably because he dares not imply that 20% is the same as $\frac{20}{100}$, and then, in a second attempt, he awkwardly extends the line of the fraction $\frac{40}{100}$ to arrive at the unexpected (but correct) fraction $\frac{20 \times 40}{100}$.

[8] The transcript has been translated from Norwegian by the author. Transcription codes:

…	Pause up to 3 seconds.
[text]	Clarification of wording.
⟨Text⟩	Account of nonverbal action.
[]	Inarticulate or inaudible utterance.
[⁝]	Omitted turns.
(NN: interjection)	Interjection by NN during another person’s turn.

[9] Based on, among other things, experiences with the session analyzed here, the teachers of the class started to schedule regular whole-class sessions succeeding small-group work, with the aim of institutionalizing the knowledge at stake.

[10] In his famous satire *Candide* (1759), although he was not an ignoramus in mathematics, Voltaire (1694–1778) made fun of algebra, writing in chapter 22 about Candide’s red sheep that the Academy of Sciences at Bordeaux “proposed, as a prize subject for the year, to prove why the wool of this sheep was red; and the prize was adjudged to a northern sage, who demonstrated by A plus B, minus C, divided by Z, that the sheep must necessarily be red…” (Voltaire, 1759, p. 63). Maybe “Northern sages” are no longer what they used to be!

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