

AN EMBEDDING OF THE UNIT BALL THAT DOES NOT EMBED INTO A LOEWNER CHAIN

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ABSTRACT. We construct a holomorphic embedding $\phi : \mathbb{B}^3 \rightarrow \mathbb{C}^3$ such that $\phi(\mathbb{B}^3)$ is not Runge in any strictly larger domain. As a consequence, $\mathcal{S} \neq \mathcal{S}^1$ for $n = 3$.

1. INTRODUCTION

Recall that a Loewner chain is a family $f_t : \mathbb{B}^n \rightarrow \mathbb{C}^n$ of holomorphic injections, $f_t(0) = 0$, $f'_t(0) = e^t \cdot \text{id}$, $t \in [0, \infty)$, with $f_t(\mathbb{B}^n) \subseteq f_s(\mathbb{B}^n)$ for $t \leq s$. We let \mathcal{S} denote the set of all univalent maps $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ with $f(0) = 0$, $f'(0) = \text{id}$, we let \mathcal{S}^1 denote the set of all $f \in \mathcal{S}$ such that f embeds into a Loewner chain, *i.e.*, $f = f_0$ where $(f_t)_{t \geq 0}$ is a Loewner chain, and finally we let \mathcal{S}^0 denote the set of all $f \in \mathcal{S}^1$ for whom we require that the family $(e^{-t} f_t)_{t \geq 0}$ is normal.

In one variable, the three sets coincide, and they are all compact. On the other hand, in higher dimensions, the sets \mathcal{S} and \mathcal{S}^1 are certainly not compact, as can be seen as a consequence of the automorphism group of \mathbb{C}^n being huge for $n \geq 2$. On the other hand, it is known that \mathcal{S}^0 is compact, and so we get the chain of inclusions

$$\mathcal{S}^0 \subsetneq \mathcal{S}^1 \subseteq \mathcal{S}. \tag{1.1}$$

However, if $f \in \mathcal{S}^1$, there exist $\psi \in \mathcal{I}(\mathbb{C}^n)$ (the set of entire injective maps), and $g \in \mathcal{S}^0$ such that $f = \psi \circ g$, and so we may say that \mathcal{S}^1 splits (see e.g. [2], Theorem 2.6.),

$$\mathcal{S}^1 = \mathcal{I}(\mathbb{C}^n) \circ \mathcal{S}^0. \tag{1.2}$$

The background for this article is that it has been unknown whether it is also the case that $\mathcal{S} = \mathcal{I}(\mathbb{C}^n) \circ \mathcal{S}^0$, or equivalently, whether $\mathcal{S} = \mathcal{S}^1$ (this problem was mentioned and discussed in [1]). In this context, the following closely related problem was recently posed by F. Bracci: Let $f \in \mathcal{S}$. Does there exist a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n$ such that $f(\mathbb{B}^n)$ is Runge in Ω ? This turns out not to be the case.

Theorem 1.1. *For any $\epsilon > 0$ there exists a continuous injective map $\phi : \overline{\mathbb{B}^3} \rightarrow \mathbb{C}^3$ with $\phi \in \mathcal{O}(\mathbb{B}^3)$, and such that*

- (i) $\|\phi - \text{id}\|_{\overline{\mathbb{B}^3}} < \epsilon$, and
- (ii) if $\phi(\mathbb{B}^3) \subset \Omega$ is a Runge pair, then $\phi(\mathbb{B}^3) = \Omega$.

Since the conditions in Docquier-Grauert [3] (Definition 20) are satisfied for the increasing family $(f_t(\mathbb{B}^n))_{0 \leq t \leq t_0}$ for any fixed t_0 , and for any Loewner chain, it follows from [3] (Satz 17–19) that each pair $(f_0(\mathbb{B}^n), f_t(\mathbb{B}^n))$ is a Runge pair, and we get our second theorem as a corollary:

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Theorem 1.2. *For $n = 3$ we have that $\mathcal{S} \neq \mathcal{S}^1$.*

2. PRELIMINARIES

The problem mentioned above was recently studied by Gaussier and Joița [4]. In particular, they studied the map

$$\phi(z) = (z_1, z_1 z_2^2 + 2z_3 z_2, z_1 z_2 + z_3), \quad (2.1)$$

although we have here changed coordinates to have a fixed point at the origin. (This map was constructed by John Wermer [5],[6] to produce a non Runge embedded polydisk in \mathbb{C}^3 .) It is straight forward to check that the map ϕ is injective holomorphic on the half space

$$H := \{z \in \mathbb{C}^3 : \operatorname{Re}(z_3) < 0\}.$$

For $0 < p < 1/4$ we set

$$D_p := \{z \in \mathbb{C}^3 : 2\operatorname{Re}(z_3) + |z_3|^2 + p(|z_1|^2 + |z_2|^2) < 0\}. \quad (2.2)$$

Then D_p is biholomorphic to the unit ball, $0 \in bD_p$, and $D_p \subset H$.

The result obtained by Gaussier and Joița is the following: For $r > 0$ sufficiently small there exists $0 < \alpha < r$ such that the set

$$S_{r,\alpha} := \{z \in \mathbb{C}^3 : |z_1| = r, z_2 = 0, z_3 = \alpha\}$$

is contained in $\phi(D_p)$. Note however that none of the disks

$$D_{r,\alpha} := \{z \in \mathbb{C}^3 : |z_1| < r, z_2 = 0, z_3 = \alpha\}$$

are contained in $\phi(D_p)$; more specifically the point $(0, 0, \alpha)$ is not contained in $\overline{\phi(D_p)}$, since ϕ restricted to the z_3 -coordinate line is the identity. The following is a consequence.

Proposition 2.1. *(Gaussier-Joița) If $\phi(D_p) \subset \Omega$ and if Ω contains an open neighbourhood of q , then $\phi(D_p)$ is not Runge in Ω .*

Our approach to prove Theorem 1.1 is to produce an embedding which has similar "bad" boundary points everywhere on the boundary of the embedded ball. More specifically, by a "bad" boundary point we will mean the following.

Definition 2.2. Let $\Omega \subset \mathbb{C}^n$ be a domain. We will say that a point $q \in b\Omega$ is W(ermer)-degenerate, if for any $\delta > 0$ there exists an embedded holomorphic disk $D \subset B_\delta(q)$ such that $bD \subset \Omega$ and $D \not\subset \overline{\Omega}$.

3. MODIFICATION OF THE WERMER MAP

We let $B \subset \mathbb{C}^3$ denote the translated unit ball $B = \{z \in \mathbb{C}^3 : 2\operatorname{Re}(z_3) + \|z\|^2 < 0\}$, and we let $B' \subset \mathbb{C}^3$ denote the ball which is scaled by a factor two, $B' = \{z \in \mathbb{C}^3 : 2\operatorname{Re}(z_3) + (1/2)\|z\|^2 < 0\}$.

Proposition 3.1. *Let $\{\alpha_1, \dots, \alpha_n\} \subset \overline{B} \setminus \{0\}$ and let $\epsilon > 0$. Then there exists an injective continuous map $\psi : \overline{B}' \rightarrow \mathbb{C}^3$ with $\psi \in \mathcal{O}(\mathbb{B}^3)$ such that the following holds*

- (i) $\|\psi - \operatorname{id}\|_{\overline{B}'} < \epsilon$,
- (ii) $(\psi - \operatorname{id})(z) = O(\|z - \alpha_j\|^3)$ for $j = 1, \dots, n$, and
- (iii) $\psi(0) = 0$, and 0 is W-degenerate for $\psi(B)$.

Proof. We will compose the map

$$\phi(z_1, z_2, z_3) = (z_1, z_1 z_2^2 + 2z_3 z_2, z_1 z_2 + z_3) \quad (3.1)$$

with several holomorphic embeddings to achieve the claims of the theorem.

For $N \in \mathbb{N}$ we set

$$f_N(z_3) = \frac{1}{2z_3} + e^{N(2z_3)} \left(1 - \frac{1}{2z_3}\right),$$

and

$$h_N^\delta(z_3) = \frac{1}{2\delta} \frac{e^{N(2z_3)} + (2\delta - 1)}{1 + (2\delta - 1)e^{N(2z_3)}} + \frac{1}{2\delta}.$$

The maps h_N^δ map the left half space to the disk of radius $1/2\delta$ centred at the point $1/2\delta$, and $-\infty$ is mapped to the point 1. So for a fixed δ we have that $h_N^\delta \rightarrow 1$ uniformly on compact subsets of the left half space, as $N \rightarrow \infty$.

Then we set

$$\begin{aligned} F_1^\delta(z) &= (\delta z_1, \delta z_2, z_3), \\ F_2^N(z) &= (z_1, z_2 f_N(z_3), z_3), \\ F_3^{N,\delta}(z) &= (z_1 h_N^\delta(z_3), z_2 h_N^\delta(z_3), z_3). \end{aligned}$$

Then we set

$$F_{N_1, N_2, \delta_1, \delta_2} = (F_1^{\delta_1})^{-1} \circ \phi \circ F_2^{N_1} \circ F_3^{N_2, \delta_2} \circ F_1^{\delta_1}.$$

Lemma 3.2. *We have that $F_{N_1, N_2, \delta_1, \delta_2} : H \rightarrow \mathbb{C}^3$ is injective holomorphic for all sufficiently large $N_1 \in \mathbb{N}$. Moreover, we have that $F_{N_1, N_2, \delta_1, \delta}$ converges uniformly to the identity as $N_1 \rightarrow \infty$, $\delta_1 \rightarrow 0$, $\delta_2 \rightarrow 0$ and $N_2 \rightarrow \infty$, and we may arrange that 0 is W -degenerate for $F_{N_1, N_2, \delta_1, \delta_2}(B)$. (We note that the rate of convergence of each of the quantities, depends on the previous one.)*

Proof. To prove that $F_{N_1, N_2, \delta_1, \delta_2} : H \rightarrow \mathbb{C}^3$ is injective, we need to check that f_N and h_N^δ are both non-zero on $\tilde{H} = \{\operatorname{Re}(z_3) < 0\}$ for sufficiently large $N \in \mathbb{N}$. We leave it to the reader to check that h_N^δ maps \tilde{H} to the disk of radius $1/2\delta$ centred at the point $1/2\delta$.

Suppose that $f_N(z_3) = 0$. Then $z_3 \neq 1/2$. Multiplying by $2z_3$ we get that

$$\begin{aligned} 1 + e^{N(2z_3)}(2z_3 - 1) = 0 &\Rightarrow 2Nz_3 + \log(1 - 2z_3) = 0 \\ &\Rightarrow 2z_3 + \log(1 - 2z_3)/N = 0. \end{aligned}$$

The last expression converges uniformly to the function $2z_3$ on compact subsets of $\{\operatorname{Re}(z_3) < 1/4\}$, so for a given compact set K containing $z_3 = 0$, the only zero on K is the point $z_3 = 0$.

Next we check the convergence to the identity. We have

$$G(z) := F_2^N(F_3^{\delta_2}(F_1^{\delta_1}(z))) = (\delta_1 z_1 h_{N_2}^{\delta_2}(z_3), \delta_1 z_2 h_{N_2}^{\delta_2}(z_3) f_{N_1}(z_3), z_3).$$

Further

$$\begin{aligned} \phi(G(z)) &= (\delta_1 z_1 h_{N_2}^{\delta_2}(z_3), \delta_1^3 z_1 z_2^2 (h_{N_2}^{\delta_2}(z_3))^3 f_{N_1}(z_3)^2 + \delta_1 z_2 h_{N_2}^{\delta_2}(z_3) f_{N_1}(z_3) (2z_3), \\ &\quad \delta_1^2 z_1 z_2 h_{N_2}^{\delta_2}(z_3)^2 f_{N_1}(z_3) + z_3), \end{aligned}$$

and

$$F_{N_1, N_2, \delta_1, \delta_2}(z) = (z_1 h_{N_2}^{\delta_2}(z_3), \delta_1^2 z_1 z_2^2 (h_{N_2}^{\delta_2}(z_3))^3 f_{N_1}(z_3)^2 + z_2 h_{N_2}^{\delta_2}(z_3) f_{N_1}(z_3) (2z_3), \\ \delta_1 z_1 z_2 h_{N_2}^{\delta_2}(z_3)^2 f_{N_1}(z_3) + z_3).$$

We now explain how to choose all the constants to get convergence to the identity. Note that $f_{N_1}(z_3)2z_3$ is bounded independently of N_1 and that $f_{N_1}(z_3)2z_3 \rightarrow 1$ uniformly on compact subsets of $B' \setminus \{0\}$ as $N_1 \rightarrow \infty$. So fix a large N_1 . This will cause the other terms containing $f_{N_1}(z_3)$ to grow, but this growth may now be eliminated by choosing δ_1 small. Next, before choosing δ_2 we consider the image $G(B)$ near the origin after such a choice is made. Note that $f_{N_1}(0) = (1 - N_1)$ and that $h_{N_2}^{\delta_2}(0) = 1/\delta_2$. This implies that $G(B)$ has a defining function

$$2\operatorname{Re}(z_3) + |z_3|^2 + \delta_2/\delta_1 |z_1|^2 + (\delta_2^2/(N_1 - 1)^2 \delta_1) |z_2|^2 + O(|z_3|(|z_1|^2 + |z_2|^2)) < 0.$$

So if we choose δ_2 sufficiently small, we see that $D_p \subset G(B)$, which will cause the origin to be a W-degenerate point for $F_{N_1, N_2, \delta_1, \delta_2}(B)$. Choosing a small δ_2 will cause growth in all the other terms containing $h_{N_2}^{\delta_2}$, but this is finally "localised" to the origin by choosing a sufficiently large N_2 . \square

Due to the lemma, we have now proved the proposition except for the claim (ii). However, it is easy to explicitly construct an interpolation operator depending continuously on the input, that corrects the map at the points $\alpha_1, \dots, \alpha_n$. \square

4. PROOF OF THEOREM 1.1

Let $\{\alpha_0, \alpha_1, \alpha_2, \dots\} \subset b\mathbb{B}^3$ be a dense set of points. For $R > 1$ we will let $B(j, R)$ denote the ball in \mathbb{C}^3 containing \mathbb{B}^3 with the common boundary point α_j . Then, for $\delta > 0, n \in \mathbb{N}$, we set

$$\Omega(R, n, \delta) = \mathbb{B}^3(\delta) \setminus \cup_{j=1}^n B(j, R)^c.$$

We will construct by induction embedded holomorphic disks $D_j \in \mathbb{C}^3$, $\delta_j, \epsilon_j > 0$, and injective continuous maps $\phi_j : \overline{\Omega}(1 + 1/j, j, \delta_j) \rightarrow \mathbb{C}^3$, $\psi_j \in \mathcal{O}(\Omega(1 + 1/j, j, \delta_j))$, such that the following holds

- (a_n) $\|\phi_j - \phi_{j-1}\|_{\overline{\mathbb{B}^3}} < \epsilon_j$ for $j = 1, 2, \dots, n$ ($\phi_0 = \text{id}$),
- (b_n) $\phi_j(\alpha_k) = \phi_{j-1}(\alpha_k)$ for $k = 0, 1, 2, \dots, j-1, j = 1, 2, \dots, n$,
- (c_n) $D_j \subset D_{(1/2)^j}(\alpha_{j-1})$ for $j = 1, \dots, n$,
- (d_n) $bD_j \subset \overline{\phi_n(\mathbb{B}^3)}$ for $j = 1, \dots, n$, and
- (e_n) $D_j \not\subset \overline{\phi_n(\mathbb{B}^3)}$ for $j = 1, \dots, n$.

As a preliminary choice of $\{\epsilon_j\}$ we set $\epsilon_j = \epsilon \cdot (1/2)^{j+1}$. This is just to ensure (i) in the theorem, as our plan is to define

$$\phi := \lim_{j \rightarrow \infty} \phi_j, \tag{4.1}$$

after we explain the inductive procedure. Each ϵ_j will however be further decreased throughout the process. Note in particular, that if the sequence decreases sufficiently fast, then the map $\phi : \overline{\mathbb{B}} \rightarrow \mathbb{C}^3$ will be injective. To start the induction, we let ϕ_1 be the map furnished by

Proposition 3.1, creating a W -degenerate point at α_0 , and such that (a_1) and (b_1) hold. This means that there exists D_1 such that (c_1) – (e_1) hold.

Assume now that (a_n) – (e_n) hold for some $n \geq 1$. Decrease ϵ_j for $j > n$ such that any limit ϕ defined as in (4.1) will satisfy

- (d_∞) $bD_j \subset \phi(\mathbb{B}^3)$ for $j = 1, \dots, n$, and
- (e_∞) $D_j \not\subset \overline{\phi(\mathbb{B}^3)}$ for $j = 1, \dots, n$.

Next we let $\tilde{\phi}_{n+1} : \overline{\Omega(1 + 1/(n+1), n+1, \delta_{n+1})} \rightarrow \overline{\Omega(1 + 1/n, n, \delta_n)}$ be a map furnished by Proposition 3.1, creating a W -degenerate point at α_n , and such that (a_{n+1}) and (b_{n+1}) holds for the composition $\phi_{n+1} = \phi_n \circ \tilde{\phi}_{n+1}$. Note that $\Omega(1 + 1/(n+1), n+1, \delta_{n+1}) \subset \Omega(1 + 1/n, n, \delta_n)$ as soon as $\delta_{n+1} < \delta_n$, and note that the existence of $\tilde{\phi}_{n+1}$ uses both the approximation property and the interpolation properties at the points $\alpha_0, \dots, \alpha_{n-1}$. Finally choose a disk D_{n+1} such that (c_{n+1}) – (e_{n+1}) hold.

This completes the induction step, and we now define ϕ is in (4.1). To complete the proof, assume that $\phi(\mathbb{B}^3) \subset \Omega$, and suppose there exists a point $p \in b\phi(\mathbb{B}^3)$ and a $\delta > 0$ such that $B_\delta(p) \subset \Omega$. Then by (c_∞) there exists a disk $D_j \subset B_\delta(p)$ which has the properties (d_∞) – (e_∞) , which implies that $\phi(\mathbb{B}^3)$ is not Runge in Ω . \square

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