Optimal reliability growth program for repairable and warranted products

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SUMMARY & CONCLUSIONS

Considering a two-dimensional and non-renewing free repair warranty (NFRW) policy, this paper proposes a modified reliability growth model to support the decision making of the manufacturer on the choice of a reliability growth test program for repairable products with a two-fold Weibull lifetime distribution, which has a bathtub-shaped failure rate function. Excess usage is used as a stress to trigger failures and the effect of usage acceleration on product reliability is modeled by the Accelerated Failure Time (AFT) approach. Learning from test failures is incorporated into periodic fixes through which the overall product failure intensity is reduced through discrete steps. The optimal test duration and the optimal number of fixes are both obtained such that the expected total cost per product to the manufacturer is minimized while achieving the stated reliability growth target at the same time.

Results of the illustrative case show that the optimal reliability growth test setting is governed by a sensitive trade-off between the test cost incurred and the reduction in the warranty cost. A sensitivity analysis on several key input parameters of the proposed model reveals the importance of conducting reliability growth, especially for products with heavy usage intensity and high repair cost during field use. It is also found with a higher failure learning level, the optimal reliability growth test program yields more prominent cost reduction and reliability improvement.

1 *INTRODUCTION*

All products are unreliable in the sense that they degrade with age and/or usage, and ultimately fail. Competitive market environment and consumer expectations press manufacturers to introduce new products with high performance and reliability, at the same time providing better warranty service and post-sale support. New products are usually an improvement over earlier products with changes to design. However initial prototypes of new generation products invariably have reliability and performance deficiencies that generally could not be foreseen and eliminated in early design stages. A warranty is a contractual agreement offered by the manufacturer to rectify any problems (such as the item not performing as expected, failures of components, etc.) that the customer experiences over the specified warranty coverage. Offering any type of warranty policy incurs additional costs to the manufacturer due to warranty claims servicing. The so-called warranty cost can be reduced through reliability improvement during development phase before launching products into market.

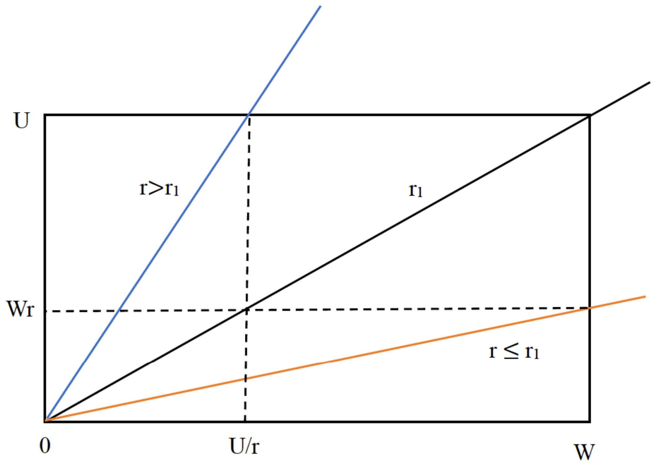
Two basic approaches are commonly used to improve product reliability during pre-launch phase. The first utilizes redundancy that involves the application of a module of replicated components as opposed to a single item. The second approach involves research and development (R&D) effort where the product is subjected to a reliability growth test to assess and improve reliability. The reliability growth is achieved through a test-analyze-and-fix (TAAF) process in an iterative manner. During this process, the product is subjected to increasing levels of stress until a failure appears. Should the failure occur, the failures data including modes of failure, time to failure and other relevant information are collected and analyzed to identify the failure mechanisms and causes. Fix is then implemented to product design effort to reduce the failure intensity of that particular failure mode. As this process is repeated, more failure modes are identified and corrected, resulting in the decrease of the overall product failure rate. Upon completion of development, due to enhanced product reliability, reward can be reflected in increase in sales, decrease in warranty costs, etc.

Up to now, a number of reliability growth models have been proposed, which can be broadly categorized into two types-discrete and continuous models. Discrete models involve discrete data and are concerned with incremental improvements in reliability as a result of design changes [1]. Continuous models are used in the context of continuous variables and attempt to describe the reliability improvement as a function of the total expected test duration [2, 3].

Building reliability into a new product is costly, while the consequence of not achieving reliability target can be even costlier. A well-defined reliability growth test program is supposed to achieve the reliability target with the cost incurred as low as possible [4]. So far in the literature very limited attention has been paid to the integrated research that links reliability growth within the developmental phase and warranty service during the field operation phase. Murthy and Nguyen treated the development as occurring according to a Non-homogeneous Poisson process (NHPP) and developed three different stochastic models to determine the optimal reliability improvement by taking the impact of reliability on the expected warranty cost into account [5]. The models suffer from the limitation that the failure rate is unbounded at the start of development test and zero after infinite development period, which is actually unrealistic in real life. On that basis, Hussain and Murthy made a further contribution by proposing a modified stochastic model where the uncertain outcome of the product development process was captured through a "black-box" approach [6]. They determined the optimal development period to achieve a sensible trade-off between the development cost and the resulting reduction in the expected warranty cost. These works have focused on failures indexed by a single timescale due to a single failure mode, and been limited to one-dimensional warranty policies. A product is usually at risk of failure from several competing failure modes. In addition, for products sold with a two-dimensional warranty, different users have various usage rates such that heterogeneous usage intensities need to be incorporated into reliability growth modeling.

So far, more attention has been paid to test resources allocation with the objective of product reliability maximization [7,8,9,10]. Distinguishing the existing literature, in this paper, considering cost minimization, we deal with a reliability growth test program for repairable products subject to specified reliability growth target and sold with a two-dimensional warranty policy. The product has a two-fold Weibull lifetime distribution with a bathtub-shaped failure rate function. Usage rate is utilized as a stress level and accelerated to induce failures with a faster pace. Effect of usage rate variation on failures is modeled by Accelerated Failure Time (AFT) approach. Whenever the product fails during test, it is minimally repaired and test continues. Fixes (corrective actions) are implemented periodically. Once a fix takes place, the “failure learning” result for each failure mode, that is modeled based on the number of mode-specified failures occurring within the previous fix interval, will be incorporated to reduce the failure intensity of that particular mode accordingly. Based on these settings, the expected total cost per tested product to the manufacturer, consisting of both the test cost and warranty cost, can be estimated. The optimal reliability growth test program including the test duration and the number of fixes is derived for cost minimization, assuring that the reliability growth requirement is met at the same time.

2 *NOTATIONS*



W,U	warranty time and usage limits
x,R	product cumulative age and random usage rate
r_0	nominal usage rate
$g(r)$	probability density function of usage rate R
$h(x\mid r),\lambda(x\mid r)$	product conditional hazard function and failure intensity function
$\alpha_1,\alpha_2;\beta_1,\beta_2$	scale and shape parameters for the two-fold Weibull failure distribution
α_1^m,α_2^m	minimum achievable value of the scale parameters α_1 and α_2
ρ,η	accelerated coefficients under each failure mode
τ	Reliability growth test duration
n	number of fixes
z	accelerated usage rate implemented during the test
b	failure learning level
$\alpha_i^j\left(\tau,n\right)$	value of scale parameter α_i after the *j*th fix
p_i^j	the fix effectiveness factor (FEF) of the *j*th fix for failure mode *i*
$\lambda_i(x;\alpha_i^j\left(\tau,n\right)\mid z)$	mode-specific failure intensity after *j*th fix
$E[N_t\left(\tau,n\right)]$	expected number of test failures
$E\left[N_w(\tau,n)\right]$	expected number of warranty failures during field use
C_s	set-up cost of the test per product
C_t	test cost per unit time
C_f	mean cost of each fix
C_a,C_r	mean repair costs to rectify a test failure and a warranty failure, respectively
$EC_t\left(\tau,n\right)$	expected cost incurred by reliability growth test
$EC_w\left(\tau,n\right)$	Expected cost incurred by warranty servicing
$TC\left(\tau,n\right)$	expected total cost to the manufacturer per product
$E\left[\square\right]$	expectation of the variable in the bracket

3 *MODEL FORMULATION*

3.1 *Warranty policy and item failures*

The tested product is repairable and sold with a two-dimensional and non-renewing free-repair warranty (NFRW) policy, under which the manufacturer rectifies all failures occurring the warranty coverage at no cost to the customer. The rectangular warranty region is considered here, as illustrated in Figure 1. Let W and U denote the age and usage limits, respectively. Define $r_1=U/W$. The warranty expires when either the product age reaches the limit W if $r \le r_1$ or the total usage exceeds the level U if $r > r_1$, whichever occurs first from the time of the product being purchased.

Figure.1. Product two-dimensional warranty coverage

Under two-dimensional warranty, failures can be viewed as random events occurring within the warranty region, and modeled by a counting process characterized by a failure intensity function which is dependent on both age and usage. To model the item failures, we use one-dimensional approach which treats the random usage rate as a covariate, conditioning on which the two-dimensional failure process is reduced to a one-dimensional one. An assumption adopted here is that the usage rate over time is constant for each customer, but varies across customer population. Let R be the non-negative random usage rate, $g(r)$ be the density function of R , and r be a realization of R. We further assume the manufacturer knows this distribution either through historical information or from a detailed market survey. For $R = r$, the conditional hazard function for the time to first failure is a non-decreasing function of the product age x and the field usage rate r.

The accelerated failure time (AFT) approach is used to characterize the effect of usage rate on the degradation of the product. During the design phase, the product has a desired reliability at a nominal usage rate r_0. When the real usage rate r differs from this nominal value, the reliability of the product will be affected. As the usage rate increases, the rate of degradation increases and this accelerates the time to failure in turn. As a result, the product reliability decreases (increases) as the usage rate increases (decreases).

Conditional on $R = r$, the tested product lifetime follows an additive Weibull distribution which combines two Weibull distributions; one has a decreasing hazard function and the other has an increasing hazard function. It has the conditional hazard function with the following form

$$h(x;\alpha_1,\alpha_2 \mid r) = \beta_1 \alpha_1^{\beta_1} (\frac{r}{r_0})^{\rho\beta_1} x^{\beta_1 - 1} + \beta_2 \alpha_2^{\beta_2} (\frac{r}{r_0})^{\eta\beta_2} x^{\beta_2 - 1} \quad (1)$$

for $x \ge 0, \alpha_1, \alpha_2, \rho, \eta > 0, 0 < \beta_2 < 1 < \beta_1$. α_1 and β_1 are the scale and shape parameters of the increasing failure rate usually caused by material fatigue or component aging, while α_2 and β_2 are the ones of the decreasing failure rate usually caused by design faults and initial problems. ρ and η represent two different accelerated coefficients under each failure mode.

The subsequent failures depend on the type of repair action performed. We confine our attention to minimal repair during reliability growth test process and warranty coverage, therefore the conditional failure intensity function $\lambda(x;\alpha_1,\alpha_2 \mid r)$ has the same form as the hazard function $h(x;\alpha_1,\alpha_2 \mid r)$ given by Equation (1).

One of the most widely used reliability measure in practice is the cumulative mean time between failures (MTBF). Suppose the time required to rectify a failure is very short compared to the mean time to failure (MTTF), the MTBF can be therefore obtained as

$$MTBF = \int_0^\infty \int_0^\infty x\lambda\left(x;\alpha_1,\alpha_2 \mid r\right) e^{-\int_0^\infty \lambda(x;\alpha_1,\alpha_2 \mid r)dx} g\left(r\right) dxdr. \quad (2)$$

3.2 Reliability growth modeling

A *test-find-fix-test* scheme is considered, under which the test duration τ consists of n periodic fix intervals. During the test process, the product usage is accelerated at rate z $(z > r_0)$ to induce failures with a faster pace. Within each fix interval, failures are minimally repaired without affecting the product failure intensity. Let α_1^m and α_2^m denote the minimum achievable value of the scale parameters in the product two-fold Weibull failure distribution, when infinite test time and maximum fix effort are assumed. Each fix does not remove a failure mode completely, but rather reduces the failure intensity of that failure mode with an adjusted fix effectiveness factor (FEF), which depends on the number of test failures occurred within the previous fix interval. This is reflected by the reduction in the values of α_1 and α_2 after each fix is implemented. One probabilistic approach - (p,q) rule [11] is used to model the effect of fix on the mode-specific failure intensity. That is, the scale parameter α_i $(i = 1,2)$ after the *j*th fix activity ($j = 1,...,n$) is modeled as

$$\alpha_i^j\left(\tau,n\right) = \alpha_i^{j-1}\left(\tau,n\right) - [\alpha_i^{j-1}\left(\tau,n\right) - \alpha_i^m] \cdot p_i^j \quad (3)$$

of which α_i^0 is the initial value of α_i before the reliability growth test starts. p_i^j is defined as the FEF of the *j*th fix for failure mode i, that is modeled as

$$p_i^j = 1 - [1 + E[N_i(S_j \mid z) - N_i(S_{j-1} \mid z)]]^{-b} \quad (4)$$

where $b \ge 0$ is the magnitude of the learning effect due to test failures. Since the second term on the right is always less than or equal to 1. The higher the value of b, the more effective the failure learning and the corresponding fix activity.

S_j denotes the time instant of the *j*th fix with $S_0 = 0$. $E[N_i(S_j \mid z) - N_i(S_{j-1} \mid z)]$ is the expected number of test failures during the *j*th fix interval due to failure mode *i*, which constitutes a NHPP with the mode-specific failure intensity function $\lambda_i(x;\alpha_i^{j-1}\left(\tau,n\right) \mid z)$ and is given by

$$E[N_i(S_j \mid z) - N_i(S_{j-1} \mid z)] = \int_{(j-1)\frac{\tau}{n}}^{j\frac{\tau}{n}} \lambda_i[x;\alpha_i^{j-1}\left(\tau,n\right) \mid z]dx \quad (5)$$

under the periodic fix scheme. Therefore, the expected number of test failures is denoted by $E[N_t\left(\tau,n\right)]$ and obtained as

$$E[N_t\left(\tau,n\right)] = \sum_{i=1}^{2} E\left[N_i(S_j \mid z) - N_i(S_{j-1} \mid z)\right]. \quad (6)$$

3.3 Failures within warranty coverage

After reliability growth test, the product conditional failure intensity function becomes

$$\lambda\left[x;\alpha_1\left(\tau,n\right),\alpha_2\left(\tau,n\right) \mid r\right] = \beta_1\left[\alpha_1\left(\tau,n\right)\right]^{\beta_1} (\frac{r}{r_0})^{\rho\beta_1} x^{\beta_1 - 1} + \beta_2\left[\alpha_2\left(\tau,n\right)\right]^{\beta_2} (\frac{r}{r_0})^{\eta\beta_2} x^{\beta_2 - 1}. \quad (7)$$

The warranty period W_r conditional on usage rate r is given by W when $r \le r_1$ and U/r when $r > r_1$. Let

$E[N_w(\tau,n)]$ denote the expected number of warranty failures, which is given by

$$E\left[N_w(\tau,n)\right] = \int_0^\infty \{\int_0^{W_r} \lambda\left[x;\alpha_1\left(\tau,n\right),\alpha_2(\tau,n)\,|\,r\right]dx\} g(r)dr. \quad (8)$$

4 COST ANALYSIS

The expected reliability growth test cost per product depends on the test duration τ and the number of fixes n. Let C_s be the set-up cost of test per product, C_t be the test cost per unit time, C_f be the mean cost of each fix and C_a be the average repair cost to rectify a failure occurred during the reliability growth test. Since there are $N_t\left(\tau,n\right)$ test failures, the expected test cost per product denoted by $EC_t\left(\tau,n\right)$ is given by

$$EC_t\left(\tau,n\right) = C_s + C_t \tau z^{\psi} + nC_f + C_a E\left[N_t\left(\tau,n\right)\right] \quad (9)$$

where ψ is the elasticity of the accelerated usage rate implemented during the test.

Similarly, there are $N_w(\tau,n)$ warranty failures within the warranty coverage, thus the expected warranty cost per product denoted by $EC_w\left(\tau,n\right)$ is given by

$$EC_w\left(\tau,n\right) = C_r E\left[N_w\left(\tau,n\right)\right] \quad (10)$$

where C_r is the mean cost of each minimal repair for a warranty failure with $C_a < C_r$.

Let $TC\left(\tau,n\right)$ denote the expected total cost per product to the manufacturer, which is the sum of the expected reliability growth test cost and the expected warranty cost and expressed as

$$TC\left(\tau,n\right) = EC_t\left(\tau,n\right) + EC_w\left(\tau,n\right). \quad (11)$$

The reliability growth test decision is to derive the optimal test duration τ^* and the number of fixes n^* to minimize the expected total cost per product $TC\left(\tau,n\right)$. The cost-based optimization model can be expressed as follow:

$$\begin{aligned} &\left(\tau^*,n^*\right) = \arg\min TC\left(\tau,n\right) \\ &s.t.\quad \alpha_1^m \le \alpha_1(\tau,n) \le \tilde{\alpha}_1, \\ &\qquad \alpha_2^m \le \alpha_2\left(\tau,n\right) \le \tilde{\alpha}_2, \\ &\qquad 0 < \tau \le \tilde{\tau}, \\ &\qquad 1 \le n \le N. \end{aligned} \quad (12)$$

where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are the upper limits of α_1 and α_2 after reliability growth is completed. $\tilde{\tau}$ is the possible time constraint imposed on the test duration and N is the upper limit for the number of fixes to be made. Upper limits for τ and n are necessary since a long test duration and extensive fixes will incur additional funds and prolong the time to market of the product.

Considering the special case of no reliability growth test ($\tau = 0$ and $n = 0$), then there is

$$E\left[N_w\left(0,0\right)\right] = \int_0^\infty \left(\int_0^{W_r} \lambda\left(x;\alpha_1^0,\alpha_2^0\,|\,r\right)dx\right) g(r)dr. \quad (13)$$

The expected total cost per product without reliability growth is simply the warranty cost that is given by

$$TC\left(0,0\right) = C_r E\left[N_w(0,0)\right] \quad (14)$$

where $TC\left(0,0\right)$ represents the upper bound of the expected total cost per product to the manufacturer, which can also be called as the *benchmark cost*. Once the expected total cost per product with reliability growth exceeds $TC\left(0,0\right)$, the reliability growth test is not cost-efficient. In mathematical terms, reliability growth test is beneficial if

$$TC\left(\tau,n\right) < TC\left(0,0\right)$$

or

$$\begin{aligned} &C_s + C_t \tau z^{\psi} + nC_f + C_a \sum_{i=1}^{2} E\left[N_i\left(S_j\,|\,z\right) - N_i\left(S_{j-1}\,|\,z\right)\right] + \\ &C_r \int_0^\infty \{\int_0^{W_r} \lambda\left[x;\alpha_1\left(\tau,n\right),\alpha_2(\tau,n)\,|\,r\right]dx\} g(r)dr \\ &< C_r \int_0^\infty \left(\int_0^{W_r} \lambda\left(x;\alpha_1^0,\alpha_2^0\,|\,r\right)dx\right) g\left(r\right)dr \end{aligned}$$

The effect of performing the optimal reliability test program $\left(\tau^*,n^*\right)$ on the cost reduction is measured by the percentage parameter Δ given by

$$\Delta = \frac{TC\left(0,0\right) - TC\left(\tau^*,n^*\right)}{TC\left(0,0\right)}$$

5 APPLICATION

In this section, we demonstrate how the proposed reliability growth test plan can be applied to the case of developing a next generation choke valve. From previous product development experience and historical data, the prototype has a two-fold Weibull failure intensity function expressed in Equation (1). The following values are considered for the model parameters.

$\alpha_1 = \alpha_1^0 = 0.24$ $\quad\alpha_2 = \alpha_2^0 = 0.4$ $\quad\alpha_1^m = 0.06$ $\quad\alpha_2^m = 0.19$
$\tilde{\alpha}_1 = 0.11$ $\quad\tilde{\alpha}_2 = 0.22$ $\quad\beta_1 = 2.0$ $\quad\beta_2 = 0.6$
$\rho = 1.8$ $\quad\eta = 1.2$ $\quad r_0 = 1.$

The customer usage rate is Gamma distributed with the density function

$$g\left(r\right) = \frac{1}{k_2^{k_1}\Gamma\left(k_1\right)} r^{k_1-1} e^{-\frac{r}{k_2}}$$

where $k_1 = 0.3$ is the shape parameter and $k_2 = 5.85$ is the scale parameter. $\Gamma\left(k_1\right)$ is the Gamma function evaluated at k_1.

The values for the remaining model parameters are presented as follow.

1. The warranty age and usage limits are $W = 2$ (years) and $U = 4\times10^4$ (km).
2. The accelerated usage rate performed during test is $z = 7.0$, and the failure learning level is $b = 3$.
3. The cost parameters are set as $C_s = 2.0$, $C_t = 1$, $C_f = 5$, $C_a = 15$, $C_r = 120$ and $\psi = 0.8$. The unit of money is US dollar ($) in this example.

From the above-mentioned values of Weibull failure intensity parameters, we can find that the initial MTBF of the valve is 1.01 year. The MTBF achieved by the reliability growth test is expected to be 2.04 years at least. The minimum achievable value of MTBF through reliability growth is 4.05 years.

A MATLAB software program for the minimization of $TC(\tau,n)$ is written. The grid search is done with τ incremented in steps of 0.01 over the range of 0-1 and n incremented in steps of 1 over the range of 0-12. The corresponding optimal test duration and the optimal number of fixes are obtained and presented in Table 1. Under the parameter setting, the optimal test duration is $\tau^* = 0.32$ and the optimal number of fixes is $n^* = 3$, resulting in the minimum cost being 155.47, which is reduced by 60.00% compared to $TC(0,0)$ being 388.75. After reliability growth, the values of scale parameters in the Weibull lifetime distribution are reduced to be $\alpha_1(0.32,3) = 0.07$ and $\alpha_2(0.32,3) = 0.14$, which are both lower than $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ respectively. The MTBF of the valve after reliability growth test is obtained to be 3.08 years, which means the reliability growth requirement has been met.

Table 1: Optimal reliability growth test program

Option	(τ^*,n^*)	$TC(\tau^*,n^*)$	$\alpha_1(\tau^*,n^*)$	$\alpha_2(\tau^*,n^*)$	$MTBF(\tau^*,n^*)$	Cost reduction Δ
Reliability growth	$(0.32,3)$	155.47	0.07	0.14	3.08	60.00%
No reliability growth	−	388.75	−	−	1.01	−

The optimal reliability growth test program mainly depends on the failure learning level, accelerated usage rate conducted during test, product usage intensity as well as cost parameters, etc. We examine the effect of changing values of several key parameters on the optimal solutions. The analysis is done by changing one parameter and keeping other parameters fixed. The results are illustrated in Table 2.

Table 2: Effects of several parameters on the optimal solution

Parameters	(τ^*,n^*)	$TC(\tau^*,n^*)$	$\alpha_1(\tau^*,n^*)$	$\alpha_2(\tau^*,n^*)$	$MTBF(\tau^*,n^*)$	$TC(0,0)$	Cost reduction Δ
Failure learning level, b							
1	(0.42, 5)	224.79	0.087	0.220	2.31	388.75	42.18%
2	(0.36, 4)	176.96	0.078	0.166	2.80	388.75	54.48%
3	(0.32, 3)	155.47	0.074	0.142	3.08	388.75	60.00%
4	(0.29, 3)	142.33	0.072	0.126	3.30	388.75	63.39%
5	(0.23, 2)	133.03	0.068	0.125	3.42	388.75	65.78%
Accelerated usage rate during test, z							
4	(0.84, 3)	155.47	0.076	0.132	3.14	388.75	60.00%
5	(0.57, 3)	155.28	0.076	0.136	3.10	388.75	60.06%
6	(0.42, 3)	155.32	0.075	0.139	3.09	388.75	60.05%
7	(0.32, 3)	155.47	0.074	0.142	3.08	388.75	60.00%
8	(0.25, 3)	155.71	0.075	0.146	3.03	388.75	59.95%
Usage rate distribution parameter, k_1							
0.1	(0.17, 3)	81.90	0.109	0.170	2.29	83.47	1.88%
0.2	(0.24, 3)	121.24	0.088	0.154	2.70	195.52	37.99%
0.3	(0.32, 3)	155.47	0.074	0.142	3.08	388.75	60.00%
0.4	(0.38, 4)	181.51	0.069	0.133	3.31	460.76	60.61%
0.5	(0.42, 4)	203.62	0.066	0.129	3.43	579.13	64.84%
Unit repair cost under test, C_a							
5	(0.43, 3)	128.32	0.067	0.132	3.34	388.75	66.99%
10	(0.36, 3)	143.00	0.070	0.138	3.23	388.75	63.22%
15	(0.32, 3)	155.47	0.074	0.142	3.08	388.75	60.00%
20	(0.30, 4)	166.37	0.076	0.141	3.05	388.75	57.20%
25	(0.27, 4)	176.18	0.080	0.145	2.93	388.75	54.68%
Unit field repair cost under warranty, C_r							
80	(0.28, 3)	120.74	0.080	0.148	2.90	259.17	53.41%
100	(0.30, 3)	138.34	0.077	0.145	2.99	323.96	57.30%
120	(0.32, 3)	155.47	0.074	0.142	3.08	388.75	60.00%
140	(0.35, 4)	172.01	0.071	0.135	3.23	453.55	62.07%
160	(0.37, 4)	188.07	0.069	0.133	3.31	518.34	63.72%

The manufacturer benefits from enhanced learning from failures occurred during test. The higher failure learning level results in shorter test duration and less number of fix activities, which can be observed through the decrease of both τ^* and n^*. For the manufacturer, the best choice is always to conduct reliability growth. The expected total cost under optimal reliability growth test setting goes down, as well as the failure distribution parameters $\alpha_1(\tau^*,n^*)$ and $\alpha_2(\tau^*,n^*)$. As a result, the $MTBF\left(\tau^*,n^*\right)$ increases with b. The similar trend can be observed in the percentage of cost reduction Δ.

It is observed that higher value of accelerated usage rate conducted in the test z will lead to shorter test duration. While compared to wear-out failures, enhanced usage rate exposes less number of infant mortalities. This is reflected by the fluctuation of $\alpha_1\left(\tau^*,n^*\right)$ and the increase of $\alpha_2\left(\tau^*,n^*\right)$. As a result, the MTBF after reliability growth decreases gradually with z.

It is noted that if the scale parameter k_1 in the usage rate distribution function is relatively small such as $k_1 = 0.1$, the benefit from reliability growth is not significant so much since the resulting warranty cost is relatively low itself. While as k_1 increases to 0.2, the reliability improvement reflected by $\alpha_1\left(\tau^*,n^*\right)$, $\alpha_2\left(\tau^*,n^*\right)$ and $MTBF\left(\tau^*,n^*\right)$ is more apparent. At the same time, there is a great saving in the expected total cost (reflected by Δ), and this in turn allows more fixes and longer test duration to reduce the overall product failure intensity.

As the unit repair cost for a test failure C_a increases, the optimal test duration τ^* goes down steadily, while the optimal number of fixes n^* increases slowly. As expected, the reduction in the mode-specific failure intensity function reflected by $\alpha_1\left(\tau^*,n^*\right)$ and $\alpha_2\left(\tau^*,n^*\right)$, as well as the increase in $MTBF\left(\tau^*,n^*\right)$ are crippled by the larger value of C_a. Also, the percentage of cost reduction Δ decreases since the minimum expected total cost $TC\left(\tau^*,n^*\right)$ increases with C_a.

The increase of the unit repair cost for a warranty failure C_r causes the warranty servicing cost rising continuouly. The optimal test duration and the optimal number of fixes are allowed to extend when C_r increases. The reductions in $\alpha_1\left(\tau^*,n^*\right)$ and $\alpha_2\left(\tau^*,n^*\right)$ and the increase in $MTBF\left(\tau^*,n^*\right)$ are observable. Due to higher unit repair cost within warranty coverage, it is more beneficial for the manufactuerer to perform reliability growth test, as the reduction in the warranty cost exceeds the additional test cost incurred apparently.

SUMMARY

The analysis in this study is a first step toward modeling accelerated reliability growth for product sold with a two-dimensional warranty policy. There could be several possible topics for future research. The analysis so far has assumed the used rate conducted during the test is constant with repect to time. Other types of usage rate acceleration can be considered such as step usage, cycling usage and random usage, etc. It is also important to quantify the benefits of learning since even though we assumed failure learning happens naturally, sometimes it may need to be trained at certain cost. In such cases, the benefit of learning should be higher than the cost of implementing it. At last but not least, general repair strategies (additional to minimal repair strategy) with post-launch phase can be introduced further as well.

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