Nonlinear Analysis 195 (2020) 111721

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

A control problem related to the parabolic dominative p-Laplace equation

Fredrik Arbo Høeg *, Eero Ruosteenoja

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

ARTICLE INFO

Article history:
Received 13 May 2019
Accepted 29 November 2019
Communicated by Enrico Valdinoci

MSC:
35K20
91A22

Keywords:
Parabolic equations
Dominative p-Laplacian
Optimal control
Viscosity solutions

ABSTRACT

We show that value functions of a certain time-dependent control problem in $\Omega \times (0,T)$, with a continuous payoff F on the parabolic boundary, converge uniformly to the viscosity solution of the parabolic dominative p-Laplace equation

$$2(n+p)u_t = \Delta u + (p-2)\lambda_n(D^2u),$$

with the boundary data F. Here $2 < p < \infty$, and $\lambda_n(D^2u)$ is the largest eigenvalue of the Hessian D^2u.

1. Introduction

In this paper we give a control problem interpretation for the parabolic dominative p-Laplace equation

$$2(n+p)u_t = \mathcal{D}_p u \qquad \text{in} \quad \Omega_T. \tag{1.1}$$

Here $\Omega_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying a uniform exterior sphere condition, and

$$\mathcal{D}_p u := (\lambda_1 + \cdots + \lambda_{n-1}) + (p-1)\lambda_n = \Delta u + (p-2)\lambda_n,$$

where $2 < p < \infty$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Hessian D^2u. The operator $\mathcal{D}_p$ is called *the dominative p-Laplacian*, introduced by Brustad [3,4] and later studied by Brustad, Lindqvist and Manfredi [5] and Høeg [9] in the elliptic case. The dominative p-Laplacian explains the superposition principle of the p-Laplace equation, see [7,13] for more about this property. The operator $\mathcal{D}_p$ is sublinear,

* Corresponding author.
E-mail addresses: fredrik.hoeg@ntnu.no (F.A. Høeg), eero.k.ruosteenoja@ntnu.no (E. Ruosteenoja).

https://doi.org/10.1016/j.na.2019.111721

so it is convex, and Eq. (1.1) is uniformly parabolic. By Theorem 3.2 in [19], viscosity solutions of (1.1) are in $C^{2+\alpha,\frac{2+\alpha}{2}}(\Omega_T)$ for some $\alpha > 0$.

Let u be a viscosity solution of (1.1) with a given continuous boundary data F on $\partial_p\Omega_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$. By [6], the solution is unique. In Section 3 we see that for $\varepsilon > 0$ and the boundary data F, there is a unique Borel-measurable function u_ε satisfying a *dynamic programming principle* (hereafter DPP)

$$\begin{aligned} u_\varepsilon(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y, t-\varepsilon^2)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{u_\varepsilon(x+\varepsilon\sigma, t-\varepsilon^2) + u_\varepsilon(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] \quad \text{in } \Omega_T. \end{aligned} \tag{1.2}$$

Here $B_\varepsilon(x) \subset \mathbb{R}^n$ is a ball centered at x with the radius ε, in the first term we have an average integral, and in the second term the supremum is taken over all unit vectors in $\mathbb{R}^n$. In Theorem 4.3 we show that $u_\varepsilon \to u$ uniformly when $\varepsilon \to 0$. The idea of the proof is to first show that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded and asymptotically equicontinuous, and use a variant of the Arzelá–Ascoli theorem to see that solutions of the DPP converge uniformly to some continuous function. To show that the uniform limit is the viscosity solution of (1.1), we make use of an asymptotic mean value formula

$$\begin{aligned} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} v(y, t-\varepsilon^2)\, dy \\ & \quad + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] \\ & = v(x,t) + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p v(x,t) - 2(n+p) v_t(x,t)) + o(\varepsilon^2), \end{aligned} \tag{1.3}$$

which is valid for *all* functions $v \in C^{2,1}(\Omega_T)$, see Theorem 2.1.

It turns out that the solution u_ε of DPP (1.2) is the value of the following time-dependent control problem. Let us denote $\alpha = \frac{p-2}{p+n}, \beta = \frac{n+2}{p+n}$, and place a token at $(x_0, t_0) \in \Omega_T$. The controller tosses a biased coin with probabilities α and β. If she gets tails (with probability β), the game state moves according to the uniform probability density to a point $x_1 \in B_\varepsilon(x_0)$. If the coin toss is heads (with probability α), the controller chooses a unitary vector $\sigma \in \mathbb{R}^n$. The position of the token is then moved to $x_1 = x_0 + \varepsilon\sigma$ or $x_1 = x_0 - \varepsilon\sigma$ with equal probabilities. After this step, the position of the token is now at (x_1, t_1), where $t_1 = t_0 - \varepsilon^2$. The game continues from (x_1, t_1) according to the same rules yielding a sequence of game states

$$(x_0, t_0), (x_1, t_1), (x_2, t_2), \ldots$$

The game is stopped when the token is moved outside of Ω_T for the first time and we denote this point by (x_τ, t_τ). The controller is then paid the amount $F(x_\tau, t_\tau)$. Naturally, the controller aims to maximize her payoff, and heuristically, the rules of the game can be read from the DPP (1.2).

We remark that the scaling of the time derivative in Eq. (1.1) is just a matter of convenience. For the equation $u_t = \mathcal{D}_p u$ we would define a game with the same rules as before, except that we would have $t_{j+1} = t_j - \frac{\varepsilon^2}{2(n+p)}$ for every step in the game, see also Remark 2.4.

This control problem has some similarities with two-player zero-sum *tug-of-war* games, which were introduced by Peres, Schramm, Sheffield and Wilson [17,18] and later studied from different perspectives, see e.g. [1,11,15]. Time-dependent tug-of-war games, having connections to parabolic equations with the normalized p-Laplacian, were studied in [8,14,16], whereas two-player games for equations $u_t = \lambda_j(D^2u)$, $j \in \{1, \ldots, n\}$, were recently formulated in [2]. For a deterministic game-theoretic approach to parabolic equations, we refer to [10].

This paper is organized as follows. In Section 2 we prove the asymptotic mean value formula (1.3). In Section 3 we show that the value of the control problem satisfies the DPP (1.2). Finally, in Section 4 we show that value functions converge uniformly to the viscosity solution of (1.1) when $\varepsilon \to 0$.

2. Asymptotic mean value formula

Theorem 2.1. *Let $v : \Omega_T \to \mathbb{R}$ be in $C^{2,1}(\Omega_T)$. Then it satisfies the asymptotic mean value formula (1.3).*

Proof. Averaging the Taylor expansion

$$\begin{aligned} v(y, t-\varepsilon^2) = v(x,t) + \langle Dv(x,t), (y-x)\rangle + \frac{1}{2}\langle D^2 v(x,t)(y-x), (y-x)\rangle \\ - \varepsilon^2 v_t(x,t) + o(|y-x|^2 + \varepsilon^2) \end{aligned}$$

over the ball $B_\varepsilon(x)$ and calculating

$$\fint_{B_\varepsilon(x)} \langle Dv(x,t), (y-x)\rangle \, dy = 0$$

and

$$\fint_{B_\varepsilon(x)} \langle D^2 v(x,t)(y-x), (y-x)\rangle \, dy = \frac{\varepsilon^2}{n+2}\Delta v(x,t),$$

we obtain

$$\begin{aligned} &\fint_{B_\varepsilon(x)} v(y, t-\varepsilon^2)\, \mathrm{d}y \\ &= v(x,t) + \frac{\varepsilon^2}{2(n+2)}\Delta v(x,t) - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \end{aligned} \tag{2.4}$$

Next we take an arbitrary unit vector σ and write the Taylor expansions for $v(x+h, t-\varepsilon^2)$ with $h = \varepsilon\sigma$ and $h = -\varepsilon\sigma$ to obtain

$$\begin{aligned} v(x+\varepsilon\sigma, t-\varepsilon^2) = v(x,t) + \langle Dv(x,t), \varepsilon\sigma\rangle + \frac{1}{2}\langle D^2 v(x,t)\varepsilon\sigma, \varepsilon\sigma\rangle \\ - \varepsilon^2 v_t(x,t) + o(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} v(x-\varepsilon\sigma, t-\varepsilon^2) = v(x,t) - \langle Dv(x,t), \varepsilon\sigma\rangle + \frac{1}{2}\langle D^2 v(x,t)(-\varepsilon\sigma), (-\varepsilon\sigma)\rangle \\ - \varepsilon^2 v_t(x,t) + o(\varepsilon^2), \end{aligned}$$

which yield

$$\begin{aligned} &\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \\ &= v(x,t) + \frac{\varepsilon^2}{2}\langle D^2 v(x,t)\sigma, \sigma\rangle - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \end{aligned}$$

Taking the supremum over all $|\sigma| = 1$ gives

$$\begin{aligned} &\sup_{|\sigma|=1}\left[\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2}\right] \\ &= v(x,t) + \frac{\varepsilon^2}{2}\lambda_n - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \end{aligned} \tag{2.5}$$

By multiplying Eqs. (2.4) and (2.5) by $\frac{n+2}{p+n}$ and $\frac{p-2}{p+n}$ respectively, we get

$$\begin{aligned} &\frac{n+2}{p+n}\fint_{B_\varepsilon(x)} v(y, t-\varepsilon^2)\, dy \\ &\quad + \frac{p-2}{p+n}\sup_{|\sigma|=1}\left[\frac{v(x+\varepsilon\sigma, t-\varepsilon^2) + v(x-\varepsilon\sigma, t-\varepsilon^2)}{2}\right] \\ &= v(x,t) + \frac{\varepsilon^2}{2(n+p)}(\mathcal{D}_p v(x,t) - 2(n+p)v_t(x,t)) + o(\varepsilon^2). \quad \square \end{aligned}$$

Next we define viscosity solutions for Eq. (1.1).

Definition 2.2. An upper semicontinuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity subsolution to the equation $2(n+p)u_t = \mathcal{D}_p u$ in Ω_T if for all $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ such that

(i) $u(x_0, t_0) = \phi(x_0, t_0)$,
(ii) $\phi(x, t) > u(x, t)$ for $(x, t) \in \Omega_T,\ (x, t) \neq (x_0, t_0)$,

it holds $2(n+p)\phi_t(x_0, t_0) \leq \mathcal{D}_p \phi(x_0, t_0)$.

A lower semicontinuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity supersolution to the equation $2(n+p)u_t = \mathcal{D}_p u$ in Ω_T if for all $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ such that

(i) $u(x_0, t_0) = \phi(x_0, t_0)$,
(ii) $\phi(x, t) < u(x, t)$ for $(x, t) \in \Omega_T,\ (x, t) \neq (x_0, t_0)$,

it holds $2(n+p)\phi_t(x_0, t_0) \geq \mathcal{D}_p \phi(x_0, t_0)$.

A continuous function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution to equation $2(n+p)u_t = \mathcal{D}_p u$ in Ω_T if it is both a subsolution and a supersolution.

Because viscosity solutions of (1.1) are in $C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_T)$ for some $\alpha > 0$ (see Section 1), we get the following corollary.

Corollary 2.3. *Let u be a viscosity solution of (1.1). Then it satisfies an asymptotic mean value formula*

$$\begin{aligned} u(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u(y, t-\varepsilon^2)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{u(x+\varepsilon\sigma, t-\varepsilon^2) + u(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right] + o(\varepsilon^2). \end{aligned} \tag{2.6}$$

Remark 2.4. Our scaling of the time variable is for convenience. The same idea would give for viscosity solutions of

$$u_t = \mathcal{D}_p u$$

an asymptotic mean value formula

$$\begin{aligned} u(x,t) = {} & \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u\Big(y, t - \frac{\varepsilon^2}{2(n+p)}\Big)\, dy \\ & + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{u(x+\varepsilon\sigma, t - \frac{\varepsilon^2}{2(n+p)}) + u(x-\varepsilon\sigma, t - \frac{\varepsilon^2}{2(n+p)})}{2} \right] + o(\varepsilon^2). \end{aligned}$$

3. Control problem formulation

In this section we show that the value of the control problem described in Section 1 satisfies the DPP (1.2). Since the game token may be placed outside of $\overline{\Omega}_T$, we denote the compact parabolic boundary strip of width $\varepsilon > 0$ by

$$\Gamma_\varepsilon = \left(S_\varepsilon \times \left[-\varepsilon^2, 0\right]\right) \cup \left(\Omega \times \left[-\varepsilon^2, 0\right]\right),$$

where

$$S_\varepsilon = \left\{ x \in \mathbb{R}^n \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \leq \varepsilon \right\}.$$

Throughout this section, we are given a continuous function

$$F : \Gamma_\varepsilon \to \mathbb{R}.$$

Our control problem with the payoff F was formulated in Section 1. The process is stopped when the token hits the boundary strip Γ_ε for the first time at, say $(x_\tau, t_\tau) \in \Gamma_\varepsilon$, and then the controller earns the amount $F(x_\tau, t_\tau)$.

Next we define the stochastic vocabulary for the control problem. A *strategy* is a rule which gives, at each step of the game, a direction σ,

$$S(t_0, x_0, x_1, \dots, x_k) = \sigma \in \mathbb{R}^n, \quad |\sigma| = 1.$$

Here, S is a Borel measurable function. Let $A \subset \Omega_T \cup \Gamma_\varepsilon$ be a measurable set. Given a sequence of token positions $(x_0, t_0), (x_1, t_1), \dots, (x_k, t_k)$ and a strategy S, the next position of the token is distributed according to the transition probability

$$\begin{aligned}\pi_S\left((x_0, t_0), (x_1, t_1), \dots, (x_k, t_k), A\right) &= \beta \frac{\left|A \cap \left(B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\}\right)\right|}{|B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\}|} \\ &\quad + \frac{\alpha}{2}\delta_{(x_k+\varepsilon\sigma, t_k-\varepsilon^2)}(A) + \frac{\alpha}{2}\delta_{(x_k-\varepsilon\sigma, t_k-\varepsilon^2)}(A)\end{aligned}$$

where in the first term we use the n-dimensional Lebesgue measure, and in the last terms $\delta_{(y,s)}(B) = 1$ if $(y, s) \in B$ and 0 otherwise.

For a starting point (x_0, t_0), a strategy S and the corresponding transition probabilities, we can use Kolmogorov's extension theorem to determine a unique probability measure $\mathbb{P}_S^{(x_0,t_0)}$ in the space of all game sequences denoted H^∞. The expected payoff is then

$$\mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] = \int_{H^\infty} F(x_\tau, t_\tau)\, d\mathbb{P}_S^{(x_0,t_0)},$$

and the value of the game for the controller is

$$u^\varepsilon(x_0, t_0) = \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)].$$

Since F is bounded and

$$\tau \le \frac{T}{\varepsilon^2} + 1,$$

the value of the game is well defined. From the definition we immediately get the following comparison principle.

Proposition 3.1. *Fix $\varepsilon > 0$. Let u^ε be the value of the game with the payoff F_1, and v^ε the value of the game with the payoff F_2. Assume that $F_1 \ge F_2$ on Γ_ε. Then $u^\varepsilon \ge v^\varepsilon$ in Ω_T.*

Our aim is to show that the value function u^ε satisfies the DPP with the boundary data F.

Definition 3.2. A Borel measurable function u_ε satisfies the dynamic programming principle, abbreviated DPP, in Ω_T, with the boundary data F, if

$$\begin{aligned} u_\varepsilon(x,t) &= \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y, t-\varepsilon^2)\, dy \\ &\quad + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{u_\varepsilon(x+\varepsilon\sigma, t-\varepsilon^2) + u_\varepsilon(x-\varepsilon\sigma, t-\varepsilon^2)}{2}\right] \quad \text{in } \Omega_T \\ u_\varepsilon(x,t) &= F(x,t) \quad \text{on } \Gamma_\varepsilon. \end{aligned}$$

Lemma 3.3. *There is a unique Borel measurable function u_ε satisfying the DPP. Moreover, u_ε is lower semi-continuous.*

Proof. The existence and uniqueness of such a function u_ε can be seen from the following argument. Given F on Γ_ε, we can determine $u_\varepsilon(x,t)$ for all $x \in \Omega$ and $0 < t < \varepsilon^2$. We want to continue this process, but we need to make sure that the function is lower semi-continuous or at least Borel measurable. The following argument is from personal communication with Brustad, Lindqvist, and Manfredi. In general, when u is any bounded and lower semi-continuous function, then by using Fatou's lemma,

$$\frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u(y, t-\varepsilon^2)\, dy + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[\frac{u(x+\varepsilon\sigma, t-\varepsilon^2) + u(x-\varepsilon\sigma, t-\varepsilon^2)}{2} \right]$$

is again bounded and lower semi-continuous. This gives a lower semi-continuous function u_ε defined for all $x \in \Omega$ and $0 < t < \varepsilon^2$. Continuing this process until $t = T$ gives the desired function. □

Lemma 3.4. *Let u_ε be the unique function satisfying the DPP of Definition 3.2 with the boundary data F on Γ_ε, and let u^ε be the value of the game with the payoff F. Then*

$$u_\varepsilon = u^\varepsilon.$$

Proof. Let $(x_0, t_0) \in \Omega_T$. We aim to show that $u_\varepsilon(x_0,t_0) = u^\varepsilon(x_0,t_0)$. Assume that the game starts at $(x_0,t_0) \in \Omega_T$.

First we assume that the controller uses an arbitrary strategy S. Then we have for the function u_ε satisfying the DPP,

$$\begin{aligned}
&\mathbb{E}_S^{(x_0,t_0)}[u_\varepsilon(x_{k+1},t_{k+1}) | (t_0, x_0, x_1, \ldots, x_k)] = \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y, t_k - \varepsilon^2)\, dy \\
&\quad + \alpha \frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \\
&\le \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y, t_k - \varepsilon^2)\, dy \\
&\quad + \alpha \sup_{|\sigma|=1} \left[\frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \right] \\
&= u_\varepsilon(x_k, t_k).
\end{aligned}$$

This shows that $M_k := u_\varepsilon(x_k, t_k)$ is a supermartingale, so

$$\mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau) | (t_0, x_0, x_1, \ldots, x_{\tau-1})] \le u_\varepsilon(x_0, t_0)$$

by the optimal stopping theorem. Hence

$$u^\varepsilon(x_0,t_0) = \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] \le u_\varepsilon(x_0,t_0).$$

To prove the reverse inequality, we choose a strategy S_0 giving a corresponding $\sigma(x,t)$ for the controller that *almost* maximizes $u_\varepsilon(x,t)$. To be more precise, for arbitrary $\eta > 0$, the controller chooses

$$\begin{aligned}
&\frac{u_\varepsilon(x_k + \varepsilon\sigma(x_k,t_k), t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma(x_k,t_k), t_k - \varepsilon^2)}{2} \\
&\ge \sup_{|\sigma|=1} \left[\frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \right] - \eta 2^{-(k+1)}.
\end{aligned}$$

The function S_0 can be taken to be a Borel function, see Lemma 3.4 in [12].

We obtain

$$
\begin{aligned}
&\mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_{k+1},t_{k+1})-\eta 2^{-(k+1)}|(t_0,x_0,x_1,\dots,x_k)]\\
&\geq \beta \fint_{B_\varepsilon(x_k)} u_\varepsilon(y,t_k-\varepsilon^2)\,dy\\
&\quad+\alpha \sup_{|\sigma|=1}\left[\frac{u_\varepsilon(x_k+\varepsilon\sigma,t_k-\varepsilon^2)+u_\varepsilon(x_k-\varepsilon\sigma,t_k-\varepsilon^2)}{2}\right]\\
&\quad-\alpha\eta 2^{-(k+1)}-\eta 2^{-(k+1)}\\
&\geq u_\varepsilon(x_k,t_k)-\eta 2^{-k}.
\end{aligned}
$$

Hence

$$
M_k = u_\varepsilon(x_k,t_k)-\eta 2^{-k}
$$

is a submartingale. Using the optimal stopping theorem for this submartingale we find

$$
\begin{aligned}
u^\varepsilon(x_0,t_0) &= \sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau,t_\tau)] \geq \mathbb{E}_{S_0}^{(x_0,t_0)}[F(x_\tau,t_\tau)]\\
&\geq \mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_\tau,t_\tau)-\eta 2^{-k}]\\
&\geq \mathbb{E}_{S_0}^{(x_0,t_0)}[u_\varepsilon(x_0,t_0)-\eta 2^{-0}] = u_\varepsilon(x_0,t_0)-\eta.
\end{aligned}
$$

Since $\eta > 0$ was arbitrary, this proves the lemma. □

4. Convergence to the viscosity solution

In this section, we are given a continuous payoff function $F : \Gamma_1 \to \mathbb{R}$. Our goal is to show that with this payoff, value functions of our game converge uniformly to the unique viscosity solution of

$$
\begin{cases}
2(n+p)u_t = \mathcal{D}_p u & \text{in} \quad \Omega_T,\\
u = F & \text{on} \quad \partial_p\Omega_T.
\end{cases}
\tag{4.7}
$$

We will make use of the following Arzelá–Ascoli-type lemma, which has been previously used e.g. in [2,14,16]. We omit the proof, which is a modification of [15, Lemma 4.2].

Lemma 4.1. *Let $\left\{f_\varepsilon : \overline{\Omega}_T \to \mathbb{R}\right\}_{\varepsilon\in(0,1)}$ be a uniformly bounded family of functions such that for a given $\eta > 0$, there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $(x,t),(y,s) \in \overline{\Omega}_T$ with*

$$
|(x,t)-(y,s)| < r_0,
$$

it holds

$$
|f_\varepsilon(x,t)-f_\varepsilon(y,s)| < \eta.
$$

Then there exists a uniformly continuous function $f : \overline{\Omega}_T \to \mathbb{R}$ and a subsequence, still denoted by (f_ε), such that $f_\varepsilon \to f$ uniformly in $\overline{\Omega}_T$ as $\varepsilon \to 0$.

For the next lemma, we assume that the domain Ω satisfies a *uniform exterior sphere condition.* That is, we assume that there is $\delta > 0$ such that for any $y \in \partial\Omega$, there is an open ball $B_\delta \subset \mathbb{R}^n \setminus \Omega$ with the radius δ so that $\overline{B}_\delta \cap \overline{\Omega} = \{y\}$.

Lemma 4.2. *The family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ of value functions of the game satisfies the assumptions of Lemma 4.1.*

Proof. Since $|u_\varepsilon(x,t)| \le \max_{\Gamma_1}|F|$ for all $(x,t) \in \overline{\Omega}_T$ and $\varepsilon \in (0,1)$, the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded.

Fix $\eta > 0$. Since the payoff function F is uniformly continuous on Γ_1, there is $\gamma > 0$ so that when $(x,t),(y,s) \in \Gamma_1$ with $|(x,t)-(y,s)| < \gamma$, it holds $|F(x,t)-F(y,s)| < \frac{\eta}{2}$. We prove the asymptotic equicontinuity of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ in four steps. In all steps we have $\varepsilon < \varepsilon_0$ and $|(x,t)-(y,s)| < r_0$. The precise choices of ε_0 and r_0 clarify during the proof. We will denote by $C_1, C_2, \dots$ constants larger than 1 which may depend only on n, δ, and the diameter of Ω.

Step 1. If $(x,t),(y,s) \in \partial_p\Omega_T$, then

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| = |F(x,t) - F(y,s)| < \eta$$

when $r_0 < \gamma$.

Step 2. Suppose that $(x,t) \in \Omega_T$ and $(y,0) \in \Gamma_\varepsilon$. Let us start the game from $(x_0,t_0) = (x,t)$ with an arbitrary strategy S. We obtain

$$\begin{aligned}
&\mathbb{E}_S^{(x_0,t_0)}[|x_k - x_0|^2 \,|\, (t_0, x_0, \dots, x_{k-1})] \\
&= \frac{\alpha}{2}(|(x_{k-1}+\sigma\varepsilon) - x_0|^2 + |(x_{k-1}-\sigma\varepsilon) - x_0|^2) + \beta \fint_{B_\varepsilon(x_{k-1})} |y - x_0|^2\, dy \\
&\le \alpha(|x_{k-1}-x_0|^2 + \varepsilon^2) + \beta(|x_{k-1}-x_0|^2 + C_1\varepsilon^2) \\
&\le |x_{k-1}-x_0|^2 + C_1\varepsilon^2.
\end{aligned}$$

Hence,

$$M_k := |x_k - x_0|^2 - C_1 k\varepsilon^2$$

is a supermartingale, and the optimal stopping theorem gives

$$\mathbb{E}_S^{(x_0,t_0)}[|x_\tau - x_0|^2] \le |x_0 - x_0|^2 + C_1\varepsilon^2\mathbb{E}_S^{(x_0,t_0)}[\tau] \le C_1(r_0 + \varepsilon_0^2).$$

Here, we used the fact that the stopping time $\tau \le \frac{t_0}{\varepsilon^2} + 1$ for a game starting at t_0 and in this case $t_0 \le r_0$. Since this is true for all strategies, it holds

$$\sup_S \mathbb{E}_S^{(x_0,t_0)}[|x_\tau - x_0|^2] \le C_1(r_0 + \varepsilon_0^2),$$

which yields

$$|u_\varepsilon(x_0,t_0) - u_\varepsilon(x_0,0)| = |\sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau,t_\tau)] - F(x_0,0)| < \frac{\eta}{2},$$

when r_0, ε_0 are chosen so that $C_1(r_0 + \varepsilon_0^2) < \gamma^2$.

The triangle inequality finishes the argument. Recalling that $(x_0,t_0) = (x,t)$, we have

$$|u_\varepsilon(x,t) - u_\varepsilon(y,0)| \le |u_\varepsilon(x,t) - F(x,0)| + |F(x,0) - F(y,0)| < \eta.$$

Step 3. Suppose that $(x,t) \in \Omega_T$ and $(y,s) \in \partial_p\Omega_T$ with $y \in \partial\Omega$. Since the domain Ω satisfies the uniform exterior sphere condition with δ, there is a ball $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ with $\partial B_\delta(z) \cap \overline{\Omega} = \{y\}$.

We use a barrier argument. In an annulus of $\mathbb{R}^n$, define a function w as

$$\begin{cases} w(x) = -a|x-z|^2 - b|x-z|^{-\xi} + c & \text{in } B_R(z) \setminus \overline{B}_\delta(z), \\ w = 0 & \text{on } \partial B_\delta(z), \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial B_R(z), \end{cases}$$

where $\frac{\partial w}{\partial \nu}$ is the normal derivative, and R is chosen so that $\Omega \subset B_R(z)$. The exponent $\xi = n + p - 4 > 0$, since $p > 2$ and we may assume that $n \geq 2$ (1-dimensional case is essentially a random walk in an open interval). The positive constants a, b, c are specified below. The function w satisfies

$$\Delta w(x) = -2an + b\xi n|x - z|^{-\xi-2} - b\xi(\xi + 2)|x - z|^{-\xi-2},$$

$$\lambda_n(D^2 w(x)) = -2a + b\xi|x - z|^{-\xi-2},$$

hence

$$\mathcal{D}_p w = -2a(n + p - 2) \quad \text{in } B_R(z) \setminus \overline{B}_\delta(z), \tag{4.8}$$

and it can be extended as a solution to the same equations in $B_{R+\varepsilon}(z) \setminus \overline{B}_{\delta-\varepsilon}(z)$ so that Eq. (4.8) holds also near the boundaries. It satisfies an estimate

$$w(x) \leq C_2(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x) + o(1)$$

for any $x \in B_R(z) \setminus B_\delta(z)$. Here $o(1) \to 0$ when $\varepsilon \to 0$.

Let us consider for a moment an elliptic game starting at $x_0 = x$ and played by the rules of our game without a time-dependence in the annulus $B_R(z) \setminus \overline{B}_\delta(z)$, with a special rule that if we are at, say x_k, a possible random move is chosen from $B_\varepsilon(x_k) \cap B_R(z)$ according to the uniform probability density, and also the controller cannot exit $B_R(z)$. The game ends when the token enters the ball $\overline{B}_\delta(z)$. Because of the random moves, the game ends almost surely in a finite time. Define a stopping time for this game as τ^*,

$$\tau^* = \inf\{k \ : \ x_k \in \overline{B}_\delta(z)\}.$$

Let S be an arbitrary strategy for the controller. The Taylor expansion for w gives

$$\begin{aligned}
&\frac{1}{2}(w(x_{k-1} + \varepsilon\sigma) + w(x_{k-1} - \varepsilon\sigma)) \\
&\quad = w(x_{k-1}) + \frac{1}{2}\varepsilon^2 \langle D^2 w(x_{k-1})\sigma, \sigma\rangle + o(\varepsilon^2) \\
&\quad \leq w(x_{k-1}) + \frac{1}{2}\varepsilon^2 \lambda_n(D^2 w(x_{k-1})) + o(\varepsilon^2),
\end{aligned}$$

since the first order terms vanish,

$$\langle Dw(x_{k-1}), \varepsilon\sigma\rangle + \langle Dw(x_{k-1}), -\varepsilon\sigma\rangle = 0.$$

Moreover, since w is radially increasing, it holds

$$⨍_{B_\varepsilon(x_{k-1}) \cap B_R(z)} w(y)\,\mathrm{d}y \leq w(x_{k-1}) + \frac{\varepsilon^2}{2(n+2)} \Delta w(x_{k-1}) + o(\varepsilon^2).$$

By choosing the constant a properly,

$$M_k := w(x_k) + k\varepsilon^2$$

is a supermartingale. Indeed, we have

$$\begin{aligned}
\mathbb{E}_S^{x_0}[M_k \,|\, x_0, \ldots, x_{k-1}] &= \frac{\alpha}{2}(w(x_{k-1} + \varepsilon\sigma) + w(x_{k-1} - \varepsilon\sigma)) \\
&\quad + \beta ⨍_{B_\varepsilon(x_{k-1}) \cap B_R(z)} w(y)\,dy + k\varepsilon^2 \\
&\leq w(x_{k-1}) + \frac{\varepsilon^2}{2(p+n)} \mathcal{D}_p w(x_{k-1}) + k\varepsilon^2 + o(\varepsilon^2)
\end{aligned}$$

$$
\begin{aligned}
&= w(x_{k-1}) - \frac{n+p-2}{n+p} a\varepsilon^2 + k\varepsilon^2 + o(\varepsilon^2) \\
&\le w(x_{k-1}) + (k-1)\varepsilon^2,
\end{aligned}
$$

by choosing for example $a = 2\frac{n+p}{n+p-2}$ and assuming that $o(\varepsilon^2) < \varepsilon^2$. The choice of a determines the other constants b and c: The Neumann and Dirichlet boundary conditions of the barrier function w are satisfied by choosing $b = (2a/\xi)R^{\xi+2}$ and $c = a\delta^2 + b\delta^{-\xi}$.

By the optimal stopping theorem, we have

$$\mathbb{E}_S^{x_0}[w(x_{\tau*}) + \tau^*\varepsilon^2] \le w(x_0),$$

that is,

$$\mathbb{E}_S^{x_0}[\tau^*] \le \frac{w(x_0)}{\varepsilon^2} \le \frac{C_2(R/\delta)\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

where we used $|\mathbb{E}_S^{x_0}[w(x_{\tau*})]| \le o(1)$.

Now we come back to our game, starting at $(x_0, t_0) = (x, t)$, again with an arbitrary strategy S. Since it holds $|x_0 - y| \ge \operatorname{dist}(\partial B_\delta(z), x_0)$, for the stopping time of our game we now have an estimate

$$
\begin{aligned}
\mathbb{E}_S^{(x_0,t_0)}[\tau] &\le \mathbb{E}_S^{(x_0,t_0)}[\tau^*] \\
&\le \frac{C_2(R/\delta)\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2} \\
&\le \frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2}.
\end{aligned}
$$

By using the same martingale argument as in Step 2 but replacing x_0 by y, we have

$$
\begin{aligned}
\mathbb{E}_S^{(x_0,t_0)}[|x_\tau - y|^2] &\le |x_0 - y|^2 + C_1\varepsilon^2 \mathbb{E}_S^{(x_0,t_0)}[\tau] \\
&\le |x_0 - y|^2 + C_1\varepsilon^2 \frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2} \\
&\le |x_0 - y|^2 + C_3(|x_0 - y| + o(1)) \\
&< r_0^2 + C_3(r_0 + o(1)) < \left(\frac{\gamma}{2}\right)^2,
\end{aligned}
$$

when ε_0, r_0 are chosen so that $C_3(r_0 + o(1)) < \left(\frac{\gamma}{4}\right)^2$ and $r_0^2 < \left(\frac{\gamma}{4}\right)^2$. This also gives

$$|\mathbb{E}_S^{(x_0,t_0)}[t_\tau] - t_0| < \left(\frac{\gamma}{4}\right)^2.$$

Hence, we have

$$|u_\varepsilon(x_0, t_0) - u_\varepsilon(y, t_0)| = |\sup_S \mathbb{E}_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] - F(y, t_0)| < \frac{\eta}{2},$$

and recalling that $(x_0, t_0) = (x, t)$ the triangle inequality gives

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \le |u_\varepsilon(x,t) - F(y,t)| + |F(y,t) - F(y,s)| < \eta.$$

Step 4. Finally, suppose that $(x,t), (y,s) \in \Omega_T$. This is an argument based on translation invariance and comparison principle. Let r_0, ε_0 satisfy the conditions of the previous steps. Define an inner ε-strip I_ε by

$$I_\varepsilon := \{(z,r) \in \overline{\Omega}_T \ : \ \operatorname{dist}((z,r), \partial_p\Omega_T) \le r_0\}.$$

If $(x,t) \in I_\varepsilon$, there is a point $(x', t') \in \partial_p\Omega_T$ such that $|(x,t) - (x',t')| \le r_0$. Then from the conclusions of the previous steps we obtain

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \le |u_\varepsilon(x,t) - F(x',t')| + |F(x',t') - u_\varepsilon(y,s)| < \eta.$$

The argument is identical if $(y,s) \in I_\varepsilon$, so it remains to study the case $(x,t),(y,s) \in \Omega_T \setminus I_\varepsilon$. We may assume that $t \leq s$. Define functions F_1, F_2 on the strip I_ε as follows,

$$F_1(z,r) = u_\varepsilon(z-x+y, r-t+s) - \eta, \quad F_2(z,r) = u_\varepsilon(z-x+y, r-t+s) + \eta.$$

Then

$$F_1(z,r) \leq u_\varepsilon(z,r) \leq F_2(z,r)$$

for all $(z,r) \in I_\varepsilon$. Let u_ε^1 be the value function of the game in $\Omega_T \setminus I_\varepsilon$ with the payoff F_1 on I_ε, and u_ε^2 the value function of the game in $\Omega_T \setminus I_\varepsilon$ with the payoff F_2 on I_ε. By the uniqueness of the value function, we have for all $(z,r) \in \Omega_T \setminus I_\varepsilon$

$$\begin{aligned} u_\varepsilon^1(z,r) &= u_\varepsilon(z-x+y, r-t+s) - \eta, \\ u_\varepsilon^2(z,r) &= u_\varepsilon(z-x+y, r-t+s) + \eta. \end{aligned}$$

By the comparison principle, see Proposition 3.1, we have

$$\begin{aligned} u_\varepsilon(x,t) &\geq u_\varepsilon^1(x,t) = u_\varepsilon(y,s) - \eta, \\ u_\varepsilon(x,t) &\leq u_\varepsilon^2(x,t) = u_\varepsilon(y,s) + \eta. \quad \square \end{aligned}$$

From the previous lemmas it follows that if (u_{ε_j}) is a sequence of value functions with $\varepsilon_j \to 0$ and $(u_{\varepsilon_{j_k}})$ is an arbitrary subsequence, then this subsequence has a subsequence converging uniformly to v. Hence, the sequence (u_{ε_j}) converges to v uniformly, and we write $u_\varepsilon \to v$ to simplify the notation. It remains to show that the function v is the solution of (4.7).

Theorem 4.3. *The uniform limit $v = \lim_{\varepsilon \to 0} u_\varepsilon$ is the unique viscosity solution of (4.7).*

Proof. By uniqueness of viscosity solutions (see [6]), it is sufficient to show that v is a viscosity solution of (4.7). To this end, let $\phi \in C^2$ touch v from above at $(x_0,t_0) \in \Omega_T$,

$$0 = (v-\phi)(x_0,t_0) > (v-\phi)(x,t)$$

for all (x,t) close to (x_0,t_0). From the definition of supremum, given $\delta_\varepsilon > 0$, there are points $(x_\varepsilon, t_\varepsilon)$ close to (x_0,t_0) such that

$$u_\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon) \geq u_\varepsilon(y,s) - \phi(y,s) - \delta_\varepsilon$$

for all (y,s) in a neighborhood of $(x_\varepsilon,t_\varepsilon)$. Using the fact that $u_\varepsilon \to v$ uniformly and $v-\phi$ is a continuous function with a maximum point at (x_0,t_0), we see that $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ as $\varepsilon \to 0$.

Since $\phi \in C^2(\Omega_T)$, Theorem 2.1 gives

$$\begin{aligned} &\beta \fint_{B_\varepsilon(x_\varepsilon)} \phi(y, t_\varepsilon - \varepsilon^2)\, dy \\ &\quad + \alpha \sup_{|\sigma|=1} \left[\frac{\phi(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\ &= \phi(x_\varepsilon,t_\varepsilon) + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon,t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon,t_\varepsilon)) + o(\varepsilon^2). \end{aligned}$$

We can now estimate

$$
\begin{aligned}
&\beta \fint_{B_\varepsilon(x_\varepsilon)} u_\varepsilon(y, t_\varepsilon - \varepsilon^2)\, dy \\
&\quad + \alpha \sup_{|\sigma|=1} \left[\frac{u_\varepsilon(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + u_\varepsilon(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
&\leq u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \beta \fint_{B_\varepsilon(x)} \phi(y, t_\varepsilon - \varepsilon^2)\, dy \\
&\quad + \alpha \sup_{|\sigma|=1} \left[\frac{\phi(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
&= u_\varepsilon(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\end{aligned}
$$

As the function u_ε satisfies the DPP, we are left with

$$
0 < \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (\mathcal{D}_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
$$

Choose now $\delta_\varepsilon = o(\varepsilon^2)$. Dividing by ε^2 and letting $\varepsilon \to 0$ gives

$$
2(n+p)\phi_t(x_0, t_0) \leq \mathcal{D}_p \phi(x_0, t_0),
$$

which shows that v is a viscosity subsolution. To show that v is a viscosity supersolution is analogous. □

Acknowledgments

E.R. is supported by the Magnus Ehrnrooth Foundation, Finland. The authors would like to thank Peter Lindqvist and Tommi Brander for useful discussions.

References

[1] S. Armstrong, C. Smart, A finite difference approach to the infinity Laplace equation and tug-of-war games, Trans. Amer. Math. Soc. 364 (2) (2012) 595–636.
[2] P. Blanc, C. Esteve, J.D. Rossi, The evolution problem associated with eigenvalues of the Hessian, 2019, ArXiv preprint arXiv:1901.01052.
[3] K. Brustad, Superposition of p-superharmonic functions, 2017, ArXiv preprint arXiv:1705.08203.
[4] K. Brustad, Sublinear elliptic operators, 2018, ArXiv preprint arXiv:1802.04297.
[5] K. Brustad, P. Lindqvist, J.J. Manfredi, A discrete stochastic interpretation of the Dominative p-Laplacian, 2018, ArXiv preprint arXiv:1809.00714.
[6] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[7] M.G. Crandall, J. Zhang, Another way to say harmonic, Trans. Amer. Math. Soc. 355 (1) (2003) 241–263.
[8] J. Han, Local Lipschitz regularity for functions satisfying a time-dependent dynamic programming principle, 2018, ArXiv preprint arXiv:1812.00646.
[9] F.A. Høeg, Concave power solutions of the Dominative p-Laplace equation, 2019, ArXiv preprint arXiv:1901.07053.
[10] R.V. Kohn, S. Serfaty, A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations, Comm. Pure Appl. Math. 63 (10) (2010) 1298–1350.
[11] M. Lewicka, Random tug of war games for the p-Laplacian: $1 < p < \infty$, 2018, ArXiv preprint arXiv:1810.03413.
[12] M. Lewicka, J.J. Manfredi, The obstacle problem for the p-laplacian via optimal stopping of tug-of-war games, Probab. Theory Related Fields 162 (1–2) (2017) 349–378.
[13] P. Lindqvist, J.J. Manfredi, Note on a remarkable superposition for a nonlinear equation, Proc. Amer. Math. Soc. 136 (1) (2008) 133–140, (electronic).
[14] J.J. Manfredi, M. Parviainen, J.D. Rossi, An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games, SIAM J. Math. Anal. 42 (5) (2010) 2058–2081.
[15] J.J. Manfredi, M. Parviainen, J.D. Rossi, On the definition and properties of p-harmonious functions, Ann. Sc. Norm. Super Pisa Cl. Sci. 11 (2) (2012) 215–241.

[16] M. Parviainen, E. Ruosteenoja, Local regularity for time-dependent tug-of-war games with varying probabilities, J. Differential Equations 261 (2) (2016) 1357–1398.
[17] Y. Peres, O. Schramm, S. Sheffield, D.B. Wilson, Tug-of-war and the infinity Laplacian, J. Amer. Math. Soc. 22 (1) (2009) 167–210.
[18] Y. Peres, S. Sheffield, Tug-of-war with noise: a game-theoretic view of the p-Laplacian, Duke Math. J. 145 (1) (2008) 91–120.
[19] L. Wang, On the regularity theory of fully nonlinear parabolic equations. II, Comm. Pure Appl. Math. 45 (2) (1992) 141–178.