Sense and reference of signifiers for elements of polygons

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This paper is based mainly on a planning meeting involving teachers and researchers before classroom sessions in a Norwegian school where 7-8-year-old children are working with concepts connected to geometrical shapes. During the planning meeting different conjectures were made about possible confusion that might occur concerning the names of the parts (elements) of a polygon (edges and vertices). Comparing these conjectures to what actually happened in the classroom, it turns out that the group of experienced teachers and teacher educators were not able to foresee all the nuances in the children's sense and reference concerning concepts connected to polygons. The findings suggest that these concepts may be more challenging than previously assumed. The classroom sessions are designed using principles from Brousseau's Theory of didactical situations (TDS).

Keywords: Register, sense and reference, polygons, milieu.

Introduction

The project Language Use and Development in the Mathematics Classroom (LaUDiM) is an intervention study carried out in collaboration between researchers at the Norwegian University of Science and Technology and two local primary schools in the period 2014-2018. A central part of the project is to design and implement teaching sequences. The design process takes place in close collaboration between researchers and teachers. The teachers are responsible for the implementation in their respective classes, with researchers present in the classroom. The core of the project is twofold: to study pupils' development and use of mathematical language to express their ideas, and their use of language in arguing and justification; as well as to study and develop teachers' mathematics teaching practices. The design of each teaching sequence is guided by principles from *The theory of didactical situations, TDS* (Brousseau, 1997). A teaching sequence typically consists of a planning meeting between teachers and researchers followed by two or three classroom sessions. Between the classroom sessions there are reflection sessions to discuss what happened in the classroom and, if necessary, revise the plans.

This paper is mainly based on data from the planning meeting preceding one particular sequence of teaching sessions where the main aim is pupils' development of a precise mathematical language for describing elements and properties of polygons. The actual classroom sessions were reported on in (Rønning & Strømskag, 2017). In the present paper, some of the findings from the first classroom session are mentioned in order to be able to reflect on what happened in the planning meeting. Also the reflection session after the first classroom session will be mentioned. The paper will shed light on the importance of the planning session and how different discourses influence the planning, as well as deepen the knowledge about the complexity regarding the, apparently simple, concepts involved. We will in particular compare teachers and researchers' conjectures about pupils' language use about polygons to what actually happened in the classroom.

Theoretical framework

In this paper we see a word as a sign, or signifier, and, in the language of Frege (1892) we connect the sign with its *Sinn* and its *Bedeutung.* The German words *Sinn* and *Bedutung* can be seen translated into English in different ways. We shall follow the translation in (Geach & Black, 1960), using the words *sense* and *reference*, respectively. Here, the reference is the object that the sign refers to and the sense comprises all thoughts and ideas connected to the sign. In his work, Frege gives examples to show that two different signs may have the same reference but different sense. One example used by Frege is the following, from Euclidian geometry. Let *a, b* and *c* be the medians of a Euclidian triangle. Let *A* be the intersection point between *a* and *b,* and *B* be the intersection point between *b* and *c.* Then *A* and *B* have different sense but from a theorem in Euclidian geometry the points *A* and *B* coincide, so *A* and *B* have the same reference (see e.g. Geach & Black, 1960, p. 57).

A distinction similar to Frege's sense vs. reference is made by Peirce in his definition of a sign: "A *sign* is a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent.* This thing is called the *object* of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called the *interpretant* of the sign" (Peirce, 1998, p. 13). Also Ogden and Richards (1923/1948) describe a similar model when they talk about a *symbol* as being connected to a *referent* via a *thought* or *reference.* They state that "[b]etween the *symbol* and the *referent* there is no relevant relation other than the indirect one, which consists in its being used by someone to stand for a referent" (p. 11). They further state that "[a] true symbol = one which correctly records an adequate reference. … [W]hen it will cause a similar reference to occur in a suitable interpreter" (p. 102). A given word is therefore a true symbol if it causes the same reference in different interpreters. Successful communication in a mathematics classroom depends on symbols causing the same reference in the actors involved, usually teachers and pupils.

Halliday (1979) defines the term register as "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (p. 195). So, changing between registers can mean that the same word gets both a different sense and a different reference. The mathematical register is characterised by the property that words have very precise sense and reference. Sometimes the same word may be used both in the mathematical register and in the register of everyday language but with different sense and reference. This feature is language specific so that for a given word it can be part of both the mathematical and the everyday register in one language but translating into another language may lead to different words in different registers. When the same word is used in both the mathematical and the everyday register in a language, it is of particular importance to be aware of possible discrepancies in sense and reference because such a word may not be seen as a challenge in the learning process.

Design and research question

Each teaching sequence in the project starts with a planning meeting where teachers and researchers work together to plan the activities for two or, in this case three, classroom sessions. In this planning meeting TDS plays an important role. TDS provides a model for instructional design, where a didactical situation is designed with a problem that has the target piece of knowledge as an optimal solution (Brousseau, 1997). The didactical situation consists of four phases—action, formulation, validation, and institutionalisation—designed with milieus so as to make the knowledge (necessary to solve the given problem) progress from informal knowledge to increasingly formal (mathematical) knowledge. More precisely, in action, the pupils use an *implicit strategy* to solve the problem given to them; in formulation, they need an *explicit strategy* because they are supposed to use the knowledge to make another pupil solve the problem; in validation, the knowledge needs to be *mathematical knowledge* in order to jusitfy that the solution is valid in general; and, in institutionalisation, the teacher decontextualises the knowledge reached to become *scholarly knowledge*. For a more detailed account, and exemplification, of the evolution of knowledge and the teacher's role in the different phases, see (Strømskag, 2017).

A teaching sequence was designed according to the teachers' and researchers' expectations regarding pupils' sense and reference of words connected to polygons (see Rønning & Strømskag, 2017). However, the data collected during implementation of the sequence gave us insight into the pupils' *actual* sense and reference of the words in question. This design was instrumental for creating opportunities for revealing the challenges of the learning process. For this paper however, the concepts from Brousseau's theory will not play a significant role.

Between the first and the second classroom session follows a reflection session, where adjustments are made for the second session. Researchers present in the classroom interact with the pupils but are not directly involved in the teaching. The planning meeting and classroom activities are video recorded, and the reflection meeting is audio recorded. After completing a cycle of planning, reflection and classroom sessions, teachers and researchers meet to watch parts of the video recordings from the classroom. This is also video recorded, but is not used as data for this paper.

The topic of polygons turned out to be interesting for several reasons. It demonstrated how the phenomenon that words (signs) may have different reference in the mathematical and everyday registers constrained pupils' conceptual development and although parts of this phenomenon could be foreseen in the planning meeting, unexpected observations were made, showing that there are challenges involved in the apparently simple concepts involved when talking about polygons.

This paper is addressing the following research question: *To what extent do teachers' and researchers' expectations regarding pupils' sense and reference of words connected to polygons fit with actual experiences from the classroom?*

The main data are the transcribed video recordings from the planning meeting. In a line by line analysis of the transcription, we look for evidence of contributions from the different participants in the meeting. Then these contributions are compared to what actually happened in the classroom at one of the schools. Here we will refer to findings in (Rønning & Strømskag, 2017).

The Norwegian language and the mathematical terms involved

The relevance of the work with the topic of polygons is connected to particular features of the Norwegian language. In Norwegian, polygons are named after the number of edges, using ordinary number words. That is, an *"n-gon"* is called an "*n-edge",* where n can be three, four, …. The Norwegian word for *edge* is *kant.* For $n \geq 5$ this is in line with English, except that the English words are based in Greek (e.g. pentagon), and therefore they have no obvious meaning for young children. In the paper, we will use the Norwegian word *kant* (plural *kanter*) and whenever this word occurs, it is written in italics. In everyday language, the word is used in expressions like "falling off the *kant* of the cliff", "walking along the *kant* of the road", "sitting on the *kant* of the table". *Edge* is also a word in English everyday language but *vertex* is a mathematical term, not used in everyday language. The Norwegian language does not have a strictly mathematical term for vertex. The word *hjørne* is used (*corner* in English). In the sequel, we will use *hjørne* (plural *hjørner*), written in italics. Since the names for the figures are made up of everyday terms, it is generally believed that learning the names of and distinguishing between different polygons is an easy task. In German, the situation is very much like in Norwegian, except there the focus is not on the edges but on the vertices, i.e. a *firkant* (four edge) is referred to as a *Viereck,* meaning "four corner". Since the number of edges is equal to the number of vertices ($V = E$) it does not matter whether the edges or the corners are referred to, the number word connected to the figure will be the same.

Another feature is that the edges may be referred to by different words depending on the situation. For instance, when stating the formula for the area of a square, it is often said "*side* times *side*". Then the word "*side*" (or "sidekant") is seen as a variable denoting the length of the line segment making up the edge. The word *side* corresponds to the English word 'side'.

The planning meeting

A starting point for the discussion is that the participants (teachers and reseachers) present their experiences with the topic and also some evidence of confusion regarding language. Present at the planning meeting were the teachers Ruth and Pam, and the researchers Anne, Becky and Cathy.

Ruth:	We mostly talk about *kanter* so far. There is not so much talking about *hjørner.*
Anne.:	Or they do not agree on what a *kant* really is. Is this the *kant* (strikes along the side of the table), or is this the *kant* (marks the transition between the table top and the side).
Ruth:	We have discussed this a little in the staff room. Is it *side* or is it *kant*? Because when you talk about the area of a square, for instance, then it is like *side* times *side,* and why is it then called *side* when it is called *trekant* [triangle] and *firkant* [quadrilateral] and how many *kanter* has a … ? But when you come to three dimensional figures, then the *kant* is in a way, when you see from one side, then it is one of the sides.
Pam:	*Sidekant* I think it is called.
Ruth:	Yes, because you also have the concept *sideflate* [face]*,* and that is on a three-dimensional figure. And that is the *side,* that is what we call the *side.* So, I think it is best that the line in a two-dimensional figure is called a *kant.*

Ruth has observed that the same object in a geometric figure may be given different names in different contexts. When talking about the line segments making up a polygon, the word *kant* is used but when the length of the line segment is referred to, e.g. for calculating the area of a square, the word *side* is used, as in the formula *side* times *side.* This indicates that the words *kant* and *side* have different sense, although they refer to the same element of the polygon, i.e. they have the same reference. It could be argued that in the formula *side* times *side* the reference is not the actual line segment itself, but a number representing the length of the line segment. In the beginning, Ruth also admits that they have not talked so much about *hjørner,* mostly about *kanter*. This may be because it is the word *kant* that appears in the names of the polygons. Ruth also refers to the three-dimensional situation where the word *sideflate* is used for the face (F) but also sometimes just *side*, for short. She finds this confusing and suggests to stick to the word *kant*.

Anne:	A *hjørne* is where two *kanter* meet.
Becky:	Yes, so then at least we agree.
Ruth:	This is a much simpler definition.
Anne:	Yes, but it isn't. When we discuss this with student teachers, they do not agree on what a *kant* is. Some envisage, yes, a *trekant,* it has three *kanter,* yes, what I would call *kanter,* and like you Pam, call *sidekanter,* but it also has three *hjørner.* So if you say *trekant* and think about *hjørner*, there is no mismatch.
Pam:	And I heard a mother asking, when we had about *hjørner* and *kanter* on the working plan; *hjørner* AND *kanter,* but isn't that the same? That is the same, so why?
Cathy:	I have not at all thought like this. I have thought like you, a *hjørne* is a *hjørne.*
Pam:	Yes, and that is correct. But I think many have a wrong opinion about this.

Anne gives a precise mathematical definition of the term *hjørne,* and both Becky and Ruth seem to be happy about this, recognising it as a simple definition. However, Anne claims that it may not be that simple because she has evidence from student teachers not agreeing on what a *kant* actually is. They say *kant* and mean *hjørne,* and because of the relation $V = E$ for polygons they get away with it. This is supported by Pam who has evidence from a mother of one of the pupils claiming that *hjørne* and *kant* are two words for the same thing. Cathy has not reflected on this being an issue. This conversation shows evidence that the word *kant* may be used by different people with different sense and different reference and also that some people think that the two words are two different signs with both the same sense and reference. The fact that the group discussing this is a group consisting of experienced teachers and teacher educators indicates that the topic may not be as simple as it may seem at a first glance.

There is agreement that one should stick to the concepts *kant* and *hjørne,* and that it is important to stick to one word in the beginning, and Ruth says that *kant* seems to be the most precise word.

The reflection session

The reflection session is based on experiences from the first classroom session and took place immediately after the classroom session. The classroom session is discussed in (Rønning & Strømskag, 2017). These are some of the observations made.

- Most pupils used the word *kanter* to refer to the vertices but there were also some who used it correctly, to refer to the edges.
- Some pupils refer to a vertex as a *hjørne* when they approach it from the inside and as a *kant* when they approach it from the outside.
- In a quadrilateral with one reflex angle and three acute angles, the vertex at the reflex angle is sometimes referred to as a *hjørne* and a vertex at an acute angle is referred to as a *kant*.

The classroom session revealed that for most of the pupils the sense of the word *kant* was something sharp. This was emphasised when they talked about *kanter* as "the pointed parts [spissene]". The word *hjørne* was not often in use but when used, it had the sense of a space, something one can stay in, corresponding to the expression "stand in the corner" [stå i hjørnet]. It can therefore be said that the pupils use two different words for the same reference but each word has its own sense which is connected to the size of the angle at the vertex, or whether they see the vertex from the inside or from the outside.

Pam was surprised that they did not use the words correctly. They mainly used the word *kant* with the reference expected of the word *hjørne* and paid very little attention to the actual *kanter,* i.e. the edges. Diana, a researcher who was not present at the planning meeting, suggests to use *sidekant* about what they call *strek* [line]. Then there is a discussion about how the words *kant* and *hjørne* could be connected to everyday settings. Diana had observed in the classroom that some pupils talked about a "soft *kant*"*,* and said that that is a *hjørne* (with reference to a part of a local shopping centre)*,* but a "real *kant*" is sharp. For 3D objects it makes sense to talk about an edge (*kant*) as being sharp. This may transfer to 2D objects. One of the tasks for the pupils was that they should write the name of a given geometrical figure and explain why it had this name. Based on the pupils' workings it seemed that they used the words in a correct way. They could draw e.g. a quadrilateral and write "this is a *firkant* because it has four *kanter*". Here the reference of the word *firkant* is the drawing of a quadrilateral, as it should be, and the sense of the word is that the figure has four *kanter*. This is also correct, and the written statement will not reveal the misconceptions.

In the reflection session, it is discussed how the activity could be developed to overcome the misconception. In the first classroom session, it turned out that using the abstract geometrical figures as reference context was inadequate to stimulate the correct concepts connected to the words *kant* and *hjørne.* Therefore, the teacher suggested to use a mini-pitch as a reference context. This was assumed to be a context from the pupils' everyday experience and at the same time a context with properties sufficiently similar to an abstract geometrical figure, in this case a rectangle. Then the idea was presented to take the pupils out into the mini-pitch in the school yard and give commands like "go to the *hjørne*", "walk along the *kant*" and similar expressions.

Discussion

The target knowledge of the teaching sequence was that the pupils should develop the scientific language for naming 2D shapes and become aware that these names are based on the number of edges (*kanter*) in the shape. To know the difference between edges (*kanter*) and corners/vertices (*hjørner*) will then also be part of the target knowledge.

The planning session revealed certain conjectures about the use of the words *kant* and *hjørne.* The different participants in the session had different expectations, based on different experiences. The teachers had experienced confusion based on previous work with pupils and also based on conversations with parents. Teacher educators in mathematics had experienced confusion among student teachers, whereas the general educators were not necessarily aware that these words could be a source of confusion. For most adults, at least adults with some mathematical background, it is so obvious what is a *hjørne* and what is a *kant* that they do not envisage any problems with this. And also, the connection $V = E,$ ensures that the naming of geometrical figures will be consistent even if there is confusion about what is what. But when the sign is broken down into components, *tre* = three and *kanter* = edges, we see that the sense and reference are not as desired.

Kant	Sense	Reference
Mathematical discourse	A straight line segment	E.g. *AB, BC, CD* and *AD* in quadrilateral *ABCD.*
Pupils' discourse	Something sharp A real *kant* is sharp The pointed parts	E.g. *A, B, C* and *D* in a convex quadrilateral *ABCD,* when seen from the outside.

Table 1. Sense and reference of *kant*

Hjørne	Sense	Reference
Mathematical discourse	A point where two *kanter* meet	E.g. *A, B, C* and *D* in quadrilateral *ABCD.*
Pupils' discourse	A spacious area A soft *kant*	E.g. *A, B, C* and *D* in a convex quadrilateral *ABCD,* when seen from the inside. A non-convex vertex.

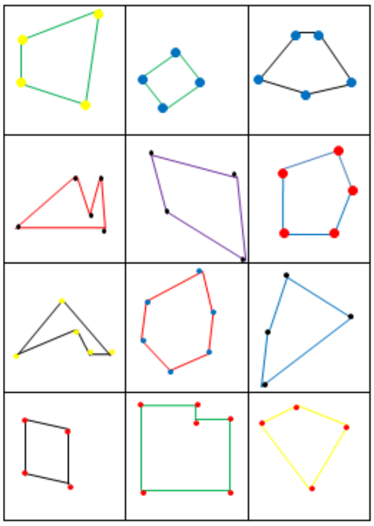
Table 2. Sense and reference of *hjørne*

Even if some confusion was expected, the classroom sessions brought new knowledge to the sense and reference of the words *kant* and *hjørne.* This means that even for such seemingly simple mathematical concepts as here, the group of experienced teachers and teacher educators could not fully forecast how the terms would be handled by the pupils. Using the terms from Ogden and Richards (1923/1948) we may say that the words *kant* and *hjørne* did not function as true symbols because they did not cause the same reference in all persons involved.

The insight we got during the classroom sessions was used to redesign the adidactical situation with respect to the conventional sense and reference of the words *kant* and *hjørne.* The desired statuses of knowledge in formulation and validation, as described above, were decisive for the team in designing a material milieu and a game (for acting on the milieu) with adidactical potential regarding the target knowledge, so the pupils could know whether their responses were adequate or not. The new material milieu is shown in Figure 1. It consists of 12 tiles, where on one side of the tile was depicted a polygon where the edges had one colour and the vertices were marked with another colour. On the reverse of the tile was written descriptions like e.g. "*firkant* (quadrilateral) with blue *sidekanter*". This game was played in pairs of pupils both having the full set of tiles. One pupil reads the text and the other one is supposed to pick the correct shape. After picking he/she can turn the tile and read the text to see if the correct shape has been picked.

The conditions described in this paper indicate that there is a need to address the concepts *hjørne* and *kant* in Norwegian schools. The naming of figures by counting the *kanter* using ordinary Norwegian number words, like in *firkant*, may indicate that there is no need to go further into the topic. Also, the books that are used for the pupils seem to take the concepts for granted, there is never any indication of what is actually *hjørne* and what is *kant*. The discussion in the planning session also shows that there is varying consciousness about the topic among teachers and researchers. And those that are conscious about it are so because they have previous experiences indicating that there may be some confusion.

Distinguishing between edges and vertices can also be seen to be important for future learning, e.g. about polyhedra, where the number of edges and the number of vertices are not the same.



Femkant (pentagon) with blue *hjørner*	*Firkant* (quadrilateral) with green *sidekanter* and blue *hjørner*	*Firkant* (quadrilateral) with yellow *hjørner*
Femkant (pentagon) with blue *sidekanter*	*Firkant* (quadrilateral) with purple *sidekanter*	*Femkant* (pentagon) with red *sidekanter*
Firkant (quadrilateral) with blue *sidekanter*	*Sekskant* (hexagon) with red *sidekanter*	*Femkant* (pentagon) with yellow *hjørner*
Firkant (quadrilateral) with yellow *sidekanter*	*Sekskant* (hexagon) with red *hjørner*	*Firkant* (quadrilateral) with black *sidekanter*

Figure 1. Material milieu for classification of polygons

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