An Adaptive Observer Design for 2×2 Semi-linear Hyperbolic Systems using Distributed Sensing

Haavard Holta and Ole Morten Aamo

***Abstract*— We design an adaptive model-based observer for state and parameter estimation in 2×2 semi-linear hyperbolic systems with uncertain parameters where we assume that one of the two distributed states is available through distributed sensing. The uncertainties appear in the equation for the unmeasured distributed state and may be non-linear in the unmeasured state, although linearly parameterized. The adaptive law is designed using a Lyapunov approach and expressed in terms of known signals by utilizing the specific model structure which gives rise to a general solution strategy valid for a large class of non-linear source terms.**

I. Introduction

1) Background: Hyperbolic systems are used to model various flow and transport phenomena like electrical transmission lines, gas pipelines, propagation of light in optical fibers, flow in blood vessels, the motion of chemicals in plug flow reactors or the propagation of epidemics, to mention a few (see [1] for an overview). 2×2 hyperbolic systems in particular can be used to model single-phase flows. Relevant to the problem considered in this paper is an application in offshore oil well drilling where drilling mud is circulated down the drill string and used for (among other things) pressure control. When drilling into an oil reservoir, a pressure difference between reservoir and drilling mud will result in an in- or out-flow of fluids which, if not counteracted, might have a severe effect on operational efficiency and/or safety. Precise control of down-hole pressure is therefore essential, but complicated for systems without reliable measurements, possibly nonlinear dynamics and uncertain elements such as for instance friction coefficients for the flow. In such cases, a model-based adaptive observer might be used to obtain sufficiently accurate state and parameter estimates for down-hole pressure control.

Observers for 2×2 hyperbolic systems with boundary sensing only can be designed using the infinite dimensional backstepping approach [2]. In the adaptive case, this technique is utilized in [3], [4] for the case of having uncertain boundary parameters collocated and anti-collocated with the sensed boundary, respectively. Estimation of uncertain in-domain parameters, assuming full state feedback, is studied in [4], and assuming only boundary sensing, but in the context of control, in [5]. All of the aforementioned observers are designed for linear systems. In the present paper, we consider a semi-linear system where the source terms may be non-linear functions of the state. To achieve this, we assume that one of the two distributed states is available through distributed sensing. In the drilling application, this corresponds to a distributed pressure measurement throughout the domain, which is possible in an approximate manner due to so-called wired pipe technology.

A common technique to derive adaptive observers, is to transform the system into a *canonical* form for which already known design methods are applicable. This approach was used for adaptive control design of hyperbolic PDEs in [5], [6]. Non-linear finite dimensional systems that can be transformed to the canonical form is characterized in [7]. However, if the transformations depend on the unknown parameter, estimates of the original states can only be reconstructed if the system is persistently excited (PE) [8]. For systems satisfying certain special conditions on the non-linearities, it is possible to design adaptive observers without the use of transformation techniques. In [9], the non-linearities satisfy a Lipschitz condition and a high-gain is used to dominate the non-linear terms. Yet another approach is to assume that the non-linearities satisfy a sector condition. Inspired by the circle criterion [10], an adaptive observer design for systems with parametric uncertainties in the unmeasured state dynamics and non-linearities satisfying a sector condition is presented in [11], [12]. Essential to this design is a change of coordinates which makes it possible to design an adaptive law that is implementable even though it is driven by unmeasured signals. In the current paper, this method is extended to the hyperbolic PDE setting. Since the state is now distributed and a function of one spatial variable in addition to time, the extension turns out to be non-trivial.

2) Notation and preliminaries: The set of non-negative real numbers is denoted $\mathbb{R}^+$. A function $u:[0,1] \to \mathbb{R}$ is said to be in $L_2(0,1)$ if

$$\sqrt{\int_0^1 u^2(x)dx} < \infty. \tag{1}$$

For $u_1, u_2 \in L_2(0,1)$ the inner product is

$$\langle u_1, u_2 \rangle := \int_0^1 u_1(x)u_2(x)dx \tag{2}$$

with the associated norm $||u|| = \sqrt{\langle u, u \rangle}$.

For $f:\mathbb{R}^+ \to \mathbb{R}$, we use the spaces

$$f \in \mathscr{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt \right)^{\frac{1}{p}} < \infty \tag{3}$$

for $p \geq 1$ with the particular case $f \in \mathscr{L}_\infty \leftrightarrow \sup_{t \geq 0} |f(t)| < \infty$.

The authors are with the Department of Engineering Cybernetics. Norwegian University of Science and Technology, Trondheim N-7491, Norway (e-mail: haavard.holta@ntnu.no; aamo@ntnu.no).

Economic support from The Research Council of Norway and Equinor ASA through project no. 255348/E30 Sensors and models for improved kick/loss detection in drilling (Semi-kidd) is gratefully acknowledged

The partial derivative of a function is denoted with a subscript, for example $u_t(x,t) = \frac{\partial}{\partial t}u(x,t)$. For a function of one variable, the derivative is denoted using a prime, that is $f'(x) = \frac{d}{dx}f(x)$. The dot notation is reserved for the derivative of functions of time only; $\dot{f}(t) = \frac{d}{dt}f(t)$.

An operator $\Xi : L_2(0,1) \to \mathbb{R}$ is called *Fréchet differentiable* at $u \in L_2(0,1)$ if there exists a bounded linear operator $D_u\Xi : L_2(0,1) \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|\Xi[u+h] - \Xi[u] - D_u\Xi[h]|}{||h||} = 0 \tag{4}$$

for $h \in L_2(0,1)$. If such a bounded linear operator exists, it is unique and we call $D_u\Xi$ the *Fréchet derivative* of Ξ at u.

3) Problem statement: We study systems in the form

$$\begin{aligned} p_t(x,t) + aq_x(x,t) =& f(p(x,t),x) && (5a)\\ q_t(x,t) + bp_x(x,t) =& \phi^T(q(x,t),x)\theta && (5b)\\ q(0,t) =& q_{x_0}(t) && (5c)\\ p(1,t) =& p_{x_1}(t) && (5d)\\ p(x,0) =& p_{t_0}(x) && (5e)\\ q(x,0) =& q_{t_0}(x) && (5f) \end{aligned}$$

where $p,q : [0,1]\times\mathbb{R}^+ \to \mathbb{R}$ are the system states, $a,b \in \mathbb{R}$ are known parameters such that the eigenvalues of the coefficient matrix (the coefficients of the spatial first derivatives) are distinct and real (that is $ab > 0$), $\theta \in \mathbb{R}^n$ is assumed to be unknown, and the source terms $f : \mathbb{R}\times[0,1] \to \mathbb{R}$ and $\phi : \mathbb{R}\times[0,1] \to \mathbb{R}^n$ may be non-linear functions. The linearity in the first order derivatives combined with the nonlinear source terms and real eigenvalues make the system (5) of type *semi-linear hyperbolic*.

System (5) is usually written in *characteristic form* in terms of its *Riemann coordinates* (u,v), which can be obtained through a linear change of variables $(p,q) \to (u,v)$ (see e.g. [13]). However, in this paper we assume that the state p is measured (for all $x \in [0,1]$ at every time), while q is unknown, which makes the (p,q) state representation the most convenient form. In addition, we assume the following.

Assumption 1: The uncertain source term $\phi^T\theta$ in (5b) satisfies the sector condition $(\phi(u_1,x) - \phi(u_2,x))^T\theta(u_1-u_2) \le 0$ for any $u_1,u_2 \in \mathbb{R}$ and $x \in [0,1]$. Furthermore, $u \in L_2(0,1) \Rightarrow \phi(u,\cdot) \in L_2(0,1)$ and $||u|| \in \mathscr{L}_\infty \Rightarrow ||\phi(u,\cdot)|| \in \mathscr{L}_\infty$.

Assumption 2: The boundary conditions q_{x_0}, p_{x_1} and initial conditions p_{t_0}, q_{t_0} are restricted to signals such that (5) has a unique, bounded, solution $p(\cdot,t), q(\cdot,t) \in L_2(0,1)$.

II. Observer design

Consider the coordinate transformation $(p,q) \to (p,\zeta)$ defined by

$$\zeta(x,t) = q(x,t) + lp(x,t) \tag{6}$$

where $l \in \mathbb{R}$ is a design parameter selected such that $la > 0$. In terms of the new coordinates, system (5) takes the form

$$\begin{aligned} p_t(x,t) =& lap_x(x,t) - a\zeta_x(x,t) + f(p(x,t),x) && (7a)\\ \zeta_t(x,t) =& -la\zeta_x(x,t) + (l^2a-b)p_x(x,t) \\ &+ lf(p(x,t),x) + \phi^T(q(x,t),x)\theta && (7b)\\ \zeta(0,t) =& q_{x_0}(t) + lp(0,t) && (7c)\\ p(1,t) =& p_{x_1}(t) && (7d)\\ p(x,0) =& p_{t_0}(x) && (7e)\\ \zeta(x,0) =& q_{t_0}(x) + lp_{t_0}(x) =: \zeta_{t_0}(x). && (7f) \end{aligned}$$

Based on this, we design a reduced order observer for q as

$$\begin{aligned} \hat{\zeta}_t(x,t) =& -la\hat{\zeta}_x(x,t) + (l^2a-b)p_x(x,t) \\ &+ lf(p(x,t),x) + \phi^T(\hat{q}(x,t),x)\hat{\theta}(t) && (8a)\\ \hat{\zeta}(0,t) =& q_{x_0}(t) + lp(0,t) && (8b)\\ \hat{\zeta}(x,0) =& \hat{\zeta}_{t_0}(x) && (8c)\\ \hat{q}(x,t) =& \hat{\zeta}(x,t) - lp(x,t) && (8d) \end{aligned}$$

where $\hat{\theta}$ is a parameter estimate. Defining the state estimation error $\tilde{\zeta}(x,t) = \zeta(x,t) - \hat{\zeta}(x,t)$ and parameter estimation error $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$ yields the error dynamics

$$\begin{aligned} \tilde{\zeta}_t(x,t) =& -la\tilde{\zeta}_x(x,t) + \phi^T(\hat{q}(x,t),x)\tilde{\theta}(t) \\ &+ \left(\phi^T(q(x,t),x) - \phi^T(\hat{q}(x,t),x)\right)\theta && (9a)\\ \tilde{\zeta}(0,t) =& 0 && (9b)\\ \tilde{\zeta}(x,0) =& \zeta_{t_0}(x) - \hat{\zeta}_{t_0}(x) =: \tilde{\zeta}_{t_0}(x). && (9c) \end{aligned}$$

Or equivalently, since $\tilde{q}(x,t) := q(x,t) - \hat{q}(x,t) = \tilde{\zeta}(x,t)$, we have

$$\begin{aligned} \tilde{q}_t(x,t) =& -la\tilde{q}_x(x,t) + \phi^T(\hat{q}(x,t),x)\tilde{\theta}(t) \\ &+ \left(\phi^T(q(x,t),x) - \phi^T(\hat{q}(x,t),x)\right)\theta && (10a)\\ \tilde{q}(0,t) =& 0. && (10b)\\ \tilde{q}(x,0) =& \tilde{\zeta}_{t_0}(x). && (10c) \end{aligned}$$

To study the stability of the state estimation error system (10), consider the Lyapunov function candidate

$$V_0(t) = \frac{1}{2}\int_0^1 W(x)\tilde{q}^2(x,t)dx \tag{11}$$

where W can be any differentiable function satisfying $W(x) > 0$ and $W'(x) \le -cW(x)$ for all $x \in [0,1]$ and some $c > 0$. Differentiating with respect to time gives

$$\begin{aligned} \dot{V}_0(t) =& -la\int_0^1 W(x)\tilde{q}(x,t)\tilde{q}_x(x,t)dx \\ &+ \int_0^1 W(x)\left(\phi^T(q(x,t),x) - \phi^T(\hat{q}(x,t),x)\right)\theta\tilde{q}(x,t)dx \\ &+ \int_0^1 W(x)\tilde{q}(x,t)\phi^T(\hat{q}(x,t),x)\tilde{\theta}(t)dx. && (12) \end{aligned}$$

Integrating by parts, using the sector condition $\left(\phi^T(q(x,t),x) - \phi^T(\hat{q}(x,t),x)\right)\theta\tilde{q}(x,t) \le 0$ (Assumption 1), and boundary condition (10b) yields

$$\dot{V}_0(t) \leq la\int_0^1 W'(x)\tilde{q}^2(x,t)dx + \int_0^1 W(x)\tilde{q}(x,t)\phi^T(\hat{q}(x,t),x)\tilde{\theta}(t)dx. \quad (13)$$

Remark 1: Without any parametric uncertainties, that is $\tilde{\theta}=0$, the upper bound (13) simplifies to

$$\dot{V}_0(t) \leq -lacV_0(t). \quad (14)$$

Then, since $lac>0$ by design, the origin of (10) is exponentially stable in the L_2-sense with convergence rate proportional to $|l|$.

III. ADAPTIVE LAW

To make the system robust to parametric uncertainties, we need a scheme to update the parameter estimate $\hat{\theta}$. We augment the Lyapunov function candidate with terms quadratic in the parameter estimation error and select the adaptive law using a passivity argument: Let

$$V(t) = V_0(t) + \frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t) \quad (15)$$

where the adaptation gain $\Gamma>0$ is diagonal.

Lemma 1: The observer (8) and the adaptive law

$$\dot{\hat{\theta}}(t) = \Gamma\int_0^1 W(x)\tilde{q}(x,t)\phi(\hat{q}(x,t),x)dx \quad (16)$$

for any initial estimate $\hat{\theta}(0)=\hat{\theta}_{t_0}$ and diagonal $\Gamma>0$ provide the following properties

$$||\tilde{q}||, \hat{\theta}, \dot{\hat{\theta}} \in \mathscr{L}_\infty \quad (17a)$$

$$||\tilde{q}||, \dot{\hat{\theta}} \in \mathscr{L}_2 \quad (17b)$$

$$||\tilde{q}|| \to 0. \quad (17c)$$

Proof: Differentiating (15) with respect to time yields

$$\dot{V}(t) = \dot{V}_0(t) - \dot{\hat{\theta}}^T\Gamma^{-1}\tilde{\theta}(t). \quad (18)$$

In view of (13) and the properties of $W(x)$, selecting the adaptive law according to (16) renders the Lyapunov function negative semidefinite with upper bound

$$\dot{V}(t) \leq la\int_0^1 W'(x)\tilde{q}^2(x,t)dx \leq -lacV_0(t) \quad (19)$$

and $||\tilde{q}||, \hat{\theta} \in \mathscr{L}_\infty$ follows. Furthermore, from $V>0, \dot{V}\leq 0$ we have that $\lim_{t\to\infty} V(t) = V(\infty)$ exists and therefore

$$lac\int_0^\infty V_0(t)dt \leq V(0) - V(\infty) \quad (20)$$

which implies $||\tilde{q}|| \in \mathscr{L}_2$. From (16), we have that $|\dot{\hat{\theta}}| \leq \Gamma||W||||\tilde{q}||||\phi||$. Boundedness of $||q||$ (Assumption 2) and $||\tilde{q}||$ imply boundedness of $||\hat{q}||$ and in turn $||\phi||$ by Assumption 1. $\dot{\hat{\theta}} \in \mathscr{L}_\infty \cap \mathscr{L}_2$ then follows. Lastly, from (18) and (19) we get

$$\dot{V}_0(t) \leq -lacV_0(t) + \dot{\hat{\theta}}^T(t)\Gamma^{-1}\tilde{\theta}(t) \quad (21)$$

which shows that $\dot{V}_0(t)$ is upper bounded by some constant. By Lemma 5 in Section VII-A $V_0, ||\tilde{q}|| \to 0$. ■

The adaptive law in Lemma 1 is expressed in terms of $\tilde{q}$ which is unknown. Hence, it is not possible to implement (16) directly. Next, we will show that it is possible to represent (16) in terms of only measured and estimated signals by exploiting the structure of (5).

IV. FILTER AND OPERATOR DESIGN

To ease the notation, for each element ϕ_i, $i=1,\ldots,n$ in ϕ and $u \in L_2(0,1)$ let

$$\Phi_i[u](x) := -\int_x^1 W(\xi)\phi_i(u(\xi),\xi)d\xi. \quad (22)$$

For each parameter $\hat{\theta}_i$ in $\hat{\theta}$ and diagonal element Γ_i in Γ, the adaptive law (16) can be written in the form $\dot{\hat{\theta}}_i(t) = -\dot{\tilde{\theta}}_i(t) = -\Gamma_i\langle\Phi_i'[\hat{q}(\cdot,t)], \tilde{q}(\cdot,t)\rangle$. Alternatively, since $\tilde{q}(0,t) = \Phi_i[\hat{q}(\cdot,t)](1) = 0$, integrating by parts yields

$$\dot{\hat{\theta}}_i(t) = -\Gamma_i\langle\Phi_i'[\hat{q}(\cdot,t)], \tilde{q}(\cdot,t)\rangle = \Gamma_i\langle\Phi_i[\hat{q}(\cdot,t)], \tilde{q}_x(\cdot,t)\rangle. \quad (23)$$

Let the signal $\sigma_i : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\sigma_i(t) := \theta_i + \Xi_i[\hat{q}(\cdot,t)] \quad (24)$$

where Ξ_i is a possibly nonlinear operator to be specified. Based on (24), we set

$$\hat{\theta}_i(t) = \hat{\sigma}_i(t) - \Xi_i[\hat{q}(\cdot,t)] \quad (25)$$

where the estimate $\hat{\sigma}_i : \mathbb{R}^+ \to \mathbb{R}$ of σ_i is specified in the following lemma.

Lemma 2: If some operator $\Xi_i : L_2(0,1) \to \mathbb{R}$ satisfying

$$D_u\Xi_i[h] = \langle\eta_i[u], h\rangle \quad (26a)$$

$$\Xi_i[\hat{q}(\cdot,0)] = 0 \quad (26b)$$

where

$$\eta_i[u](x) := \frac{\Gamma_i}{la}\Phi_i[u](x) \quad (27)$$

for all $u,h \in L_2(0,1)$ can be found, then $\hat{\theta}_i$ calculated using (25) with $\hat{\sigma}_i$ defined by

$$\dot{\hat{\sigma}}_i(t) = \langle\eta_i[\hat{q}(\cdot,t)], -bp_x + \phi^T(\hat{q}(\cdot,t),x)\hat{\theta}(t)\rangle \quad (28a)$$

$$\hat{\sigma}_i(0) = \hat{\theta}_{i,t_0} \quad (28b)$$

and $\hat{q}$ generated from (8), satisfies (16).

Proof: Differentiating (24) with respect to time by introducing the *Fréchet derivative* of Ξ_i yields

$$\dot{\sigma}_i(t) = \frac{d}{dt}\Xi_i[\hat{q}(\cdot,t)] = D_{\hat{q}}\Xi_i[\hat{q}_t(\cdot,t)] = D_{\hat{q}}\Xi_i[\hat{\zeta}_t(\cdot,t) - lp_t(\cdot,t)]. \quad (29)$$

From the condition (26), we get

$$\dot{\sigma}_i(t) = \langle\eta_i[\hat{q}(\cdot,t)], \hat{\zeta}_t(\cdot,t) - lp_t(\cdot,t)\rangle \quad (30)$$

which, after inserting the dynamics (5a) and (8a), reads

$$\dot{\sigma}_i(t) = \langle\eta_i[\hat{q}(\cdot,t)], -bp_x(\cdot,t) + \phi^T(\hat{q}(\cdot,t),\cdot)\hat{\theta}(t) + la\tilde{q}_x(\cdot,t)\rangle. \quad (31)$$

From (24) and (25), we see that the error $\tilde{\sigma}_i = \sigma_i - \hat{\sigma}_i$ satisfies $\tilde{\theta}_i = \tilde{\sigma}_i$, so that differentiating with respect to time and inserting the dynamics (28) and (31) yield

$$\dot{\tilde{\theta}}_i(t) = \langle \eta_i[\hat{q}(\cdot,t)], la\tilde{q}_x(\cdot,t)\rangle. \tag{32}$$

In view of (27), the dynamics (32) and (23) are the same. Furthermore, the condition (26b) and (25) give $\hat{\theta}_i(0) = \hat{\sigma}_i(0)$ so that the initial condition (28b) implies $\hat{\theta}_i(0) = \hat{\theta}_{i,t_0}$. ∎

To implement the estimate $\hat{\theta}$ from (28) we need to evaluate $\Xi_i[\hat{q}]$. As a simple example, consider the case where ϕ_i is constant and independent of q. For that case, it can easily be verified that $\Xi_i[\hat{q}(\cdot,t)] = \Gamma\langle\Phi_i[\hat{q}(\cdot,t)], \hat{q}(\cdot,t)\rangle$ satisfies the definition (4) with $D_{\hat{q}}\Xi_i[h]$ given in (26). For the general case with non-constant ϕ_i, finding an explicit expression turns out to be hard. However, Ξ_i is not needed at every point in $L_2(0,1)$ but only along the trajectory $\hat{q}(\cdot,t)$. Therefore, Ξ_i can be evaluated at $\hat{q}(\cdot,t)$ by integrating (26a) along a line, as demonstrated in the next section.

V. Evaluating the operator Ξ

Let $\hat{q}_0, \hat{q}_1$ be arbitrary functions in $L_2(0,1)$. We seek a method to calculate the incremental value $\Xi_i[\hat{q}_1] - \Xi_i[\hat{q}_0]$. To that end, let $S : [0,1] \to L_2(0,1)$ be given by

$$S(\gamma) = \hat{q}_0 + \gamma(\hat{q}_1 - \hat{q}_0). \tag{33}$$

Evaluating Ξ_i at $S(\gamma)$, differentiating with respect to γ and inserting (26) yield

$$\begin{aligned}\frac{d}{d\gamma}\Xi_i[S(\gamma)] =& D_{S(\gamma)}\Xi_i[S'(\gamma)]\\ =& \langle\eta_i[S(\gamma)], S'(\gamma)\rangle\\ =& \langle\eta_i[S(\gamma)], \hat{q}_1 - \hat{q}_0\rangle.\end{aligned} \tag{34}$$

Integrating from $\gamma = 0$ to $\gamma = 1$ gives

$$\Xi_i[S(1)] = \Xi_i[S(0)] + \int_0^1 \langle\eta_i[S(\gamma)], \hat{q}_1 - \hat{q}_0\rangle d\gamma. \tag{35}$$

Finally, since $S(1) = \hat{q}_1$ and $S(0) = \hat{q}_0$, we obtain

$$\Xi_i[\hat{q}_1] = \Xi_i[\hat{q}_0] + \int_0^1 \langle\eta_i[\hat{q}_0 + \gamma(\hat{q}_1 - \hat{q}_0)], \hat{q}_1 - \hat{q}_0\rangle d\gamma. \tag{36}$$

Choosing $\hat{q}_1(x) = \hat{q}(x,t)$ for any $t \geq 0$, $\hat{q}_0(x) = \hat{q}(x,0)$, and considering the fact that $\Xi_i[\hat{q}(0)] = 0$ yield an expression for the operator evaluated at the current state. The result is summarized in the following lemma.

Lemma 3: For any $t \geq 0$, the conditions of Lemma 2 are satisfied with

$$\begin{aligned}\Xi_i[\hat{q}(\cdot,t)] = \int_0^1 \langle\eta_i[\hat{q}(\cdot,0) + \gamma(\hat{q}(\cdot,t) - \hat{q}(\cdot,0))],\\ \hat{q}(\cdot,t) - \hat{q}(\cdot,0)\rangle d\gamma.\end{aligned} \tag{37}$$

Remark 2: The scheme (36) generating $\Xi_i[\hat{q}_1]$ utilizes the fact that the path integral between $\Xi_i[\hat{q}_1]$ and $\Xi_i[\hat{q}_0]$ is path independent.

Computing (37) at every time step in a computer implementation is computationally expensive. However, for some special classes of source terms ϕ_i, the computation simplifies.

Lemma 4: If ϕ_i is in the form

$$\phi_i(\gamma\hat{q}(x,t), x) = \rho_i(\gamma)\phi_i(\hat{q}(x,t), x) \tag{38}$$

for some function $\rho_i : [0,1] \to \mathbb{R}$ then the operator Ξ_i can be evaluated at $\hat{q}(\cdot,t)$ as

$$\Xi_i[\hat{q}(\cdot,t)] = \alpha_i + \beta_i\langle\eta_i[\hat{q}(\cdot,t)], \hat{q}(\cdot,t)\rangle \tag{39}$$

where

$$\alpha_i = -\int_0^1 \rho_i(\gamma)d\gamma\langle\eta_i[\hat{q}(\cdot,0)], \hat{q}(\cdot,0)\rangle \tag{40a}$$

$$\beta_i = \int_0^1 \sigma_i(\gamma)d\gamma. \tag{40b}$$

Proof: From (36), letting $\hat{q}_1(x) = \hat{q}(x,t)$ and $\hat{q}_0 = 0$, we have

$$\Xi_i[\hat{q}(\cdot,t)] = \Xi_i[0] + \int_0^1 \langle\eta_i[\gamma\hat{q}(\cdot,t)], \hat{q}(\cdot,t)\rangle d\gamma \tag{41}$$

and from letting $\hat{q}_1(x) = \hat{q}(x,0)$ and $\hat{q}_0 = 0$, we have

$$\Xi_i[\hat{q}(\cdot,0)] = \Xi_i[0] + \int_0^1 \langle\eta_i[\gamma\hat{q}(\cdot,0)], \hat{q}(\cdot,0)\rangle d\gamma = 0. \tag{42}$$

Thus, combining (41) and (42) and using (26b), we get

$$\begin{aligned}\Xi_i[\hat{q}(\cdot,t)] = -\int_0^1 \langle\eta_i[\gamma\hat{q}(\cdot,0)], \hat{q}(\cdot,0)\rangle d\gamma\\ + \int_0^1 \langle\eta_i[\gamma\hat{q}(\cdot,t)], \hat{q}(\cdot,t)\rangle d\gamma.\end{aligned} \tag{43}$$

From (38) it follows that

$$\begin{aligned}\eta_i[\gamma\hat{q}(\cdot,t)](x) =& \frac{\Gamma_i}{la}\Phi_i[\gamma\hat{q}(\cdot,t)](x)\\ =& -\frac{\Gamma_i}{la}\int_x^1 W(\xi)\phi_i(\gamma\hat{q}(\xi,t),\xi)d\xi\\ =& -\frac{\Gamma_i}{la}\int_x^1 W(\xi)\rho_i(\gamma)\phi_i(\hat{q}(\xi,t),\xi)d\xi\\ =& -\rho_i(\gamma)\frac{\Gamma_i}{la}\int_x^1 W(\xi)\phi_i(\hat{q}(\xi,t),\xi)d\xi\\ =& \rho_i(\gamma)\eta_i[\hat{q}(\cdot,t)](x),\end{aligned} \tag{44}$$

so that

$$\begin{aligned}\Xi_i[\hat{q}(\cdot,t)] = -\int_0^1 \rho_i(\gamma)d\gamma\langle\eta_i[\hat{q}(\cdot,0)], \hat{q}(\cdot,0)\rangle\\ + \int_0^1 \rho_i(\gamma)d\gamma\langle\eta_i[\hat{q}(\cdot,t)], \hat{q}(\cdot,t)\rangle.\end{aligned} \tag{45}$$

∎

Theorem 1: Consider the observer (8) and the parameter update law (25) with $\hat{\sigma}$ satisfying (28), and Ξ computed with (37) or with (39) if the condition (38) is satisfied. Then, the state estimation error $\tilde{q}$ and parameter estimate $\tilde{\theta}$ satisfy properties (17).

Proof: The conditions in Lemma 2 are satisfied with the operators computed according to Lemma 3. By Lemma 2, the estimates generated from (25) satisfy (16) and so, by Lemma 1, (17) holds. By Lemma 4, (37) and (39) are equivalent if (38) holds. ∎

VI. Simulation and Discussion

The system (5) was simulated in MATLAB together with the state and parameter estimation scheme consisting of the observer dynamics (8), the filter dynamics (28) and parameter estimates (25).

The state and observer dynamics are implemented using the *method of lines*. That is, where the spatial derivatives are approximated using finite differences and the resulting ODE is solved using a *Runge-Kutta* method. The integrals are approximated as a finite number $N \geq 1$ of trapezoids using the trapezoidal rule

$$\int_0^1 f(x)dx \approx \frac{1}{N}\sum_{k=0}^{N-1}\frac{f(x_{k+1})+f(x_k)}{2}. \tag{46}$$

In order to use the method of lines, system (5) in (p,q)-coordinates is first transformed to Riemann coordinates (u,v), and only transformed back before plotting. The system is simulated for 5 seconds with 200 spatial discretization points. The system functions and parameters are selected as follows

$$a = 4 \tag{47a}$$
$$b = 4 \tag{47b}$$
$$\theta = \begin{bmatrix}5 & 7\end{bmatrix}^T \tag{47c}$$
$$\phi(q(x,t),x) = \begin{bmatrix}-q(x,t)x^2 & -q(x,t)|q(x,t)|x\end{bmatrix}^T \tag{47d}$$
$$f(p(x,t),x) = p(x,t)+x \tag{47e}$$
$$q_0(t) = 1 \tag{47f}$$
$$p_{x_1}(t) = \sin(2t) \tag{47g}$$
$$l = 0.8 \tag{47h}$$
$$W(x) = 2-x \tag{47i}$$
$$\Gamma = \mathrm{diag}(5,5). \tag{47j}$$

The initial conditions are selected as

$$p_{t_0}(x) = p_{x_1}(0)+2(1-x) \tag{48a}$$
$$q_{t_0}(x) = q_{x_0}(0)+2x \tag{48b}$$
$$\hat{\zeta}_{t_0}(x) = l p_{t_0}(x) \tag{48c}$$
$$\hat{\theta}_{t_0} = [0,0]. \tag{48d}$$

With the selected ϕ in (47d), we have

$$\phi(\gamma\hat{q}(x,t),x) = \mathrm{diag}(\gamma,\gamma^2)\phi(\hat{q}(x,t),x) \tag{49}$$

which shows that the condition (38) in Lemma 4 is satisfied with $\rho_1(\gamma) = \gamma$ and $\rho_2(\gamma) = \gamma^2$. The operator Ξ_i can be evaluated using (39) with

$$\beta_1 = \int_0^1 \gamma d\gamma = \frac{1}{2} \tag{50a}$$
$$\beta_2 = \int_0^1 \gamma^2 d\gamma = \frac{1}{3} \tag{50b}$$

and since $\hat{q}(x,0) = 0$, $\alpha_1 = \alpha_2 = 0$. That is,

$$\Xi_1[\hat{q}(\cdot,t)] = \frac{1}{2}\langle\eta_1[\hat{q}(\cdot,t)],\hat{q}(\cdot,t)\rangle \tag{51a}$$
$$\Xi_2[\hat{q}(\cdot,t)] = \frac{1}{3}\langle\eta_2[\hat{q}(\cdot,t)],\hat{q}(\cdot,t)\rangle. \tag{51b}$$

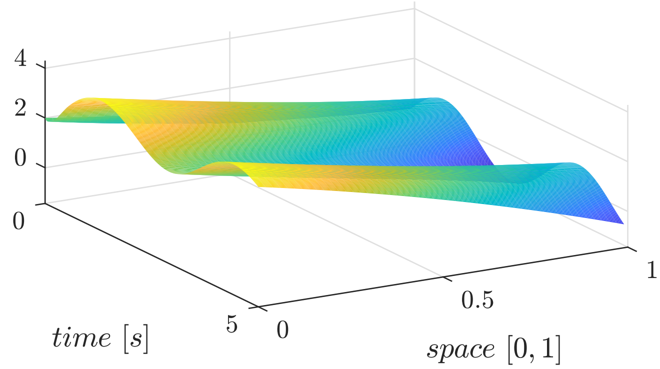


Fig. 1: State $p(x,t)$.

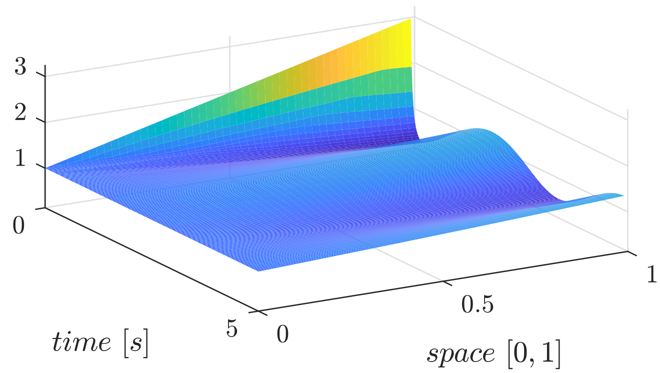


Fig. 2: State $q(x,t)$.

Figures 1 and 2 show that with the selected sinusoidal boundary condition, the system states fluctuate in the range $[0,3]$. The observer scheme is able to estimate the unknown state q and the estimation error $\tilde{q}$ converges asymptotically to zero, as can be seen in Figures 3 and 4. The parameter estimate $(\hat{\theta}_1,\hat{\theta}_2)$ converges to the true parameter value (θ_1,θ_2) as can be seen in Figure 5.

From the upper bound on the Lyapunov function derivative in (13), we see that the convergence rate of $||\tilde{q}||^2$ is proportional to the design parameter l. And from the state estimate definition (8d), we see that l sets the relative weighting between the observer variable $\hat{\zeta}$ and the measured state signal p. So, if the measurements are reliable, a high design parameter l can be used to achieve fast convergence. A system dependent on uncertain measurements on the other hand, will require more time to achieve the same level of accuracy.

VII. Concluding Remarks and Further Work

We have designed an adaptive observer for 2×2 hyperbolic PDEs estimating both system parameters and state. The method is a generalization of previous work on adaptive observers for finite dimensional systems to the infinite dimensional setting. The structure in both the ODE design and the current design for hyperbolic PDEs is the same in that one state variable is known and where the governing dynamics are without any uncertain parameter. This allowed us to relate the unknown state variable to the dynamics of the known state. For the ODE case, this relationship is trivial. In the PDE case,

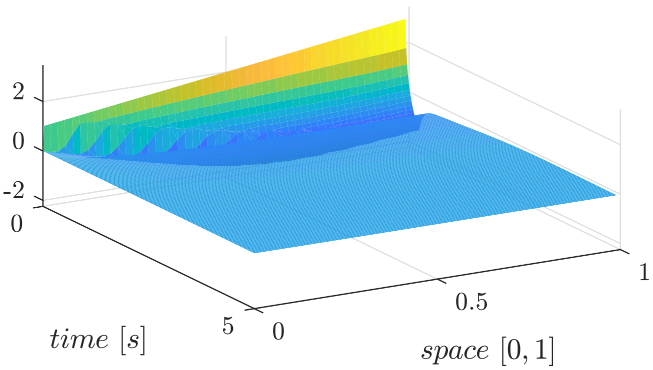


Fig. 3: State estimation error $\tilde{q}$.

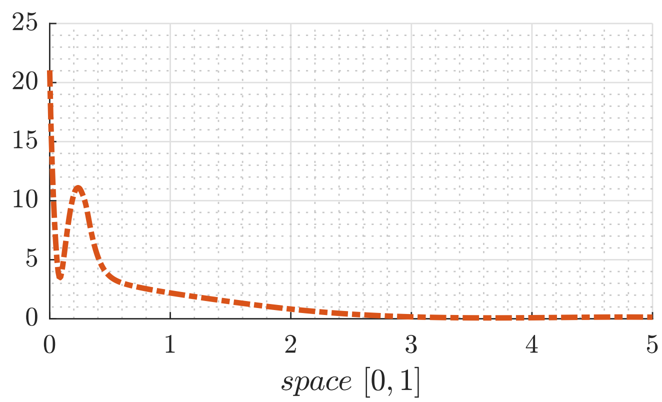
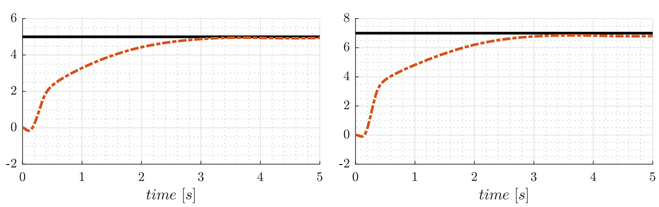


Fig. 4: State estimation error $||\tilde{q}||$.



(a) $\hat{\theta}_1$ vs. θ_1.

(b) $\hat{\theta}_2$ vs. θ_2.

Fig. 5: Parameter estimates (red dotted) vs. true parameters (solid black).

the system variables are related through spatial derivatives which are distributed in space. The resulting condition on the adaptive law involved finding a non-linear operator satisfying a condition given in terms of the Fréchet derivative of the operator. In contrast to the ODE case, we were only able to find an explicit solution for the trivial case where the source terms are constant. This forced us, at the expense of increased computational complexity, to study the incremental value of the operator between two arbitrary states. The resulting method is applicable to a large class of systems without relying on any problem specific solution method. Furthermore, for some special classes of source terms, the path integral is separable into a constant part and a time-varying part independent of the along-path variable, which simplifies the on-line computation.

As mentioned, we are restricted to systems without any parametric uncertainties in the dynamics governing the measured state variable. An area of further work is to generalize the design to systems with both uncertain parameters and the unknown state variable appearing in both equations. Possible extensions also include a generalization to $m+n$ systems, or a redesign using a more general coefficient matrix with measurements taken as linear combinations of the system states. The parameter convergence properties of the proposed method should also be investigated further.

APPENDIX

A. Convergence lemma

Lemma 5 (Lemma 3.1 from [14]): Let g be a real valued function defined for $t \geq 0$. Suppose:

1) $g(t) \geq 0$ for all $t \in [0, \infty)$,
2) $g(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that $\dot{g}(t) \leq M$, for all $t \geq 0$,
3) $g \in \mathscr{L}_1$.

Then

$$\lim_{t \to \infty} g(t) = 0. \tag{52}$$

REFERENCES

[1] G. Bastin and J.-M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*. Springer, 2016, vol. 88.

[2] R. Vazquez, M. Krstic, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, Dec. 2011, pp. 4937–4942.

[3] H. Anfinsen, M. Diagne, O. M. Aamo, and M. Krstic, "An adaptive observer design for $n+1$ coupled linear hyperbolic PDEs based on swapping," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3979–3990, Dec. 2016.

[4] H. Anfinsen and O. M. Aamo, "Boundary parameter and state estimation in 2×2 linear hyperbolic PDEs using adaptive backstepping," in *Proceedings of the 55th Conference on Decision and Control*, 2016, pp. 2054–2060.

[5] ——, "Adaptive output-feedback stabilization of linear 2×2 hyperbolic systems using anti-collocated sensing and control," *Systems & Control Letters*, vol. 104, pp. 86–94, 2017.

[6] P. Bernard and M. Krstic, "Adaptive output-feedback stabilization of non-local hyperbolic PDEs," *Automatica*, vol. 50, no. 10, pp. 2692–2699, Oct. 2014.

[7] R. Marino and P. Tomei, "Global adaptive observers for nonlinear systems via filtered transformations," *IEEE Transactions on Automatic Control*, vol. 37, no. 8, pp. 1239–1245, Aug. 1992.

[8] ——, *Nonlinear control design: geometric, adaptive and robust*. Prentice Hall London, 1995, vol. 1.

[9] G. Besanon, Q. Zhang, and H. Hammouri, "High-gain observer based state and parameter estimation in nonlinear systems," in *Proceedings of the 6th IFAC Symposium on Nonlinear Control Systems 2004*, vol. 37, no. 13, 2004, pp. 327 – 332.

[10] M. Arcak and P. Kokotovic, "Nonlinear observers: a circle criterion design," in *Proceedings of the 38th IEEE Conference on Decision and Control*, vol. 5. IEEE, Dec. 1999, pp. 4872–4876.

[11] Ø. N. Stamnes, J. Zhou, O. M. Aamo, and G. Kaasa, "Adaptive observer design for nonlinear systems with parametric uncertainties in unmeasured state dynamics," in *Proceedings of the 48h IEEE Conference on Decision and Control*, Dec. 2009, pp. 4414–4419.

[12] Ø. N. Stamnes, O. M. Aamo, and G. Kaasa, "Adaptive redesign of nonlinear observers," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1152–1157, May 2011.

[13] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*. Springer-Verlag, Berlin,, 2010.

[14] W.-J. Liu and M. Krstic, "Adaptive control of burgers' equation with unknown viscosity," *International Journal of Adaptive Control and Signal Processing*, vol. 15, no. 7, pp. 745–766, 2001.