FLAT BUNDLES OVER SOME COMPACT COMPLEX MANIFOLDS

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Abstract. We construct examples of flat fiber bundles over the Hopf surface such that the total spaces have no pseudoconvex neighborhood basis, admit a complete Kähler metric, or are hyperconvex but have no nonconstant holomorphic functions. For any compact Riemannian surface of positive genus, we construct a flat $\mathbb{P}^1$ bundle over it and a Stein domain with real analytic bundary in it whose closure does not have pseudoconvex neighborhood basis. For a compact complex manifold with positive first Betti number, we construct a flat bundle over it such that the total space is hyperconvex but admits no nonconstant holomorphic functions.

1. Introduction

The aim of the present note is to construct flat bundles over some compact complex manifolds such that the total spaces can be used as examples for some problems in several complex variables.

It is known that there exists a bounded pseudoconvex domain D in $\mathbb{C}^n$ ($n > 1$) with smooth boundary such that its closure has no pseudoconvex (or Stein) neighborhood basis [2]. But if a bounded pseudoconvex domain in $\mathbb{C}^n$ has real analytic boundary, its closure has a pseudoconvex neighborhood basis [3]. So it is natural to ask whether the closure of a bounded pseudoconvex or Stein domain with real analytic boundary in a complex manifold has apseudoconvex neighborhood basis. In [4], Diederich and Fornæss constructed a domain Ω with real analytic Levi-flat boundary in the total space B of a flat $\mathbb{P}^1$ bundle over a Hopf surface such that its closure does not have a pseudoconvex neighborhood basis.

The domain Ω mentioned in the previous paragraph is a flat disc bundle over the Hopf surface. Note that the Hopf surface is not Kähler. It is natural to ask whether Ω admits a Kähler metric. It is observed in [4] that Ω is biholomorphic to the product $A \times \mathbb{C}^2\backslash\{0\}$, where A is an annulus. By a basic result from complex geometry, Ω admits a complete Kähler metric.

In this note, inspired by the work in [4], we present a general framework to construct flat fiber bundles over complex manifolds, based on discrete group actions. We then show some basic properties of these bundles and develop an idea of duality, with the product structure of Ω mentioned above as a special case. As one of the applications, we construct a flat $\mathbb{C}^*$-bundle over the Hopf surface such that the total space has a complete Kähler metric.

In the above example, B is not projective and Ω is not Stein. For any compact Riemannian surface of positive genus, we will construct a flat $\mathbb{P}^1$ bundle over it such that the total space is a projective manifold, and a Stein domain with real analytic boundary in the total space whose closure does not have a pseudoconvex neighborhood basis.

Finally, we will construct a hyperconvex complex manifold which does not admit any nonconstant holomorphic function. More precisely, for any compact complex manifold X with positive first Betti number and any $n \geq 1$, we construct a flat B^n-bundle over X such that the total space Ω admits a real analytic bounded exhaustive plurisubharmonic function ρ whose complex Hessian has n-positive eigenvalues, but Ω admits no nonconstant holomorphic function, where B^n is the unit ball in $\mathbb{C}^n$.

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2. The general framework

In the section, we set up a general framework for constructing flat fiber bundles, and prove a result about the existence of complete Kähler metrics that will be used a few times later.

2.1. **Construction of flat bundles.** Let $\tilde{X}$ be a complex manifold. Let $\Gamma \subset Aut(\tilde{X})$ be a discrete subgroup in the automorphism group $Aut(\tilde{X})$ of $\tilde{X}$ that acts freely on $\tilde{X}$. Then $X := \tilde{X}/\Gamma$ is a complex manifold. Let F be another complex manifold. Let $\sigma : \Gamma \to Aut(F)$ be a group morphism. We can construct a flat fiber bundle $p : B \to X$ with fiber F as follows, where flatness means that the transition functions are locally constant. First let $\tilde{B} = \tilde{X} \times F$ be the trivial bundle over $\tilde{X}$ and $\tilde{p} : \tilde{B} \to \tilde{X}$ be the natural projection. Then we can define an action of Γ on $\tilde{B}$ as follows:

$$\gamma \cdot (z, v) = (\gamma z, \sigma(\gamma)v), \ (z, v) \in \tilde{X} \times F.$$

It is obvious that the action of Γ on $\tilde{B}$ is free and properly discontinuous. The quotient $B := \tilde{B}/\Gamma$ is a complex manifold and gives a flat fiber bundle over X with fibers biholomorphic to F. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{\pi}} & B \\ \Big\downarrow{\scriptstyle \tilde{p}} & & \Big\downarrow{\scriptstyle p} \\ \tilde{X} & \xrightarrow{\pi} & X \end{array},$$

where $\tilde{\pi}$ is the quotient map, and p is defined as $p(\tilde{\pi}(z, v)) = \pi(z)$.

Let $v \in F$ and let $s_v : \tilde{X} \to \tilde{B}$ be the constant section given by $s_v(z) = (z, v)$ for all $z \in \tilde{X}$. We consider the map $\tilde{\pi} \circ s_v : \tilde{X} \to B$ and let $X_v := \tilde{\pi} \circ s_v(\tilde{X}) \subset B$. Note also that there is an action of Γ on F induced by $\sigma : \Gamma \to Aut(F)$. We have the following:

Lemma 2.1. *With the above notations, we have*

1. *If the action of* Γ *on* F *is free, then the map* $\tilde{\pi} \circ s_v : \tilde{X} \to B$ *is injective;*
2. *If the action of* Γ *on* F *is properly discontinuous, then the map* $\tilde{\pi} \circ s_v : \tilde{X} \to B$ *is proper;*
3. *If the action of* Γ *on* F *is free and properly discontinuous, then the map* $\tilde{\pi} \circ s_v : \tilde{X} \to B$ *is a proper holomorphic embedding;*
4. *If* $v \in F$ *is fixed by* Γ, *then* X_v *is biholomorphic to* X*;*
5. *For any* $v, w \in F$, *if* $X_v \cap X_w \neq \emptyset$, *we have* $X_v = X_w$*; and* $X_v = X_w$ *if and only if* $w = \sigma(\gamma)v$ *for some* $\gamma \in \Gamma$.

Proof. For simplicity, we denote $\tilde{\pi}(z,v)$ by $[z,v]$ for $(z,v)\in\tilde{B}$.

(1) if $[z,v]=[z',v]$, there is some $\gamma\in\Gamma$ such that $z'=\gamma z$ and $v=\sigma(\gamma)v$. We then have $\gamma=Id$ and hence $z=z'$ since the action of Γ on F is free. So $\tilde{\pi}\circ s_v$ is injective.

(2) Recall that the action of Γ on F is called properly discontinuous if for any compact sets $K,L\subset F$, the set $\{\gamma\in\Gamma;\gamma K\cap L\neq\emptyset\}$ is always finite. we need to prove that if $z_n\to\infty$ in $\tilde{X}$, then $[z_n,v]\to\infty$ in B. If it is not the case, then we can find a sequence $\{z_n\}$ in $\tilde{X}$ such that $z_n\to\infty$ but $\{[z_n,v]\}$ lies in some compact set K of B. Let $K_1\subset\tilde{X},K_2\subset F$ be compact sets such that $K\subset\tilde{\pi}(K_1\times K_2)$. Then for each z_n, we can find $\gamma_n\in\Gamma$ such that $\gamma_n(z_n)\in K_1,\sigma(\gamma_n)(v)\in K_2$. From the first inclusion, we see that $\{\gamma_n;n\geq 1\}$ is infinite; but from the second inclusion, we see that $\{\gamma_n;n\geq 1\}$ is a finite set since the action of Γ on F is properly discontinuously. Contradiction.

(3) this is a direct consequence of (1) and (2).

(4) this is obvious.

(5) assume $X_v\cap X_w\neq\emptyset$, then we have $[z_1,v]=[z_2,w]$ for some $z_1,z_2\in\tilde{X}$. this implies $(z_2,w)=(\gamma z_1,\sigma(\gamma)v)$ for some $\gamma\in\Gamma$. So for any $z\in\tilde{X}$, we have $[z,v]=[\gamma z,\sigma(\gamma)v]=[\gamma z,w]\in X_w$. Hence we have $X_v\subset X_w$. We can prove $X_w\subset X_v$ in the same way. The second statement in (5) is obvious.

□

From (5) in the above lemma, we see that B is foliated by X_v for $v\in F$, and the parameter space of the leaves is naturally identified to the quotient space F/Γ.

The above construction can be extended to some continuous group actions, namely, replacing Γ by some continuous subgroup of $Aut(\tilde{X})$. But we will not carry out the details of this case in this note.

2.2. **Existence of Kähler metrics.** We now prove a result about the existence of Kähler metrics that will be used several times later.

Proposition 2.2. *Let $p:X\to T$ be a holomorphic submersion, where X and T are compact complex manifolds. Assume that there is a holomorphic line bundle L over X such that $L|_{X_t=p^{-1}(t)}$ is ample for all $t\in T$. Then X is Kähler if T is Kähler , and X is projective if T is projective.*

We first proof a Lemma.

Lemma 2.3. *Let X,T,L as in Proposition 2.2. Then there is a Hermitian metric h on L such that $Ric(L,h)|_{X_t}$ is strictly positive for all $t\in T$, where $Ric(L,h)$ is the Ricci curvature form of L with respect to h.*

Proof. Replacing L by some powers of it, we may assume for simplicity that L is very ample on each fiber X_t.

Instead of considering L itself, we will consider the dual bundle L^* of L. Fix an arbitrary point $t_0\in T$, there is a neighborhood U of t_0 in T such that there is a fiber preserving diffeomorphism

$$\phi:U\times X_{t_0}\to p^{-1}(U)$$

such that ϕ is the identity on X_{t_0}.

Let $\tilde{L} = \phi^{-1}L|_{p^{-1}(U)}$, then $\tilde{L}$ is a smooth complex line bundle over $U \times X_{t_0}$ and $\tilde{L}|_{X_{t_0}} = L|_{X_{t_0}}$. Since $L|_{X_{t_0}}$ is very ample, there is a holomorphic map $g : X_{t_0} \to \mathbb{P}^\infty$ such that $L|_{X_{t_0}} = g^{-1}O(1)$, where $O(1)$ is the dual of the tautological line bundle $O(-1)$ over $\mathbb{P}^\infty$. So $\tilde{L}^*|_{X_{t_0}} = g^{-1}O(-1)$.

We know assume that U is diffeomorphic to a ball. It is known from topology that $\tilde{L}^*$ is isomorphic as topological complex line bundles to $G^{-1}O(-1)$ for some smooth map $G : U \times X_{t_0} \to \mathbb{P}^\infty$. The restriction $G|_{X_{t_0}}$ and g are homotopic. So G is homotopic to the map $g \circ p_1 : U \times X_{t_0} \to \mathbb{P}^\infty$, where $p_1 : U \times X_{t_0} \to X_{t_0}$ is the natural projection. It follows that $\tilde{L}^*$ is isomorphic to $p_1^{-1}\tilde{L}^*|_{X_{t_0}} = p_1^{-1}L^*|_{X_{t_0}}$ as smooth complex line bundles.

There is a canonical Hermitian metric h_0 on $O(-1)$ whose curvature form is strictly negative. This induces a smooth Hermitian metric say h_U^* on $L^*|_{p^{-1}(U)}$. The curvature form of $(L^*|_{p^{-1}(U)}, h_U^*)$ is strictly negative on X_{t_0}. By continuity and by contracting U if necessary, we see that the curvature form of $(L^*|_{p^{-1}(U)}, h_U^*)$ is strictly negative on X_t for all $t \in U$.

So there is a finite open cover $\{U_\alpha\}$ of T and Hermitian metrics h_α^*'s on $L^*|_{U_\alpha}$ for each α such that $Ric(L^*|_{U_\alpha}, h_\alpha^*)$ is strictly negative on X_t for all $t \in U_\alpha$. Let $\{\rho_\alpha\}$ be a partition of unity of T with respect to the open cover $\{U_\alpha\}$, then $h^* = \sum_\alpha h_\alpha^*$ is a Hermitian metric on L^* such that $Ric(L^*, h^*)$ is strictly negative along X_t for all $t \in T$. Let h be the metric on L dual to h^*, then $Ric(L, h)$ is strictly positive along X_t for all $t \in T$. □

We now give the proof of Proposition 2.2.

Proof. We give the proof that X is projective if T is projective. The proof of the first statement is similar.

By Lemma 2.3, there is a Hermitian metric h on L such that the curvature form $Ric(L, h)$ is positive on $X_t = X_{p^{-1}(t)}$ for all $t \in T$.

Let L_0 be a positive line bundle over T with a Hermitian metric h_0 such that $Ric(L_0, h_0)$ is positive. Then $Ric(L, h) + Np^*Ric(L_0, h_0)$ is positive for $N >> 1$. Let $\tilde{L}_0 = p^{-1}L_0$ be the pull back of L_0, which is a line bundle over X. Then $h + Np^*h_0$ is a Hermitian metric on $L + Np^{-1}L_0$ whose curvature form is $Ric(L, h) + Np^*Ric(L_0, h_0)$. It follows that $L + Np^{-1}L_0$ is a positive line bundle over X. By Kodaria's embedding theorem, X is a projective manifold. □

Let X be a compact complex manifold. Let $p : E \to X$ be a holomorphic vector bundle of rank r over X. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X such that $E|_{U_\alpha}$ is trivial for all α. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \subset Aut(\mathbb{C}^r)$ be the transition functions of E. Then E can be constructed by gluing all $U_\alpha \times \mathbb{C}^r$ via the transition functions $g_{\alpha\beta}$. We now view $\mathbb{C}^r$ as a subset of the projective space $\mathbb{P}^r$. Then we have a natural inclusion from $GL(r, \mathbb{C})$ to $Aut(\mathbb{P}^r) = PGL(r + 1, \mathbb{C})$. So the transition function $g_{\alpha\beta}$ of E induce a fiber bundle E_∞ over X with fibers biholomorphic to $\mathbb{P}^r$. We have a natural inclusion $E \subset E_\infty$.

Let $D = E_\infty \backslash E$. Then D is a divisor of E_∞. Let L be the line bundle over E_∞ associated to the divisor D. Then the restriction of L on each fiber of $E_\infty \to X$ is positive. By Proposition 2.2, we have the following

Lemma 2.4. *If X is a Kähler manifold, then E_∞ is a Kähler manifold; if X is a projective manifold, then E_∞ is a projective manifold.*

Another result that will be used repeatedly is the following

Lemma 2.5. *Let X be a compact Kähler manifold or a Stein manifold and $A \subset X$ be a closed analytic subset. Then $X \backslash A$ admits a complete Kähler metric.*

For the proof of Lemma 2.5, see [5].

3. Flat bundles over the Hopf surface

3.1. **Disc bundle over the Hopf surface I.** We denote $\mathbb{C}^2 \setminus \{0\}$ by $\mathbb{C}^2_*$. Let $\gamma_n \in Aut(\mathbb{C}^2_*)$ be the map given by $z \to 2^n z$, and let $\Gamma = \{\gamma_n; n \in \mathbb{Z}\}$. Then Γ acts freely and discontinuously on $\mathbb{C}^2_*$. The quotient space $H = \mathbb{C}^2_*/\Gamma$ is called the Hopf surface, which is a compact complex manifold of dimension 2. Since $\pi_1(H) \approx \Gamma \approx \mathbb{Z}$ is commutative, the first homology group of H is $H_1(H, \mathbb{Z}) = \pi_1(H)/[\pi_1(H), \pi_1(H)] = \pi_1(H) \approx \mathbb{Z}$. So by Hodge theory for compact Kähler manifolds, we know that H does not admit any Kähler metric.

Let $\phi \in Aut(\Delta)$ be an automorphism of the unit disc Δ given by $\phi(v) = \frac{v+1/2}{1+v/2}$, and let $\Gamma' = \{\phi^n; n \in \mathbb{Z}\}$. Then Γ' also acts freely and properly discontinuously on Δ. The map $\sigma : \Gamma \to \Gamma'$ given by $\gamma_n \mapsto \phi^n$ is obviously a group isomorphism. Since elements in Γ' are fractional transformations on the projective line $\mathbb{P}^1$, we can also view Γ' as a subgroup of $Aut(\mathbb{P}^1)$.

Now we apply the construction in §3.2 to our special case by setting $\tilde{X} = \mathbb{C}^2_*$, $X = H$, and $F = \mathbb{P}^1$. Let $\tilde{B} = \mathbb{C}^2_* \times \mathbb{P}^1$ be the trivial $\mathbb{P}^1$ bundle over $\mathbb{C}^2_*$. Then the action of Γ on $\mathbb{C}^2_*$ lifts to an action on $\tilde{B}$ as follows:

$$\gamma_n \cdot (z, v) = (2^n z, \phi^n(v)),\ (z, v) \in \mathbb{C}^2_* \times \mathbb{P}^1.$$

Let $B = \tilde{B}/\Gamma$ and let $\tilde{\pi} : \tilde{B} \to B$ be the quotient map. For simplicity, we denote $\tilde{\pi}(z, v)$ by $[z, v]$ for $(z, v) \in \tilde{B}$. Then the map $p : B \to H$ given by $[z, v] \mapsto \pi(z)$ realize B as a flat $\mathbb{P}^1$ bundle over H, where $\pi : \mathbb{C}^2_* \to H$ is the quotient map. In summery, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{\pi}} & B \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \mathbb{C}^2_* & \xrightarrow{\pi} & H \end{array} .$$

We decompose $\mathbb{P}^1$ into three Γ'-invariant pieces Δ, Δ', S^1, where S^1 is the unit circle, and Δ' is the complement of the closed unit disc. It is clear that Γ' acts on Δ and Δ' freely and properly discontinuously. The action of Γ' on $S^1 \backslash \{\pm 1\}$ is also free and properly discontinuous. We have $\phi(\pm 1) = \pm 1$, and for any $v \in \mathbb{P}^1 \backslash \{\pm 1\}$, $\lim_{n \to +\infty} \phi^n(v) = +1$, $\lim_{n \to -\infty} \phi^n(v) = -1$.

Let $\tilde{\Omega} = \mathbb{C}^2_* \times \Delta, \tilde{\Omega}' = \mathbb{C}^2_* \times \Delta', \tilde{M} = \mathbb{C}^2_* \times S^1$. Then $\tilde{\Omega}, \tilde{\Omega}'$ are Γ-invariant open subsets of $\tilde{B}$, and $\tilde{M}$ is a Γ-invariant Levi-flat hypersurface of $\tilde{B}$. Let $\Omega = \tilde{\Omega}/\Gamma, \Omega' = \tilde{\Omega}'/\Gamma, M = \tilde{M}/\Gamma$, which are subsets of $B = \tilde{B}/\Gamma$. Both Ω and Ω' are flat disc bundles over H, and M is a flat circle bundle over H.

For $v \in \mathbb{P}^1$, as in the previous section, let $s_v : \mathbb{C}^2_* \to \tilde{B}$ be the constant section given by $s_v(z) = (z, v)$, and denote by $X_v \subset B$ the image of $\tilde{\pi} \circ s_v$. It is clear that $X_v \subset \Omega, \Omega', \Omega_0$ if $v \in \Delta, \Delta', S^1$ respectively. By Lemma 2.1, for $v \in \Delta,\ (\Delta')$, the map $\tilde{\pi} \circ s_v : \mathbb{C}^2_* \to \Omega$ $(\tilde{\pi} \circ s_v : \mathbb{C}^2_* \to \Omega')$ is a proper holomorphic embedding. We have the following

Theorem 3.1 ([4]). *Ω is a pseudoconvex domain in B. But there does not exist an increasing sequence of relatively compact pseudoconvex domains Ω_n in Ω such that $\cup\Omega_n = \Omega$.*

Proof. The boundary of Ω in B is M, which is a real analytic compact Levi-Flat hypersurface in B. So Ω is a pseudoconvex domain in B. If there is an increasing sequence of relatively compact pseudoconvex domains Ω_n such that $\cup\Omega_n = \Omega$, then $\Omega_n \cap X_0$ is an increasing sequence of relatively compact pseudoconvex domains in X_0 such that $\cup(\Omega_n \cap X_0) = X_0$. Note that since X_0 is biholomorphic to $\mathbb{C}^2_*$, this is impossible by Hartogs extension theorem. □

For $v = \pm 1$, it is clear that $X_v \subset M$ are biholomorphic to the Hopf surface H. For $v \in \mathbb{P}^1 \backslash \{\pm 1\}$, by the property of the action of Γ' on $\mathbb{P}^1$, we see that $\tilde{\pi} \circ s_v(z)$ approaches X_{-1} as $z \to \infty$, and $\tilde{\pi} \circ s_v(z)$ approaches X_1 as $z \to 0$.

Theorem 3.2 ([4]). *The closure $\overline{\Omega} = \Omega \cup M$ of Ω in B has no pseudoconvex neighborhood basis. More precisely, there does not exist a decreasing sequence of pseudoconvex domains Ω_n in B such that $\cap\Omega_n = \overline{\Omega}$.*

Proof. We argue by contradiction. Suppose that Ω_n is a decreasing sequence of pseudoconvex domains Ω_n in B such that $\cap\Omega_n = \overline{\Omega}$. By Lemma 2.1, $\tilde{\pi} \circ s_\infty : \mathbb{C}^2_* \to \Omega'$ is a proper holomorphic embedding. We identify X_∞ with $\mathbb{C}^2_*$ by identifying $[z, \infty]$ with z. Since $\tilde{\pi} \circ s_\infty(z) \to X_1 \subset M$ as $z \to \infty$, $\Omega_n \cap X_\infty$ is a pseudoconvex domain in $\mathbb{C}^2_*$ that contains a neighborhood of $\infty \in \mathbb{C}^2$. However, by Hartogs extension theorem, a pseudoconvex domain in $\mathbb{C}^2$ that contains a neighborhood of ∞ has to be equal to the whole $\mathbb{C}^2$. Contradiction. □

Let $\Delta_r = \{v \in \mathbb{C}; |v| < r\}$. One may wonder whether $\Omega_r := \tilde{\pi}(\mathbb{C}^2_* \times \Delta_r)(r > 1)$ is a pseudoconvex neighborhood basis of Ω. However, from Lemma 2.1, it is easy to see that $\Omega_r = B$ for any $r > 1$.

3.2. **Disc bundle over the Hopf surface II.** We will follow the notations in §3.1. As we mentioned, the Hopf surface H does not admit any Kähler metric. So it is very natural to ask whether the flat disc bundle Ω constructed in §3.1 admits a Kähler metric. In this subsection, we will prove that Ω admits a complete Kähler metric. The essential observation here is that Ω is biholomorphic to a product. The product structure of Ω was observed and stated as a remark in [4]. We will provide the details of its proof here, by developing an idea of duality.

We have seen that Ω is foliated by closed submanifolds X_v that are all biholomorphic to $\mathbb{C}^2_*$ with $v \in \Delta$, and X_v and X_w are equal if and only if v, w lie in the same orbit with respect to the action of Γ' on Δ. However, just from this fact, it is not obvious whether Ω is biholomorphic to a $\mathbb{C}^2_*$ bundle over Δ/Γ'. But we will show that it is indeed the case.

Note that Γ' acts on Δ freely and properly discontinuously. The quotient space $A = \Delta/\Gamma'$ is a Riemann surface which is clearly not compact.

Lemma 3.3. *The quotient space A is holomorphically equivalent to an annulus $A_r := \{v \in \mathbb{C}; r < |v| < 1\}$ for some $r \in (0, 1)$.*

Proof. Note that the fundamental group of A is isomorphic to Γ', which is commutative. By Theorem IV.6.8 in [6], A is biholomorphic to some A_r or the punctured disc Δ^*. Note that the automorphism group of Δ^* is just the circle group, and the automorphism group of A_r have two connected components. Let $G = \{\phi_r(z) =$

$\frac{z+r}{1+rz}; r \in (-1,1)\} \subset Aut(\Delta)$. Then G/Γ' is naturally identified to a subgroup of $Aut(A)$ which is clearly isomorphic to S^1. Let $\tau \in Aut(\Delta)$ be given by $\tau(z) = -z$. Then we have $\tau\phi = -\phi^{-1}$. So τ also induces an automorphism of A, which is also denoted by τ. It is easy to check that $\phi_r \circ \tau^{-1} = \phi_r \circ \tau = -\phi_{-r} \notin G$ for any $r \neq 0$. So $\tau \notin G/\Gamma'$, which implies that $Aut(A)$ is not connected. So A has to be biholomorphic to some A_r. □

Recall the construction of Ω. We have a group Γ acting freely and properly discontinuously on $\mathbb{C}^2_*$, and a group Γ' acting freely and properly discontinuously on Δ. The map $\sigma : \Gamma \to \Gamma'$ given by $\gamma_n \mapsto \phi^n$ is a group isomorphism and induces an action of Γ on Δ, and further induces an action of Γ on $\tilde{\Omega} = \mathbb{C}^2_* \times \Delta$. Then $\Omega := \tilde{\Omega}/\Gamma$ is naturally realized as a flat Δ-bundle over the Hopf surface $H = \mathbb{C}^2_*/\Gamma$.

Now the key observation is that Ω can also be constructed by revising the roles of $\mathbb{C}^2_*$ and Δ, and the roles of Γ and Γ'. Note that the group isomorphism $\sigma^{-1} : \Gamma' \to \Gamma$ induces an action of Γ' on $\mathbb{C}^2_*$ and further induces an action of Γ' on $\Delta \times \mathbb{C}^2_*$, which is of course can be identified to $\mathbb{C}^2_* \times \Delta$. Now it is obvious that Ω can be naturally identified to $\Delta \times \mathbb{C}^2_*/\Gamma'$. But now by the construction in §, Ω is realized as a flat $\mathbb{C}^2_*$-bundle over $A = \Delta/\Gamma'$. Since the action of Γ on $\mathbb{C}^2_*$ is linear, Ω can be extended to a rank two holomorphic vector bundle, say E, over A by adding the zero section E_0 to Ω.

We now apply the famous Oka-Grauert principle to E, which states that E is trivial as a holomorphic vector bundle if and only if it is trivial as a complex topological bundle (see [7]). Note that the isomorphism class of a complex topological bundle of rank 2 over A is determined by a homotopic class of continuous maps from A to the classifying space $BU(2) = G_2(\mathbb{C}^\infty)$ (see Theorem 23.10 in [1]). Note that $BU(2)$ is simply connected. Since A is biholomorphic to some A_r by Lemma 3.3, all continuous maps from A to $BU(2)$ are homotopically trivial. So all complex topological vector bundle over A are trivial. Hence E is a holomorphically trivial vector bundle over A. We get the following result, which is stated in [4] (Remarks. c) in [4]).

Theorem 3.4. *Ω is biholomorphic to the product manifold $A \times \mathbb{C}^2_*$.*

Note that $A \times \mathbb{C}^2$ is a Stein manifold, by Lemma 2.5, we get

Theorem 3.5. *There is a complete Kähler metric on Ω.*

The $\mathbb{P}^1$-bundle B over H has no Kähler metric since it contains the compact submanifolds $X_{\pm 1}$, which are biholomorphic to the Hopf surface. Furthermore it seems that B is not bimeromorphic to a compact Kähler manifold. It is also unknown whether there exists a fiber bundle over H with compact fibers such that the total space is Kähler.

3.3. **Disc bundle over the Hopf surface III.** In this subsection, we construct a disc bundle over the Hopf surface H such that the total space has a real analytic exhaustive bounded plurisubharmonic function but has no nonconstant holomorphic functions. We will follow the general idea discussed in §3.2.

We again denote $\mathbb{C}^2\backslash\{0\}$ by $\mathbb{C}^2_*$, and let $\Gamma = \{\gamma_n : z \mapsto 2^n z; n \in \mathbb{Z}\} \subset Aut(\mathbb{C}^2_*)$. The quotient space $H = \mathbb{C}^2_*/\Gamma$ is the Hopf surface.

Fix an irrational real number λ and let $\alpha = e^{\lambda 2\pi i}$. Then $\phi(v) = \alpha v$ is a rotation of $\mathbb{P}^1$ and let $\Gamma' = \{\phi^n; n \in \mathbb{Z}\}$. Note that Γ' can also be viewed as an automorphism group of any disc centered at the origin.

The map $\sigma : \Gamma \to \Gamma'$; $\gamma_n \mapsto \phi^n$ is a group isomorphism, and induces an action of Γ on $\mathbb{P}^1$, and further induces an action of Γ on the product $\tilde{B} = \mathbb{C}^2_* \times \mathbb{P}^1$. Let $B = \tilde{B}/\Gamma$, which is a flat $\mathbb{P}^1$-bundle over H. Let $\tilde{\Omega} = \mathbb{C}^2_* \times \Delta$ and let $\Omega = \tilde{\Omega}/\Gamma$, then Ω is a disc bundle over H and is a domain in B with real analytic pseudoconvex boundary. Our aim is to prove the following

Proposition 3.6. *There is a real analytic bounded exhaustive plurisubharmonic function on Ω; but there is no nonconstant holomorphic function on Ω.*

Proof. Let $\tilde{\rho} : \tilde{\Omega} \to \mathbb{R}$ be the function given by $(z, v) \mapsto |v|^2$. It is clear that $\tilde{\rho}$ is a Γ-invariant p.s.h function on $\tilde{\Omega}$. So it induces a p.s.h function on Ω which is exhaustive on Ω.

For the second statement, assume f is a holomorphic function on Ω. Let $\tilde{f}$ be the pull back of f on $\tilde{\Omega} = \mathbb{C}^2_* \times \Delta$. Then $\tilde{f}$ is invariant under the action of Γ. More precisely, for all $(z, v) \in \mathbb{C}^2_* \times \Delta$, for any $n \in \mathbb{Z}$, we have $\tilde{f}(2^n z, \alpha^n v) = \tilde{f}(z, v)$.

By Riemann's removable singularity theorem, $\tilde{f}$ can be extended to a holomorphic function on $\mathbb{C}^2 \times \Delta$. We expand $\tilde{f}$ near (0,0,0) as

$$\tilde{f}(z_1, z_2, v) = \sum_{i,j,k \geq 0} a_{ijk} z_1^i z_2^j v^k.$$

Then we have

$$\tilde{f}(2^n z_1, 2^n z_2, \alpha^n v) = \sum_{i,j,k \geq 0} a_{ijk} 2^{n(i+j)} \alpha^{nk} z_1^i z_2^j vk.$$

From the invariance of $\tilde{f}$, we get

$$a_{ijk} = a_{ijk} 2^{n(i+j)} \alpha^{nk}, \text{ for all } i, j, k \geq 0,\ n \in \mathbb{Z}.$$

Since α is irrational, we get $a_{ijk} = 0$ if $i + j + k > 0$, and hence $\tilde{f}$ is constant. So f is constant. □

3.4. **$\mathbb{C}^*$-bundle over the Hopf surface.** In this subsection, we construct a flat $\mathbb{C}^*$-bundle over the Hopf surface H such that the total space has a complete Kähler metric.

Recall that $\Gamma = \{\gamma_n : z \mapsto 2^n z; n \in \mathbb{Z}\}$ acts freely and properly discontinuously on $\mathbb{C}^2_*$. We define an action of Γ on $\mathbb{P}^1$ by associating γ_n to the map $v \mapsto 2^n v, v \in \mathbb{P}^1$. Then $\mathbb{C}^* \subset \mathbb{P}^1$ is a Γ-invariant open subset and Γ acts on $\mathbb{C}^*$ freely and properly discontinuously. The quotient space $S = \mathbb{C}^*/\Gamma$ is an elliptic curve.

We can now define an action of Γ on $\tilde{B} = \mathbb{C}^2_* \times \mathbb{P}^1$ by acting on both factors. Let $B = \tilde{B}/\Gamma$ be the quotient space. Let $\tilde{\Omega} = \mathbb{C}^2_* \times \mathbb{C}^*$ and $\Omega = \tilde{\Omega}/\Gamma$. Similarly to the consideration in §3.2, we can look at Ω in two ways. One is to view it as a $\mathbb{C}^*$-bundle over the Hopf surface H, and the other is to view it as a $\mathbb{C}^2_*$ bundle over S. If taking the second viewpoint, Ω can be extended to a rank two holomorphic vector bundle say E over S by adding the zero section.

By Lemma 2.4, E_∞ (see §3.2 for definition) is a compact Kähler manifold. Note that $\Omega \subset E_\infty$ is the complement of an analytic set in E_∞. By Lemma 2.5, Ω has a complete Kähler metric.

Theorem 3.7. *The flat $\mathbb{C}^*$-bundle Ω over the Hopf surface H has a complete Kähler metric.*

We take a more careful look at B, which is a compactification of Ω. Ω is a Zariski open set of B and the complement $B\backslash\Omega = X_0 \cup X_\infty$, where $X_0 = \{[z,0]; z \in \mathbb{C}^2_*\}$ and $X_\infty = \{[z,\infty]; z \in \mathbb{C}^2_*\}$ are two copies of the Hopf surface $H = \mathbb{C}^2_*/\Gamma$. On the other hand, by identifying $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$, we can compactify E naturally to a $\mathbb{P}^2$-bundle E_∞ over S as shown in §3.2. E_∞ is a compactification of Ω that is different from B. The complement $E_\infty\backslash\Omega$ of Ω in E_∞ also contains two connected components, namely the zero section E_0 of E and $D_\infty := E_\infty\backslash E$ which is a $\mathbb{P}^1$-bundle over the elliptic curve S.

It is interesting to compare X_∞ and D_∞, which are connected boundary components in the two different compactifications of Ω. As mentioned above, D_∞ is a $\mathbb{P}^1$-bundle over S. Recall that $H = \mathbb{C}^2_*/\Gamma$, so it can be fibred over $\mathbb{C}^2_*/\mathbb{C}^*$ with fibers biholomorphic to $\mathbb{C}^*/\Gamma = S$. So X_∞ is an S-bundle over $\mathbb{P}^1$. So it seems that X_∞ and D_∞ has some duality relation, which deserves a further study.

There is no Kähler metric on B since it contains H as a submanifold. But E_∞ is projective manifold by Lemma 2.4. One can prove that B and E_∞ are not bimeromorphic. In fact, if B can not be bimeromorphic to any Kähler manifold. If this is not true, there will be a Kähler current say T on B. It is then clear that the direct image p_*T is a Kähler current on H, recalling that B is a fiber bundle over H and $p : B \to H$ is the projection. Then p_*T gives a nonzero class in $H^2(H)$. This is a contradiction since the second cohomology group of H vanishes.

4. Flat bundles over compact Riemann surfaces

4.1. **Flat bundles over elliptic curves.** The aim is to construct compact projective manifolds B and some Stein domain Ω in B with real analytic boundary such that the closure of Ω in B has no pseudoconvex neighborhood basis. We will realize B as a flat $\mathbb{P}^1$ bundle over elliptic curves.

Let Γ be a lattice in $\mathbb{C}$, then $\Sigma = \mathbb{C}/\Gamma$ is an elliptic curve. Let $\tilde{\sigma} : \Gamma \to \mathbb{Z}$ be a surjective group morphism. Then $\sigma : \Gamma \to Aut(\mathbb{P}^1)$ given by $\gamma \mapsto \phi^{\tilde{\sigma}(\gamma)}$ is a group morphism, where $\phi(v) = \frac{v+1/2}{1+v/2}$ is an automorphism of $\mathbb{P}^1$. Through σ, Γ acts on $\mathbb{P}^1$ and further acts on $\tilde{B} := \mathbb{C} \times \mathbb{P}^1$ by acting on each factor. The quotient space $B = \tilde{B}/\Gamma$ is a complex manifold, which is a $\mathbb{P}^1$-bundle over Σ. Let $\tilde{\pi} : \tilde{B} \to B$ and $\pi : \mathbb{C} \to \Sigma$ be the quotient maps, and let $\tilde{p} : \tilde{B} \to \mathbb{C}$ and $p : B \to \Sigma$ be the bundle maps. We have $p \circ \tilde{\pi} = \pi \circ \tilde{p}$.

Since the action of Γ on $\mathbb{P}^1$ fixes ± 1, by the same argument as in proof of Lemma 2.4, we can prove that B is a projective manifold.

Let $\Delta' = \mathbb{P}^1\backslash\overline{\Delta}$. Let $\tilde{\Omega} = \mathbb{C} \times \Delta, \tilde{\Omega}' = \mathbb{C} \times \Delta'$. Let $\Omega = \tilde{\Omega}/\Gamma, \Omega' = \tilde{\Omega}'/\Gamma$, which are disc bundles over Σ. The boundary of Ω and Ω' in B are equal and equal to $\mathbb{C} \times S^1/\Gamma$, which is a real analytic Levi-flat hypersurface in B.

The quotient space $\mathbb{C}/\ker\sigma$ is biholomorphic to $\mathbb{C}^*$, and the action of $\Gamma/\ker\sigma$ on $\mathbb{C}/\ker\sigma$ corresponds to multiplications on $\mathbb{C}^*$. By the discussion in §3.2, Ω and Ω' are biholomorphic to $\mathbb{C}^*$ bundles over the annulus $A := \Delta/\sigma(\Gamma)$, whose transition functions are given by multiplications on $\mathbb{C}^*$. So Ω and Ω' can be extended to holomorphic line bundles over A by adding zero sections. By the Oka-Grauert principle and by the same argument as in the proof of Theorem 3.4, both Ω and Ω' are biholomorphic to the product $A \times \mathbb{C}^*$, and hence are Stein.

Theorem 4.1. *The flat $\mathbb{P}^1$-bundle B over Σ is projective. The domain Ω in B is Stein and its boundary is a real analytic Levi-flat hypersurface in B. $\overline{\Omega}$ has no pseudoconvex neighborhood basis in B.*

Proof. The first statement follows from the above discussion. For the last statements, assume D is a small pseudoconvex neighborhood of $\overline{\Omega}$ in B. Then $D \cap \Omega'$ is a pseudoconvex domain in Ω'. By construction, $\Omega' \backslash D$ is a compact set. This is impossible since Ω' is Stein. □

4.2. **Flat bundles over compact Riemann surfaces of positive genus.** Applying Mok's solution of the Serre problem for open Riemann surfaces [9], we can generalize the construction in §4.1 to all compact Riemann surfaces of positive genus. The aim is to construct compact projective manifolds B and some Stein domain Ω in B with real analytic boundary such that the closure of Ω in B has no pseudoconvex neighborhood basis. Here B is realized as a flat $\mathbb{P}^1$-bundle over a compact Riemann surfaces and Ω is the corresponding disc bundle in B.

Let X be a compact Riemann surface with genus $g \geq 1$. Let $\tilde{X}$ be the universal covering of X with $X = \tilde{X}/\Gamma'$, where $\Gamma' \subset Aut(\tilde{X})$ is isomorphic to the fundamental group of X. By the uniformization theorem, $\tilde{X}$ is biholomorphic to $\mathbb{C}$ if $g = 1$ and biholomorphic to Δ if $g > 1$.

It is well known from topology that $H_1(X, \mathbb{Z}) \approx \Gamma'/[\Gamma', \Gamma']$, where $[\Gamma', \Gamma']$ is the normal subgroup of Γ' generated by all elements of the form $aba^{-1}b^{-1}$ with $a, b \in \Gamma'$ (see Theorem 2A.1 in [8]). Note that $H^1(X, \mathbb{Z}) \approx \mathbb{Z}^{2g}$, there is a surjective group morphism $\sigma' : \Gamma' \to \mathbb{Z}$.

Let $D = \tilde{X}/\ker \sigma'$ and let $\Gamma = \Gamma'/\ker \sigma'$. Then D is an open Riemann surface and Γ is isomorphic to $\mathbb{Z}$ which acts freely and properly discontinuously on D. The quotient space D/Γ is just X.

Let γ be a generator of Γ. Let $\sigma : \Gamma \to Aut(\mathbb{P}^1)$ be the group morphism given by $\gamma^n \mapsto \phi^n$, where $\phi(v) = \frac{v+1/2}{1+v/2}$ is an automorphism of $\mathbb{P}^1$. Then σ induces an action of Γ on $\mathbb{P}^1$ and further induces an action of Γ on $\tilde{B} = D \times \mathbb{P}^1$. The quotient $B = \tilde{B}/\Gamma$ if a flat $\mathbb{P}^1$-bundle over X.

Let $\Delta' = \mathbb{P}^1 \backslash \overline{\Delta}$. Let $\tilde{\Omega} = D \times \Delta, \tilde{\Omega}' = D \times \Delta'$. Let $\Omega = \tilde{\Omega}/\Gamma, \Omega' = \tilde{\Omega}'/\Gamma$, which are disc bundles over X. The boundary of Ω and Ω' in B are equal and equal to $D \times S^1/\Gamma$, which is a real analytic Levi-flat hypersurface in B.

Lemma 4.2. *B is a projective manifold.*

Proof. Viewing B as a $\mathbb{P}^1$-bundle over X, the transition functions preserves $1 \in \mathbb{P}^1$. Let L_1 be the line bundle over B corresponding to the divisor $X \times \{1\}$. Then L_1 is positive when restricted to each fiber of $B \to X$. By Proposition 2.2, B is projective. □

Lemma 4.3. *Both Ω and Ω' are Stein manifolds.*

Proof. It is clear that Ω and Ω' are biholomorphic. From the discussion in §3.2, Ω is a flat disc bundle over X. Since $\sigma(\Gamma) = \{\phi^n; n \in \mathbb{Z}\}$ acts freely and properly discontinuously on Δ, Ω can also be realized as a flat D-bundle over $A = \Delta/\Gamma$. We have proven that A is biholomorphic to an annulus in $\mathbb{C}$ in Lemma 3.3. By Mok's solution of the Serre's problem for open Riemann surfaces in [9], Ω is a Stein manifold. □

Theorem 4.4. *The closure $\overline{\Omega}$ of Ω in B has no pseudoconvex neighborhood basis.*

Proof. Assume that W is a small pseudoconvex neighborhood of $\overline{\Omega}$ in B. Then $W \cap \Omega'$ is a pseudoconvex domain in Ω'. By construction, $\Omega' \backslash W$ is a compact set. This is impossible since Ω' is Stein. □

5. Flat bundles over compact manifolds with positive first Betti numbers

In this section, for any compact complex manifold X with positive first Betti number, we construct a flat B^n-bundle over X such that the total space Ω admits a real analytic bounded exhaustive plurisubharmonic function ρ whose complex Hessian has n-positive eigenvalues, but Ω admits no nonconstant holomorphic function, where B^n is the unit ball in $\mathbb{C}^n$. The construction is a generalization of that in §3.3.

We first consider the case that $n = 1$. Let X be a compact complex manifold with first Betti number $b_1(X) \geq 1$. Let $\tilde{X}$ be the universal covering of X with $X = \tilde{X}/\Gamma'$, where $\Gamma' \subset Aut(\tilde{X})$ is isomorphic to the fundamental group of X.

We have $H_1(X, \mathbb{Z}) \approx \Gamma'/[\Gamma', \Gamma']$, where $[\Gamma', \Gamma']$ is the normal subgroup of Γ' generated by all elements of the form $aba^{-1}b^{-1}$ with $a, b \in \Gamma'$ (see Theorem 2A.1 in [8]). Since $b_1(X) \geq 1$, there is a surjective group morphism $\sigma' : \Gamma' \to \mathbb{Z}$.

Let $D = \tilde{X}/\ker \sigma'$ and let $\Gamma = \Gamma'/\ker \sigma'$. Then D is a noncompact manifold and Γ is isomorphic to $\mathbb{Z}$ which acts freely and properly discontinuously on D. The quotient space D/Γ is just X.

Let γ be a generator of Γ. Let $\sigma : \Gamma \to Aut(\mathbb{P}^1)$ be the group morphism given by $\sigma(\gamma)(v) = \alpha v$, where $\alpha = e^{\lambda 2\pi i}$ for a fixed irrational real number λ. Then σ induces an action of Γ on $\mathbb{P}^1$ and further induces an action of Γ on $\tilde{B} = D \times \mathbb{P}^1$. The quotient $B = \tilde{B}/\Gamma$ if a flat $\mathbb{P}^1$-bundle over X.

Let $\tilde{\Omega} = D \times \Delta, \tilde{\Omega}' = D \times \Delta'$. Let $\Omega = \tilde{\Omega}/\Gamma$, which is a disc bundle over X. The aim is to prove the following

Theorem 5.1. *B is a projective manifold if X is projective. There is a real analytic exhaustive plurisubharmonic function on Ω. There is no nonconstant holomorphic function on Ω.*

Proof. The first statement can be proved by the same argument as in the proof of Lemma 4.2.

For the second statement, let $\tilde{\rho}(z, v) = |v|^2$. Then $\tilde{\rho}$ is a real analytic function on $\tilde{\Omega} = D \times \Delta$ which is invariant under the action of Γ, and hence induces a plurisubharmonic function say ρ on Ω, which is obviously exhaustive.

For the last statement, since D/Γ is compact, there is a compact set $F \subset D$ such that $\cup_{n\in\mathbb{Z}} \gamma^n F = D$. Let $D_v = D \times \{v\} \subset \tilde{\Omega}$ for $v \in \Delta$ and $S_r = \{v' \in \Delta; |v'| = r\}$ for $r \in (0, 1)$. Assume f is a holomorphic function on $\Omega = \tilde{\Omega}/\Gamma$. Let $\tilde{f}$ be the pull back of f to $\tilde{\Omega}$. Then $\tilde{f}$ is invariant under the action of Γ, that is, $\tilde{f}(\gamma^n z, \alpha^n v) = \tilde{f}(z, v), (z, v) \in D \times \Delta$ for all $n \in \mathbb{Z}$. It suffices to prove that $\tilde{f}$ is constant.

Let $(z_0, v_0) \in F \times S_r$ such that $\tilde{f}(z_0, v_0) = \sup_{(z,v)\in F\times S_r} |\tilde{f}(z, v)|$. For any $(z, v_0) \in D_{v_0}$, there exists n such that $(\gamma^n z, \alpha^n v_0) \subset F \times S_r$. Note that $\tilde{f}$ is invariant under the action of Γ, $\tilde{f}|_{D_{v_0}}$ attains its maximum at (z_0, v_0). By the maximum principle for holomorphic functions, $\tilde{f}|_{D_{v_0}}$ is constant and identically equals to $\tilde{f}(z_0, v_0)$.

We go further to prove that $\tilde{f}$ is constant on $D \times S_r$. By assumption, we also have $\tilde{f}(z_0, v_0) = \sup_{(z,v)\in D\times S_r} |\tilde{f}(z,v)|$. For any $n \in \mathbb{Z}$, we have $\tilde{f}(\gamma^n z_0, \alpha^n v_0) = \tilde{f}(z_0, v_0)$. So $\tilde{f}|_{D_{\alpha^n v_0}}$ attains its maximum $\tilde{f}(z_0, v_0)$ at $(\gamma^n z_0, \alpha^n v_0)$. Again by the maximum principle, $\tilde{f}|_{D_{\alpha^n v_0}} \equiv \tilde{f}(z_0, v_0)$. So $\tilde{f}$ takes constant value on $\cup_{n\in\mathbb{Z}} D_{\alpha^n v_0}$. Since λ is irrational, $\{\alpha^n v_0; n \in \mathbb{Z}\}$ is dense in S_r. By continuity, $\tilde{f}$ is constant on $D \times S_r$. By the identity theorem for holomorphic functions, $\tilde{f}$ is constant on $D \times \Delta$. □

In the above construction, we can replace Δ by the unit ball B^n of dimension $n \geq 1$, and replace the action of S^1 on Δ by the natural action of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ on B^n. Note that there exist $\alpha_1, \cdots, \alpha_n \in S^1$ such that $\{(\alpha_1^k, \cdots, \alpha_n^k) | k \in \mathbb{Z}\}$ is dense in T^n. By the same argument, we can prove the following theorem which answers affirmatively a problem proposed by professor Xiangyu Zhou (private communication).

Theorem 5.2. *For any compact complex manifold X with positive first Betti number and any $n \geq 1$, there is a flat B^n-bundle Ω over X such that:*

(1) *there is a real analytic bounded exhaustive plurisubharmonic function ρ on Ω such that $\partial\bar{\partial}\rho$ has n-positive eigenvalues;*
(2) *Ω admits no nonconstant holomorphic function.*

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