A domain with non-plurisubharmonic squeezing function

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Abstract. We construct a strictly pseudoconvex domain with smooth boundary whose squeezing function is not plurisubharmonic.

1. Introduction

In this paper we are dealing with the properties of squeezing functions on domains. The idea of using this concept goes back to the papers [LSY1] and [LSY2] where a new notion of holomorphic homogeneous regular domains was introduced. The last kind of domains can be seen as a generalization of Teichmüller spaces, and, as it was shown in [LSY1], [LSY2] and [Ye], they admit many nice geometric and analytic properties.

Motivated by the mentioned above works [LSY1] and [LSY2], Deng, Guan and Zhang in [DGZ1] introduced the notion of squeezing functions defined for arbitrary bounded domains:

Definition. Let Ω be a bounded domain in $\mathbb{C}^n$. For $p \in \Omega$ and a holomorphic embedding $f : \Omega \to \mathbb{B}^n$ satisfying $f(p) = 0$ we set

$$S_\Omega(p, f) := \mathrm{sup}\{\mathrm{r} > 0 : \mathrm{r}\mathbb{B}^\mathrm{n} \subset \mathrm{f}(\Omega)\},$$

and then we set

$$S_\Omega(p) := \mathrm{sup}_\mathrm{f}\{\mathrm{S}_\Omega(\mathrm{p}, \mathrm{f})\},$$

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where the supremum is taken over all holomorphic embeddings $f : \Omega \to \mathbb{B}^n$ with $f(p) = 0$ and $\mathbb{B}^n$ is representing the unit ball in $\mathbb{C}^n$. The function S_Ω is called the squeezing function of Ω.

Properties of the squeezing function for different classes of domains were then studied in [DGZ1], [DGZ2] and [KZ]. Moreover, using the results of [DFW], sharp estimates not only for the squeezing functions, but also for the Carathéodory, Sibony and Azukawa metrics near the boundary of a given strictly pseudoconvex domain were obtained in [FW]. Similar results for the Bergman metric are given in [DF].

On the other hand, in many cases functions which are naturally defined on pseudoconvex domains enjoy plurisubharmonicity properties (see, for example, [Ya] and [B]). That is why a few years ago the following question was raised:

Is it always true that the squeezing function of a strictly pseudoconvex domain with smooth boundary is plurisubharmonic?

The main result of this paper gives a negative answer to the question and can be formulated as follows.

Theorem. *There exists a bounded strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^2$ whose squeezing function is not plurisubharmonic.*

2. Preliminaries

First we briefly recall the definitions of the Kobayashi and Carathéodory metrics. Let Δ denote the unit disc, and let $\mathcal{O}(M, N)$ denote the set of holomorphic maps from M to N. For a domain $\Omega \subset \mathbb{C}^n$ we consider an arbitrary point $p \in \Omega$ and an arbitrary vector $\xi \in \mathbb{C}^n$.

- *Kobayashi metric* $K_\Omega(p, \xi)$. We define

  $K_\Omega(p, \xi) = \inf\{|\alpha|;\ \exists\ \mathrm{f} \in \mathcal{O}(\Delta, \Omega)\ \ \mathrm{f}(0) = \mathrm{p},\ \alpha \mathrm{f}'(0) = \xi\}$.

- *Carathéodory metric* $C_\Omega(p, \xi)$. We define

  $C_\Omega(p, \xi) = \sup\{|\mathrm{f}'(\mathrm{p})(\xi)|;\ \exists\ \mathrm{f} \in \mathcal{O}(\Omega, \Delta)\ \ \mathrm{f}(\mathrm{p}) = 0\}$.

Observe that the above definitions imply directly the next well known properties of metrics.

Monotonicity of Metrics. *Let $\Omega_1 \subset \Omega_2$ be bounded domains in $\mathbb{C}^n$, p be a point in Ω_1 and ξ be an arbitrary vector in $\mathbb{C}^n$. Then the following properties hold true*

$$K_{\Omega_1}(p, \xi) \geq K_{\Omega_2}(p, \xi) \quad \text{and} \qquad C_{\Omega_1}(p, \xi) \geq C_{\Omega_2}(p, \xi).$$

We will also need the following two statements which one easily gets from the definitions (detailed proofs of them can be found in [DGZ1]).

Lemma 1. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then for all $z \in \Omega$ and all $\xi \in \mathbb{C}^n$ one has*

$$S_\Omega(p) K_\Omega(p, \xi) \leq C_\Omega(p, \xi) \leq K_\Omega(p, \xi).$$

Lemma 2. *The squeezing function S_Ω of any bounded domain Ω in $\mathbb{C}^n$ is continuous.*

The last statement implies, in particular, the following property (a slightly weaker result was stated as Theorem 2.1 in [DGZ2], but a slight modification of the proof presented there gives actually the stronger statement as it is formulated below).

Lemma 3. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then for any compact set $K \subset \Omega$ and any $\epsilon > 0$ there exists $\delta > 0$ such that for each subdomain $\widetilde{\Omega}$ of Ω, $K \subset \widetilde{\Omega}$, having the property that $b\widetilde{\Omega} \subset U_\delta(b\Omega)$ one has $|S_\Omega(p) - S_{\widetilde{\Omega}}(p)| < \epsilon$ for every $p \in K$. Here by $U_\delta(b\Omega)$ is denoted the δ-neighbourhood of the boundary $b\Omega$ of Ω.*

Now we give some estimates on the Carathéodory and Kobayashi metrics of some special domains.

Lemma 4. *Let $0 < a < 1 < b < +\infty$ be given numbers. For each $m \in \mathbb{N}$, consider the domain*

$$\Omega'_m := \{(z, w) \in \mathbb{C}^2 : a < |z| < b, |w| < 1, |w| < |z|^{-m}\}.$$

Then there exists $C > 0$ such that $C_{\Omega'_m}(p, \xi) \leq C$ for $p = (1, 0)$, $\xi = (1, 1)$ and all $m \in \mathbb{N}$.

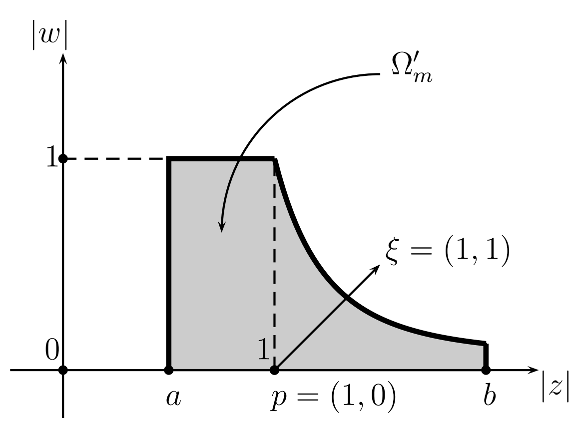

Figure 1: The domain Ω'_m.

Proof. Consider an arbitrary function $f \in \mathcal{O}(\Omega'_m, \Delta)$ such that $f(p) = 0$. Observe that the restriction f_v of f to the vertical disc $\Delta_v := \{z = 1\} \times \{|w| < 1\} = \{z = 1\} \cap \Omega'_m$ centered at p is a holomorphic function from Δ_v to Δ having the property $f_v(p) = 0$. Then, by the Schwarz lemma, one has

$$|f'(p)(0,1)| = |\frac{\partial f}{\partial w}(p)| = |f'_v(p)| \leq 1.$$

Similarly, for the restriction f_h of f to the horizontal disc

$$\Delta_h := \{|z-1| < \mathrm{min}(1-\mathrm{a}, \mathrm{b}-1)\} \times \{\mathrm{w} = 0\} \subset \Omega'_\mathrm{m} \cap \{\mathrm{w} = 0\}$$

we have that $f_h : \Delta_h \to \Delta$ is a holomorphic function such that $f_h(p) = 0$. Hence, in view of the Schwarz lemma, one also has

$$|f'(p)(1,0)| = |\frac{\partial f}{\partial z}(p)| = |f'_h(p)| \leq \frac{1}{\mathrm{min}(1-\mathrm{a}, \mathrm{b}-1)}.$$

Therefore

$$|f'(p)(1,1)| = |\frac{\partial f}{\partial z}(p) + \frac{\partial f}{\partial w}(p)| \leq |\frac{\partial f}{\partial z}(p)| + |\frac{\partial f}{\partial w}(p)| \leq \frac{1}{\mathrm{min}(1-\mathrm{a}, \mathrm{b}-1)} + 1 =: C.$$

Since f was an arbitrary function from $\mathcal{O}(\Omega'_m, \Delta)$ such that $f(p) = 0$, we finally conclude that for the Carathéodory metric the estimate $C_{\Omega'_m}(p, \xi) \leq C$ holds true for all $m \in \mathbb{N}$. $\square$

Lemma 5. *For each $m \in \mathbb{N}$, consider the domain*

$$\Omega''_m := \{(z, w) \in \mathbb{C}^2 : |w| < 1, |w| < |z|^{-m}\}.$$

Then $K_{\Omega''_m}(p, \xi) \geq \sqrt{\frac{m}{2}}$ for $p = (1, 0)$, $\xi = (1, 1)$ and each $m \in \mathbb{N}$.

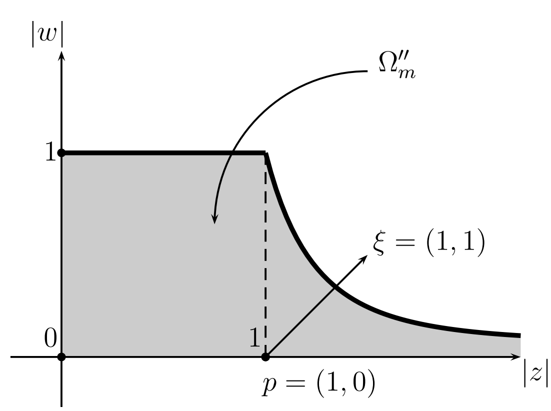

Figure 2: The domain Ω''_m.

Proof. Consider an arbitrary map $f \in \mathcal{O}(\Delta, \Omega''_m)$ such that $f(0) = p$ and $\alpha f'(0) = \xi = (1,1)$ for some α. Then f can be represented by

$$f(\zeta) = (z(\zeta), w(\zeta)) = (1 + \frac{1}{\alpha}\zeta + a_2\zeta^2 + ..., \frac{1}{\alpha}\zeta + b_2\zeta^2 + ...),$$

where $\zeta \in \Delta$. Since, by the definition of Ω''_m, one has $|wz^m| < 1$, it follows that

$$1 > |(\frac{1}{\alpha}\zeta + b_2\zeta^2 + ...)(1 + \frac{1}{\alpha}\zeta + a_2\zeta^2 + ...)^m| = |\frac{1}{\alpha}\zeta + (b_2 + \frac{m}{\alpha^2})\zeta^2 + ...|.$$

Then, from the Schwarz type bound for higher order coefficients (see Theorem 2 in [R] for a relatively recent generalization of the classical Schwarz inequality to similar bounds for all coefficients of the Taylor expansion), we get that

$$|b_2 + \frac{m}{\alpha^2}| \leq 1. \tag{1}$$

Since, by the definition of Ω''_m, one also has

$$|\frac{1}{\alpha}\zeta + b_2\zeta^2 + ...| = |w| \leq 1,$$

we conclude from the mentioned above Schwarz type bound for the higher order coefficients that

$$|b_2| \leq 1. \tag{2}$$

Combining estimates (1) and (2), we get

$$|\frac{m}{\alpha^2}| \leq 2 \quad \Rightarrow \quad |\alpha| \geq \sqrt{\frac{m}{2}},$$

which gives the desired estimate $K_{\Omega''_m}(p, \xi) \geq \sqrt{\frac{m}{2}}$ for each $m \in \mathbb{N}$. $\square$

3. Example

We first construct an auxiliary domain which we will denote by Ω. Let $a > 1$ be an arbitrary number, which will be fixed in what follows, and let $1 < a_1 < a_2 < ... < a_k < ... < a$ be a sequence (which will also be fixed) such that $\lim_{k\to\infty} a_k = a$. We define Ω as the set of points $(z, w) \in \{\frac{1}{a} < |z| < a\} \times \mathbb{C}_w$ satisfying the following conditions:

$$\begin{cases} |w| < B_k\,|z|^{n_k}\ , & \text{for } \frac{1}{a_{k+1}} < |z| \leq \frac{1}{a_k},\ k = 1, 2, 3, ..., \\ \quad |w| < 1 \quad , & \text{for } \frac{1}{a_1} < |z| \leq a_1, \\ |w| < B_k\,|z|^{-n_k}, & \text{for } a_k \leq |z| < a_{k+1},\ k = 1, 2, 3, ... \end{cases}$$

The numbers n_k and B_k will be defined inductively so that $B_1 = 1$, and for each $k \in \mathbb{N}, k \geq 2$, one has $n_k > n_{k-1}$ and $B_{k-1}\, {a_k}^{-n_{k-1}} = B_k\, {a_k}^{-n_k}$ (the last condition guarantees that the functions defining Ω will match at the points a_k and $\frac{1}{a_k}$, $k \in \mathbb{N}$) and, moreover, the inequality $S_\Omega(p_k) < \frac{1}{k}$ for the squeezing function on Ω at the point $p_k = (a_k, 0)$ holds true for every $k \in \mathbb{N}$.

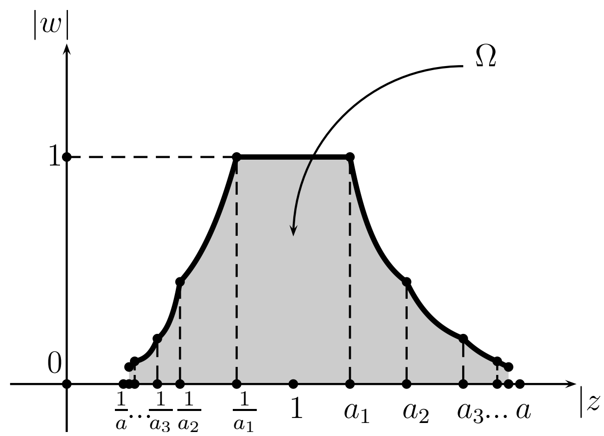

Figure 3: The auxiliary domain Ω.

The starting point of our inductive construction is the definition of Ω over the annulus $\{\frac{1}{a_1} < |z| \leq a_1\}$ by the inequality $|w| < 1$. Now we describe the inductive step of this construction. Assume that the part Ω_k of the domain Ω over the annulus $\{\frac{1}{a_k} < |z| < a_k\}$ is already constructed, i.e., we have already defined the numbers n_q, B_q for $q = 1, 2, ..., k-1$. For being able to find suitable values of n_k and B_k, we first make a biholomorphic change of coordinates F_k in $\mathbb{C}^* \times \mathbb{C}$:

$$z \to \frac{z}{a_k} =: z',\ w \to w\frac{{a_k}^{n_{k-1}}}{B_{k-1}}\Big(\frac{z}{a_k}\Big)^{n_{k-1}} =: w'.$$

Observe that in new coordinates (z', w') the part of the domain $F_k(\Omega)$ over the annulus $\{\frac{a_{k-1}}{a_k} \leq |z'| < 1\}$ is defined by $|w'| < 1$ and the part of $F_k(\Omega)$ over the annulus $\{1 \leq |z'| < \frac{a_{k+1}}{a_k}\}$ is defined by $|w'| < |z'|^{-(n_k - n_{k-1})}$, where n_k still has to be chosen. Note also that the domain

$$F_k(\Omega \cap (\{a_{k-1} < |z| < a_{k+1}\} \times \mathbb{C}_w)) =$$

$$= \{(z', w') : \frac{a_{k-1}}{a_k} < |z'| < \frac{a_{k+1}}{a_k}, |w'| < 1, |w'| < |z'|^{-(n_k - n_{k-1})}\}$$

has the form Ω'_m (see Lemma 4 for the description of Ω'_m) with $m = n_k - n_{k-1}, a = \frac{a_{k-1}}{a_k}, b = \frac{a_{k+1}}{a_k}$ and it is a proper subdomain of the domain $F_k(\Omega)$. Moreover, since for each $k \in \mathbb{N}$ the inequality $n_k > n_{k-1}$ holds, the domain $F_k(\Omega)$ will be contained in the domain Ω''_m (see Lemma 5 for the description of Ω''_m) with $m = n_k - n_{k-1}$.

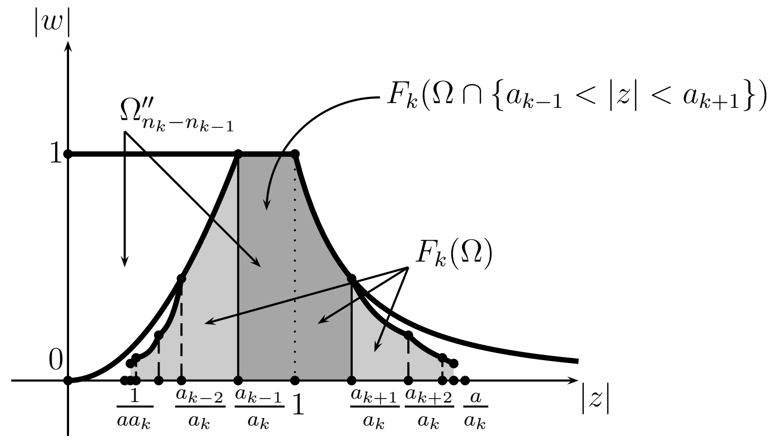

Figure 4: The domains $F_k\big(\Omega\cap\{a_k<|z|<a_{k+1}\}\big)$, $F_k(\Omega)$ and $\Omega''_{n_k-n_{k-1}}$.

Hence, in view of monotonicity of the Carathéodory metric and Lemma 4, one has

$$C_{F_k(\Omega)}(p,\xi)\le C_{\Omega'_m}(p,\xi)\le C_k$$

for $p=(1,0)$, $\xi=(1,1)$ and all $m\in\mathbb{N}$. We also have from monotonicity of the Kobayashi metric and Lemma 5 that

$$K_{F_k(\Omega)}(p,\xi)\ge K_{\Omega''_m}(p,\xi)\ge\sqrt{\frac{m}{2}}$$

for $p=(1,0)$, $\xi=(1,1)$ and each $m\in\mathbb{N}$. It follows then from Lemma 1 that

$$S_{F_k(\Omega)}(p)\le\frac{C_{F_k(\Omega)}(p,\xi)}{K_{F_k(\Omega)}(p,\xi)}\le C_k\sqrt{\frac{2}{m}}$$

and hence $S_{F_k(\Omega)}(p)<\frac{1}{k}$ for $n_k>n_{k-1}+2k^2C_k^2$. If we choose now n_k satisfying the last inequality, then, using the condition $B_{k-1}\,{a_k}^{-n_{k-1}}=B_k\,{a_k}^{-n_k}$, we can easily compute $B_k=B_{k-1}\,{a_k}^{n_k-n_{k-1}}$. Finally, note that, in view of biholomorphic invariance of the squeezing function,

$$S_\Omega(a_k)=S_{F_k(\Omega)}(p)<\frac{1}{k},$$

for each $k\in\mathbb{N}$. This completes the inductive step of our construction of the auxiliary domain Ω.

Now we are ready to construct a strictly pseudoconvex domain with non-plurisubharmonic squeezing function. Note first that Ω is pseudoconvex by construction. Observe also that, since the map $z\to\frac{1}{z}, w\to w$ is a biholomorphic automorphism of Ω, and, since the squeezing function is biholomorphically invariant, one has

$$S_\Omega\Big(\frac{1}{a_k}\Big)=S_\Omega(a_k)<\frac{1}{k},$$

for each $k \in \mathbb{N}$. Take now $p = (1, 0) \in \Omega$, denote $c := S_\Omega(p) > 0$ and fix from now on a number $k \in \mathbb{N}$ so large that $\frac{1}{k} < c$. Then, using Lemma 3 with $\epsilon < \frac{1}{2}\,(c - \frac{1}{k})$, we approximate the domain Ω from inside by a strictly pseudoconvex smoothly bounded domain $\widetilde{\Omega}$ (one can obviously choose this domain to be also circular in z and w) so well that for every point q of the set

$$(\{|z| = \frac{1}{a_k}\} \times \{w = 0\}) \cup (\{|z| = a_k\} \times \{w = 0\}) \subset \widetilde{\Omega} \cap \{w = 0\}$$

one has

$$S_{\widetilde{\Omega}}(q) < \frac{1}{k} + \epsilon < c - \epsilon < S_{\widetilde{\Omega}}(p).$$

This means that the maximum principle for the restriction of the function $S_{\widetilde{\Omega}}(\cdot)$ to the annulus $\{\frac{1}{a_k} \le |z| \le a_k\} \times \{w = 0\} \subset \widetilde{\Omega} \cap \{w = 0\}$ does not hold and, hence, the function $S_{\widetilde{\Omega}}(\cdot)$ cannot be plurisubharmonic. Thus $\widetilde{\Omega}$ is a strictly pseudoconvex domain as desired. The proof of the Theorem is now completed. $\square$

Remark. In the proof above instead of using Lemma 3 it is enough to use the weaker statement of Theorem 2.1 from [DGZ2] at the points $\frac{1}{a_k}$, a_k and p and the circular invariance of the domain $\widetilde{\Omega}$ and the squeezing function $S_{\widetilde{\Omega}}(\cdot)$.

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