d-abelian quotients of (d+2)-angulated categories

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Abstract

Let \mathscr{T} be a triangulated category. If T is a cluster tilting object and $I = [\operatorname{add} T]$ is the ideal of morphisms factoring through an object of $\operatorname{add} T$, then the quotient category \mathscr{T}/I is abelian. This is an important result of cluster theory, due to Keller–Reiten and König–Zhu. More general conditions which imply that \mathscr{T}/I is abelian were determined by Grimeland and the first author.

Now let \mathscr{T} be a suitable (d+2)-angulated category for an integer $d \ge 1$. If T is a cluster tilting object in the sense of Oppermann–Thomas and $I = [\operatorname{add} T]$ is the ideal of morphisms factoring through an object of $\operatorname{add} T$, then we show that \mathscr{T}/I is d-abelian.

The notions of (d + 2)-angulated and *d*-abelian categories are due to Geiss–Keller– Oppermann and Jasso. They are higher homological generalisations of triangulated and abelian categories, which are recovered in the special case d = 1. We actually show that if $\Gamma = \operatorname{End}_{\mathscr{T}} T$ is the endomorphism algebra of T, then \mathscr{T}/I is equivalent to a *d*-cluster tilting subcategory of mod Γ in the sense of Iyama; this implies that \mathscr{T}/I is *d*-abelian. Moreover, we show that Γ is a *d*-Gorenstein algebra.

More general conditions which imply that \mathscr{T}/I is *d*-abelian will also be determined, generalising the triangulated results of Grimeland and the first author.

Keywords: Cluster tilting object, d-abelian category, d-cluster tilting subcategory, d-representation finite algebra, (d + 2)-angulated category, functorially finite subcategory, Gorenstein algebra, higher homological algebra, quotient category, quotient functor

0. Introduction

It is an important result of cluster theory that certain quotients of triangulated categories are abelian. This is stated in theorems by Keller–Reiten, König–Zhu, and in [5,

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thm. 1], which will be generalised here to (d+2)-angulated and d-abelian categories, the basic objects of higher homological algebra.

A. Classic background

Let \mathscr{T} be a k-linear Hom-finite triangulated category over a field k, and let $T \in \mathscr{T}$ be an object with endomorphism algebra $\Gamma = \operatorname{End}_{\mathscr{T}} T$. Denote by \mathscr{D} the essential image of the functor $\mathscr{T}(T, -) : \mathscr{T} \to \operatorname{mod} \Gamma$.

Recall the notion of cluster tilting objects (also known as maximal 1-orthogonal objects), which was introduced by Iyama, see [12, def. 3.1]. In our setup, T is cluster tilting if it satisfies:

$$\operatorname{add} T = \{X \in \mathscr{T} \mid \mathscr{T}(T, \Sigma X) = 0\} = \{X \in \mathscr{T} \mid \mathscr{T}(X, \Sigma T) = 0\},\$$

where Σ is the suspension functor of \mathscr{T} . When T is cluster tilting, each $X \in \mathscr{T}$ permits what might be called a T-presentation: A triangle $T_1 \to T_0 \to X \to \Sigma T_1$ with the T_i in add T. See [11, sec. 2.1, proposition] and [12, lem. 3.2.1].

It follows that a cluster tilting object T satisfies the following conditions:

- (a) Let $T_1 \xrightarrow{f} T_0$ be a right minimal morphism in add T. Then each completion of f to a triangle $T_1 \xrightarrow{f} T_0 \to X \xrightarrow{h} \Sigma T_1$ in \mathscr{T} satisfies $\mathscr{T}(T,h) = 0$.
- (b) Let $X \in \mathscr{T}$ be indecomposable with $\mathscr{T}(T, X) \neq 0$. Then there exists a triangle $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ in \mathscr{T} which satisfies $\mathscr{T}(T, h) = 0$.

Note that (a) and (b) do not imply that T is cluster tilting, see [5, exa. 18]. We are interested in (a) and (b) because of the following result:

Theorem 0.1 ([5, thm. 1]). Conditions (i) and (ii) below are equivalent.

- (i) The functor $\mathscr{T}(T,-): \mathscr{T} \to \operatorname{mod} \Gamma$ is essentially surjective (in other words: $\mathscr{D} = \operatorname{mod} \Gamma$), and it is full.
- (ii) T satisfies conditions (a) and (b).
 - If (i) holds, then $\mathscr{T}(T,-):\mathscr{T}\to \mathrm{mod}\,\Gamma$ induces an equivalence of categories

$$\mathscr{T}/I \xrightarrow{\sim} \mod \Gamma$$

where I is the ideal of morphisms f such that $\mathscr{T}(T, f) = 0$. In other words, the triangulated category \mathscr{T} has an abelian quotient \mathscr{T}/I .

If T is cluster tilting, then more is true. The following is a combination of [11, sec. 2.1] and [12, cors. 4.4 and 4.5]:

Theorem 0.2 (Keller–Reiten and König–Zhu). Assume that T is cluster tilting. Then:

- (i) The functor $\mathscr{T}(T,-): \mathscr{T} \to \operatorname{mod} \Gamma$ is essentially surjective (in other words: $\mathscr{D} = \operatorname{mod} \Gamma$).
- (ii) The functor $\mathscr{T}(T,-)$ induces an equivalence of categories

$$\mathscr{T}/[\operatorname{add}\Sigma T] \xrightarrow{\sim} \operatorname{mod}\Gamma.$$

- (iii) Γ is a 1-Gorenstein algebra, that is, each injective module has projective dimension ≤ 1 , and each projective module has injective dimension ≤ 1 .
- (iv) If the global dimension of Γ is finite, then it is at most 1.

The purpose of this paper is to generalise Theorems 0.1 and 0.2 to (d+2)-angulated categories.

B. Primer on (d+2)-angulated and d-abelian categories

The notions of (d + 2)-angulated and *d*-abelian categories were introduced by Geiss-Keller–Oppermann in [4, def. 2.1] and Jasso in [10, def. 3.1]. They are the basic objects of higher homological algebra. For d = 1 they specialise to triangulated and abelian categories. For general values of *d*, they are defined in terms of (d + 2)-angles, *d*-kernels, and *d*-cokernels; these are longer complexes with properties resembling those of triangles, kernels, and cokernels.

Many examples of (d + 2)-angulated and d-abelian categories are known, see for instance [4], [10], [14], and Section 7, and there are strong links to higher dimensional combinatorics.

The notion of cluster tilting object can be generalised to (d+2)-angulated categories:

Definition 0.3 (Cluster tilting objects in the sense of [14, def. 5.3]). An object T of a (d+2)-angulated category \mathscr{T} with d-suspension functor Σ^d is called <u>cluster tilting in the</u> sense of Oppermann–Thomas if:

- (i) $\mathscr{T}(T, \Sigma^d T) = 0.$
- (ii) Each $X \in \mathscr{T}$ occurs in a (d+2)-angle

$$T_d \to T_{d-1} \to \cdots \to T_1 \to T_0 \xrightarrow{f_0} X \xrightarrow{h} \Sigma^d T_d$$

with $T_i \in \operatorname{add} T$ for $0 \leq i \leq d$.

C. This paper

This paper generalises Theorems 0.1 and 0.2 to (d+2)-angulated categories. We first fix the notation. Concrete examples of the following setup are provided in Section 7.

Setup 0.4. The rest of the paper assumes the following setup: k is an algebraically closed field, $d \ge 1$ is an integer, \mathscr{T} is a k-linear Hom-finite (d+2)-angulated category with split idempotents. The d-suspension functor of \mathscr{T} is denoted by Σ^d . We assume that \mathscr{T} has a Serre functor S, that is, an autoequivalence for which there are natural equivalences $D\mathscr{T}(X,Y) \cong \mathscr{T}(Y,SX)$, where $D(-) = \operatorname{Hom}_k(-,k)$ is the k-linear duality functor.

We let $T \in \mathscr{T}$ be an object with endomorphism algebra $\Gamma = \operatorname{End}_{\mathscr{T}} T$. By \mathscr{D} we denote the essential image of the functor $\mathscr{T}(T, -) : \mathscr{T} \to \operatorname{mod} \Gamma$, where $\operatorname{mod} \Gamma$ is the category of finite dimensional right Γ -modules.

Observe that since \mathscr{T} is k-linear and Hom-finite, it is a Krull–Schmidt category. \Box

Our first main result is a higher homological generalisation of Theorem 0.1, which can be recovered by setting d = 1. Conditions (a), (a'), (strong a), (strong a'), and (b) in the theorem are higher homological versions of conditions (a) and (b) on page 2. We do not state them here, but refer to Definition 3.1.

Theorem 0.5. Conditions (i), (ii), and (iii) below are equivalent.

- (i) \mathscr{D} is a d-cluster tilting subcategory of mod Γ (see Definition 1.1 below) and the functor $\mathscr{T}(T,-): \mathscr{T} \to \operatorname{mod} \Gamma$ is full.
- (ii) T satisfies conditions (a), (a'), and (b) in Definition 3.1.
- (iii) T satisfies conditions (strong a), (strong a'), and (b) in Definition 3.1.

If (i) holds, then \mathscr{D} is a *d*-cluster tilting subcategory of mod Γ , hence *d*-abelian by [10, thm. 3.16]. Moreover, $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod }\Gamma$ induces an equivalence of categories

$$\mathscr{T}/I \xrightarrow{\sim} \mathscr{D},$$

where I is the ideal of morphisms f such that $\mathscr{T}(T, f) = 0$. In other words, the (d+2)-angulated category \mathscr{T} has a d-abelian quotient \mathscr{T}/I .

Let us remark that the implication (iii) \Rightarrow (ii) in the theorem is clear by Definition 3.1, since conditions (strong a), (strong a') are explicitly stronger versions of conditions (a), (a'). The implications (ii) \Rightarrow (i) and (i) \Rightarrow (iii) will be proved in Sections 4 and 5, respectively.

Our second main result is a higher homological generalisation of Theorem 0.2, which can be recovered by setting d = 1. Note that the following was obtained in a special case in the first part of [14, thm. 5.6].

Theorem 0.6. Assume that T is cluster tilting in the sense of Oppermann–Thomas, see Definition 0.3. Then:

- (i) \mathscr{D} is a d-cluster tilting subcategory of mod Γ .
- (ii) The functor $\mathscr{T}(T,-)$ induces an equivalence of categories

$$\mathscr{T}/[\operatorname{add}\Sigma^d T] \xrightarrow{\sim} \mathscr{D}$$

- (iii) Γ is a d-Gorenstein algebra, that is, each injective module has projective dimension $\leq d$, and each projective module has injective dimension $\leq d$.
- (iv) If the global dimension of Γ is finite, then it is at most d.

From Theorem 0.6 follows the next result, which was obtained in a special case in the second part of [14, thm. 5.6]. The notion of (weakly) d-representation finite algebras was defined in [8, def. 2].

Corollary 0.7. Assume that T is cluster tilting in the sense of Oppermann–Thomas, see Definition 0.3, and that \mathscr{T} has finitely many indecomposable objects up to isomorphism. Then:

- (i) Γ is weakly d-representation finite.
- (ii) If Γ has finite global dimension, then it is d-representation finite.

The paper is organised as follows: Section 1 provides some lemmas on *d*-cluster tilting subcategories of mod Γ . Section 2 provides some lemmas on the functor $\mathscr{T}(T, -)$. Section 3 states conditions (a), (a'), (b), (strong a), and (strong a'), and provides a connection to cluster tilting in the sense of Oppermann–Thomas. Sections 4 and 5 prove the implications (ii) \Rightarrow (i) and (i) \Rightarrow (iii) in Theorem 0.5. Section 6 proves Theorem 0.6 and Corollary 0.7. Section 7 provides two classes of examples, the first of which shows how Theorem 0.6 and Corollary 0.7 imply [14, thm. 5.6].

1. Lemmas on *d*-cluster tilting subcategories of mod Γ

The results of this section do not require Γ to arise as in Setup 0.4; they are valid for any finite dimensional k-algebra.

Definition 1.1 (*d*-cluster tilting subcategories, [7, def. 1.1]). Let $\mathscr{X} \subseteq \text{mod } \Gamma$ be a full subcategory.

(i) \mathscr{X} is weakly *d*-cluster tilting if

$$\mathscr{X} = \{ X \in \text{mod}\,\Gamma \mid \text{Ext}_{\Gamma}^{i}(X,\mathscr{X}) = 0 \text{ for } 1 \leqslant i \leqslant d-1 \} \\ = \{ X \in \text{mod}\,\Gamma \mid \text{Ext}_{\Gamma}^{i}(\mathscr{X},X) = 0 \text{ for } 1 \leqslant i \leqslant d-1 \}.$$

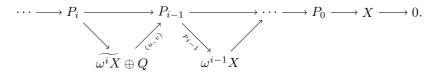
(ii) \mathscr{X} is *d*-cluster tilting if it is weakly *d*-cluster tilting and functorially finite in mod Γ .

A module $X \in \text{mod } \Gamma$ is called <u>*d*-cluster tilting</u> if add X is a *d*-cluster tilting subcategory.

Setup 1.2. From now on, $\mathscr{X} \subseteq \text{mod } \Gamma$ is a *d*-cluster tilting subcategory. Note that \mathscr{X} is a *d*-abelian category by [10, thm. 3.16].

Lemma 1.3. For $1 \leq i \leq d-1$ and $X \in \mathscr{X}$, the *i*th syzygy $\omega^i X$, as defined by a minimal projective resolution of X, has no non-zero projective summands.

Proof. Assume to the contrary that $\omega^i X = \omega^i X \oplus Q$ with Q non-zero projective. Consider the augmented minimal projective resolution with syzygies:



Since $X, Q \in \mathscr{X}$, we have $\operatorname{Ext}_{\Gamma}^{i}(X, Q) = 0$. Hence the map $(0, 1_{Q}) : \omega^{i}X \oplus Q \to Q$ must factor through (u, v), so there is a map $w : P_{i-1} \to Q$ with $(0, 1_{Q}) = w \circ (u, v)$. In particular $1_{Q} = w \circ v$, whence $1_{Q} - wv = 0$, so $v \notin \operatorname{rad}_{\operatorname{mod}\Gamma}$. This contradicts that p_{i-1} is a projective cover.

Lemma 1.4. Let $X \in \mathscr{X}$ have the augmented minimal projective resolution

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to X \to 0.$$

- (i) If $2 \leq j \leq d$ then f_j is left minimal.
- (ii) If X has no non-zero projective summands, then f_1 is left minimal.

Proof. (i): Suppose $g: P_{j-1} \to P_{j-1}$ satisfies $gf_j = f_j$. Let $p_{j-1}: P_{j-1} \to \omega^{j-1}X$ be the projective cover of the (j-1)th syzygy. Since $(g-1_{P_{j-1}})f_j = 0$, there must exist $h: \omega^{j-1}X \to P_{j-1}$ such that $g-1_{P_{j-1}} = hp_{j-1}$. In other words, $g=1_{P_{j-1}} + hp_{j-1}$. But Lemma 1.3 implies that p_{j-1} is in the radical, so g is invertible.

(ii): Use the same argument as for (i) with f_1 in place of f_j .

Lemma 1.5. If $X \in \mathscr{X}$ has the augmented projective resolution

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0,$$

then

$$P_{r-1} \xrightarrow{f_{r-1}} P_{r-2} \xrightarrow{f_{r-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0 \to \cdots \to 0$$

is a d-cokernel of f_r in \mathscr{X} for each $1 \leq r \leq d$. (For the definition of d-cokernels see [10, def. 2.2/.)

Proof. By the definition of *d*-cokernels, we must show that the complex

$$P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} P_{r-2} \xrightarrow{f_{r-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0 \to \cdots \to 0$$

becomes exact when we apply the functor $\mathscr{X}(-,Y)$ for $Y \in \mathscr{X}$. Since \mathscr{X} is a full subcategory, this amounts to the complex becoming exact when we apply the functor $\operatorname{Hom}_{\Gamma}(-,Y)$ for $Y \in \mathscr{X}$. This is true because $\operatorname{Ext}^{i}_{\Gamma}(X,Y) = 0$ for $1 \leq i \leq d-1$ since $X, Y \in \mathscr{X}.$

Lemma 1.6. Let

$$\varepsilon = \left[0 \to X \xrightarrow{f^{-1}} Y^0 \xrightarrow{f^0} \cdots \xrightarrow{f^{d-2}} Y^{d-1} \xrightarrow{f^{d-1}} Z \to 0 \right]$$

be a d-extension in $mod \Gamma$.

- (i) Suppose $\operatorname{Ext}^{i}_{\Gamma}(Y^{i}, X) = 0$ for $1 \leq i \leq d-1$. If ε represents 0 in $\operatorname{Ext}^{d}_{\Gamma}(Z, X)$, then f^{-1} is a split monomorphism.
- (ii) Suppose $\operatorname{Ext}_{\Gamma}^{i}(Z, Y^{d-i}) = 0$ for $1 \leq i \leq d-1$. If ε represents 0 in $\operatorname{Ext}_{\Gamma}^{d}(Z, X)$, then f^{d-1} is a split epimorphism.

Proof. We show (i) only, (ii) being dual. Let

$$0 \to X \xrightarrow{g^{-1}} I^0 \xrightarrow{g^0} I^1 \to \cdots$$

be an augmented injective resolution. We use it to define the cozysygies $\sigma^i X$ for $i \ge 0$ which satisfy

$$\operatorname{Ext}^{1}_{\Gamma}(Y^{i}, \sigma^{i-1}X) = \operatorname{Ext}^{i}_{\Gamma}(Y^{i}, X) = 0$$
(1.1)

for $1 \leq i \leq d-1$. We can construct the following commutative diagram.

If ε represents 0 in $\operatorname{Ext}_{\Gamma}^{d}(Z, X)$, then h factors through g. Using Equation (1.1) repeatedly, we can then construct the following homotopy.

Then $s^0 f^{-1} = 1_X$ so f^{-1} is a split monomorphism.

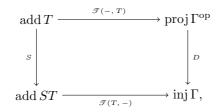
2. Lemmas on the functor $\mathscr{T}(T, -)$

The results of this section do not require the full assumptions on \mathscr{T} made in Setup 0.4; they are valid if \mathscr{T} is a k-linear Hom-finite category with a Serre functor S.

Lemma 2.1. (i) The functor $\mathscr{T}(T, -)$ restricts to an equivalence add $T \to \operatorname{proj} \Gamma$.

(ii) The functor $\mathscr{T}(T, -)$ restricts to an equivalence add $ST \to inj \Gamma$.

Proof. Part (i) is classic. For part (ii) note that the Serre functor S gives the following commutative square of functors,



where $\operatorname{proj} \Gamma^{\operatorname{op}}$ is the category of projective finite dimensional left Γ -modules, and the functor $D(-) = \operatorname{Hom}_k(-, k)$ denotes k-linear equivalence. The functors S and D in the diagram are equivalences, and it is classic that so is $\mathscr{T}(-, T)$. Hence the functor $\mathscr{T}(T, -)$: add $ST \to \operatorname{inj} \Gamma$ is an equivalence.

Lemma 2.2. For $T' \in \operatorname{add} T$ and $X \in \mathscr{T}$, the induced maps

- (i) $\mathscr{T}(T', X) \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T, T'), \mathscr{T}(T, X)),$
- (ii) $\mathscr{T}(X, ST') \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T, X), \mathscr{T}(T, ST'))$

are bijective.

Proof. (i): Fixing X, the map in (i) is a natural transformation of additive functors of $T' \in \text{add } T$. Hence it is enough to show bijectivity for T' = T, where the map is

$$\mathscr{T}(T,X) \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T,T),\mathscr{T}(T,X)) = \operatorname{Hom}_{\Gamma}(\Gamma,\mathscr{T}(T,X)).$$

This is bijective since it can be identified with the identity map on $\mathscr{T}(T, X)$.

(ii): The Serre functor S is an autoequivalence so $\Gamma = \mathscr{T}(ST, ST)$. An argument analogous to that in (i) shows that the induced map

$$\mathscr{T}(X, ST') \to \operatorname{Hom}_{\Gamma^{\operatorname{op}}}(\mathscr{T}(ST', ST), \mathscr{T}(X, ST))$$

$$(2.1)$$

is bijective. However, there are further bijections

$$\operatorname{Hom}_{\Gamma^{\operatorname{op}}}(\mathscr{T}(ST',ST),\mathscr{T}(X,ST)) \to \operatorname{Hom}_{\Gamma}(D\mathscr{T}(X,ST),D\mathscr{T}(ST',ST)) \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T,X),\mathscr{T}(T,ST')),$$
(2.2)

by k-linear and Serre duality. Using the natural property of the constituent morphisms, it can be checked that the composition of (2.1) and (2.2) is the map in (ii) which is hence bijective.

Lemma 2.3. Assume that $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod}\,\Gamma$ is a full functor. If $X \in \mathscr{T}$ is indecomposable and $\mathscr{T}(T, X)$ is a projective Γ -module, then $X \in \text{add}\,T$.

Proof. When $\mathscr{T}(T,X)$ is projective, Lemma 2.1(i) implies that there is some object $T' \in \operatorname{add} T$ such that $\mathscr{T}(T,T') \cong \mathscr{T}(T,X)$. Since $\mathscr{T}(T,-)$ is full, we can find morphisms $T' \xrightarrow{f} X \xrightarrow{g} T'$ which are mapped to inverse isomorphisms by $\mathscr{T}(T,-)$. In other words, $\mathscr{T}(T,gf) = \mathscr{T}(T,g)\mathscr{T}(T,f) = 1_{\mathscr{T}(T,T')}$.

It follows from Lemma 2.1(i) that $gf = 1_{T'}$. Hence T' is a direct summand of X. But X is indecomposable so in fact $X \cong T' \in \text{add } T$.

Lemma 2.4. If \mathscr{T} has finitely many indecomposable objects, then so does \mathscr{D} .

Proof. Since \mathscr{D} is the essential image of $\mathscr{T}(T, -)$, each indecomposable object $M \in \mathscr{D}$ has the form $M \cong \mathscr{T}(T, X_1 \oplus \cdots \oplus X_n) \cong \mathscr{T}(T, X_1) \oplus \cdots \oplus \mathscr{T}(T, X_n)$, where the X_i are indecomposable objects of \mathscr{T} . Since M is indecomposable, precisely one summand is non-zero, so $M \cong \mathscr{T}(T, X)$ for an indecomposable object $X \in \mathscr{T}$. Since \mathscr{T} has finitely many indecomposable objects up to isomorphism, it follows that so does \mathscr{D} .

Proposition 2.5. Assume that \mathscr{T} has weak kernels and weak cokernels. Then \mathscr{D} is functorially finite in mod Γ .

Proof. Existence of left \mathscr{D} -approximations: Let $M \in \text{mod }\Gamma$ have the projective presentation

$$\mathscr{T}(T,T_1) \xrightarrow{\mathscr{Y}(T,f)} \mathscr{T}(T,T_0) \xrightarrow{u} M \to 0,$$

cf. Lemma 2.1(i), and let

$$T_1 \xrightarrow{J} T_0 \xrightarrow{g} X$$

be a weak cokernel. Use $\mathscr{T}(T, -)$ to get the following commutative diagram in mod Γ ,

where v exists because M is a cokernel while $\mathscr{T}(T,g) \circ \mathscr{T}(T,f) = \mathscr{T}(T,gf) = 0$. We will show that $v: M \to \mathscr{T}(T,X)$ is a left \mathscr{D} -approximation of M.

Let $w: M \to \mathscr{T}(T, Y)$ be a homomorphism in mod Γ and consider the composition $wu: \mathscr{T}(T, T_0) \to \mathscr{T}(T, Y)$. Lemma 2.2(i) gives $wu = \mathscr{T}(T, h)$ for some $h: T_0 \to Y$. Then $hf: T_1 \to Y$ satisfies $\mathscr{T}(T, hf) = wu\mathscr{T}(T, f) = w \circ 0 = 0$ whence hf = 0 by Lemma 2.2(i). So h factors through g and hence $\mathscr{T}(T, h) = wu$ factors through $\mathscr{T}(T, g) = vu$. Since u is an epimorphism, this means that w factors through v as desired.

Existence of right \mathscr{D} -approximations: Let $N \in \text{mod}\,\Gamma$ have the injective copresentation

$$0 \to N \to \mathscr{T}(T, ST^0) \xrightarrow{\mathscr{T}(T, f)} \mathscr{T}(T, ST^1),$$

cf. Lemma 2.1(ii), and let

$$Y \to ST^0 \xrightarrow{f} ST^1$$

be a weak kernel. Dually to the above, one shows that there is a right \mathscr{D} -approximation $v: \mathscr{T}(T,Y) \to N$.

3. The conditions (a), (a'), (b), (c), (strong a), (strong a')

Recall that we still assume Setup 0.4. This section introduces the conditions (a), (a'), (b), (c), (strong a), (strong a'), and shows how they are linked to cluster tilting in the sense of Oppermann–Thomas.

Definition 3.1. The following are conditions which can be imposed on the object T:

(a) Suppose that $M \in \text{mod } \Gamma$ satisfies $\text{Ext}_{\Gamma}^{j}(\mathscr{D}, M) = 0$ for $1 \leq j \leq d-1$, and that $T_1 \xrightarrow{f} T_0$ is a morphism in add T for which

$$\mathscr{T}(T,T_1) \xrightarrow{\mathscr{Y}(T,f)} \mathscr{T}(T,T_0) \to M \to 0$$

is a minimal projective presentation in mod Γ . Then there exists a completion of f to a (d+2)-angle in \mathscr{T} ,

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} \Sigma^d T_1,$$

which satisfies $\mathscr{T}(T, h_i) = 0$ for some $1 \leq i \leq d+1$.

(a') Suppose that $N \in \text{mod } \Gamma$ satisfies $\text{Ext}_{\Gamma}^{j}(N, \mathscr{D}) = 0$ for $1 \leq j \leq d-1$, and that $ST_{1} \xrightarrow{g} ST_{0}$ is a morphism in add ST for which

$$\begin{array}{c} 0 \to N \to \mathscr{T}(T, ST_1) \xrightarrow{\mathscr{T}(T,g)} \mathscr{T}(T, ST_0) \\ 9 \end{array}$$

is a minimal injective copresentation in mod Γ . Then there exists a completion of g to a (d+2)-angle in \mathscr{T} ,

$$\Sigma^{-d}ST_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} ST_1 \xrightarrow{g} ST_0,$$

which satisfies $\mathscr{T}(T, h_i) = 0$ for some $1 \leq i \leq d+1$.

(b) Suppose that $X \in \mathscr{T}$ is indecomposable and satisfies $\mathscr{T}(T, X) \neq 0$. Then there exists a (d+2)-angle in \mathscr{T} ,

$$T_d \to \cdots \to T_0 \to X \xrightarrow{h} \Sigma^d T_d,$$

with $T_i \in \operatorname{add} T$ for $0 \leq i \leq d$, which satisfies $\mathscr{T}(T,h) = 0$.

(c) $\{X \in \mathscr{T} \mid \mathscr{T}(T, X) = 0\} = \operatorname{add} \Sigma^d T.$

Stronger versions of (a) and (a') are also useful.

(strong a) The same as condition (a), except that in the last line we require $\mathscr{T}(T, h_d) = 0.$

(strong a') The same as condition (a'), except that in the last line we require
$$\mathscr{T}(T, h_2) = 0.$$

Having stated the conditions, the implication (iii) \Rightarrow (ii) in Theorem 0.5 is clear. The other implications in the theorem will be proved in Sections 4 and 5.

Lemma 3.2. T is cluster tilting in the sense of Oppermann–Thomas (see Definition 0.3) if and only if it satisfies (a), (a'), (b), and (c).

Proof. "If": Assume that T satisfies (a), (a'), (b), and (c). Definition 0.3(i) is immediate from (c).

To establish Definition 0.3(ii), note that, since the set of (d+2)-angles is closed under direct sums by [4, def. 2.1, (F1)(a)], we can assume that $X \in \mathscr{T}$ is indecomposable. If $\mathscr{T}(T, X) = 0$ then $X \in \operatorname{add} \Sigma^d T$ by (c), so the trivial (d+2)-angle

$$\Sigma^{-d} X \to 0 \to \dots \to 0 \to X \xrightarrow{1_X} X$$

can be used in Definition 0.3(ii). If $\mathscr{T}(T, X) \neq 0$, then the (d+2)-angle from (b) can be used in Definition 0.3(ii).

"Only if": Assume that T is cluster tilting in the sense of Oppermann–Thomas.

Suppose that we are in the situation of (a). Then there is a morphism $T_1 \xrightarrow{f} T_0$ in add T which we complete to a (d+2)-angle in \mathscr{T} :

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} \Sigma^d T_1.$$

Then $\mathscr{T}(T, h_1) = 0$ since $\mathscr{T}(T, \Sigma^d T) = 0$, so T satisfies (a). A dual argument show that T satisfies (a').

To show that T satisfies (b), we can use the (d + 2)-angle from Definition 0.3(ii), where $\mathscr{T}(T,h) = 0$ since $\mathscr{T}(T,\Sigma^d T) = 0$.

To show (c), let $X \in \mathscr{T}$ be given with $\mathscr{T}(T, X) = 0$. Then $\mathscr{T}(T_0, X) = 0$ for each $T_0 \in \operatorname{add} T$. In particular, the morphism $T_0 \to X$ in the (d+2)-angle from Definition 0.3(ii) is zero, so h is a split monomorphism whence $X \in \operatorname{add} \Sigma^d T$. Conversely, let $X \in \operatorname{add} \Sigma^d T$ be given. Then $\mathscr{T}(T, X) = 0$ since $\mathscr{T}(T, \Sigma^d T) = 0$.

4. Proof of the implication (ii) \Rightarrow (i) in Theorem 0.5

Recall that we still assume Setup 0.4. After providing the necessary ingredients, this section ends with a proof of the implication (ii) \Rightarrow (i) in Theorem 0.5.

Lemma 4.1. Let $X \in \mathscr{T}$ be indecomposable with $\mathscr{T}(T, X) \neq 0$ and consider a (d+2)angle satisfying the requirements in (b). If we apply the functor $\mathscr{T}(T, -)$ to all terms
but the last, then we get a complex

$$\mathscr{T}(T, T_d) \to \cdots \to \mathscr{T}(T, T_0) \to \mathscr{T}(T, X)$$

which is part of an augmented projective resolution of $\mathscr{T}(T,X)$ over Γ .

Proof. By [4, prop. 2.5(a)], the complex is exact. Since $\mathscr{T}(T,h) = 0$, the last morphism is surjective. By Lemma 2.1(i), the Γ -modules $\mathscr{T}(T,T_i)$ are projective.

Lemma 4.2. Let $X \in \mathscr{T}$ be indecomposable with $\mathscr{T}(T, X) \neq 0$ and consider a (d+2)angle satisfying the requirements in (b). Then $h \in \operatorname{rad}_{\mathscr{T}}$.

Proof. Suppose $h \notin \operatorname{rad}_{\mathscr{T}}$. If we write h as a matrix H of morphisms from the indecomposable object X to the indecomposable summands of $\Sigma^d T_d$, then one of the entries of H is invertible, say H_i . Let $f: T \to X$ be a morphism. Then hf = 0 by (b), so in particular $H_i f = 0$ whence f = 0. Hence $\mathscr{T}(T, X) = 0$, a contradiction.

Lemma 4.3. If T satisfies (b), then \mathcal{D} is closed under direct summands.

Proof. Consider an object $\mathscr{T}(T,X)$ of \mathscr{D} . Suppose $\mathscr{T}(T,X) = M' \oplus M''$ for some $M', M'' \in \text{mod } \Gamma$. We will show $M' \in \mathscr{D}$.

Let X_i denote the indecomposable direct summands of X. We can obviously drop each X_i which is mapped to zero by $\mathscr{T}(T, -)$, so can assume $\mathscr{T}(T, X_i) \neq 0$ for each *i*. Applying (b) and Lemma 4.2 to each X_i and taking the direct sum of the resulting (d+2)-angles shows that there is a (d+2)-angle

$$T_d \to \cdots \to T_0 \xrightarrow{g} X \xrightarrow{h} \Sigma^d T_d$$

with $T_i \in \operatorname{add} T$ for each i and $h \in \operatorname{rad}_{\mathscr{T}}$.

Consider the induced algebra homomorphism

$$\pi: \mathscr{T}(X, X) \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T, X), \mathscr{T}(T, X)).$$

If $x \in \mathscr{T}(X, X)$ is in the kernel of π , then xg = 0. Then x factors through h by [4, prop. 2.5(a)] whence $x \in \operatorname{rad}_{\mathscr{T}}$. Hence $\operatorname{Ker} \pi$ is contained in $\operatorname{rad}_{\mathscr{T}}(X, X) = \operatorname{rad}_{\mathscr{T}}(X, X)$, so idempotents lift through π by the combination of [1, cor. I.2.3] and [13, thm. (21.28)].

Hence the projection $e : \mathscr{T}(T, X) \to \mathscr{T}(T, X)$ onto the direct summand M' can be lifted to an idempotent morphism $f : X \to X$. Then f is split by assumption, so f is the projection onto a direct summand X' of X, and it follows that $\mathscr{T}(T, X') = M'$ whence $M' \in \mathscr{D}$.

Proposition 4.4. If T satisfies (b), then $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod } \Gamma$ is a full functor.

Proof. Let $u : \mathscr{T}(T, X) \to \mathscr{T}(T, Y)$ be a morphism in mod Γ . We must find $f \in \mathscr{T}(X, Y)$ with $\mathscr{T}(T, f) = u$. Without loss of generality, we can assume that X and Y are indecomposable.

If $\mathscr{T}(T, X) = 0$ or $\mathscr{T}(T, Y) = 0$, then we can set f = 0.

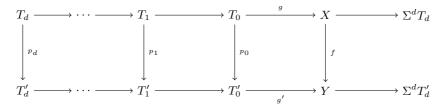
If $\mathscr{T}(T,X) \neq 0$ and $\mathscr{T}(T,Y) \neq 0$, then (b) gives two (d+2)-angles in \mathscr{T} ,

$$T_d \to \dots \to T_0 \xrightarrow{g} X \xrightarrow{h} \Sigma^d T_d,$$
$$T'_d \to \dots \to T'_0 \xrightarrow{g'} Y \xrightarrow{h'} \Sigma^d T'_d,$$

with $T_i, T'_i \in \operatorname{add} T$ and $\mathscr{T}(T, h) = \mathscr{T}(T, h') = 0$. Applying the functor $\mathscr{T}(T, -)$ gives the beginning of two augmented projective resolutions by Lemma 4.1. Hence the comparison theorem for projective resolutions gives the following commutative diagram.

$$\begin{aligned} \mathscr{T}(T,T_d) &\longrightarrow \cdots \longrightarrow \mathscr{T}(T,T_1) & \longrightarrow \mathscr{T}(T,T_0) \xrightarrow{\mathscr{T}(T,g)} \mathscr{T}(T,X) & \longrightarrow 0 \\ \\ \downarrow^{v_d} & \downarrow^{v_1} & \downarrow^{v_0} & \downarrow^{u} \\ \mathscr{T}(T,T'_d) & \longrightarrow \cdots & \longrightarrow \mathscr{T}(T,T'_1) & \longrightarrow \mathscr{T}(T,T'_0) \xrightarrow{\mathscr{T}(T,g')} \mathscr{T}(T,Y) & \longrightarrow 0 \end{aligned}$$

By Lemma 2.1(i) the second square from the right can be lifted to \mathscr{T} . Completing to a morphism of (d+2)-angles gives the following commutative diagram.



The first diagram gives

$$u\mathscr{T}(T,g) = \mathscr{T}(T,g')v_0 = (*)$$

We know $v_0 = \mathscr{T}(T, p_0)$ so have

$$(*) = \mathscr{T}(T, g')\mathscr{T}(T, p_0) = \mathscr{T}(T, f)\mathscr{T}(T, g),$$

where the last equality is by the second diagram. Since $\mathscr{T}(T,g)$ is surjective, it follows that $u = \mathscr{T}(T,f)$.

Proposition 4.5. If T satisfies (b), then \mathscr{D} is a d-rigid subcategory of mod Γ , that is, $\operatorname{Ext}^{i}_{\Gamma}(\mathscr{D}, \mathscr{D}) = 0$ for $1 \leq i \leq d-1$.

Proof. It is enough to see that if $X, Y \in \mathcal{T}$ are indecomposable, then

$$\operatorname{Ext}^{i}_{\Gamma}(\mathscr{T}(T,X),\mathscr{T}(T,Y)) = 0 \quad \text{for} \quad 1 \leq i \leq d-1.$$

$$(4.1)$$

This is clear for $\mathscr{T}(T, X) = 0$, so we can assume $\mathscr{T}(T, X) \neq 0$. Condition (b) gives a (d+2)-angle $T_d \to \cdots \to T_0 \to X \xrightarrow{h} \Sigma^d T_d$, and Lemma 4.1 implies that

$$\mathscr{T}(T, T_d) \to \cdots \mathscr{T}(T, T_0)$$

are the first d + 1 terms of a projective resolution of $\mathscr{T}(T, X)$. Hence the homology groups of the complex

$$\operatorname{Hom}_{\Gamma}(\mathscr{T}(T,T_0),\mathscr{T}(T,Y)) \to \dots \to \operatorname{Hom}_{\Gamma}(\mathscr{T}(T,T_d),\mathscr{T}(T,Y))$$
(4.2)

are the Ext groups in Equation (4.1). But Lemma 2.2(i) says that (4.2) is isomorphic to

$$\mathscr{T}(T_0, Y) \to \cdots \to \mathscr{T}(T_d, Y)$$

which is exact by [4, prop. 2.5(a)]. Hence Equation (4.1) is satisfied.

Proposition 4.6. (i) Assume T satisfies (a) and (b).

If $M \in \text{mod } \Gamma$ satisfies $\text{Ext}^i_{\Gamma}(\mathscr{D}, M) = 0$ for $1 \leq i \leq d-1$, then $M \in \mathscr{D}$.

(ii) Assume T satisfies (a') and (b).

If $N \in \text{mod } \Gamma$ satisfies $\text{Ext}^i_{\Gamma}(N, \mathscr{D}) = 0$ for $1 \leq i \leq d-1$, then $N \in \mathscr{D}$.

Proof. (i): By Lemma 2.1(i) we can pick a morphism $T_1 \xrightarrow{f} T_0$ in add T such that

$$\mathscr{T}(T,T_1) \xrightarrow{\mathscr{T}(T,f)} \mathscr{T}(T,T_0) \to M \to 0$$

is a minimal projective presentation in mod Γ . By (a) there exists a (d+2)-angle in \mathscr{T} ,

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} \Sigma^d T_1,$$

such that $\mathscr{T}(T,h_i) = 0$ for some $1 \leq i \leq d+1$. There is an induced long exact sequence

$$\mathscr{T}(T,T_1) \xrightarrow{\mathscr{T}(T,f)} \mathscr{T}(T,T_0) \xrightarrow{\mathscr{T}(T,h_{d+1})} \mathscr{T}(T,X_d) \xrightarrow{\mathscr{T}(T,h_d)} \cdots \cdots \cdots \xrightarrow{\mathscr{T}(T,h_2)} \mathscr{T}(T,X_1) \xrightarrow{\mathscr{T}(T,h_1)} \mathscr{T}(T,\Sigma^d T_1).$$

If $\mathscr{T}(T, h_{d+1}) = 0$ then $\mathscr{T}(T, f)$ is surjective whence M = 0 so $M \in \mathscr{D}$.

If $\mathscr{T}(T, h_d) = 0$ then $M \cong \mathscr{T}(T, X_d)$ so $M \in \mathscr{D}$.

If $\mathscr{T}(T,h_i)=0$ for some $1\leqslant i\leqslant d-1,$ then the long exact sequence induces an exact sequence

$$0 \to M \xrightarrow{\mu} \mathscr{T}(T, X_d) \to \dots \to \mathscr{T}(T, X_i) \to 0.$$

This is a (d-i)-extension representing an element in $\operatorname{Ext}_{\Gamma}^{d-i}(\mathscr{T}(T, X_i), M)$. This Ext is zero by the assumption on M. It follows from Lemma 1.6(i) that μ is split injective. So M is a direct summand of $\mathscr{T}(T, X_d)$ which is in \mathscr{D} , so $M \in \mathscr{D}$ by Lemma 4.3.

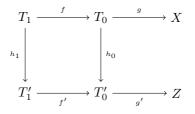
(ii): This is proved dually to (i).

Proof of Theorem 0.5, the implication (ii) \Rightarrow (i): Under condition (ii) in Theorem 0.5, the functor $\mathscr{T}(T,-): \mathscr{T} \to \mod \Gamma$ is full by Proposition 4.4, and its essential image \mathscr{D} is *d*-cluster tilting in mod Γ by Propositions 2.5, 4.5, and 4.6.

5. Proof of the implication (i) \Rightarrow (iii) in Theorem 0.5

Recall that we still assume Setup 0.4. After providing the necessary ingredients, this section ends with a proof of the implication (i) \Rightarrow (iii) in Theorem 0.5.

Lemma 5.1. Assume that $\mathscr{T}(T,-): \mathscr{T} \to \text{mod}\,\Gamma$ is a full functor, and that we are given the following commutative diagram in \mathscr{T} .



Suppose the following are satisfied:

- (i) $T_0, T'_1 \in \operatorname{add} T$.
- (ii) h_1 and h_0 are isomorphisms.
- (iii) The first row is part of a (d+2)-angle in \mathcal{T} , with g left minimal.
- (iv) $\mathscr{T}(T,g')$ is a weak cohernel of $\mathscr{T}(T,f')$ in \mathscr{D} .

Then there exists a split monomorphism $v: X \to Z$ completing to a larger commutative diagram.

Suppose we also have:

(v) g' is left minimal.

Then v is an isomorphism.

Proof. Condition (iv) implies $\mathscr{T}(T, g'f') = \mathscr{T}(T, g')\mathscr{T}(T, f') = 0$ whence g'f' = 0 by condition (i) and Lemma 2.2(i). Hence $g'h_0f = g'f'h_1 = 0$ and it follows from condition (iii) and [4, prop. 2.5(a)] that there is a morphism $v: X \to Z$ such that

$$g'h_0 = vg. (5.1)$$

We will show that v is a split monomorphism.

Condition (ii) says that f' and f are isomorphic in the morphism category of \mathscr{T} , so hence $\mathscr{T}(T, f')$ and $\mathscr{T}(T, f)$ are isomorphic in the morphism category of mod Γ . Since $\mathscr{T}(T, f')$ has weak cokernel $\mathscr{T}(T, g')$, it follows that $\mathscr{T}(T, f)$ has weak cokernel $\mathscr{T}(T, g')\mathscr{T}(T, h_0) = \mathscr{T}(T, g'h_0)$. Then $\mathscr{T}(T, g)\mathscr{T}(T, f) = \mathscr{T}(T, gf) = 0$ implies that there is $\phi : \mathscr{T}(T, Z) \to \mathscr{T}(T, X)$ such that $\mathscr{T}(T, g) = \phi \circ \mathscr{T}(T, g'h_0)$. Since $\mathscr{T}(T, -)$ is full, there exists some $w : Z \to X$ such that $\mathscr{T}(T, w) = \phi$, and it follows that $\mathscr{T}(T, g) = \mathscr{T}(T, wg'h_0)$. By condition (i) and Lemma 2.2(i) this implies

$$g = wg'h_0. \tag{5.2}$$

Equations (5.1) and (5.2) imply g = wvg. Condition (iii) says that g is left minimal, so wv is an isomorphism. In particular, v is a split monomorphism.

Now suppose that g' is left minimal. Then so is $g'h_0$ since h_0 is an isomorphism by condition (ii). Equations (5.1) and (5.2) imply $vwg'h_0 = g'h_0$, so vw is an isomorphism. We already proved that so is wv, so v is an isomorphism.

Proposition 5.2. If $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod} \Gamma$ is a full functor and \mathscr{D} is a d-cluster tilting subcategory of mod Γ , then T satisfies (strong a) and (strong a').

Proof. Suppose that we are in the situation of (strong a), that is, $M \in \text{mod }\Gamma$ satisfies

$$\operatorname{Ext}_{\Gamma}^{j}(\mathscr{D}, M) = 0 \quad \text{for} \quad 1 \leq j \leq d-1, \tag{5.3}$$

and $T_1 \xrightarrow{f} T_0$ is a morphism in add T for which

$$\mathscr{T}(T,T_1) \xrightarrow{\mathscr{T}(T,f)} \mathscr{T}(T,T_0) \to M \to 0$$
 (5.4)

is a minimal projective presentation in $\operatorname{mod} \Gamma$.

Complete f to a (d+2)-angle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} \Sigma^d T_1,$$

with $h_i \in \operatorname{rad}_{\mathscr{T}}$ for $2 \leq i \leq d$, see [14, lem. 5.18(2)]. Then h_i is left minimal for $3 \leq i \leq d+1$ by [3, lem. 2.11]. We will show $\mathscr{T}(T, h_d) = 0$, thereby establishing (strong a).

Since \mathscr{D} is a *d*-cluster tilting subcategory, Equation (5.3) implies $M \in \mathscr{D}$. Hence there is $Z \in \mathscr{T}$ with $M \cong \mathscr{T}(T, Z)$. Since $\mathscr{T}(T, -)$ is full, there is a diagram

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} Z$$

which $\mathscr{T}(T, -)$ maps to (5.4). We get a diagram,

which satisfies conditions (i)–(iv) in Lemma 5.1. The lemma provides a split monomorphism $v: X_d \to Z$ satisfying $vh_{d+1} = g$.

To show $\mathscr{T}(T, h_d) = 0$, let $a: T \to X_d$ be given. Consider $va: T \to Z$. Since $\mathscr{T}(T, g)$ is surjective, there is $u: T \to T_0$ such that gu = va. Thus $vh_{d+1}u = gu = va$ whence $h_{d+1}u = a$ because v is a split monomorphism. But then $h_da = h_dh_{d+1}u = 0 \circ u = 0$ as desired, where we used [4, prop. 2.5(a)].

Condition (strong a') is established by a dual argument.

Proposition 5.3. If $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod} \Gamma$ is a full functor and \mathscr{D} is a d-cluster tilting subcategory of mod Γ , then T satisfies (b).

Proof. Let $X \in \mathscr{T}$ be indecomposable with $\mathscr{T}(T, X) \neq 0$.

If $\mathscr{T}(T, X)$ is a projective Γ -module, then $X \in \operatorname{add} T$ by Lemma 2.3, so the trivial (d+2)-angle $0 \to \cdots \to 0 \to X \xrightarrow{1_X} X \to \Sigma^d 0$ can be used in (b).

Suppose that $\mathscr{T}(T, X)$ is not a projective Γ -module. By Lemmas 2.1(i) and 2.2(i), the augmented minimal projective resolution of $\mathscr{T}(T, X)$ can be written in the form

$$\cdots \xrightarrow{\mathscr{T}(T, f_{d+1})} \mathscr{T}(T, T_d) \xrightarrow{\mathscr{T}(T, f_d)} \cdots \xrightarrow{\mathscr{T}(T, f_1)} \mathscr{T}(T, T_0) \xrightarrow{\mathscr{T}(T, f_0)} \mathscr{T}(T, X) \to 0$$

with the T_i in add T. The morphism f_d can be completed to a (d+2)-angle, which is the first row in the following diagram.

We will use Lemma 5.1 repeatedly. We start by verifying conditions (i)-(v) in the lemma for some of the objects and morphisms in the diagram.

(i): We already know that the T_i are in add T.

(ii): The identity morphisms in the diagram are isomorphisms.

(iii): When constructing the (d + 2)-angle in the first row of the diagram, we can assume $g_i \in \operatorname{rad}_{\mathscr{T}}$ for $0 \leq i \leq d-2$ by [14, lem. 5.18(2)]. Then g_i is left minimal for $1 \leq i \leq d-1$ by [3, lem. 2.11]. Moreover, $\mathscr{T}(T, f_d)$ is a morphism in a minimal projective resolution so is right minimal. By Lemma 2.1(i), so is f_d . Then $\Sigma^d f_d$ is right minimal, forcing $h \in \operatorname{rad}_{\mathscr{T}}$ and hence g_0 left minimal by [3, lem. 2.11]. Summing up, g_i is left minimal for $0 \leq i \leq d-1$.

(iv): The complex

$$\mathscr{T}(T, T_{d-1}) \xrightarrow{\mathscr{T}(T, f_{d-1})} \cdots \xrightarrow{\mathscr{T}(T, f_1)} \mathscr{T}(T, T_0) \xrightarrow{\mathscr{T}(T, f_0)} \mathscr{T}(T, X) \to 0$$

is a *d*-cokernel of $\mathscr{T}(T, f_d)$ by Lemma 1.5. In particular, $\mathscr{T}(T, f_i)$ is a weak cokernel of $\mathscr{T}(T, f_{i+1})$ for $1 \leq i \leq d-1$, and $\mathscr{T}(T, f_0)$ is a cokernel of $\mathscr{T}(T, f_1)$.

(v): Lemma 1.4 says that $\mathscr{T}(T, f_i)$ is left minimal for $1 \leq i \leq d$. By Lemma 2.1(i) this implies that f_i is left minimal for $1 \leq i \leq d$.

We can now use Lemma 5.1 repeatedly to get the following commutative diagram.

In the final step, we only know that g_0 is left minimal, not that f_0 is left minimal, so Lemma 5.1 only gives a split monomorphism $v: Y \hookrightarrow X$. However, X is indecomposable so v is either zero or an isomorphism. If it were zero, then the rightmost commutative square in the diagram would force $f_0 = 0$ whence $\mathscr{T}(T, f_0) = 0$, contradicting that $\mathscr{T}(T, f_0)$ is a surjection onto the non-zero module $\mathscr{T}(T, X)$. It follows that v is an isomorphism.

Hence the rightmost commutative square in the diagram implies that $\mathscr{T}(T, g_0)$ is surjective, and so $\mathscr{T}(T, h) = 0$ by [4, prop. 2.5(a)]. Hence the (d+2)-angle in the first row of the diagram can be used in (b).

Proof of Theorem 0.5, the implication (i) \Rightarrow (iii): Assuming condition (i) in Theorem 0.5, the object *T* satisfies (strong a) and (strong a') by Proposition 5.2, and (b) by Proposition 5.3.

6. Proof of Theorem 0.6 and Corollary 0.7

Recall that we still assume Setup 0.4.

Proof of Theorem 0.6:

(i): This follows from Theorem 0.5 and Lemma 3.2.

(ii): Theorem 0.5 and Lemma 3.2 show that $\mathscr{T}(T, -) : \mathscr{T} \to \text{mod}\,\Gamma$ is full, so (ii) amounts to showing that if $g: X \to Y$ is a morphism in \mathscr{T} , then

$$\mathscr{T}(T,g) = 0 \Leftrightarrow g$$
 factors through an object in add $\Sigma^d T$.

To show \Rightarrow , consider the (d+2)-angle in Definition 0.3(ii). Since $T_0 \in \operatorname{add} T$, the condition $\mathscr{T}(T,g) = 0$ implies $gf_0 = 0$. By [4, prop. 2.5(a)] there is $g' : \Sigma^d T_d \to Y$ such that g = g'h. That is, g has been factored through $\Sigma^d T_d \in \operatorname{add} \Sigma^d T$. The implication \Leftarrow is clear since $\mathscr{T}(T, \Sigma^d T) = 0$ by Definition 0.3(i).

(iii): Recall that S is the Serre functor of \mathscr{T} . By Definition 0.3(ii) there is a (d+2)-angle in \mathscr{T} ,

$$T_d \to \cdots \to T_0 \to ST \to \Sigma^d T_d,$$

with the T_i in add T. Applying $\mathscr{T}(T, -)$ gives a sequence in mod Γ ,

$$\mathscr{T}(T,\Sigma^{-d}ST) \to \mathscr{T}(T,T_d) \to \dots \to \mathscr{T}(T,T_0) \to \mathscr{T}(T,ST) \to \mathscr{T}(T,\Sigma^dT_d),$$

which is exact by [4, prop. 2.5(a)]. By Serre duality we have

$$\mathscr{T}(T, \Sigma^{-d}ST) \cong D\mathscr{T}(T, \Sigma^{d}T) = 0 \text{ and } \mathscr{T}(T, ST) \cong D\mathscr{T}(T, T) = D\Gamma$$

as right- Γ -modules. Moreover, $\mathscr{T}(T, \Sigma^d T_d) = 0$ by Definition 0.3(i). The sequence hence reads

$$0 \to \mathscr{T}(T, T_d) \to \cdots \to \mathscr{T}(T, T_0) \to (D\Gamma)_{\Gamma} \to 0.$$

This provides a projective resolution of $(D\Gamma)_{\Gamma}$ with at most d + 1 non-zero projective modules. Consequently, each injective right Γ -module has projective dimension $\leq d$.

The opposite category $\mathscr{T}^{^{\mathrm{op}}}$ is (d+2)-angulated, and T is a cluster tilting object in the sense of Oppermann–Thomas of $\mathscr{T}^{^{\mathrm{op}}}$ with endomorphism algebra $\operatorname{Ext}_{\mathscr{T}^{^{\mathrm{op}}}} T = \Gamma^{^{\mathrm{op}}}$. Applying to this setup what we already proved shows that each injective right $\Gamma^{^{\mathrm{op}}}$ module has projective dimension $\leq d$. That is, each injective left Γ -module has projective dimension $\leq d$.

The statements about the injective dimension of projective modules follow by k-linear duality.

(iv): It is well-known that if the global dimension of Γ is finite, then it is equal to the projective dimension of $(D\Gamma)_{\Gamma}$, so (iv) follows from (iii).

Proof of Corollary 0.7:

(i): Lemma 2.4 says that \mathscr{D} has finitely many indecomposable objects, so $\mathscr{D} = \operatorname{add} M$ for some $M \in \operatorname{mod} \Gamma$. Theorem 0.6(i) says that \mathscr{D} is a *d*-cluster tilting subcategory of $\operatorname{mod} \Gamma$, so M is a *d*-cluster tilting module. Hence Γ is weakly *d*-representation finite.

(ii): If Γ has finite global dimension, then the global dimension is at most d by Theorem 0.6(iv). Since Γ is weakly d-representation finite by (i), it is then d-representation finite.

7. Two classes of examples

We conclude with two classes of examples. The first shows how Theorem 0.6 and Corollary 0.7 imply [14, thm. 5.6]. Recall that the notion of d-representation finite algebras was defined in [8, def. 2], and that large classes of such algebras exist, see for instance [6, thm. 3.11] and [8, sec. 5].

Example 7.1. Let Λ be a *d*-representation finite *k*-algebra. In [14, sec. 5] was constructed a so-called (d+2)-angulated cluster category \mathcal{O}_{Λ} . Let $T \in \mathcal{O}_{\Lambda}$ be a cluster tilting object in the sense of Oppermann–Thomas with endomorphism algebra $\Gamma = \operatorname{End}_{\mathcal{O}_{\Lambda}} T$.

Our results apply to this situation because the conditions of Setup 0.4 are satisfied: The category \mathcal{O}_{Λ} is k-linear Hom-finite with split idempotents by construction, see [14, thms. 5.14 and 5.25], and it has a Serre functor by [14, thm. 5.2]. Observe that \mathcal{O}_{Λ} has finitely many indecomposable objects by [14, thm. 5.2(1)].

We can recover the results of [14, thm. 5.6] on \mathscr{O}_{Λ} , T, and Γ as follows: Consider the functor $\mathscr{O}_{\Lambda}(T, -) : \mathscr{O}_{\Lambda} \to \mod \Gamma$. Theorem 0.6 says that its essential image \mathscr{D} is *d*-cluster tilting in mod Γ , that $\mathscr{O}_{\Lambda}(T, -)$ induces an equivalence of categories

$$\mathscr{O}_{\Lambda}/[\operatorname{add}\Sigma^{d}T] \xrightarrow{\sim} \mathscr{D},$$

and that Γ is *d*-Gorenstein. Corollary 0.7 says that Γ is weakly *d*-representation finite, and that if it has finite global dimension, then it is *d*-representation finite.

Example 7.2. Let Λ be a *d*-representation finite *k*-algebra. Let \mathscr{F} be the unique *d*-cluster tilting subcategory of mod Λ , and consider the full subcategory

$$\overline{\mathscr{F}} = \operatorname{add}\{\Sigma^{id}F \mid i \in \mathbb{Z}, F \in \mathscr{F}\}\$$

of the derived category $\mathscr{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. It is clearly invariant under Σ^d , and it is a *d*-cluster tilting subcategory of $\mathscr{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ by [7, thm. 1.21], so it is a (d+2)-angulated category by [4, thm. 1].

Our results apply to this situation because the conditions of Setup 0.4 are satisfied: The category $\overline{\mathscr{F}}$ is k-linear Hom-finite with split idempotents because it is a full subcategory of $\mathscr{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ closed under direct summands, and [9, thm. 3.1] implies that the Serre functor S_D of $\mathscr{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ restricts to a Serre functor S of $\overline{\mathscr{F}}$.

Set $T = \Lambda_{\Lambda}$. Then $\overline{\mathscr{F}}(T, -)$ is a restriction of $\operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)}(\Lambda_{\Lambda}, -)$, so the endomorphism algebra $\operatorname{End}_{\overline{\mathscr{F}}}T$ is

$$\overline{\mathscr{F}}(T,T) \cong \Lambda$$
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and the functor $\overline{\mathscr{F}}(T,-):\overline{\mathscr{F}}\to \operatorname{mod}\Lambda$ can be identified with

$$\mathrm{H}^{0}: \overline{\mathscr{F}} \to \mathrm{mod}\,\Lambda.$$

By definition, each object of $\overline{\mathscr{F}}$ is a finite direct sum of the form

$$\overline{F} = \bigoplus_i \Sigma^{id} F_i$$

and $\mathrm{H}^{0}(\overline{F}) = F_{0}$. It follows that the essential image of $\overline{\mathscr{F}}(T, -) = \mathrm{H}^{0}(-)$ is \mathscr{F} , which is *d*-cluster tilting in mod Λ , and that $\overline{\mathscr{F}}(T, -) = \mathrm{H}^{0}(-)$ is full.

By Theorem 0.5 the object T satisfies (strong a), (strong a'), and (b). However, T does not satisfy (c) since $\Sigma^{2d}T$ is mapped to 0 by $\overline{\mathscr{F}}(T,-) = \mathrm{H}^{0}(-)$, but is not in $\mathrm{add} \Sigma^{d}T$.

Finally, the category $\overline{\mathscr{F}}$ is stable under the functor $S_d = S_D \Sigma^{-d}$ by [9, thms. 2.16 and 2.21], where S_D is again the Serre functor of $\mathscr{D}^{\rm b}(\operatorname{mod} \Lambda)$. The functor S_d plays the role of AR translation of $\overline{\mathscr{F}}$; in particular, it is an autoequivalence of $\overline{\mathscr{F}}$. It follows that for each integer ℓ , the object $S_d^{\ell}T$ also satisfies (strong a), (strong a'), and (b), but not (c).

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