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Discussion of 'Nonparametric generalized fiducial inference for survival functions under censoring'

G. Taraldsen and B.H. Lindqvist

Department of Mathematical Sciences
Norwegian University of Science and Technology
NTNU, NO-7491 Trondheim, Norway

Gunnar.Taraldsen@ntnu.no and Bo.Lindqvist@ntnu.no

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Abstract

The following discussion is inspired by the paper *Nonparametric generalized fiducial inference for survival functions under censoring* by Cui and Hannig. The discussion consists of comments on the results, but also indicates it's importance more generally in the context of fiducial inference. A two page introduction to fiducial inference is given to provide a context.

Keywords: Foundations and philosophical topics (62A01); Bayesian; Fiducial; Frequentist

1 Fiducial inference

We expect that many readers are not familiar with fiducial inference. This is in contrast to the well founded alternatives given by Bayesian and classical inference known to every statistician today. Fiducial inference has not yet been established as a general theory, but there has been considerable progress on this during the last decades, as also demonstrated by Cui and Hannig (2019). To discuss their contribution we need to provide a context given by fiducial inference as we see it today.

The original fiducial argument of Fisher (1930, p.532) starts by considering the relation

$$u = F(x) \tag{1}$$

where F is the cumulative distribution function for the observation x. Fisher considers in particular the case where x is the empirical correlation of a sample of size n from the bivariate Gaussian distribution. In this case F is strictly decreasing from 1 down to 0 as a function of the unknown correlation θ. From this, Fisher argues that $1 - F(x \mid \theta)$ is the cumulative fiducial distribution for θ, and that $\pi^x(\theta) = -\partial_\theta F(x \mid \theta)$ is the fiducial density of θ given x.

Fisher's argument uses the fact that equation (1) gives a correspondence between a uniform law for u and the sampling law for x. The argument explains, in fact, that the percentiles of the fiducial distribution give confidence intervals, and hence that the fiducial distribution is a confidence distribution in this case. Even though Fisher himself abandoned this interpretation in later works, it must be seen as one of the pioneering works that lead to the theory of confidence intervals and hypothesis testing as used today. It is, as far as we know, the first paper that calculates exact confidence intervals and explain them as such.

Fiducial inference, in the version considered here, is given by replacing the relation (1) by a *fiducial model*

$$x = \theta u \tag{2}$$

This economic notation is used by Dawid and Stone (1982, p.1055) when they define a *functional model*. It is a generalization of the *structural models* of Fraser (1968) who considers the case where the model space Ω_Θ is a group, and θu is the action of θ on u. Cui and Hannig (2019, eq.1) refer to equation (2) as a *data generating equation*. Samples from a known distribution for u gives samples from the distribution of the observation x. In modern statistics, the possibility of simulating data from a statistical model is most central, and any such algorithm is in fact a fiducial model.

Equation (1) can be inverted to give $x = \theta u = F^{-1}(u)$, where F depends on θ. Fisher's initial model is hence a special case of a fiducial model. Consider for a moment the following problem:

> *The observation x is given and known to be generated from the fiducial model (2) by sampling u from a known distribution. How would You quantify Your uncertainty about the unknown model parameter θ?*

It is clear that both u and θ are still uncertain, and it is reasonable, we claim, to quantify these uncertainties by a joint distribution for (u, θ) such that equation (2) holds. Define $\theta = xu^{-1}$ to be a measurable selection solution of equation (2) for those (x, u) that allows a solution. Assume, as we will exemplify below, that there exists a fiducial distribution for u^x derived from the original distribution of u and the observation x. A fiducial distribution for the model θ can then be defined to be the distribution of

$$\theta^x = x\,(u^x)^{-1} \tag{3}$$

The fiducial distribution quantifies the uncertainty of θ given the assumed fiducial model and given the observation x. This interpretation of the fiducial is what Fisher (1973, p.54-55) aimed at in his final writing on this:

> *By contrast, the fiducial argument uses the observations only to change the logical status of the parameter from one in which nothing is known of it, and no*

probability statement about it can be made, to the status of a random variable having a well-defined distribution.

The correlation coefficient example treated initially by Fisher is such that the fiducial equation (2) defines a one-one correspondence between any two variables when the third is fixed. In this case, a simple fiducial model, the distribution of u^x can be set equal to the original distribution of u. Fiducial samples are obtained simply by solving the fiducial equation for each sample u and returning the solution $\theta^x = xu^{-1}$.

Another example is given by $x = \theta u = \theta + u$, where θ is an element of a subspace Ω_Θ of a Hilbert space Ω_X. An important class of problems is obtained by letting Ω_Θ be the image space of the design matrix in linear regression. In this case, the fiducial equation will fail to have solutions for all (x, u). Let P be the orthogonal projection on Ω_Θ, and let $Q = 1 - P$. Define the law of u^x to be the conditional law of u given $Qu = Qx$. The fiducial is then $\theta^x = x[u^x]^{-1} = x - u^x$.

The previous example includes the general case of a location parameter, and in particular inference based on sampling from the Gaussian distribution with unknown mean and known variance. As demonstrated by Fraser (1968), this can be seen as a particular case of a group Ω_Θ acting on the observation space Ω_X, and cases with unknown variance can also be included by considering other group actions. It follows in these cases, as also for the simple fiducial models, that the fiducial is a confidence distribution. Furthermore, Taraldsen and Lindqvist (2013) have proved that classical optimal actions, if they exist, are determined by the fiducial if the loss is invariant. Incidentally, the previous also exemplify a nonparametric fiducial in the sense given by an infinite dimensional Ω_Θ.

The previous indicate that a fiducial model (2) can be used to obtain a distribution with interpretation similar to a Bayesian posterior as intended originally by Fisher. It also show that confidence distributions and classical optimal actions can be obtained by fiducial arguments. Finally, a fiducial model (2) can also be used as a method for sampling from a Bayesian posterior. In a Bayesian set-up the joint distribution of (u, θ) is specified, and the distribution of u used above must be identified with the conditional distribution of u given θ. Sampling from the posterior can be done by sampling u conditionally given x and then θ given (u, x). In the case of group actions with prior equal to the right invariant prior this gives that the posterior coincides with the fiducial.

2 The results in the paper and future research

Cui and Hannig (2019) consider failure distributions based on right censored data in a nonparametric case. For simplicity, and since we will focus on the theoretical principles, we will focus on the uncensored case. Before leaving the censored case we will emphasize its importance in applications, and add, as we see it, that the fiducial model for this case is most natural. The ease of including this in the analysis is by itself a most convincing argument for the success of fiducial inference as demonstrated by Cui and Hannig (2019).

The obvious choice, in retrospect, is to base nonparametric fiducial inference on Fisher's original fiducial relation in equation (1). The data is given by an ordered sample x that obeys the fiducial relation $u_i = F(x_i)$, or equivalently the fiducial model

$x_i = F^{-1}(u_i)$. Here $u_1 \leq \cdots \leq u_n$ is the order statistic of a random sample from the uniform distribution on $[0, 1]$. A fiducial distribution for the unknown cumulative distribution function F is given by a measurable selection solution of this fiducial relation. We can and will restrict attention to the case where it is assumed that F is absolutely continuous in accordance with Cui and Hannig (2019, Assumption 2). In this case it follows hence that the fiducial distribution for u^x equals the original distribution for u as in Fishers original fiducial argument for the correlation coefficient. In contrast to Fishers original argument there is here an infinity of possible randomized measurable selection solutions. It can, additionally, be observed that the given fiducial model is equivalent with a group model $x = \theta v$: Ω_Θ is the group of increasing and differentiable transformations θ of the positive real line and $v_1 \leq \cdots \leq v_n$ is the order statistic of a random sample from the standard exponential distribution.

A particular absolutely continuous fiducial F^I is determined by log-linear interpolation as described by Cui and Hannig (2019). This gives fiducial distributions for any parameters of interest, and in particular for $F(x)$ for a fixed x and the percentiles x_α for a fixed α. The case with k samples can be treated similarly by the joint fiducial for $F_1, \ldots, F_k$. It is straightforward, in principle, to calculate corresponding fiducial intervals or regions and corresponding fiducial p-values. This is exemplified by Cui and Hannig (2019) by a series of examples for $k = 1, 2$, and good frequentist properties are demonstrated as compared with existing methodology. The group model structure opens the question: Is optimal equivariant inference possible?

The demonstrations, and the previous two paragraphs, constitute, in our opinion, the main message of the paper. Many more examples can, and should, be published based on concrete applied problems, and the indicated natural route for nonparametric inference. An alternative approach is to take your favorite book on nonparametric inference and implement and experiment with corresponding fiducial solutions.

Proofs of stated coverage in the finite sample case are absent, but for $k > 1$ this can be expected to be a long standing challenge as even the Behrens-Fisher problem remains unsettled. The $k = 1$ case seems possible to analyse completely, and the methodology should then be compared with similar results for the uncensored case presented by Schweder and Hjort (2016, Chap.11). It should be noted that Schweder and Hjort (2016) only consider confidence distributions for real valued parameters, and not for the unknown F itself. It is, in fact, unknown if the fiducial for F is a confidence distribution in a strict sense. The group model structure gives a starting point for investigating this further.

All of these questions are related to the choice of a measurable selection solution. Is there a natural choice? Is there a best choice? This question should be investigated in concrete data situations. It can be observed that the choice F^I is quick and convenient, but each realization is so special that it is not realistic in most situations. An alternative, which is still quick and convenient, is given by monotonic spline interpolation. The fiducial distribution given by F^I has defects when considered as a fiducial distribution for F, but the simulations demonstrate that resulting finite dimensional fiducials of certain focus parameters have excellent properties.

In summary, what is the possible role of the fiducial argument and distribution? The following Bayesian-Fiducial-Frequentist list give guidance:

(B) Alternative algorithms for Bayesian analysis.

(F) A posterior fiducial state interpreted as Fisher intended.

(F) Alternative algorithms for frequentist analysis.

All of this, seen in retrospect, is excellently presented and exemplified by Fraser (1968) for classical linear models. We believe that Cui and Hannig (2019) have taken the first important step for similar results in the nonparametric case. Their main technical result proves that the nonparametric fiducial is asymptotically a confidence distribution.

3 Conclusion

We take the opportunity of expressing our thanks for the invitation to comment on the interesting and thought-provoking paper by Cui and Hannig (2019). This paper will serve as motivation for further developments of the theory of fiducial inference as initiated by Fisher in the *Inverse probability* paper from 1930. The importance of the 1930 paper by Fisher, lies, according to Fisher (1950), in retrospect, in setting forth a new mode of reasoning from observations to their hypothetical causes. We congratulate Cui and Hannig with a successful demonstration of a fiducial argument in a nonparametric problem. In conclusion, we can wholeheartedly and repeatedly agree with Efron (1998, p.107):

> *This is all quite speculative, but here is a safe prediction for the 21st century: statisticians will be asked to solve bigger and more complicated problems. I believe that there is a good chance that objective Bayes methods will be developed for such problems, and that something like fiducial inference will play an important role in this development. Maybe Fisher's biggest blunder will become a big hit in the 21st century!*

Additionally, we believe that the addition of nonparametric fiducial inference, as introduced by Cui and Hannig (2019), will play an important part of this adventure.

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