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MODELS FOR DENSE MULTILANE VEHICULAR TRAFFIC*

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Abstract. We study vehicular traffic on a road with multiple lanes and dense, unidirectional traffic following the traditional Lighthill–Whitham–Richards model where the velocity in each lane depends only on the density in the same lane. The model assumes that the tendency of drivers to change to a neighboring lane is proportional to the difference in velocity between the lanes. The model allows for an arbitrary number of lanes, each with its distinct velocity function. The resulting model is a well-posed weakly coupled system of hyperbolic conservation laws with a Lipschitz continuous source. We show several relevant bounds for solutions of this model that are not valid for general weakly coupled systems. Furthermore, by taking an appropriately scaled limit as the number of lanes increases, we derive a model describing a continuum of lanes, and show that the N-lane model converges to a weak solution of the continuum model.

Key words. Lighthill–Whitham–Richards model, multilane traffic flow, continuum limit

AMS subject classifications. 35L60, 35L65, 35L67, 82B21

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1. Introduction.

The Lighthill–Whitham–Richards (LWR) model for unidirectional traffic on a single road (see [16, 19]) reads

$$u_t + (uv(u))_x = 0, \tag{1.1}$$

where $u = u(t, x)$ denotes the density of vehicles at the position x and time t, and $v = v(u)$ is a given velocity function. The LWR model expresses conservation of vehicles and is a well-established model for dense unidirectional single lane vehicular traffic on a homogeneous road without exits and entries. Furthermore, it serves as the standard textbook example to gain intuition regarding the behavior of solutions of scalar one-dimensional hyperbolic conservation laws; see, e.g., [13].

Given the importance of vehicular traffic modeling in modern society, it is no wonder that the LWR model has been generalized to describe several important scenarios in dense traffic flow. Indeed, "traffic hydrodynamics" has become a research field in its own right, where the flow of vehicles is modeled by conservation laws or balance equations. In the general context, the LWR model is the simplest model among the many hydrodynamic traffic models. Among the other models often used is the Aw–Rascle model [1], which is a system of conservation laws where the velocity v is not a given function of u, but satisfies a second conservation law. It is thus considerably more complicated than the simple LWR model. For a general introduction to how conservation laws are used in traffic modeling, see [12, 4] and the many references therein.

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In this paper we introduce a new model for multilane dense vehicular traffic where the underlying model for each lane remains the LWR model. Our basic assumption is that drivers prefer to drive faster, and that the tendency of a vehicle to change lanes is proportional to the difference in velocity between neighboring lanes. If (1.1) describes the density of vehicles in a particular lane, the multilane behavior is described by a source term, accounting for lane changes. The result is thus a system of weakly coupled scalar conservation laws.

More precisely, consider two lanes denoted 1 and 2. The model we study reads

$$\begin{aligned}\partial_t u_1 + \partial_x(u_1 v_1(u_1)) &= -S(u_1,u_2),\\ \partial_t u_2 + \partial_x(u_2 v_2(u_2)) &= S(u_1,u_2),\end{aligned}$$

where the change of lanes is codified in

$$S(u_1,u_2) = K(v_2(u_2) - v_1(u_1)) \cdot \begin{cases} u_1 & v_2(u_2) \geq v_1(u_1),\\ u_2 & v_2(u_2) < v_1(u_1),\end{cases}$$

where K is a constant of proportionality. Here u_i denotes the density in lane i. The system constitutes a weakly coupled 2×2 system of one-dimensional hyperbolic conservation laws, and there is ample theory available for systems of this type; see section 2. The system readily generalizes to an arbitrary number of lanes; see section 3. We show that the general system with N lanes has a unique entropy solution, and that the solution is well posed in the sense that one has a surprising L^1 stability,

$$\sum_{i=1}^{N} \|u_i(t) - \bar{u}_i(t)\|_{L^1(\mathbb{R})} \leq \sum_{i=1}^{N} \|u_{i,0} - \bar{u}_{i,0}\|_{L^1(\mathbb{R})},$$

for two solutions u_i and $\bar{u}_i$; see Theorems 3.2 and 3.3. Note that the L^1 stability *does not* hold, in general, for systems of balance laws, that is, hyperbolic conservation laws with source.

The model invites considering the continuum limit where the number of lanes increases to infinity. We organize the parallel lanes along the x-axis, and measure the distance between the lanes along the y-axis. The distance between the lanes is scaled as $\Delta y = 1/N$, where N denotes the number of lanes. For simplicity we assume that the velocity function is given by $v_i(u) = -k(y_i)g(u)$ for all $u \in [0,1]$, where $y_i = i\Delta y$, and $-g(u)$ is the velocity function. We scale the function such that $g(0) = -1$ and $g(1) = 0$.

We consider given initial data $u_0 \colon \mathbb{R}\times[0,1] \to [0,1]$, where the initial data for lane i is $u_{i,0}$, given by (4.21) and with solution u_i. We interpolate this function to $u_{\Delta y}$ where $u_{\Delta y} \colon [0,\infty)\times\mathbb{R}\times[0,1] \to [0,1]$. We assume that k is smooth and positive with $k'(0) = k'(1) = 0$. In Theorem 4.2 we show, provided the constant K scales as $1/\Delta y^2$, that $u_{\Delta y} \to u$ where u is a weak solution of

$$\begin{cases} u_t + kf(u)_x + (k'f(u))_y = \left(kug_y\right)_y,\\ g(u)_y|_{y=0,1} = 0,\\ u|_{t=0} = u_0,\end{cases} \tag{1.2}$$

where the *flux function* f is defined as $f(u) = uv(u)$. This equation is an interesting anisotropic and degenerate parabolic equation with nontrivial boundary conditions in the y-direction.

There is a plethora of approaches to the modeling of multilane dense traffic. See [17] for an early approach, and consult [18] for a survey up to 2010 of various models for lane changing. Microscopic models based on kinetic theory are studied in [14, 15]. A macroscopic model derived from individual behavior can be found in [6]. A multilane model where the focus is on the total vehicle density across all lanes is analyzed in [5]. A rather different approach is taken in [7], where traffic is studied as a two-dimensional flow problem, and in [8, 9] the analysis is extended to the second-order Aw–Rascle model and a hybrid stochastic kinetic model, respectively. The approach in [3] is more similar to the analysis presented here, however, with a different source term.

The novel model we present here is conceptually simple, captures an essential aspect of lane changing, while at the same time allowing for a rigorous mathematical treatment.

One could, of course, let the lanes have finite length. This would entail prescribing boundary conditions for each lane. Boundary conditions for scalar conservation laws are (by now) well understood, though quite technical; see, e.g., the pioneering paper [2]. Therefore, we have chosen to work with lanes of infinite length in this paper.

The rest of this paper is organized as follows: In section 2 we detail the two-lane case, and show that $u_i \in [0, 1]$ is an invariant region. In section 3 we state the N-lane model, and prove a number of estimates on the solution. Finally, in section 4, we study the limit as $N \to \infty$. Analogously to the analysis of numerical schemes for degenerate parabolic equations, we establish enough estimates on the solution, enabling us to conclude that a limit exists, and that this limit is a weak solution of a degenerate convection-diffusion equation. All sections are illustrated by numerical examples.

2. A continuum model for two-lane vehicular traffic.

Consider a road with two lanes, each with its own velocity function. The lanes are homogeneous, and traffic on the road is unidirectional. We assume that the vehicular traffic is dense, allowing for a continuum formulation. Let u_i and $v_i = v_i(u_i)$ denote the density and velocity, respectively, in lane i.

In this paper we focus on the interaction between the two lanes. We assume that drivers prefer to drive in the faster lane, and the tendency of a vehicle to change lanes is proportional to the difference in velocity. Thus the flow *from* lane 1 *to* lane 2 equals

$$
\begin{aligned}
S(u_1, u_2) &= K(v_2(u_2) - v_1(u_1)) \cdot \begin{cases} u_1 & v_2(u_2) \geq v_1(u_1), \\ u_2 & v_2(u_2) < v_1(u_1), \end{cases} \\
&= K\left[(v_2(u_2) - v_1(u_1))^+ u_1 - (v_2(u_2) - v_1(u_1))^- u_2\right], \qquad (2.1)
\end{aligned}
$$

where K is a constant, $(a)^+ = \max\{a, 0\}$ and $(a)^- = -\min\{a, 0\}$. The flow *from* lane 2 *to* lane 1 equals $-S(u_1, u_2)$. The classical LWR model implies the following model describing the two-lane traffic:

$$
\partial_t u_1 + \partial_x (u_1 v_1(u_1)) = -S(u_1, u_2), \qquad (2.2a)
$$

$$
\partial_t u_2 + \partial_x (u_2 v_2(u_2)) = S(u_1, u_2), \qquad (2.2b)
$$

where x is the position along the road and t denotes time. This 2×2 system of hyperbolic conservation laws is weakly coupled with a Lipschitz continuous source term.

The velocities $v_i = v_i(u_i)$ are strictly decreasing positive functions, and we assume that they are scaled such that $v_1(1) = v_2(1) = 0$. For simplicity, we scale space and time such that $K = 1$.

It is well known that this system, in general, only allows for weak solutions $u_i \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, the set of integrable functions of finite total variation; see, e.g., [13]. Furthermore, the issue of uniqueness of the solution is nontrivial and one needs to require that the solution satisfies an entropy condition.

DEFINITION 2.1. *Let $v_i = v_i(u_i)$ be strictly decreasing positive functions such that $v_1(1) = v_2(1) = 0$. Assume that $u_{i,0} \in L^1([0,1]) \cap BV([0,1])$ for $i = 1, 2$. We say that $u = \{u_1, u_2\}$, where $u_i \in C([0,\infty); L^1(\mathbb{R}))$ with $u_i(t,\cdot) \in BV(\mathbb{R})$ for $t \in [0,\infty)$ for $i = 1, 2$ is a weak solution of (2.2) with initial data $u_{i,0}$ if*

$$\int_0^\infty \int_{\mathbb{R}} \big(u_1\varphi_t + u_1 v_1(u_1)\varphi_x - S(u_1,u_2)\varphi\big)\,dxdt + \int_{\mathbb{R}} u_{1,0}\varphi|_{t=0}\,dx = 0,$$
$$\int_0^\infty \int_{\mathbb{R}} \big(u_2\varphi_t + u_2 v_2(u_2)\varphi_x + S(u_1,u_2)\varphi\big)\,dxdt + \int_{\mathbb{R}} u_{2,0}\varphi|_{t=0}\,dx = 0$$

for all compactly supported test functions $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R})$.

The solution is called an entropy solution if

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \big(\eta(u_1)\varphi_t + q_1(u_1)\varphi_x\big)\,dxdt + &\int_{\mathbb{R}} \eta(u_{1,0})\varphi|_{t=0}\,dx \\ &\geq \int_0^\infty \int_{\mathbb{R}} \eta'(u_1)\varphi S(u_1,u_2)\,dxdt,\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \big(\eta(u_2)\varphi_t + q_2(u_2)\varphi_x\big)\,dxdt + &\int_{\mathbb{R}} \eta(u_{2,0})\varphi|_{t=0}\,dx \\ &\geq -\int_0^\infty \int_{\mathbb{R}} \eta'(u_2)\varphi S(u_1,u_2)\,dxdt\end{aligned} \tag{2.3b}$$

for all twice differentiable convex functions η where q_i satisfies $q_i'(u) = \eta'(u)f_i'(u)$ with $f_i(u) = uv_i(u)$, and for all compactly supported nonnegative test functions $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R})$, $\varphi \geq 0$.

Remark 2.2. By a density argument it suffices that (2.3) holds for η of the form $\eta(u) = |u-k|$ for all constants $k \in \mathbb{R}$; see [13, Remark 2.1]. In that case $q_i(u) = \operatorname{sign}(u-k)\,(f_i(u) - f_i(k))$.

Remark 2.3. The existence and uniqueness of entropy solutions to (2.2) follows by Theorem 3.2.

Throughout the paper, we will use the following notation:

$$a^\pm = \frac{1}{2}\big(\,|a| \pm a\big), \quad H(a) = \mathbf{1}_{[0,\infty)}(a), \tag{2.4}$$

where $\mathbf{1}_M$ is the indicator (characteristic) function of a set M. Note that

$$\begin{aligned}&0 \leq a^\pm \leq |a|\,, \quad |a| = a^+ + a^-, \quad a = a^+ - a^-, \quad a^+a^- = 0, \quad (\mp a)^- = (\pm a)^+,\\ &H(x) + H(-x) = 1, \quad (x^+)' = H(x), \quad (x^-)' = -H(-x),\ x \neq 0.\end{aligned}$$

We shall also employ the convention that C denotes a "generic" finite positive constant, independent of critical parameters, whose actual value may change from one occurrence to the next. Similarly, we use C_α to denote a positive function $C_\alpha < \infty$ for $\alpha < \infty$.

This model (2.2) has the natural invariant region $u \in [0,1]$. This is the content of the following lemma.

LEMMA 2.4. *Let $u = \{u_1, u_2\}$ be an entropy solution in the sense of Definition 2.1, with initial data $u_{i,0}$ for $i = 1, 2$. If $u_{i,0}(x) \in [0,1]$ for almost all x and $i = 1, 2$, then $u_i(t,x) \in [0,1]$ for almost all x and for $t > 0$.*

Proof. To show that $u_i \geq 0$ if $u_{i,0} \geq 0$, we use the entropy $\eta(u) = u^-$. Then

$$\partial_t (u_i)^- + \partial_x q_i^-(u_i) \leq (-1)^{i+1} H(-u_i) S(u_1, u_2)$$

in $\mathcal{D}'$ for $i = 1, 2$. We use a nonnegative test function $\varphi(x,t) \approx \mathbf{1}_{[0,\tau]}$ to find that

$$\int_{\mathbb{R}} (u_i(\tau,x))^- \, dx \leq \int_{\mathbb{R}} (u_{i,0}(x))^- \, dx + (-1)^{i+1} \int_0^\tau \int_{\mathbb{R}} H(-u_i) S(u_1,u_2) \, dx dt.$$

Adding these two equations and using that $(u_{i,0})^- = 0$, we get

$$\int_{\mathbb{R}} (u_1(\tau,x))^- + (u_2(\tau,x))^- \, dx \leq \int_0^\tau \int_{\mathbb{R}} r(u_1,u_2) \, dx dt,$$

with

$$r(u_1,u_2) = S(u_1,u_2)(H(-u_1) - H(-u_2)).$$

We have that

$$\begin{aligned} & r(u_1,u_2) \\ & \quad = \begin{cases} 0, & u_1 < 0 \text{ and } u_2 < 0, \\ 0, & u_1 > 0 \text{ and } u_2 > 0, \\ -\Big[(v_2(u_2) - v_1(u_1))^+ u_1 - (v_2(u_2) - v_1(u_1))^- u_2 \Big], & u_2 \leq 0 < u_1, \\ \Big[(v_2(u_2) - v_1(u_1))^+ u_1 - (v_2(u_2) - v_1(u_1))^- u_2 \Big], & u_1 \leq 0 < u_2, \end{cases} \\ & \quad \leq \begin{cases} 0, & u_1 < 0 \text{ and } u_2 < 0, \\ 0, & u_1 > 0 \text{ and } u_2 > 0, \\ -(v_2(u_2) - v_1(u_1))^+ u_1, & u_2 \leq 0 < u_1, \\ -(v_2(u_2) - v_1(u_1))^- u_2, & u_1 \leq 0 < u_2, \end{cases} \\ & \quad \leq 0. \end{aligned}$$

Hence $u_i(\tau,x) \geq 0$ for almost all x.

Similarly, by using the convex entropy $\eta(u) = (u-1)^+$ we get

$$\partial_t (u_i - 1)^+ + \partial_x q_i^+(u_i) \leq (-1)^i H(u_i - 1) S(u_1,u_2)$$

in $\mathcal{D}'$, the set of distributions. By the same argument as before, we arrive at

$$\int_{\mathbb{R}} \left[(u_1(\tau,x) - 1)^+ + (u_2(\tau,x) - 1)^+ \right] \, dx \leq \int_0^\tau \int_{\mathbb{R}} r(u_1,u_2) \, dx dt,$$

with

$$r(u_1,u_2) = S(u_1,u_2) \left(H(u_2 - 1) - H(u_1 - 1) \right).$$

We have that

$$
\begin{aligned}
&r(u_1,u_2)\\
&\quad=\begin{cases}
0, & u_1<1 \text{ and } u_2<1,\\
0, & u_1>1 \text{ and } u_2>1,\\
-\Big[\left(v_2(u_2)-v_1(u_1)\right)^+u_1-\left(v_2(u_2)-v_1(u_1)\right)^-u_2\Big], & u_2\le 1<u_1,\\
\Big[\left(v_2(u_2)-v_1(u_1)\right)^+u_1-\left(v_2(u_2)-v_1(u_1)\right)^-u_2\Big], & u_1\le 1<u_2,
\end{cases}\\
&\quad=\begin{cases}
0, & u_1<1 \text{ and } u_2<1,\\
0, & u_1>1 \text{ and } u_2>1,\\
-\left(v_2(u_2)-v_1(u_1)\right)^+u_1, & u_2\le 1<u_1,\\
-\left(v_2(u_2)-v_1(u_1)\right)^-u_2, & u_1\le 1<u_2,
\end{cases}\\
&\quad\le 0
\end{aligned}
$$

if u_1 and u_2 are nonnegative. Here we used that $v(u)<0$ if $u>1$. ∎

Remark 2.5. There are also other invariant regions for this equation. If

$$v_2\left(u_{2,0}(x)\right)\ge v_1\left(u_{1,0}(x)\right),$$

then

$$v_2\left(u_2(t,x)\right)\ge v_1\left(u_1(t,x)\right)$$

for $t>0$. This can be shown using similar arguments that are used in the proof of Lemma 2.4.

2.1. An example. We finish our discussion of the two-lane case by exhibiting an example. The velocities on the two roads are

$$v_1(u)=1.5(1-u)\quad\text{and}\quad v_2(u)=2.5(1-u), \tag{2.5}$$

and the initial data

$$u_{1,0}(x)=u_{2,0}(x)=\sin^2(\pi x/2). \tag{2.6}$$

Of course, we do not have entropy solutions in closed form, so instead we use a numerical approximation generated by the Engquist–Osher scheme with 800 grid points and periodic boundary conditions in the interval $[0,2]$. Figure 1 shows the computed solution at $t=0.375$, $t=0.75$, $t=1.125$, and $t=1.5$. For comparison, we have also included the single lane model with the (average of v_1 and v_2) speed $v(u)=2(1-u)$. We see that there is the expected change of lanes to the faster lane, and that a shock builds up in the fast lane to the left of the shock in the slow lane.

3. Multilane model. The model (2.2) can be generalized to an arbitrary number of lanes. Consider a road with N lanes. Traffic is unidirectional and dense. Each lane has its specific velocity function v_i depending only on the density in that lane; thus $v_i=v_i(u_i)$, where u_i is the density in lane i.

Assume that drivers prefer to drive in the faster lane, and this tendency increases with the velocity difference with adjacent lanes. Thus the flow *from* lane i *to* lane $i+1$ equals

$$S_i(u_i,u_{i+1})=\Big[\left(v_{i+1}(u_{i+1})-v_i(u_i)\right)^+u_i-\left(v_{i+1}(u_{i+1})-v_i(u_i)\right)^-u_{i+1}\Big],$$

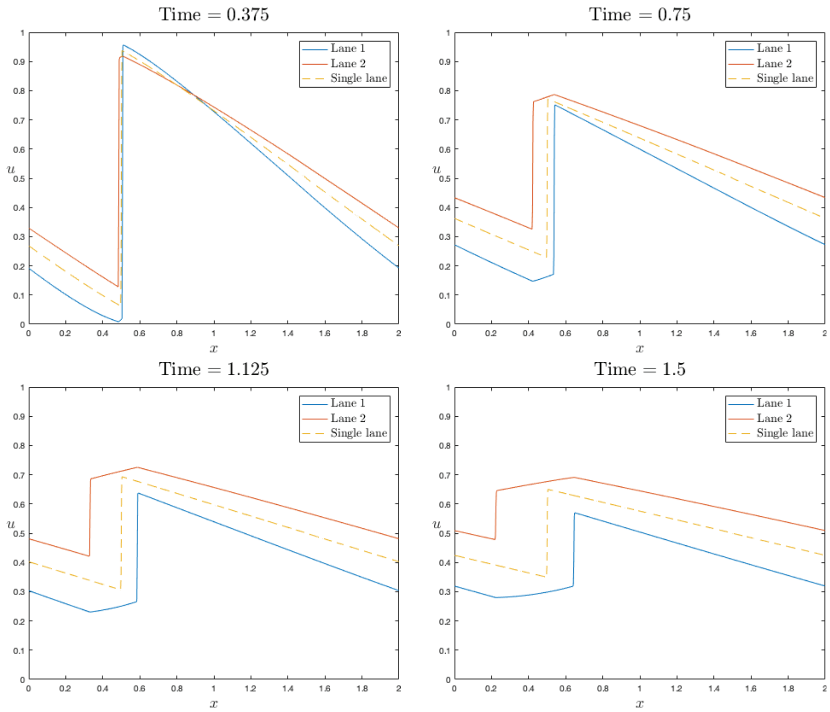

FIG. 1. *The computed solutions of* (2.2) *with* v_1 *and* v_2 *given by* (2.5) *and initial data given by* (2.6).

where we have taken the constant K equal for all lanes, and furthermore scaled time such that this constant of proportionality is one. We then get, in the analogous manner to the derivation of (2.2), that

$$\partial_t u_i + \partial_x \left(u_i v_i(u_i)\right) = S_{i-1}(u_{i-1}, u_i) - S_i(u_i, u_{i+1}), \qquad i = 1, \dots, N, \tag{3.1}$$

coupled with the boundary conditions

$$S_0(u_0, u_1) = S_N(u_N, u_{N+1}) = 0. \tag{3.2}$$

DEFINITION 3.1. *Let* $v_i = v_i(u_i)$ *be Lipschitz continuous functions, and assume that* $u_{i,0} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *for* $i = 1, \dots, N$. *We say that* $u = \{u_i\}_i$ *with* $u_i \in C([0,\infty); L^1(\mathbb{R}))$ *is a weak solution of* (3.1) *with initial data* $u_{i,0}$ *if*

$$\int_0^\infty \int_{\mathbb{R}} \left(u_i \varphi_t + u_1 v_i(u_i) \varphi_x + \left(S_i(u_i, u_{i+1}) - S_{i-1}(u_{i-1}, u_i)\right) \varphi\right) dx dt \\ + \int_{\mathbb{R}} u_{i,0} \varphi|_{t=0} \, dx = 0, \quad i = 1, \dots, N,$$

for all compactly supported test functions $\varphi \in C^\infty([0,\infty) \times \mathbb{R})$.

It is an entropy solution if

$$\int_0^\infty \int_{\mathbb{R}} \left(\eta(u_i)\varphi_t + q_i(u_i)\varphi_x\right) dx dt + \int_{\mathbb{R}} \eta(u_{i,0})\varphi|_{t=0} \, dx \\ \geq \int_0^\infty \int_{\mathbb{R}} \eta'(u_i) \left(S_i(u_i, u_{i+1}) - S_{i-1}(u_{i-1}, u_i)\right) \varphi \, dx dt, \quad i = 1, \dots, N, \tag{3.3}$$

for all convex functions η, and for all nonnegative test functions $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R})$. Here q_i is defined by $q_i'(u) = \eta'(u) f_i'(u)$ with $f_i(u) = uv_i(u)$.

The well posedness of the system of equations (3.1) is ensured by the following general theorem from [11]; see also [10].

THEOREM 3.2 (see [10, Theorem 3.13]). *Assume that v_i and $u_{i,0}$ are as in Definition 3.1. Then there exists a unique entropy solution $u = \{u_i\}_{i=1}^N$. Furthermore, if $\bar u = \{\bar u_i\}_{i=1}^N$ is another entropy solution with initial data $\{\bar u_{i,0}\}_{i=1}^N$, then*

$$\begin{aligned}(3.4)\quad &\sum_{i=1}^N \|u_i(t,\cdot\,) - \bar u_i(t,\cdot\,)\|_{L^1(\mathbb{R})} \\ &\qquad\qquad \le \sqrt{N}\exp\left(2N\sup_i \|S_i\|_{\mathrm{Lip}}\, t\right)\sum_{i=1}^N \|u_{i,0} - \bar u_{i,0}\|_{L^1(\mathbb{R})}\,.\end{aligned}$$

A fundamental property of hyperbolic conservation law is the L^1 contractivity of solutions in the sense that the spatial L^1-norm of the difference between two entropy solutions at a specific time does not increase in time. This property is, in general, lost for weakly coupled systems, or for scalar conservation laws with a source. The general bound (3.4) does not imply L^1 contractivity. However, for system (3.1), the special form of the source yields L^1 contractivity for the whole solution, as the next theorem shows.

THEOREM 3.3. *Consider two entropy solutions $u = \{u_i\}_{i=1}^N$ and $\bar u = \{\bar u_i\}_{i=1}^N$ of (3.1) with initial data $u_0 = \{u_{i,0}\}$ and $\bar u_0 = \{\bar u_{i,0}\}$, respectively. Then we have*

$$\sum_{i=1}^N \int_{\mathbb{R}} |u_i(x,t) - \bar u_i(x,t)|\, dx \le \sum_{i=1}^N \int_{\mathbb{R}} |u_{i,0}(x) - \bar u_{i,0}(x)|\, dx. \tag{3.5}$$

Proof. By using Kružkov's doubling of variables technique we get

$$\begin{aligned}&\partial_t |u_i - \bar u_i| + \partial_x [\mathrm{sign}\,(u_i - \bar u_i)\,(f_i(u_i) - f_i(\bar u_i))] \\ &\le -\,\mathrm{sign}\,(u_i - \bar u_i)\,[S_i(u_i,u_{i+1}) - S_i(\bar u_i,\bar u_{i+1}) - (S_{i-1}(u_{i-1},u_i) - S_{i-1}(\bar u_{i-1},\bar u_i))]\end{aligned}$$

in $\mathcal{D}'$. Subtracting the equation for u_i and adding the equation for $\bar u_i$ we arrive at

$$\begin{aligned}&\partial_t (u_i - \bar u_i)^+ + \partial_x [H(u_i - \bar u_i)\,(f_i(u_i) - f_i(\bar u_i))] \\ &\quad \le -H(u_i - \bar u_i)\,[S_i(u_i,u_{i+1}) - S_i(\bar u_i,\bar u_{i+1}) - (S_{i-1}(u_{i-1},u_i) - S_{i-1}(\bar u_{i-1},\bar u_i))]\end{aligned}$$

in $\mathcal{D}'$. Now let ω_ε be a standard Friedrichs mollifier in one variable, and let $\psi_\varepsilon(x)$ be a smooth function with compact support satisfying

$$\begin{gathered}\psi_\varepsilon(x) = \psi_\varepsilon(-x), \quad 0 \le \psi_\varepsilon(x) \le 1, \\ \psi_\varepsilon(x) = 1 \quad \text{for } |x| < 1/\varepsilon, \quad \psi_\varepsilon(x) = 0 \quad \text{for } |x| > 2/\varepsilon, \quad \text{and} \quad |\psi_\varepsilon'(x)| \le 2\varepsilon.\end{gathered}$$

As a test function, choose

$$\varphi_\varepsilon(x,t) = \big(\omega_\varepsilon * \mathbf{1}_{[0,\tau]}(t)\big)\,\psi_\varepsilon(x),$$

where $*$ denotes convolution. Then pass to the limit $\varepsilon \downarrow 0$ to infer that

$$\begin{aligned}
&\int_{\mathbb{R}} (u_i(x,\tau) - \bar{u}_i(x,\tau))^+ \, dx \\
&\quad \le \int_{\mathbb{R}} (u_i(x,0) - \bar{u}_i(x,0))^+ \, dx + \int_0^\tau \int_{\mathbb{R}} H\left(u_i - \bar{u}_i\right) \\
&\qquad \times \left[S_{i-1}(u_{i-1}, u_i) - S_{i-1}(\bar{u}_{i-1}, \bar{u}_i) - (S_i(u_i, u_{i+1}) - S_i(\bar{u}_i, \bar{u}_{i+1}))\right] dxdt.
\end{aligned} \tag{3.6}$$

Recall that

$$S_i(a,b) = (v_{i+1}(b) - v_i(a))^+ \, a - (v_{i+1}(b) - v_i(a))^- \, b.$$

Now

$$\begin{aligned}
\frac{\partial S_i}{\partial a} &= (v_{i+1}(b) - v_i(a))^+ \\
&\quad - \left(H\left(v_{i+1}(b) - v_i(a)\right) a + H\left(-(v_{i+1}(b) - v_i(a))\right) b\right) v_i'(a) \ge 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial S_i}{\partial b} &= -\left(v_{i+1}(b) - v_i(a)\right)^- \\
&\quad + \left(H\left(v_{i+1}(b) - v_i(a)\right) a + H\left(-(v_{i+1}(b) - v_i(a))\right) b\right) v_{i+1}'(b) \\
&\le 0.
\end{aligned}$$

So if $u_i > \bar{u}_i$,

$$\begin{aligned}
&S_{i-1}(u_{i-1}, u_i) - S_{i-1}(\bar{u}_{i-1}, \bar{u}_i) - (S_i(u_i, u_{i+1}) - S_i(\bar{u}_i, \bar{u}_{i+1})) \\
&\qquad \le S_{i-1}(u_{i-1}, \bar{u}_i) - S_{i-1}(\bar{u}_{i-1}, \bar{u}_i) - (S_i(u_i, u_{i+1}) - S_i(u_i, \bar{u}_{i+1})) \\
&\qquad \le c \max\left\{u_{i-1}, \bar{u}_{i-1}\right\} (u_{i-1} - \bar{u}_{i-1})^+ + c \max\left\{u_{i+1}, \bar{u}_{i+1}\right\} (u_{i+1} - \bar{u}_{i+1})^+ \\
&\qquad \le c\left[(u_{i-1} - \bar{u}_{i-1})^+ + (u_{i+1} - \bar{u}_{i+1})^+\right],
\end{aligned}$$

since u_i and $\bar{u}_i$ are in $[0,1]$, and where $0 < c < |v_i'|$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^N H\left(u_i - \bar{u}_i\right) \left[S_{i-1}(u_{i-1}, u_i) - S_{i-1}(\bar{u}_{i-1}, \bar{u}_i) - (S_i(u_i, u_{i+1}) - S_i(\bar{u}_i, \bar{u}_{i+1}))\right] \\
&\qquad\qquad \le 2c \sum_{i=1}^N \left(u_i - \bar{u}_i\right)^+ .
\end{aligned}$$

Define

$$\Theta(t) = \int_{\mathbb{R}} \sum_{i=1}^N \left(u_i(x,t) - \bar{u}_i(x,t)\right)^+ \, dx;$$

then (3.6) and the above inequality imply that

$$\Theta(T) \le \Theta(0) + 2c \int_0^t \Theta(t)\, dt.$$

Gronwall's inequality then implies that

$$\Theta(T) \le \Theta(0) e^{2cT}.$$

Thus if $\Theta(0) = 0$, i.e., $u_{i,0}(x) \le \bar{u}_{i,0}(x)$ for a.e. x, then $\Theta(T) = 0$ for $T > 0$, i.e., $u_i(x,T) \le \bar{u}_i(x,T)$ for a.e. x.

By the Crandall–Tartar lemma [13, Lemma 2.13], this implies L^1 contractivity; i.e., if u and $\bar{u}$ are entropy solutions to (3.1) with initial data u_0 and $\bar{u}_0$, then (3.5) holds for $t > 0$. ∎

One way to enforce the boundary conditions (3.2) is to define $u_0(x,t) = u_1(t,x)$, $v_0(u) = v_1(u)$, $u_{N+1}(x,t) = u_N(x,t)$, and $v_{N+1}(u) = v_N(u)$. Henceforth we will use this convention.

COROLLARY 3.4. *Let $u = \{u_i\}_{i=1}^N$ be a solution of* (3.1) *with initial data $u_0 = \{u_{i,0}\}_{i=1}^N$, in the sense of Definition* 3.1. *Then we have*

$$\sum_{i=1}^{N-1} \|u_{i+1}(\,\cdot\,,t) - u_i(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le \sum_{i=1}^{N-1} \|u_{i+1,0} - u_{i,0}\|_{L^1(\mathbb{R})}\,. \tag{3.7}$$

Furthermore, we have

$$\sum_{i=1}^{N} |u_i(\,\cdot\,,t)|_{BV(\mathbb{R})} \le \sum_{i=1}^{N} |u_{i,0}|_{BV(\mathbb{R})}\,. \tag{3.8}$$

In addition,

$$\sum_{i=1}^{N} \|u_i(\,\cdot\,,t+h) - u_i(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le \sum_{i=1}^{N} \|u_i(\,\cdot\,,h) - u_i(\,\cdot\,,0)\|_{L^1(\mathbb{R})}\,. \tag{3.9}$$

Proof. Setting $\bar{u}_{i,0} = u_{i+1,0}$ in Theorem 3.3 for $i = 1,\dots,N$ yields (3.7). Similarly, defining $\bar{u}_{i,0}(x) = u_{i,0}(x+h)$, using (3.5), and sending h to zero gives (3.8). To obtain time continuity we define $\bar{u}_{i.0}(x) = u_i(x,h)$, to get (3.9). ∎

We also note the following useful estimates. Define $f_i(u) = uv_i(u)$ and $\Delta_i^- a_i = a_i - a_{i-1}$, divide (3.9) by h, and let $h \downarrow 0$ to find that

$$\sum_{i=1}^{N} \left\| f_i(u_i)_x - \Delta^- S_i(u_i,u_{i+1}) \right\|_{L^1(\mathbb{R})} \le \sum_{i=1}^{N} \left\| f_i(u_{i,0})_x - \Delta^- S_i(u_{i,0},u_{i+1,0}) \right\|_{L^1(\mathbb{R})}\,. \tag{3.10}$$

If we assume that the quantity on the left is bounded by C, then we get

$$\sum_{i=1}^{N} \|u_i(\,\cdot\,,t+h) - u_i(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le Ch. \tag{3.11}$$

Furthermore, we have the useful observation

$$\sum_{i=1}^{N} \left\| \Delta^- S_i((u_i,u_{i+1})(\,\cdot\,,t)) \right\|_{L^1(\mathbb{R})} \le C + \sum_{i=1}^{N} |f_i(u_{i,0})|_{BV(\mathbb{R})}\,. \tag{3.12}$$

3.1. An example. We also include here an example. For $i = 1,\dots,8$ we set $u_{i,0}(x) = \sin^2(\pi x/2)$ and define

$$v_i(u) = k_i(1-u), \quad k_i = \frac{13}{12} + \frac{i-1}{4}, \qquad i = 1,\dots,8. \tag{3.13}$$

Also, in this case the depicted solutions were calculated with the Engquist–Osher scheme with 800 grid points and periodic boundary conditions in the interval $[0, 2]$. Figure 2 shows the computed solutions at $t = 0.375$, $t = 0.75$, $t = 1.125$, and $t = 1.5$. We see the expected change of lanes to the faster lanes, and that a shock builds up in the faster lanes to the left of the slower lanes.

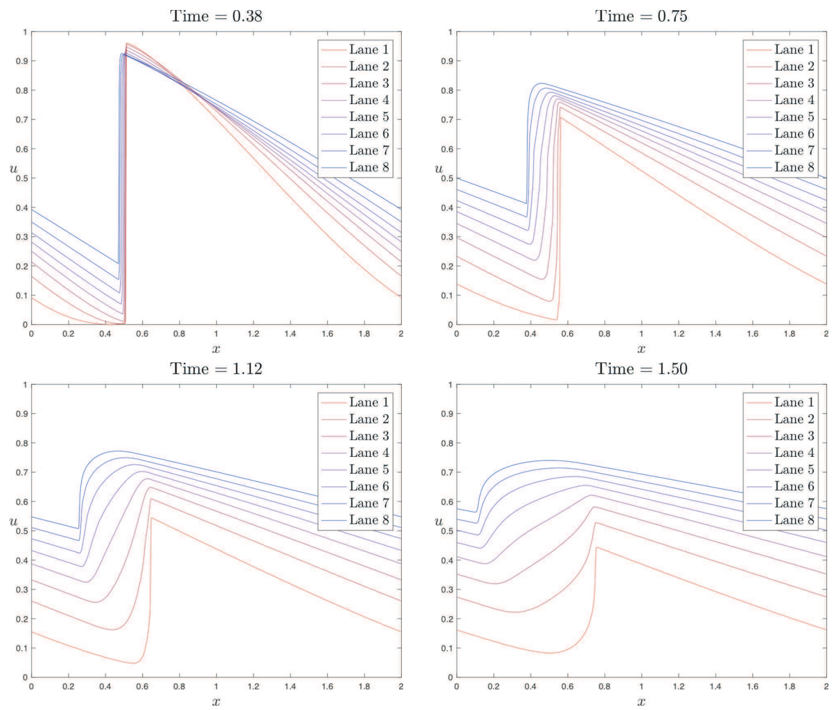

FIG. 2. *The solution of* (3.1) *with* $N = 8$*, and* v_i *given by* (3.13)*. Upper left:* $t = 0.38$*. Upper right:* $t = 0.75$*. Lower left:* $t = 1.12$*. Lower right* $t = 1.50$.

4. Infinitely many lanes—the continuum limit. It is natural, at least mathematically, to consider the case where the lanes increase in number while at the same time get closer. Our aim in this section is therefore to investigate limit as $N \to \infty$ in the system in the previous section.

To this end we let (the number of lanes) N be a positive integer and set $\Delta y = 1/N$. Let $y_i = (i - 1/2)\Delta y$ for $i = 1, \ldots, N$. We shall also use the "divided difference" notation

$$D^{\pm} a_i = \pm \frac{a_{i\pm 1} - a_i}{\Delta y}.$$

For simplicity, we restrict our presentation to the case where $v_i(u) = -k(y_i)g(u)$, where g is a differentiable function with $g'(u) > 0$, $g(0) = -1$, and $g(1) = 0$. Define $f(u) = -ug(u)$. Throughout we will use the notation $f_i = f(u_i)$, $g_i = g(u_i)$, and $k_i = k(y_i)$. Now we reintroduce the scaling constant K in (2.1), and set $K = \kappa/\Delta y^2$. For the reader's convenience we set $\kappa = 1$. Thus, for $i = 1, \ldots, N$, u_i is the unique entropy (in the sense of Definition 3.1) solution of the balance equation

$$\partial_t u_i + k_i \partial_x f(u_i) = \frac{1}{\Delta y^2} \left[S_{i-1}(u_{i-1}, u_i) - S_i(u_i, u_{i+1}) \right], \tag{4.1}$$

with the boundary conditions

$$u_0 = u_1, \quad u_{N+1} = u_N, \quad k_0 = k_1, \quad \text{and} \quad k_{N+1} = k_N.$$

It is also useful to define the function $u_{\Delta y}(t, x, y)$ by

$$u_{\Delta y}(t, x, y) = \begin{cases} u_i(t, x) & \text{if } y \in [y_{j-1/2}, y_{j+1/2}), \ \ i = 1, \dots, N-1, \\ u_N(t, x) & \text{if } y \in [y_{N-1/2}, 1]. \end{cases} \tag{4.2}$$

Next we shall show that the family $\{u_{\Delta y}\}_{\Delta y = 1/N}$, $N \in \mathbb{N}$, is compact, and that the limit is a weak solution to (1.2), in the sense of the following definition.

Definition 4.1. *Set $\Omega = \mathbb{R} \times [0, 1]$ and $\Omega_T = [0, T] \times \Omega$. Let $k = k(y)$ be as above, in particular $k'(0) = k'(1) = 0$. We say that $u \in C([0, \infty); L^1(\Omega))$, such that $uu_y \in L^2(\Omega_T)$, is a weak solution to*

$$\begin{cases} u_t + kf(u)_x + (k'f(u))_y = (kug_y)_y\,, & t > 0, \quad (x, y) \in \mathbb{R} \times (0, 1), \\ g(u)_y = 0, \quad x \in \mathbb{R}, \quad y = 0, \quad y = 1, \\ u(0, x, y) = u_0(x, y), \quad (x, y) \in \mathbb{R} \times (0, 1), \end{cases}$$

if for all test functions $\varphi \in C_0^\infty(\Omega_T)$,

$$\begin{aligned} \int_{\Omega_T} \big(u\varphi_t + kf(u)\varphi_x + k'f(u)\varphi_y\big)\, dydxdt &= \int_{\Pi_T}\int_0^1 kug'(u)u_y\varphi_y\, dydxdt \\ &\quad + \int_\Omega u(T, x, y)\varphi(T, x, y)\, dxdy - \int_\Omega u_0\varphi(0, x, y)\, dxdy. \end{aligned}$$

The next theorem is the main result of this section.

Theorem 4.2. *Let $k \in C^2([0, 1])$ such that $k'(0) = k'(1) = 0$, and $k(y) > 0$ for all $y \in [0, 1]$, and assume that $g = g(u)$ is a strictly increasing differentiable function such that $g(0) = -1$ and $g(1) = 0$.*

Assume that $u_0 \in L^1(\Omega) \cap BV(\Omega)$, and let $u_{\Delta y}$ be defined as in (4.2) where u_i solves (4.1) for $i = 1, \dots, N$.

Then there exists a sequence $N_j \to \infty$ and correspondingly $\Delta y_j = 1/N_j \to 0$ such that the sequence of solutions $\left\{u_{\Delta y_j}\right\}_{j=1}^\infty$ has a limit, i.e.,

$$u = \lim_{j\to\infty} u_{\Delta y_j} \quad \text{in } C([0, \infty); L^1(\Omega)).$$

The limit u is a weak solution according to Definition 4.1.

We also have the regularity estimate

$$\left\|u^2(y_1) - u^2(y_2)\right\|^2_{L^2([0,T]\times\mathbb{R})} \le C\,|y_1 - y_2|\,, \quad y_1, y_2 \in [0, 1]. \tag{4.3}$$

Proof. We first show compactness by a series of estimates, and then proceed to show that any limit is a weak solution.

The right-hand side of (4.1) equals

$$
\begin{aligned}
\frac{1}{\Delta y^2}(S_{i-1}-S_i) &= u_i D^+D^-(k_ig_i) + \underbrace{D^+u_i\left(D^+k_ig_i\right)^+ - D^-u_i\left(D^-k_ig_i\right)^-}_{b_i} \\
&= D^+\left(u_iD^-(k_ig_i)\right) - D^+u_iD^-k_ig_i \\
&\qquad + D^+u_i\left(D^+k_ig_i\right)^+ - D^-u_i\left(D^-k_ig_i\right)^- \\
&= D^+\left(u_iD^-(k_ig_i)\right) + \Delta_i^+u_i\left(D^+k_ig_i\right)^- - D^-u_i\left(D^-k_ig_i\right)^- \\
&= D^+\left(u_iD^-(k_ig_i)\right) + \Delta y\, D^-\left(\left(D^+u_i\right)\left(D^+k_ig_i\right)^-\right). \qquad (4.4)
\end{aligned}
$$

Thus (4.1) reads

$$
\partial_t u_i + k_i\partial_x f(u_i) = D^+\left(u_iD^-(k_ig_i)\right) + \Delta y\, D^-\left(\left(D^+u_i\right)\left(D^+k_ig_i\right)^-\right) \qquad (4.5)
$$

for $i=1,\dots,N$, and we have the boundary values

$$
D^-(k_1g_1) = D^+(k_Ng_N) = 0. \qquad (4.6)
$$

Remark 4.3. Observe that the above term b_i is an upwind discretization of the transport term corresponding to au_y, with $a=(kg)_y$.

Similarly to (4.4), we also get the expression

$$
\frac{1}{\Delta y^2}\left(S_{i-1}-S_i\right) = D^-\left(u_iD^+\left(k_ig_i\right)\right) + \Delta y D^-\left(\left(D^+u_i\right)\left(D^+k_ig_i\right)^+\right). \qquad (4.7)
$$

Recall (3.3) with $\eta(u)=u^2/2$ and φ an approximation to $\mathbf{1}_{[0,T]}$. That gives

$$
\begin{aligned}
&\frac12\int_{\mathbb{R}}(u_i(x,T))^2\,dx \\
&\quad\le \frac12\int_{\mathbb{R}}(u_{i,0}(x))^2\,dx \\
&\qquad + \int_{\Pi_T}\left(u_iD^+\left(u_iD^-(k_ig_i)\right) + \Delta y\, u_i\, D^-\left(D^+u_i\left(D^+k_ig_i\right)^-\right)\right)dxdt,
\end{aligned}
$$

where $\Pi_T=[0,T]\times\mathbb{R}$. We can sum this for $i=1,\dots,N$, multiply with Δy, and do a summation by parts to get

$$
\begin{aligned}
&\frac12\Delta y\sum_{i=1}^N\int_{\mathbb{R}}(u_i(x,T))^2\,dx \qquad (4.8)\\
&+\Delta y\sum_{i=1}^N\int_{\Pi_T}u_iD^-(k_ig_i)D^-u_i\,dxdt + \Delta y^2\sum_{i=1}^N\int_{\Pi_T}\left(D^+k_ig_i\right)^-\left(D^+u_i\right)^2\,dxdt \\
&\qquad\qquad\le \frac12\Delta y\sum_{i=1}^N\int_{\mathbb{R}}(u_{i,0}(x))^2\,dx.
\end{aligned}
$$

It will be useful to lower bound the last two terms on the left-hand side.

Recall first that

$$
0\le u_i\le 1,\quad \left|D^+k_i\right|\le C,\quad\text{and}\quad \Delta y\sum_{i=1}^N\int_{\Pi_T}\left|D^+u_i\right|\,dxdt\le C \qquad (4.9)
$$

for some constant C independent of Δy. Using this and the fact that $\max_{u\in[0,1]}|g(u)|$ is bounded, as well as

$$\Delta y \left|D^{\pm} u_i\right| \leq C, \tag{4.10}$$

we have that

$$\begin{aligned}\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left|g_i D^+ k_i\right| \left(D^+ u_i\right)^2 \, dxdt &\leq C\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left(D^+ u_i\right)^2 \, dxdt \\ &\leq C\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|D^+ u_i\right| \, dxdt \leq C. \end{aligned} \tag{4.11}$$

Furthermore, note that the same argument yields

$$\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|u_i g_{i-1} (D^- k_i)(D^- u_i)\right| \, dxdt \leq C\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|D^- u_i\right| \, dxdt \leq C. \tag{4.12}$$

Observe that

$$D^+ k_i g_i = k_{i+1} D^+ g_i + g_i D^+ k_i,$$

and then use the inequality $(a+b)^- \geq a^- - |b|$. Thus, since $g' > 0$,

$$\begin{aligned}\left(D^+ k_i g_i\right)^- \left(D^+ u_i\right)^2 &\geq k_{i+1} \left(D^+ g_i\right)^- (D^+ u_i)^2 - \left|g_i D^+ k_i\right| \left(D^+ u_i\right)^2 \\ &\geq c \left((D^+ u_i)^-\right)^3 - \left|g_i D^+ k_i\right| \left(D^+ u_i\right)^2, \end{aligned}$$

where $0 < c \leq \min_i k_i \min_u g'(u)$. Similarly,

$$D^-(k_i g_i) = k_i D^- g_i + g_{i-1} D^- k_i,$$

and therefore,

$$u_i D^-(k_i g_i)(D^- u_i) \geq k_i u_i (D^- g_i)(D^- u_i) - \left|u_i g_{i-1}(D^- k_i)(D^- u_i)\right|.$$

Note that due to the monotonicity of g we have for some $\tilde{u}$ between u_i and u_{1-1},

$$k_i u_i D^- g_i D^- u_i = k_i u_i g'(\tilde{u})(D^- u_i)^2 \geq c u_i (D^- u_i)^2 \geq 0.$$

We can now estimate the last two terms of the left-hand side of (4.8) from below. More precisely,

$$\begin{aligned} c\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left((D^+ u_i)^-\right)^3 \, dxdt - \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left|g_i D^+ k_i\right| \left(D^+ u_i\right)^2 \, dxdt \\ + c\Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^- u_i\right)^2 \, dxdt - \Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|u_i g_{i-1} D^- k_i D^- u_i\right| \, dxdt \\ \leq \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left(D^+ k_i g_i\right)^- \left(D^+ u_i\right)^2 \, dxdt + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i D^-(k_i g_i) D^- u_i \, dxdt \\ \leq \frac{1}{2} \Delta y \sum_{i=1}^{N} \int_{\mathbb{R}} (u_i(x,T))^2 \, dx + \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left(D^+ k_i g_i\right)^- \left(\Delta_i^+ u_i\right)^2 \, dxdt \end{aligned}$$

$$
\begin{aligned}
&\quad + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i D^-(k_i g_i) D^- u_i \, dxdt \\
&\leq \frac{1}{2} \Delta y \sum_{i=1}^{N} \int_{\mathbb{R}} (u_{i,0}(x))^2 \, dx,
\end{aligned}
$$

which we can rewrite as

$$
\begin{aligned}
c\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left((D^+ u_i)^-\right)^3 \, dxdt &+ c\Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^- u_i\right)^2 \, dxdt \\
&\leq \frac{1}{2} \Delta y \sum_{i=1}^{N} \int_{\mathbb{R}} (u_{i,0}(x))^2 \, dx + \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left|g_i D^+ k_i\right| \left(D^+ u_i\right)^2 \, dxdt \\
&\quad + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|u_i g_{i-1} (D^- k_i)(D^- u_i)\right| \, dxdt \\
&\quad + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left|u_i g_{i-1} (D^- k_i)(D^- u_i)\right| \, dxdt \\
&\leq C,
\end{aligned}
$$

using (4.11) and (4.12).

This implies that

$$
\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left((D^+ u_i)^-\right)^3 \, dxdt \leq C \tag{4.13}
$$

and

$$
\Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^- u_i\right)^2 \, dxdt \leq C. \tag{4.14}
$$

Observe that by (4.10), (4.13) follows from (4.14), viz.

$$
\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left((D^+ u_i)^-\right)^3 \, dxdt \leq \Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^- u_i\right)^2 \, dxdt \leq C.
$$

By the same procedure, starting with (4.5) but using the alternate form (4.7) of the right-hand side, we arrive at the bounds

$$
\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left((D^+ u_i)^+\right)^3 \, dxdt \leq C \tag{4.15}
$$

and

$$
\Delta y \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^+ u_i\right)^2 \, dxdt \leq C. \tag{4.16}
$$

Combining the two bounds (4.13) and (4.15) we get

$$
\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left|D^+ u_i\right|^3 \, dxdt \leq C. \tag{4.17}
$$

In a similar manner, we find

$$\Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left|D^- u_i\right|^3 \, dxdt \le C. \tag{4.18}$$

The other two bounds, (4.14) and (4.16), can be used for a continuity estimate. Write $u_{i-1/2} = (u_i + u_{i-1})/2$ and compute for $\ell \ge m$

$$\begin{aligned}
\frac{1}{2}\left|u_\ell^2 - u_m^2\right| &= \frac{\Delta y}{2}\left|\sum_{i=m+1}^{\ell} D^- u_i^2\right| \\
&= \Delta y \left|\sum_{i=m+1}^{\ell} u_{i-1/2} D^- u_i\right| \\
&\le \left(\Delta y \sum_{i=m+1}^{\ell} u_{i-1/2}\right)^{1/2} \left(\Delta y \sum_{i=m+1}^{\ell} u_{i-1/2}\left(D^- u_i\right)^2\right)^{1/2} \\
&\le \sqrt{\Delta y(\ell - m)} \left(\frac{\Delta y}{2} \sum_{i=1}^{N} u_i \left((D^- u_i)^2 + (D^+ u_i)^2\right)\right)^{1/2}.
\end{aligned}$$

Squaring and integrating over $[0,T] \times \mathbb{R}$ gives

$$\int_{\Pi_T} \left(u_\ell^2 - u_m^2\right)^2 \, dxdt \le C(\ell - m)\Delta y, \quad \ell \ge m. \tag{4.19}$$

By direct computations we have that

$$\frac{1}{2} D^- u_i^2 = u_{i-1/2} D^- u_i = u_i D^- u_i - \frac{\Delta y}{2}\left(D^- u_i\right)^2,$$

which gives

$$\begin{aligned}
\left(u_i D^- u_i\right)^2 &= \frac{1}{4}\left(D^- u_i^2\right)^2 + \Delta y u_i \left(D^- u_i\right)^3 - \frac{\Delta y^2}{4}\left(D^- u_i\right)^4 \\
&\le \frac{1}{4}\left(D^- u_i^2\right)^2 + \Delta y \left|D^- u_i\right|^3.
\end{aligned}$$

Multiplying with Δy, summing over i, and integrating in x, t gives the bound, using (4.19) with $m = i-1$, $\ell = i$, and (4.18),

$$\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left(u_i D^- u_i\right)^2 \, dxdt \le C. \tag{4.20}$$

Note that this also follows from (4.14), using that $u_i \in [0,1]$.

Convergence. We assume that $u_0 \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ is such that $0 \le u_0(x,y) \le 1$ and that $u_0 \in L^1 \cap BV$. Now we assume that the initial data $u_{i,0}$ is such that there is a function $u_0(x,y)$ such that

$$u_{i,0}(x) = \frac{1}{\Delta y} \int_{y_{i-1/2}}^{y_{i+1/2}} u_0(x,y) \, dy \in L^1(\mathbb{R}) \quad \text{for } i = 1, \ldots, N, \tag{4.21}$$

where $\Delta y = 1/N$ and $y_{i-1/2} = (i-1)\Delta y$. Furthermore, $0 \le u_0(x,y) \le 1$. Since $u_0 \in BV(\mathbb{R}\times[0,1])$,

$$\Delta y \sum_{i=1}^{N} |u_{i,0}|_{BV(\mathbb{R})} + \Delta y \int_{\mathbb{R}} \sum_{i=1}^{N} \left|D^{\pm} u_{i,0}\right| \, dx \le C$$

for some constant C which is independent of Δy. For convenience, we have set $u_{0,0} = u_{0,1}$ and $u_{0,N+1} = u_{0,N}$.

We assume that $k \in C^1([0,1])$ is given, such that $k'(0) = k'(1) = 0$, and $k(y) > 0$ for $y \in [0,1]$. Define $k_i = k(y_i)$. Let $u_i(t,x)$ be the entropy solutions to (4.5) with the boundary conditions

$$D^- k_1 = D^+ k_N = 0, \quad D^- u_1 = D^+ u_N = 0,$$

which actually is a special case of (4.6). Then we define

$$u_{\Delta y}(t,x,y) = u_i(t,x) \quad \text{for } y \in [y_{i-1/2}, y_{i+1/2}),$$

for $i = 1,\dots,N-1$ and $u_{\Delta y}(t,x,y) = u_N(t,x)$ if $y \in [y_{N-1/2}, 1]$. We have that $0 \le u_{\Delta y}(t,x,y) \le 1$, $\|u_{\Delta y}(t,\,\cdot\,,\,\cdot\,)\|_{L^1(\mathbb{R}\times[0,1])} = \|u_0\|_{L^1(\mathbb{R}\times[0,1])}$, and, using the bounds (3.7) and (3.8), $\|u_{\Delta y}(t,\,\cdot\,,\,\cdot\,)\|_{BV(\mathbb{R}\times[0,1])} \le C$, where C is independent of Δy. Furthermore, using (4.19),

$$\|u_{\Delta y}(t,\,\cdot\,,\,\cdot\,) - u_{\Delta y}(s,\,\cdot\,,\,\cdot\,)\|_{L^1(\mathbb{R}\times[0,1])} \le C\,|t-s|\,,$$

where C is independent of Δy. This is sufficient to conclude that there are a function $u \in C([0,\infty); L^1(\mathbb{R}\times[0,1]))$ and a sequence $\{\Delta y_j\}_{j=0}^{\infty}$, $\Delta y_j \to 0$ as $j \to \infty$, such that

$$u = \lim_{j\to\infty} u_{\Delta y_j} \quad \text{in } C([0,\infty); L^1(\mathbb{R}\times[0,1])).$$

Furthermore, we have that $D^- u_{\Delta y_j} \rightharpoonup u_y$; therefore, $u_{\Delta y_j} D^- u_{\Delta y_j} \rightharpoonup u u_y$. The bound (4.20) ensures that $u u_y \in L^2([0,T]\times\mathbb{R}\times[0,1])$.

The aim is now to show that the limit u is a weak solution in the above sense. Since u_i is a weak solution of (4.5), we have

$$\int_{\Pi_T} \left(u_i \varphi_t + k_i f_i \varphi_x - D^+\left((D^- k_i) f_i\right)\varphi\right) dx dt \tag{4.22}$$

$$= -\int_{\Pi_T} D^+\left(k_{i-1} u_i D^- g_i\right) \varphi \, dx dt \tag{4.23}$$

$$- \Delta y \int_{\Pi_T} D^-\left(D^+ u_i \left(D^+ k_i g_i\right)^-\right) \varphi \, dx dt \tag{4.24}$$

$$+ \int_{\mathbb{R}} u_i(T,x)\varphi(T,x)\, dx dy - \int_{\mathbb{R}} u_{i,0}\varphi(0,x)\, dx \tag{4.25}$$

for $i = 1,\dots,N$. We use $\varphi = \varphi^i$, where

$$\varphi^i(t,x) = \frac{1}{\Delta y}\int_{y_{i-1/2}}^{y_{i+1/2}} \varphi(t,x,y)\, dy$$

for a suitable test function φ. Next, we multiply with Δy and sum over $i = 1,\dots,N$ and do a summation by parts on the terms which have $D^{\pm}(\cdots)$. This will give us the weak formulation for $u_{\Delta y}$. For simplicity we assume that $\Delta y = \Delta y_j$, so that the whole sequence converges. Term by term we get

$$\begin{aligned}\Delta y \sum_{i=1}^{N} (4.22) &= \Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left(u_i \varphi_t^i + k_i f_i \varphi_x^i + (D^- k_i) f_i D^- \varphi^i \right) \, dxdt \\ &\longrightarrow \int_{\Pi_T} \Big(\int_0^1 u\varphi_t + kf\varphi_x + k' f \varphi_y \, dy \Big) \, dxdt\end{aligned}$$

as $\Delta y \to 0$.

Turning to (4.23), we have that

$$D^- g_i = g'(\tilde{u}_{i-1/2}) D^- u_i = g'(u_i) D^- u_i + g''(\xi_{i-1/2})(u_i - \tilde{u}_{i-1/2}) D^- u_i,$$

where $\tilde{u}_{i-1/2}$ is between u_i and u_{i-1} and $\xi_{i-1/2}$ is between u_i and $\tilde{u}_{i-1/2}$. Therefore,

$$\begin{aligned}\Delta y \sum_{i=1}^{N} (4.23) &= \Delta y \sum_{i=1}^{N} \int_{\Pi_T} k_{i-1} u_i g'(u_i) D^- u_i D^- \varphi^i \, dxdt \\ &\quad + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} k_{i-1} u_i g''(\xi_{i-1/2}) \left(u_i - \tilde{u}_{i-1/2} \right) D^- u_i D^- \varphi^i \, dxdt. \qquad (4.26)\end{aligned}$$

The last term here vanishes as $\Delta y \to 0$ since

$$\begin{aligned}|(4.26)| &\le C \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} u_i \left(D^- u_i \right)^2 \left| D^- \varphi^i \right| \, dxdt \\ &\le C \Delta y,\end{aligned}$$

where we used (4.14). Hence

$$\Delta y \sum_{i=1}^{N} (4.23) \longrightarrow \int_{\Omega_T} k u g'(u) u_y \varphi_y \, dydxdt$$

as $\Delta y \to 0$.

Now for (4.24), we have

$$\begin{aligned}\Delta y \left| \sum_{i=1}^{N} (4.24) \right| &\le \Delta y^2 \sum_{i=1}^{N} \int_{\Pi_T} \left| \Delta_i^+ u_i \right| \left(k_{i-1} \left| D^+ g_i \right| + |g_i| \left| D^+ k_i \right| \right) \left| D^+ \varphi^i \right| \, dxdt \\ &\le C \Delta y \left(\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left(D^+ u_i \right)^2 \, dxdt + \Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left| D^+ u_i \right| \, dxdt \right) \\ &\le C \Delta y \left(\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left| D^+ u_i \right| \, dxdt \right)^{1/2} \left(\Delta y \sum_{i=1}^{N} \int_{\Pi_T} \left| D^+ u_i \right|^3 \, dxdt \right)^{1/2} \\ &\qquad + C \Delta y \\ &\le C \Delta y \left(\frac{1}{\sqrt{\Delta y}} + 1 \right),\end{aligned}$$

using (4.9), (4.17), and interpolation between L^1 and L^3. Thus $\Delta y \left| \sum_{i=1}^{N} (4.24) \right| \to 0$ as $\Delta y \to 0$.

It is straightforward to show that

$$\Delta y \sum_{i=1}^{N} (4.25) \;\longrightarrow\; \int_\Omega u(T,x,y)\varphi(T,x,y)\,dydx - \int_\Omega u_0(x,y)\varphi(0,x,y)\,dydx.$$

Hence, the limit u is a weak solution. ∎

Remark 4.4. The arguments in this section would also hold if one replaced the conservation laws on the left side of (3.1) with a numerical scheme for a scalar conservation law using a monotone numerical flux function consistent with $u_i v_i(u_i)$. Then one could derive the analogous bounds to show the convergence to a weak solution as was done here.

4.1. An example. To illustrate the continuum limit, we have tested the "same" initial value problem as in sections 2.1 and 3.1. The relevant data are

$$u_0(x,y) = \sin^2(\pi x/2), \quad x \in \mathbb{R},\ y \in (0,1),$$

and

$$k(y) = 1 + 2y, \quad y \in (0,1), \quad v(y,u) = k(y)(1-u). \tag{4.27}$$

We have used $\Delta y = 1/60$ (i.e., 60 lanes) and solved (4.1) using the Engquist–Osher scheme with 800 grid points and periodic boundary conditions in the interval $[0,2]$. Figure 3 shows the computed density u at four different times. We observe that Figures 1 and 2 are "discretizations" of the same problem. To obtain Figure 1 we used $N = 2$, and to obtain Figure 2 we used $N = 8$. Hence all three figures can be viewed as picturing approximations of the same function.

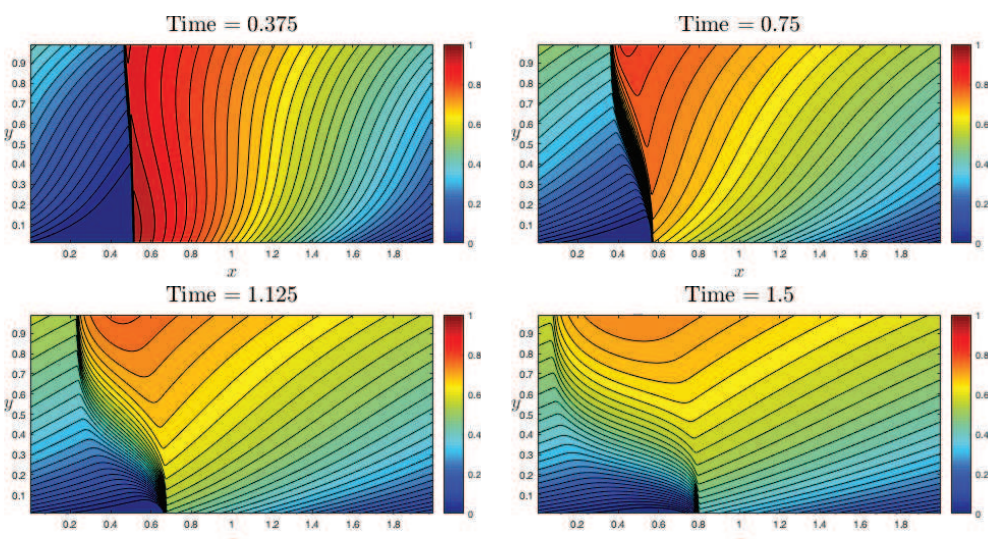

FIG. 3. *The solution of* (4.1) *with* $N = 60$*, and* $v(y,u)$ *given by* (4.27)*. Upper left:* $t = 0.375$*. Upper right:* $t = 0.75$*. Lower left:* $t = 1.125$*. Lower right:* $t = 1.5$.

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